CMB Observables for inferring properties of reionization

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Contents

Chapter 1

A Brief introduction to polarization

A lighter introduction to polarization states.
Derivation of Stokes parameters
This section is referenced from S. Chandrashekhar's Radiative Transfer:

The motivation of this excercise: Right now I will not investing in the question as to how a scattering process renders an unpolarized light into a state of partial polarization. In this section our motive is to obtain the stokes parameters starting from an electric field.

Description of an elliptically polarized beam: Before introducing the mathematical representation of elliptically polarized beam, we should note two characteristics of an elliptically polarized beam:

- Ratio of amplitudes of the components in any two direction at right angles to each other are absolute constants.
- Difference in phases of the components in any two direction at right angles to each other are absolute constants.

Mathematically this is represented as:

$$\xi_l = \xi_l^{(0)} \sin(\omega t - \epsilon_l) \quad \text{and} \quad \xi_r = \xi_r^{(0)} \cos(\omega t - \epsilon_r) \tag{1.1}$$

Add a figure of waveform along l and r. This will highlight the phase shift ϵ_l and ϵ_r.

Refer to Figure 1.1. Here, ξ_l and ξ_r are components of electric(or magnetic) field in the $l - r$ plane. In the equation (1.1) note that $\frac{\xi_l^{(0)}}{\xi_r^{(0)}} =$ constant and $\epsilon_l - \epsilon_r =$ constant owing to the requirement of characters of the elliptical beam described earlier.

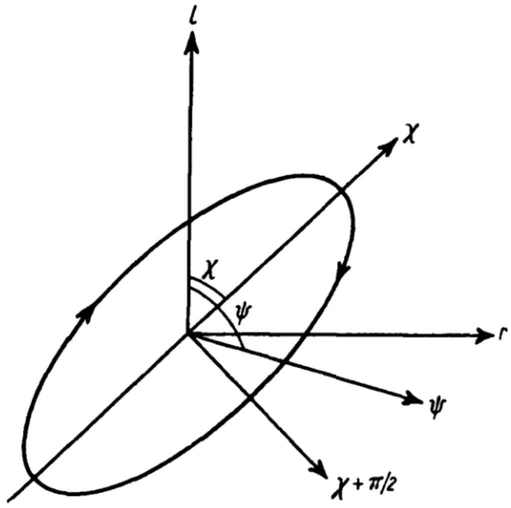

Figure 1.1: Represenation of polarized beam on an $l - r$ basis

If the semi major axis of the ellipse is at angle χ from the direction l, then we can transport the basis from $l - r$ plane to $\chi - \chi + \frac{\pi}{2}$ plane by a rotation operator such that

$$\begin{bmatrix} \xi_l \\ \xi_r \end{bmatrix} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix} \begin{bmatrix} \xi_\chi \\ \xi_{\chi+\frac{\pi}{2}} \end{bmatrix} \tag{1.2}$$

Once our basis vectors align with the semi-major and semi-minor axes, the description of the electric field ellipse is reduced to

$$\xi_\chi = \xi^{(0)} \cos\beta \sin\omega t \text{ and } \xi_{\chi+\frac{\pi}{2}} = \xi^{(0)} \sin\beta \cos\omega t \tag{1.3}$$

Here β is the angle whose tangent refers to the ratio of the axes of the ellipse traced by the end point of the electric vector. Refer to Figure 1.2. Such that magnitude $\beta \in [0, \frac{\pi}{2}]$ and the sign of β is positive or negative as the polarization is right handed or left-handed.

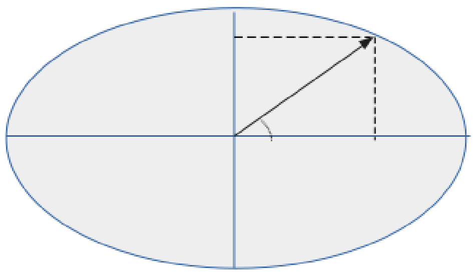

Figure 1.2: Representation of ellipse on an $\chi - \chi + \frac{\pi}{2}$ basis

Additionally, $\xi^{(0)}$ is defined as

$$\left[\xi^{(0)}\right]^2 = \left[\xi_l^{(0)}\right]^2 + \left[\xi_r^{(0)}\right]^2 = I_l + I_r = I \tag{1.4}$$

where I is in intensity of the beam.

