HW 3 SOLUTIONS

1/ $x_{c1}(t) = \sin 20\pi t$

$x_1[n] = \sin \frac{\pi n}{5}$

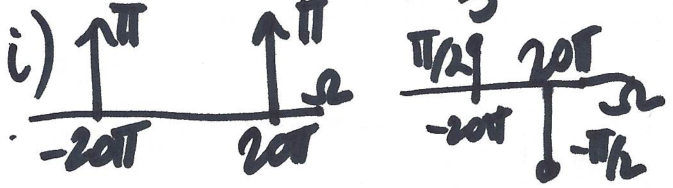
a) $\Omega_0 \; 20\pi$, $\quad w_0 = \frac{\pi}{5}$ $\Big\}$ $\Omega_0 T_1 = w_0$

$20\pi \, T_1 = \frac{\pi}{5}$

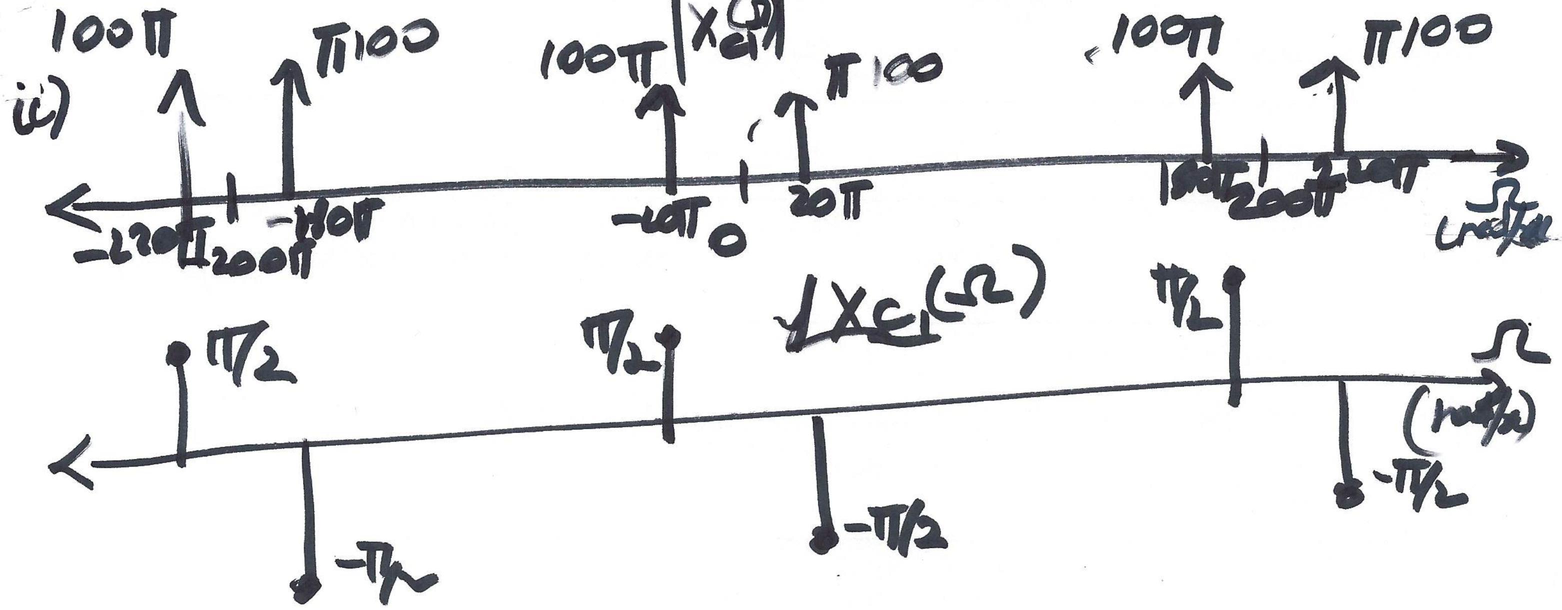
$T_1 = \frac{1}{100}$

$\Omega_s = \frac{2\pi}{T_1} = 200\pi$

b) i)

