Static admissibility #1

A prismatic concrete column of mass density ρ supports its own weight (height of the column is H, squared cross section, bottom cross section is centered on point $O = (0, 0, 0)$, axis of the column is the vertical direction $(O,\underline{e}_2)$). We assume that the solid is subjected to a uniform gravitational body force of magnitude g per unit mass.

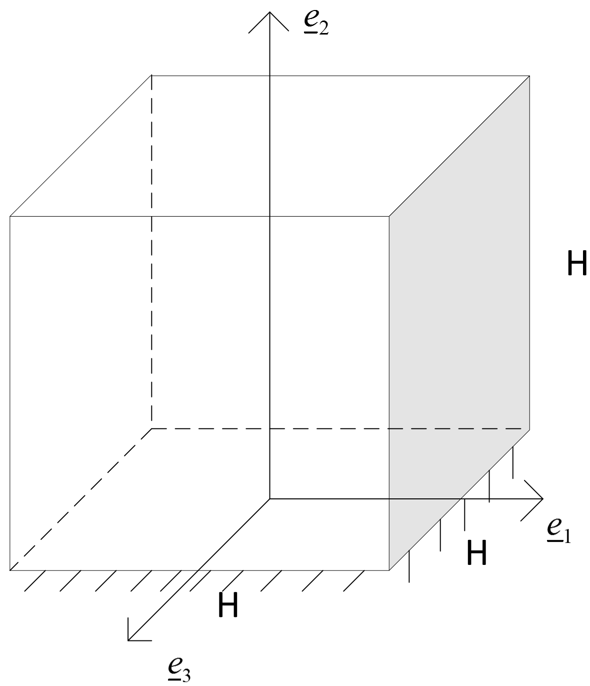

Figure 1: Sketch of the problem

All surfaces are free of traction except the bottom surface which is perfectly clamped.

Question: Write all the equations defining static admissibility for $\underline{\underline{\sigma}}$ and expand them.

Answer: Set of static admissibility Find $\underline{\underline{\sigma}}$ such that $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$, $\forall n \in \Omega$

Interior equilibrium equation:

$$\underline{\mathrm{div}\underline{\underline{\sigma}}} - \rho g \underline{e}_2 = \underline{0}, \ \forall m \in \Omega \tag{1}$$

Boundary conditions:

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0}, \ \forall n \in S_1^- \cup S_1^+ \cup S_2^+ \cup S_3^- \cup S_3^+ \tag{2}$$

The above equations can be expanded as follows:

Interior equilibrium equation:

$$\begin{aligned} &\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0, \ (1a) \\ &\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} - \rho g = 0, \ (1b) \\ &\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0, \ (1c) \end{aligned} \tag{3}$$

Boundary conditions:

$$\text{For surface } S_1^- : \begin{aligned} \sigma_{11}(-H/2, y, z) = 0 \\ \sigma_{21}(-H/2, y, z) = 0 \\ \sigma_{31}(-H/2, y, z) = 0 \end{aligned} \tag{4}$$

$$\text{For surface } S_1^+ : \begin{aligned} \sigma_{11}(H/2, y, z) = 0 \\ \sigma_{21}(H/2, y, z) = 0 \\ \sigma_{31}(H/2, y, z) = 0 \end{aligned} \tag{5}$$

$$\text{For surface } S_2^+ : \begin{aligned} \sigma_{12}(x, H, z) = 0 \\ \sigma_{22}(x, H, z) = 0 \\ \sigma_{32}(x, H, z) = 0 \end{aligned} \tag{6}$$

$$\text{For surface } S_3^- : \begin{aligned} \sigma_{13}(x, y, -H/2) = 0 \\ \sigma_{23}(x, y, -H/2) = 0 \\ \sigma_{33}(x, y, -H/2) = 0 \end{aligned} \tag{7}$$

$$\text{For surface } S_3^+ : \begin{aligned} \sigma_{13}(x, y, H/2) = 0 \\ \sigma_{23}(x, y, H/2) = 0 \\ \sigma_{33}(x, y, H/2) = 0 \end{aligned} \tag{8}$$

Question: Can the following stress field be a viable solution for the problem:

$$\sigma_{22} = -\rho g (H - x_2) \quad ; \quad \text{otherwise} \quad \sigma_{ij} = 0 \tag{9}$$

Answer: First, lets check if the **interior equilibrium equation** is satisfied:

$$\begin{aligned}
&\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 + 0 + 0 = 0 \\
&\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} - \rho g = 0 + \rho g + 0 - \rho g = 0 \\
&\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 + 0 + 0 = 0
\end{aligned} \tag{10}$$

From above, we can see that the internal equilibrium equation is satisfied.

Then, lets check if the **boundary condition** is satisfied:
For the surface S_1^-, S_1^+, S_3^- and S_3^+, it doesn't have the stress component σ_{22}. And other stress components are all zero, so they are satisfied. For the surface S_2^+, we have $\sigma_{22} = -\rho g(H - H) = 0$ which means it also satisfies the free boundary condition. Therefore, the proposed stress field could be a viable solution.