TBC 603 Computer Based Numerical and Statistical Techniques

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Unit – 1

Floating Point Arithmetic

- The term floating point refers to the fact that a number's radix point (decimal point, or, more commonly in computers, binary point) can "float";

that is, it can be placed anywhere relative to the significant digits of the number.

Representation of Floating point number–

- The following description explains terminology and primary details of IEEE 754 binary floating point representation. The discussion confines to single and double precision formats.
- Usually, a real number in binary will be represented in the following format,

  ImIm-1…I2I1I0.F1F2…FnFn-1
- Where Im and Fn will be either 0 or 1 of integer and fraction parts respectively.
- A finite number can also represented by four integers components, a sign (s), a base (b), a significant (m), and an exponent (e). Then the numerical value of the number is evaluated as

  (-1) s x m x be ________ Where m < |b|
- Depending on base and the number of bits used to encode various components, the IEEE 754 standard defines five basic formats. Among the five formats, the binary32 and the binary64 formats are single precision and double precision formats respectively in which the base is 2.

One More Somple Way to Understand –

An n-digit floating point number in base β has the form

x = ±(0.d1d2 · · · dn)β × β e

where
0.d1d2 · · · dn is a β-fraction called the mantissa and e is an integer called the exponent. Such a floating point number is called normalised if
d1 6= 0, or else, d1 = d2 = · · · = dn =.
The exponent e is limited to a range m < e < M Usually, m = −M.

Operations–

- A floating-point operation is any mathematical operation (such as +, -, *, /) or assignment that involves floating-point numbers (as opposed to binary integer operations).

- Floating-point numbers have decimal points in them. The number 2.0 is a floating-point number because it has a decimal in it. The number 2 (without a decimal point) is a binary integer.

- Floating-point operations involve floating-point numbers and typically take longer to execute than simple binary integer operations. For this reason, most embedded applications avoid wide-spread usage of floating-point math in favor of faster, smaller integer operations.

Normalization–

- Many floating point representations have an implicit hidden bit in the mantissa. This is a bit which is present virtually in the mantissa, but not stored in memory because its value is always 1 in a normalized number.

- We say that the floating point number is normalized if the fraction is at least 1/b, where b is the base. In other words, the mantissa would be too large to fit if it were multiplied by the base. Non-normalized numbers are sometimes called de-normal; they contain less precision than the representation normally can hold.
- If the number is not normalized, then you can subtract 1 from the exponent while multiplying the mantissa by the base, and get another floating point number with the same value. Normalization consists of doing this repeatedly until the number is normalized. Two distinct normalized floating point numbers cannot be equal in value.

Pitfalls of floating point representation–

- Current critical systems often use a lot of floating-point computations, and thus the testing or static analysis of programs containing floating-point operators has become a priority. However, correctly defining the semantics of common implementations of floating-point is tricky, because semantics may change according to many factors beyond source-code level, such as choices made by compilers. We here give concrete examples of problems that can appear and solutions for implementing in analysis software.

Error in numerical computation–

- Often in Numerical Analysis we have to make approximations to numerically compute things. That's the thing; in Calculus we can take limits and arrive at exact results, but when we use a computer to calculate, say something simple like a derivative, we can't take an infinite limit so we have to approximate the answer, and therefore, it has error. Computers are limited in their capacity to store and calculate the precision and magnitude of numbers.
- We can characterize error in measurements and computations with respect to their accuracy and their precision. Accuracy refers to how closely a calculated value agrees with the true value. Precision refers to how closely calculated values agree with each other. Inaccuracy (also called bias) is a systematic deviation from the true values. Imprecision (also called uncertainty) refers to how close together calculated results are to each other. We use the term error to represent both inaccuracy and imprecision in our results.
- The relationship between the exact result and the approximation can be formulated as true value=approximation+error

Iterative methods

- Iterative method. In computational mathematics, an iterative method is a mathematical procedure that uses an initial guess to generate a sequence of improving approximate solutions for a class of problems, in which the nth approximation is derived from the previous ones.

