

# ECON 340

## Economic Research Methods

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Lecture 20  
Quadratic and Log functional forms

# Remember Calculus?

For a function

$$y = f(x)$$

The derivative denoted by:

$$\frac{dy}{dx} \quad \text{or} \quad f'(x)$$

captures how the value of the function changes due to a small change in  $x$ .

# Rules of Differentiation

- $y = a \rightarrow \frac{dy}{dx} = 0$
- $y = ax \rightarrow \frac{dy}{dx} = a$
- $y = ax^b \rightarrow \frac{dy}{dx} = abx^{b-1}$
- $y = f(x) + g(x) \rightarrow \frac{dy}{dx} = f'(x) + g'(x)$

Examples:  $y = 10$ ,  $y = 5x$ ,  $y = 8x^3$ ,  $y = 3x^2 + 4$

# Rules of Differentiation

- Derivative of a log function:

$$y = \log(x) \rightarrow \frac{dy}{dx} = \frac{1}{x}$$

- Chain rule:

$$z = f(y), y = g(x) \rightarrow \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

*Examples:*

$$y = 2 + 3 \cdot \log(x), y = \log(z) \ \& \ z = x^2, y = \log(x^2), y = \log(f(x))$$

# Fitting a Line

Linear relationship (with some error):

$$Y = \beta_0 + \beta_1 X + u$$

Taking the conditional expectation:

$$E(Y|X) = \beta_0 + \beta_1 X + E(u|X)$$

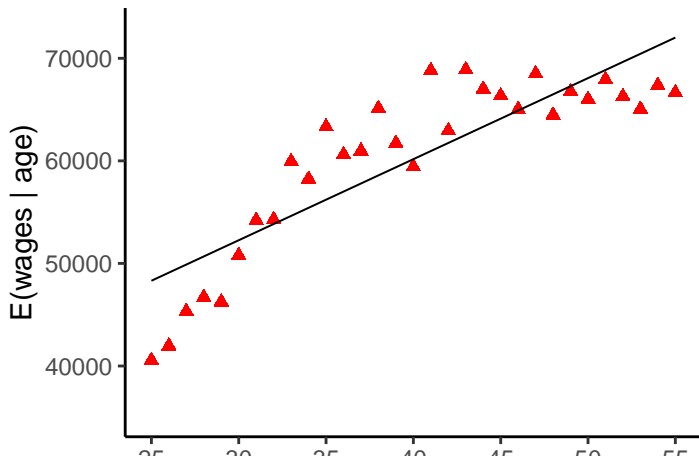
With  $E(u|X) = 0$ ,

$$E(Y|X) = \beta_0 + \beta_1 X$$

OLS fits a linear line between average  $Y$  at each  $X$  and  $X$ .

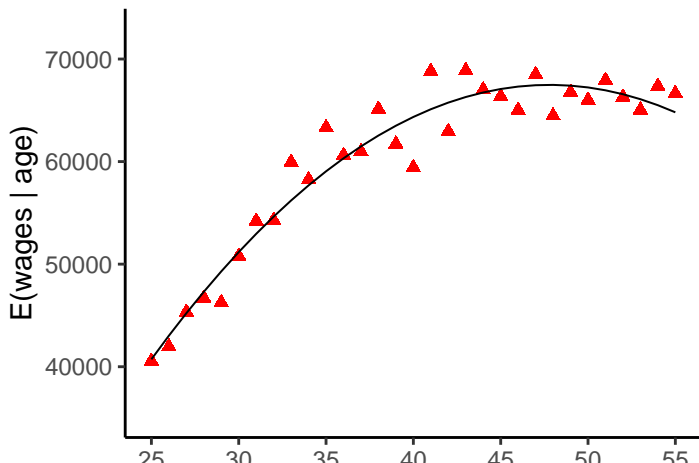
# ACS Data: Wages and Age

Is the relationship really linear?



# ACS Data: Wages and Age

Does this model have a better  $R^2$ ?



# Fitting a Line

Linear relationship:

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$$

Take the derivative:

$$\frac{d\hat{Y}}{dX} = \hat{\beta}_1 \rightarrow d\hat{Y} = \hat{\beta}_1 dX$$

Can think of  $d$  as 'change in': One unit change in  $X$ , associated with  $\beta_1$  units change in  $Y$ .

Impact of  $X$  on  $Y$  constant with  $X$ .



# Fitting a Curve

Quadratic relationship:

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2$$

Take the derivative:

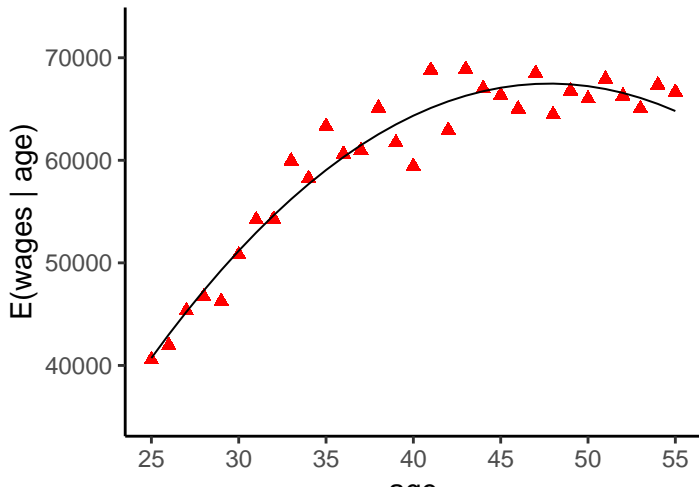
$$\frac{d\hat{Y}}{dX} = \hat{\beta}_1 + 2\hat{\beta}_2 X$$

Now the impact of  $X$  on  $Y$  changes with  $X$ .

