Practice:
Static and Kinematic Admissibility

Problem 2

Let us consider a sphere with cavity (exterior radius R_e and internal radius R_i). The external surface S_e is free of loading and the interior surface S_i is submitted to a pressure p_0. The material is assumed to be isotropic linear elastic with young modulus E and Poisson ratio ν.

Question 1: Write the static admissibility equations!

Answer: Find $\underline{\underline{\sigma}}$ symmetric such that

$$\begin{aligned}
&(a)\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \quad \forall\ M \in S_e \quad \text{Boundary equation} \\
&(b)\ \underline{\underline{\sigma}} \cdot \underline{n} = -p_0 \underline{n} \quad \forall\ M \in S_i \quad \text{Boundary equation} \\
&(c)\ \underline{\mathrm{div}\underline{\underline{\sigma}}} = \underline{0} \quad \forall\ M \in \Omega \quad \text{Internal equilibrium}
\end{aligned} \tag{1}$$

Question 2: Please expand this equation using spherical coordinates system!

Answer: In spherical coordinates,

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\varphi} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta\varphi} \\ \sigma_{r\varphi} & \sigma_{\theta\varphi} & \sigma_{\varphi\varphi} \end{bmatrix} \tag{2}$$

From Eq. (a),

$$\sigma_{rr}(r = R_e, \theta, \varphi) = \sigma_{r\theta}(r = R_e, \theta, \varphi) = \sigma_{r\varphi}(r = R_e, \theta, \varphi) = 0 \tag{3}$$

From Eq. (b),

$$\begin{aligned}
&-\sigma_{rr}(r = R_i, \theta, \varphi) = p_0 \\
&-\sigma_{r\theta}(r = R_i, \theta, \varphi) = 0 \\
&-\sigma_{r\varphi}(r = R_i, \theta, \varphi) = 0
\end{aligned} \tag{4}$$

From Eq. (c) $\rightarrow$ See Textbook equation 1.11a (divergence with spherical coordinates)

Question 3: Let's assume that the following stress tensor

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{bmatrix} \tag{5}$$

with

$$\sigma_{rr} = -A\left[\frac{B}{r^3} - 1\right] \tag{6}$$

and

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = A\left[\frac{B}{2r^3} + 1\right] \tag{7}$$

Please calculate the coefficient A and B so that $\underline{\underline{\sigma}}$ could be a solution of the problem!

Answer: Using the proposed stress tensor, and expanding equation (c), we get

$$\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = 0 \\ &0 = 0 \\ &0 = 0 \end{aligned} \tag{8}$$

We now check if $\underline{\underline{\sigma}}$ ca fulfill the static admissibility equations.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{r} = \frac{3AB}{r^4} - \frac{3AB}{r^4} = 0 \rightarrow \text{(c) is OK. Interior equilibrium is satisfied.} \tag{9}$$

$$\text{(a)} \rightarrow -A\left[\frac{B}{R_e^3} - 1\right] = 0 \rightarrow B = R_e^3 \tag{10}$$

$$\text{(b)} \rightarrow -A\left[\frac{R_i^3}{R_e^3} - 1\right] = p_0 \rightarrow A = p_0 \frac{R_e^3}{R_i^3 - R_e^3} \tag{11}$$

All the other conditions concerning $\sigma_{\theta r}$ and $\sigma_{r\varphi}$ are automatically satisfied due to the chosen stress tensor. Thus, a statically admissible stress field is:

$$\underline{\underline{\sigma}} = \begin{bmatrix} p_0 \frac{R_e^3}{R_i^3 - R_e^3}\left[\frac{R_i^3}{r} - 1\right] & 0 & 0 \\ 0 & -p_0 \frac{R_e^3}{R_e^3 - R_i^3}\left[\frac{R_e^3}{2r^3} + 1\right] & 0 \\ 0 & 0 & -p_0 \frac{R_e^3}{R_e^3 - R_i^3}\left[\frac{R_e^3}{2r^3} + 1\right] \end{bmatrix} \tag{12}$$

Question 4: Please calculate the strain tensor $\underline{\underline{\varepsilon}}$? Does it satisfy the compatibility equation? Conclusion?

