Homework 5 (DRAFT)

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Contents

1 Problem 1

Exercise 2 in Section 9.6

1.1 Solution

Let the instrument recording of voltage measurements sampled at $F_s = 50\text{Hz}$ be $\tilde{y}(t)$ with $N = 2000$ measurements, which was provided an plotted below in figure 1.

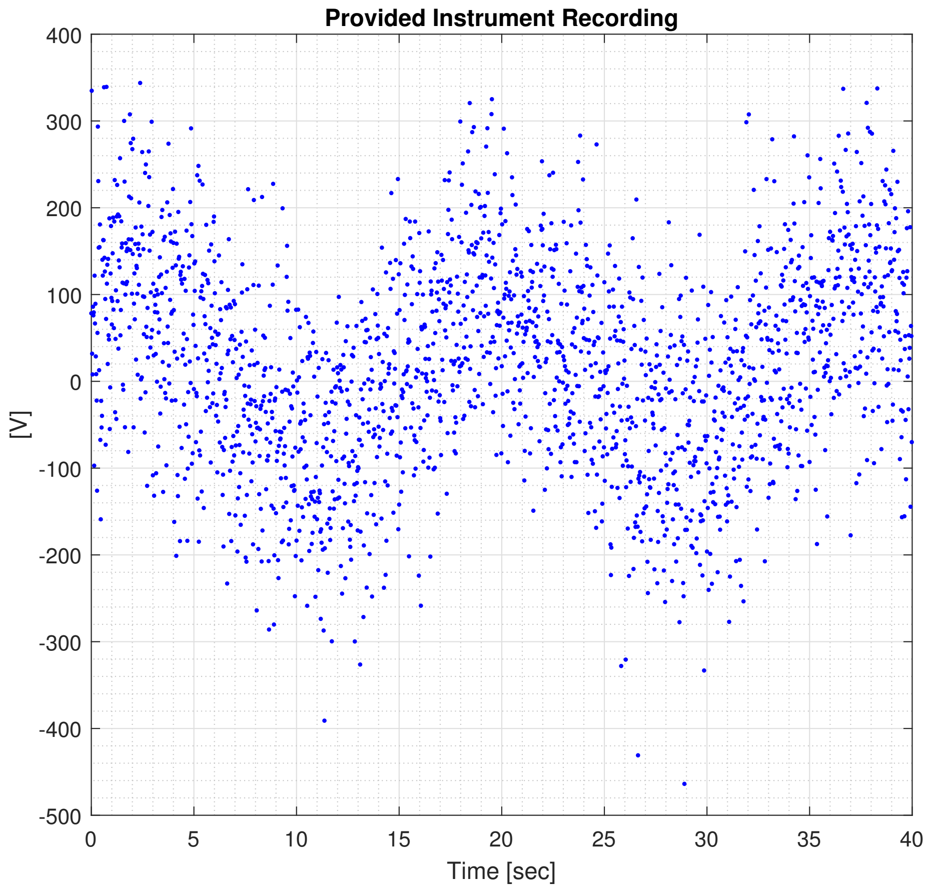

Figure 1: Instrument Recording

The model of this recording is

$$y(t) = A \sin\left(2\pi f_0 t + \chi^2\right) + c + s\eta(t)$$

with unknown parameters A, f_0, χ^2, c which are the amplitude, frequency, phase shift, and DC offset respectively. In addition, the scale of the standard deviation is also unknown.

For this non-linear inverse problem, the model is

$$\boldsymbol{m} = \begin{bmatrix} A \\ f_0 \\ \chi^2 \\ c \end{bmatrix}$$

and a starting model needs to better determined from the available data. To do so, I first started out with computing and plotting the power spectral density (PSD) to find a suitable value for f_0 as shown in figure 2.

