Menu

- Priority Queues
- Heaps
- Heapsort

Priority Queue

A data structure implementing a set S of elements, each associated with a key, supporting the following operations:

insert(S, x) : set S return element of S with largest key

max(S) :

extract_max(S) : return element of S with largest key and remove it from S

increase_key(S, x, k) : increase the value of element x' s key to new value k

insert element x into (assumed to be as large as current value)

Source: MIT 6.006- Introduction to Algorithms

Heap

- Implementation of a priority queue
- An array, visualized as a nearly complete binary tree
- Max Heap Property: The key of a node is $\geq$ than the keys of its children

(Min Heap defined analogously)

16	14	10	8	7	9	3	2	4	1

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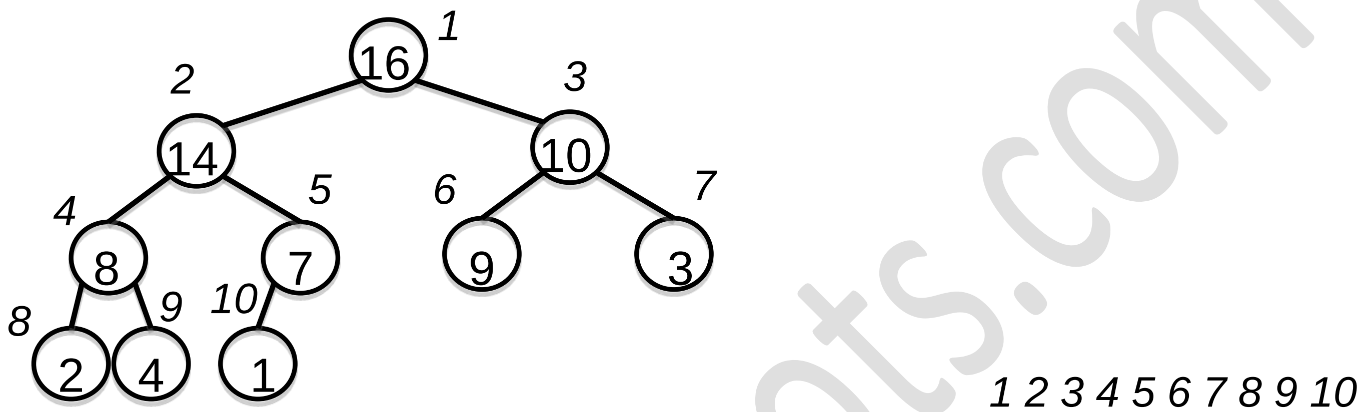

Heap as a Tree

root of tree: first element in the array, corresponding to i = 1 parent(i) =i/2: returns index of node's parent left(i)=2i:

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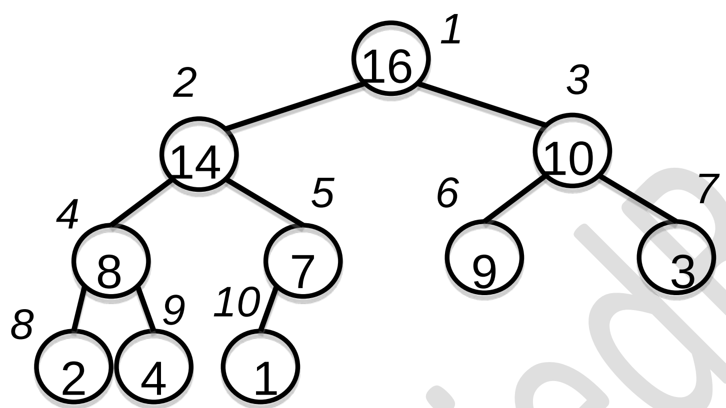
returns index of node's left child right(i)=2i+1: returns index of node's right child



16	14	10	8	7	9	3	2	4	1

1 2 3 4 5 6 7

8 9 10

No pointers required! Height of a binary heap is O(lg n)

Heap Operations

build_max_heap : produce a max-heap from an unordered array

max_heapify : correct a single violation of the heap property in a subtree at its root

insert, extract_max, heapsort

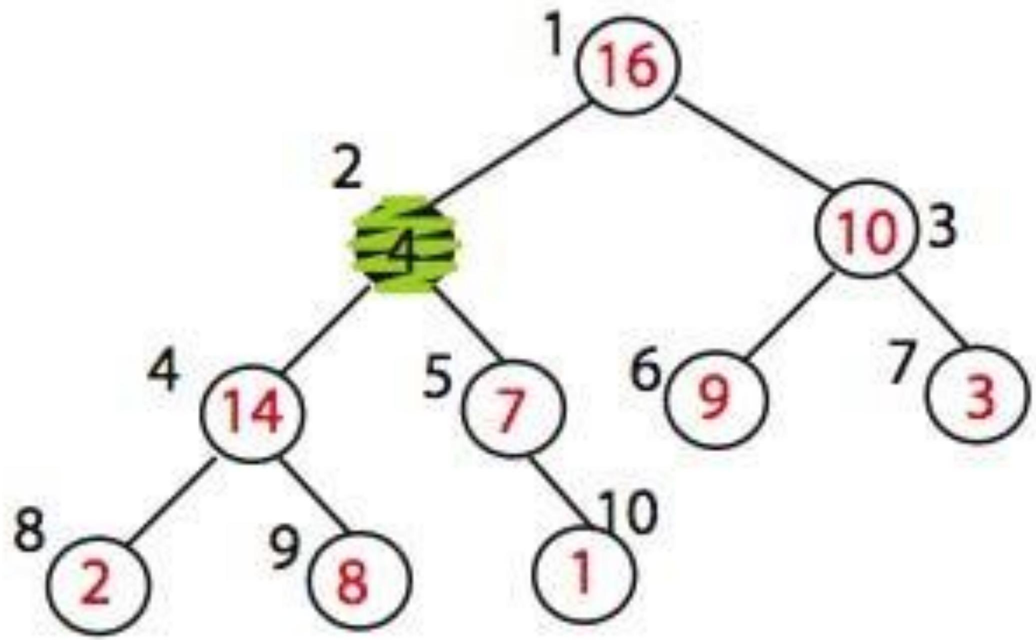
Source: MIT 6.006- Introduction to Algorithms

