

Determinant and Matrix Inversion

ECON 441: Introduction to Mathematical Economics

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Determinant $|A|$ is a unique scalar associated with a square matrix A .

Determinant of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Can be calculated as:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Find the determinant of A and B given below:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Is A nonsingular? What about B ? Check if you get the same answer by reducing A and B to their echelon form and then finding the rank.

Determinant of a 3×3 matrix:

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

Find the determinant for:

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Properties of Determinants

1. $|A| = |A'|$
2. Interchanging rows or columns will alter the sign but not the value
3. Multiplication of any one row (or one column) by a scalar k will change the value of the determinant k -fold
4. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the determinant unaltered
5. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Verify the above properties for a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The **minor** of the element a_{ij} , denoted by $|M_{ij}|$ is obtained by deleting the i th row and j th column of the matrix and taking the determinant of the resulting matrix.

Whereas, **cofactor** $|C_{ij}|$ is defined as:

$$|C_{ij}| = (-1)^{i+j} |M_{ij}|$$

Example.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor for the element a_{12} :

$$|M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Cofactor for the element a_{12} :

$$|C_{12}| = (-1)^{(1+2)} |M_{12}| = -|M_{12}|$$

Find $|C_{31}|$, $|C_{32}|$, and $|C_{33}|$ for

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Determinant for an $n \times n$ matrix is given by:

$$|A| = \sum_{i=1}^n a_{ij} |C_{ij}| = \sum_{j=1}^n a_{ij} |C_{ij}|$$

The first expression corresponds to expanding with respect to the j th column, while the second expression is the expression for the determinant when expanding with respect to the i th row.

Find the determinant of $|A|$ by expanding with the third row.

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

To find the inverse of a nonsingular matrix A take the transpose of its cofactor matrix $C = [|C_{ij}|]$ to find the adjoint of A and divide it by the determinant of A .

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

Adjoint of a nonsingular $n \times n$ matrix

$$\text{adj} A = C' = \begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \vdots & \vdots & \dots & \vdots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}$$

Find the inverse of

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$