

Practice:

Static and Kinematic Admissibility

Problem 2

Let us consider a sphere with cavity (exterior radius R_e and internal radius R_i). The external surface S_e is free of loading and the interior surface S_i is submitted to a pressure p_0 . The material is assumed to be isotropic linear elastic with young modulus E and Poisson ratio ν .

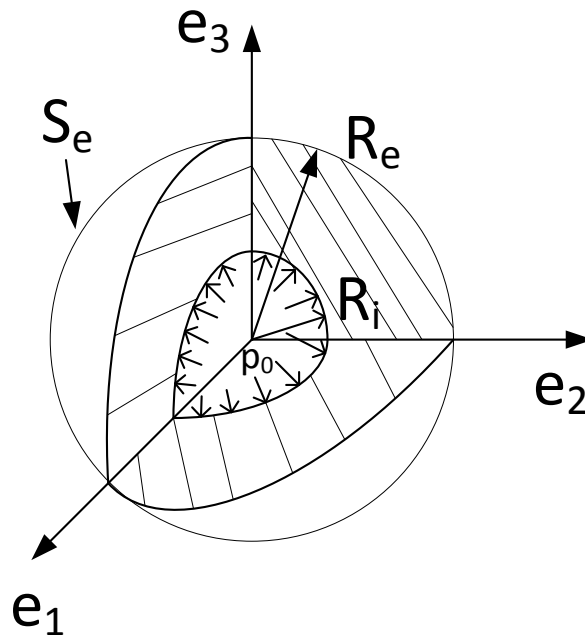


Figure 1: Configuration of problem 2

Question 1: Write the static admissibility equations!

Answer: Find $\underline{\underline{\sigma}}$ symmetric such that

- (a) $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \quad \forall M \in S_e$ Boundary equation
- (b) $\underline{\underline{\sigma}} \cdot \underline{n} = -p_0 \underline{n} \quad \forall M \in S_i$ Boundary equation
- (c) $\text{div} \underline{\underline{\sigma}} = \underline{0} \quad \forall M \in \Omega$ Internal equilibrium

Question 2: Please expand this equation using spherical coordinates system!

Answer: In spherical coordinates,

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\varphi} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta\varphi} \\ \sigma_{r\varphi} & \sigma_{\theta\varphi} & \sigma_{\varphi\varphi} \end{bmatrix} \quad (2)$$

From Eq. (a),

$$\sigma_{rr}(r = R_e, \theta, \varphi) = \sigma_{r\theta}(r = R_e, \theta, \varphi) = \sigma_{r\varphi}(r = R_e, \theta, \varphi) = 0 \quad (3)$$

From Eq. (b),

$$\begin{aligned} -\sigma_{rr}(r = R_i, \theta, \varphi) &= p_0 \\ -\sigma_{r\theta}(r = R_i, \theta, \varphi) &= 0 \\ -\sigma_{r\varphi}(r = R_i, \theta, \varphi) &= 0 \end{aligned} \quad (4)$$

From Eq. (c) \rightarrow See Textbook equation 1.11a (divergence with spherical coordinates)

Question 3: Let's assume that the following stress tensor

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{bmatrix} \quad (5)$$

with

$$\sigma_{rr} = -A \left[\frac{B}{r^3} - 1 \right] \quad (6)$$

and

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = A \left[\frac{B}{2r^3} + 1 \right] \quad (7)$$

Please calculate the coefficient A and B so that $\underline{\underline{\sigma}}$ could be a solution of the problem!

Answer: Using the proposed stress tensor, and expanding equation (c), we get

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \quad (8)$$

We now check if $\underline{\underline{\sigma}}$ ca fulfill the static admissibility equations.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = \frac{3AB}{r^4} - \frac{3AB}{r^4} = 0 \rightarrow (c) \text{ is OK. Interior equilibrium is satisfied.} \quad (9)$$

$$(a) \rightarrow -A \left[\frac{B}{R_e^3} - 1 \right] = 0 \rightarrow B = R_e^3 \quad (10)$$

$$(b) \rightarrow -A \left[\frac{R_i^3}{R_e^3} - 1 \right] = p_0 \rightarrow A = p_0 \frac{R_e^3}{R_i^3 - R_e^3} \quad (11)$$

