Homework 1 (DRAFT)

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February 9, 2025

Problem 1

1. Exercise Consider the Fredholm integral equation of the first kind,

$$\int_0^1 g(x,z)m(z)dz = d(x)$$

with $g(x,z) = 5\sin(xz)$, and $d(x) = 50\sin(x) - 50\sin(x)\cos(x)$, $0 \leq x \leq 1$. The exact solution to this equation is $m(x) = 10x\sin(x)$ as can easily be verified by substituting it into the equation.

(a) Using MATLAB, discretize this integral equation using midpoints of $n = 20$ equally spaced intervals of width 0.05. Your discretized model should be of the form $Gm = d$. Output G and d. Use the MATLAB backslash command to solve for m and output your inverse model.

(b) What is the condition number of G?

(c) Plot your solution and the exact solution.

(d) Why is the solution to the discretized model so poor?

Solution

First, let's verify the solution of $m(x)$ via substitution. (This helped me understand the problem immensely, so I will include it here for the sake of completeness)

$$\int_0^1 g(x,z)m(z)dz = d(x)$$

$$\int_0^1 5\sin(xz)m(z)dz = 50\sin(x) - 50\sin(x)\cos(x)$$

$$\int_0^1 5\sin(xz)m(z)dz = \int_0^1 5\sin(xz)m(x)dz$$

$$= \int_0^1 5\sin(xz)\left(10x\sin(x)\right)dz$$

$$= 50x\sin(x)\int_0^1 \sin(xz)dz$$

$$= -\frac{50x\sin(x)}{x}\left[\cos(xz)\right]|_0^1$$

$$= -\frac{50x\sin(x)}{x}\left(\cos(x) - \cos(0)\right)$$

$$= 50\sin(x)\left(1 - \cos(x)\right)$$

$$= 50\sin(x) - 50\sin(x)\cos(x) = d(x) \quad \checkmark$$

Part A

Discretizing the given integral using 20 midpoints on the interval $0 \leq z \leq 1$ such that the index j is a member of the set $\{j \in \mathbb{Z} : 1 \leq j \leq 20\}$. Each discrete sample of z_j is picked using the midpoint rule such that

$$\Delta z = \frac{1}{20}$$

$$z_j = \frac{\Delta z}{2} + (j-1)\,\Delta z$$

Approximating the given Fredholm integral equation of the first kind leads to

$$\int_0^1 g(x,z)m(z)dz = d(x)$$

$$\int_0^1 5\sin(xz)m(z)dz \approx \sum_{j=1}^{20} 5\sin(xz_j)m(z_j)\Delta z$$

$$\approx 5\sin(xz_1)m(z_1)\Delta z + 5\sin(xz_2)m(z_2)\Delta z + \cdots + 5\sin(xz_{20})m(z_{20})\Delta z$$

To fit this discrete numerical integration to the form $\boldsymbol{Gm} = \boldsymbol{d}$, let the variable x using the same midpoint rule be sampled at 20 equally spaced such that the index i is a member of set $\{x \in \mathbb{Z} : 1 \leq i \leq 20\}$. (Note: A bold symbol indicates a vector quantity) This above summation can be expressed as a linear system of equations such that

$$\begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,20} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,20} \\ \vdots & \vdots & \ddots & \vdots \\ g_{20,1} & g_{20,2} & \cdots & g_{20,20} \end{bmatrix} \begin{bmatrix} m(z_1) \\ m_(z_2) \\ \vdots \\ m(z_{20}) \end{bmatrix} = \begin{bmatrix} d(x_1) \\ d(x_2) \\ \vdots \\ d(x_{20}) \end{bmatrix}$$

where,

$$G \in \mathbb{R}^{20\times 20} : g_{i,j} = 5\sin(x_i z_j)\Delta z$$

$$\boldsymbol{d} \in \mathbb{R}^{20} : d_i = 50\sin(x_i) - 50\sin(x_i)\cos(x_i)$$

The vector $\boldsymbol{m} \in \mathbb{R}^{20}$ can be solved as

$$\boldsymbol{m} = G^{-1}\boldsymbol{d}$$

Constructing these vectors and matrices in MATLAB® (provided in file `prob1.m`), this results in the following quantities for G and $\boldsymbol{m}$.

