Practice:
Static and Kinematic Admissibility

Problem 2

Let us consider a sphere with cavity (exterior radius R_e and internal radius R_i). The external surface S_e is free of loading and the interior surface S_i is submitted to a pressure p_0. The material is assumed to be isotropic linear elastic with young modulus E and Poisson ratio ν.

Question 1: Write the static admissibility equations!

Answer: Find $\underline{\underline{\sigma}}$ symmetric such that

$$
\begin{aligned}
&\text{(a) } \underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \quad \forall\, M \in S_e \quad \text{Boundary equation} \\
&\text{(b) } \underline{\underline{\sigma}} \cdot \underline{n} = -p_0\underline{n} \quad \forall\, M \in S_i \quad \text{Boundary equation} \\
&\text{(c) } \underline{\mathrm{div}\underline{\underline{\sigma}}} = \underline{0} \quad \forall\, M \in \Omega \quad \text{Internal equilibrium}
\end{aligned} \tag{1}
$$

Question 2: Please expand this equation using spherical coordinates system!

Answer: In spherical coordinates,

$$
\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\varphi} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta\varphi} \\ \sigma_{r\varphi} & \sigma_{\theta\varphi} & \sigma_{\varphi\varphi} \end{bmatrix} \tag{2}
$$

From Eq. (a),

$$
\sigma_{rr}(r = R_e, \theta, \varphi) = \sigma_{r\theta}(r = R_e, \theta, \varphi) = \sigma_{r\varphi}(r = R_e, \theta, \varphi) = 0 \tag{3}
$$

From Eq. (b),

$$
\begin{aligned}
&-\sigma_{rr}(r = R_i, \theta, \varphi) = p_0 \\
&-\sigma_{r\theta}(r = R_i, \theta, \varphi) = 0 \\
&-\sigma_{r\varphi}(r = R_i, \theta, \varphi) = 0
\end{aligned} \tag{4}
$$

From Eq. (c) $\rightarrow$ See Textbook equation 1.11a (divergence with spherical coordinates)

Question 3: Let's assume that the following stress tensor

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{bmatrix} \tag{5}$$

with

$$\sigma_{rr} = -A\left[\frac{B}{r^3} - 1\right] \tag{6}$$

and

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = A\left[\frac{B}{2r^3} + 1\right] \tag{7}$$

Please calculate the coefficient A and B so that $\underline{\underline{\sigma}}$ could be a solution of the problem!

Answer: Using the proposed stress tensor, and expanding equation (c), we get

$$\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = 0 \\ &0 = 0 \\ &0 = 0 \end{aligned} \tag{8}$$

We now check if $\underline{\underline{\sigma}}$ ca fulfill the static admissibility equations.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = \frac{3AB}{r^4} - \frac{3AB}{r^4} = 0 \rightarrow \text{(c) is OK. Interior equilibrium is satisfied.} \tag{9}$$

$$\text{(a)} \rightarrow -A\left[\frac{B}{R_e^3} - 1\right] = 0 \rightarrow B = R_e^3 \tag{10}$$

$$\text{(b)} \rightarrow -A\left[\frac{R_i^3}{R_e^3} - 1\right] = p_0 \rightarrow A = p_0 \frac{R_e^3}{R_i^3 - R_e^3} \tag{11}$$

All the other conditions concerning $\sigma_{\theta r}$ and $\sigma_{r\varphi}$ are automatically satisfied due to the chosen stress tensor. Thus, a statically admissible stress field is:

$$\underline{\underline{\sigma}} = \begin{bmatrix} p_0 \frac{R_e^3}{R_i^3 - R_e^3}\left[\frac{R_i^3}{r} - 1\right] & 0 & 0 \\ 0 & -p_0 \frac{R_e^3}{R_e^3 - R_i^3}\left[\frac{R_e^3}{2r^3} + 1\right] & 0 \\ 0 & 0 & -p_0 \frac{R_e^3}{R_e^3 - R_i^3}\left[\frac{R_e^3}{2r^3} + 1\right] \end{bmatrix} \tag{12}$$

Question 4: Please calculate the strain tensor $\underline{\underline{\varepsilon}}$? Does it satisfy the compatibility equation? Conclusion?

