ECON 441

Introduction to Mathematical Economics

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Midterm Review

Numbers, Sets, and Functions

Sets

- $\mathbb{R}$ is the set of real numbers (rational and irrational)
- $x \in \mathbb{R}$ to denote x belongs to the set of real numbers
- Consider the universe of all real numbers, set A is given by:

$$A = \{x | x > 0\}$$

- What is the complement of A?

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$$A = \{x | x > 0\}$$

- What is the complement of A?

$$A^c = \{x | x \leqslant 0\}$$

Set Relations and Operations

Consider the following sets:

$$A = \{x | x > 0\}$$
$$B = \{x | x \text{ is a positive integer}\}$$
$$C = \{x | 1 < x < 5\}$$

- Is $A = B$? Is $B \subset A$?
- Are C and B disjoint sets?
- What is $A \cap B$? What about $A \cup B$?
- What is $B \cap C$?

Subsets

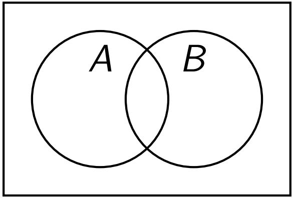
- $\emptyset$: empty or null set
- What are all possible subsets of

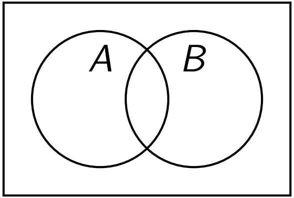
$$S = \{a, b, c\}$$

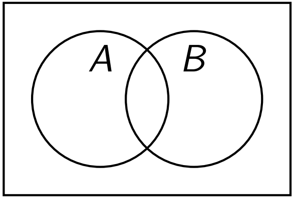
$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$

Always 2^n subsets. Here $n = 3$, so 8 subsets.

Set Operations: Venn Diagrams







Laws of Set Operations

- Commutative law

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Cartesian Product

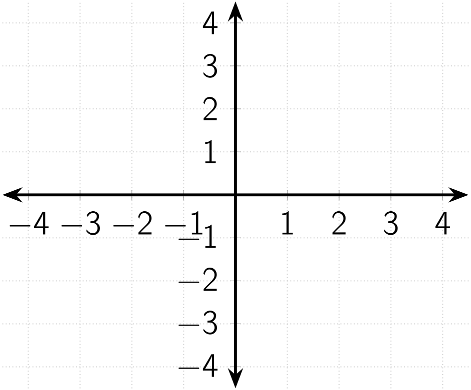
$$A = \{1, 2\} \quad B = \{3, 4\}$$

Cartesian Product: set of all possible ordered pairs

$$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

Cartesian Plane

$$\mathbb{R}^2 = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$$

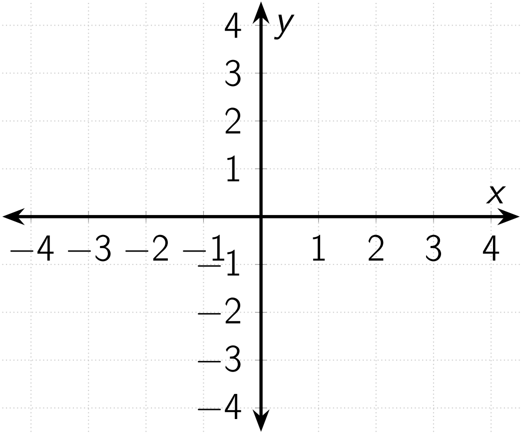


Can have $\mathbb{R}^3, \mathbb{R}^4, \ldots, \mathbb{R}^n$

Relations

Relation: subset of the Cartesian product

Example. $\{(x,y) | y \leqslant x\}$



Functions

Function: a relation where for each x there is a unique y

$$f : X \to Y, \quad y = f(x)$$

Examples. $y = x, y = x^2, y = 2x + 3$

X : domain, Y : codomain, $f(X)$: range

Most functions we will encounter, $f : \mathbb{R}^k \to \mathbb{R}$

Inverse of a function

Function $y = f(x)$ has an inverse if it is a one-to-one mapping, i.e. each value of y is associated with a unique value of x.