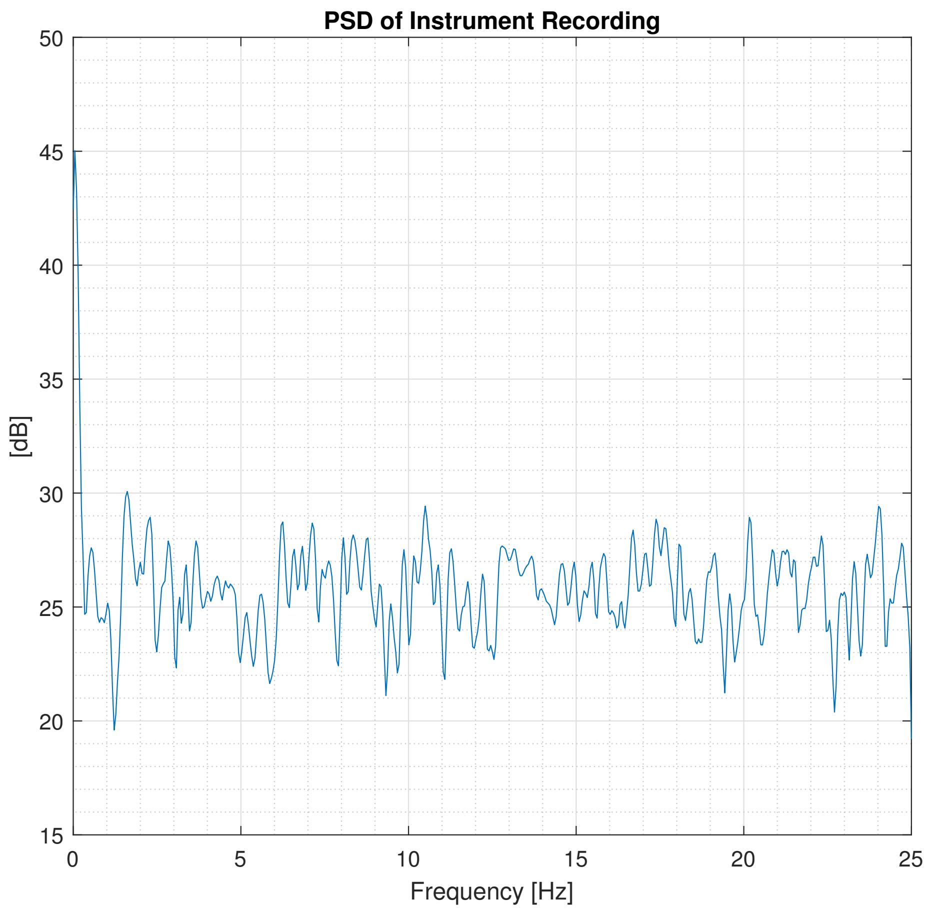

Figure 2: Instrument Recording PSD

The peak of the PSD occurs at frequency $f_0 \approx 0.049$Hz. The other initial guesses for the other parameters were formulated accordingly below.

$$c_0 = \frac{1}{N} \sum_{i=1}^{N} \tilde{y}\left(t_i\right)$$

$$A_0 = \frac{\max\left(\tilde{y}(t) - c_0\right) - \min\left(\tilde{y}(t) - c_0\right)}{2}$$

The final model parameter χ^2 was simply chosen by trail and error such that the initial model appeared to have a decent match as demonstrated in figure 3.