All the other conditions concerning $\sigma_{\theta r}$ and $\sigma_{r\varphi}$ are automatically satisfied due to the chosen stress tensor. Thus, a statically admissible stress field is:

$$\underline{\underline{\sigma}} = \begin{bmatrix} p_0 \frac{R_e^3}{R_i^3 - R_e^3} \left[\frac{R_i^3}{r} - 1 \right] & 0 & 0 \\ 0 & -p_0 \frac{R_e^3}{R_i^3 - R_e^3} \left[\frac{R_e^3}{2r^3} + 1 \right] & 0 \\ 0 & 0 & -p_0 \frac{R_e^3}{R_i^3 - R_e^3} \left[\frac{R_e^3}{2r^3} + 1 \right] \end{bmatrix} \quad (12)$$

Question 4: Please calculate the strain tensor $\underline{\underline{\varepsilon}}$? Does it satisfy the compatibility equation? Conclusion?

Answer: Based on the constitutive equation, we can calculate the strains:

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E} \underline{\underline{\sigma}} - \frac{\nu}{E} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} \quad (13)$$

$$\text{tr} \underline{\underline{\sigma}} = -p_0 \frac{R_e^3}{R_i^3 - R_e^3} \quad (14)$$

Thus, we get $\varepsilon_{r\theta} = \varepsilon_{\theta\varphi} = \varepsilon_{\varphi\theta} = 0$. Let us note that

$$A_1 = p_0 \frac{R_e^3}{R_i^3 - R_e^3} \quad (15)$$

We get:

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \left(\frac{1+\nu}{E} \right) A_1 \left[\frac{R_e^3}{r} - 1 \right] + \frac{\nu}{E} A_1 & 0 & 0 \\ 0 & - \left(\frac{1+\nu}{E} \right) A_1 \left[\frac{R_e^3}{2r^3} + 1 \right] + \frac{\nu}{E} A_1 & 0 \\ 0 & 0 & - \left(\frac{1+\nu}{E} \right) A_1 \left[\frac{R_e^3}{2r^3} + 1 \right] + \frac{\nu}{E} A_1 \end{bmatrix} \quad (16)$$

In case of complete spherical symmetry, the compatibility equation can be simplified (see Textbook equation 2.14b):

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} + \frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = 0 \quad (17)$$

Let us verify if this equation is satisfied by $\underline{\underline{\varepsilon}}$.

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} = 3A_1 \left(\frac{1+\nu}{E} \right) \frac{R_e^3}{2r^4} \quad (18)$$

$$\frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = -3A_1 \left(\frac{1+\nu}{E} \right) \frac{R_e^3}{2r^4} \quad (19)$$

The two equations above satisfied the compatibility equation. Thus it will be possible to integrate $\underline{\underline{\varepsilon}}$ to determine a displacement field \underline{u} . Considering there is no additional kinematic condition (no kinematic boundary condition), we got the exact solution of the problem.

Problem 3

Let us consider a domain Ω , simply supported by a rigid basis (Γ). Ω is submitted to gravity volume forces $\underline{f} = \rho g \underline{y}$ and to uniform pressure $p > 0$ over every vertical side. We assume plane stresses in the plane of $x - y$ and the material is isotropic linear elastic with Young modulus E and Poisson ratio ν .