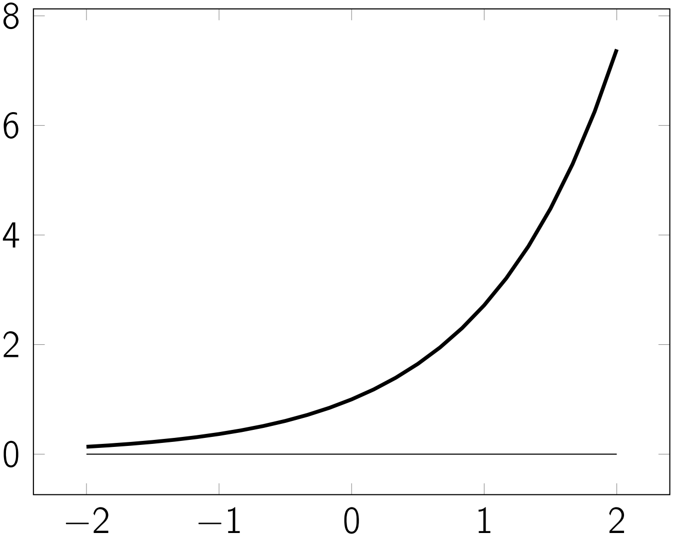
Inverse function

$$x = f^{-1}(y)$$

returns the value corresponding value of x for each y.

One-to-one mapping unique to strictly monotonic functions

$y = exp(x)$



Logarithmic Function

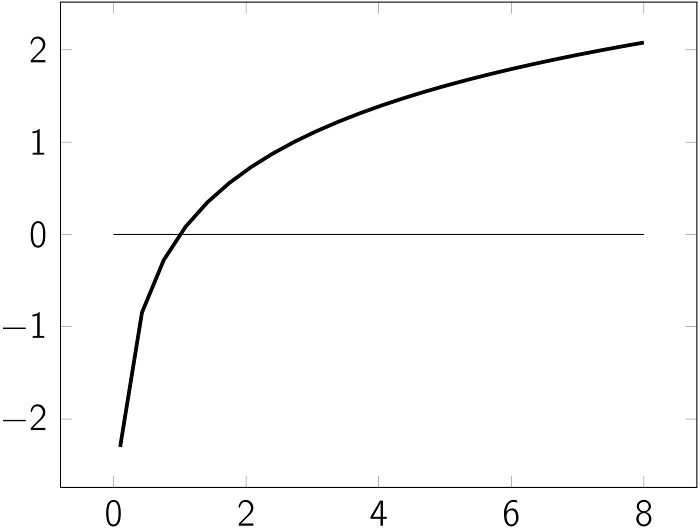
Since the exponential function is a monotonic function, its inverse exists.

The inverse of the exponential function is called the log or logarithmic function.

For the natural exponential function:

$$y = e^t \rightarrow \log_e y = ln(y)$$

$y = ln(x)$



Rules for Logarithmic Functions

$$\ln(uv) = \ln u + \ln v$$

$$\ln(u/v) = \ln u - \ln v$$

$$\ln u^a = a \ln u$$

Linear Algebra

Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

Can write it more compactly

$$A = \left[a_{ij}\right] \quad i = 1, 2, ..., m; j = 1, 2, ..., n$$

Square matrices: $m = n$

Matrices

Two matrices are *equal* if all their elements are identical.

Example.

$$A = \begin{bmatrix} 1 & 8 \\ 4 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 4 & 2 \end{bmatrix}$$

So $A = B$ if and only if $a_{ij} = b_{ij}$ for all i, j

Matrix Addition and Subtraction

- How to add or take the difference between two matrices?

  $\rightarrow$ Element-by-element

  $\rightarrow$ Matrices have to have same dimension

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

- What is $A + B$ and $A - B$?

Scalar Multiplication

How to multiply a scalar to a matrix?

$$\lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

What is $2B$ and $A - 2B$?

Matrix Multiplication

Only possible to multiply two matrices, $A_{m\times n}$ and $B_{p\times q}$ to get AB if $n = p$ i.e.

$$\text{number of columns in } A = \text{ number of rows in } B$$

Example. $A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} 1 & 8 & 1 \\ -2 & 3 & 1 \end{bmatrix}_{2\times 3}$

Can do AB, but cannot do BA. Dimension of AB is 2×3.