Answer: Based on the constitutive equation, we can calculate the strains:

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}\mathrm{tr}\underline{\underline{\sigma}}\,\underline{\underline{1}} \tag{13}$$

$$\mathrm{tr}\underline{\underline{\sigma}} = -p_0 \frac{R_e^3}{R_e^3 - R_i^3} \tag{14}$$

Thus, we get $\varepsilon_{r\theta} = \varepsilon_{\theta\varphi} = \varepsilon_{\varphi\theta} = 0$. Let us note that

$$A_1 = p_0 \frac{R_e^3}{R_e^3 - R_i^3} \tag{15}$$

We get:

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{r} - 1\right] + \frac{\nu}{E}A_1 & 0 & 0 \\ 0 & -\left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{2r^3} + 1\right] + \frac{\nu}{E}A_1 & 0 \\ 0 & 0 & -\left(\frac{1+\nu}{E}\right) A_1 \left[\frac{R_e^3}{2r^3} + 1\right] + \frac{\nu}{E}A_1 \end{bmatrix} \tag{16}$$

In case of complete spherical symmetry, the compatibility equation can be simplified (see Textbook equation 2.14b):

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} + \frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = 0 \tag{17}$$

Let us verify if this equation is satisfied by $\underline{\underline{\varepsilon}}$.

$$\frac{d\varepsilon_{\varphi\varphi}}{dr} = 3A_1 \left(\frac{1+\nu}{E}\right) \frac{R_e^3}{2r^4} \tag{18}$$

$$\frac{\varepsilon_{\varphi\varphi} - \varepsilon_{rr}}{r} = -3A_1 \left(\frac{1+\nu}{E}\right) \frac{R_e^3}{2r^4} \tag{19}$$

The two equations above satisfied the compatibility equation. Thus it will be possible to integrate $\underline{\underline{\varepsilon}}$ to determine a displacement field $\underline{u}$. Considering there is no additional kinematic condition (no kinematic boundary condition), we got the exact solution of the problem.

Problem 3

Let us consider a domain Ω, simply supported by a rigid basis (Γ). Ω is submitted to gravity volume forces $\underline{f} = \rho g \underline{y}$ and to uniform pressure $p > 0$ over every vertical side. We assume plane stresses in the plane of $x - y$ and the material is isotropic linear elastic with Young modulus E and Poisson ratio ν.

Question 1: Write down the full problem! (kinematic, static and constitutive equations)

Answer: Kinematic admissibility:

Let us not $\underline{u}$ the displacement vector such that $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\underline{u} = u\underline{x} + v\underline{y}$

$$\begin{cases} v = 0, \quad \forall M \in S_y^+ \\ \underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}^t\right), \quad \text{compatibility} \end{cases}$$

Static admissibility: Find $\underline{\underline{\sigma}}$ symmetric such that

$$\left.\begin{aligned} &(1)\ \underline{\underline{\sigma}} \cdot \underline{n} = -p\underline{n} \quad \forall\ M \in S_x^+ \\ &(2)\ \underline{\underline{\sigma}} \cdot \underline{n} = -p\underline{n} \quad \forall\ M \in S_x^- \\ &(3)\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \quad \forall\ M \in S_y^- \\ &(4)\ \left(\underline{\underline{\sigma}} \cdot \underline{n}\right) \cdot \underline{t} = \underline{0} \quad \forall\ M \in S_y^+ \end{aligned}\right\}\text{Boundary equation} \tag{20}$$

$$\underline{\text{div}\underline{\underline{\sigma}}} + \rho g \underline{y} = \underline{0} \quad \forall\ M \in \Omega \quad \text{Internal equilibrium} \tag{21}$$

