

ECON 441

Introduction to Mathematical Economics

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Lecture 10
Constrained Optimization

Utility Maximization

Maximize:

$$f(x_1, x_2) = x_1 x_2$$

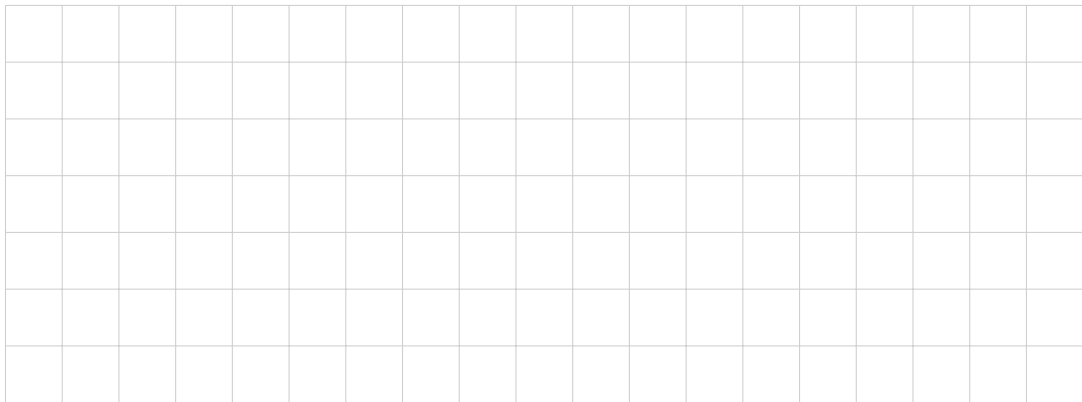
Subject to the budget constraint:

$$2x_1 + x_2 = 40$$

To find x_1^* and x_2^* , we could substitute $x_2 = 40 - 2x_1$ and use our first-order and second-order conditions.

Utility Maximization

$$f(x_1, x_2) = x_1(40 - 2x_1) = 40x_1 - 2x_1^2$$



Constrained Maximization

Substitution doesn't always work, get's complicated with many variables and constraints.

Thankfully, we have the method of Lagrange multiplier.

Lagrange function:

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

λ is the Lagrange multiplier.

Now apply FOC for the case of 3 variables (x_1, x_2, λ) .

Utility Maximization

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

First-order conditions:

$$\frac{\partial L}{\partial x_1} = x_2 - 2\lambda = 0$$

$$\frac{\partial L}{\partial x_2} = x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 40 - 2x_1 - x_2 = 0$$

Lagrange-Multiplier Method

Given an objective function

$$y = f(x_1, x_2)$$

subject to the constraint

$$g(x_1, x_2) = c$$

where c is a constant. We can write the Lagrange function as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$$

Lagrange-Multiplier Method

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$$

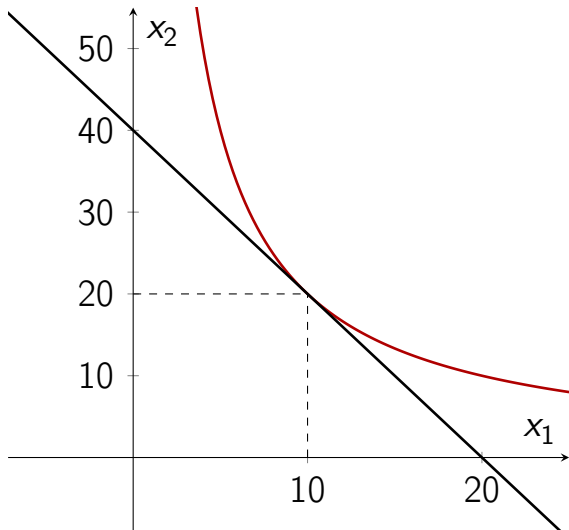
First-order conditions:

$$\frac{\partial L}{\partial x_1} = f_1(x_1^*, x_2^*) - \lambda^* \cdot g_1(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2(x_1^*, x_2^*) - \lambda^* \cdot g_2(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial \lambda} = c - g(x_1^*, x_2^*) = 0$$

Utility Maximization



At the optimal point:

$$\frac{\partial f / \partial x_1}{\partial f / \partial x_2} = \frac{\partial g / \partial x_1}{\partial g / \partial x_2}$$

Balance trade-off:
optimize f vs satisfy g

Interpretation of the Lagrange Multiplier

At the optimum:

$$L(x_1^*, x_2^*, \lambda^*) = f(x_1^*, x_2^*) + \lambda^*[c - g(x_1^*, x_2^*)]$$

What if c changes by a little bit?

