Conditions for small perturbations

We consider the transformation $\underline{x} = \underline{\Phi}(\underline{X}, t)$ defined by:

$$
\begin{aligned}
x_1 &= X_1 + X_2 && (1)\\
x_2 &= X_2 && (2)\\
x_3 &= X_3 && (3)
\end{aligned}
$$

Question 1: Make a graphical representation of the reference configuration and of the deformed configuration. Calculate the gradient of the transformation, $\underline{\underline{F}}$.

Question 2: Is this an homogenous transformation? It is acceptable from a physical point of view?

Question 3: Calculate the expansion of the following vectors: $\underline{e}_1$, $\underline{e}_2$ and $\frac{\underline{e}_1+\underline{e}_2}{\sqrt{2}}$

Question 4: Calculate $\underline{\underline{e}}$ and $\underline{\underline{\varepsilon}}$? Can we consider we are in small perturbations?

Question 5: Consider the transformation $x_1 = X_1 + \alpha \cdot X_2$, $x_2 = X_2$ and $x_3 = X_3$. What is the condition for having small perturbations?

Solution to "Conditions for small pertubations"

Question 1: Make a graphical representation of the reference configuration and of the deformed configuration. Calculate the gradient of the transformation, $\underline{\underline{F}}$.

Answer:

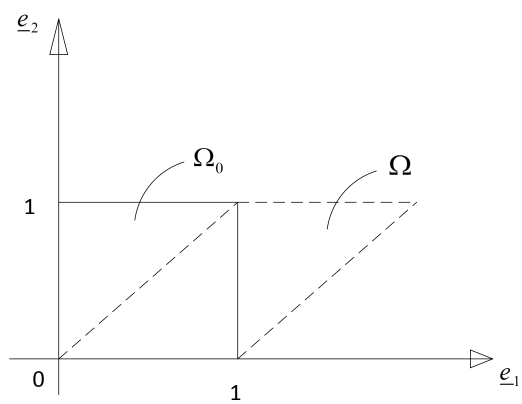

Figure 1: Original and deformed configuration

The gradient of the transformation $\underline{\underline{F}}$ also known as deformation gradient can be calculated as follows.

$$\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \frac{\partial \underline{\phi}}{\partial \underline{X}} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

Question 2: Is this an homogenous transformation? It is acceptable from a physical point of view?

Answer: This is a homogeneous transformation since $\underline{\underline{F}}$ does not depend on $\underline{X}$ which means $\underline{\underline{F}}$ is a constant everywhere (homogeneous field). The determinant of deformation gradient $\underline{\underline{F}}$ has a physical significance i.e. the ratio between the volume after deformation to initial volume as shown below

$$|\Omega| = J \quad |\Omega_0| \tag{5}$$

where

$$J = det\underline{\underline{F}} \tag{6}$$

In this particular case, $detF = 1$ which means that there is no volume change (no volume variation). And yes, it is acceptable from a physical point of view to have a constant volume. The transformation is not physically accepted if the $detF \leq 0$.

Question 3: Calculate the expansion of the following vectors: $\underline{V}_1 = \underline{e}_1$, $\underline{V}_2 = \underline{e}_2$ and $\underline{V}_3 = \frac{\underline{e}_1+\underline{e}_2}{\sqrt{2}}$

Answer: Let say $\underline{v}$ is the vector after transformation and $\underline{V}$ is undeformed vector. By definition,

$$\underline{v} \cdot \underline{v} = \underline{V} \cdot \underline{\underline{C}} \cdot \underline{V} \quad \forall \underline{V} \rightarrow \underline{v} \tag{7}$$

Then the expansion can be calculated as follow:

$$\frac{|\underline{v}|}{|\underline{V}|} = \frac{\sqrt{\underline{V} \cdot \underline{\underline{C}} \cdot \underline{V}}}{\sqrt{\underline{V} \cdot \underline{V}}} = \lambda_{\underline{V}} \tag{8}$$

with

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

For $\underline{V}_1 = \underline{e}_1$, then $\lambda_{\underline{V}_1} = \sqrt{\frac{\underline{e}_1 \cdot \underline{\underline{C}} \underline{e}_1}{\underline{e}_1 \cdot \underline{e}_1}} = 1 \rightarrow$ No length change.

For $\underline{V}_2 = \underline{e}_2$, then $\lambda_{\underline{V}_2} = \sqrt{\frac{\underline{e}_2 \cdot \underline{\underline{C}} \underline{e}_2}{\underline{e}_2 \cdot \underline{e}_2}} = \sqrt{2} \rightarrow$ The length changes $\sqrt{2}$ times the initial length.

For $\underline{V}_3 = \frac{\underline{e}_1+\underline{e}_2}{\sqrt{2}}$, then $\lambda_{\underline{V}_3} = \sqrt{\frac{\underline{e}_3 \cdot \underline{\underline{C}} \underline{e}_3}{\underline{e}_3 \cdot \underline{e}_3}} = \sqrt{\frac{5}{2}} \rightarrow$ The length changes $\sqrt{\frac{5}{2}}$ times the initial length.

Question 4: Calculate $\underline{\underline{e}}$ and $\underline{\underline{\varepsilon}}$? Can we consider we are in small perturbations?

Answer: The Green-Lagrange strain tensor can be calculated as follows

$$\underline{\underline{e}} = \frac{1}{2}\{\underline{\underline{C}} - \underline{\underline{1}}\} = \frac{1}{2}(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{1}}) = \frac{1}{2}\left(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

The linearized strain $\underline{\underline{\varepsilon}}$ can be calculated as follows

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{\nabla}}\underline{\xi} + \underline{\underline{\nabla}}^T \underline{\xi}) \tag{11}$$

where $\underline{\xi}$ the displacement vector that can be calculated as follows

$$\underline{x} = \underline{X} + \underline{\xi} \tag{12}$$

$$\underline{\xi} = \underline{x} - \underline{X} = \begin{pmatrix} X_2 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

$$\underline{\underline{\nabla}}\xi = \frac{\partial \xi_i}{\partial X_j} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

Finally,

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{\nabla}\xi} + \underline{\underline{\nabla}}^T\underline{\xi}) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

By comparing $\underline{\underline{\varepsilon}}$ and $\underline{\underline{e}}$, since $\underline{\underline{\varepsilon}} \neq \underline{\underline{e}}$, then this is not a small perturbation case.

Question 5: Consider the transformation $x_1 = X_1 + \alpha \cdot X_2$, $x_2 = X_2$ and $x_3 = X_3$. What is the condition for having small perturbations?

Answer: Note that the transformation above has the same pattern as previous transformation. The only difference is that now $x_1 = X_1 + \alpha X_2$ instead of $x_1 = X_1 + X_2$. Therefore, we can the displacement vector $\underline{\xi}$ as below

$$\underline{\xi} = \begin{pmatrix} \alpha X_2 \\ 0 \\ 0 \end{pmatrix} \tag{16}$$

and

$$\underline{\underline{\nabla}\xi} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

Then to be small perturbation, we should have $||\underline{\underline{\nabla}\xi}|| = \sqrt{\alpha^2} << 1$. In this situation,

$$\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{\nabla}\xi} + \underline{\underline{\nabla}}^T\underline{\xi}) + \frac{1}{2}\underline{\underline{\nabla}}^T\underline{\underline{\nabla}\xi} \tag{18}$$

where the first term is small strain term and the second term is second order.

$$\underline{\underline{e}} = \frac{1}{2}\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$