ECON 441

Introduction to Mathematical Economics

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Lecture 7: Calculus

Elasticity

Demand curve:

$$Q(p) = \frac{c}{p^{\alpha}}$$

Partial Elasticities

Production function:

$$F(K, L) = AK^{\alpha}L^{\beta}$$

Find ε_{QK} and ε_{QL}.

Total Differential

For a function of n variables

$$y = f\left(x_1, x_2, \cdots, x_n\right)$$

Total differential:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = \sum_{i=1}^{n} f_i dx_i$$

I am using ∂ to differentiate partial derivatives from total derivatives. In particular,

$$\frac{\partial f}{\partial x_i} = \left.\frac{df}{dx_i}\right|_{\textit{other variables are constant}}$$

Total Derivative

For a function of n variables

$$y = f\left(x_1, x_2, \cdots, x_n\right)$$

$$\frac{df}{dt} = f_1 \cdot \frac{dx_1}{dt} + f_2 \cdot \frac{dx_2}{dt} + \cdots + f_n \cdot \frac{dx_n}{dt}$$

Total Derivative

Given the function

$$y = f(x_1, x_2)$$

We are interested in how y changes with respect to x_1, but x_2 also depends of x_1

$$x_2 = g(x_1)$$

Total derivative with respect to x_1:

$$\frac{dy}{dx_1} = f_1 + f_2 \cdot g'(x_1)$$

Example

Utility from consumption (C) and leisure hours (L).

$$U = U(C, L) = \ln C + \ln L$$

Budget constraint: $C = w(T - L)$ where w is the hourly wage and T is total hours. How does utility change due to change in leisure hours?

Implicit Functions

Explicit function:

$$y = f\left(x_1, x_2, \cdots, x_n\right)$$

Implicit function:

$$F\left(y, x_1, x_2, \cdots, x_n\right) = 0$$

Example

Implicit function:

$$F(x, y) = y - 3x^2 = 0$$

Corresponding explicit function:

$$y = f(x) = 3x^2$$

However, not all implicit functions have a corresponding explicit function. E.g. $F(x, y) = x^2 + y^2 - 9 = 0$

Implicit Function Theorem

Given,

$$F\left(x, y\right) = 0$$

If the following conditions are met:

- F_y and F_x are continuous, and
- At some point $\left(a, b\right)$, F_y is non-zero

Then in a neighborhood around $\left(a, b\right)$, an implicit function exists. Moreover, this function is continuous and has continuous partial derivatives.

Derivatives of Implicit Functions

Total differentiating F, we have $dF = 0$, or

$$F_y dy + F_1 dx_1 + \cdots + F_n dx_n = 0$$

Suppose that only y and x_1 are allowed to vary:

$$\frac{\partial y}{\partial x_1} = -\frac{F_1}{F_y}.$$

In the simple case where the given equation is $F(y, x) = 0$, the rule gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Example

Given the following function, let's find $\partial y/\partial x$ and $\partial y/\partial z$.

$$F(x, y, z) = x^3 z^2 + y^3 + 4xyz = 0$$

Another Example

Estimate the following model for demand for fast food:

$$orders = \beta_0 + \beta_1 price + \beta_2 quality + \varepsilon$$

What is the interpretation of β_1?

Another Example (cont.)

What if instead we estimate:

$$\ln(orders) = \beta_0 + \beta_1 \ln(price) + \beta_2 quality + \varepsilon$$

What is the interpretation of β_1?

Another related example

Production function:

$$Y = AL^{\alpha}K^{\beta}$$

To estimate the elasticities from data:

$$\ln Y = \ln A + \alpha \ln L + \beta \ln K + \varepsilon$$

Find the Derivative by Taking the Log

$$\text{Demand:} \quad Q(p) = \frac{c}{p^{\alpha}}$$

Integral Calculus

Inverse of Differentiation

Path of population over time:

$$P(t) = 2t^{0.5}$$

Rate of change of population:

$$P'(t) = \frac{dP}{dt} = t^{-0.5}$$

But what if instead we were given $P'(t)$ and were tasked with finding $P(t)$.