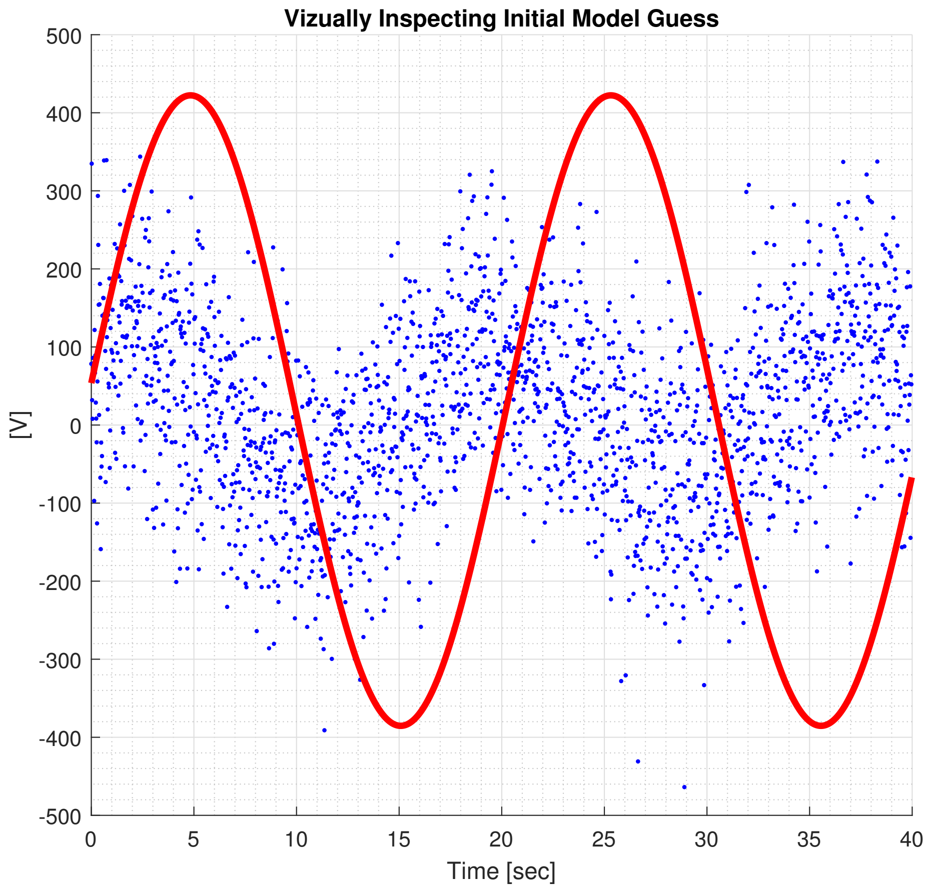

Figure 3: Initial Model Guess

This resulted in the initial model below.

$$\boldsymbol{m}^{(0)} = \begin{bmatrix} 201.902 \\ 0.049 \\ 0.087 \\ 18.502 \end{bmatrix}$$

To prepare for using the Levenberg-Marquardt (LM) algorithm, the Jacobian of $y(t)$ was required. The partial derivative of $y(t)$ with respect to each model parameter is below.

$$\frac{\partial y}{\partial A} = \sin\left(2\pi f_0 t + \chi^2\right)$$

$$\frac{\partial y}{\partial f_0} = 2\pi A t \cos\left(2\pi f_0 t + \chi^2\right)$$

$$\frac{\partial y}{\partial \chi^2} = A \cos\left(2\pi f_0 t + \chi^2\right)$$

$$\frac{\partial y}{\partial c} = 1$$

These partial derivatives were verified to Wolfram Alpha to ensure I didn't make a mistake in computing the partial derivatives. To formulate the Jacobian J in MATLAB®, each column of J corresponds to the partial derivatives above and the partial derivatives are evaluated at each value of t_i. This results in $J \in \mathbb{R}^{2000\times 4}$.

The `lm()` function from the `Lib` directory was used to estimate the model. The tolerance was set to $1e-12$ with a maximum of 100 iterations. This resulted in the model below.

$$\boldsymbol{m}_{LM} = \begin{bmatrix} xxx.xxx \\ xxx.xxx \\ xxx.xxx \\ xxx.xxx \end{bmatrix}$$

These models resulted in the following comparison to the provided data in figure 4.

