ME 211A: Quadratic Transformation

September 17, 2018

quadratic transformation

We consider the transformation $\underline{x} = \underline{\Phi}(\underline{X}, t)$ defined by:

$$x_1 = X_1 + 0.1X_2^2 + 10 \tag{1}$$
$$x_2 = \beta X_1^2 + X_2 + 10 \tag{2}$$
$$x_3 = X_3 \tag{3}$$

Question: Calculate the gradient of the transformation?

Question: Make a graphical representation of the reference configuration and of the deformed configuration for some specific values ($\beta = 0$, $\beta = 0.025$, ...)?

Question: What is the value of the local variation of volume?

Question: What is the maximum admissible value for β so the transformation has any physycal sense?

Answer: The gradient of transformation?

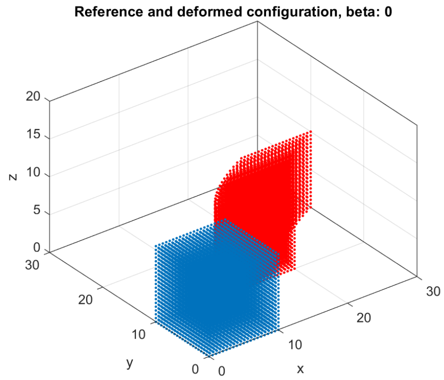
$$\underline{\underline{F}} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \frac{\partial \underline{x}}{\partial \underline{X}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0.2x_2 & 0 \\ 2\beta x_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer: We define the original shape as a $10 \times 10 \times 10mm^3$ cube. Fig. 1 (a) gives the original and deformed shape with $\beta = 0$. Since the coordinate in the third direction is not changed, we can see it as a 2D deformation problem and make the comparison more clear, like in Fig. 1 (b). Fig. 2 (a)-(d) shows the reference and deformed configuration with $\beta = 0, 0.01, 0.025, 0.05$ respectively.

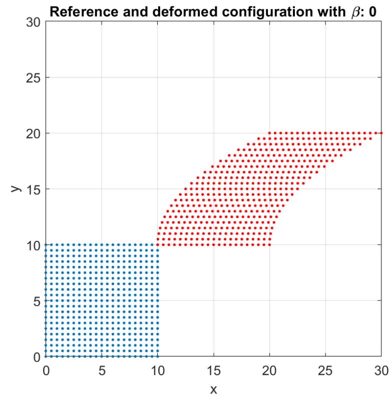
Answer: The local variation of volume is the determinant of $\underline{\underline{F}}$ at every point of the reference configuration. Fig. 3 (a)-(d) show the local variation of volume with $\beta = 0, 0.01, 0.025, 0.05$ respectively.

With $\beta = 0$, the determinant of $\underline{\underline{F}} = 1$, thus the volume will not change.

With $\beta = 0.025$, the determinant of $\underline{\underline{F}} = 0$ at the edge point $(10, 10)$.

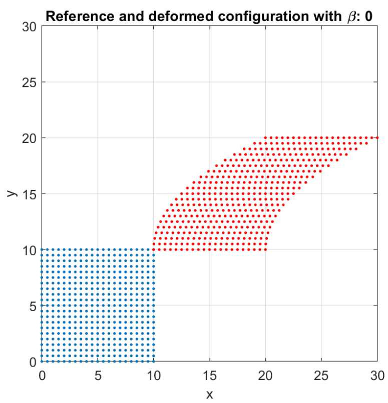


(a) $\beta = 0$

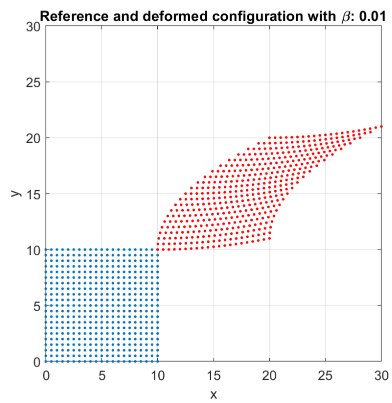


(b) $\beta = 0.01$

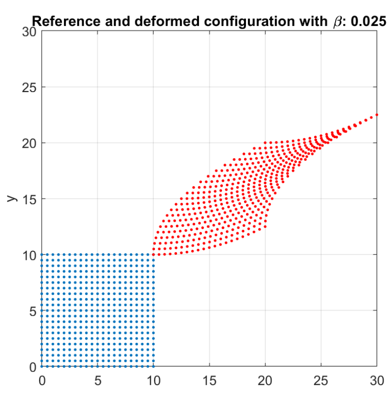
Figure 1: 3D and 2D plot of original and deformed shape at $\beta = 0$.