Matrix Multiplication

So how to actually multiply these matrices?

$$C = AB$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Element c_{ij} obtained by multiplying term-by-term the entries of the ith row of A and jth column of B.

Matrix Multiplication

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 1 & 8 & 1 \\ -2 & 3 & 1 \end{bmatrix}_{2\times 3}$$

Vectors

- Matrices with only one column: column vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Matrices with only one row: row vectors

$$x' = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

Inner Product

Inner product of two vectors each with n elements:

$$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Example.

$$u = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Linear Dependence

A set of vectors is said to be *linearly dependent* if and only if any one of them can be expressed as a linear combination of the remaining vectors.

Example.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Identity Matrices

Square matrix with 1s in its *principal diagonal* and 0s elsewhere

A 2×2 identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A 3×3 identity matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrices

Acts like 1,

$$AI = IA = A$$

Example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

Transpose of a Matrix

Transpose of A: interchange rows and columns (A')

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

Properties of Transposes

$$(A')' = A$$

$$(A + B)' = A' + B'$$

$$(AB)' = B'A'$$

Example: $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$

Inverse of a Matrix

For a **square** matrix A, it's inverse A^{-1} is defined as:

$$AA^{-1} = A^{-1}A = I$$

Squareness is a *necessary* condition not a *sufficient* condition

If a matrix's inverse exists, it's called a **nonsingular** matrix

Properties of Inverses

$$\left(A^{-1}\right)^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A')^{-1} = \left(A^{-1}\right)'$$

Conditions for Nonsingularity

Squareness is *necessary* but not *sufficient*

Sufficient condition for nonsingularity:

> *Rows or columns are linearly independent*

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Conditions for Nonsingularity

Squareness is *necessary* but not *sufficient*

Sufficient condition for nonsingularity:

> *Rows or columns are linearly independent*

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

A is singular, B is nonsingular.

Rank of a Matrix

Rank of a matrix $=$ maximum number of linearly independent rows

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Rank of A? Rank of B?

Full rank = all rows linearly independent =nonsingular matrix

Determinant

Determinant $|A|$ is a unique scalar associated with a *square* matrix A.

$|A| = 0$ for a singular matrix.

Determinant of a 2×2 Matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Can be calculated as:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a 3×3 Matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of a $n \times n$ Matrix

A *minor* of the element a_{ij}, denoted by $|M_{ij}|$ is obtained by deleting the ith row and jth column.

Cofactor C_{ij} is defined as:

$$|C_{ij}| = (-1)^{i+j}|M_{ij}|$$

Then,

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$$

Find the Determinant

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Properties of Determinants

1. $|A| = |A'|$
2. Interchanging rows or columns will alter the sign but not the value
3. Multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold
4. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the determinant unaltered
5. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Matrix Inversion

Adjoint of a nonsingular $n \times n$ matrix

$$adjA = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \vdots & \vdots & \dots & \vdots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} AdjA$$

Find the Inverse

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

A Simple Economic Model

Two equations in two unknowns:

$$\begin{aligned} q + 2p &= 100 \\ q - 3p &= 20 \end{aligned}$$

Can write this as:

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Solution of Linear-Equation System

$$Ax = b$$

Pre-multiply both sides by A^{-1},

$$A^{-1}Ax = A^{-1}b \quad \implies x = A^{-1}b$$

What happens if A is singular? Infinite solutions.

Homogeneous equation system

A homogeneous equation system is given by

$$Ax = 0$$

If A is nonsingular, $x^* = A^{-1}0 = 0$.

If A is singular there can be infinite number of solutions (this is true for any system of equations).

An Application of Matrix Algebra

Linear regression model:

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + \varepsilon_i$$

Denote,

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times 1} \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}_{(k+1)\times 1} \quad \boldsymbol{X} = \begin{bmatrix} 1 & X_{1,1} & \ldots & X_{k,1} \\ 1 & X_{1,2} & \ldots & X_{k,2} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & X_{1,n} & \ldots & X_{k,n} \end{bmatrix}_{n\times(k+1)} \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}$$

Can write the model as:

$$Y = X\beta + \varepsilon$$

An Application of Matrix Algebra

Linear regression model:

$$Y = X\beta + \varepsilon$$

Ordinary least squares estimator: β that minimizes

$$\varepsilon'\varepsilon = (Y - X\beta)'(Y - X\beta)$$

Few lines of calculus:

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

Another Application of Matrix Algebra

In network theory, represent network as a matrix.