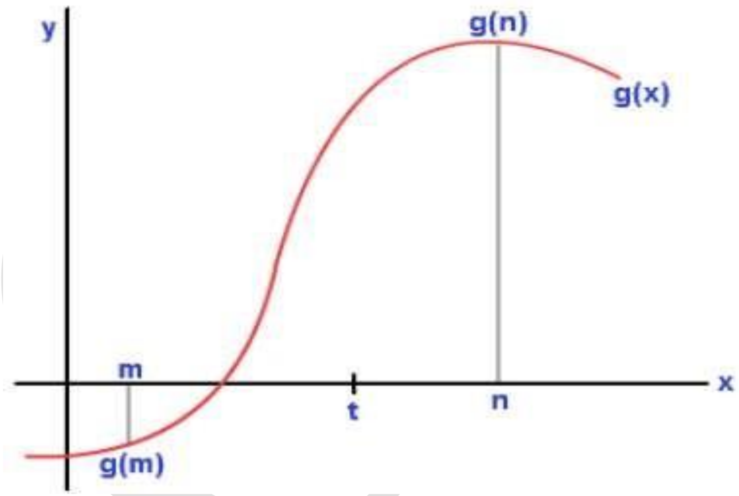
Bisection methods–

- The bisection method in mathematics is a root-finding methodthat repeatedly bisects an interval and then selects a subinterval in which a root must lie for further processing. ... The method is also called the interval halving method, the binary searchmethod, or the dichotomy method.

How To Solve

- The Bisection method is a approximation method to find the roots of an equation by continuously dividing an interval. It will divide the interval in halves until the resulting interval found, which is extremely small.

There is no any specific formula to find the root of a function using bisection method:



For the ith iteration, the interval width is:

ΔΔxii = 1212 = 0.5 and ΔΔxi−1i−1 = (0.5)ii(n-m); where n > m

So the new midpoint is xii = mi−1i−1 + ΔΔxii

for i = 1, 2, 3, ..., n

-
- elow are certain steps to get the solution for the function.
  For a continuous function $g(x)$

  Step 1: Find two points, say m and n st $m < n$ and $g(m) * g(n) < 0$

  Step 2: Find the midpoint of m and n, say t.

  Step 3: t is root of function if $g(t) = 0$, else follow the next step.

  Step 4: Divide the interval [m, n]. If $g(t) * g(n) < 0$, let $m = t$, else if $g(t) * g(m) < 0$ then let $n = t$.

  Step 5: Repeat above two steps until $g(t) = 0$.
- Example: Find the root of the polynomial, $g(x) = x33 - 5 + 3x$ using bisection method. Where $m = 1$ and $n = 2$?

  Solution:
  First find the value of $g(x)$ at $m = 1$ and $n = 2$

  $g(1) = 133 - 5 + 3*1 = -1 < 0$

  $g(2) = 233 - 5 + 3*2 = 9 > 0$

  Since function is continuous, its root lies in the interval [1, 2].

  Let t be the average of the interval i.e $t = 1+221+22 = 1.5$

  The value of the function at t is

  $g(1.5) = (1.5)33 - 5 + 3*(1.5) = 2.875$

  As $g(t)$ is negative so $n = 2$ is replaced with $t = 1.5$ for the next iteration. Make sure that $g(m)$ and $g(n)$ have opposite signs.

Below table contains nine iterations of the function -

Iteration	m	n	t	g(m)	g(n)	g(t)
1	1	2	1.5	-1	9	2.875
2	1	1.5	1.25	-1	2.875	0.703125
3	1	1.25	1.125	-1	0.703125	-0.201171875
4	1.125	1.25	1.1875	-0.201171875	0.703125	0.237060546875
5	1.125	1.1875	1.15625	-0.201171875	0.237060546875	0.014556884765625
6	1.125	1.15625	1.140625	-0.201171875	0.014556884765625	-0.0941429138183594
7	1.140625	1.15625	1.1484375	-0.0941429138183594	0.014556884765625	-0.0400032997131348
8	1.1484375	1.15625	1.15234375	-0.0400032997131348	0.014556884765625	-0.0127759575843811
9	1.15234375	1.15625	1.154296875	-0.0127759575843811	0.014556884765625	0.000877253711223602

Therefore we chose m = 1.15234375 to be our approximated solution.