Using the MATLAB® syntax `m = G \d`, the following outputs are produced and printed to the command window.

```
**********************
Part A
**********************

G Matrix
  Columns 1 through 10

    0.0031    0.0094    0.0156    0.0219    0.0281    0.0344    0.0406    0.0469    0.0531    0.0594
    0.0094    0.0281    0.0469    0.0656    0.0844    0.1031    0.1219    0.1406    0.1593    0.1781
    0.0156    0.0469    0.0781    0.1094    0.1406    0.1718    0.2031    0.2343    0.2655    0.2967
    0.0219    0.0656    0.1094    0.1531    0.1968    0.2405    0.2842    0.3279    0.3715    0.4151
    0.0281    0.0844    0.1406    0.1968    0.2530    0.3092    0.3653    0.4214    0.4774    0.5334
    0.0344    0.1031    0.1718    0.2405    0.3092    0.3778    0.4463    0.5147    0.5830    0.6513
    0.0406    0.1219    0.2031    0.2842    0.3653    0.4463    0.5271    0.6079    0.6884    0.7688
    0.0469    0.1406    0.2343    0.3279    0.4214    0.5147    0.6079    0.7008    0.7935    0.8859
    0.0531    0.1593    0.2655    0.3715    0.4774    0.5830    0.6884    0.7935    0.8982    1.0025
    0.0594    0.1781    0.2967    0.4151    0.5334    0.6513    0.7688    0.8859    1.0025    1.1186
    0.0656    0.1968    0.3279    0.4587    0.5893    0.7194    0.8490    0.9780    1.1064    1.2340
    0.0719    0.2156    0.3591    0.5023    0.6451    0.7873    0.9289    1.0698    1.2097    1.3487
    0.0781    0.2343    0.3902    0.5458    0.7008    0.8552    1.0087    1.1612    1.3126    1.4627
    0.0844    0.2530    0.4214    0.5893    0.7565    0.9228    1.0881    1.2522    1.4148    1.5758
    0.0906    0.2717    0.4525    0.6327    0.8120    0.9903    1.1673    1.3427    1.5164    1.6880
    0.0969    0.2905    0.4836    0.6760    0.8675    1.0576    1.2461    1.4328    1.6173    1.7993
    0.1031    0.3092    0.5147    0.7194    0.9228    1.1247    1.3246    1.5223    1.7174    1.9096
    0.1094    0.3279    0.5458    0.7626    0.9780    1.1915    1.4028    1.6113    1.8168    2.0188
    0.1156    0.3466    0.5768    0.8058    1.0331    1.2582    1.4806    1.6998    1.9154    2.1269
    0.1219    0.3653    0.6079    0.8490    1.0881    1.3246    1.5580    1.7877    2.0131    2.2337

  Columns 11 through 20

    0.0656    0.0719    0.0781    0.0844    0.0906    0.0969    0.1031    0.1094    0.1156    0.1219
    0.1968    0.2156    0.2343    0.2530    0.2717    0.2905    0.3092    0.3279    0.3466    0.3653
    0.3279    0.3591    0.3902    0.4214    0.4525    0.4836    0.5147    0.5458    0.5768    0.6079
    0.4587    0.5023    0.5458    0.5893    0.6327    0.6760    0.7194    0.7626    0.8058    0.8490
    0.5893    0.6451    0.7008    0.7565    0.8120    0.8675    0.9228    0.9780    1.0331    1.0881
    0.7194    0.7873    0.8552    0.9228    0.9903    1.0576    1.1247    1.1915    1.2582    1.3246
    0.8490    0.9289    1.0087    1.0881    1.1673    1.2461    1.3246    1.4028    1.4806    1.5580
    0.9780    1.0698    1.1612    1.2522    1.3427    1.4328    1.5223    1.6113    1.6998    1.7877
    1.1064    1.2097    1.3126    1.4148    1.5164    1.6173    1.7174    1.8168    1.9154    2.0131
    1.2340    1.3487    1.4627    1.5758    1.6880    1.7993    1.9096    2.0188    2.1269    2.2337
    1.3607    1.4866    1.6113    1.7350    1.8575    1.9787    2.0985    2.2169    2.3338    2.4491
    1.4866    1.6232    1.7584    1.8923    2.0245    2.1551    2.2839    2.4108    2.5358    2.6586
    1.6113    1.7584    1.9038    2.0474    2.1889    2.3283    2.4654    2.6001    2.7323    2.8618
    1.7350    1.8923    2.0474    2.2001    2.3504    2.4979    2.6427    2.7844    2.9230    3.0582
    1.8575    2.0245    2.1889    2.3504    2.5088    2.6639    2.8155    2.9634    3.1074    3.2473
    1.9787    2.1551    2.3283    2.4979    2.6639    2.8258    2.9835    3.1367    3.2852    3.4287
    2.0985    2.2839    2.4654    2.6427    2.8155    2.9835    3.1464    3.3040    3.4559    3.6020
    2.2169    2.4108    2.6001    2.7844    2.9634    3.1367    3.3040    3.4649    3.6193    3.7667
    2.3338    2.5358    2.7323    2.9230    3.1074    3.2852    3.4559    3.6193    3.7749    3.9225
    2.4491    2.6586    2.8618    3.0582    3.2473    3.4287    3.6020    3.7667    3.9225    4.0689
```