Max_heapify

- **Assume that the trees rooted at left(*i*) and right(*i*) are max-heaps**

- If element A[*i*] violates the max-heap property, correct violation by "trickling" element A[*i*] down the tree, making the subtree rooted at index *i* a max-heap

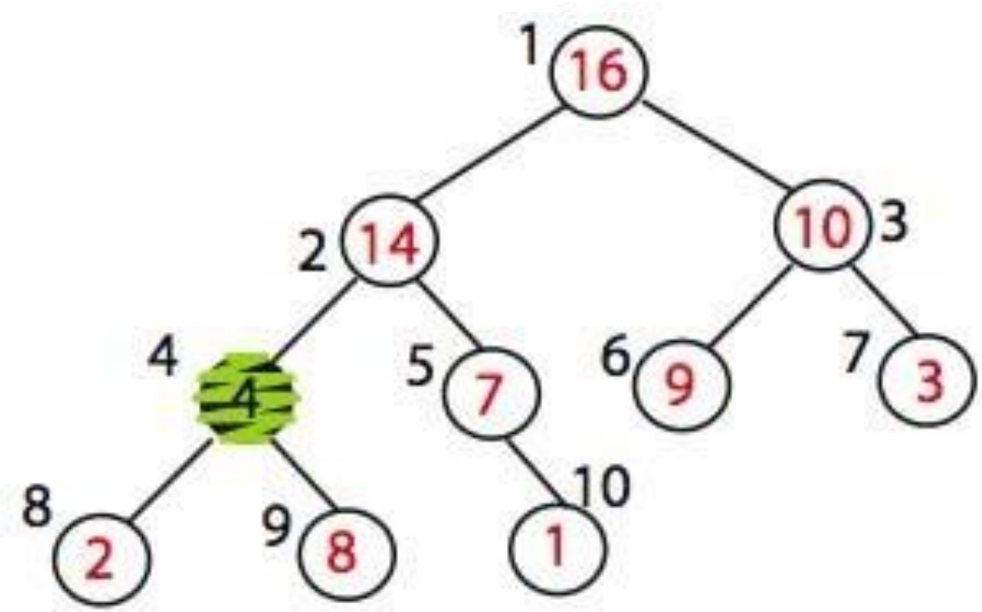
Max_heapify (Example)



MAX_HEAPIFY (A,2)
heap_size[A] = 10

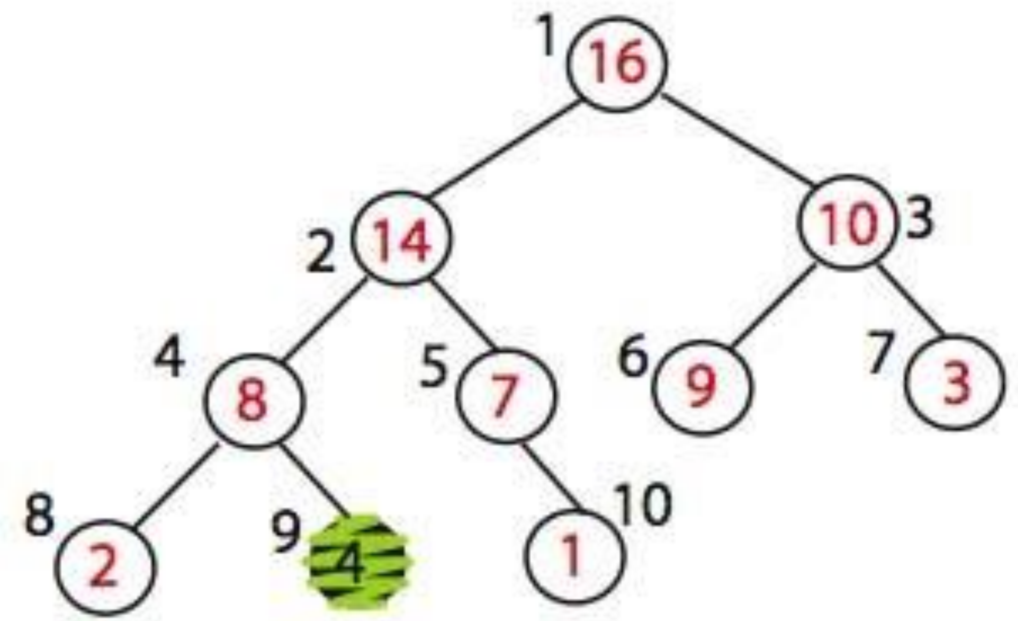
Max_heapify (Example)

