Homework 6

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1 Problem 2

Exercise 10 in Section 11.6

1.1 Solution

1.1.1 Part A

The first step is to convert the grid of χ^2 values from the previous homework problem into the likelihood values $L(\boldsymbol{m}|\boldsymbol{d})$. Recall the surface of chi^2 values from the previous problem below.

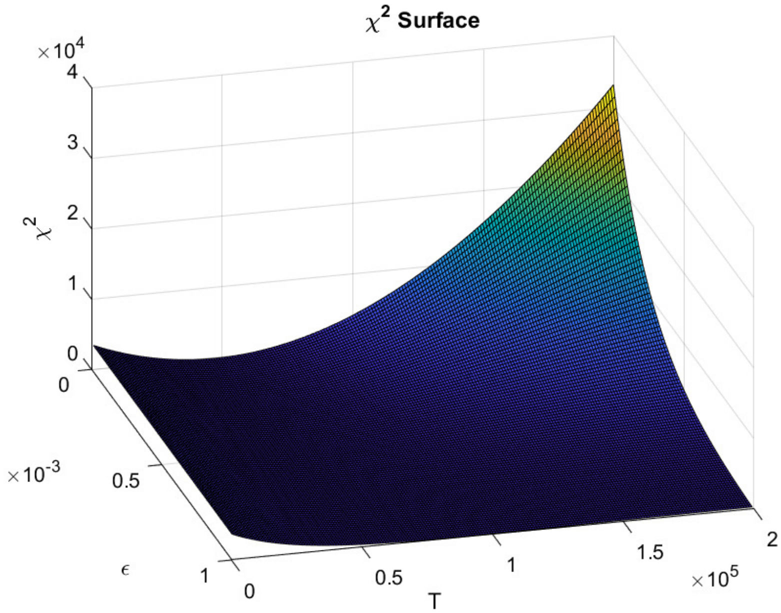

Figure 1: Chi-Squared Surface

The likelihood is equal to $L(\boldsymbol{m}|\boldsymbol{d}) = f(\boldsymbol{d}|\boldsymbol{m}) = f(d_1|\boldsymbol{m})\, f(d_1|\boldsymbol{m}) \ldots f(d_m|\boldsymbol{m})$.
The likelihood of a given data point d_i is below.

$$f(d_i|\boldsymbol{m}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(G(\boldsymbol{m})_i - d_i)^2}{2\sigma^2}}$$

For m data points, the product of each $f(d_i|\boldsymbol{m})$ is also shown below, which differs slightly from equation 11.10 since our data points in this problem have different sigma values.

$$L(\boldsymbol{m}|\boldsymbol{d}) = f(\boldsymbol{d}|\boldsymbol{m}) = \left[\prod_{i=1}^{m} \frac{1}{\sigma_i\sqrt{2\pi}}\right] \left[e^{-\sum_{i=1}^{m} \frac{(G(\boldsymbol{m})_i - d_i)^2}{2\sigma^2}}\right]$$

Recall that $\chi^2 = \sum_{i=1}^{m} \frac{(G(\boldsymbol{m})_i - d_i)^2}{\sigma^2}$ per equation 2.20, which can be substituted in the above expression.

$$L(\boldsymbol{m}|\boldsymbol{d}) = \left[\prod_{i=1}^{m} \frac{1}{\sigma_i\sqrt{2\pi}}\right]\left[e^{\frac{-\chi^2}{2}}\right]$$

Doing so allows us to convert our grid of chi^2 values into likelihood values. Given that the grid consists of so many points, the likelihood at each point in the grid is incredibly small.