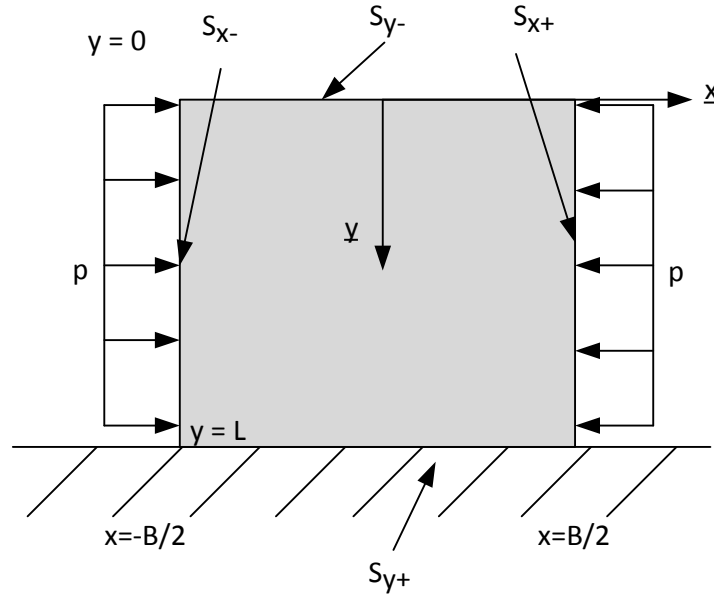


Figure 2: Configuration of problem 3

Question 1: Write down the full problem! (kinematic, static and constitutive equations)

Answer: Kinematic admissibility:

Let us not \underline{u} the displacement vector such that $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\underline{u} = u\underline{x} + v\underline{y}$

$$\begin{cases} v = 0, & \forall M \in S_y^+ \\ \underline{\underline{\epsilon}} = \frac{1}{2} (\underline{\nabla} \underline{u} + \underline{\nabla} \underline{u}^t), & \text{compatibility} \end{cases} \quad (20)$$

Static admissibility: Find $\underline{\underline{\sigma}}$ symmetric such that

$$\left. \begin{aligned} (1) \quad \underline{\underline{\sigma}} \cdot \underline{n} &= -p\underline{n} \quad \forall M \in S_x^+ \\ (2) \quad \underline{\underline{\sigma}} \cdot \underline{n} &= -p\underline{n} \quad \forall M \in S_x^- \\ (3) \quad \underline{\underline{\sigma}} \cdot \underline{n} &= \underline{0} \quad \forall M \in S_y^- \\ (4) \quad (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{t} &= \underline{0} \quad \forall M \in S_y^+ \end{aligned} \right\} \text{Boundary equation} \quad (21)$$

$$\underline{\underline{\text{div}}} \underline{\underline{\sigma}} + \rho g \underline{y} = \underline{0} \quad \forall M \in \Omega \quad \text{Internal equilibrium} \quad (22)$$

As plane stress are assumed, the equation can be reduced to:

$$\left. \begin{aligned} \sigma_{xx,x} + \sigma_{xy,y} &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + \rho g &= 0 \\ \sigma_{xz} = \sigma_{yz} = \sigma_{zz} &= 0 \end{aligned} \right\} \forall M \in \Omega \quad (23)$$

Constitutive equation: The 3D constitutive equation is:

$$\underline{\underline{\epsilon}} = \frac{1 + \nu}{E} \underline{\underline{\sigma}} - \frac{\nu}{E} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} \quad (24)$$

Here, we can use the plane stress assumption ($\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$)

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} = \left(\frac{1 + \nu}{E} \right) \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (25)$$

As a consequence, we can also write this equation in 2D:

$$\begin{cases} \underline{\underline{\epsilon}}^{2D} = \frac{1 + \nu}{E} \underline{\underline{\sigma}}^{2D} - \frac{\nu}{E} \text{tr} \underline{\underline{\sigma}}^{2D} \underline{\underline{1}}^{2D} \\ \epsilon_{xz} = \epsilon_{yz} = 0 \\ \epsilon_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}) \quad \text{the out of plane strain is not equal to zero} \end{cases} \quad (26)$$

Question 2: We assume an Airy function $\phi(x, y) = ax^2 + cy^2 + hy^3$. Find a, c, h so we define a statistically admissible stress field.

Answer: We want to use an approach based on Airy function but we have to be careful in case of volume force.

Let us assume the volume force density \underline{f} derive from a potential:

$$\underline{f} = \underline{\text{grad}} V \quad (27)$$

For example here: $V = \rho g y \Rightarrow \underline{f} = \underline{\text{grad}} V = \begin{pmatrix} 0 \\ \rho g \end{pmatrix}$

In this case:

$$\begin{cases} \frac{\partial(\sigma_{xx} + V)}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial(\sigma_{yy} + V)}{\partial y} = 0 \end{cases} \quad (28)$$

and $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ can be defined based on the Airy function by the following equation:

$$\sigma_{xx} + V = \frac{\partial^2 \phi}{\partial y^2}; \quad \sigma_{yy} + V = \frac{\partial^2 \phi}{\partial x^2}; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (29)$$