Node 10 is the left child of node 5 but is drawn to the right for convenience



Exchange A[2] with A[4]
Call MAX_HEAPIFY(A,4)
because max_heap property is violated

Max_heapify (Example)



Exchange A[4] with A[9]
No more calls

Time=? O(log n)

Max_Heapify Pseudocode

l = left(i) r

= right(i)

if (l <= heap-size(A) and A[l] > A[i])

then largest = l else largest = i if (r <=

heap-size(A) and A[r] > A[largest])

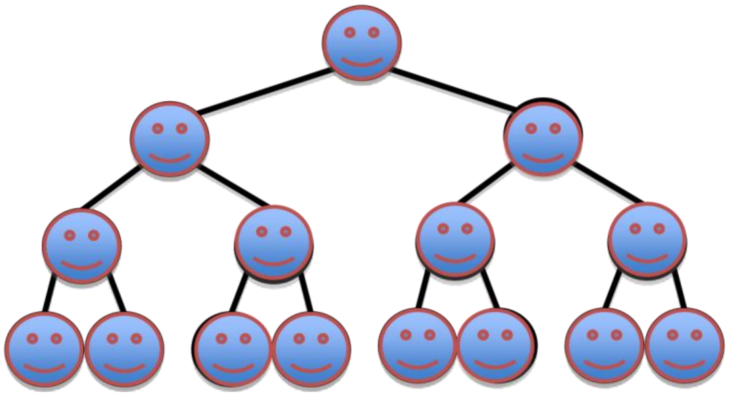
then largest = r if largest = i

then exchange A[i] and A[largest]

Max_Heapify(A, largest)

Build_Max_Heap(A)

Converts A[1…n] to a max heap



Build_Max_Heap(A): for i=n/2 downto 1 do Max_Heapify(A, i)

Source: MIT 6.006- Introduction to Algorithms

Why start at n/2?

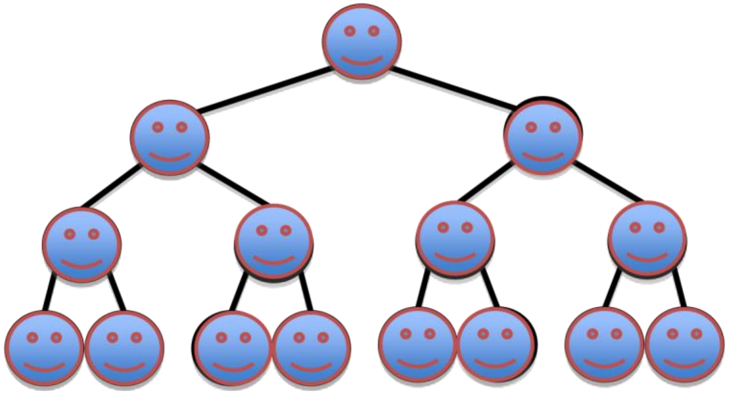
Because elements A[n/2 + 1 … n] are all leaves of the tree
2i > n, for i > n/2 + 1

Time=?   O(n log n) via simple analysis

Build_Max_Heap(A) Analysis

Converts A[1…*n*] to a max heap

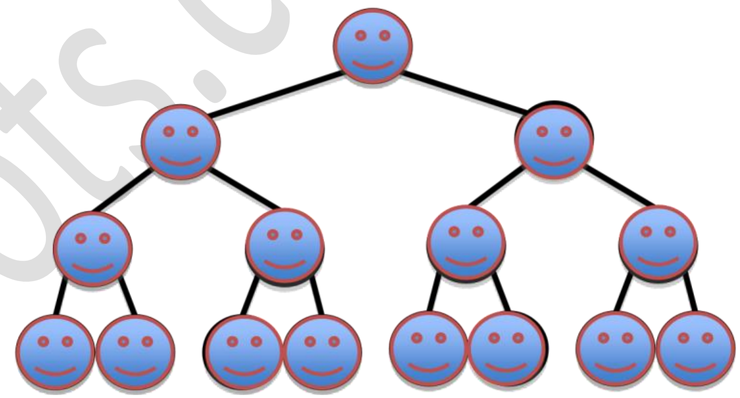
Build_Max_Heap(A):   for i=n/2
downto 1   do
Max_Heapify(A, i)



Observe however that Max_Heapify takes O(1) for time for nodes that are one level above the leaves, and in general, O(*l*) for the nodes that are *l* levels above the leaves. We have n/4 nodes with level 1, n/8 with level 2, and so on till we have one root node that is lg n levels above the leaves.

Build_Max_Heap(A) Analysis

Converts A[1…*n*] to a max heap



Build_Max_Heap(A): for i=n/2 downto 1 do Max_Heapify(A, i)

Total amount of work in the for loop can be summed as:

$$n/4\ (1\ c) + n/8\ (2\ c) + n/16\ (3\ c) + \ldots + 1\ (\lg n\ c)$$

Setting $n/4 = 2^k$ and simplifying we get: $c\ 2^k(1/2^0 + 2/2^1 + 3/2^2 + \ldots (k+1)/2^k)$

The term is brackets is bounded by a constant!