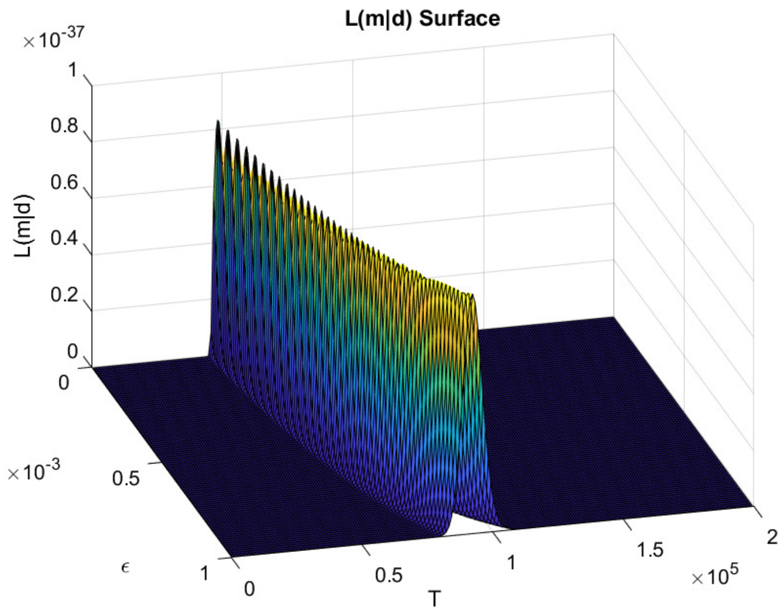

Figure 2: Likelihood Surface

The posterior distribution is given by the expression below:

$$q(\boldsymbol{m}|\boldsymbol{d}) = \frac{p(\boldsymbol{m})\, f(\boldsymbol{d}|\boldsymbol{m})}{\int_{\text{all models}} f(\boldsymbol{d}|\boldsymbol{m})\, p(\boldsymbol{m})\, d\boldsymbol{m}}$$

The prior is a uniform distribution $P(\epsilon, T) = 1$ which simplifies the above expression as shown below.

$$q(\boldsymbol{m}|\boldsymbol{d}) = \frac{f(\boldsymbol{d}|\boldsymbol{m})}{\int_{\text{all models}} f(\boldsymbol{d}|\boldsymbol{m})\, d\boldsymbol{m}}$$

In essence, the posterior distribution is $f(\boldsymbol{d}|\boldsymbol{m})$ normalized by the volume under the two-dimensional surface. The volume can be approximated as:

$$q\left(\boldsymbol{m}|\boldsymbol{d}\right) \approx \frac{f\left(\boldsymbol{d}|\boldsymbol{m}\right)}{\sum f\left(\boldsymbol{d}|\boldsymbol{m}\right) d\epsilon dT}$$

The resulting posterior distribution is shown in figure 3;

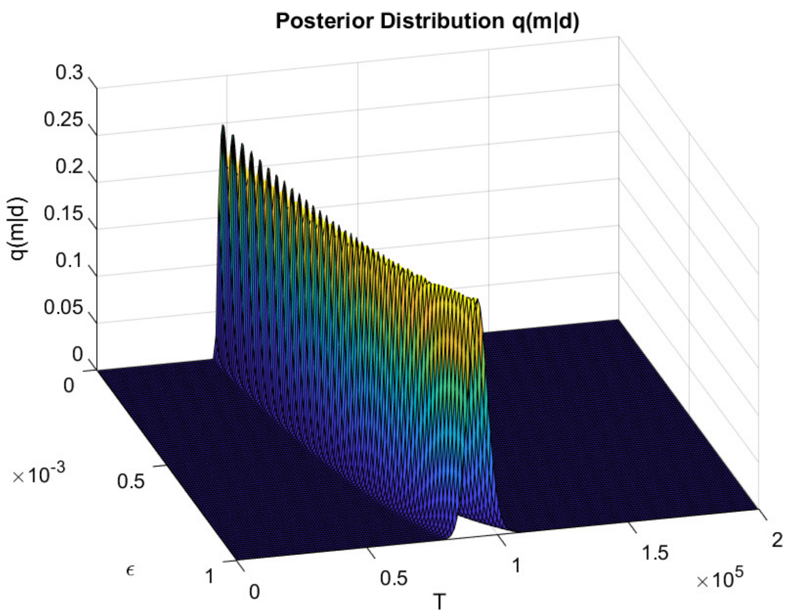

Figure 3: Posterior Distribution

1.1.2 Part B

Given the joint probability $q\left(\boldsymbol{m}|\boldsymbol{d}\right) = p\left(\epsilon, T\right)$, each marginal probability for the two random variables are below.

$$p_\epsilon(\epsilon) = \int_T p\left(\epsilon, T\right) dT$$

$$p_T(T) = \int_\epsilon p\left(\epsilon, T\right) d\epsilon$$

The marginal probability density functions (technically, its a probability mass function) are in figure 4.

