Problem 1 We consider the transformation defined by

$$\begin{cases} x_1 = X_1 + X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

1. Calculate the gradient of the transformation.

2. Is this an homogeneous transformation? Calculate the volume variation.

3. Calculate the expansion of the following vectors:

$$\underline{e}_1 \; ; \; \underline{e}_2 \; ; \; \frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_2)$$

5. Calculate $\underline{\underline{e}}$, $\underline{\underline{\varepsilon}}$? Can we consider we are in small perturbations?

6. Consider the transformation

$$\begin{cases} x_1 = X_1 + \alpha X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

What is the condition for having small perturbations?

$\boxed{\text{Problem 1}}$ We study the transformation described on fig 1, $\underline{x} = \underline{\phi}(\underline{X})$

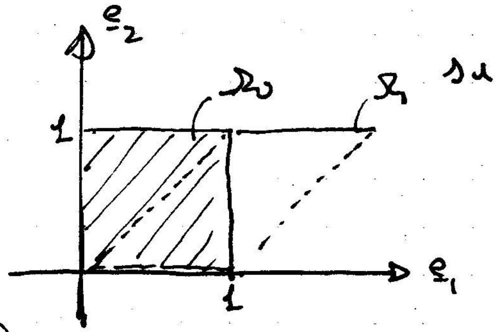

fig 1.

such that:

$$\begin{cases} x_1 = X_1 + X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

① Calculate the gradient of the transformation $\underline{\underline{F}}$?

$$\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \frac{\partial \underline{\phi}}{\partial \underline{X}} = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Carthesien basis.

② Is this transformation reasonable from a physic point of view? Calculate the volume variation?

$|\Omega| = J\,|\Omega_0|$ with $J = \det \underline{\underline{F}} = 1 \rightarrow$ no volume variation.

③ Is this an homogeneous transformation?

· Yes, $\underline{\underline{F}}$ does not depend on $(\underline{X})$. | straight lines remain straight. In other words, we can write:

$\underline{x} = \underline{\underline{F}} \cdot \underline{X} + \underline{c}$ with $\underline{\underline{F}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

④ Calculate the expansion of the following vectors:

$$\underbrace{\underline{e}_1}_{\underline{V}_1} \; ; \; \underbrace{\underline{e}_2}_{\underline{V}_2} \; ; \; \underbrace{\frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_2)}_{\underline{V}_3}$$

by definition: $\forall \underline{V} \to \underline{v}$: $\underline{v} \cdot \underline{v} = \underline{V} \cdot \underline{\underline{C}} \cdot \underline{V}$

Then, the expansion $\dfrac{|\underline{v}|}{|\underline{V}|} = \dfrac{\sqrt{\underline{V} : \underline{\underline{C}} \cdot \underline{V}}}{\sqrt{\underline{V} \cdot \underline{V}}} = \lambda_{\underline{V}}$ with $\underline{\underline{C}} = {}^t\underline{\underline{F}} \cdot \underline{\underline{F}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

for $\underline{V}_1 = \underline{e}_1$ $\quad \lambda_{\underline{V}_1} = \sqrt{\dfrac{\underline{e}_1 \cdot \underline{\underline{C}} \cdot \underline{e}_1}{\underline{e}_1 \cdot \underline{e}_1}} = 1 \rightsquigarrow$ No length variation.

for $\underline{V}_2 = \underline{e}_2$ $\quad \lambda_{\underline{V}_2} = \sqrt{\dfrac{\underline{e}_2 \cdot \underline{\underline{C}} \cdot \underline{e}_2}{\underline{e}_2 \cdot \underline{e}_2}} = \sqrt{2} \rightsquigarrow$ See figure.

for $\underline{V}_3 = \frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_2)$ $\quad \lambda_{\underline{V}_3} = \sqrt{\dfrac{\underline{V}_3 \cdot \underline{\underline{C}} \cdot \underline{V}_3}{\underline{V}_3 \cdot \underline{V}_3}} = \sqrt{\dfrac{5}{2}}$

) Calculate the linearized strain tensor $\underline{\underline{\varepsilon}}$?

$$\underline{\underline{e}} = \frac{1}{2} \left\{ \underline{\underline{C}} - \underline{\underline{1}} \right\} = \text{Green-Lagrange tensor} = \underbrace{\frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{General deformation tensor}}$$

Now, if we calculate $\underline{\underline{\varepsilon}}$, we have to start with the displacement:

$$\underline{x} = \underline{X} + \underline{\xi} \quad \text{with } \underline{\xi} = \begin{pmatrix} X_2 \\ 0 \\ 0 \end{pmatrix} \rightarrow \underline{\underline{\nabla}}\, \underline{\xi} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then: $\underline{\underline{\varepsilon}} = \frac{1}{2} \left\{ \underline{\underline{\nabla}}\, \underline{\xi} + {}^t\underline{\underline{\nabla}}\, \underline{\xi} \right\} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

In this situation, $\underline{\underline{e}} \neq \underline{\underline{\varepsilon}} \Rightarrow \underline{\underline{\varepsilon}}$ can not be a good approximation the strain, as we don't work in small perturbations.

Indeed: what would be the framework to have $\underline{\underline{e}} \sim \underline{\underline{\varepsilon}}$?

Let's assume a small perturbation: $\begin{cases} x_1 = X_1 + \alpha X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$

Then: $\underline{\xi}$ (displacement) $= \begin{pmatrix} \alpha X_2 \\ 0 \\ 0 \end{pmatrix}$ and $\underline{\underline{\nabla}}\,\underline{\xi} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\|\underline{\underline{\nabla}}\,\underline{\xi}\| = \sqrt{\underline{\underline{\nabla}}\underline{\xi} : \underline{\underline{\nabla}}\underline{\xi}} = \sqrt{\alpha^2} \ll 1$. Then, to be in small perturbation we should have $|\alpha| \ll 1$. In this situation:

$$\underline{\underline{e}} = \frac{1}{2}\left\{ \underline{\underline{\nabla}}\,\underline{\xi} + {}^t\underline{\underline{\nabla}}\,\underline{\xi} \right\} + \frac{1}{2}\,{}^t\underline{\underline{\nabla}}\underline{\xi} \cdot \underline{\underline{\nabla}}\underline{\xi}$$

$$\underline{\underline{e}} = \underbrace{\frac{1}{2}\left\{ \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}}_{\underline{\underline{\varepsilon}}} + \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{second order}}$$