This means that Build_Max_Heap is O(n)

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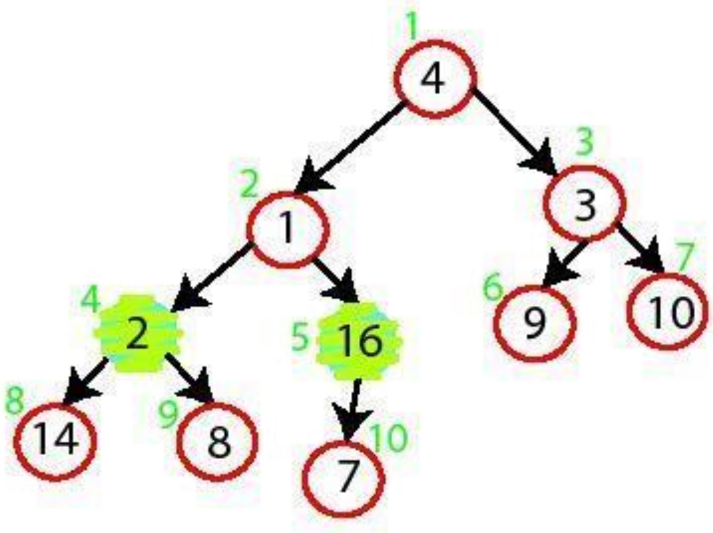
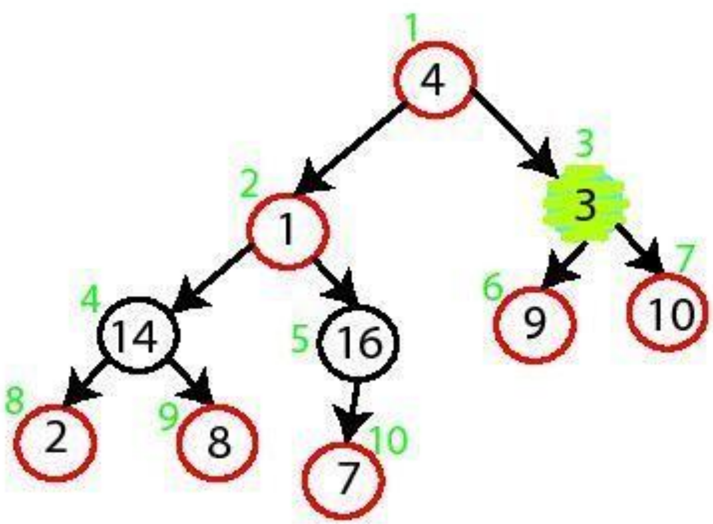
Build-Max-Heap Demo

A									
4	1	3	2	16	9	10	14	8	7



MAX-HEAPIFY (A,5)
no change
MAX-HEAPIFY (A,4)
Swap A[4] and A[8]



MAX-HEAPIFY (A,3)
Swap A[3] and A[7]

Build-Max-Heap Demo

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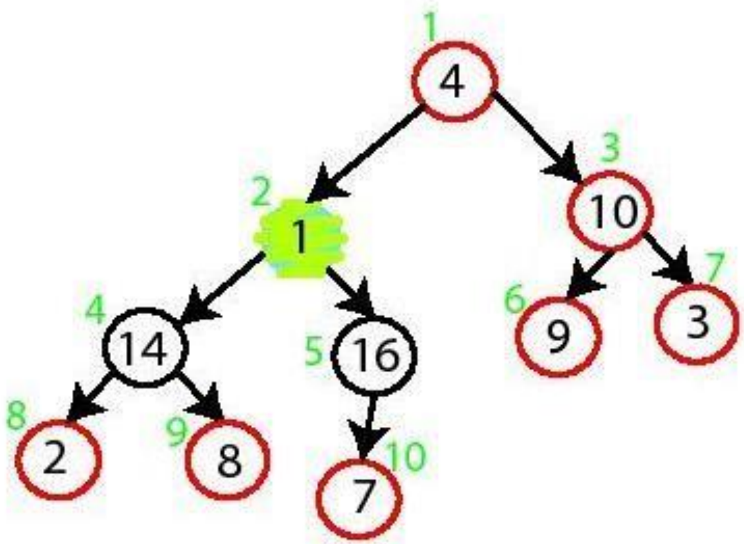
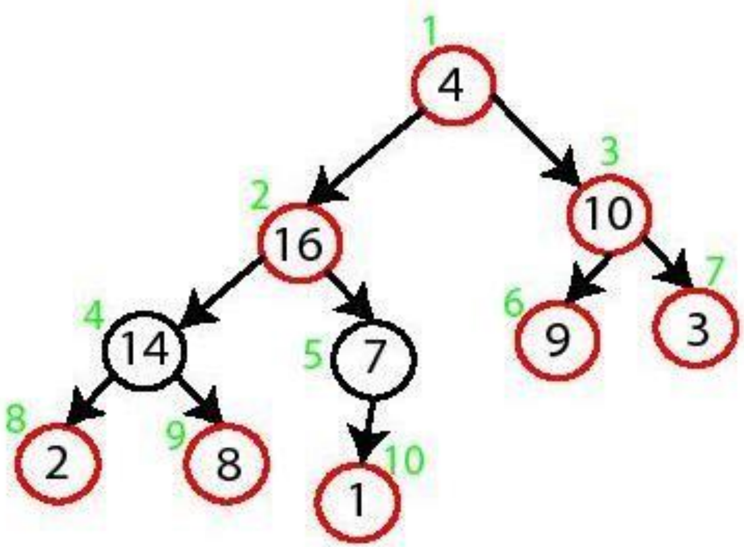