Inverse of Differentiation

Note that,

$$P'(t) = \frac{dP}{dt} = t^{-0.5}$$

is the derivative of $P(t) = 2t^{0.5}$, but also of $P(t) = 2t^{0.5} + 30$.

Generally, at best, we can find the following from just $P'(t)$:

$$P(t) = 2t^{0.5} + c$$

However, if we have an initial condition such as $P(0) = 50$, we can also find c.

Integration

- Integration is the reverse of differentiation
- If $f(x)$ is the derivative of $F(x)$, we can *integrate* $f(x)$ to find $F(x)$

$$\frac{d}{dx}F(x) = f(x) \Rightarrow \int f(x)dx = F(x) + c$$

- Rules of integration follow from rules of differentiation

Rules of Integration

Power Rule

$$\int x^n dx = \frac{1}{n+1} \cdot x^{n+1} + c \quad (n \neq -1)$$

Examples:

$$\int x^3 dx, \qquad \int x dx, \qquad \int 1 dx$$

Rules of Integration

Exponential Rule

$$\int e^x dx = e^x + c$$

Log Rule

$$\int \frac{1}{x} dx = \ln x + c \quad (x > 0)$$

Rules of Integration

<u>Integral of a sum</u>

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

<u>Integral of a multiple</u>

$$\int kf(x)dx = k \int f(x)dx$$

Example:

$$\int (x^2 + 3x + 1)dx$$

Definite Integrals

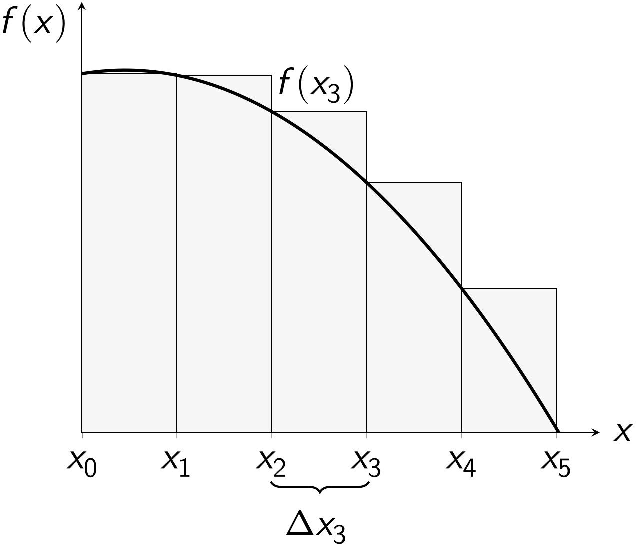
Definite integral:

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

Example:

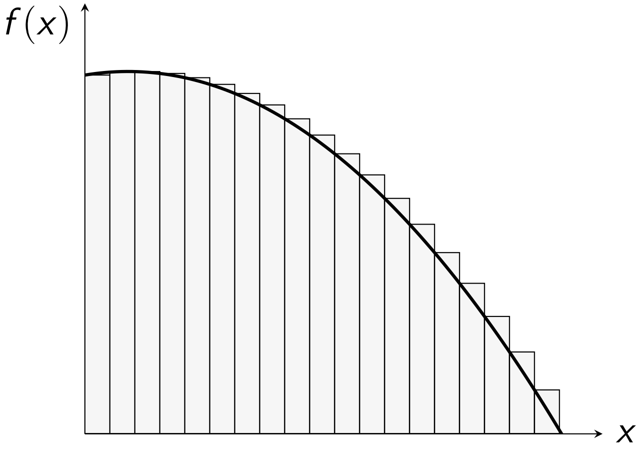
$$\int_1^3 2x^2 =$$

Area under the curve



$$Area \approx \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Area under the curve



$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$
$$= \int_{x_1}^{x_n} f(x)\,dx$$

References and Homework

- References: Sections 8.5 and Sections 14.1-14.3
- Homework problems:
    - Ex 8.5: 1, 2(d), 3 (a)
    - Ex 14.2: 1 (a), (c), (d)
    - Ex 14.3: 1 (a) (e), 2 (a) (d), 5
- Next week: Review class
- Midterm is in two weeks (10/17)