

# ECON 441

## Introduction to Mathematical Economics

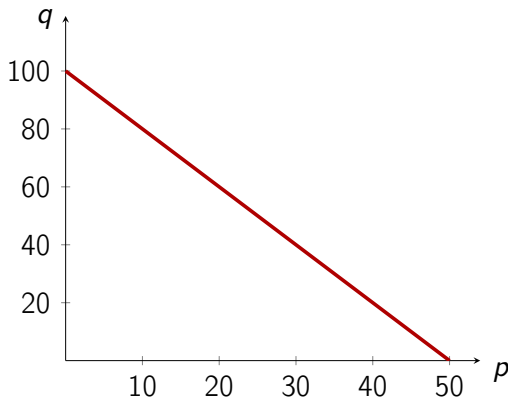
Div Bhagia

Lecture 2: Linear Algebra

# A Simple Economic Model

$q$ : quantity of hats,  $p$ : price of a single hat

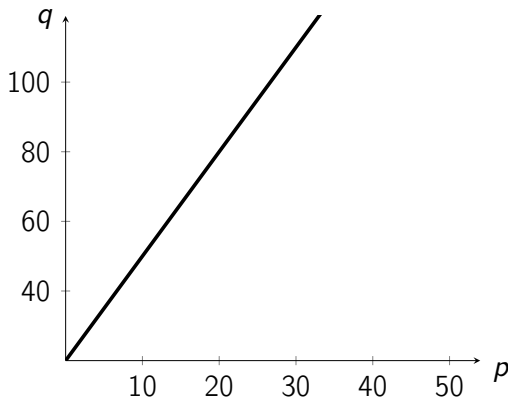
Demand for hats:  $q = 100 - 2p$



# A Simple Economic Model

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Supply for hats:  $q = 20 + 3p$



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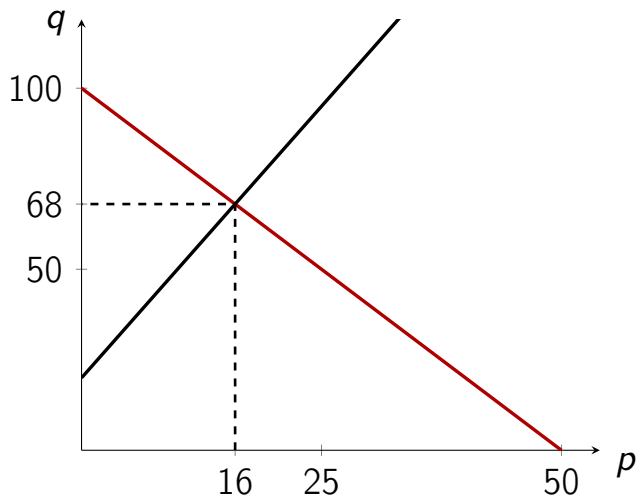
$$100 - 2p = 20 + 3p \rightarrow p^* = \$16$$

What is the quantity traded at this price?

$$q^* = 100 - 2 \times 16 = 20 + 3 \times 16 = 68$$

$q^*$  and  $p^*$  are determined simultaneously.

# Equilibrium



# Matrix Algebra

We solved a *system* of two (linear) equations in two variables.

Complex economic models: multiple equations with multiple variables

Hard to just wing it... Enter, Matrix Algebra!

Matrix Algebra can help us write complex system of equations compactly and solve them



# A Simple Economic Model

$q$ : quantity of hats,  $p$ : price of a single hat

Demand for hats:  $q = 100 - 2p$

Supply for hats:  $q = 20 + 3p$

Rewrite the two equations:

$$q + 2p = 100$$

$$q - 3p = 20$$

# A Simple Economic Model

Two equations in two unknowns:

$$q + 2p = 100$$

$$q - 3p = 20$$

Can write this as:

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

These arrays are called matrices.

# Today

- Matrices: Addition, Subtraction, and Scalar Multiplication
- Matrix Multiplication
- Vectors
- Identity and Null Matrices
- Transpose and Inverse of a Matrix

Textbook reference: 4.1-4.6

# Matrices

A *matrix* is a rectangular array of numbers, parameters, or vectors.

Example.  $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 6 \end{bmatrix}$

Dimensions of matrix:

- Number of rows ( $m$ )
- Number of columns ( $n$ )

# Matrices

A matrix with  $m$  rows and  $n$  columns is referred to as an  $m \times n$  matrix

What's the dimension of  $A$ ?

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# Matrices

A matrix with  $m$  rows and  $n$  columns is referred to as an  $m \times n$  matrix

What's the dimension of  $A$ ?

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 6 \end{bmatrix}_{2 \times 3}$$

# Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Can write it more compactly

$$A = [a_{ij}] \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n$$

# Matrices

*Square matrix*: equal number of rows and columns

Example.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$



# Matrices

Two matrices are *equal* if all their elements are identical.

Example.

$$A = \begin{bmatrix} 1 & 8 \\ 4 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 4 & 2 \end{bmatrix}$$

So  $A = B$  if and only if  $a_{ij} = b_{ij}$  for all  $i, j$

# Matrix Addition and Subtraction

- How to add or take the difference between two matrices?
  - Element-by-element
  - Matrices have to have same dimension

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

- What is  $A + B$  and  $A - B$ ?

# Scalar Multiplication

How to multiply a scalar to a matrix?

$$\lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

What is  $2B$  and  $A - 2B$ ?

# Matrix Multiplication

A whole new animal...