Interpretation of the Lagrange Multiplier

At the optimum:

$$L(x_1^*, x_2^*, \lambda^*) = f(x_1^*, x_2^*) + \lambda^* [c - g(x_1^*, x_2^*)]$$

What if c changes by a little bit?

$$\begin{aligned} \frac{dL^*}{dc} &= f_1 \cdot \frac{dx_1^*}{dc} + f_2 \cdot \frac{dx_2^*}{dc} + \lambda^* \left[1 - g_1 \cdot \frac{dx_1^*}{dc} - g_2 \cdot \frac{dx_2^*}{dc} \right] \\ &= \underbrace{(f_1 - \lambda^* \cdot g_1)}_0 \cdot \frac{dx_1^*}{dc} + \underbrace{(f_2 - \lambda^* \cdot g_2)}_0 \cdot \frac{dx_2^*}{dc} + \lambda^* = \lambda^* \end{aligned}$$

Utility Maximization

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

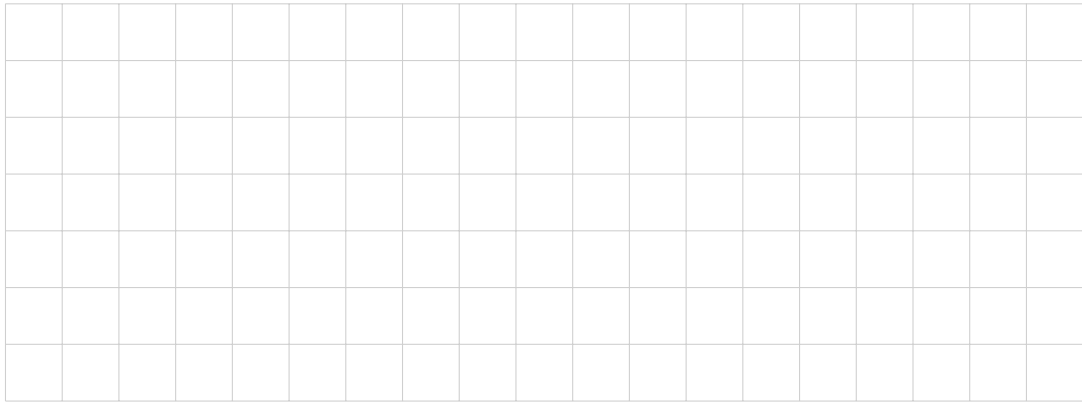
At the optimum:

$$x_2^* = 2x_1^*, \quad 2x_1^* + x_2^* = 40 \quad \rightarrow \quad x_1^* = 10, x_2^* = 20, \lambda^* = 10$$

If my income increases by \$1, by how much does my utility increase? $\lambda^* = 10$