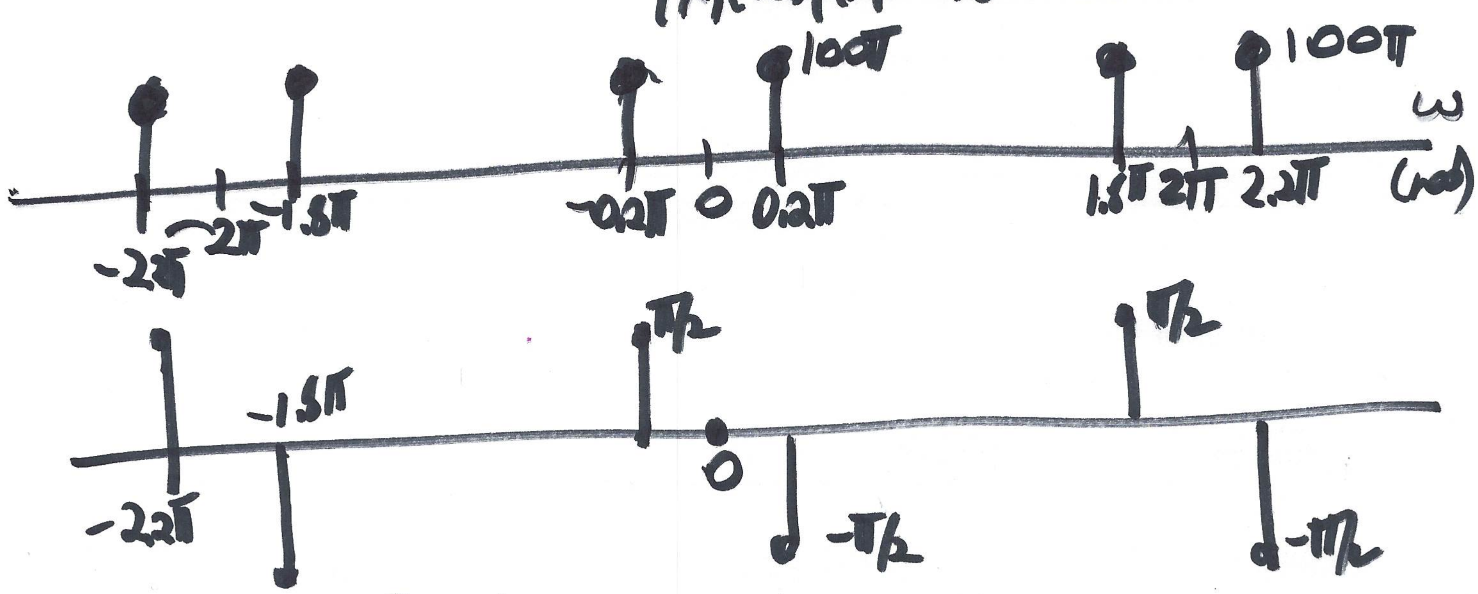


ii)



iii) The corresponding $X_1(e^{jw})$ for $T = \frac{1}{100}$ is:

$|X_1(\omega)|$ or $|X_1(e^{j\omega})|$



Sample calculation: $\Omega_0 T_1 = \omega_0 \quad \frac{20\pi}{100} = 0.2\pi$

c) Let T_1' be another sampling period that yields the same $X_1(e^{jw})$. Then,

$$\underbrace{2\pi k - w_0' = -w_0}_{(\star)} \quad \text{where } w_0 = 20\pi \cdot T_s \text{ and } w_0' = 20\pi T_s'$$