Regula – Falsi method–

- The Regula falsi method is an oldest method for computing the real roots of an algebraic equation. This below worksheet helps you to understand how to compute the roots of an algebraic equation using Regula falsi method. The practice problems along with this worksheet improve your problem solving capabilities when you try on your own.

Examples:

Find the root between (2,3) of x3+ - 2x - 5 = 0, by using regular falsi method.

Given

f(x) = x3 - 2 x - 5 f(2) = 23 - 2
(2) - 5 = -1 (negative) f(3) = 33 -
2 (3) - 5 = 16 (positive)

Let us take a= 2 and b= 3.
The first approximation to root is x1 and is given by x1 = (a f(a) - b f(b))/(f(b)-f(a)) =(2 f(3)- 3 f(2))/(f(3) - f(2))
=(2 x 16 - 3 (-1))/ (16- (-1))
= (32 + 3)/(16+1) =35/17
= 2.058

Now f(2.058) = 2.0583 - 2 x 2.058 - 5
= 8.716 - 4.116 - 5
= - 0.4
The root lies between 2.058 and 3

Taking a = 2.058 and b = 3. we have the second approximation to the root given by x2 = (a f(a) - b f(b))/(f(b)-f(a))
= (2.058 x f(3) - 3 x f(2.058)) /(f(3) - f(2.058))
= (2.058 x 16 -3 x -0.4) / (16 - (-0.4))
= 2.081

Now f(2.081) = 2.0812 - 2 x 2.081 - 5
= -0.15
The root lies between 2.081 and 3
Take a = 2.081 and b = 3
The third approximation to the root is given by x3
= (a f(a) - b f(b))/(f(b)-f(a))
= (2.089 X 16 - 3 x (-0.062))/ (16 - (-0.062))
= 2.093
The root is 2.09

Newton – Raphson method–

- The Newton-Raphson method, or Newton Method, is a powerful technique for solving equations numerically. Like so much of the differential calculus, it is based on the simple idea of linear approximation. The Newton Method, properly used, usually homes in on a root with devastating efficiency.

Let x_0 be a good estimate of r and let $r = x_0 + h$. Since the true root is r, and $h = r - x_0$, the number h measures how far the estimate x_0 is from the truth.

Since h is 'small,' we can use the linear (tangent line) approximation to conclude that

$$0 = f(r) = f(x_0 + h) \approx f(x_0) + hf'(x_0),$$

and therefore, unless $f'(x_0)$ is close to 0,

$$h \approx - f(x_0)/ f'(x_0) .$$

It follows that $r = x_0 + h \approx$

$x_0 - f(x_0) / f'(x_0)$.

Our new improved (?) estimate x1 of r is therefore given by

$x1 = x_0 - f(x_0) / f'(x\ )$.

The next estimate x is obtained from x_1 in exactly the same way as x_1 was obtained from x : $x_2 = x_1 - f(x\ )\ f\ (x\ )$.

Continue in this way. If x_n is the current estimate, then the next estimate x_n+1 is given by $x_n+1 = x_n - f(x_n) / f'(x_n)$

Unit – 2

Simultaneous Linear Equations Solution of systems of linear equations–

- The purpose of this section is to look at the solution of simultaneous linear equations. We will see that solving a pair of simultaneous equations is equivalent to finding the location of the point of intersection of two straight lines.

$$2x - y = 3$$

This is a linear equation. It is a linear equation because there are no terms involving x^2, y^2 or x Xy, or indeed any higher powers of x and y. The only terms we have got are terms in x, terms in y and some numbers.