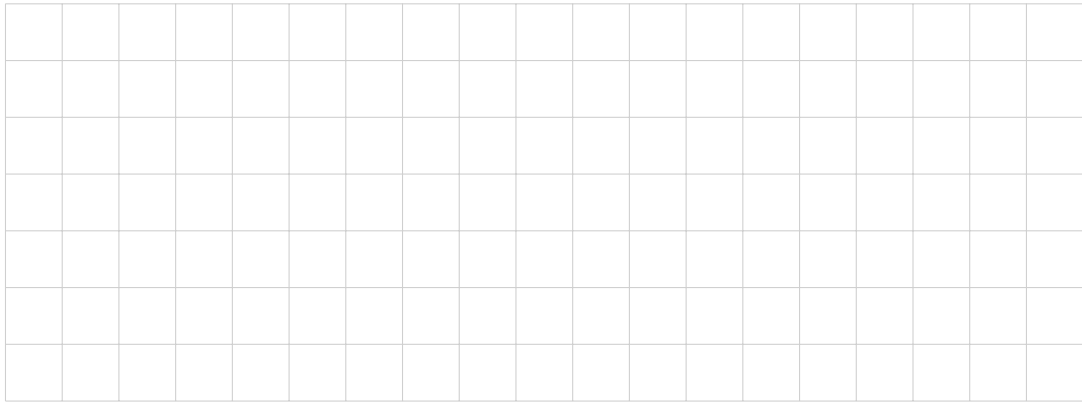
Example I

$$\max_{\{x_1, x_2\}} x_1^2 x_2 \quad s.t. \quad 2x_1^2 + x_2^2 = 3$$



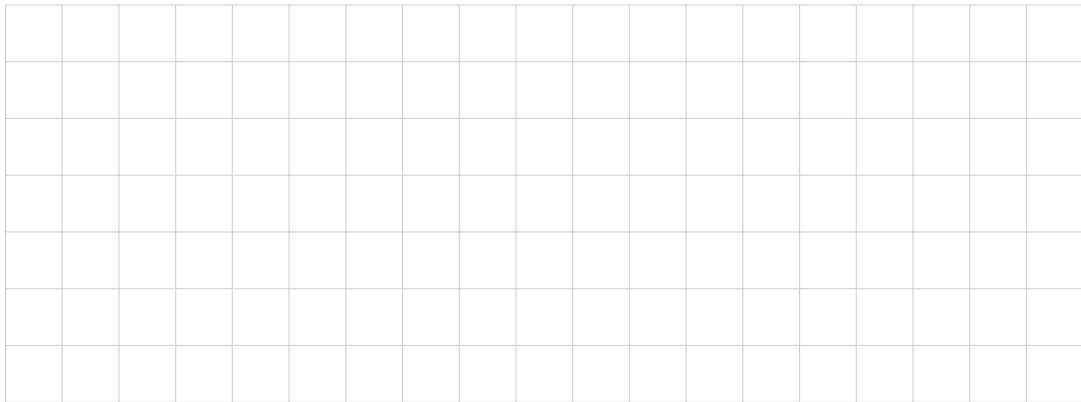
Example II

$$\max_{\{x,y\}} x^a y^b \quad s.t. \quad x + y = 10$$



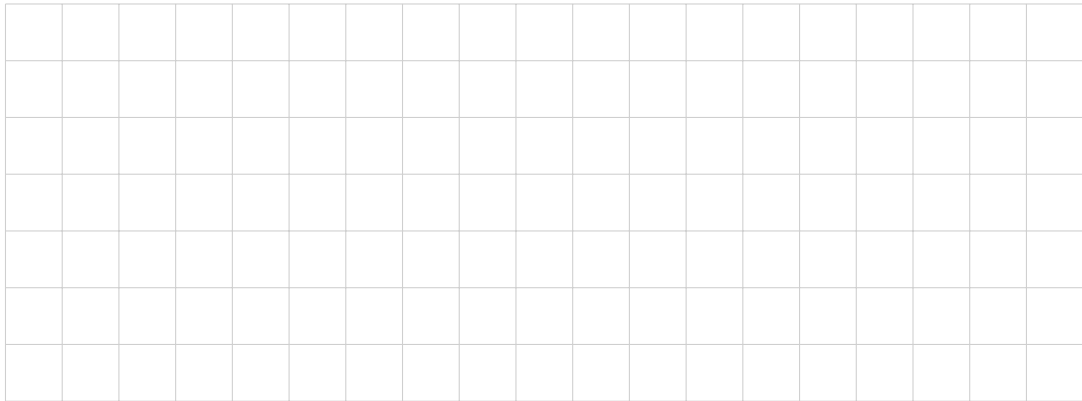
Example III

$$\max_{\{x_1, x_2\}} x_1^{1/3} x_2^{2/3} \quad s.t. \quad x_1 + 4x_2 = 30$$



More Generally

$$\max_{\{x_1, x_2\}} x_1^\alpha x_2^\beta \quad s.t. \quad p_1 x_1 + p_2 x_2 = m$$



Multiple Variables

Objective function:

$$\text{Maximize } f(x_1, x_2, \dots, x_n) \quad \text{subject to} \quad g(x_1, x_2, \dots, x_n) = c$$

Lagrange function:

$$L = f(x_1, x_2, \dots, x_n) + \lambda [c - g(x_1, x_2, \dots, x_n)]$$

First-order conditions:

$$L_i = f_i(x_1, x_2, \dots, x_n) - \lambda g_i(x_1, x_2, \dots, x_n) \quad [i = 1, 2, \dots, n]$$

$$L_\lambda = c - g(x_1, x_2, \dots, x_n)$$

Multiple Constraints

Maximize $f(x_1, x_2, \dots, x_n)$, subject to

$$g(x_1, x_2, \dots, x_n) = c \quad \text{and} \quad h(x_1, x_2, \dots, x_n) = d$$

Lagrange function:

$$L = f(x_1, x_2, \dots, x_n) + \lambda [c - g(x_1, x_2, \dots, x_n)] + \mu [d - h(x_1, x_2, \dots, x_n)]$$