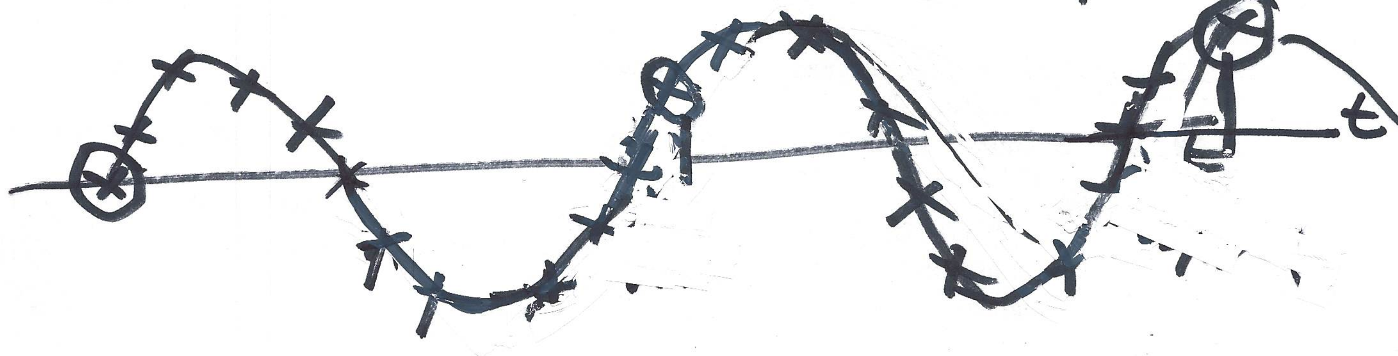
Let $k=1$ and $T_1 = 1/100$

$$2\pi - 20\pi \cdot T_s' = -20\pi \cdot \underbrace{T_s}_{T_1 = 1/100} = -0.2\pi$$

$$2 - 20 T_s' = -0.2 \qquad T_s' = 11/100$$

$(\star)$ This condition implies the component at -20π is replaced by the higher frequency "aliased" component from the k^{th} replica under another sampling period T_s'.

What is happening in time?



$T_s = \frac{1}{100}$ sec or $f_s = 100$ Hz, $f_0 = 10$ Hz so 10 samples/cycle are taken yielding the "X" points, but $T_s = 11/100$ gives the same discrete time sequence depicted above with ⊗ points. Note that other solutions for $k = 2, 3, 4 ..$ exist using $(\star)$ $2\pi k - 20\pi T_s' = -0.2\pi$

$2\pi f_0 = 20\pi$

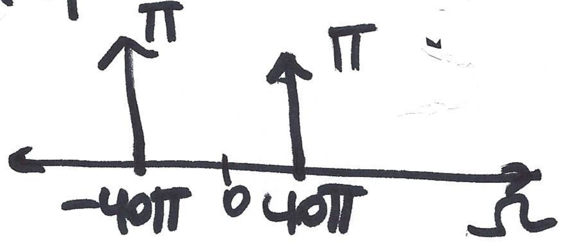
2) $x_2(t) = \cos 40\pi t$
$x_2[n] = \cos\left(\frac{2\pi}{5} n\right)$

