Homework 3

Foundations of Algorithms

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1. Electronic Submission
2. Consider the following inputs:
   $a = < 2, 4, 6 >$
   $v = 8$
   Since $l = 0$ and $h = 2$
   So, $\widehat{m} = 1$

			l	h	$\widehat{m}$
search (<2, 4, 6>, 8)	=	searchHelp (<2, 4, 6>, 8; 0, 2)	0	2	1
	=	searchHelp (<2, 4, 6>, 8; 1, 2)	1	2	1
	=	searchHelp (<2, 4, 6>, 8; 1, 2)	1	2	1
	=	searchHelp (<2, 4, 6>, 8; 1, 2)	1	2	1
	=	…	.	.	.
	=	…	.	.	.

   Since the values of $\widehat{m}$, h and l are the same in every iteration after the first iteration this will lead into an infinite loop.

3. a. $A_{1,1} = 1, A_{1,2} = 3, \ A_{2,1} = 7, A_{2,2} = 5$
   $B_{1,1} = 6, B_{1,2} = 8, B_{2,1} = 4, B_{2,2} = 2$
   The ten sums are
   $S_1 = B_{1,2} - B_{2,2} = 8 - 2 = 6$
   $S_2 = A_{1,1} + A_{1,2} = 1 + 3 = 4$
   $S_3 = A_{2,1} + A_{2,2} = 7 + 5 = 12$
   $S_4 = B_{2,1} - B_{1,1} = 4 - 6 = -2$
   $S_5 = A_{1,1} + \ A_{2,2} = 1 + 5 = 6$
   $S_6 = B_{1,1} + B_{2,2} = 6 + 2 = 8$
   $S_7 = A_{1,2} - A_{2,2} = 3 - 5 = -2$
   $S_8 = B_{2,1} + B_{2,2} = 4 + 2 = 6$
   $S_9 = A_{1,1} - A_{2,1} = 1 - 7 = -6$
   $S_{10} = B_{1,1} + B_{1,2} = 6 + 8 = 14$

   The seven products are

   $P_1 = A_{1,1} * S_1 = 1 * 6 = 6$
   $P_2 = S_2 * B_{2,2} = 4 * 2 = 8$
   $P_3 = S_3 * B_{1,1} = 12 * 6 = 72$
   $P_4 = A_{2,2} * S_4 = 5 * -2 = -10$
   $P_5 = S_5 * S_6 = 6 * 8 = 48$
   $P_6 = S_7 * B_8 = -2 * 6 = -12$
   $P_7 = S_{9,} * S_{10} = -6 * 14 = -84$

The final Matrix elements are: -
$C_{1,1} = P_5 + P_4 - P_2 + P_6 = 48 + (-10) - 8 + (-12) = 18$
$C_{1,2} = P_1 + P_2 = 6 + 8 = 14$
$C_{2,1} = P_3 + P_4 = 72 + (-10) = 62$
$C_{2,2} = P_5 + P_1 - P_3 - P_7 = 48 + 6 - 72 - (-84) = 66$

The final matrix is: -
$\begin{pmatrix} 18 & 14 \\ 62 & 66 \end{pmatrix}$

b. The functional pseudo code is as follows: -

$strassen(A, B)$:
    $if\ n\ ==\ 1$:
        $return\ A * B$
    $else$:
        $P_1 \to strassen(A_{1,1}; B_{1,2} - B_{2,2})$
        $P_2 \to strassen(A_{1,1} + A_{1,2}; B_{2,2})$
        $P_3 \to strassen(A_{2,1} + A_{2,2}; B_{1,1})$
        $P_4 \to strassen(A_{2,2}; B_{2,1} - B_{1,1})$
        $P_5 \to strassen(A_{1,1} + A_{2,2}; B_{1,1} + B_{2,2})$
        $P_6 \to strassen(A_{1,2} - A_{2,2}; B_{2,1} + B_{2,2})$
        $P_7 \to strassen(A_{1,1} - A_{2,1}; B_{1,1} + B_{1,2})$
        $C_{11} \to P_5 + P_4 - P_2 + P_6$
        $C_{12} \to P_1 + P_2$
        $C_{21} \to P_3 + P_4$
        $C_{22} \to P_5 + P_1 - P_3 + P_7$
        Output C
    $End\ If$