Answer: Based on the constitutive equation, we can calculate the strains:

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}\text{tr}\underline{\underline{\sigma}}\underline{\underline{1}} \tag{13}$$

$$\text{tr}\underline{\underline{\sigma}} = -p_0\frac{R_e^3}{R_e^3 - R_i^3} \tag{14}$$

Thus, we get $\varepsilon_{r\theta} = \varepsilon_{\theta\varphi} = \varepsilon_{\varphi\theta} = 0$. Let us note that

$$A_1 = p_0\frac{R_e^3}{R_e^3 - R_i^3} \tag{15}$$

We get:

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{r} - 1\right] + \frac{\nu}{E}A_1 & 0 & 0 \\ 0 & -\left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{2r^3} + 1\right] + \frac{\nu}{E}A_1 & 0 \\ 0 & 0 & -\left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{2r^3} + 1\right] + \frac{\nu}{E}A_1 \end{bmatrix} \tag{16}$$

In case of complete spherical symmetry, the compatibility equation can be simplified (see Textbook equation 2.14b):

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} + \frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = 0 \tag{17}$$

Let us verify if this equation is satisfied by $\underline{\underline{\varepsilon}}$.

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} = 3A_1\left(\frac{1+\nu}{E}\right)\frac{R_e^3}{2r^4} \tag{18}$$

$$\frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = -3A_1\left(\frac{1+\nu}{E}\right)\frac{R_e^3}{2r^4} \tag{19}$$

The two equations above satisfied the compatibility equation. Thus it will be possible to integrate $\underline{\varepsilon}$ to determine a displacement field $\underline{u}$. Considering there is no additional kinematic condition (no kinematic boundary condition), we got the exact solution of the problem.

Problem 3

Let us consider a domain Ω, simply supported by a rigid basis (Γ). Ω is submitted to gravity volume forces $\underline{f} = \rho g \underline{y}$ and to uniform pressure $p > 0$ over every vertical side. We assume plane stresses in the plane of $x - y$ and the material is isotropic linear elastic with Young modulus E and Poisson ratio ν.

Question 1: Write down the full problem! (kinematic, static and constitutive equations)

Question 2: We assume an Airy function $\phi(x, y) = ax^2 + cy^2 + hy^3$. Find a, c, h so we define a statistically admissible stress field.

Question 3: Find the related strain field using the constitutive equation. Does that satisfy the compatibility equation? Conclusion?

Question 4: Integrate the strain field to get the displacement at every point. Can we find the exact solution of the problem using this Airy function?

Problem 4

Let us consider a classical test in material engineering that make possible to identify the oedometric modelus $\hat{E}$. A cylinder material domain Ω is compressed due to the prescribed motion of the top surface $\underline{u} = -U\underline{e}_z$. All surfaces are assumed to be perfect contact without friction. Note that $\underline{P} = -P\underline{e}_z$ the resulting force that is necessary to press the materials (Young modulus E and Poisson ratio ν.

Question 1: Write the complete problem: kinematic + static + constitutive equation!

Answer: Kinematic admissibility:

$$\underline{u}\cdot\underline{n} = 0 \quad \forall \underline{M} \in S_{Lat} \rightarrow u_r = 0 \quad \forall \underline{M} \in S_{Lat} \tag{20}$$

$$\underline{n}\cdot\underline{u} = -U \quad \forall \underline{M} \in S_z^+ \rightarrow u_z = -U \quad \forall \underline{M} \in S_z^+ \tag{21}$$

$$\underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2}(\underline{\underline{\nabla}}\underline{u} + \underline{\underline{\nabla}}^T\underline{u}) \quad \forall \underline{M} \in \Omega \tag{22}$$

$$\underline{u}\cdot\underline{n} = 0 \quad \forall \underline{M} \in S_z^- \rightarrow u_z = 0 \quad \forall \underline{M} \in S_z^- \tag{23}$$

Static admissibility:

$$\underline{\mathrm{div}\underline{\underline{\sigma}}} = \underline{0} \quad \forall \underline{M} \in \Omega \tag{24}$$

or in expanded form as below

$$\begin{aligned} &\sigma_{rr},_r + \frac{1}{r}(\sigma_{r\theta},_\theta + \sigma_{rr} - \sigma_{\theta\theta}) + \sigma_{rz},_z = 0 \\ &\sigma_{\theta r},_r + \frac{1}{r}(\sigma_{\theta\theta},_\theta + \sigma_{r\theta} + \sigma_{\theta r}) + \sigma_{\theta z},_z = 0 \qquad \rightarrow \qquad \forall \underline{M} \in \Omega \\ &\sigma_{zr},_r + \frac{1}{r}(\sigma_{z\theta},_\theta + \sigma_{zr}) + \sigma_{zz},_z = 0 \end{aligned} \tag{25}$$

Contact without friction:

$$(\underline{\underline{\sigma}}\cdot\underline{t})\cdot\underline{t} = 0 \quad \forall \underline{M} \in S_z^+ \rightarrow \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^+ \tag{26}$$

Contact without friction:

$$(\underline{\underline{\sigma}}\cdot\underline{t})\cdot\underline{t} = 0 \quad \forall \underline{M} \in S_z^- \rightarrow \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^- \tag{27}$$

Contact without friction:

$$(\underline{\underline{\sigma}}\cdot\underline{t})\cdot\underline{t} = 0 \quad \forall \underline{M} \in S_{Lat} \rightarrow \sigma_{\theta r} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_{Lat} \tag{28}$$

Constitutive equations:

$$\underline{\underline{\sigma}} = \lambda \mathrm{tr}\underline{\underline{\varepsilon}}\underline{\underline{1}} + 2\mu\underline{\underline{\varepsilon}} \quad \forall \underline{M} \in \Omega \tag{29}$$

Question 2: Assume that $\underline{u} = -\frac{U}{h} z \underline{e}_z$. Is this kinematically admissible?