a) $\Omega_0 T_2 = w_0$ $\quad 40\pi T_2 = \frac{2\pi}{5}$

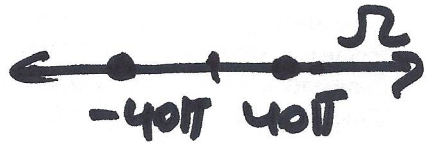
$T_2 = 1/100$

$\Omega_s = 2\pi / T_s = 200\pi$

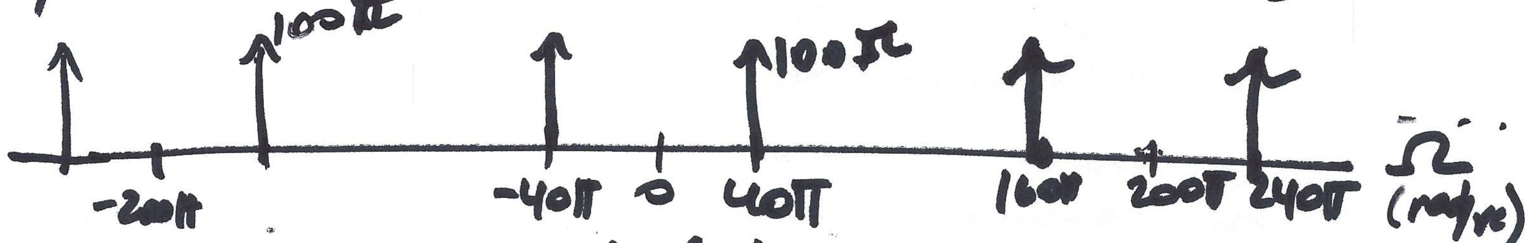
b) i) $|X(\Omega)|$



$\angle X_2(\Omega)$



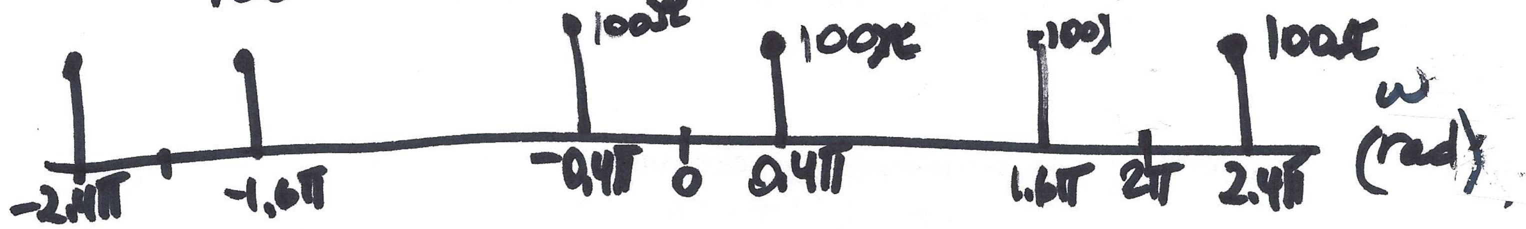
ii) $|X_2(\Omega)|$ $\quad T_2 = 1/100$



$\angle X(\Omega)$

iii) The corresponding $X_2(e^{jw})$ for $T = 1/100$

$40\pi \cdot \frac{1}{100} = 0.4\pi$



c) For cosine(.) there are two cases

a) $2\pi k - w_0' = 0.4\pi$ and b) $2\pi m - w_0' = -0.4\pi$
where $w_0' = 40\pi T_2'$, $\quad$ where $w_0' = 40\pi T_2'$

2c continued

For case a $\quad 2\pi k - 40\pi T_2' = 0.4\pi \Rightarrow \dfrac{2k\pi - 0.4\pi}{40\pi} = T_2'$

$$T_2' = \frac{5k-1}{100} = \left\{\frac{4}{100}, \frac{9}{100}, \frac{14}{100}\right.$$

For case b $\quad 2\pi k - 40\pi T_2' = -0.4\pi \Rightarrow \dfrac{2k\pi + 0.4\pi}{40\pi} = T_2'$

$$T_2' = \frac{5k+1}{100} = \left\{\frac{6}{100}, \frac{11}{100}, \frac{16}{100}, \ldots\right.$$