If $\phi(x, y) = ax^2 + cy^2 + hy^3$

$$\begin{aligned}\sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} - V = 2C + 6hy - \rho gy \rightarrow \sigma_{xx,x} = 0 \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} - V = 2a - \rho gy \rightarrow \sigma_{yy,y} = -\rho g \\ \sigma_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} = 0\end{aligned}\tag{30}$$

$$\text{Interior equation: } \begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 \rightarrow \text{OK} \\ \sigma_{xy,x} + \sigma_{yy,y} + \rho g = 0 \rightarrow \text{OK} \end{cases}\tag{31}$$

Boundary static equation:

$$\begin{aligned}(1) &\Rightarrow \begin{cases} \sigma_{xx}(x = B/2, y) = -p, & \forall y \rightarrow \begin{cases} 2c = -p \\ 6h - \rho g = 0 \end{cases} \rightarrow \begin{cases} c = -p/2 \\ h = \frac{\rho g}{6} \end{cases} \\ \sigma_{xy}(x = B/2, y) = 0, & \forall y \quad \text{OK} \end{cases} \\ (2) &\Rightarrow \begin{cases} \sigma_{xx}(x = -B/2, y) = -p, & \forall y \quad \text{OK} \\ \sigma_{xy}(x = -B/2, y) = 0, & \forall y \quad \text{OK} \end{cases} \\ (3) &\Rightarrow \begin{cases} \sigma_{yy}(y = 0, x) = 0, & \forall x \rightarrow 2a = 0 \rightarrow a = 0 \\ \sigma_{xy}(y = 0, x) = 0, & \forall x \quad \text{OK} \end{cases} \\ (4) &\Rightarrow \{ \sigma_{xy}(y = L, x) = 0, \quad \forall x \quad \text{OK} \end{aligned}\tag{32}$$

Finally, we have $a = 0$; $c = -p/2$; $h = \rho g/6$, and we get the following statistically admissible stress field:

$$\begin{cases} \sigma_{xx} = -p \\ \sigma_{yy} = -\rho gy \\ \sigma_{xy} = 0 \\ \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \end{cases}\tag{33}$$

Question 3: Find the related strain field using the constitutive equation. Does that satisfy the compatibility equation? Conclusion?

Answer: As a consequence, using the constitutive equation:

$$\begin{cases} \epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} = -\frac{p}{E} + \frac{\nu\rho g}{E}y \\ \epsilon_{yy} = -\frac{\rho g}{E}y + \frac{\nu p}{E} \\ \epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0 \\ \epsilon_{zz} = \frac{\nu}{E}(p + \rho gy) \end{cases}\tag{34}$$

Check the compatibility:

The compatibility equation are obviously satisfied as we just have linear relations (and the compatibility equations involves the second derivation).

Question 4: Integrate the strain field to get the displacement at every point. Can we find the exact solution of the problem using this Airy function?

Answer:

$$\begin{aligned}
u_{,x} &= -\frac{p}{E} + \frac{\nu\rho g}{E}y \rightarrow u = \left(-\frac{p}{E} + \frac{\nu\rho g}{E}y\right)x + f(y) \\
v_{,y} &= -\frac{\rho g}{E}y + \frac{\nu p}{E} \rightarrow v = -\frac{\rho g}{2E}y^2 + \frac{\nu p}{E}y + g(x) \\
\epsilon_{xy} &= \frac{1}{2}(u_{,y} + v_{,x}) = 0 \rightarrow \frac{\nu\rho g}{E}x + \frac{\partial f(y)}{y} + \frac{\partial g(x)}{x} = 0
\end{aligned} \tag{35}$$

$$\begin{aligned}
\frac{\partial f(y)}{y} &= \lambda \rightarrow f(y) = \lambda y + \beta \\
\frac{\nu\rho g}{E}x + \frac{\partial g(x)}{x} &= -\lambda \rightarrow g(x) = -\frac{\nu\rho g}{2E}x^2 - \lambda x + \gamma
\end{aligned} \tag{36}$$

Thus:

$$\begin{cases} u = \left(-\frac{p}{E} + \frac{\nu\rho g}{E}y\right)x + \lambda y + \beta \\ v = -\frac{\rho g}{2E}y^2 + \frac{\nu p}{E}y - \frac{\nu\rho g}{2E}x^2 - \lambda x + \gamma \end{cases} \tag{37}$$

Introducing the kinematic boundary equations:

$$v(x, y = L) \Rightarrow \text{this solution can not be satisfied} \tag{38}$$

We do not have the exact solution.