First-order conditions:

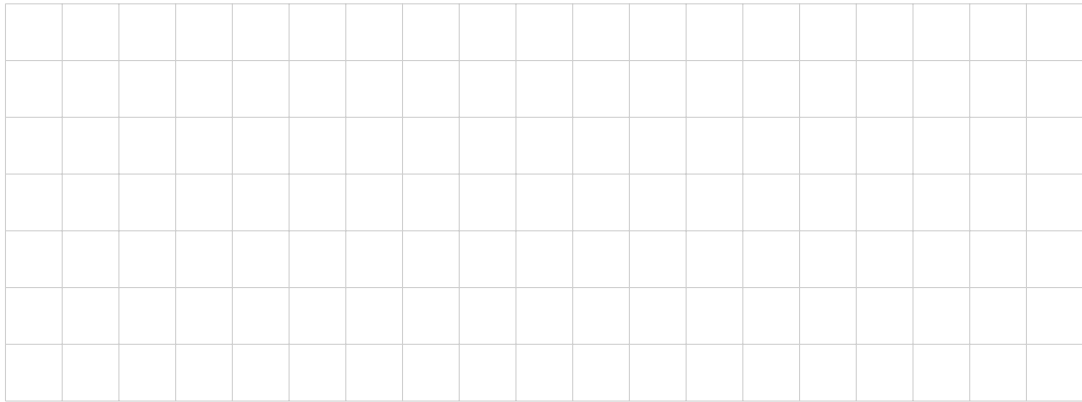
$$L_\lambda = c - g(x_1, x_2, \dots, x_n) = 0$$

$$L_\mu = d - h(x_1, x_2, \dots, x_n) = 0$$

$$L_i = f_i(x_1, x_2, \dots, x_n) - \lambda g_i(x_1, x_2, \dots, x_n) - \mu h_i(x_1, x_2, \dots, x_n) = 0$$

Example IV

$$\max_{\{x,y,z\}} \quad yz + xz \quad s.t. \quad y^2 + z^2 = 1 \quad and \quad xz = 3$$



Intertemporal Utility Maximization

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \quad 0 < \beta < 1$$

- $y_1, y_2 > 0$: income in period 1 and 2
- Income you save s in period 1 earns interest $r > 0$
- In which case,

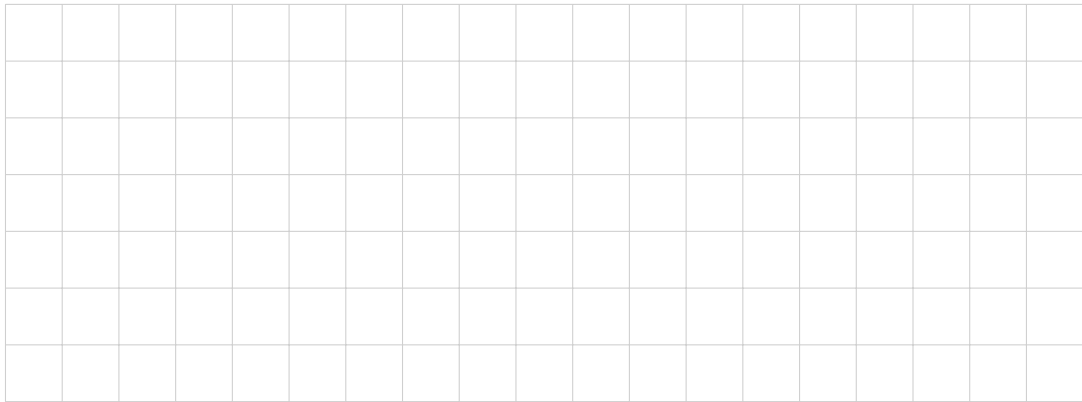
$$c_1 + s = y_1 \quad c_2 = y_2 + (1 + r)s$$

- Combining these constraints:

$$c_1 + \frac{1}{1+r}c_2 = \underbrace{y_1 + \frac{1}{1+r}y_2}_{m \equiv \text{present-discounted income}}$$

Intertemporal Utility Maximization

$$\max_{\{c_1, c_2\}} U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \quad s.t. \quad c_1 + \frac{1}{1+r} c_2 = m$$



References and Homework Problems

- References for today: Sections 12.1 and 12.2
- Homework: Exercise 12.2: Questions 1–4 + Examples from today's class