Time depiction on the following page

3) $x_c(t) = x_{c_1}(t) + x_{c_2}(t)$

$$x_c(t) = \sin 20\pi t + \cos 40\pi t$$

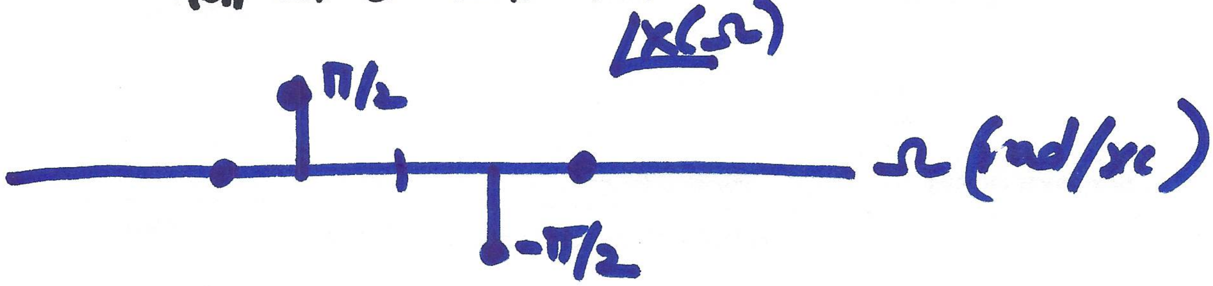
and

$$x[n] = \sin\frac{\pi}{5}n + \cos\frac{2\pi}{5}n$$

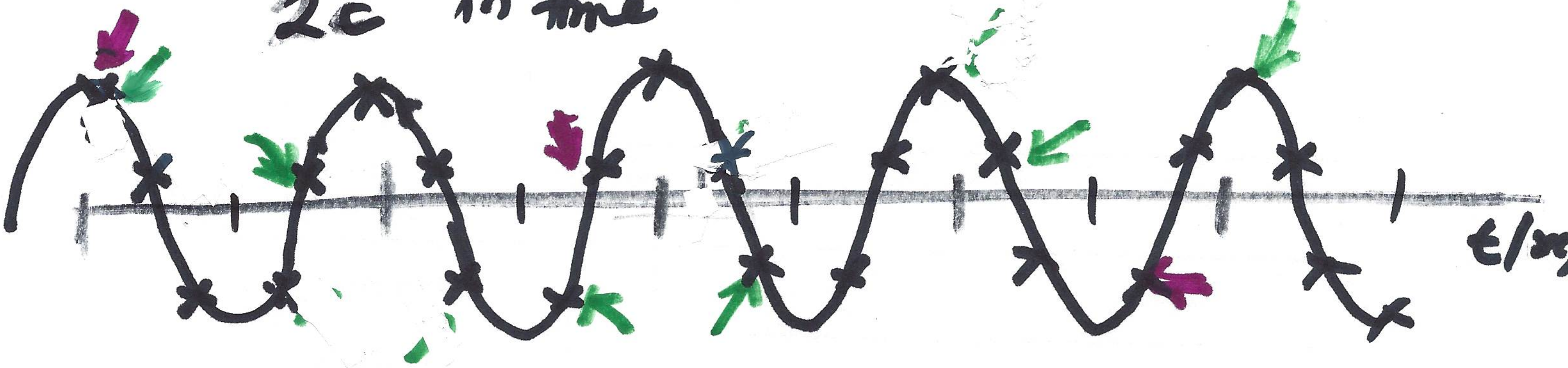
a) To generate the discrete sequence $x[n]$, $T_s = \frac{1}{100}$

b)

i)