c. $C_{2,1} = P_3 + P_4$
$= S_3 * B_{1,1} + A_{2,2} * S_4$
$= (A_{2,1} + A_{2,2}) * B_{1,1} + A_{2,2} * (B_{2,1} - B_{1,1})$
$= A_{2,1} * B_{1,1} + A_{2,2} * B_{1,1} + A_{2,2} * B_{2,1} - A_{2,2} * B_{1,1}$
$= A_{2,1} * B_{1,1} + A_{2,2} * B_{2,1}$

d. $C_{2,2} = P_5 + P_1 - P_3 - P_7$
$= S_5 * S_6 + A_{1,1} * S_1 - S_3 * B_{1,1} + S_{9,} * S_{10}$
$= \left(A_{1,1} + A_{2,2}\right) * \left(B_{1,1} + B_{2,2}\right) + A_{1,1} * \left(B_{1,2} - B_{2,2}\right) - \left(A_{2,1} + A_{2,2}\right) * B_{1,1} - (A_{1,1} - A_{2,1}) * (B_{1,1} + B_{1,2})$
$= A_{1,1} * B_{1,1} + A_{2,2} * B_{1,1} + A_{1,1} * B_{2,2} + A_{2,2} * B_{2,2} + A_{1,1} * B_{1,2} - A_{1,1} * B_{2,2} - A_{2,1} * B_{1,1} - A_{2,2} * B_{1,1} - A_{1,1} * B_{1,1} + A_{2,1} * B_{1,1} - A_{1,1} * B_{1,2} + A_{2,1} * B_{1,2}$
$= A_{2,2} * B_{2,2} + A_{2,1} * B_{1,2}$

e. Given that: $T(n) = 7T\left(\frac{n}{2}\right) + \frac{9}{2}n^2$
Assuming $n = 2^m$
$T(2^m) = 7T\left(\frac{2^m}{2}\right) + \frac{9}{2}2^{m2}$
$= 7T(2^{m-1}) + \frac{9}{2}(2^m)^2$
$= 7\left(7T(2^{m-2}) + \frac{9}{2}(2^{m-1})^2\right) + \frac{9}{2}(2^m)^2$
$= 7\left(7\left(7T(2^{m-3}) + \frac{9}{2}(2^{m-2})^2\right) + \frac{9}{2}(2^{m-1})^2\right) + \frac{9}{2}(2^m)^2$

$= 7^3T(2^{m-3}) + 7^2\frac{9}{2}(2^{m-2})^2 + 7^1\frac{9}{2}(2^{m-1})^2 + 7^0\frac{9}{2}(2^m)^2 \leftarrow$

$Identifying\ the\ pattern$

$= 7^kT\left(2^{m-k}\right) + 7^{k-1}\frac{9}{2}\left(2^{m-(k-1)}\right)^2 + 7^{(k-2)}\frac{9}{2}\left(2^{m-(k-2)}\right)^2 + 7^{m-m}\frac{9}{2}\left(2^{m-(k-m)}\right)^2$

$= 7^kT\left(2^{m-k}\right) + \sum_{i=0}^{k-1}\frac{9}{2}7^i(2^{m-1})^2$

Substituting $k = m$

$= 7^mT(2^{m-m}) + \sum_{i=0}^{m-1}\frac{9}{2}7^i(2^{m-1})^2$

$= 7^mT(2^0) + \frac{9}{2}\sum_{i=0}^{m-1}7^i(2^{m-1})^2$

$= 7^mT(1) + \frac{9}{2}\sum_{i=0}^{m-1}7^i\left(\frac{2^m}{2^i}\right)^2$

$= 7^m + \frac{9}{2}\sum_{i=0}^{m-1}7^i\,(2^m)^2/(2^i)^2$

$= 7^m + \frac{9}{2}(2^m)^2\sum_{i=0}^{m-1}7^i\,/2^{2^i}$

$= 7^m + \frac{9}{2}(2^m)^2\sum_{i=0}^{m-1}7^i\,/4^i$

$= 7^m + \frac{9}{2}(2^m)^2\sum_{i=0}^{m-1}\left(\frac{7}{4}\right)^i$

$= 7^m + \frac{9}{2}(2^m)^2\left(\frac{\left(\left(\frac{7}{4}\right)^m - 1\right)}{\left(\frac{7}{4}\right) - 1}\right)$

