# ECON 340
# Economic Research Methods

Div Bhagia

Lecture 20
Quadratic and Log functional forms

# Remember Calculus?

For a function

$$y = f(x)$$

The derivative denoted by:

$$\frac{dy}{dx} \quad or \quad f'(x)$$

captures how the value of the function changes due to a small change in $x$.

# Rules of Differentiation

- $y = a \rightarrow \dfrac{dy}{dx} = 0$
- $y = ax \rightarrow \dfrac{dy}{dx} = a$
- $y = ax^b \rightarrow \dfrac{dy}{dx} = abx^{b-1}$
- $y = f(x) + g(x) \rightarrow \dfrac{dy}{dx} = f'(x) + g'(x)$

*Examples:* $y = 10, \quad y = 5x, \quad y = 8x^3, \quad y = 3x^2 + 4$

# Rules of Differentiation

- Derivative of a log function:

$$y = log(x) \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$$

- Chain rule:

$$z = f(y),\ y = g(x) \quad \rightarrow \quad \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

*Examples*:

$y = 2 + 3 \cdot log(x),\ y = log(z)\ \&\ z = x^2,\ y = log(x^2),\ y = log(f(x))$

# Fitting a Line

Linear relationship (with some error):

$$Y = \beta_0 + \beta_1 X + u$$

Taking the conditional expectation:

$$E(Y|X) = \beta_0 + \beta_1 X + E(u|X)$$

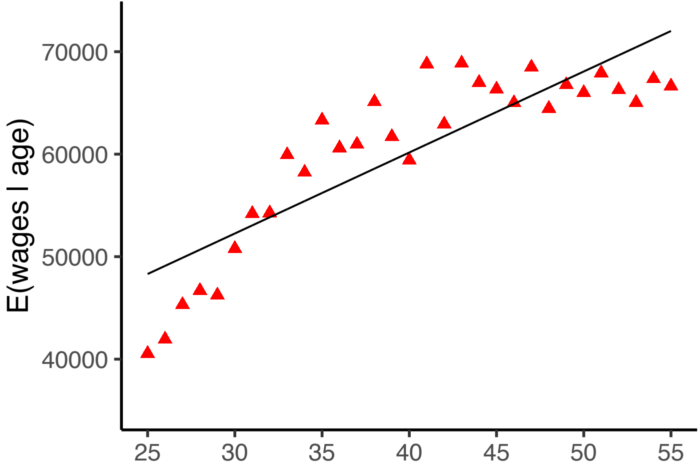
With $E(u|X) = 0$,

$$E(Y|X) = \beta_0 + \beta_1 X$$

OLS fits a linear line between average $Y$ at each $X$ and $X$.

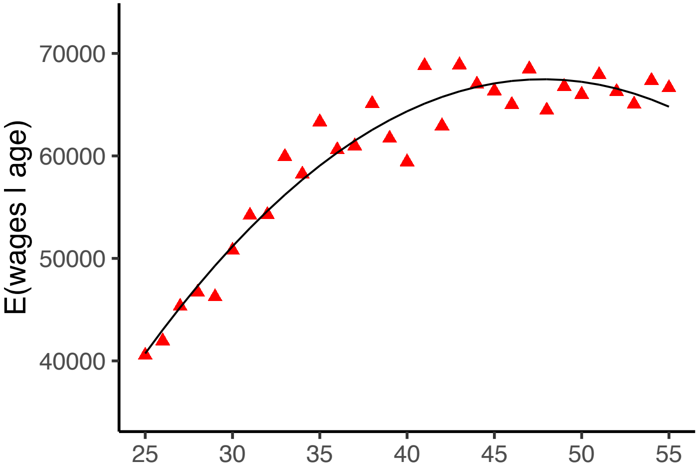
# ACS Data: Wages and Age

Is the relationship really linear?



# ACS Data: Wages and Age

Does this model have a better $R^2$?



# Fitting a Line

Linear relationship:

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$$

Take the derivative:

$$\frac{d\hat{Y}}{dX} = \hat{\beta}_1 \rightarrow d\hat{Y} = \hat{\beta}_1 dX$$

Can think of $d$ as 'change in': One unit change in $X$, associated with $\beta_1$ units change in $Y$.

Impact of $X$ on $Y$ constant with $X$.

# Fitting a Curve

Quadratic relationship:

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2$$

Take the derivative:

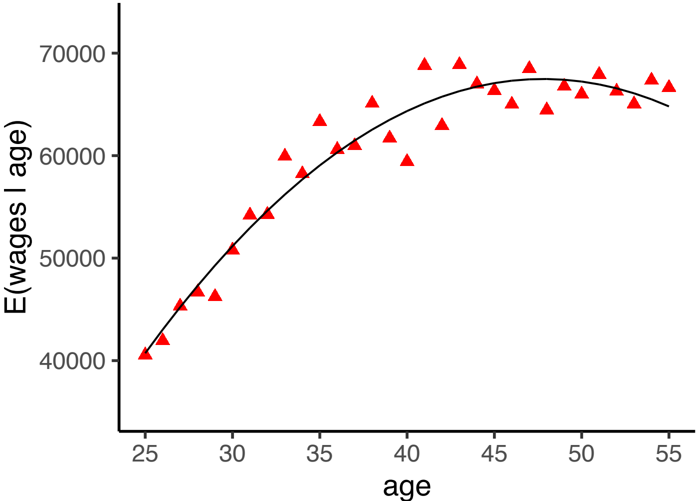
$$\frac{d\hat{Y}}{dX} = \hat{\beta}_1 + 2\hat{\beta}_2 X$$

Now the impact of $X$ on $Y$ changes with $X$.

