

Practice:

Static and Kinematic Admissibility

1 Problem 2

Let us consider a sphere with cavity (exterior radius R_e and internal radius R_i). The external surface S_e is free of loading and the interior surface S_i is submitted to a pressure p_0 . The material is assumed to be isotropic linear elastic with young modulus E and Poisson ratio ν .

Question 1: Write the static admissibility equations!

Answer: Find $\underline{\underline{\sigma}}$ symmetric such that

$$\begin{aligned} \text{(a)} \quad \underline{\underline{\sigma}} \cdot \underline{n} &= \underline{0} \quad \forall M \in S_e \quad \text{Boundary equation} \\ \text{(b)} \quad \underline{\underline{\sigma}} \cdot \underline{n} &= -p_0 \underline{n} \quad \forall M \in S_i \quad \text{Boundary equation} \\ \text{(c)} \quad \text{div} \underline{\underline{\sigma}} &= \underline{0} \quad \forall M \in \Omega \quad \text{Internal equilibrium} \end{aligned} \tag{1}$$

Question 2: Please expand this equation using spherical coordinates system!

Answer: In spherical coordinates,

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\varphi} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta\varphi} \\ \sigma_{r\varphi} & \sigma_{\theta\varphi} & \sigma_{\varphi\varphi} \end{bmatrix} \tag{2}$$

From Eq. (a),

$$\sigma_{rr}(r = R_e, \theta, \varphi) = \sigma_{r\theta}(r = R_e, \theta, \varphi) = \sigma_{r\varphi}(r = R_e, \theta, \varphi) = 0 \tag{3}$$

From Eq. (b),

$$\begin{aligned} -\sigma_{rr}(r = R_i, \theta, \varphi) &= p_0 \\ -\sigma_{r\theta}(r = R_i, \theta, \varphi) &= 0 \\ -\sigma_{r\varphi}(r = R_i, \theta, \varphi) &= 0 \end{aligned} \tag{4}$$

From Eq. (c) \rightarrow See Textbook equation 1.11a (divergence with spherical coordinates)

Question 3: Let's assume that the following stress tensor

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{bmatrix} \quad (5)$$

with

$$\sigma_{rr} = -A \left[\frac{B}{r^3} - 1 \right] \quad (6)$$

and

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = A \left[\frac{B}{2r^3} + 1 \right] \quad (7)$$

Please calculate the coefficient A and B so that $\underline{\underline{\sigma}}$ could be a solution of the problem!

Answer: Using the proposed stress tensor, and expanding equation (c), we get

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \quad (8)$$

We now check if $\underline{\underline{\sigma}}$ ca fulfill the static admissibility equations.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = \frac{3AB}{r^4} - \frac{3AB}{r^4} = 0 \rightarrow (c) \text{ is OK. Interior equilibrium is satisfied.} \quad (9)$$

$$(a) \rightarrow -A \left[\frac{B}{R_e^3} - 1 \right] = 0 \rightarrow B = R_e^3 \quad (10)$$

$$(b) \rightarrow -A \left[\frac{R_i^3}{R_e^3} - 1 \right] = p_0 \rightarrow A = p_0 \frac{R_e^3}{R_i^3 - R_e^3} \quad (11)$$

All the other conditions concerning $\sigma_{\theta r}$ and $\sigma_{r\varphi}$ are automatically satisfied due to the chosen stress tensor. Thus, a statically admissible stress field is:

$$\underline{\underline{\sigma}} = \begin{bmatrix} p_0 \frac{R_e^3}{R_i^3 - R_e^3} \left[\frac{R_i^3}{r} - 1 \right] & 0 & 0 \\ 0 & -p_0 \frac{R_e^3}{R_i^3 - R_e^3} \left[\frac{R_e^3}{2r^3} + 1 \right] & 0 \\ 0 & 0 & -p_0 \frac{R_e^3}{R_i^3 - R_e^3} \left[\frac{R_e^3}{2r^3} + 1 \right] \end{bmatrix} \quad (12)$$

Question 4: Please calculate the strain tensor $\underline{\underline{\varepsilon}}$? Does it satisfy the compatibility equation? Conclusion?