$= 7^m + \frac{9}{2}(2^m)^2\left(\frac{4}{3}\right)\left(\left(\frac{7}{4}\right)^m - 1\right)$

$= (2^m)^\beta + \frac{9}{2}(2^m)^2\left(\frac{4}{3}\right)\left((2^m)^\alpha - 1\right)$

$where\ \alpha = \lg\left(\frac{7}{4}\right) = 0.8073549\ and\ \beta = \lg(7) = 2.8073549$

$= (n)^\beta + \frac{9}{2}(n)^2\left(\frac{4}{3}\right)\left((n)^\alpha - 1\right)$

Hence $T(n) = O(n^{2.81})$

f. Assuming $2^{k-1} < n < N = 2^k$

Now, we must extend original $n * n$ matrices to $N * N$ matrices by adding zeros for the Strassen's algorithm to work on next matrix.

To remove all the zeroes we need $O(n^2)$

Since $2^{k-1} < n$ it implies that $N < 2n$.

Therefore the runtime becomes is $\theta\left((2n)^{lg7}\right) = \theta\left(2^{lg7}n^{lg7}\right) = \theta\left(n^{lg7}\right)$

g. The three multiplications are

$A = (a+b)*(c+d) = ac + ad + bc + bd$

$B = ac$

$C = bd$

The complex number is then achieved by

$(B - C) + (A - B - C)i$

4. $T(1) = 0, T(n) = T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + T\left(\left\lceil\frac{n}{2}\right\rceil\right) + n$

a. For every even $n \in N$ there exists a $k \in N$ such that $n = 2k$

$\left(\left\lfloor\frac{n+1}{2}\right\rfloor\right) = \left(\left\lfloor\frac{2k+1}{2}\right\rfloor\right)$

$= \left(\left\lfloor k + \frac{1}{2}\right\rfloor\right)$

$= k$

$= \lceil k\rceil$

$= \left\lceil\frac{2k}{2}\right\rceil$

$= \left\lceil \frac{n}{2} \right\rceil$ since $n = 2k$

For every odd $n \in N$ there exists a $k \in N$ such that $n = 2k$+1

$\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) = \left(\left\lfloor \frac{2k+1+1}{2} \right\rfloor\right)$
$= \left(\left\lfloor \frac{(2k+2)}{2} \right\rfloor\right)$
$= (\lfloor k+1 \rfloor)$
$= k+1$
$= \left\lceil k + \frac{1}{2} \right\rceil$
$= \left\lceil \frac{2k+1}{2} \right\rceil$
$= \left\lceil \frac{n}{2} \right\rceil$ since $n = 2k$+1

Hence for every $\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) = \left\lceil \frac{n}{2} \right\rceil$

b. For every even $n \in N$ there exists a $k \in N$ such that $n = 2k$

$\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = \left(\left\lfloor \frac{2k}{2} \right\rfloor + 1\right)$
$= (\lfloor k \rfloor + 1)$
$= k+1$
$= \left\lceil k + \frac{1}{2} \right\rceil$
$= \left\lceil \frac{2k+1}{2} \right\rceil$
$= \left\lceil \frac{n+1}{2} \right\rceil$ since $n = 2k$

For every odd $n \in N$ there exists a $k \in N$ such that $n = 2k$+1

$\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = \left(\left\lfloor \frac{2k+1}{2} \right\rfloor + 1\right)$
$= \left(\left\lfloor k + \frac{1}{2} \right\rfloor + 1\right)$
$= \lfloor k \rfloor + 1$
$= k+1$
$= \lceil k+1 \rceil$
$= \left\lceil \frac{2k+2}{2} \right\rceil$
$= \left\lceil \frac{2k+1+1}{2} \right\rceil$
$= \left\lceil \frac{n+1}{2} \right\rceil$ since $n = 2k$

Hence for every $\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = \left\lceil \frac{n+1}{2} \right\rceil$

c. Let $D(n) = T(n+1) - T(n)$. Prove that $D(1) = 2, D(n) = D\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$.