MAX-HEAPIFY (A,2)
Swap A[2] and A[5]
Swap A[5] and A[10]

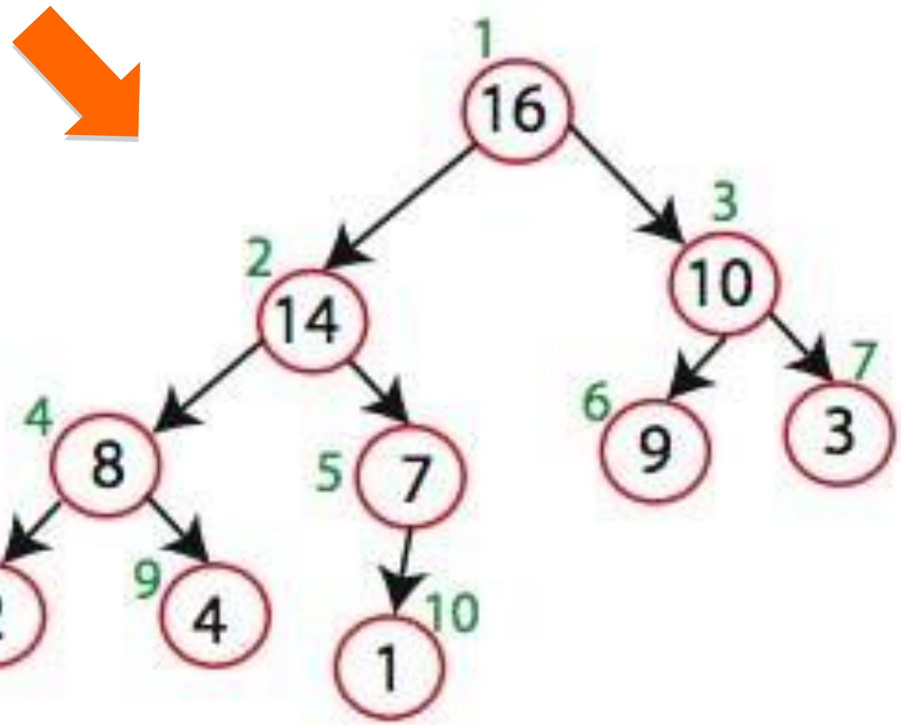


MAX-HEAPIFY (A,1)
Swap A[1] with A[2]
Swap A[2] with A[4]
Swap A[4] with A[9]

Build-Max-Heap

Source: MIT 6.006- Introduction to Algorithms

A										
	4	1	3	2	16	9	10	14	8	7



Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

Heap-Sort

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1. Build Max Heap from unordered array;

2. Find maximum element A[1];

3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!

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Heap-Sort

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4. Discard node n from heap (by decrementing heap-size variable)

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Heap-Sort

4. Discard node *n* from heap
   (by decrementing heap-size variable)
5. New root may violate max heap property, but its children are max heaps. Run max_heapify to fix this.

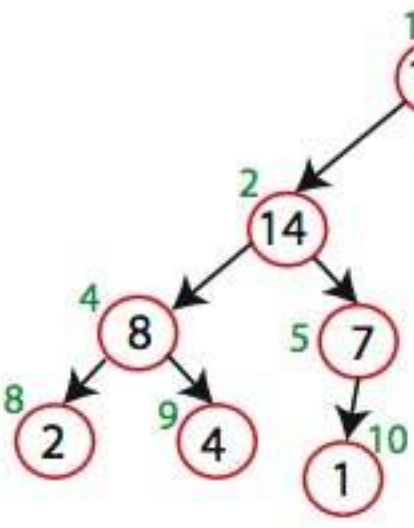
Sorting Strategy:

1. Build Max Heap from unordered array;
2. Find maximum element A[1];
3. Swap elements A[*n*] and A[1]: now max element is at the end of the array!

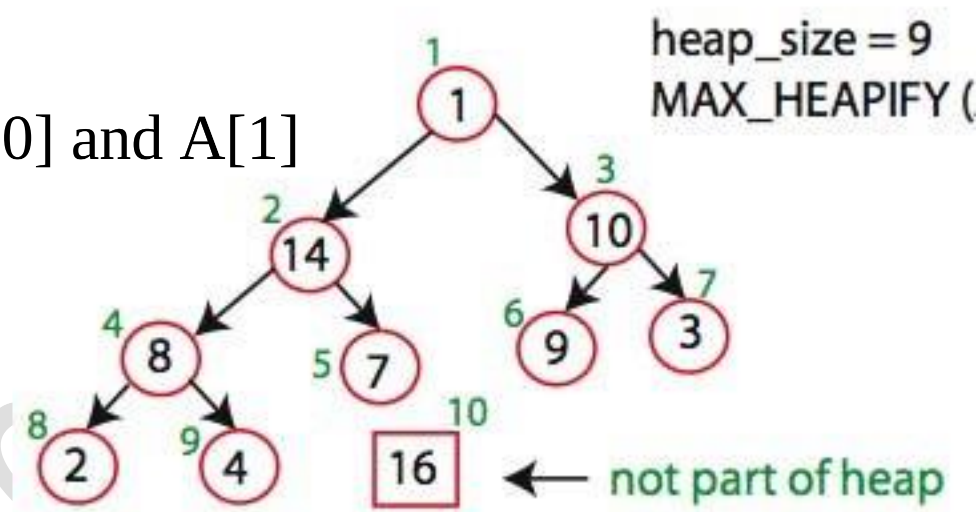
Heap-Sort

4. Discard node *n* from heap
   (by decrementing heap-size variable)
5. New root may violate max heap property, but its children are max heaps. Run max_heapify to fix this.
6. Go to Step 2 unless heap is empty.

