

# Conditions for small perturbations

We consider the transformation  $\underline{x} = \underline{\Phi}(\underline{X}, t)$  defined by:

$$x_1 = X_1 + X_2 \quad (1)$$

$$x_2 = X_2 \quad (2)$$

$$x_3 = X_3 \quad (3)$$

**Question 1:** Make a graphical representation of the reference configuration and of the deformed configuration. Calculate the gradient of the transformation,  $\underline{\underline{F}}$ .

**Question 2:** Is this an homogenous transformation? It is acceptable from a physical point of view?

**Question 3:** Calculate the expansion of the following vectors:  $\underline{e}_1$ ,  $\underline{e}_2$  and  $\frac{\underline{e}_1 + \underline{e}_2}{\sqrt{2}}$

**Question 4:** Calculate  $\underline{\underline{e}}$  and  $\underline{\underline{\epsilon}}$ ? Can we consider we are in small perturbations?

**Question 5:** Consider the transformation  $x_1 = X_1 + \alpha \cdot X_2$ ,  $x_2 = X_2$  and  $x_3 = X_3$ . What is the condition for having small perturbations?

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## Solution to "Conditions for small perturbations"

**Question 1:** Make a graphical representation of the reference configuration and of the deformed configuration. Calculate the gradient of the transformation,  $\underline{\underline{F}}$ .

**Answer:** The gradient of the transformation  $\underline{\underline{F}}$  also known as deformation gradient can be calculated as follows.

$$\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \frac{\partial \phi}{\partial \underline{X}} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4)$$

**Question 2:** Is this an homogenous transformation? It is acceptable from a physical point of view?

**Answer:** This is a homogeneous transformation since  $\underline{\underline{F}}$  does not depend on  $\underline{X}$  which means  $\underline{\underline{F}}$  is a constant everywhere (homogeneous field). The determinant of deformation gradient  $\underline{\underline{F}}$  has a physical significance i.e. the ratio between the volume after deformation to initial volume as shown below

$$|\Omega| = J \quad |\Omega_0| \quad (5)$$

where

$$J = \det \underline{\underline{F}} \quad (6)$$

In this particular case,  $\det F = 1$  which means that there is no volume change (no volume variation). And yes, it is acceptable from a physical point of view to have a constant volume. The transformation is not physically accepted if the  $\det F \leq 0$ .

**Question 3:** Calculate the expansion of the following vectors:  $\underline{V}_1 = \underline{e}_1$ ,  $\underline{V}_2 = \underline{e}_2$  and  $\underline{V}_3 = \frac{\underline{e}_1 + \underline{e}_2}{\sqrt{2}}$

**Answer:** Let say  $\underline{v}$  is the vector after transformation and  $\underline{V}$  is undeformed vector. By definition,

$$\underline{v} \cdot \underline{v} = \underline{V} \cdot \underline{\underline{C}} \cdot \underline{V} \quad \forall \underline{V} \rightarrow \underline{v} \quad (7)$$

Then the expansion can be calculated as follow:

$$\frac{|\underline{v}|}{|\underline{V}|} = \frac{\sqrt{\underline{V} \cdot \underline{\underline{C}} \cdot \underline{V}}}{\sqrt{\underline{V} \cdot \underline{V}}} = \lambda_{\underline{V}} \quad (8)$$

with

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (9)$$

For  $\underline{V}_1 = \underline{e}_1$ , then  $\lambda_{\underline{V}_1} = \sqrt{\frac{\underline{e}_1 \cdot \underline{C} \underline{e}_1}{\underline{e}_1 \cdot \underline{e}_1}} = 1 \rightarrow$  No length change.

For  $\underline{V}_2 = \underline{e}_2$ , then  $\lambda_{\underline{V}_2} = \sqrt{\frac{\underline{e}_2 \cdot \underline{C} \underline{e}_2}{\underline{e}_2 \cdot \underline{e}_2}} = \sqrt{2} \rightarrow$  The length changes  $\sqrt{2}$  times the initial length.

For  $\underline{V}_3 = \frac{\underline{e}_1 + \underline{e}_2}{\sqrt{2}}$ , then  $\lambda_{\underline{V}_3} = \sqrt{\frac{\underline{e}_3 \cdot \underline{C} \underline{e}_3}{\underline{e}_3 \cdot \underline{e}_3}} = \sqrt{\frac{5}{2}} \rightarrow$  The length changes  $\sqrt{\frac{5}{2}}$  times the initial length.

**Question 4:** Calculate  $\underline{\underline{e}}$  and  $\underline{\underline{\varepsilon}}$ ? Can we consider we are in small perturbations?

**Answer:** The Green-Lagrange strain tensor can be calculated as follows

$$\underline{\underline{e}} = \frac{1}{2} \{ \underline{C} - \underline{1} \} = \frac{1}{2} (\underline{F}^T \underline{F} - \underline{1}) = \frac{1}{2} \left( \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (10)$$

The linearized strain  $\underline{\underline{\varepsilon}}$  can be calculated as follows

$$\underline{\underline{\varepsilon}} = \frac{1}{2} (\underline{\nabla} \underline{\xi} + \underline{\nabla}^T \underline{\xi}) \quad (11)$$

where  $\underline{\xi}$  the displacement vector that can be calculated as follows

$$\underline{x} = \underline{X} + \underline{\xi} \quad (12)$$

$$\underline{\xi} = \underline{x} - \underline{X} = \begin{pmatrix} X_2 \\ 0 \\ 0 \end{pmatrix} \quad (13)$$

$$\underline{\nabla} \underline{\xi} = \frac{\partial \xi_i}{\partial X_j} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (14)$$

Finally,

$$\underline{\underline{\varepsilon}} = \frac{1}{2} (\underline{\nabla} \underline{\xi} + \underline{\nabla}^T \underline{\xi}) = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (15)$$

By comparing  $\underline{\underline{\varepsilon}}$  and  $\underline{\underline{e}}$ , since  $\underline{\underline{\varepsilon}} \neq \underline{\underline{e}}$ , then this is not a small perturbation case.

**Question 5:** Consider the transformation  $x_1 = X_1 + \alpha \cdot X_2$ ,  $x_2 = X_2$  and  $x_3 = X_3$ . What is the condition for having small perturbations?

**Answer:** Note that the transformation above has the same pattern as previous transformation. The only difference is that now  $x_1 = X_1 + \alpha X_2$  instead of  $x_1 = X_1 + X_2$ . Therefore, we can the displacement vector  $\underline{\xi}$  as below

$$\underline{\xi} = \begin{pmatrix} \alpha X_2 \\ 0 \\ 0 \end{pmatrix} \quad (16)$$

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and

$$\underline{\underline{\nabla}}\xi = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (17)$$

Then to be small perturbation, we should have  $||\underline{\underline{\nabla}}\xi|| = \sqrt{\alpha^2} \ll 1$ . In this situation,

$$\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{\nabla}}\xi + \underline{\underline{\nabla}}^T\xi) + \frac{1}{2}\underline{\underline{\nabla}}^T\underline{\underline{\nabla}}\xi \quad (18)$$

where the first term is small strain term and the second term is second order.

$$\underline{\underline{e}} = \frac{1}{2} \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (19)$$