Derivation of Vieta's theorem for solving quadratic equations

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February 9 2025

1 Introduction

If you try to find how to solve quadratic equations via Vieta's theorem, you probably won't find anything but brute-force solutions. But in this paper we will derive from the theorem an elegant formula for finding roots.

2 Derivation

We have a system of equations like this

$$\begin{cases} x_1 + x_2 = d \\ x_1 x_2 = h \end{cases} \tag{1}$$

We make a system with x and y, where $x = x_1$ and $y = x_2^{-1}$, and then we have a system like this

$$\begin{cases} x + \frac{1}{y} = d \\ \frac{x}{y} = h \end{cases} \tag{2}$$

Express x through y and h

$$x = hy \tag{3}$$

And replace x in the first equation with its expression

$$hy + y = d \tag{4}$$

Now we can solve the equation to find y and we get

$$y = \frac{d}{h+1} \tag{5}$$

And now we can find the formula to find x_1

$$x_1 = y^{-1} = \frac{h+1}{d} \tag{6}$$

Now x_2 can be expressed as

$$x_2 = d - x_1 \text{ or } \frac{h}{x_1} \tag{7}$$

We'll take the formula with h divided by x_1.
Now we can shorten the formulas by replacing d and h with $-\frac{b}{a}$ and $\frac{c}{a}$

$$x_1 = \frac{\frac{c}{a}+1}{-\frac{b}{a}} = (\frac{c+a}{a})(-\frac{a}{b}) = -\frac{c+a}{b}$$
$$x_2 = \frac{\frac{c}{a}}{-\frac{c+a}{b}} = (\frac{c}{a})(-\frac{b}{c+a}) = -\frac{bc}{a(c+a)}$$

Thus we have obtained an elegant formula for finding roots in a quadratic equation that looks like this

$$x_1 = -\frac{c+a}{b}$$
$$x_2 = -\frac{bc}{a(c+a)}$$
$$\textbf{Q.E.D} \tag{8}$$