ME 211A: Infinitesimal deformation/transformation

September 28, 2018

quadratic transformation

In a plane $(\underline{O}, \underline{e}_1, \underline{e}_2)$, we consider an homogeneous transformation $\underline{x} = \underline{\Phi}(\underline{X}, t)$ made of the composition of:
a pure expansion about the axis $(\underline{O}, \underline{e}_1)$ with a dilatation factor λ,
a rotation of angle θ about the third axis $(\underline{O}, \underline{e}_3)$.

Question: Calculate the gradient of the transformation?

Answer: $\underline{\Phi}(\underline{X}, t)$ is defined by:

$$\begin{aligned} x_1 &= \lambda X_1 \cos\theta + X_2 \sin\theta \\ x_2 &= -\lambda X_1 \sin\theta + X_2 \cos\theta \\ x_3 &= X_3 \end{aligned} \tag{1}$$

So the gradient of transformation is:

$$\underline{\underline{F}} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \frac{\partial \underline{x}}{\partial \underline{X}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} \lambda\cos\theta & \sin\theta & 0 \\ -\lambda\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Question: Calculate $\underline{\underline{C}}$.

Answer:

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} \lambda^2 \cos^2\theta + \sin^2\theta & (1-\lambda^2)\sin\theta\cos\theta & 0 \\ (1-\lambda^2)\sin\theta\cos\theta & \lambda^2 sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Question: Calculate $\underline{\underline{e}}$.

Answer:

$$\underline{\underline{e}} = \frac{1}{2}\left(\underline{\underline{C}} - \underline{\underline{1}}\right) = \frac{1}{2} \begin{bmatrix} (\lambda^2-1)\cos^2\theta & (1-\lambda^2)\sin\theta\cos\theta & 0 \\ (1-\lambda^2)\sin\theta\cos\theta & (\lambda^2-1)\, sin^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Question: Calculate the displacement $\underline{\xi}$ and its gradient.

Answer:

$$\underline{\xi} = \underline{x} - \underline{X} = \begin{bmatrix} (\lambda\cos\theta - 1)\, X_1 + X_2 \sin\theta \\ -\lambda X_1 \sin\theta + X_2 \,(\cos\theta - 1) \\ 0 \end{bmatrix} \tag{5}$$

$$\underline{\underline{\nabla\xi}} = \begin{bmatrix} \lambda\cos\theta - 1 & \sin\theta & 0 \\ -\lambda\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

Question: What is the condition for having small deformation $\left\|\underline{\underline{e}}\right\| \ll 1$? What is the physical interpretation for this condition?

Answer:

$$\begin{aligned} &\left\|\underline{\underline{e}}\right\| = \left(\lambda^2 - 1\right)^2 \left(\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\right) \\ &= \left(\lambda^2 - 1\right)^2 \left(\sin^2\theta + \cos^2\theta\right)^2 \\ &= \left(\lambda^2 - 1\right)^2 \ll 1 \end{aligned} \tag{7}$$

Thus, $\lambda \approx 1$ is the condition for having small deformation. The physical interpretation for this condition is that the expansion about axis $(\underline{O}, \underline{e}_1)$ has to be around 1.

Question: Explain what happens in the specific configuration $\lambda = 1, \theta = \pi/2$?

Answer:

It's rigid body rotation. It rotates $\pi/2$.