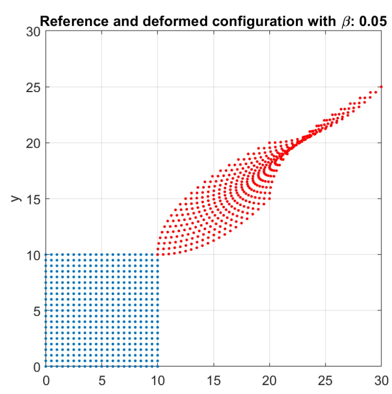
(a) $\beta = 0$



(b) $\beta = 0.01$

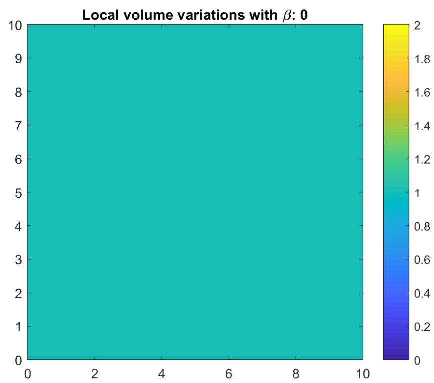


(c) $\beta = 0.025$

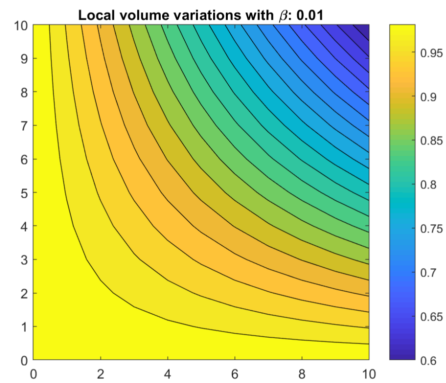


(d) $\beta = 0.05$

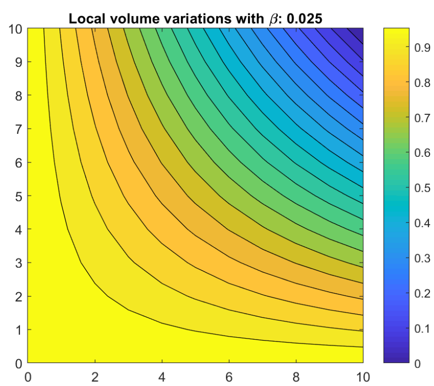
Figure 2: 2D (x-y) plot of original and deformed shape.



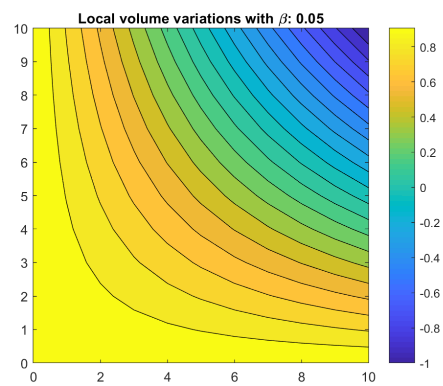
(a) $\beta = 0$



(b) $\beta = 0.01$



(c) $\beta = 0.025$



(d) $\beta = 0.05$

Figure 3: Local variation of volume with different β.

Answer: To make the transformation physical sounds, the determinant of $\underline{\underline{F}}$ should be larger than zero. Otherwise, the deformed volume will be negative, which doesn't make sense.

$$\left|\underline{\underline{F}}\right| = 1 - 0.4\beta x_1 x_2 > 0 \tag{4}$$

Thus,

$$\beta x_1 x_2 < 2.5 \tag{5}$$

and also, $\max\left(x_1 x_2\right) = 100$ in our case. So β should satisfy:

$$\beta < 0.025 \tag{6}$$

That's why when choose $\beta = 0.05$, there is some region that the local variation of volume is less than zero.