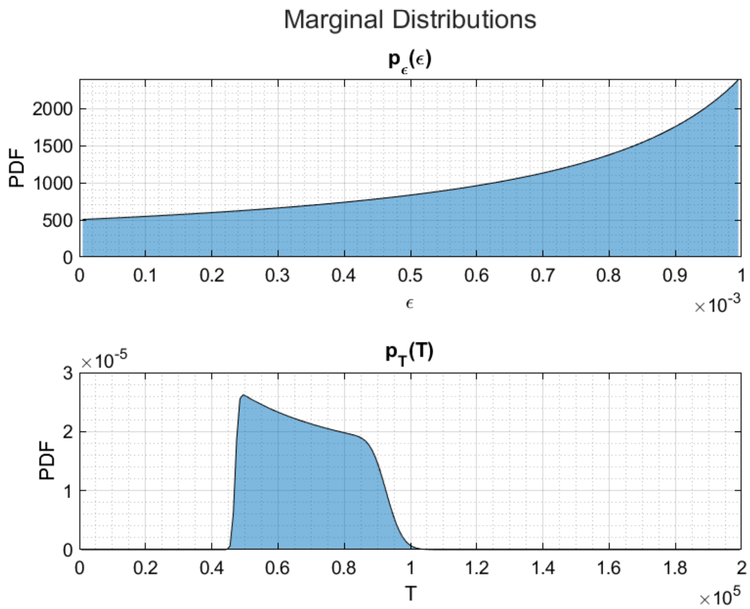

Figure 4: Marginal Distributions

1.1.3 Part C

Now we are provided a new prior, which is shown in figure 5.

$$p\left(\boldsymbol{m}\right) = p\left(\epsilon,\, T\right) \propto e^{-\frac{\left(\epsilon - 0.0005^2\right)^2}{2\cdot 0.0002^2}}$$

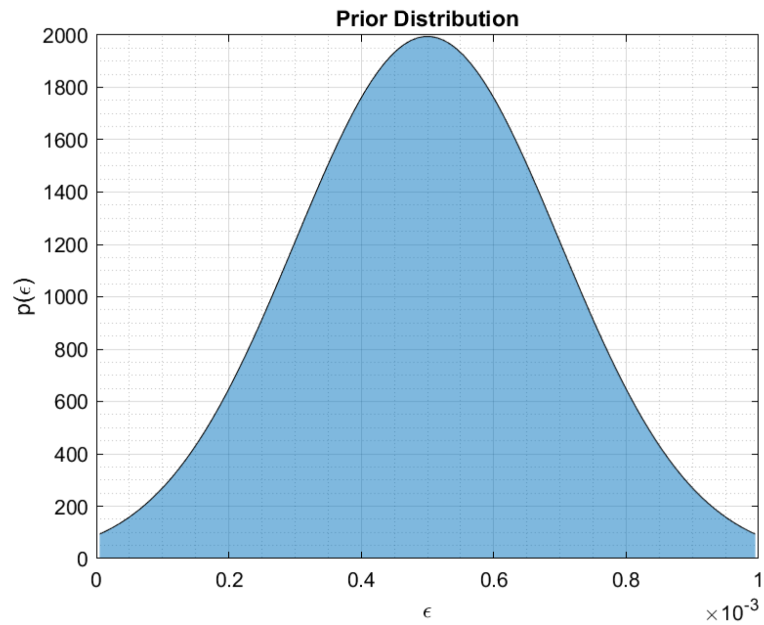

Figure 5: Prior Distribution

To compute the new posterior distribution, consider that the posterior distribution is proportional to the product of the prior distribution and the likelihood. Since we are considering the proportionality, we can drop the scale factor term on the likelihood function.

$$q(\boldsymbol{m}|\boldsymbol{d}) \propto p(\boldsymbol{m})\, f(\boldsymbol{d}|\boldsymbol{m})$$

$$\propto e^{-\frac{(\epsilon-0.0005)^2}{2\cdot 0.0002^2}}\, e^{\frac{-\chi^2}{2}}$$

The grid of likelihood values $f(\boldsymbol{d}|\boldsymbol{m})$ were each multiplied by the probability density function output related to the value of ϵ associated with the grid. This resulted in a new joint probability density function which is shown in figure 6.