Example: Solve simultaneously for x and y: x

+ y=10

x − y =2

This means that we must find values of x and y that will solve both equations. We must find two numbers whose sum is 10 and whose difference is 2. The two numbers, obviously, are 6 and 4:

6 + 4=10

6 − 4=2

Let us represent the solution as the ordered pair (6, 4).
Now, these two equations -- x

+ y=10

x − y =2

-- are linear equations (Lesson 33). Hence, the graph of each one is a straight line. Here are the two graphs:

Gauss elimination Direct method–

Gaussian elimination is a method of solving a linear system $A\mathbf{x} = \mathbf{b}$ (consisting of m equations in n unknowns) by bringing the augmented matrix

$$[A \ \ \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{mn} & d_m \end{array}\right].$$

of equations in

to an upper triangular form

This elimination process is also called the forward elimination method.

The following examples illustrate the Gauss elimination procedure.

EXAMPLE : Solve the linear system by Gauss elimination method.

$$y + z = 2$$

$$2x + 3z = 5$$

$$x + y + z = 3$$

Solution: In this case, the augmented matrix is $\begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. 'he method proceeds along the following steps.

Interchange 1^{st} and 2^{nd} equation (or R_{12}).

$$\begin{aligned} 2x + 3z &= 5 \\ y + z &= 2 \\ x + y + z &= 3 \end{aligned} \qquad \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

Divide the 1^{st} equation by 2 (or $R_1(1/2)$).

$$\begin{aligned} x + \tfrac{3}{2}z &= \tfrac{5}{2} \\ y + z &= 2 \\ x + y + z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

Add -1 times the 1^{st} equation to the 3^{rd} equation (or $R_{31}(-1)$).

$$\begin{aligned} x + \tfrac{3}{2}z &= \tfrac{5}{2} \\ y + z &= 2 \\ y - \tfrac{1}{2}z &= \tfrac{1}{2} \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Add -1 times the 2^{nd} equation to the 3^{rd} equation (or $R_{32}(-1)$).

$$\begin{aligned} x + \tfrac{3}{2}z &= \tfrac{5}{2} \\ y + z &= 2 \\ -\tfrac{3}{2}z &= -\tfrac{3}{2} \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}.$$

1. Interchange

2. Divide the

3. Add

4. Add

5. Multiply the 3^{rd} equation by $\frac{-2}{3}$ (or $R_3(-\frac{2}{3})$).

$$\begin{aligned} x + \tfrac{3}{2}z &= \tfrac{5}{2} \\ y + z &= 2 \\ z &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The last equation gives $z = 1,$ the second equation now gives $y = 1.$ finally the first equation gives $x = 1.$ Hence the set of solutions is $(x, y, z)^t = (1, 1, 1)^t.$ A UNIQUE SOLUTION.

Pivoting–

- The main draw back of the above elimination process is division by the diagonal term while converting the augmented matrix into upper triangular form. If the diagonal element is zero or a vanishingly very small then the elements of the rows below this diagonal become very large in magnitude and difficult to handle because of the finite storage capacity of the computers. Alternative is to convert the system such that the element which has large magnitude in that column comes at the pivotal position i.e., the diagonal position.

Partial Pivoting : If only row interchanging is used to bring the element of large magnitude of the pivotal column to the pivotal position at each step of diagonalization then such a process is called partial pivoting. In this process the matrix may have larger element in non-pivotal column(the column where the pivot is there) but the largest element in the pivotal column only brought to pivotal (or diagonal) position in this process by making use of row transformations.

Complete Pivoting : In this process the largest element(in magnitude) of the whole coefficient matrix A is first brought at 1x 1 position of the coefficeint matrix and then leaving the first row and first column, the largest among the remaining elements is brought to the pivotal 2 x2 position and so on by using both row and column transformations, is called complete pivoting. During row transformations the last column of the augmented matrix also has to be considered but this column is not considered to find the largest element in magnitude. Since the column transformations are also allowed in this process, there will be a change in the position of the individual elements of the unknown vector X. Hence in the end the elements of the unknown vector X has to be rearranged by applying inverse column transformations in reverse order to all the column transformations preformed.

