ECON 441

Introduction to Mathematical Economics

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Lecture 6: Calculus

Example

Total cost: $C = C(Q)$

Marginal cost: $MC = C'(Q)$

Average cost:

$$AC = \frac{C(Q)}{Q}$$

When is $\frac{dAC}{dQ}$ positive?

Example

Revenue: $R = f(Q)$

Output: $Q = g(L)$

How does revenue change due to a change in labor input L?

$$\frac{dR}{dL} = \frac{dR}{dQ} \cdot \frac{dQ}{dL}$$

Exponential Functions

The exponential or power function can be represented as:

$$y = f(t) = b^t \quad (b > 1)$$

where b denotes a fixed base of the exponent.

A more generalized version can be written as:

$$y = ab^{ct}$$

Natural Exponential Function

Natural exponential function: Base is a special mathematical constant called Euler's number $e = 2.71828...$

$$y = ae^{rt}$$

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Natural exponential function: Base is a special mathematical constant called Euler's number $e = 2.71828\ldots$

$$y = ae^{rt}$$

Where did this number e come from?

It can be shown:

$$e \equiv \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Natural Exponential Function

Jacob Bernoulli discovered this constant in 1683 while studying a question about compound interest.

Logarithmic Function

Since the exponential function is a monotonic function, its inverse exists.

The inverse of the exponential function is called the log or logarithmic function.

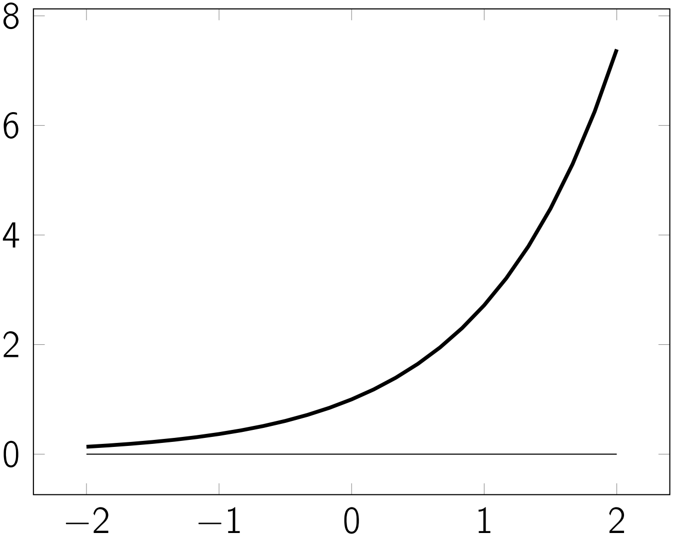
For the exponential function:

$$y = b^t \rightarrow log_b(y) = t$$

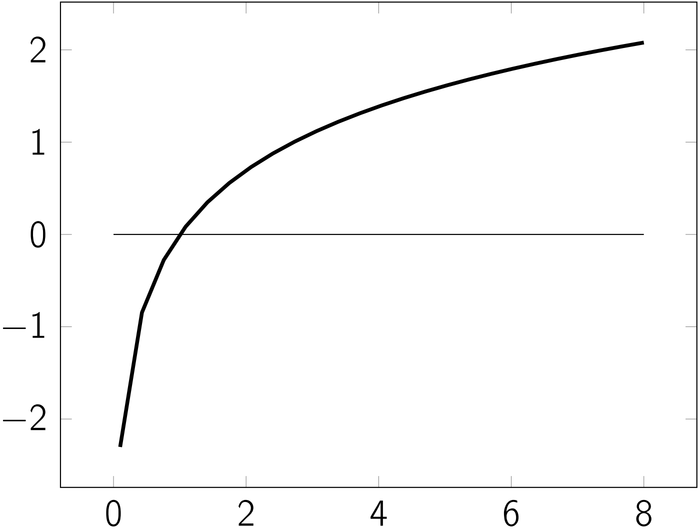
For the natural exponential function:

