Homework 1

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Problem 1

1. Exercise Consider the Fredholm integral equation of the first kind,

$$\int_0^1 g(x,z)m(z)dz = d(x)$$

with $g(x,z) = 5\sin(xz)$, and $d(x) = 50\sin(x) - 50\sin(x)\cos(x)$, $0 \leq x \leq 1$. The exact solution to this equation is $m(x) = 10x\sin(x)$ as can easily be verified by substituting it into the equation.

(a) Using MATLAB, discretize this integral equation using midpoints of $n = 20$ equally spaced intervals of width 0.05. Your discretized model should be of the form $Gm = d$. Output G and d. Use the MATLAB backslash command to solve for m and output your inverse model.

(b) What is the condition number of G?

(c) Plot your solution and the exact solution.

(d) Why is the solution to the discretized model so poor?

Solution

First, let's verify the solution of $m(x)$ via substitution. (This helped me understand the problem immensely, so I will include it here for the sake of completeness)

$$\int_0^1 g(x,z)m(z)dz = d(x)$$

$$\int_0^1 5\sin(xz)m(z)dz = 50\sin(x) - 50\sin(x)\cos(x)$$

$$\int_0^1 5\sin(xz)m(z)dz = \int_0^1 5\sin(xz)m(x)dz$$

$$= \int_0^1 5\sin(xz)\left(10x\sin(x)\right)dz$$

$$= 50x\sin(x)\int_0^1 \sin(xz)dz$$

$$= -\frac{50x\sin(x)}{x}\left[\cos(xz)\right]\big|_0^1$$

$$= -\frac{50x\sin(x)}{x}\left(\cos(x) - \cos(0)\right)$$

$$= 50\sin(x)\left(1 - \cos(x)\right)$$

$$= 50\sin(x) - 50\sin(x)\cos(x) = d(x) \quad \checkmark$$

Part A

Now, suppose that I do not know $m(x)$ for the purposes of this question. Discretizing the given integral using 20 midpoints such that the index j is a member of the set $\{j \in \mathbb{Z} : 1 \leq j \leq 20\}$. Approximating the given Fredholm integral equation of the first kind leads to

$$\int_0^1 g(x,z)m(z)dz = d(x)$$

$$\int_0^1 5\sin(xz)m(z)dz \approx \sum_{j=1}^{20} 5\sin(xz_j)m(z_j)\Delta z$$

$$\approx 5\sin(xz_1)m(z_1)\Delta z + 5\sin(xz_2)m(z_2)\Delta z + \cdots + 5\sin(xz_{20})m(z_{20})\Delta z$$

To fit this discrete numerical integration to the form $\boldsymbol{Gm} = \boldsymbol{d}$, let the variable x be sampled at 20 equally spaced such that the index i is a member of set $\{x \in \mathbb{Z} : 1 \leq i \leq 20\}$. (Note: A bold symbol indicates a vector quantity) This above summation can be expressed as a linear system of equations such that

$$\begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,20} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,20} \\ \vdots & \vdots & \ddots & \vdots \\ g_{20,1} & g_{20,2} & \cdots & g_{20,20} \end{bmatrix} \begin{bmatrix} m(z_1) \\ m_(z_2) \\ \vdots \\ m(z_{20}) \end{bmatrix} = \begin{bmatrix} d(x_1) \\ d(x_2) \\ \vdots \\ d(x_{20}) \end{bmatrix}$$

where,

$$G \in \mathbb{R}^{20\times 20} : g_{i,j} = 5\sin(x_i z_j)\Delta z$$

$$\boldsymbol{d} \in \mathbb{R}^{20} : d_i = 50\sin(x_i) - 50\sin(x_i)\cos(x_i)$$

The vector $\boldsymbol{m} \in \mathbb{R}^{20}$ can be solved as

$$\boldsymbol{m} = G^{-1}\boldsymbol{d}$$

Constructing these vectors and matrices in MATLAB®(provided in file `prob1.m`), this results in the following quantities for G and $\boldsymbol{m}$.