Practice:
Tube under internal pressure

Let us consider a circular tube with height h, and internal radius r, and thickness e. The tube is imposed to internal pressure p (see Fig. 1). Note that gravity and atmospheric pressure are neglected.

Figure 1: Circular tube under internal pressure

Question 1: Please verify that a static solution exist! Assume that $\underline{\underline{\sigma}}$ only depends on r (the polar radius)

Answer: Existence $\rightarrow$ the structure should be under global equilibrium. (We will come back on this when we discuss about variational formulations)

$$\underline{R}_{ext\rightarrow\Omega} = \underline{0} \quad \text{sum of external forces} \tag{1}$$

$$\underline{M}_{ext\rightarrow\Omega} = \underline{0} \quad \text{sum of external moments} \tag{2}$$

$$
\begin{aligned}
\underline{R}_{ext\to\Omega} &= \int_{S_{int}} \underline{\underline{\sigma}} \cdot (-\underline{e}_r) dS = \int_{S_{int}} p\underline{e}_r dS = p\int_{z=0}^{z=h}\int_{\theta=0}^{\theta=2\pi} \underline{e}_r R d\theta dz \\
&= pRh \underbrace{\int_0^{2\pi} \underline{e}_r d\theta}_{\underline{0}} = \underline{0}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\underline{M}_{ext\to\Omega} &= \int_{S_{int}} \underline{OM} \times \underline{\underline{\sigma}}(-\underline{e}_r) dS = \int_{z=0}^{z=h}\int_{\theta=0}^{\theta=2\pi} (z\underline{e}_z + R\underline{e}_r) \times (p\underline{e}_r) R d\theta dz \\
&= pR\int_{z=0}^{z=h}\int_{\theta=0}^{\theta=2\pi} z\underline{e}_\theta d\theta dz = pR\int_{z=0}^{z=h} z \underbrace{\left(\int_{\theta=0}^{\theta=2\pi} \underline{e}_\theta d\theta\right)}_{\underline{0}} dz = \underline{0}
\end{aligned} \tag{4}
$$

Question 2: Find all the local equations satisfied by $\underline{\underline{\sigma}}(r)$ that define the static admissibility? Prove that $\sigma_{r\theta} = 0$ with the assumption $\underline{\underline{\sigma}}(r)$?

Answer:
Internal equilibrium:

$$
\underline{\mathrm{div}}\ \underline{\underline{\sigma}} = \underline{0}\ \forall M \in \Omega \tag{5}
$$

Boundary condition:

$$
\underline{\underline{\sigma}}(r) = \begin{bmatrix} \sigma_{rr}(r) & \sigma_{r\theta}(r) & \sigma_{rz}(r) \\ \sigma_{r\theta}(r) & \sigma_{\theta\theta}(r) & \sigma_{\theta z}(r) \\ \sigma_{rz}(r) & \sigma_{\theta z}(r) & \sigma_{zz}(r) \end{bmatrix} \tag{6}
$$

At upper surface S_{up}:

$$
\underline{\underline{\sigma}} \cdot \underline{n}_{|S_{up}} = \underline{\underline{\sigma}} \cdot (\underline{e}_z)_{|z=h} = \underline{0} \tag{7}
$$

$$
\sigma_{rz}(r) = \sigma_{\theta z}(r) = \sigma_{zz}(r) = 0\ \forall\ r \in [R, R+e] \tag{8}
$$

At the lower surface S_{lo}:

$$
\underline{\underline{\sigma}} \cdot \underline{n}_{|S_{lo}} = \underline{\underline{\sigma}} \cdot (-\underline{e}_z)_{|z=0} = \underline{0} \tag{9}
$$

$$
\sigma_{rz}(r) = \sigma_{\theta z}(r) = \sigma_{zz}(r) = 0\ \forall\ r \in [R, R+e] \tag{10}
$$

At the external surface of tube wall S_{ext}:

$$
\underline{\underline{\sigma}} \cdot \underline{n}_{|r=R+e} = \underline{\underline{\sigma}} \cdot (\underline{e}_r)_{|r=R+e} = \underline{0} \tag{11}
$$

$$
\sigma_{rr}(R+e) = \sigma_{r\theta}(R+e) = \sigma_{rz}(R+e) = 0\ \forall\ r \in [R, R+e] \tag{12}
$$

At the internal surface of tube wall S_{int}:

$$\underline{\underline{\sigma}} \cdot \underline{n}_{|r=R} = \underline{\underline{\sigma}} \cdot (-\underline{e}_r)_{|r=R} = p\underline{e}_r \tag{13}$$

$$\begin{aligned} \sigma_{rr}(R) &= -p \\ \sigma_{r\theta}(R) = \sigma_{rz}(R) &= 0 \end{aligned} \tag{14}$$

And therefore,

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Rewriting $\underline{\text{div}}\ \underline{\underline{\sigma}} = \underline{0}$,

$$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{16}$$

$$\sigma_{r\theta,r} + 2\frac{\sigma_{r\theta}}{r} = 0 \tag{17}$$

$$0 = 0 \tag{18}$$

From Eq. 17,

$$\frac{1}{r}(r^2\sigma_{r\theta})_{,r} = 0 \rightarrow \sigma_{r\theta} = \frac{K_1}{r} \tag{19}$$

But as obtained previously, $\sigma_{r\theta}(R+e) = 0$ and $\sigma_{r\theta}(R) = 0$. So that $K_1 = 0$ and consequently $\sigma_{r\theta} = 0$.