Figure 1: Problem 1, Part A: Operator G

Data (d)
0.0004
0.0105
0.0486
0.1330
0.2812
0.5102
0.8358
1.2727
1.8340
2.5315
3.3751
4.3726
5.5302
6.8518
8.3389
9.9912
11.8058
13.7775
15.8992
18.1611

Figure 2: Problem 1, Part A: Data $\boldsymbol{d}$

```
Model (m)
   1.0e+09

    1.0796
    0.5591
   -0.3371
   -0.2944
   -0.0508
    0.0044
    0.1532
   -0.0831
    0.1198
   -0.0394
    0.0147
   -0.1513
    0.0982
    0.0636
   -0.0668
   -0.0066
    0.0124
    0.0134
   -0.0134
    0.0031
```

Figure 3: Problem 1, Part A: Model $\boldsymbol{m}$

Part B

Using the function `cond()` in MATLAB® provides the following output.

```
**********************
Part B
**********************

Condition number of G: 1.403249e+19
```

Figure 4: Problem 1, Part B: Condition Number

Clearly, the operator G is ill-conditioned.

Part C

Figure shows the true model compared to the computed model, as well as the error between the two models.

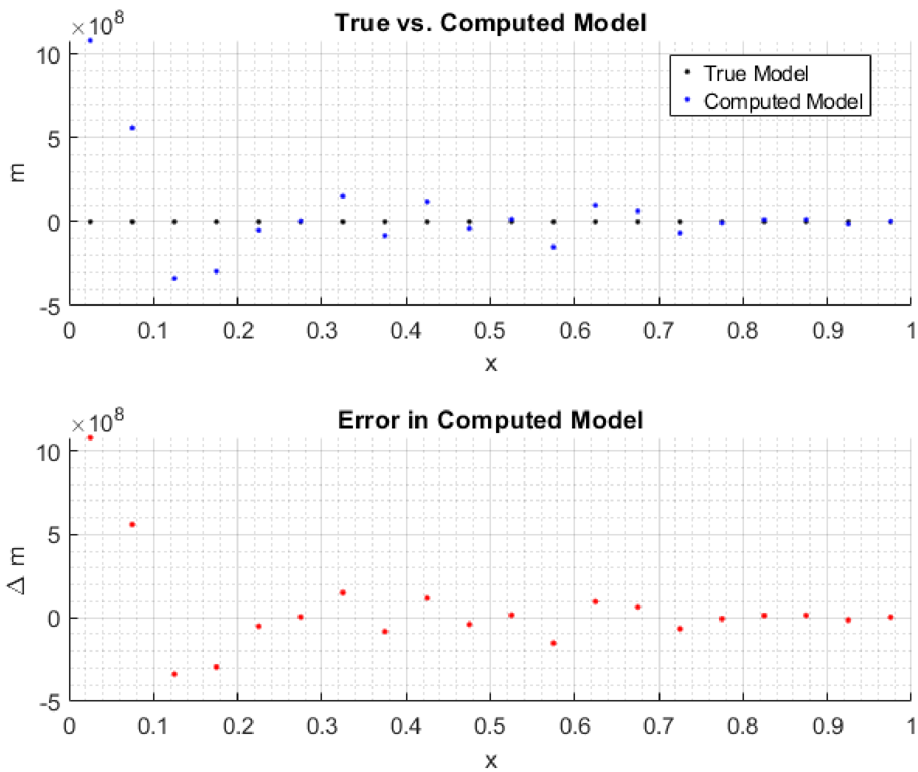

Figure 5: Problem 1, Part C: True vs. Computed Model

Part D

The solution is so poor due to the high condition number of the operator G. This means that small perturbations in the data are amplified by 19 orders of magnitude, making the solution completely unstable.

Another way to think about it, the determinate of G is incredibly close to zero, meaning that any vector transformed by this matrix is scaled down to nearly zero.

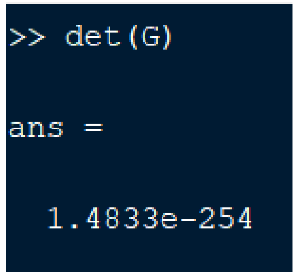

Figure 6: Problem 1, Part D: Determinate

In this case, it is likely that the floating point error in MATLAB ® kept the determinate from equaling exactly zero. While I am not eager to compute the determinate of a $\mathbb{R}^{20\times 20}$ matrix by hand, I suspect that analytically it may turn out to be zero. If that was truly the case, then it shouldn't have even been possible to compute the inverse of G to solve for the model.