As plane stress are assumed, the equation can be reduced to:

$$\left.\begin{aligned} &\sigma_{xx,x} + \sigma_{xy,y} = 0 \\ &\sigma_{xy,x} + \sigma_{yy,y} + \rho g = 0 \\ &\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \end{aligned}\right\}\forall\ M \in \Omega \tag{22}$$

Constitutive equation: The 3D constitutive equation is:

$$\underline{\underline{\epsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}\text{tr}\underline{\underline{\sigma}}\underline{\underline{1}} \tag{23}$$

Here, we can use the plane stress assumption ($\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$)

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} = \left(\frac{1+\nu}{E}\right)\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{24}$$

As a consequence, we can also write this equation in 2D:

$$\begin{cases} \underline{\underline{\epsilon}}^{2D} = \dfrac{1+\nu}{E}\underline{\underline{\sigma}}^{2D} - \dfrac{\nu}{E}\text{tr}\underline{\underline{\sigma}}^{2D}\underline{\underline{1}}^{2D} \\ \epsilon_{xz} = \epsilon_{yz} = 0 \\ \epsilon_{zz} = -\dfrac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \quad \text{the out of plane strain is not equal to zero} \end{cases} \tag{25}$$

Question 2: We assume an Airy function $\phi(x,y) = ax^2 + cy^2 + hy^3$. Find a, c, h so we define a statistically admissible stress field.

Answer: We want to use an approach based on Airy function but we have to be careful in case of volume force.
Let us assume the volume force density $\underline{f}$ derive from a potential:

$$\underline{f} = \underline{\underline{\text{grad}}}V \tag{26}$$

For example here: $V = \rho g y \Rightarrow \underline{f} = \underline{\underline{\text{grad}}}V = \begin{pmatrix} 0 \\ \rho g \end{pmatrix}$
In this case:

$$\begin{cases} \dfrac{\partial(\sigma_{xx}+V)}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial(\sigma_{yy}+V)}{\partial y} = 0 \end{cases} \tag{27}$$

and $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ can be defined based on the Airy function by the following equation:

$$\sigma_{xx} + V = \frac{\partial^2 \phi}{\partial y^2}; \quad \sigma_{yy} + V = \frac{\partial^2 \phi}{\partial x^2}; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{28}$$

If $\phi(x,y) = ax^2 + cy^2 + hy^3$

$$\begin{aligned} \sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} - V = 2C + 6hy - \rho g y \rightarrow \sigma_{xx,x} = 0 \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} - V = 2a - \rho g y \rightarrow \sigma_{yy,y} = -\rho g \\ \sigma_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} = 0 \end{aligned} \tag{29}$$

$$\text{Interior equation:} \begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 \rightarrow \text{OK} \\ \sigma_{xy,x} + \sigma_{yy,y} + \rho g = 0 \rightarrow \text{OK} \end{cases} \tag{30}$$

Boundary static equation:

$$\begin{aligned} &(1) \Rightarrow \begin{cases} \sigma_{xx}(x = B/2, y) = -p, \quad \forall y \rightarrow \begin{cases} 2c = -p \\ 6h - \rho g = 0 \end{cases} \rightarrow \begin{cases} c = -p/2 \\ h = \dfrac{\rho g}{6} \end{cases} \\ \sigma_{xy}(x = B/2, y) = 0, \quad \forall y \quad \text{OK} \end{cases} \\ &(2) \Rightarrow \begin{cases} \sigma_{xx}(x = -B/2, y) = -p, \quad \forall y \quad \text{OK} \\ \sigma_{xy}(x = -B/2, y) = 0, \quad \forall y \quad \text{OK} \end{cases} \\ &(3) \Rightarrow \begin{cases} \sigma_{yy}(y = 0, x) = 0, \quad \forall x \rightarrow 2a = 0 \rightarrow a = 0 \\ \sigma_{xy}(y = 0, x) = 0, \quad \forall x \quad \text{OK} \end{cases} \\ &(4) \Rightarrow \begin{cases} \sigma_{xy}(y = L, x) = 0, \quad \forall x \quad \text{OK} \end{cases} \end{aligned} \tag{31}$$

Finally, we have $a = 0; c = -p/2; h = \rho g/6$, and we get the following statistically admissible stress field:

$$\begin{cases} \sigma_{xx} = -p \\ \sigma_{yy} = -\rho g y \\ \sigma_{xy} = 0 \\ \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \end{cases} \tag{32}$$

Question 3: Find the related strain field using the constitutive equation. Does that satisfy the compatibility equation? Conclusion?