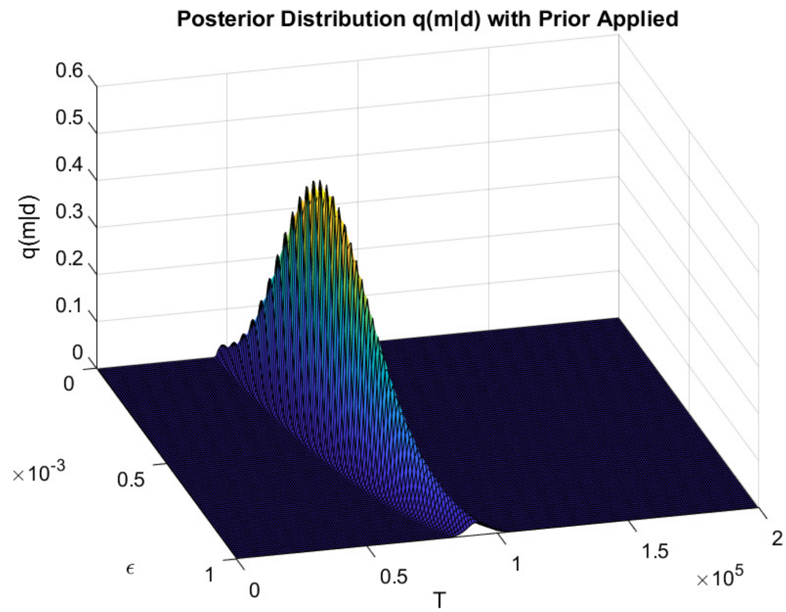

Figure 6: Posterior Distribution with Prior Applied

The newly computed marginal probabilities are also shown below in figure 7.

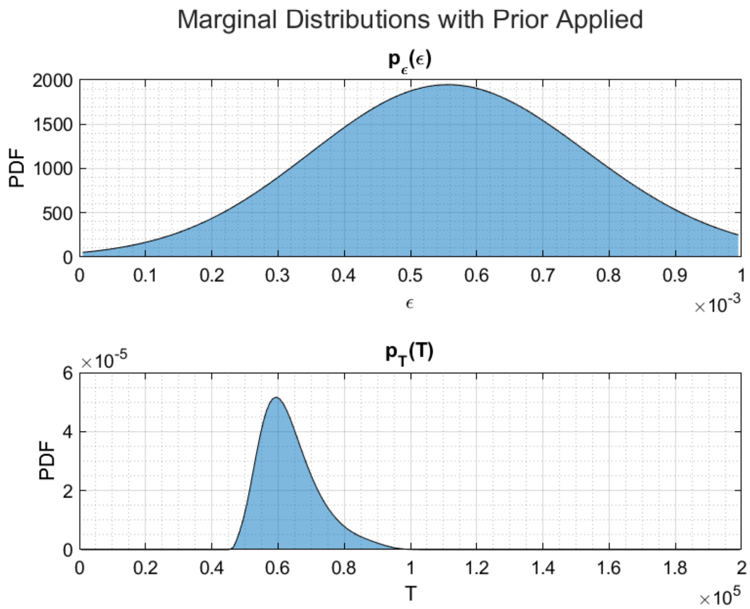

Figure 7: Marginal Distributions with Prior Applied

1.1.4 Part D

When comparing the results between using the uniformly-distributed prior versus the normally-distributed prior, it is clear that this made a key difference for the ϵ parameter. The marginal probability $p_\epsilon(\epsilon)$ *should have* appeared flat, but for some reason it showed that it was increasing. (I think there might have been a bug somewhere).

However for the normally-distributed prior case, we were able to see the mean shift in the marginal $p_\epsilon(\epsilon)$ in response to the likelihoods. It seems that the variance also grew as well. Even though the prior didn't contain any information about the T model parameter, its marginal probability density function was still reshaped into something that appears more multi-modal.