Ill conditioned system of equations–

- A system of equations is considered to be ill conditioned if a small change in the coefficient matrix or a small change in the right hand side results in a large change in the solution vector.
- It is well known that for a system of equations with an ill-conditioned matrix, an erroneous solution can be obtained which seems to satisfy the system quite well.
- Various measures of the ill-conditioning of a matrix have been proposed. For example, the condition number associated with the linear equation Ax = b gives a bound on how inaccurate the solution x will be after approximation.

  Note that this is before the effects of round-off error are taken into account; conditioning is a property of the matrix, not the algorithm or floating point accuracy of the computer used to solve the corresponding system. In particular, one should think of the condition number as being (very roughly) the rate at which the solution, x, will change with respect to a change in b. Thus, if the condition number is large, even a small error in b may cause a large error in x. On the other hand, if the condition number is small then the error in x will not be much bigger than the error in b.

Refinement of solution–

- Iterative refinement is a technique introduced by Wilkinson for reducing the roundoff error produced during the solution of simultaneous linear equations. Higher precision arithmetic is required for the calculation of the residuals.

The iterative refinement algorithm is easily described.

- Solve Ax=bAx=b, saving the triangular factors.
- Compute the residuals, r=Ax−br=Ax−b.
- Use the triangular factors to solve Ad=rAd=r.
- Subtract the correction, x=x−dx=x−d
- Repeat the previous three steps if desired.

Gauss seidal method–

- Gauss–Seidel method. In numerical linear algebra, the Gauss–Seidel method, also known as the Liebmann method or the method of successive displacement, is an iterative method used to solve a linear system of equations.

Gauss-Seidel Method

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Gauss-Seidel Method

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Algorithm

Rewriting each equation

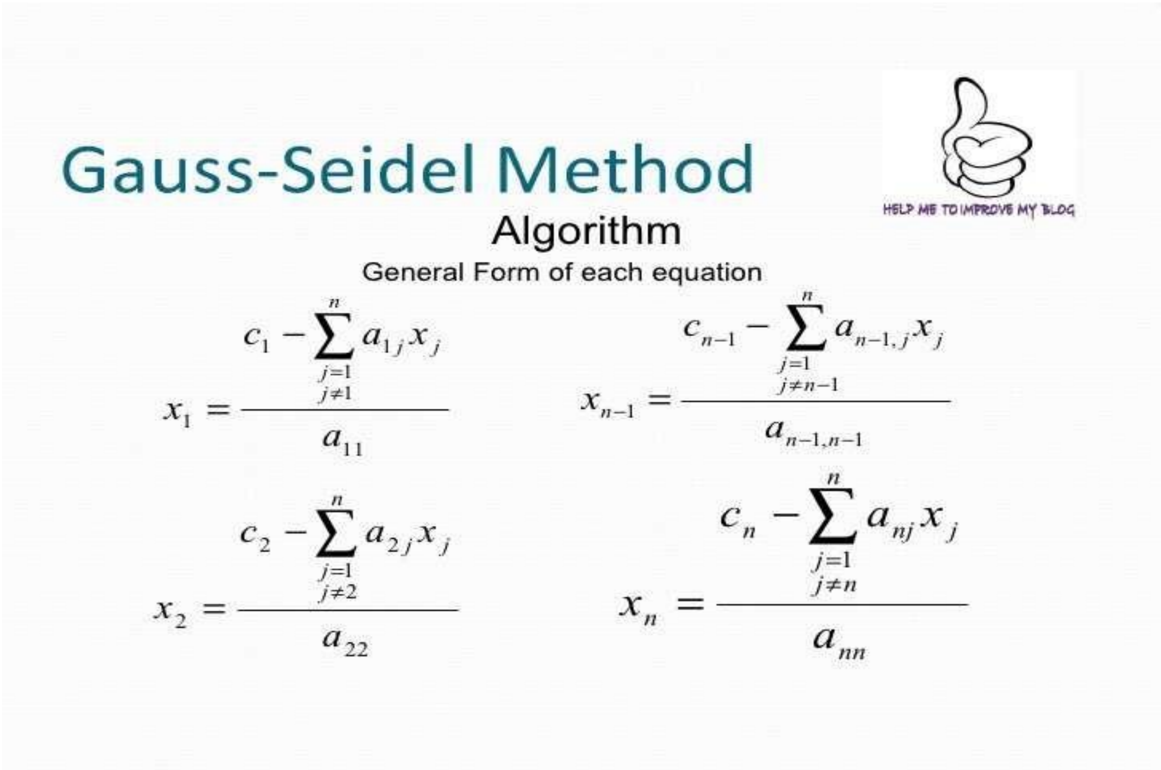
$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}} \longleftarrow \text{From Equation 1}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}} \longleftarrow \text{From equation 2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}} \longleftarrow \text{From equation n-1}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}} \longleftarrow \text{From equation n}$$