Only possible to multiply two matrices,  $A_{m \times n}$  and  $B_{p \times q}$  to get  $AB$  if  $n = p$  i.e.

number of columns in  $A$  = number of rows in  $B$

Example.  $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2 \times 3}$   $B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}_{2 \times 2}$

Cannot do  $AB$ , but can do  $BA$

# Matrix Multiplication

Another example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3 \times 1}$$

Can we multiply  $A$  and  $B$  to find  $C = AB$ ?

# Matrix Multiplication

Another example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3 \times 1}$$

Can we multiply  $A$  and  $B$  to find  $C = AB$ ?

Yes, since  $A$  has 3 rows which is equal to the number of columns in  $B$ .

Also, the dimension of  $C$  will be  $2 \times 1$ .

# Matrix Multiplication

So how to actually multiply these matrices?

$$C = AB$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

The **element**  $c_{ij}$  is obtained by multiplying term-by-term the entries of the  **$i$ th row of  $A$**  and  **$j$ th column of  $B$** .

# Matrix Multiplication: Examples

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3 \times 2}$$

Here,

$$C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}_{2 \times 2}$$



# Matrix Multiplication: Examples

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3 \times 1}$$

# Matrix Multiplication: Examples

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3 \times 1}$$

$$C = AB = \begin{bmatrix} 2 \times 1 + 3 \times -2 + 1 \times 4 \\ 4 \times 1 + -6 \times -2 + -2 \times 4 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}_{2 \times 1}$$

# Matrix Multiplication: Examples

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

# A Simple Economic Model

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

What is  $Ax$ ?

# A Simple Economic Model

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

What is  $Ax$ ?

$$Ax = \begin{bmatrix} q + 2p \\ q - 3p \end{bmatrix}$$

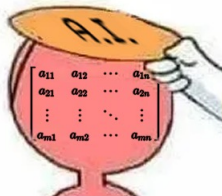
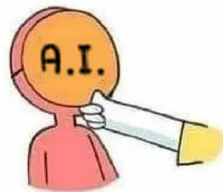
# A Simple Economic Model

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

What is  $Ax$ ?

$$Ax = \begin{bmatrix} q + 2p \\ q - 3p \end{bmatrix}$$

Setting  $Ax = b$  gives us back our demand and supply equations.



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# Vectors

- Matrices with only one column: column vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Matrices with only one row: row vectors

$$x' = [x_1 \quad x_2 \quad \dots \quad x_n]$$



# Inner Product

Inner product of two vectors each with  $n$  elements:

$$u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^n u_i v_i$$

Example.

$$u = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

# Linear Dependence

A set of vectors is said to be *linearly dependent* if and only if any one of them can be expressed as a linear combination of the remaining vectors.

Example.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

# Linear Dependence

A set of vectors is said to be *linearly dependent* if and only if any one of them can be expressed as a linear combination of the remaining vectors.

Example.

$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

# Linear Dependence

A set of  $m$ -vectors  $v_1, v_2, \dots, v_n$  is linearly dependent if and only if there exists a set of scalar  $k_1, k_2, \dots, k_n$  (not all zero) such that:

$$\sum_{i=1}^n k_i v_i = 0 \quad (m \times 1)$$

# Identity Matrices

Square matrix with 1s in its *principal diagonal* and 0s elsewhere

A  $2 \times 2$  identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A  $3 \times 3$  identity matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Identity Matrices

Acts like 1,

$$AI = IA = A$$

Example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

# Idempotent Matrices

A matrix is an *idempotent* matrix if it remains unchanged when multiplied by itself any number of times.

$A$  is idempotent if and only if  $A = A^k$ .

Is an identity matrix idempotent?

# Null Matrix

A null matrix is a matrix with all elements 0.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $A + 0 = A$
- $A0 = 0$



# Transpose of a Matrix

Transpose of A: interchange rows and columns ( $A'$ )

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

# Transpose of a Matrix

Transpose of A ( $A'$  or  $A^T$ ): interchange rows and columns

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

# Transpose of a Matrix

- A matrix  $A$  is said to be *symmetric* if

$$A' = A$$

- A matrix  $A$  is said to be *skew-symmetric* if

$$A' = -A$$

- A matrix  $A$  is said to be *orthogonal* if

$$A'A = I$$

## Example: Symmetric Matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -5 \\ 0 & -5 & 4 \end{bmatrix}$$

## Example: Skew-symmetric Matrix

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

## Example: Orthogonal Matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

# Properties of Transposes

$$(A')' = A$$

$$(A + B)' = A' + B'$$

$$(AB)' = B'A'$$

Example:  $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}$      $B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$

# Inverse of a Matrix

For a **square** matrix  $A$ , it's inverse  $A^{-1}$  is defined as:

$$AA^{-1} = A^{-1}A = I$$

Squareness is a *necessary* condition not a *sufficient* condition

If a matrix's inverse exists, it's called a **nonsingular** matrix



# Properties of Inverses

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A')^{-1} = (A^{-1})'$$

# Solution of Linear-Equation System

$$Ax = b$$

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$$Ax = b$$

Pre-multiply both sides by  $A^{-1}$ ,

$$A^{-1}Ax = A^{-1}b \implies x = A^{-1}b$$

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$$Ax = b$$

Pre-multiply both sides by  $A^{-1}$ ,

$$A^{-1}Ax = A^{-1}b \implies x = A^{-1}b$$

If  $A$  is singular, a unique solution does not exist.

# Homework Problems

- Exercise 4.2: 1, 2, 4
- Exercise 4.4: 5 (e), 7
- Exercise 4.5: 1, 4
- Exercise 4.6: 2, 6

Reminder: Quiz 1 is next Tuesday (09/05)