Remember: Derivative captures the slope of the tangent line.

# ACS Data: Wages and Age

$$\hat{wage} = -52207 + 4775.64 \cdot age - 49.493 \cdot age^2$$



# ACS Data: Wages and Age

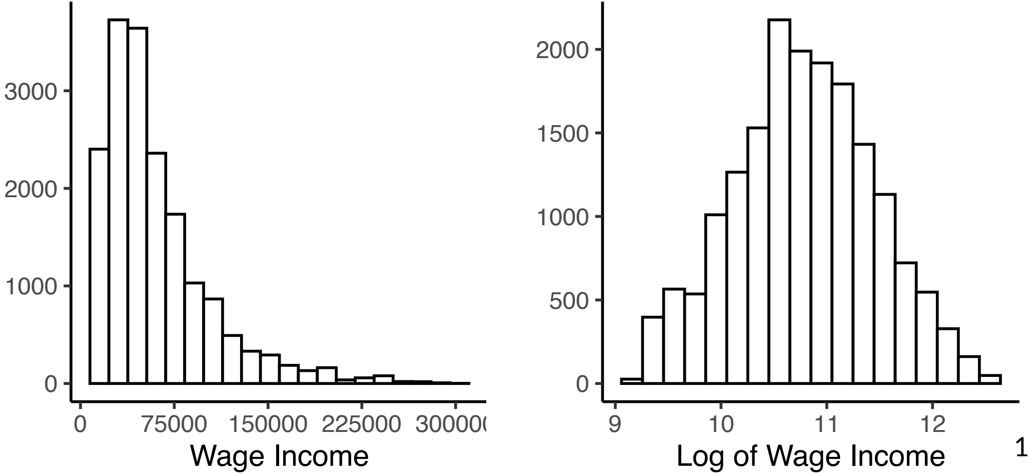
$$\widehat{wage} = -52207 + 4775.64 \cdot age - 49.493 \cdot age^2$$

How does the predicted wage change going from 30 to 31?

What about going from 50 to 51?

# Log Functional Forms

Sometimes we log transform a variable before fitting a model.
Useful if the data is skewed or has outliers.



# Log Functional Forms

- Log-transformation leads to interpretation of regression coefficients in % changes than unit changes which can sometimes be more informative

- Can think of change in log of X as the relative change in X with respect to its original value

$$\frac{d}{dX}\log(X) = \frac{1}{X} \rightarrow d\log(X) = \frac{dX}{X}$$

In which case $100 \times d\log(X)$ represents % change in $X$

# Log Functional Forms: Interpretation

Three possible models:

1. Level-Log: $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 log(X)$

Differentiating both left and right hand side with respect to $X$:

$$\frac{d\hat{Y}}{dX} = \hat{\beta}_1 \cdot \frac{1}{X} \quad \rightarrow \quad \hat{\beta}_1 = \frac{d\hat{Y}}{dX/X}$$

In which case,

$$\frac{\hat{\beta}_1}{100} = \frac{\text{unit change in } Y}{\text{\% change in } X}$$

# Level-Log Model

| | Wages |
|---|---|
| Intercept | $-57{,}224.83^{***}$ |
| | (5,008.20) |
| Log Age | $32{,}052.27^{***}$ |
| | (1,363.87) |
| Observations | 17,578 |
| $R^2$ | 0.03 |
| *Note:* | $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$ |

*1% increase in age leads to $320 increase in predicted wages.*

# Log Functional Forms: Interpretation

Three possible models:

2. Log-Level: $\hat{log}(Y) = \hat{\beta}_0 + \hat{\beta}_1 X$

$$\hat{\beta}_1 = \frac{1}{Y} \cdot \frac{dY}{dX} \rightarrow 100\hat{\beta}_1 = \frac{\text{\% change in } Y}{\text{unit change in } X}$$

# Log-Level Model

| | Log Wages |
|---|---|
| Intercept | 10.31*** |
| | (0.02) |
| Age | 0.01*** |
| | (0.001) |
| Observations | 17,578 |
| $R^2$ | 0.03 |
| *Note:* | *p<0.1; **p<0.05; ***p<0.01 |

*1 year increase in age leads to 1% increase in predicted wages.*

# Log Functional Forms: Interpretation

Three possible models:

3. Log-Log: $\log(\hat{Y}) = \hat{\beta}_0 + \hat{\beta}_1 \log(X)$

$$\hat{\beta}_1 = \frac{dY/Y}{dX/X} \rightarrow \hat{\beta}_1 = \frac{\%\text{ change in } Y}{\%\text{ change in } X}$$

# Log-Log Model

| | Log Wages |
|---|---|
| Intercept | 8.99*** |
| | (0.08) |
| Log Age | 0.49*** |
| | (0.02) |
| Observations | 17,578 |
| $R^2$ | 0.03 |
| *Note:* | *p<0.1; **p<0.05; ***p<0.01 |

*1% increase in age leads to 0.49% increase in predicted wages.*

# What's next?

- **No class on Thursday** (11/09) as I am traveling for a conference. Use this time to review the linear regression model and work on Problem Set 4.
- Problem Set 4 is due on the following Tuesday (11/14)
- Next week: Regression Analysis in R