CMB Observables for inferring properties of reionization

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Contents

Chapter 1

A Brief introduction to polarization

This section is referenced from S. Chandrashekhar's Radiative Transfer:

The motivation of this excercise: Right now I will not investing in the question as to how a scattering process renders an unpolarized light into a state of partial polarization. In this section our motive is to obtain the stokes parameters starting from an electric field.

Description of an elliptically polarized beam: Before introducing the mathematical representation of elliptically polarized beam, we should note two characteristics of an elliptically polarized beam:

- Ratio of amplitudes of the components in any two direction at right angles to each other are absolute constants.
- Difference in phases of the components in any two direction at right angles to each other are absolute constants.

Mathematically this is represented as:

$$\xi_l = \xi_l^{(0)} \sin(\omega t - \epsilon_l) \quad \text{and} \quad \xi_r = \xi_r^{(0)} \cos(\omega t - \epsilon_r) \tag{1.1}$$

Refer to Figure 1.1. Here, ξ_l and ξ_r are components of electric(or magnetic) field in the $l - r$ plane. In the equation (1.1) note that $\frac{\xi_l^{(0)}}{\xi_r^{(0)}}$ = constant and $\epsilon_l - \epsilon_r$ = constant owing to the requirement of characters of the elliptical beam described earlier.

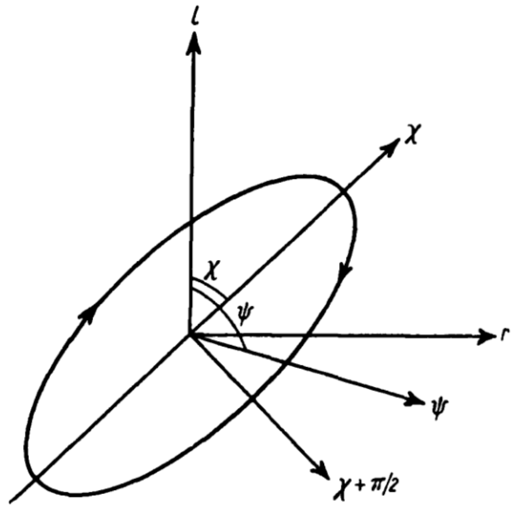

Figure 1.1: Represenation of polarized beam on an $l - r$ basis

If the semi major axis of the ellipse is at angle χ from the direction l, then we can transport the basis from $l - r$ plane to $\chi - \chi + \frac{\pi}{2}$ plane by a rotation operator such that

$$\begin{bmatrix} \xi_l \\ \xi_r \end{bmatrix} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix} \begin{bmatrix} \xi_\chi \\ \xi_{\chi+\frac{\pi}{2}} \end{bmatrix} \tag{1.2}$$

Once our basis vectors align with the semi-major and semi-minor axes, the description of the electric field ellipse is reduced to

$$\xi_\chi = \xi^{(0)} \cos\beta \sin\omega t \text{ and } \xi_{\chi+\frac{\pi}{2}} = \xi^{(0)} \sin\beta \cos\omega t \tag{1.3}$$

Here β is the angle whose tangent refers to the ratio of the axes of the ellipse traced by the end point of the electric vector. Refer to Figure 1.2. Such that magnitude $\beta \in [0, \frac{\pi}{2}]$ and the sign of β is positive or negative as the polarization is right handed or left-handed.

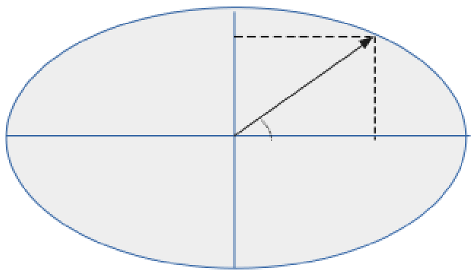

Figure 1.2: Representation of ellipse on an $\chi - \chi + \frac{\pi}{2}$ basis

Additionally, $\xi^{(0)}$ is defined as

$$\left[\xi^{(0)}\right]^2 = \left[\xi_l^{(0)}\right]^2 + \left[\xi_r^{(0)}\right]^2 = I_l + I_r = I \tag{1.4}$$

where I is in intensity of the beam.