Problem 4

Let us consider a classical test in material engineering that make possible to identify the oedometric modulus \hat{E} . A cylinder material domain Ω is compressed due to the prescribed motion of the top surface $\underline{u} = -U\underline{e}_z$. All surfaces are assumed to be perfect contact without friction. Note that $\underline{P} = -P\underline{e}_z$ the resulting force that is necessary to press the materials (Young modulus E and Poisson ratio ν).

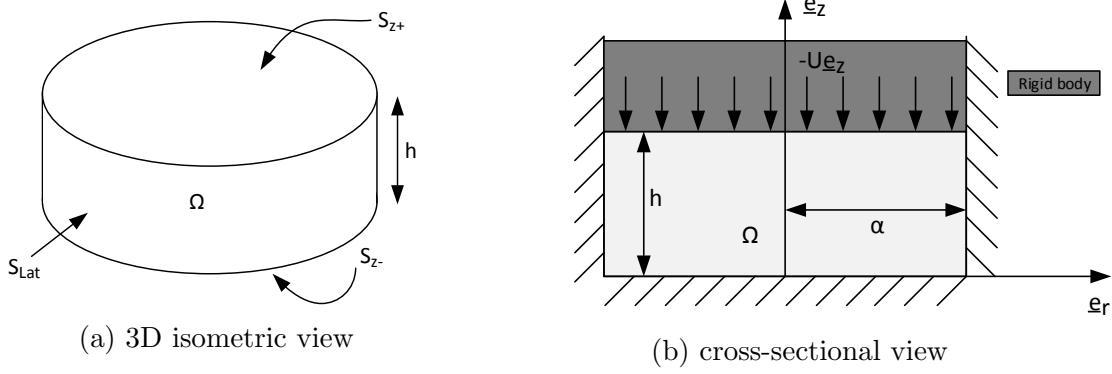


Figure 3: Oedometric compression

Question 1: Write the complete problem: kinematic + static + constitutive equation!

Answer: Kinematic admissibility:

$$\underline{u} \cdot \underline{n} = 0 \quad \forall \underline{M} \in S_{Lat} \rightarrow u_r = 0 \quad \forall \underline{M} \in S_{Lat} \quad (39)$$

$$\underline{n} \cdot \underline{u} = -U \quad \forall \underline{M} \in S_z^+ \rightarrow u_z = -U \quad \forall \underline{M} \in S_z^+ \quad (40)$$

$$\underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2}(\underline{\underline{\nabla}}\underline{u} + \underline{\underline{\nabla}}^T \underline{u}) \quad \forall \underline{M} \in \Omega \quad (41)$$

$$\underline{u} \cdot \underline{n} = 0 \quad \forall \underline{M} \in S_z^- \rightarrow u_z = 0 \quad \forall \underline{M} \in S_z^- \quad (42)$$

Static admissibility:

$$\text{div} \underline{\underline{\sigma}} = \underline{0} \quad \forall \underline{M} \in \Omega \quad (43)$$

or in expanded form as below

$$\begin{aligned} \sigma_{rr,r} + \frac{1}{r}(\sigma_{r\theta,\theta} + \sigma_{rr} - \sigma_{\theta\theta}) + \sigma_{rz,z} &= 0 \\ \sigma_{\theta r,r} + \frac{1}{r}(\sigma_{\theta\theta,\theta} + \sigma_{r\theta} + \sigma_{\theta r}) + \sigma_{\theta z,z} &= 0 \quad \rightarrow \quad \forall \underline{M} \in \Omega \\ \sigma_{zr,r} + \frac{1}{r}(\sigma_{z\theta,\theta} + \sigma_{zr}) + \sigma_{zz,z} &= 0 \end{aligned} \quad (44)$$

Contact without friction:

$$(\underline{\sigma} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_z^+ \rightarrow \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^+ \quad (45)$$