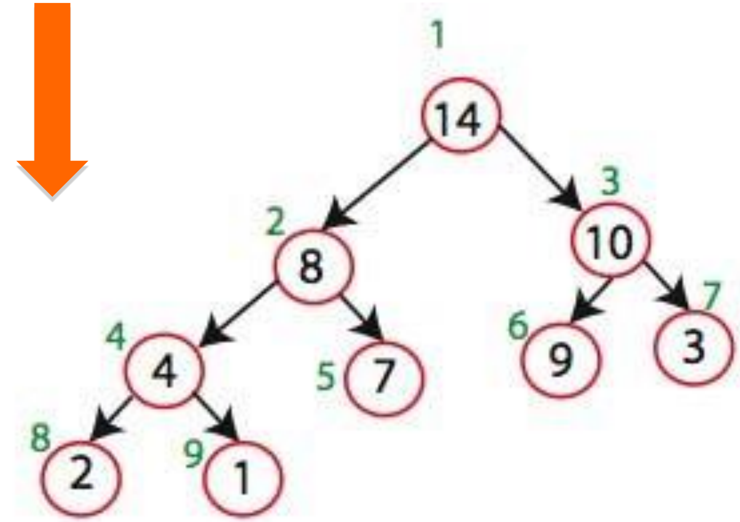
Heap-Sort Demo



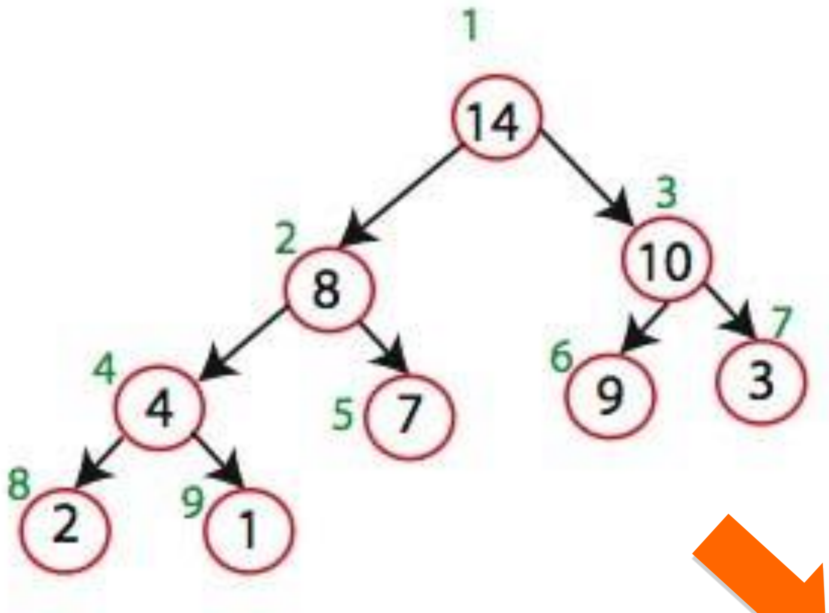
Swap A[10] and A[1]



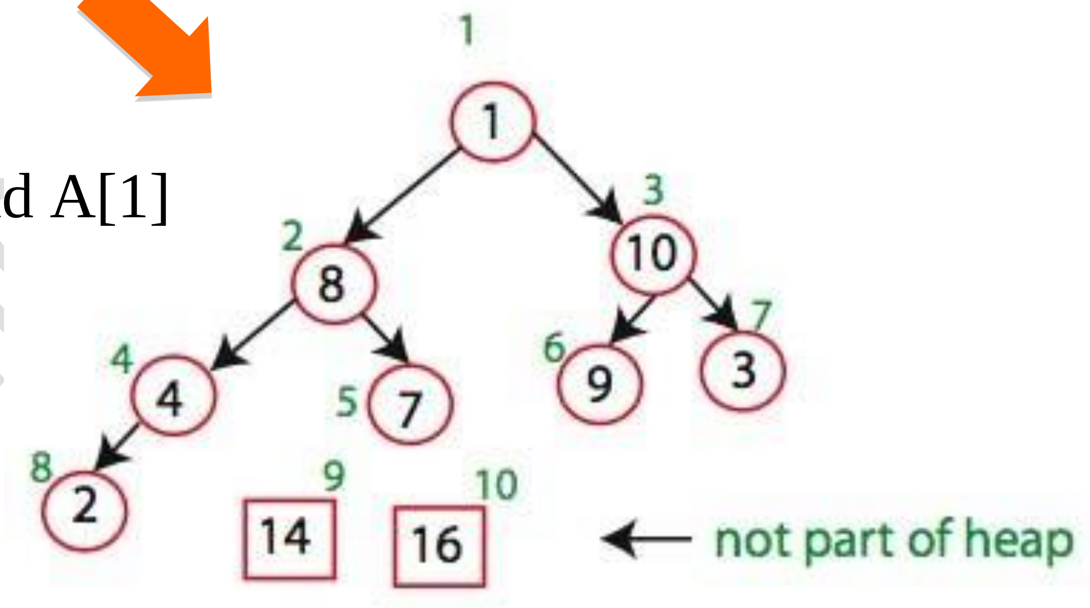
Max_heapify(A,1)