$$y = e^t \rightarrow \log_e y = ln(y)$$

$y = exp(x)$



$y = ln(x)$



Rules for Logarithmic Functions

- $\ln(uv) = \ln u + \ln v$
- $\ln(u/v) = \ln u - \ln v$
- $\ln u^a = a \ln u$

Derivatives of Exponential Functions

Derivative of the exponential function:

$$y = e^t \quad \longrightarrow \quad \frac{dy}{dt} = e^t$$

Using the chain rule:

$$y = e^{f(t)} \quad \longrightarrow \quad \frac{dy}{dt} = f'(t)e^{f(t)}$$

Derivatives of Logarithmic Functions

Derivative of the log function:

$$\frac{d}{dt} \ln t = \frac{1}{t}$$

Using the chain rule:

$$\frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)}$$

Examples

Find the derivatives for the following functions:

1. $y = e^t$
2. $y = \ln t$
3. $y = ae^{rt}$
4. $y = e^{-t}$
5. $y = \ln at$
6. $y = \ln t^c$

Partial Differentiation

For a function of several variables:

$$y = f\left(x_1, x_2, \cdots, x_n\right)$$

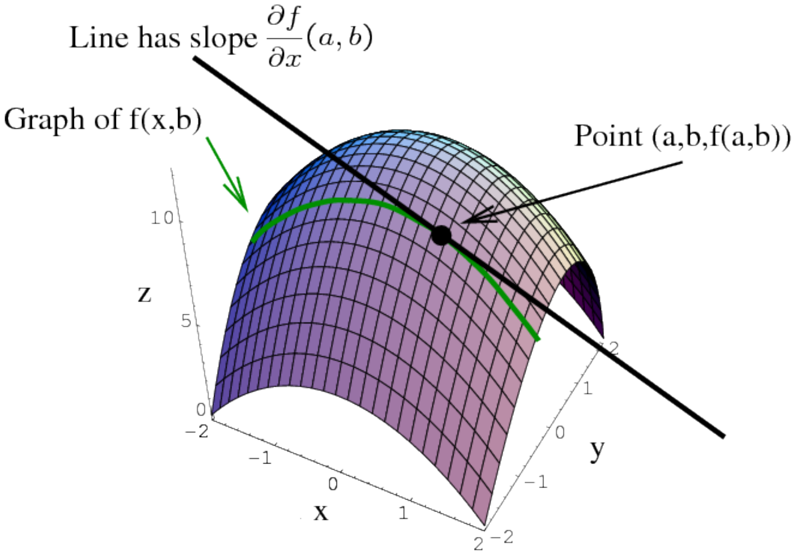
If x_1 changes by Δx_1 but all other variables remain constant:

$$\frac{\Delta y}{\Delta x_1} = \frac{f\left(x_1 + \Delta x_1, x_2, \cdots, x_n\right) - f\left(x_1, x_2, \cdots, x_n\right)}{\Delta x_1}$$

Partial derivative of y with respect to x_i:

$$\frac{\partial y}{\partial x_i} = f_i = \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i}$$

Partial Derivatives



Gradient Vector

Gradient: vector of all partial derivatives of a function

$$\nabla f(x_1, x_2, \cdots, x_n) = [f_1, f_2, \cdots, f_n]'$$

Example

$$y = f\left(x_1, x_2\right) = 3x_1^2 + x_1 x_2 + 4x_2^2$$

$$\frac{\partial y}{\partial x_1} = f_1 =$$

$$\frac{\partial y}{\partial x_2} = f_2 =$$

Example

$$y = f(u, v) = (u + 4)(3u + 2v)$$

$$\frac{\partial y}{\partial u} = f_u =$$

$$\frac{\partial y}{\partial v} = f_v =$$

Production Function

$$Q = AK^{\alpha}L^{1-\alpha}$$

Marginal product of capital (MPK):

$$\frac{\partial Q}{\partial K} = Q_K =$$

Marginal product of labor (MPL):

$$\frac{\partial Q}{\partial L} = Q_L =$$

Differentials

Note that,

$$\Delta y \equiv \left[\frac{\Delta y}{\Delta x}\right] \Delta x$$

Then for infinitesimal changes,

$$dy \equiv \left[\frac{dy}{dx}\right] dx \quad \text{or} \quad dy = f'(x)dx$$

We will call dy and dx differentials of y and x, respectively.

