Foundations of Algorithms

Homework 1

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1. Below are the functions ranked by their order of growth:
    1. $2^{2^{n+1}}$
    2. 2^{2^n}
    3. $(n+1)!$
    4. $n!$
    5. e^n
    6. $n2^n$
    7. 2^n
    8. $\left(\frac{3}{2}\right)^n$
    9. $n^{\lg\lg n} = (\lg n)^{\lg n}$
    10. $(\lg n)!$
    11. n^3
    12. $n^2 = 4^{\lg n}$
    13. $\lg(n)! = nlgn$
    14. $2^{\lg n} = n$
    15. $\left(\sqrt{2}\right)^{\lg n}$
    16. $2^{\sqrt{2\lg n}}$
    17. $lg^2 n$
    18. $\ln n$
    19. $\sqrt{\lg n}$
    20. $\ln(\ln(n))$
    21. $2^{lg^* n}$
    22. $lg^* n = lg^*(\lg n)$
    23. $\lg\,(lg^*(n))$
    24. $1 = n^{1/\lg n}$

2. $a^x > x^c > \sqrt[k]{x} > \log_b(x)$
   Assuming that all the constants are smaller than x

3. a. Proof: -

   Consider N = 1 and c = 1
   Suppose

$$n \geq N$$
$$N \leq n \rightarrow N \leq n$$
$$Nn \leq n^2$$
$$n \leq n^2$$
$$|n| \leq |n^2|$$
$$|n| \leq 1.|n^2|$$
$$|f(x)| \leq c \cdot |g(x)|$$

b. Proof: -

Consider N = 0 and c = 2

Suppose

$n \geq N$

$N \leq n \rightarrow N \leq n$

$Nn + n^2 \leq n^2 + n^2$

$Nn + n^2 \leq 2n^2$

$|n^2| \leq |2n^2|$

$|n^2| \leq 2.|n^2|$

$|f(x)| \leq c \cdot |g(x)|$

c. Proof: -

Consider N = 5 and c = 4

Suppose

$n \geq N$

$N \leq n \rightarrow N \leq n$

$Nn \leq n^2$

$3n^2 + Nn \leq n^2 + 3n^2$

$3n^2 + 5n \leq 4n^2$

$|3n^2 + 5n| \leq 4.|n^2|$

$|f(x)| \leq c \cdot |g(x)|$

4. a. Proof: -

Consider N = 1 and c = 1

Suppose

$n \geq N$

$N \leq n \rightarrow N \leq n$

$Nn^k \leq n.n^k$

$Nn^k \leq n^{k+1}$

Given that $k' \geq k$ we can replace $k+1$ as k'

$Nn^k \leq n^{k\prime}$

$|n^k| \leq 1.|n^{k'}|$

$|f(x)| \leq c \cdot |g(x)|$

b. Proof: - Given a function f for the domain D

There exists $n \in N$ and $c \in R^+$such that $x \in D, x \geq n$ implies $|f(x)| \leq c.|g(x)|$

Consider $N = n$ and $c = 2c$

Suppose

$x \geq n$

We know that $|f(x)| \leq c|g(x)|$

$|f(x)| \leq c.|g(x)| \rightarrow 2|f(x)| \leq 2c|g(x)|$

$|f(x)| + |f(x)| \leq 2c|g(x)|$

5. a. Proof: -

Consider $N = 1$ and $c = 1$

Given: $\sum_{k=2}^{n} \frac{1}{k} \leq \ln(n) - \ln(1)$

$\sum_{k=2}^{n} \frac{1}{k} + 1 \leq \ln(n) - \ln(1) + 1$

$\sum_{k=2}^{n} \frac{1}{k} + \frac{1}{1} \leq \ln(n) + 1$

$H_n \leq \ln(n) + 1$

$|H_n| \leq 1. |\ln(n) + 1|$

$|f(x)| \leq c \cdot |g(x)|$

Since $\mathrm{O}(1)$ is constant we can say that $H_n \in \mathrm{O}(\ln(n))$

b. Proof: -

Consider N=1 and c=1

Given: $\sum_{k=2}^{n} \frac{1}{k} \geq \ln(n+1) - \ln(2)$

$\sum_{k=2}^{n} \frac{1}{k} + 1 \geq \ln(n+1) - \ln(2) + 1$

$\sum_{k=2}^{n} \frac{1}{k} + \frac{1}{1} \geq \ln(n+1) - 1 + 1$

$H_n \geq \ln(n+1)$

$|H_n| \geq 1. |\ln(n+1)|$

$|f(x)| \geq c \cdot |g(x)|$

Since H_n is in $\ln(n+1)$ we can say that $H_n \in \mathrm{O}(\ln(n))$

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7. a. Proof: - By Mathematical Induction

Observe that when $n = 2$ we have

$f(2,a,b) = f(2-1,b,a+b)$from the recurrence above.

$= f(1,b,a+b)$from the recurrence above.

$= a+b$

$= b+a$

$= f(1,a,b) + f(0,a,b)$from the recurrence above

$= f(2-1,a,b) + f(2-2,a,b)$

Assume for $n = k$

$f(k,a,b) = f(k-1,a,b) + f(k-2,a,b)$

For $n = k+1$

$f(k+1,a,b) = f\big((k+1)-1,b,a+b\big)$from the recurrence above

$= f(k,b,a+b)$

$= f(k-1,b,a+b) + f(k-2,b,a+b)$from the induction hypothesis

$= f(k,a,b) + f(k-1,a,b)$from the recurrence above.

Hence for $n \in N$, if $n > 1$ then $f(n,a,b) = f(n-1,a,b) + f(n-2,a,b)$.

b. Proof: - By Strong form of induction

Observe that when $n = 0$, $F_0 = 0 = f(0{,}0{,}1)$

when $n = 1$, $F_1 = 1 = f(1{,}0{,}1)$

Assume when $n = k$, $F_k = f(k,0{,}1)$ if $k < n$

From the recurrence in question 6

$F_n = F_{n-1} + F_{n-2}$

$= f(n-1{,}0{,}1) + f(n-2{,}0{,}1)$

$= f(n,0{,}1)$ result of the previous proof

Hence, for any $n \in N$, $F_n = f(n,0{,}1)$

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