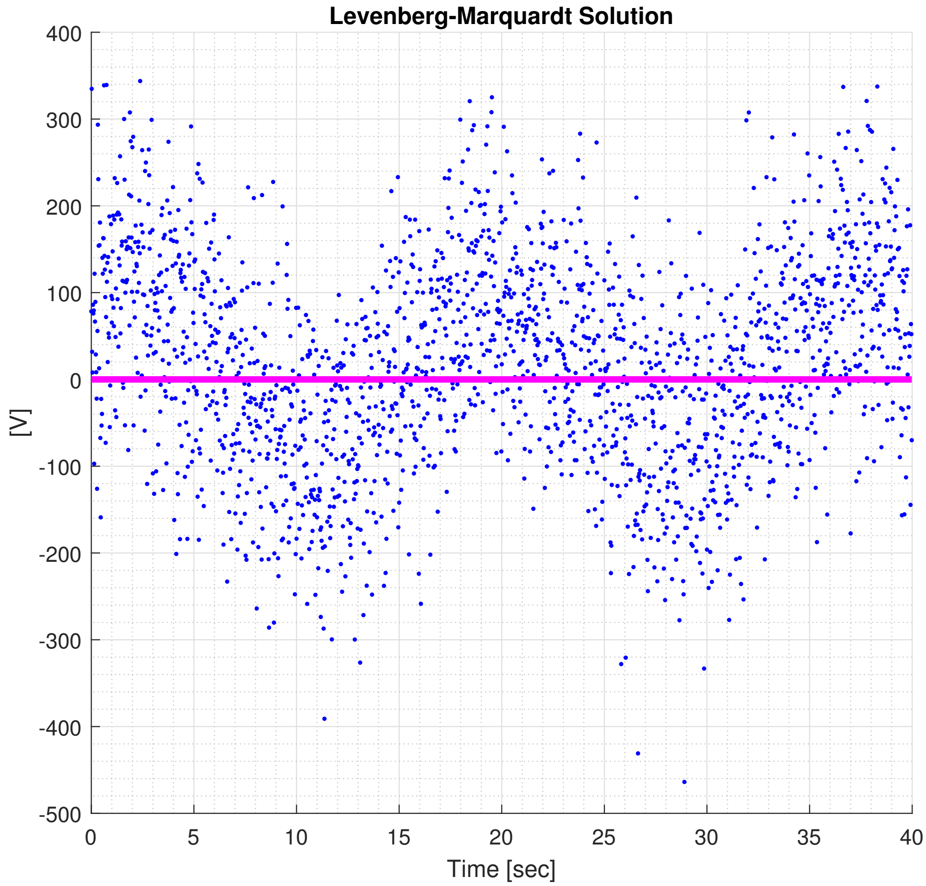

Figure 4: Levenberg-Marquardt Solution

To estimate the standard deviation s applied in the model, the residuals were computed using the model parameters from the LM solution.

$$r = \tilde{y}(t) - F(\boldsymbol{m}_{LM})$$

The MATLAB® function `std()` was called on the residuals, which resulted in $s = xxx.xxx$. Figure 5 below shows the provided data as well as the 3-sigma bound which should contain nearly all of the residuals.

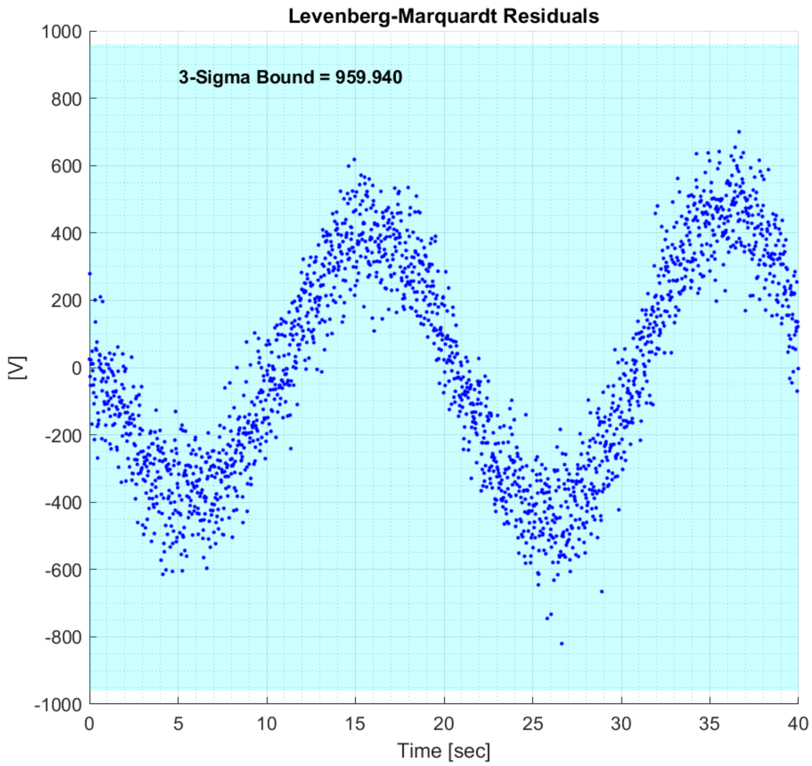

Figure 5: Levenberg-Marquardt Solution Residuals

Since the model is sinusoidal and any error in the model parameters, especially f_0 and χ^2, could leave some residual sinusoidal elements which would violate the assumption that the power in the additive white noise should be equally spread across all frequencies.