For example, consider 3 banks. Below is a matrix representing connectedness between banks:

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

M_{ij} is 1 if i has lent to j. So bank 1 has lent to bank 2 and bank 2 has lent to bank 3. What if bank 3 fails? Look at M^2.

Calculus

Differentiability and Continuity

$f'(x_0)$ exists if the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

A function $y = f(x)$ is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Continuity is a necessary condition for differentiability, but it is not sufficient.

So how to differentiate functions?

Rules of differentiation, easier than taking the limit each time

Constant function rule:
For function $f(x) = k$, $f'(x) = 0$.

Power function rule:
For function $f(x) = x^n$, $f'(x) = nx^{n-1}$.

Generalized power function rule:
For function $f(x) = cx^n$, $f'(x) = cnx^{n-1}$.

Derivatives of Exponential and Logarithmic Functions

Derivative of the exponential function:

$$y = e^x \quad \rightarrow \quad \frac{dy}{dx} = e^x$$

Derivative of the log function:

$$y = \ln x \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

Rules of Differentiation

Two or more functions of one variable

Sum-Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Rules of Differentiation

Two or more functions of one variable

Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Inverse Function Rule

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Rules of Differentiation

Functions of Different Variables

Chain Rule

For $z = f(y), \quad y = g(x)$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x)$$

Example

Total cost: $C = C(Q)$

Marginal cost: $MC = C'(Q)$

Average cost:

$$AC = \frac{C(Q)}{Q}$$

When is $\frac{dAC}{dQ}$ positive?

Example

Revenue: $R = f(Q), \quad f'(Q) > 0$

Output: $Q = g(L), \quad g'(L) > 0$

Change in revenue due to labor adjustment:

$$\frac{dR}{dL} = \frac{dR}{dQ} \cdot \frac{dQ}{dL} = f'(Q)g'(L)$$

Elasticity

Elasticity is defined as:

$$\varepsilon = \frac{\text{Percentage change in y}}{\text{Percentage change in x}} = \frac{dy/y}{dx/x}$$

We can calculate this as:

$$\varepsilon = \frac{dy}{dx} \cdot \frac{x}{y}$$

Elasticity

Elasticity:

$$\varepsilon = \frac{dy}{dx} \cdot \frac{x}{y}$$

- $|\varepsilon| > 1$, elastic
- $|\varepsilon| = 1$, unit elasticity
- $|\varepsilon| < 1$, inelastic

Example

$$C = a + bY$$

Partial Differentiation

For a function of several variables:

$$y = f(x_1, x_2, \cdots, x_n)$$

If x_1 changes by Δx_1 but all other variables remain constant:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{\Delta x_1}$$

Partial derivative of y with respect to x_i:

$$\frac{\partial y}{\partial x_i} = f_i = \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i}$$

Production Function

$$Q = AK^{\alpha}L^{1-\alpha}$$

Marginal product of capital (MPK):

$$\frac{\partial Q}{\partial K} = Q_K =$$

Marginal product of labor (MPL):

$$\frac{\partial Q}{\partial L} = Q_L =$$

Total Derivative

For a function of n variables

$$y = f\left(x_1, x_2, \cdots, x_n\right)$$

$$\frac{df}{dt} = f_1 \cdot \frac{dx_1}{dt} + f_2 \cdot \frac{dx_2}{dt} + \cdots + f_n \cdot \frac{dx_n}{dt}$$

Total Derivative

Given the function

$$y = f(x_1, x_2)$$

We are interested in how y changes with respect to x_1, but x_2 also depends of x_1

$$x_2 = g(x_1)$$

Total derivative with respect to x_1:

$$\frac{dy}{dx_1} = f_1 + f_2 \cdot g'(x_1)$$

Example

Let a production function be

$$Q(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha}$$

where

$$K(t) = K_0 + at \qquad L(t) = L_0 + bt$$

Gradient

For the function:

$$y = f(x_1, x_2, ..., x_n)$$

The gradient is given by

$$\nabla f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Derivatives of Implicit Functions