Unit – 3

Interpolation and approximation

Finite Differences–

- In mathematics, finite-difference methods (FDM) are numerical methods for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. FDMs are thus discretizationmethods.
- Finite differences play a key role in the solution of differential equations and in the formulation of interpolating polynomials. The interpolation is the art of reading between the tabular values. Also the interpolation formulae are used to derive formulae for numerical differentiation and integration.
- Three forms are commonly considered: forward, backward, and central differences.

A forward difference is an expression of the form

$\Delta_h[f](x) = f(x+h) - f(x)$.

Depending on the application, the spacing h may be variable or constant. When omitted, h is taken to be

1: $\Delta[f](x) = \Delta_1[f](x)$.

A backward difference uses the function values at x and x − h, instead of the values at x + h and x:

$\nabla\{h\}[f](x)=f(x)-f(x-h).\}$

Finally, the central difference is given by

$\{h\}[f](x)=f(x+1/2\ h)-f(x-\tfrac{1}{2}h)$.

Difference Table–

- An auxiliary table to facilitate interpolation between the numbers of the principal table giving approximate differences in values of the tabulated function corresponding to certain submultiples (such as tenths) of the constant smallest increment of the independent variable in the table.

Polynomial Interpolation–

- Polynomial interpolation. In numerical analysis, polynomial interpolation is the interpolation of a given data set by the polynomial of lowest possible degree that passes through the points of the dataset.

Newton Forward Formula–

- $\mathbf{y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)\dots(p-n+1)}{4!}\Delta^4 y_0}$

This formula is particularly useful for interpolating the values of f(x) near the beginning of the set of values given. h is called the interval of difference and u = (x – a) / h, Here a is first term.

Example :

Input : Value of Sin 52

θ°	45°	50°	55°	60°
sin θ	0.7071	0.7660	0.8192	0.8660

Output :

$x°$	Differences			
	$10^4 y$	$10^4 \Delta y$	$10^4 \Delta^2 y$	$10^4 \Delta^3 y$
45°	$\boxed{7071}$			
		$\boxed{589}$		
50°	7660		$\boxed{-57}$	
		532		$\boxed{-7}$
55°	8192		– 64	
		468		
60°	8660			

Newton Backward Formula–

$$\text{Ans}: \quad y(x_n + ph) = y(x) = y_n + \frac{p}{1}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3}\nabla^3 y_n + \ldots$$

$$-\frac{p(p-1)(p+2)\ldots(p+(n-1))}{n!}\nabla^n y_n \qquad \text{where } p = \frac{x - x_n}{h}$$

This formula is useful when the value of f(x) is required near the end of the table. h is called the interval of difference and u = (x – an) / h, Here an is last term.

Example :

Input : Population in 1925

Year (x):	1891	1901	1911	1921	1931
Population (y): (in thousands)	46	66	81	93	101

Output :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Value in 1925 is 96.8368

Central Difference Formula–

Gauss forward central difference formula

Statement: If $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ are given set of observations with common difference h and let $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$ are their corresponding values, where $y = f(x)$ be the given function then $y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p\,(p-1)(p+1)\,(p-2)}{4!}\Delta^4 y_{-2} + \ldots$ where $p = \frac{x-x_0}{h}$.