Derivative as a ratio

Given that,

$$dy \equiv \left[\frac{dy}{dx}\right] dx \quad \text{or} \quad dy = f'(x)dx$$

We can think of $f'(x)$ as a ratio of two quantities dy and dx.

Elasticity

An important quantity that economists love to calculate is the elasticity of a function.

Elasticity is defined as:

$$\varepsilon = \frac{\text{Percentage change in y}}{\text{Percentage change in x}} = \frac{dy/y}{dx/x}$$

We can calculate this as:

$$\varepsilon = \frac{dy}{dx} \cdot \frac{x}{y}$$

Elasticity

Elasticity:

$$\varepsilon = \frac{dy}{dx} \cdot \frac{x}{y}$$

- $|\varepsilon| > 1$, elastic
- $|\varepsilon| = 1$, unit elasticity
- $|\varepsilon| < 1$, inelastic

Example

$$C = a + bY$$

Total Differential

For a function of n variables

$$y = f\left(x_1, x_2, \cdots, x_n\right)$$

Total differential:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n = \sum_{i=1}^{n} f_i dx_i$$

I am using ∂ to differentiate partial derivatives from total derivatives. In particular,

$$\frac{\partial f}{\partial x_i} = \left.\frac{df}{dx_i}\right|_{\textit{other variables are constant}}$$

Total Differential

Consider a savings function:

$$S = S(Y, i)$$

where S is savings, Y is national income, and i is the interest rate.

Total differential:

$$dS = \frac{\partial S}{\partial Y} dY + \frac{\partial S}{\partial i} di$$

Example

$$y = 5x_1^2 + 3x_2$$

Total Derivative

Total differential:

$$df = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n$$

We can divide the total differential by dx_1 to get the *total derivative* of f with respect to x_1:

$$\frac{df}{dx_1} = f_1 + f_2 \cdot \frac{dx_2}{dx_1} + \cdots + f_n \cdot \frac{dx_n}{dx_1}$$

Total Derivative

Given the function

$$y = f(x_1, x_2)$$

We are interested in how y changes with respect to x_1, but x_2 also depends of x_1

$$x_2 = g(x_1)$$

We know that,

$$dy = f_1 dx_1 + f_2 dx_2$$

Dividing both sides by dx_1,

$$\frac{dy}{dx_1} = f_1 + f_2 \cdot g'(x_1) = \frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} \cdot \frac{dx_2}{dx_1}$$

A variation on the theme

For a function

$$y = f\left(x_1, x_2, w\right), \qquad x_1 = g(w), x_2 = h(w)$$

The total derivative of y is given by

$$\frac{dy}{dw} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dw} + \frac{\partial f}{\partial w}$$

Example

Let a production function be

$$Q(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha}$$

where

$$K(t) = K_0 + at \qquad L(t) = L_0 + bt$$

Another variation on the theme

If a function is given,

$$y = f\left(x_1, x_2, u, v\right)$$

with $x_1 = g(u, v)$ and $x_2 = h(u, v)$.

Then,

$$\frac{dy}{du} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial u} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial u} + \frac{\partial y}{\partial u}$$

References and Homework

References: Chapter 10 (notes are sufficient), Section 10.5, Section 7.4, Sections 8.1, 8.2, 8.4

Homework problems:

- Ex 10.5: 1, 3, 7
- Ex 7.4 1 (a) (d), 2 (a) (b), 3, 5, 7;
- Ex 8.1: 1 (a), 4, 6;
- Ex 8.2: 3 (a), 4, 5, 6, 7 (b) (f);
- Ex 8.4: 2, 4;