We used equation (1.2) to the equations (1.1) and (1.3) , expanding the equation we have

$$\xi_l = \xi^{(0)} \left(\cos\beta \cos\chi \sin\omega t - \sin\beta \sin\chi \cos\omega t\right) \tag{1.5}$$

and

$$\xi_r = \xi^{(0)} \left(\cos\beta \sin\chi \sin\omega t + \sin\beta \cos\chi \cos\omega t\right) \tag{1.6}$$

Let us simplify only equation (1.5). By substituting representation of ξ_l

assuming solving for $\omega t = 0$ we get

$$\xi_l^{(0)} \sin(\omega t - \epsilon_l) = \xi^{(0)} \left(\cos\beta \cos\chi \sin\omega t - \sin\beta \sin\chi \cos\omega t\right)$$
$$\implies \xi_l^{(0)} \sin(-\epsilon_l) = \xi^{(0)} \left(-\sin\beta \sin\chi\right)$$

assuming solving for $\omega t = \pi/2$ we get

$$\xi_l^{(0)} \sin(\omega t - \epsilon_l) = \xi^{(0)} \left(\cos\beta \cos\chi \sin\omega t - \sin\beta \sin\chi \cos\omega t\right)$$
$$\implies \xi_l^{(0)} \cos(\epsilon_l) = \xi^{(0)} \left(\cos\beta \cos\chi\right)$$

Therefore,

$$\xi_l^{(0)^2} = \xi^{(0)^2}\left(\sin^2\beta\sin^2\chi + \cos^2\beta\cos^2\chi\right) \tag{1.7}$$

Repeating the same for equation (1.6) we get the form

$$\xi_l^{(0)}\begin{bmatrix}\cos\epsilon_l\\ \sin\epsilon_l\end{bmatrix} = \xi^{(0)}\begin{bmatrix}\cos\beta\cos\chi\\ \sin\beta\sin\chi\end{bmatrix} \tag{1.8}$$

and

$$\xi_r^{(0)}\begin{bmatrix}\cos\epsilon_r\\ \sin\epsilon_r\end{bmatrix} = \xi^{(0)}\begin{bmatrix}\cos\beta\sin\chi\\ -\sin\beta\cos\chi\end{bmatrix} \tag{1.9}$$

Therefore,

$$\xi_r^{(0)^2} = \xi^{(0)^2}\left(\cos^2\beta\sin^2\chi + \sin^2\beta\cos^2\chi\right) \tag{1.10}$$

and taking ratios we get, $\tan\epsilon_l = \tan\beta\tan\chi$ and $\tan\epsilon_r = -\tan\beta\cot\chi$

1.1 Obtaining the stokes parameters

We define Stokes parameters as :

- $I = I_l + I_r$ Represents the radiation intensity
- $Q = I_l - I_r$
- $U = 2\sqrt{I_l}\sqrt{I_r}\cos(\epsilon_l - \epsilon_r)$
- $V = 2\sqrt{I_l}\sqrt{I_r}\sin(\epsilon_l - \epsilon_r)$ Represents the circular polarization

From equations (1.7) and (1.10) we get the simplified results for Stokes calculation is as follows

$$\begin{aligned}
I &\equiv \xi_l^{(0)^2} + \xi_r^{(0)^2} = \xi^{(0)^2} = I_l + I_r\\
Q &\equiv \xi_l^{(0)^2} - \xi_r^{(0)^2} = \xi^{(0)^2}\cos 2\beta\cos 2\chi = I_l - I_r\\
U &\equiv 2\xi_l^{(0)}\xi_r^{(0)}\cos(\epsilon_l - \epsilon_r) = \xi^{(0)^2}\cos 2\beta\cos 2\chi = (I_l - I_r)\tan 2\chi\\
V &\equiv 2\xi_l^{(0)}\xi_r^{(0)}\sin(\epsilon_l - \epsilon_r) = \xi^{(0)^2}\sin 2\beta = (I_l - I_r)\tan 2\beta\sec 2\chi
\end{aligned}$$