2c in time

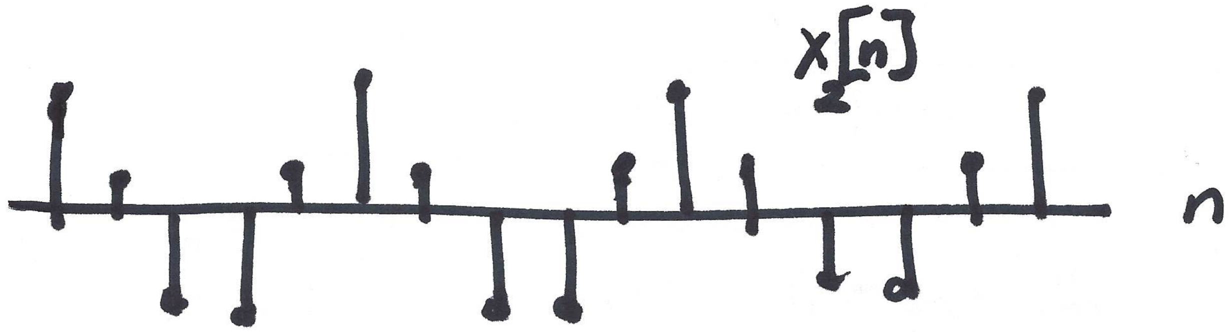


$\omega_0 = 40\pi \Rightarrow f_0 = 20\,Hz$ 20 cycles/sec

$T_s = 1/100$ sec $\Rightarrow$ 100 samples/sec

$\Rightarrow$ 5 samples/cycle

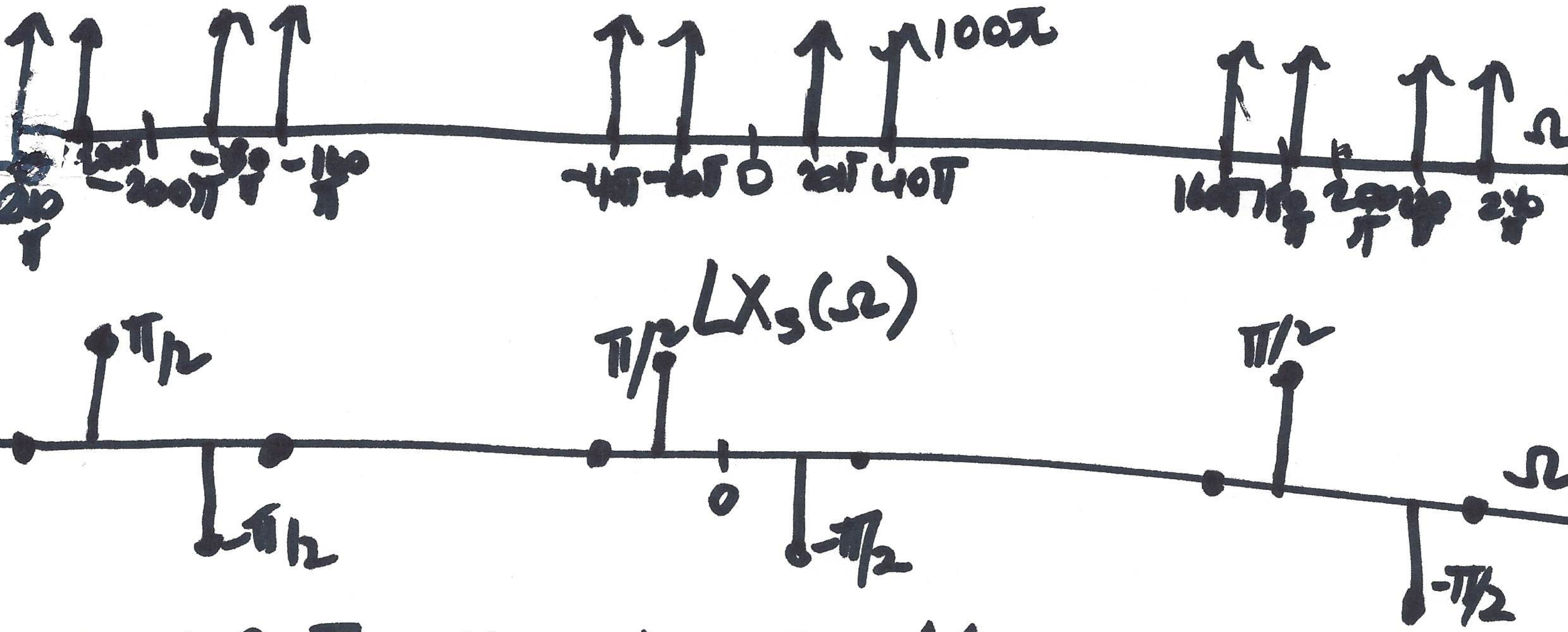
$360°/5 = 72°$ between samples



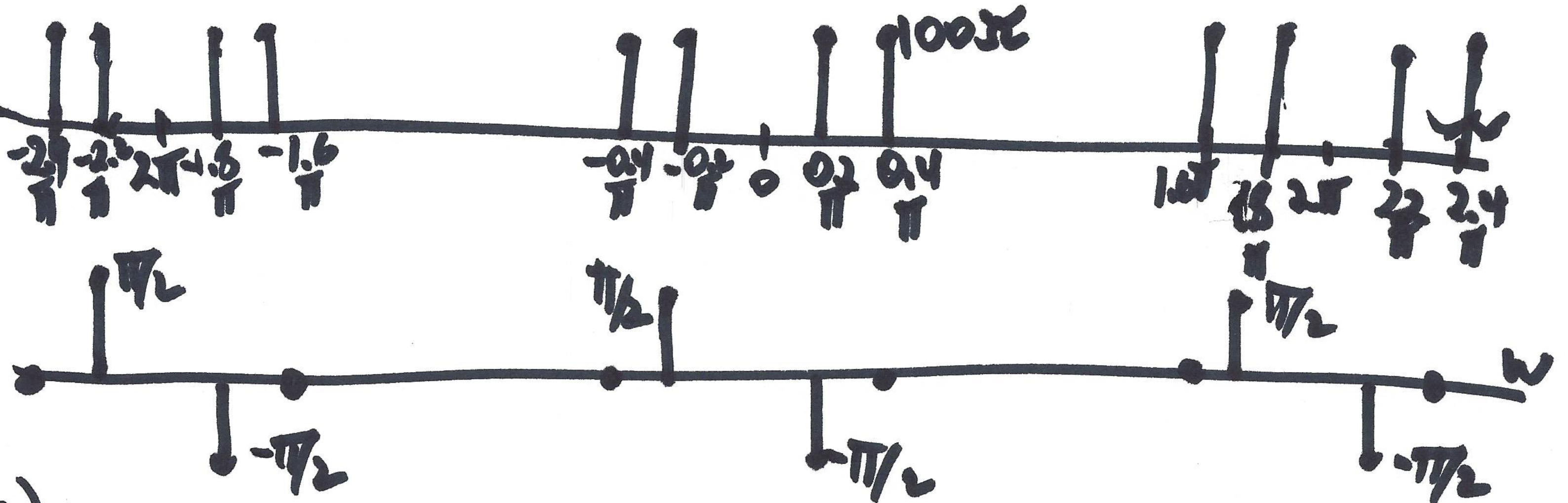
Let $k=1$ $\quad T_2' = \frac{5k-1}{100} \Rightarrow T_2' = \frac{4}{100}$ //

$k=2$ $\quad T_2' = 9/100$ //

ii)



(iii) $\Omega_0 T_s = w_0$ where $T_s = 1/100$



c)

The sampling rate that works for both $x_{c1}(t)$ and $x_{c2}(t)$ will work for $x(t) = x_{c1}(t) + x_{c2}(t)$ which is $T_s = \frac{11}{100}$.

4) $y[n] = x[n] + 3u[n+1]$

a) Linear? No.