Answer: Based on the constitutive equation, we can calculate the strains:

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E} \underline{\underline{\sigma}} - \frac{\nu}{E} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} \quad (13)$$

$$\text{tr} \underline{\underline{\sigma}} = -p_0 \frac{R_e^3}{R_e^3 - R_i^3} \quad (14)$$

Thus, we get $\varepsilon_{r\theta} = \varepsilon_{\theta\varphi} = \varepsilon_{\varphi\theta} = 0$. Let us note that

$$A_1 = p_0 \frac{R_e^3}{R_e^3 - R_i^3} \quad (15)$$

We get:

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{r} - 1\right] + \frac{\nu}{E} A_1 & 0 & 0 \\ 0 & -\left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{2r^3} + 1\right] + \frac{\nu}{E} A_1 & 0 \\ 0 & 0 & -\left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{2r^3} + 1\right] + \frac{\nu}{E} A_1 \end{bmatrix} \quad (16)$$

In case of complete spherical symmetry, the compatibility equation can be simplified (see Textbook equation 2.14b):

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} + \frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = 0 \quad (17)$$

Let us verify if this equation is satisfied by $\underline{\underline{\varepsilon}}$.

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} = 3A_1 \left(\frac{1+\nu}{E}\right) \frac{R_e^3}{2r^4} \quad (18)$$

$$\frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = -3A_1 \left(\frac{1+\nu}{E}\right) \frac{R_e^3}{2r^4} \quad (19)$$

The two equations above satisfied the compatibility equation. Thus it will be possible to integrate $\underline{\underline{\varepsilon}}$ to determine a displacement field $\underline{\underline{u}}$. Considering there is no additional kinematic condition (no kinematic boundary condition), we got the exact solution of the problem.

2 Problem 3

Let us consider a domain Ω , simply supported by a rigid basis (Γ). Ω is submitted to gravity volume forces $\underline{\underline{f}} = \rho g \underline{\underline{y}}$ and to uniform pressure $p > 0$ over every vertical side. We assume plane stresses in the plane of $x - y$ and the material is isotropic linear elastic with

Young modulus E and Poisson ratio ν .

Question 1: Write down the full problem! (kinematic, static and constitutive equations)

Question 2: We assume an Airy function $\phi(x, y) = ax^2 + cy^2 + hy^3$. Find a, c, h so we define a statistically admissible stress field.

Question 3: Find the related strain field using the constitutive equation. Does that satisfy the compatibility equation? Conclusion?

Question 4: Integrate the strain field to get the displacement at every point. Can we find the exact solution of the problem using this Airy function?

3 Problem 4

Let us consider a classical test in material engineering that make possible to identify the oedometric modulus \hat{E} . A cylinder material domain Ω is compressed due to the prescribed motion of the top surface $\underline{u} = -U\underline{e}_z$. All surfaces are assumed to be perfect contact without friction. Note that $\underline{P} = -P\underline{e}_z$ the resulting force that is necessary to press the materials (Young modulus E and Poisson ratio ν).

Question 1: Write the complete problem: kinematic + static + constitutive equation!

Answer: Kinematic admissibility:

$$\underline{u} \cdot \underline{n} = 0 \quad \forall \underline{M} \in S_{Lat} \rightarrow u_r = 0 \quad \forall \underline{M} \in S_{Lat} \quad (20)$$

$$\underline{n} \cdot \underline{u} = -U \quad \forall \underline{M} \in S_z^+ \rightarrow u_z = -U \quad \forall \underline{M} \in S_z^+ \quad (21)$$

$$\underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2}(\underline{\nabla}\underline{u} + \underline{\nabla}^T\underline{u}) \quad \forall \underline{M} \in \Omega \quad (22)$$

$$\underline{u} \cdot \underline{n} = 0 \quad \forall \underline{M} \in S_z^- \rightarrow u_z = 0 \quad \forall \underline{M} \in S_z^- \quad (23)$$

Static admissibility:

$$\underline{\text{div}}\underline{\underline{\sigma}} = \underline{0} \quad \forall \underline{M} \in \Omega \quad (24)$$

or in expanded form as below

$$\begin{aligned} \sigma_{rr,r} + \frac{1}{r}(\sigma_{r\theta,\theta} + \sigma_{rr} - \sigma_{\theta\theta}) + \sigma_{rz,z} &= 0 \\ \sigma_{\theta r,r} + \frac{1}{r}(\sigma_{\theta\theta,\theta} + \sigma_{r\theta} + \sigma_{\theta r}) + \sigma_{\theta z,z} &= 0 \quad \rightarrow \quad \forall \underline{M} \in \Omega \\ \sigma_{zr,r} + \frac{1}{r}(\sigma_{z\theta,\theta} + \sigma_{zr}) + \sigma_{zz,z} &= 0 \end{aligned} \quad (25)$$