Contact without friction:

$$(\underline{\sigma} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_z^- \rightarrow \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^- \quad (46)$$

Contact without friction:

$$(\underline{\sigma} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_{Lat} \rightarrow \sigma_{\theta r} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_{Lat} \quad (47)$$

Constitutive equations:

$$\underline{\sigma} = \lambda \text{tr} \underline{\varepsilon} \underline{1} + 2\mu \underline{\varepsilon} \quad \forall \underline{M} \in \Omega \quad (48)$$

Question 2: Assume that $\underline{u} = -\frac{U}{h} z \underline{e}_z$. Is this kinematically admissible?

Answer: We assume that

$$\underline{u} = \begin{pmatrix} 0 \\ 0 \\ -\frac{U}{h} z \end{pmatrix} = \left(-\frac{U}{h} z \right) \underline{e}_z \quad \text{Of course } \underline{u} \text{ is kinematically admissible.} \quad (49)$$

By definition (using the gradient of a vector with cylindrical coordinates):

$$\underline{\nabla} \cdot \underline{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{U}{h} \end{bmatrix} \rightarrow \underline{\varepsilon} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \quad (50)$$

with $\varepsilon_{zz} = -U/h \rightarrow$ the compatibility equation is obviously satisfied.

Question 3: Find the corresponding stress $\underline{\sigma}$. Is it statistically admissible?

Answer:

$$\underline{\sigma} = \lambda \text{tr} \underline{\varepsilon} \underline{1} + 2\mu \underline{\varepsilon} \rightarrow \underline{\sigma} = \begin{bmatrix} -\lambda \frac{U}{h} & 0 & 0 \\ 0 & -\lambda \frac{U}{h} & 0 \\ 0 & 0 & -(\lambda + 2\mu) \frac{U}{h} \end{bmatrix} \quad (51)$$

Eqs. 47, 46, and 45 are obviously satisfied. $\underline{\text{div}} \underline{\sigma} \rightarrow$ obviously equal to 0 as $\underline{\sigma}$ is constant.

Question 4: Find oedometric modulus \hat{E} as a function of E and ν .

Answer: The vertical force is equal to

$$\underline{P} = \int_{S_z^+} \underline{\sigma} \cdot \underline{e}_z dS_3^+ \quad (52)$$

$$\underline{P} = \int_{S_z^+} -(\lambda + 2\mu) \frac{U}{h} dS_3^+ \underline{e}_z = -\underline{P} \underline{e}_z \quad \text{with} \quad \underline{P} = \frac{(\lambda + 2\mu) U S}{h} \quad (53)$$

So we can identify the oedometric modulus $\hat{E} = \lambda + 2\mu$.

$$\begin{aligned} \underline{\underline{\sigma}} &= \lambda \text{tr} \underline{\underline{\varepsilon}} \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}} & \text{tr} \underline{\underline{\sigma}} &= (3\lambda + 2\mu) \text{tr} \underline{\underline{\varepsilon}} \\ \underline{\underline{\varepsilon}} &= \frac{1+\nu}{E} \underline{\underline{\sigma}} - \frac{\nu}{E} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} & \text{tr} \underline{\underline{\varepsilon}} &= \left(\frac{1-2\nu}{E}\right) \text{tr} \underline{\underline{\sigma}} \end{aligned} \quad \rightarrow \quad 3\lambda + 2\mu = \frac{E}{1-2\nu} \quad (54)$$

Or we can also write as follows:

$$\text{tr} \underline{\underline{\sigma}} = (3\lambda + 2\mu) \text{tr} \underline{\underline{\varepsilon}} \quad \rightarrow \quad \underline{\underline{\sigma}} = \frac{\lambda}{3\lambda + 2\mu} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}} \quad (55)$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2\mu} \underline{\underline{\sigma}} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} \quad (56)$$

By identification,

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \quad \text{and} \quad \frac{\lambda}{2\mu(3\lambda + 2\mu)} = \frac{\nu}{E} \quad (57)$$

$$\rightarrow \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} \quad (58)$$

Thus,

$$\hat{E} = \lambda + 2\mu = \frac{2E\nu + 2E(1-2\nu)}{2(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad (59)$$