Proof:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$\vdots$	$\vdots$				
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}			
x_0	y_0			$\Delta^3 y_{-2}$	
			$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
x_1	y_1	Δy_0		$\Delta^3 y_{-1}$	
			$\Delta^2 y_0$		
x_2	y_2	Δy_{-1}			
$\vdots$	$\vdots$				

Let us assume a polynomial equation by using the arrow marks shown in the above table.

Let $y_p = y_0 + G_1\,\Delta y_0 + G_2\,\Delta^2 y_{-1} + G_3\,\Delta^3 y_{-1} + G_4\,\Delta^4 y_{-2} + \ldots$ ---- (1)

where $G_0, G_1, G_2, \ldots$ are unknowns

$y_p = y_{p+0} = E^p y_0 = (1+\Delta)^p y_0 \quad (\because E = 1 + \Delta)$

$\Rightarrow\; y_p = \left(1 + p_{C_1}\Delta + p_{C_2}\Delta^2 + p_{C_3}\Delta^3 + \ldots + p_{C_p}\Delta^p\right) y_0$

$\Rightarrow\; y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ldots$ --- (2)

Now, $y_{-1} = y_{-1+0} = E^{-1} y_0 = (1+\Delta)^{-1} y_0$

$= (1 - \Delta + \Delta^2 - \Delta^3 + \ldots) y_0$

$\Rightarrow\; y_{-1} = y_0 - \Delta y_0 + \Delta^2 y_0 - \ldots$

Therefore, $\Delta^2\, y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \ldots$ ----- (3)

and $\quad \Delta^3\, y_{-1} = \Delta^3 y_0 - \Delta^4 y_0 + \ldots$ ----- (4)

Substituting 2, 3, 4 in 1, we get

$$y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ldots = y_0 + G_1\,\Delta y_0 + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \cdots) + G_3(\Delta^3 y_0 - \Delta^4 y_0 + \ldots) + \ldots$$

Comparing corresponding coefficients, we get

$G_1 = p\ , G_2 = \frac{p\,(p-1)}{2!}\ , -G_2 + G_3 = \frac{p\,(p-1)(p-2)}{3!}\ \Rightarrow\ G_3 = \frac{p\,(p-1)(p+1)}{3!}$

Similarly, $G_4 = \frac{p\,(p-1)(p+1)\,(p-2)}{4!}$

Substituting all these values of $G_0, G_1, G_2, \dots$ in (1), we get

$$\boldsymbol{y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p\,(p-1)(p+1)\,(p-2)}{4!}\Delta^4 y_{-2} + \dots}$$

Gauss backward central difference formula

Statement: If $\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$ are given set of observations with common difference h and let $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ are their corresponding values, where $y = f(x)$ be the given function then

$$y_p = y_0 + p\,\Delta y_{-1} + \frac{p\,(p+1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p\,(p+1)(p-1)\,(p+2)}{4!}\Delta^4 y_{-2} + \dots$$

where $p = \frac{x-x_0}{h}$.

Proof:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$\vdots$	$\vdots$				
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
x_0	y_0	Δy_{-1}		$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$
			$\Delta^2 y_{-1}$		
x_1	y_1	Δy_0		$\Delta^3 y_{-1}$	
			$\Delta^2 y_0$		
x_2	y_2	Δy_{-1}			
$\vdots$	$\vdots$				

Let us assume a polynomial equation by using the arrow marks shown in the above table.