Answer: As a consequence, using the constitutive equation:

$$\begin{cases} \epsilon_{xx} = \dfrac{1}{E}\sigma_{xx} - \dfrac{\nu}{E}\sigma_{yy} = -\dfrac{p}{E} + \dfrac{\nu\rho g}{E}y \\ \epsilon_{yy} = -\dfrac{\rho g}{E}y + \dfrac{\nu p}{E} \\ \epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0 \\ \epsilon_{zz} = \dfrac{\nu}{E}\left(p + \rho g y\right) \end{cases} \tag{33}$$

Check the compatibility:
The compatibility equation are obviously satisfied as we just have linear relations (and the compatibility equations involves the second derivation).

Question 4: Integrate the strain field to get the displacement at every point. Can we find the exact solution of the problem using this Airy function?

Answer:

$$\begin{aligned} u_{,x} &= -\frac{p}{E} + \frac{\nu\rho g}{E}y \rightarrow u = \left(-\frac{p}{E} + \frac{\nu\rho g}{E}y\right)x + f(y) \\ v_{,y} &= -\frac{\rho g}{E}y + \frac{\nu p}{E} \rightarrow v = -\frac{\rho g}{2E}y^2 + \frac{\nu p}{E}y + g(x) \\ \epsilon_{xy} &= \frac{1}{2}(u_{,y} + v_{,x}) = 0 \rightarrow \frac{\nu\rho g}{E}x + \frac{\partial f(y)}{y} + \frac{\partial g(x)}{x} = 0 \end{aligned} \tag{34}$$

$$\begin{aligned} &\frac{\partial f(y)}{y} = \lambda \rightarrow f(y) = \lambda y + \beta \\ &\frac{\nu\rho g}{E}x + \frac{\partial g(x)}{x} = -\lambda \rightarrow g(x) = -\frac{\nu\rho g}{2E}x^2 - \lambda x + \gamma \end{aligned} \tag{35}$$

Thus:

$$\begin{cases} u = \left(-\dfrac{p}{E} + \dfrac{\nu\rho g}{E}y\right)x + \lambda y + \beta \\ v = -\dfrac{\rho g}{2E}y^2 + \dfrac{\nu p}{E}y - \dfrac{\nu\rho + g}{2E}x^2 - \lambda x + \gamma \end{cases} \tag{36}$$

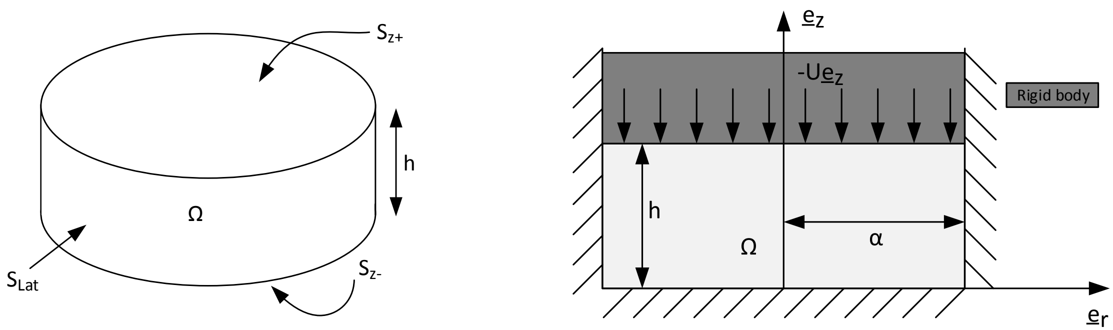
Introducing the kinematic boundary equations:

$$v(x, y = L) \Rightarrow \text{this solution can not be satisfied} \tag{37}$$

We do not have the exact solution.

