Homework 2 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Exercise 4.2

1. $A = \begin{bmatrix} 7 & -1 \\ 6 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 4 \\ 3 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 8 & 3 \\ 6 & 1 \end{bmatrix}$

   (a) $A + B = \begin{bmatrix} 7 & 3 \\ 9 & 7 \end{bmatrix}$

   (b) $C - A = \begin{bmatrix} 1 & 4 \\ 0 & -8 \end{bmatrix}$

   (c) $3A = \begin{bmatrix} 21 & -3 \\ 18 & 27 \end{bmatrix}$

   (d) $4B + 2C = \begin{bmatrix} 0 & 16 \\ 12 & -8 \end{bmatrix} + \begin{bmatrix} 16 & 6 \\ 12 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 22 \\ 24 & -6 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$

   (a) AB is defined as number of columns in A is two which is equal to the number of rows in B.

   $$AB = \begin{bmatrix} 2\times2+8\times3 & 2\times0+8\times8 \\ 3\times2+0\times3 & 3\times0+0\times8 \\ 5\times2+1\times3 & 5\times0+1\times8 \end{bmatrix} = \begin{bmatrix} 28 & 64 \\ 6 & 0 \\ 13 & 8 \end{bmatrix}$$

   Not possible to calculate BA as B has two columns, but A has three rows.

   (b) BC and CB are both defined as both have two rows and two columns.

   $$BC = \begin{bmatrix} 14 & 4 \\ 69 & 30 \end{bmatrix} \neq CB = \begin{bmatrix} 20 & 16 \\ 21 & 24 \end{bmatrix}$$

4. (a) $\begin{bmatrix} 0 & 2 \\ 36 & 20 \\ 16 & 3 \end{bmatrix}_{3\times 2}$ (b) $\begin{bmatrix} 49 & 3 \\ 4 & 3 \end{bmatrix}_{2\times 2}$

(c) $\begin{bmatrix} 3x+5y \\ 4x+2y-7z \end{bmatrix}_{2\times 1}$ (d) $[7a+c \quad 2b+4c]$

Exercise 4.4

5. (e)

$$A = \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}_{3\times 1} \qquad B = \begin{bmatrix} 3 & 6 & -2 \end{bmatrix}_{1\times 3}$$

$$C = AB = \begin{bmatrix} -6 & -12 & 4 \\ 12 & 24 & -8 \\ 21 & 42 & -14 \end{bmatrix}$$

$$D = BA = [3\times -2 + 6\times 4 + -2\times 7] = [4]$$

7. In example 5, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$. In which case,

$$x'Ax = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} a_{11}x_1 & a_{22}x_2 \end{bmatrix}_{1\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

$$= \underbrace{a_{11}x_1^2 + a_{22}x_2^2}_{\text{Weighted sum of squares}} = \sum_{i=1}^{2} a_{ii}x_i^2$$

So $x'Ax$ represents a weighted sum of squares where a_{11}, a_{22} are weights.

But now what if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. In this case,

$$\begin{aligned} x'Ax &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \\ &= \begin{bmatrix} a_{11}x_1 + a_{21}x_2 & a_{12}x_1 + a_{22}x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \\ &= a_{11}x_1^2 + a_{21}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2 \\ &= a_{11}x_1^2 + (a_{21} + a_{12})\, x_1x_2 + a_{22}x_2^2 \end{aligned}$$

So $x'Ax$ no longer represents a weighted sum of squares.

You can check that the associative law i.e.

$$(x'A)\,x = x'(Ax)$$

will apply in both cases (after all, its a law!) as all products are possible.

Exercise 4.5

1.

$$A = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 9 \\ 6 \\ 0 \end{bmatrix}_{3\times 1} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

(a) $AI = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix} = A$

(b) $IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix}$

(c) $Ix = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(d) $x'I = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

4. Let's start with a 2×2 diagonal matrix

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{bmatrix}$$

$x = x^2$ for only $x = 0, 1$ so a_{11} and a_{22} can either be 0 or 1 . So we can have the following 2×2 idempotent diagonal matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

More generally, for $n \times n$ matrix, there can be 2^n such matrices. This is because there will be n elements, each of which can take two values.

Exercise 4.6

2. $A = \begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -8 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 9 \\ 6 & 1 & 1 \end{bmatrix}$

(a) $A + B = \begin{bmatrix} 3 & -4 \\ -1 & 4 \end{bmatrix}$

$A' + B' = \begin{bmatrix} 0 & -1 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -4 & 4 \end{bmatrix}$

So, $(A + B)' = A' + B'$

(b) $AC = \begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 & 0 & 9 \\ 6 & 1 & 1 \end{bmatrix}_{2\times3} = \begin{bmatrix} 24 & 4 & 4 \\ 17 & 3 & -6 \end{bmatrix}_{2\times3}$

$C'A' = \begin{bmatrix} 1 & 6 \\ 0 & 1 \\ 9 & 1 \end{bmatrix}_{3\times2} \begin{bmatrix} 0 & -1 \\ 4 & 3 \end{bmatrix}_{2\times2} = \begin{bmatrix} 24 & 17 \\ 4 & 3 \\ 4 & -6 \end{bmatrix}_{3\times2}$

So, $(AC)' = C'A'$.

6. $A = I - X\,(X'X)^{-1}\,X'$

(a.) Say the dimension of X is $m \times n$. Then the dimension of $X'_{n\times m} X_{m\times n}$ is $n \times n$. So the dimension of $(X'X)^{-1}$ is also $n \times n$. This implies that the dimension of $X_{m\times n}\,(X'X)^{-1}_{n\times n}\,X'_{n\times m}$ is $m \times m$. Hence, $X'X$ and A must be square matrices, but X need not be square.

(b.) To prove a matrix is idempotent, we need to show $AA = A$.

$$\begin{aligned} AA &= (I - X\,(X'X)^{-1}\,X')(I - X\,(X'X)^{-1}\,X') \\ &= I - X\,(X'X)^{-1}\,X' - X\,(X'X)^{-1}\,X' + X\,\underbrace{(X'X)^{-1}\,X'X}_{I}\,(X'X)^{-1}\,X' \\ &= I - X\,(X'X)^{-1}\,X' - X\,(X'X)^{-1}\,X' + X\,(X'X)^{-1}\,X' \\ &= I - X\,(X'X)^{-1}\,X' = A \end{aligned}$$