Answer: We assume that

$$\underline{u} = \begin{pmatrix} 0 \\ 0 \\ -\frac{U}{h}z \end{pmatrix} = \begin{pmatrix} -\frac{U}{h}z \end{pmatrix} \quad \text{Of course } \underline{u} \text{ is kinematically admissible.} \tag{30}$$

By definition (using the gradient of a vector with cylindrical coordinates):

$$\underline{\underline{\nabla}} \cdot \underline{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{U}{h} \end{bmatrix} \rightarrow \underline{\underline{\varepsilon}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \tag{31}$$

with $\varepsilon_{zz} = -U/h \rightarrow$ the compatibility equation is obviously satisfied.

Question 3: Find the corresponding stress $\underline{\underline{\sigma}}$. Is it statistically admissible?

Answer:

$$\underline{\underline{\sigma}} = \lambda \mathrm{tr}\underline{\underline{\varepsilon}}\underline{\underline{1}} + 2\mu\underline{\underline{\varepsilon}} \rightarrow \underline{\underline{\sigma}} = \begin{bmatrix} -\lambda\frac{U}{h} & 0 & 0 \\ 0 & -\lambda\frac{U}{h} & 0 \\ 0 & 0 & -(\lambda+2\mu)\frac{U}{h} \end{bmatrix} \tag{32}$$

Eqs. 28, 27, and 26 are obviously satisfied. $\underline{\mathrm{div}\underline{\underline{\sigma}}} \rightarrow$ obviously equal to 0 as $\underline{\underline{\sigma}}$ is constant.

Question 4: Find oedometric modulus $\hat{E}$ as a function of E and ν.
Answer: The vertical force is equal to

$$\underline{P} = \int_{S_z^+} \underline{\underline{\sigma}} \cdot \underline{e}_z dS_3^+ \tag{33}$$

$$\underline{P} = \int_{S_z^+} -(\lambda+2\mu)\frac{U}{h} dS_3^+ \underline{e}_z = -\underline{P}\underline{e}_z \quad \text{with} \quad \underline{P} = \frac{(\lambda+2\mu)US}{h} \tag{34}$$

So we can identify the oedometric modulus $\hat{E} = \lambda + 2\mu$.

$$\begin{aligned} \underline{\underline{\sigma}} &= \lambda \mathrm{tr}\underline{\underline{\varepsilon}}\underline{\underline{1}} + 2\mu\underline{\underline{\varepsilon}} \\ \underline{\underline{\varepsilon}} &= \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}\mathrm{tr}\underline{\underline{\sigma}}\underline{\underline{1}} \end{aligned} \quad \rightarrow \quad \begin{aligned} \mathrm{tr}\underline{\underline{\sigma}} &= (3\lambda+2\mu)\ \mathrm{tr}\underline{\underline{\varepsilon}} \\ \mathrm{tr}\underline{\underline{\varepsilon}} &= (\frac{1-2\nu}{E})\ \mathrm{tr}\underline{\underline{\sigma}} \end{aligned} \quad \rightarrow \quad 3\lambda + 2\mu = \frac{E}{1-2\nu} \tag{35}$$

Or we can also write as follows:

$$\mathrm{tr}\underline{\underline{\sigma}} = (3\lambda+2\mu)\mathrm{tr}\underline{\underline{\varepsilon}} \quad \rightarrow \quad \underline{\underline{\sigma}} = \frac{\lambda}{3\lambda+2\mu}\mathrm{tr}\underline{\underline{\sigma}}\underline{\underline{1}} + 2\mu\underline{\underline{\varepsilon}} \tag{36}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2\mu}\underline{\underline{\sigma}} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\mathrm{tr}\underline{\underline{\sigma}}\underline{\underline{1}} \tag{37}$$

By identification,

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \quad \text{and} \quad \frac{\lambda}{2\mu(3\lambda+2\mu)} = \frac{\nu}{E} \tag{38}$$

$$\rightarrow \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} \tag{39}$$

Thus,

$$\hat{E} = \lambda + 2\mu = \frac{2E\nu + 2E(1-2\nu)}{2(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \tag{40}$$