So,

$D(1) = T(1+1) - T(1)$
$= T(2) - T(1)$
$= T(2) - 0$
$= T\left(\left\lfloor \frac{2}{2} \right\rfloor\right) + T\left(\left\lceil \frac{2}{2} \right\rceil\right) + 2$
$= 0 + 0 + 2$
$= 2$

$D(n) = T(n+1) - T(n)$
$= T\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n+1}{2} \right\rceil\right) + n + 1 - \left(T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + n\right)$
$= T\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n+1}{2} \right\rceil\right) + n + 1 - T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - T\left(\left\lceil \frac{n}{2} \right\rceil\right) - n$
$= T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lceil \frac{n+1}{2} \right\rceil\right) + n + 1 - T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - T\left(\left\lceil \frac{n}{2} \right\rceil\right) -$
$n \quad substituting\ from\ proof\ 4a$

$= T\left(\left\lceil\frac{n+1}{2}\right\rceil\right) - T\left(\left\lceil\frac{n}{2}\right\rceil\right) + 1$
$= T\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right) - T\left(\left\lceil\frac{n}{2}\right\rceil\right) + 1 \quad substituting\ from\ proof\ b$
$= D\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1$

d. Proof: -

Observe:

$D(1) = 2$
$= 0 + 2$
$= \lfloor \lg(1) \rfloor + 2$

Assume for,

$D(k) = \lfloor \lg(k) \rfloor + 2$

From 4c. we know that

$D(n) = D\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1$

Suppose $\left\lfloor\frac{n}{2}\right\rfloor = k$ for any $k \in N$ and $k \geq 1$

$D(n) = D(k) + 1$
$= \lfloor \lg(k) \rfloor + 2 + 1$
$= \left\lfloor \lg\left(\frac{n}{2}\right) \right\rfloor + 2 + 1$
$= \lfloor \lg n - \lg 2 \rfloor + 2 + 1$
$= \lfloor \lg n \rfloor - \lfloor 1 \rfloor + 2 + 1$
$= \lfloor \lg n \rfloor + 2$

e. $T(n) - T(1) = \sum_{k=1}^{n-1} D(k)$

$T(n) - T(1) = T(n) - T(n-1) + T(n-2) - T(n-3) \ldots \ldots - T(2) + T(2) - T(1)$
$= D(n-1) + D(n-2) + D(n-3) \ldots \ldots D(1) \quad mentioned\ in\ 4c$
$= \sum_{k=1}^{n-1} D(k)$

Since $T(1) = 0$

$T(n) = \sum_{k=1}^{n-1} D(k)$
$= \sum_{k=1}^{n-1} (\lfloor \lg k \rfloor + 2) \quad mentioned\ in\ 4d$

f. $T(n) = \sum_{k=1}^{n-1} (\lfloor \lg k \rfloor + 2)$

$= \sum_{k=1}^{n-1} (\lfloor \lg k \rfloor + 2) - (\lfloor \lg n \rfloor + 2)$
$= (\lfloor \lg 1 \rfloor + 2) + (\lfloor \lg 2 \rfloor + 2) + (\lfloor \lg 3 \rfloor + 2) \ldots \ldots (\lfloor \lg n - 1 \rfloor + 2) + (\lfloor \lg n \rfloor + 2) - (\lfloor \lg n \rfloor + 2)$
$= \lfloor \lg 1 \rfloor + \lfloor \lg 2 \rfloor + \lfloor \lg 3 \rfloor + \lfloor \lg 4 \rfloor \ldots \ldots \lfloor \lg(n-1) \rfloor + \lfloor \lg n \rfloor + 2n - (\lfloor \lg n \rfloor + 2)$
$= \lfloor \lg n! \rfloor + 2n - (\lfloor \lg n \rfloor + 2)$
$= \lfloor \lg n! \rfloor + 2n - \lfloor \lg n \rfloor - 2$
$= \lfloor \lg n! \rfloor - \lfloor \lg n \rfloor + 2(n-1)$
$= O\big(n(\lg n)\big) + O(\lg n) + O(n)$
$= O(n \lg n)$

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6.Electronice Submissions