Heap-Sort Demo

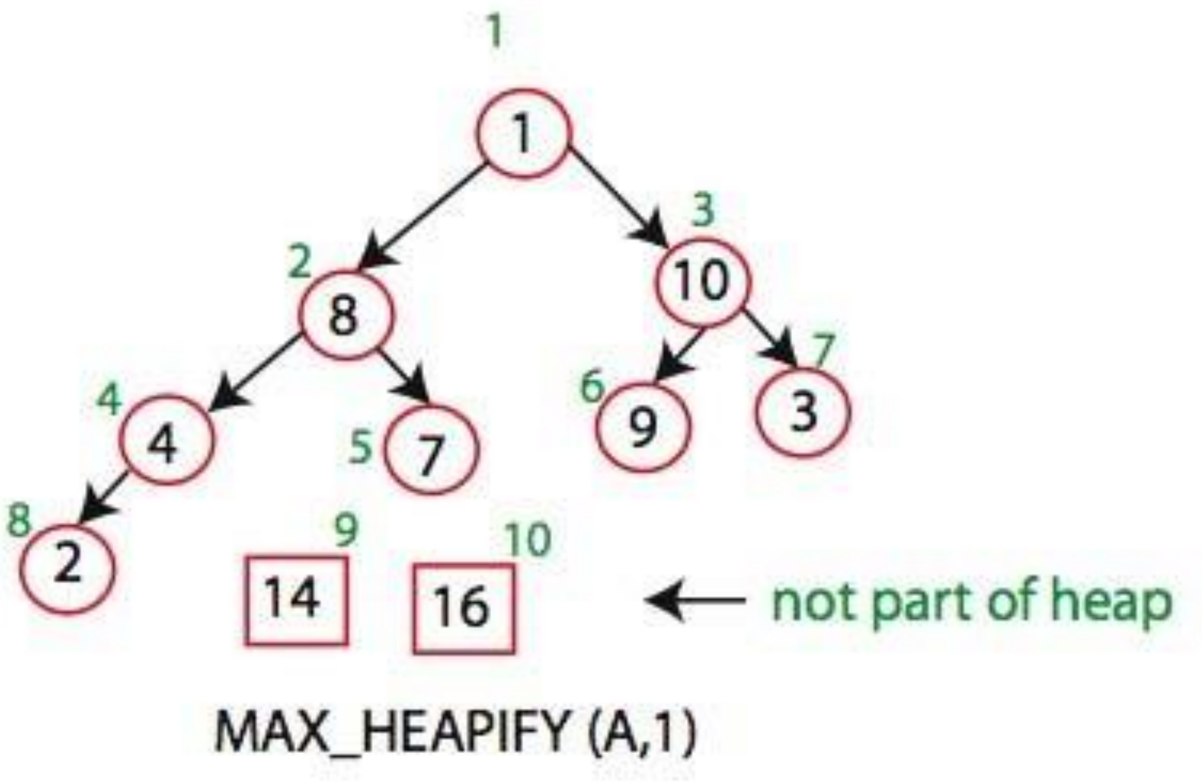


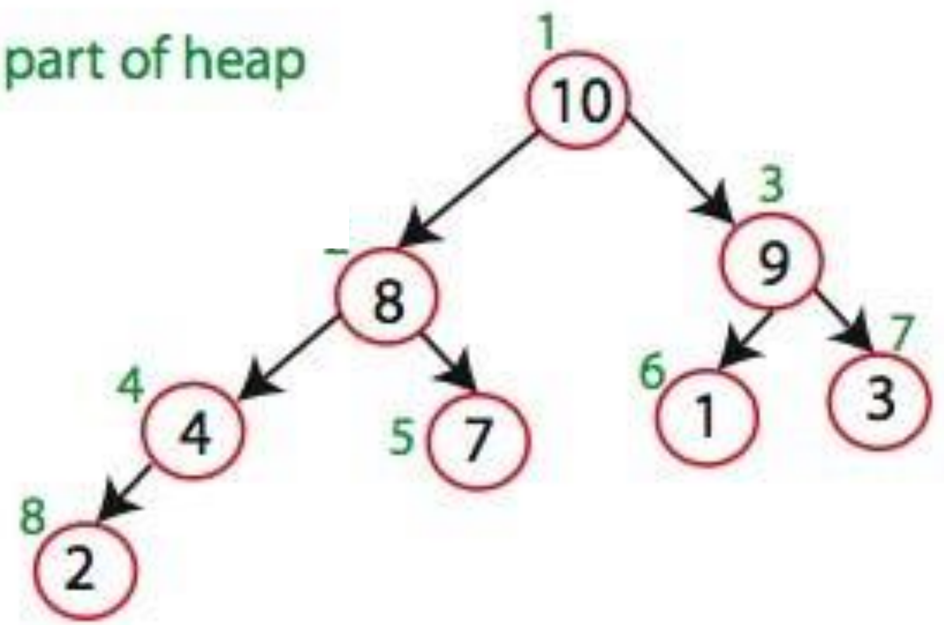
Swap A[9] and A[1]