Let $y_p = y_0 + G_1\,\Delta y_{-1} + G_2\,\Delta^2 y_{-1} + G_3\,\Delta^3 y_{-2} + G_4\,\Delta^4 y_{-2} + \dots$ ---- (1)

where $G_0, G_1, G_2, \dots$ are unknowns

$y_p = y_{p+0} = E^p y_0 = (1+\Delta)^p y_0 \qquad (\because E = 1 + \Delta)$

$\Rightarrow\ y_p = \left(1 + p_{C_1}\Delta + p_{C_2}\Delta^2 + p_{C_3}\Delta^3 + \ \dots + \ p_{C_p}\Delta^p\right) y_0$

$\Rightarrow\ y_p = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ \dots$ --- (2)

Now, $y_{-1} = y_{-1+0} = E^{-1} y_0 = (1+\Delta)^{-1} y_0$

$= (1 - \Delta + \Delta^2 - \Delta^3 + \ \dots) y_0$

$\Rightarrow\ y_{-1} = y_0 - \Delta y_0 + \Delta^2 y_0 - \ \dots$

Therefore, $\Delta y_{-1} = \Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \ \dots$ ----- (3)

$\Delta^2\, y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \ \dots$ ----- (4)

Also $y_{-2} = y_{-2+0} = E^{-2} y_0 = (1+\Delta)^{-2} y_0$

$$= (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \ \dots) y_0$$

$$\Rightarrow\ y_{-2} = y_0 - 2\Delta y_0 + 3\Delta^2 y_0 - \ \dots$$

Now, $\Delta^3\, y_{-2} = \Delta^3 y_0 - 2\Delta^4 y_0 + \ \dots$ ----- (5)

Substituting 2, 3, 4, 5 in 1, we get

$$y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ \dots\ = y_0 + G_1\,(\Delta y_0 - \Delta^2 y_0 + \Delta^3 y_0 - \ \dots\ ) + G_2(\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \cdots\ ) + G_3(\Delta^3 y_0 - 2\Delta^4 y_0 + \ \dots) + \ \dots$$

Comparing corresponding coefficients, we get

$G_1 = p\ , -G_1 + G_2 = \frac{p\,(p-1)}{2!} \ \Rightarrow\ G_2 = \frac{p\,(p+1)}{2!}$

Also, $G_1 - G_2 + G_3 = \frac{p\,(p-1)(p-2)}{3!} \Rightarrow G_3 = \frac{p\,(p+1)(p-1)}{3!}$

Similarly, $G_4 = \frac{p\,(p+1)(p-1)\,(p+2)}{4!}, \dots$

Substituting all these values of $G_0, G_1, G_2, \dots$ in (1), we get

$$\boldsymbol{y_p = y_0 + p\,\Delta y_{-1} + \frac{p\,(p+1)}{2!}\Delta^2 y_{-1} + \frac{p\,(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p\,(p+1)(p-1)\,(p+2)}{4!}\Delta^4 y_{-2} + \ \dots\ \ , p = \frac{x-x_0}{h}}$$

Interpolation with unequal interval

Langrange's interpolation–

INTERPOLATION WITH UNEQUAL INTERVALS

Lagrange's interpolation formula with unequal intervals:

- Let $y = f(x)$ be continuous and differentiable in the interval (a, b).
- Given the set of $n+1$ values $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ of x and y, where the values of x need not necessarily be equally spaced.
- It is required to find $P_n(x)$, a polynomial of degree n such that y and $P_n(x)$ agree at the tabulated points.

LAGRANGE'S INTERPOLATION

- This polynomial is given by the following formula:

$$y = f(x) \approx P_n(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0$$
$$+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots$$
$$+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n$$

NOTE:

- The above formula can be used irrespective of whether the values $x_0, x_1, \ldots, x_n$ are equally spaced or not.

Newton Divided difference formula–

Unit – 4

Statistics

Statistics and its role in decision making–

Internal and external source of data–

Formation of frequency distribution–

Types of frequency distribution–

Simple and weighted means–

Median and mode–

Unit – 5

Correlation

Significance of study of correlation–

Types of correlation–

Positive correlation–

Negative correlation–

Simple–

Partial–

Multiple correlation– Linear and non-linear correlation–

Coefficient of correlation–

Use of Regression analysis– Diffrence between correlation and regression analysis–

Regression Lines–

Regression equation of Y on X–

Regression equation of X on Y–