Total differentiating F, we have $dF = 0$, or

$$F_y dy + F_1 dx_1 + \cdots + F_n dx_n = 0$$

Suppose that only y and x_1 are allowed to vary:

$$\frac{\partial y}{\partial x_1} = -\frac{F_1}{F_y}.$$

In the simple case where the given equation is $F(y, x) = 0$, the rule gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Integration

- Integration is the reverse of differentiation
- If $f(x)$ is the derivative of $F(x)$, we can *integrate* $f(x)$ to find $F(x)$

$$\frac{d}{dx}F(x) = f(x) \Rightarrow \int f(x)dx = F(x) + c$$

- Rules of integration follow from rules of differentiation

Rules of Integration

<u>Power Rule</u>

$$\int x^n dx = \frac{1}{n+1} \cdot x^{n+1} + c \quad (n \neq -1)$$

Examples:

$$\int x^3 dx, \qquad \int x dx, \qquad \int 1 dx$$

Rules of Integration

Exponential Rule

$$\int e^x dx = e^x + c$$

Log Rule

$$\int \frac{1}{x} dx = \ln x + c \quad (x > 0)$$

Rules of Integration

Integral of a sum

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

Integral of a multiple

$$\int kf(x)dx = k \int f(x)dx$$

Example:

$$\int (x^2 + 3x + 1)dx$$

Definite Integrals

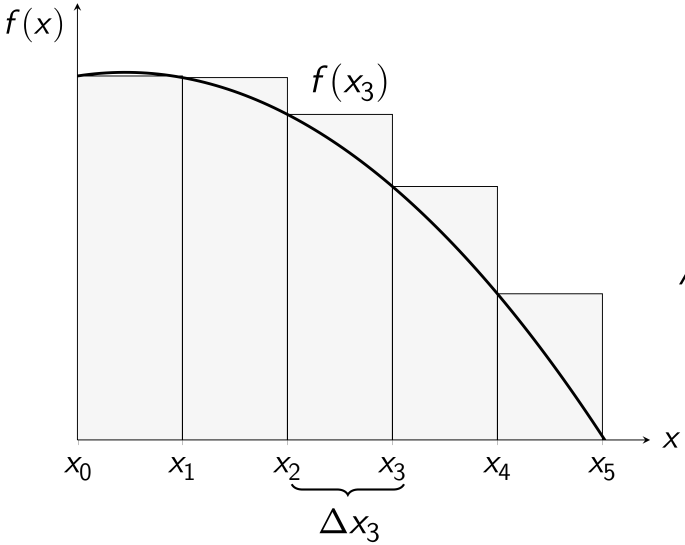
Definite integral:

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

Example:

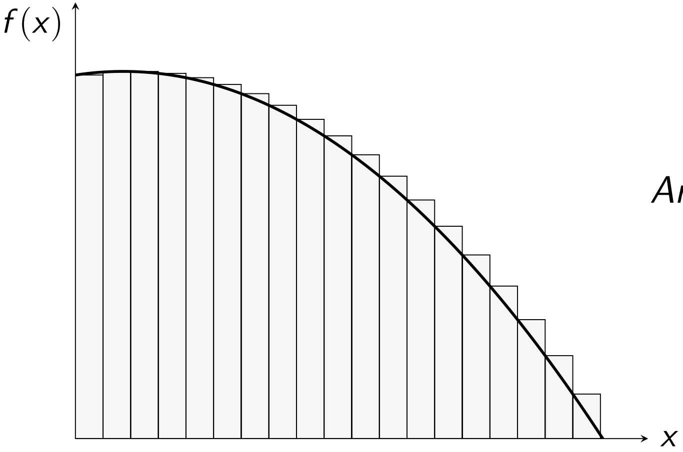
$$\int_1^3 2x^2 =$$

Area under the curve



$$Area \approx \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Area under the curve



$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$
$$= \int_{x_1}^{x_n} f(x)dx$$

Few last words

- Sample exam, midterm exams from previous semesters, and help sheet on course website
- Good luck for the exam!