Homework 3 Solutions

ECON 441: Introduction to Mathematical Economics   Instructor: Div Bhagia

Exercise 5.1

3.   (tskYes)   (tskYes)

   (tskYes)   (tskNo) the second row $= 2 \times$ first row.

4. Yes, we get the same answer.

5.   (a)

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Exchange Row 2 and 3

$$A_1 = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 0 & 0 \\ 0 & 3 & 9 \end{bmatrix}$$

New Row 2 = Row 1 + Row 2

$$A_2 = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 5 & 1 \\ 0 & 3 & 9 \end{bmatrix}$$

New Row 3 = Row 3 $-3/5 \times$ Row 2

$$A_3 = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & \frac{42}{5} \end{bmatrix}$$

The resulting echelon matrix A_3 contains 3 nonzero rows and hence A has rank 3. Since A is full-rank, it is a nonsingular matrix.

(b)

$$B = \begin{bmatrix} 0 & -1 & -4 \\ 3 & 1 & 2 \\ 6 & 1 & 0 \end{bmatrix}$$

New Row 1 = Row 2, New Row 2 = Row 3, New Row 3 = Row 1

$$B_1 = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 1 & 0 \\ 0 & -1 & -4 \end{bmatrix}$$

New Row 2 = Row 2−2 × Row 1

$$B_2 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix}$$

New Row 3 = Row 3−Row 2

$$B_3 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

The resulting echelon matrix B_3 contains only 2 nonzero rows and hence B has rank 2. Since the rank of B is less than the number of rows and columns, B is a singular matrix.

(c)

$$C = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Interchange rows 2 and 3.

$$C_1 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 8 & 0 & 0 & 8 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

New Row 2 = Row 2$-\frac{8}{7}\times$Row 1

$$C_2 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & -\frac{48}{7} & -\frac{24}{7} & 8-\frac{24}{7} \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & -\frac{48}{7} & -\frac{24}{7} & \frac{32}{7} \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

New Row 3 = Row 3$+\frac{7}{48}$ Row 2

$$C_3 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & -\frac{48}{7} & -\frac{24}{7} & \frac{32}{7} \\ 0 & 0 & \frac{3}{2} & \frac{5}{3} \end{bmatrix}$$

Rank of C is 3. The concept of nonsingularity is only defined for square matrices.

(d)

$$D = \begin{bmatrix} 2 & 7 & 9 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 5 & 9 & -3 \end{bmatrix}$$

New Row 2= Row 2$-0.5\times$ Row 1

$$D_1 = \begin{bmatrix} 2 & 7 & 9 & -1 \\ 0 & -2.5 & -4.5 & 1.5 \\ 0 & 5 & 9 & -3 \end{bmatrix}$$

New Row 3= Row 3 +2× Row 2

$$D_2 = \begin{bmatrix} 2 & 7 & 9 & -1 \\ 0 & -2.5 & -4.5 & 1.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of D is 2.

6. In the end, converting the matrix to echelon form is trying to find if any combination of rows will lead to a sum of 0, which is the definition of linear independence.

Exercise 5.2

1. (c)

$$8 \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + 3 \begin{vmatrix} 4 & 0 \\ 6 & 0 \end{vmatrix}$$

$$= 8(0-0) - 1(12-6) + 3(0-0)$$
$$= 0 - 6 + 0$$
$$= -6$$

(e)

$$a \begin{vmatrix} c & a \\ a & b \end{vmatrix} - b \begin{vmatrix} b & a \\ c & b \end{vmatrix} + c \begin{vmatrix} b & c \\ c & a \end{vmatrix}$$

$$= a\left(cb - a^2\right) - b\left(b^2 - ac\right) + c\left(ab - c^2\right)$$
$$= abc - a^3 - b^3 + abc + abc - c^3$$
$$= -a^3 - b^3 - c^3 + 3abc$$

(f)

$$x \begin{vmatrix} y & 2 \\ -1 & 8 \end{vmatrix} - 5 \begin{vmatrix} 3 & 2 \\ 9 & 8 \end{vmatrix} + 0 \begin{vmatrix} 3 & y \\ 9 & -1 \end{vmatrix}$$

$$= x(8y + 2) - 5(24 - 18) + 0(-3 - 9y)$$
$$= 8xy + 2x - 30 + 0$$
$$= 8xy + 2x - 30$$

2. $|C_{13}| : 1 + 3 = 4$ is even so +

$|C_{23}| : 2 + 3 = 5$ is odd so −

$|C_{33}|$: $3 + 3 = 6$ is even so $+$

$|C_{41}|$: $4 + 1 = 5$ is odd so $-$

$|C_{34}|$: $3 + 4 = 7$ is odd so $-$

3. Minor of a:

$$|M_{11}| = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh$$

Cofactor of a:

$$|C_{11}| = (-1)^{1+1} |M_{11}| = |M_{11}|$$

Minor of b:

$$|M_{12}| = \begin{vmatrix} d & f \\ g & i \end{vmatrix} = di - fg$$

Cofactor of b:

$$|C_{12}| = (-1)^3 |M_{12}| = -|M_{12}|$$

Minor of f:

$$|M_{23}| = \begin{vmatrix} a & b \\ g & h \end{vmatrix} = ah - bg$$

Cofactor of f:

$$|C_{23}| = (-1)^5 |M_{23}| = -|M_{23}|$$

6. Minors of third row:

$$|M_{31}| = \begin{vmatrix} 11 & 4 \\ 2 & 7 \end{vmatrix} = 69, \quad |M_{32}| = \begin{vmatrix} 9 & 4 \\ 3 & 7 \end{vmatrix} = 51, \quad |M_{33}| = \begin{vmatrix} 9 & 11 \\ 3 & 2 \end{vmatrix} = -15$$

Cofactors: $|C_{31}| = |M_{31}|, |C_{32}| = -|M_{32}|, |C_{33}| = |M_{33}|$.