Problem 4

Let us consider a classical test in material engineering that make possible to identify the oedometric modelus $\hat{E}$. A cylinder material domain Ω is compressed due to the prescribed motion of the top surface $\underline{u} = -U\underline{e}_z$. All surfaces are assumed to be perfect contact without friction. Note that $\underline{P} = -P\underline{e}_z$ the resulting force that is necessary to press the materials (Young modulus E and Poisson ratio ν.



(a) 3D isometric view

(b) cross-sectional view

Figure 1: Oedometric compression

Question 1: Write the complete problem: kinematic + static + constitutive equation!

Answer: Kinematic admissibility:

$$\underline{u} \cdot \underline{n} = 0 \quad \forall \underline{M} \in S_{Lat} \rightarrow u_r = 0 \quad \forall \underline{M} \in S_{Lat} \tag{38}$$

$$\underline{n} \cdot \underline{u} = -U \quad \forall \underline{M} \in S_z^+ \rightarrow u_z = -U \quad \forall \underline{M} \in S_z^+ \tag{39}$$

$$\underline{\underline{\varepsilon}}(\underline{u}) = \frac{1}{2}(\underline{\underline{\nabla}} u + \underline{\underline{\nabla}}^T \underline{u}) \quad \forall \underline{M} \in \Omega \tag{40}$$

$$\underline{u} \cdot \underline{n} = 0 \quad \forall \underline{M} \in S_z^- \rightarrow u_z = 0 \quad \forall \underline{M} \in S_z^- \tag{41}$$

Static admissibility:

$$\underline{\mathrm{div}\underline{\underline{\sigma}}} = \underline{0} \quad \forall \underline{M} \in \Omega \tag{42}$$

or in expanded form as below

$$\left.\begin{aligned} &\sigma_{rr},_r + \frac{1}{r}(\sigma_{r\theta},_\theta + \sigma_{rr} - \sigma_{\theta\theta}) + \sigma_{rz},_z = 0 \\ &\sigma_{\theta r},_r + \frac{1}{r}(\sigma_{\theta\theta},_\theta + \sigma_{r\theta} + \sigma_{\theta r}) + \sigma_{\theta z},_z = 0 \\ &\sigma_{zr},_r + \frac{1}{r}(\sigma_{z\theta},_\theta + \sigma_{zr}) + \sigma_{zz},_z = 0 \end{aligned}\right. \qquad \rightarrow \qquad \forall \underline{M} \in \Omega \tag{43}$$

Contact without friction:

$$(\underline{\underline{\sigma}} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_z^+ \to \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^+ \tag{44}$$

Contact without friction:

$$(\underline{\underline{\sigma}} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_z^- \to \sigma_{rz} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_z^- \tag{45}$$

Contact without friction:

$$(\underline{\underline{\sigma}} \cdot \underline{t}) \cdot \underline{t} = 0 \quad \forall \underline{M} \in S_{Lat} \to \sigma_{\theta r} = \sigma_{\theta z} = 0 \quad \underline{M} \in S_{Lat} \tag{46}$$

Constitutive equations:

$$\underline{\underline{\sigma}} = \lambda \mathrm{tr} \underline{\underline{\varepsilon}} \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}} \quad \forall \underline{M} \in \Omega \tag{47}$$

Question 2: Assume that $\underline{u} = -\frac{U}{h} z \underline{e}_z$. Is this kinematically admissible?