Contact without friction:

$$(\underline{\underline{\sigma}} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_z^+ \rightarrow \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^+ \quad (26)$$

Contact without friction:

$$(\underline{\underline{\sigma}} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_z^- \rightarrow \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^- \quad (27)$$

Contact without friction:

$$(\underline{\underline{\sigma}} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_{Lat} \rightarrow \sigma_{\theta r} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_{Lat} \quad (28)$$

Constitutive equations:

$$\underline{\underline{\sigma}} = \lambda \text{tr} \underline{\underline{\varepsilon}} \underline{1} + 2\mu \underline{\underline{\varepsilon}} \quad \forall \underline{M} \in \Omega \quad (29)$$

Question 2: Assume that $\underline{u} = -\frac{U}{h} z \underline{e}_z$. Is this kinematically admissible?

Answer: We assume that

$$\underline{u} = \begin{pmatrix} 0 \\ 0 \\ -\frac{U}{h} z \end{pmatrix} = \left(-\frac{U}{h} z \right) \quad \text{Of course } \underline{u} \text{ is kinematically admissible.} \quad (30)$$

By definition (using the gradient of a vector with cylindrical coordinates):

$$\underline{\underline{\nabla}} \cdot \underline{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{U}{h} \end{bmatrix} \rightarrow \underline{\underline{\varepsilon}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \quad (31)$$

with $\varepsilon_{zz} = -U/h \rightarrow$ the compatibility equation is obviously satisfied.

Question 3: Find the corresponding stress $\underline{\underline{\sigma}}$. Is it statistically admissible?

Answer:

$$\underline{\underline{\sigma}} = \lambda \text{tr} \underline{\underline{\varepsilon}} \underline{1} + 2\mu \underline{\underline{\varepsilon}} \rightarrow \underline{\underline{\sigma}} = \begin{bmatrix} -\lambda \frac{U}{h} & 0 & 0 \\ 0 & -\lambda \frac{U}{h} & 0 \\ 0 & 0 & -(\lambda + 2\mu) \frac{U}{h} \end{bmatrix} \quad (32)$$

Eqs. 28, 27, and 26 are obviously satisfied. $\underline{\text{div}} \underline{\underline{\sigma}} \rightarrow$ obviously equal to 0 as $\underline{\underline{\sigma}}$ is constant.

Question 4: Find oedometric modulus \hat{E} as a function of E and ν .

Answer: The vertical force is equal to

$$\underline{P} = \int_{S_z^+} \underline{\underline{\sigma}} \cdot \underline{e}_z dS_3^+ \quad (33)$$

$$\underline{P} = \int_{S_z^+} -(\lambda + 2\mu) \frac{U}{h} dS_3^+ \underline{e}_z = -\underline{P} \underline{e}_z \quad \text{with} \quad \underline{P} = \frac{(\lambda + 2\mu) U S}{h} \quad (34)$$

So we can identify the oedometric modulus $\hat{E} = \lambda + 2\mu$.

$$\begin{aligned} \underline{\underline{\sigma}} &= \lambda \text{tr} \underline{\underline{\varepsilon}} \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}} & \text{tr} \underline{\underline{\sigma}} &= (3\lambda + 2\mu) \text{tr} \underline{\underline{\varepsilon}} \\ \underline{\underline{\varepsilon}} &= \frac{1+\nu}{E} \underline{\underline{\sigma}} - \frac{\nu}{E} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} & \text{tr} \underline{\underline{\varepsilon}} &= \left(\frac{1-2\nu}{E}\right) \text{tr} \underline{\underline{\sigma}} \end{aligned} \rightarrow 3\lambda + 2\mu = \frac{E}{1-2\nu} \quad (35)$$

Or we can also write as follows:

$$\text{tr} \underline{\underline{\sigma}} = (3\lambda + 2\mu) \text{tr} \underline{\underline{\varepsilon}} \rightarrow \underline{\underline{\sigma}} = \frac{\lambda}{3\lambda + 2\mu} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}} \quad (36)$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2\mu} \underline{\underline{\sigma}} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \text{tr} \underline{\underline{\sigma}} \underline{\underline{1}} \quad (37)$$

By identification,

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \quad \text{and} \quad \frac{\lambda}{2\mu(3\lambda + 2\mu)} = \frac{\nu}{E} \quad (38)$$

$$\rightarrow \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} \quad (39)$$

Thus,

$$\hat{E} = \lambda + 2\mu = \frac{2E\nu + 2E(1-2\nu)}{2(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad (40)$$