Problem 2

2. Using the dot product, show that if $x \perp y$ (or equivalently that $x^T y = 0$, then

$$\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2.$$

This is a version of the Pythagorean theorem.

Solution

Consider column vectors of equal dimension $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$. Let $\boldsymbol{x} \perp \boldsymbol{y}$, meaning that $\boldsymbol{x}^T\boldsymbol{y} = 0$. Also, recall that the dot product operator is *additive* and *distributive*.

$$\begin{aligned}
||\boldsymbol{x}+\boldsymbol{y}||_2^2 &= ||(\boldsymbol{x}+\boldsymbol{y})^T(\boldsymbol{x}+\boldsymbol{y})||_2 \\
&= ||\boldsymbol{x}^T\boldsymbol{x}||_2 + ||\boldsymbol{x}^T\boldsymbol{y}||_2 + ||\boldsymbol{y}^T\boldsymbol{x}||_2 + ||\boldsymbol{y}^T\boldsymbol{y}||_2 \\
&= ||\boldsymbol{x}^T\boldsymbol{x}||_2 + ||\boldsymbol{x}^T\boldsymbol{y}||_2 + ||\left(\boldsymbol{x}^T\boldsymbol{y}\right)^T||_2 + ||\boldsymbol{y}^T\boldsymbol{y}||_2 \\
&= ||\boldsymbol{x}^T\boldsymbol{x}||_2 + \cancelto{0}{||\boldsymbol{x}^T\boldsymbol{y}||_2} + \cancelto{0}{||\left(\boldsymbol{x}^T\boldsymbol{y}\right)^T||_2} + ||\boldsymbol{y}^T\boldsymbol{y}||_2 \\
&= ||\boldsymbol{x}^T\boldsymbol{x}||_2 + ||\boldsymbol{y}^T\boldsymbol{y}||_2 \\
&= ||\boldsymbol{x}||_2^2 + ||\boldsymbol{y}||_2^2 \quad \checkmark
\end{aligned}$$

Problem 3

3. Suppose that X and Y are multivariate normal random vectors with a joint multivariate normal distribution with

$$E\begin{bmatrix} X \\ Y \end{bmatrix} = 0$$

and

$$\text{Cov}\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Show that $Z = X + Y$ is MVN with expected value 0 and covariance $2I$.

Solution

Given the problem statement, let random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ be of equal dimension such that

$$\boldsymbol{X},\, \boldsymbol{Y} \sim \mathrm{N}\left(0,\, I\right)$$

Creating a new random vector $\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$, the expected value of $\boldsymbol{Z}$ is

$$\begin{aligned} \mathrm{E}\left[\boldsymbol{Z}\right] &= \mathrm{E}\left[\boldsymbol{X} + \boldsymbol{Y}\right] \\ &= \mathrm{E}\left[\boldsymbol{X}\right] + \mathrm{E}\left[\boldsymbol{Y}\right] \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0} \quad \checkmark \end{aligned}$$

To determine the covariance of $\boldsymbol{Z}$, we must first find a matrix A such that $\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$. A is defined as:

$$\boldsymbol{Z} = A\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$$
$$\boldsymbol{Z} = \begin{bmatrix} I & I \end{bmatrix}\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$$

$$A := \begin{bmatrix} I & I \end{bmatrix}$$

The covariance of $\boldsymbol{Z}$ is given by the transformation $\boldsymbol{Z} = ACA^T$, which is shown below.

$$
\begin{aligned}
\operatorname{Cov}(\boldsymbol{Z}) &= A\operatorname{Cov}\left(\begin{bmatrix}\boldsymbol{X}\\ \boldsymbol{Y}\end{bmatrix}\right)A^T \\
&= \begin{bmatrix} I & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I \\ I \end{bmatrix} \\
&= \begin{bmatrix} I & I \end{bmatrix}\begin{bmatrix} I \\ I \end{bmatrix} \\
&= I + I \\
&= 2I \quad \checkmark
\end{aligned}
$$

Problem 4

4. Use the method of Lagrange multipliers to solve the problem

$$\min f(x) = x^2 - 8x + y^2 - 8y + 32$$

subject to

$$g(x) = x^2 + 4y^2 \leq 16$$

(a) Write down the Lagrange multiplier conditions as a nonlinear system of three equations in the unknowns x, y, λ.

(b) Using a symbolic computation tool (like Mathematica or Maple) solve the equations from part (a) and find the optimal solution. Give the solution in decimal form (rather than as a formula involving lots of nested square roots.) Note that you may find multiple solutions including ones that aren't actually feasible or that are not minimizers. If so, identify the solution that minimizes $f(x)$.

Solution