Answer: We assume that

$$\underline{u} = \begin{pmatrix} 0 \\ 0 \\ -\frac{U}{h} z \end{pmatrix} = \left(-\frac{U}{h} z \right) \quad \text{Of course } \underline{u} \text{ is kinematically admissible.} \tag{48}$$

By definition (using the gradient of a vector with cylindrical coordinates):

$$\underline{\underline{\nabla}} \cdot \underline{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{U}{h} \end{bmatrix} \to \underline{\underline{\varepsilon}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \tag{49}$$

with $\varepsilon_{zz} = -U/h \to$ the compatibility equation is obviously satisfied.

Question 3: Find the corresponding stress $\underline{\underline{\sigma}}$. Is it statistically admissible?

Answer:

$$\underline{\underline{\sigma}} = \lambda \mathrm{tr} \underline{\underline{\varepsilon}} \underline{\underline{1}} + 2\mu \underline{\underline{\varepsilon}} \to \underline{\underline{\sigma}} = \begin{bmatrix} -\lambda \frac{U}{h} & 0 & 0 \\ 0 & -\lambda \frac{U}{h} & 0 \\ 0 & 0 & -(\lambda + 2\mu)\frac{U}{h} \end{bmatrix} \tag{50}$$

Eqs. 46, 45, and 44 are obviously satisfied. $\underline{\mathrm{div} \underline{\underline{\sigma}}} \to$ obviously equal to 0 as $\underline{\underline{\sigma}}$ is constant.

Question 4: Find oedometric modulus $\hat{E}$ as a function of E and ν.
Answer: The vertical force is equal to

$$\underline{P} = \int_{S_z^+} \underline{\underline{\sigma}} \cdot \underline{e}_z dS_3^+ \tag{51}$$

$$\underline{P} = \int_{S_z^+} -(\lambda + 2\mu)\frac{U}{h} dS_3^+ \underline{e}_z = -\underline{P e}_z \quad \text{with} \quad \underline{P} = \frac{(\lambda + 2\mu)US}{h} \tag{52}$$

So we can identify the oedometric modulus $\hat{E} = \lambda + 2\mu$.

$$\begin{array}{l} \underline{\underline{\sigma}} = \lambda \mathrm{tr}\underline{\underline{\varepsilon}}\underline{\underline{1}} + 2\mu\underline{\underline{\varepsilon}} \\ \underline{\underline{\varepsilon}} = \dfrac{1+\nu}{E}\underline{\underline{\sigma}} - \dfrac{\nu}{E}\mathrm{tr}\underline{\underline{\sigma}}\underline{\underline{1}} \end{array} \quad \rightarrow \quad \begin{array}{c} \mathrm{tr}\underline{\underline{\sigma}} = (3\lambda + 2\mu)\ \mathrm{tr}\underline{\underline{\varepsilon}} \\ \mathrm{tr}\underline{\underline{\varepsilon}} = (\dfrac{1-2\nu}{E})\ \mathrm{tr}\underline{\underline{\sigma}} \end{array} \quad \rightarrow \quad 3\lambda + 2\mu = \frac{E}{1-2\nu} \tag{53}$$

Or we can also write as follows:

$$\mathrm{tr}\underline{\underline{\sigma}} = (3\lambda + 2\mu)\mathrm{tr}\underline{\underline{\varepsilon}} \quad \rightarrow \quad \underline{\underline{\sigma}} = \frac{\lambda}{3\lambda + 2\mu}\mathrm{tr}\underline{\underline{\sigma}}\underline{\underline{1}} + 2\mu\underline{\underline{\varepsilon}} \tag{54}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2\mu}\underline{\underline{\sigma}} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\mathrm{tr}\underline{\underline{\sigma}}\underline{\underline{1}} \tag{55}$$

By identification,

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \quad \text{and} \quad \frac{\lambda}{2\mu(3\lambda + 2\mu)} = \frac{\nu}{E} \tag{56}$$

$$\rightarrow \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} \tag{57}$$

Thus,

$$\hat{E} = \lambda + 2\mu = \frac{2E\nu + 2E(1-2\nu)}{2(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \tag{58}$$