Try homogeneity. Let $x_1[n] = a\,x[n]$ be the input to the system. Then

$$y_1[n] = x_1[n] + 3u[n+1]$$
$$= a x[n] + 3u[n+1]$$
$$\neq a\left[x[n] + 3u[n+1]\right] = a y[n].$$

This is enough to show that it is nonlinear

b) Time invariant? No, due to the $u[n+1]$ out of $x[n]$. To show, let $x_1[n] = x[n-n_0]$

Let $y_1[n] = T\left[x_1[n]\right]$

$$= x_1[n] + 3u[n+1]$$
$$= x[n-n_0] + 3u[n+1] \neq y[n-n_0]$$

because $y[n-n_0] = x[n-n_0] + 3u[n-n_0+1]$

c) Causal? Yes because the output does not depend on the future values of the input. Note that $(n+1)$ ~~if~~ is not the argument of the input $x[n]$ itself.

d) memoryless? Yes, because $y[n]$ only depends on the present value of the input $x[n]$.

e) Stable? Yes.

Let $|x[n]| \leq B_x$ then

$$|y[n]| = |x[n] + 3u[n+1]|$$
$$\leq |x[n]| + 3|u[n+1]| \quad \} \text{ triangle ineq.}$$
$$\leq B_x + 3 \cdot 1 = B_x + 3 = B_y$$

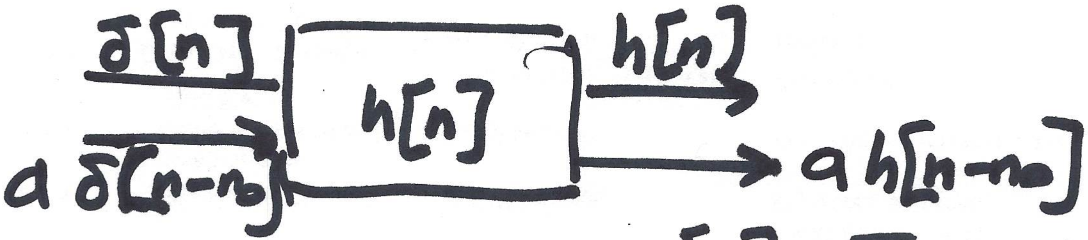
Note that $u[n+1]$ is either 0 or 1, so $|u[n+1]| \leq 1$

All bounded inputs $|x[n]| \leq B_x$ yield bounded outputs $|y[n]| \leq B_x + 3 = B_y < \infty$.

5] $h[n] = [\ldots 0 \ \tfrac{1}{2} \ \underline{1} \ \tfrac{1}{2} \ 0 \ldots]$

a) $x[n] = [\ldots 0 \ \underline{3} \ 2 \ 1 \ 0 \ldots]$

Due to LSI, the system is described the impulse response



$y[n] = \sum x(k)\, h[n-k]$

$x_0[n] = 3\delta[n] \longrightarrow 3 \cdot [\tfrac{1}{2} \ \underline{1} \ \tfrac{1}{2}] = y_0[n]$

$x_1[n] = 2\delta[n-1] \longrightarrow 2[0 \ \underline{\tfrac{1}{2}} \ 1 \ \tfrac{1}{2}] = y[n]$

$x_2[n] = 1[n-2] \longrightarrow [0 \ \underline{0} \ \tfrac{1}{2} \ 1 \ \tfrac{1}{2}] = y_2[n]$

$x[n] \rightarrow y[n] = [0 \ \tfrac{3}{2} \ \underline{4} \ 4 \ 2 \ \tfrac{1}{2}]$

Another way

	1/2	$\underline{1}$	1/2
$\underline{3}$	3/2	3	3/2
2	1	2	1
1	1/2	1	1/2

$[\tfrac{3}{2} \ 4 \ 4 \ 2 \ \tfrac{1}{2}]$

$\underline{5b}$ $h[n] = [\tfrac{1}{2} \;\; \underline{1} \;\; \tfrac{1}{2}] = \frac{1}{2}\delta[n-1] + \delta[n] + \frac{1}{2}\delta[n+1]$

$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$

Compute $y[n] = x[n] * h[n]$

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left[\frac{1}{2}\delta[n-1] + \delta[n] + \frac{1}{2}\delta[n+1]\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{n-2} u[n-2] + \left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{1}{2}\left(\frac{1}{2}\right)^{n} u[n]$$