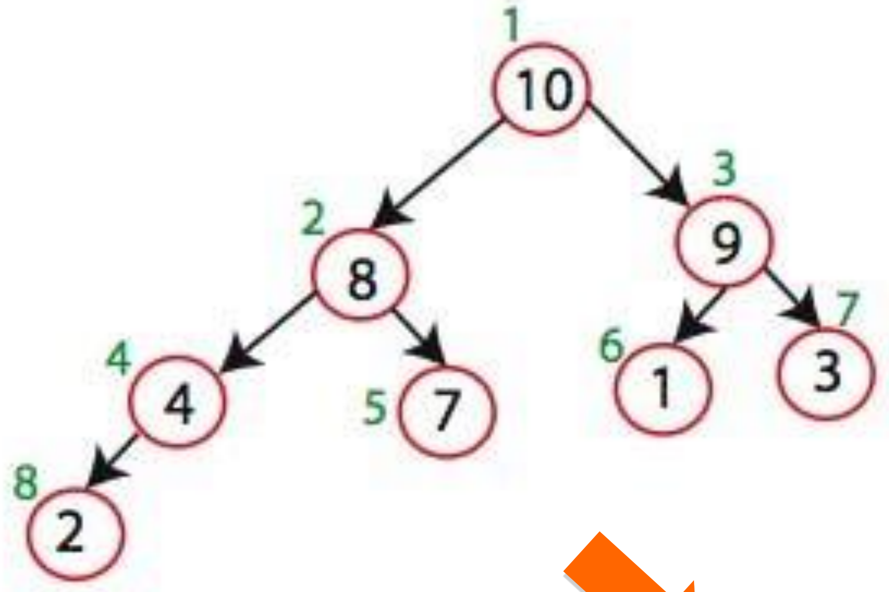
MAX_HEAPIFY (A,1)

Heap-Sort Demo

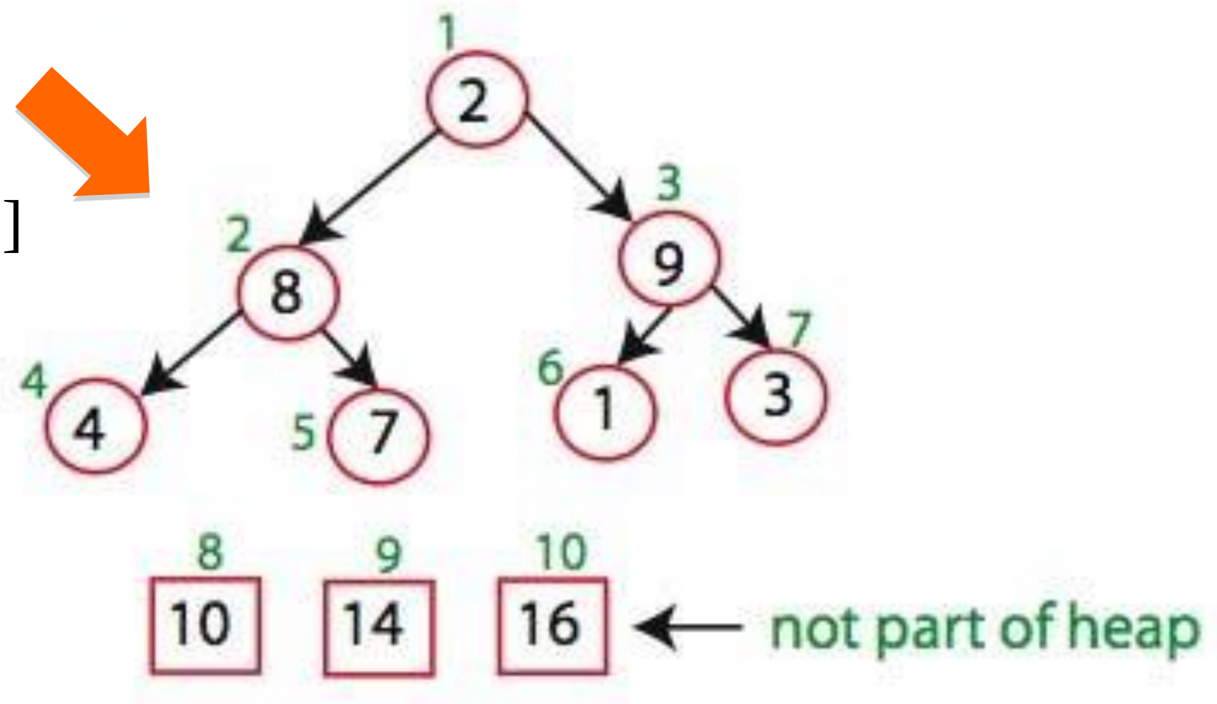




Heap-Sort Demo



Swap A[8] and A[1]



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Heap-Sort

Running time:

after n iterations the Heap is empty every iteration involves a swap and a max_heapify operation; hence it takes O(log n) time

Overall O(n log n)