Remember: Derivative captures the slope of the tangent line.

# ACS Data: Wages and Age

$$\hat{wage} = -52207 + 4775.64 \cdot age - 49.493 \cdot age^2$$



## ACS Data: Wages and Age

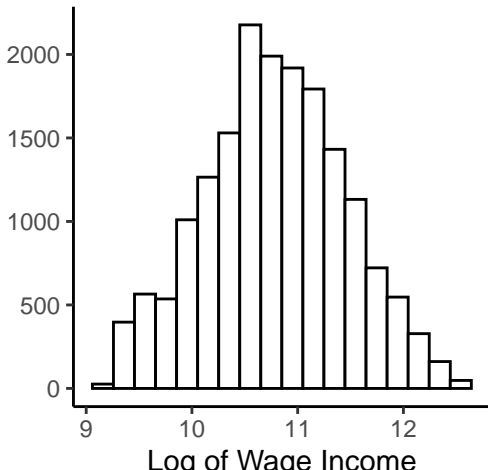
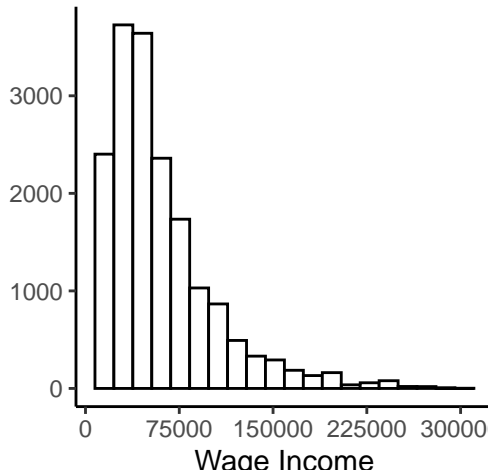
$$\widehat{wage} = -52207 + 4775.64 \cdot age - 49.493 \cdot age^2$$

How does the predicted wage change going from 30 to 31?

What about going from 50 to 51?

# Log Functional Forms

Sometimes we log transform a variable before fitting a model.  
Useful if the data is skewed or has outliers.



# Log Functional Forms

- Log-transformation leads to interpretation of regression coefficients in % changes than unit changes which can sometimes be more informative
- Can think of change in log of  $X$  as the relative change in  $X$  with respect to its original value

$$\frac{d}{dX} \log(X) = \frac{1}{X} \rightarrow d \log(X) = \frac{dX}{X}$$

In which case  $100 \times d \log(X)$  represents % change in  $X$

# Log Functional Forms: Interpretation

Three possible models:

1. Level-Log:  $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 \log(X)$

Differentiating both left and right hand side with respect to  $X$ :

$$\frac{d\hat{Y}}{dX} = \hat{\beta}_1 \cdot \frac{1}{X} \quad \rightarrow \quad \hat{\beta}_1 = \frac{d\hat{Y}}{dX/X}$$

In which case,

$$\frac{\hat{\beta}_1}{100} = \frac{\text{unit change in } Y}{\% \text{ change in } X}$$

# Level-Log Model

Wages	
Intercept	-57,224.83*** (5,008.20)
Log Age	32,052.27*** (1,363.87)
Observations	17,578
R <sup>2</sup>	0.03
Note: *p<0.1; **p<0.05; ***p<0.01	

*1% increase in age leads to \$320 increase in predicted wages.*

# Log Functional Forms: Interpretation

Three possible models:

2. Log-Level:  $\log(Y) = \hat{\beta}_0 + \hat{\beta}_1 X$

$$\hat{\beta}_1 = \frac{1}{Y} \cdot \frac{dY}{dX} \rightarrow 100\hat{\beta}_1 = \frac{\% \text{ change in } Y}{\text{unit change in } X}$$



# Log-Level Model

	Log Wages
Intercept	10.31*** (0.02)
Age	0.01*** (0.001)
Observations	17,578
R <sup>2</sup>	0.03
Note: *p<0.1; **p<0.05; ***p<0.01	

*1 year increase in age leads to 1% increase in predicted wages.*

# Log Functional Forms: Interpretation

Three possible models:

3. Log-Log:  $\log(\hat{Y}) = \hat{\beta}_0 + \hat{\beta}_1 \log(X)$

$$\hat{\beta}_1 = \frac{dY/Y}{dX/X} \rightarrow \hat{\beta}_1 = \frac{\% \text{ change in } Y}{\% \text{ change in } X}$$

# Log-Log Model

	Log Wages
Intercept	8.99*** (0.08)
Log Age	0.49*** (0.02)
Observations	17,578
R <sup>2</sup>	0.03
Note: *p<0.1; **p<0.05; ***p<0.01	

*1% increase in age leads to 0.49% increase in predicted wages.*