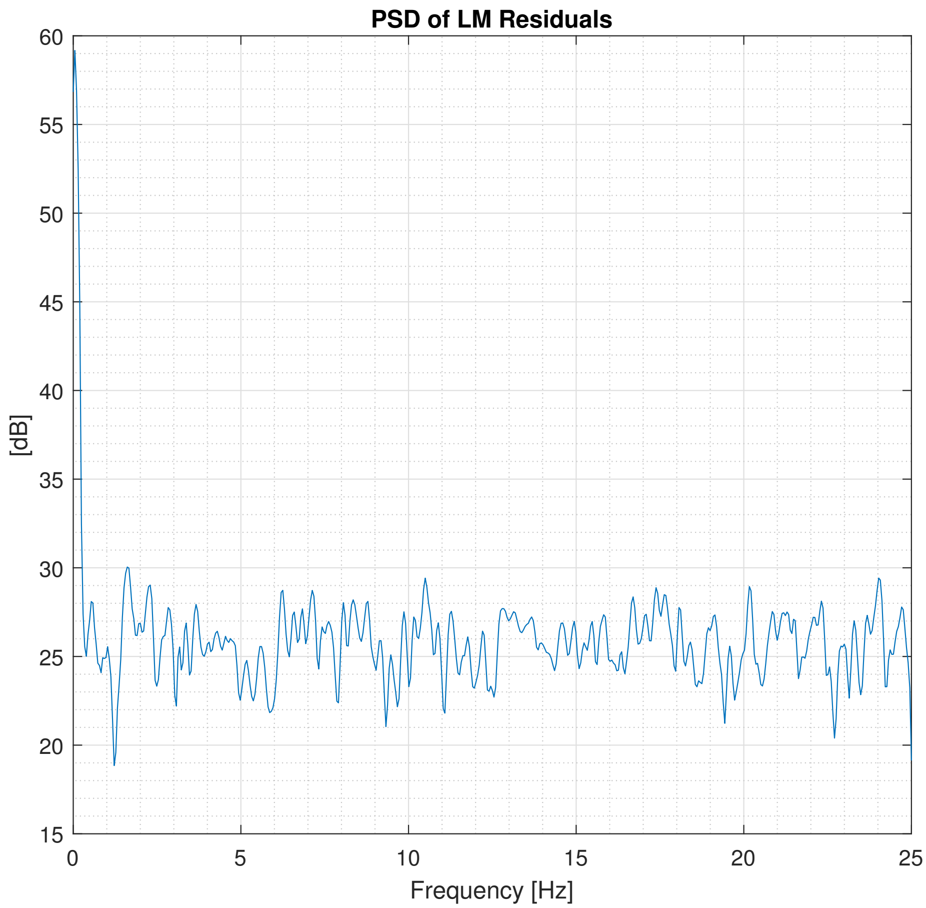

Figure 6: PSD of Levenberg-Marquardt Solution Residuals

Discuss the distribution here once the LM piece is fixed.
The model covariance matrix is estimated as $C = \left(J(\boldsymbol{m}_{LM})^T \boldsymbol{m}_{LM}\right)^{-1}$ which is shown below.

$$C = \begin{bmatrix} xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \\ xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \\ xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \\ xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \end{bmatrix}$$

The 95% confidence interval, computed as $1.96 \pm \sqrt{\text{diag}(\boldsymbol{m}_{LM})}$ is also shown below.

$$\boldsymbol{m}_{LM} = \begin{bmatrix} xxx.xxx \\ xxx.xxx \\ xxx.xxx \\ xxx.xxx \end{bmatrix} \pm \begin{bmatrix} xxx.xxx \\ xxx.xxx \\ xxx.xxx \\ xxx.xxx \end{bmatrix}$$

Then, the correlation matrix for all model parameters is below as:

$$\rho = \begin{bmatrix} xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \\ xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \\ xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \\ xxx.xxx & xxx.xxx & xxx.xxx & xxx.xxx \end{bmatrix}$$

Discuss which parameters have the strongest correlation once everything is fixed.

2 Problem 2

Exercise 8 in Section 9.6

2.1 Solution