We used equation (1.2) to the equations (1.1) and (1.3) , expanding the equation we have

$$\xi_l = \xi^{(0)} \left(\cos\beta \cos\chi \sin\omega t - \sin\beta \sin\chi \cos\omega t\right) \tag{1.5}$$

and

$$\xi_r = \xi^{(0)} \left(\cos\beta \sin\chi \sin\omega t + \sin\beta \cos\chi \cos\omega t\right) \tag{1.6}$$

Let us simplify only equation (1.5). By substituting representation of ξ_l

$$\begin{aligned}
&\text{assuming solving for } \omega t = 0 \text{ we get}\\
&\xi_l^{(0)} \sin(\omega t - \epsilon_l) = \xi^{(0)} \left(\cos\beta \cos\chi \sin\omega t - \sin\beta \sin\chi \cos\omega t\right)\\
&\implies \xi_l^{(0)} \sin(-\epsilon_l) = \xi^{(0)} \left(-\sin\beta \sin\chi\right)\\
&\text{assuming solving for } \omega t = \pi/2 \text{ we get}\\
&\xi_l^{(0)} \sin(\omega t - \epsilon_l) = \xi^{(0)} \left(\cos\beta \cos\chi \sin\omega t - \sin\beta \sin\chi \cos\omega t\right)\\
&\implies \xi_l^{(0)} \cos(\epsilon_l) = \xi^{(0)} \left(\cos\beta \cos\chi\right)
\end{aligned}$$

Therefore,

$$\xi_l^{(0)^2} = \xi^{(0)^2} \left(\sin^2\beta \sin^2\chi + \cos^2\beta \cos^2\chi\right) \tag{1.7}$$

Repeating the same for equation (1.6) we get the form

$$\xi_l^{(0)} \begin{bmatrix} \cos\epsilon_l \\ \sin\epsilon_l \end{bmatrix} = \xi^{(0)} \begin{bmatrix} \cos\beta \cos\chi \\ \sin\beta \sin\chi \end{bmatrix} \tag{1.8}$$

and

$$\xi_r^{(0)} \begin{bmatrix} \cos\epsilon_r \\ \sin\epsilon_r \end{bmatrix} = \xi^{(0)} \begin{bmatrix} \cos\beta \sin\chi \\ -\sin\beta \cos\chi \end{bmatrix} \tag{1.9}$$

Therefore,

$$\xi_r^{(0)^2} = \xi^{(0)^2} \left(\cos^2\beta \sin^2\chi + \sin^2\beta \cos^2\chi\right) \tag{1.10}$$

and taking ratios we get, $\tan\epsilon_l = \tan\beta \tan\chi$ and $\tan\epsilon_r = -\tan\beta \cot\chi$

1.1 Obtaining the stokes parameters

We define Stokes parameters as :

- $I = \xi_l^{(0)^2} + \xi_r^{(0)^2}$ Represents the radiation intensity
- $Q = \xi_l^{(0)^2} - \xi_r^{(0)^2}$
- $U = 2\xi_l^{(0)}\xi_r^{(0)} \cos(\epsilon_l - \epsilon_r)$
- $V = 2\xi_l^{(0)}\xi_r^{(0)} \sin(\epsilon_l - \epsilon_r)$ Represents the circular polarization

From equations (1.7) and (1.10) we get the simplified results for Stokes calculation as follows

$$I \equiv \xi_l^{(0)^2} + \xi_r^{(0)^2} = \xi^{(0)^2} = I_l + I_r \tag{1.11}$$

$$Q \equiv \xi_l^{(0)^2} - \xi_r^{(0)^2} = \xi^{(0)^2} \cos 2\beta \cos 2\chi = I_l - I_r \tag{1.12}$$

$$U \equiv 2\xi_l^{(0)}\xi_r^{(0)} \cos(\epsilon_l - \epsilon_r) = \xi^{(0)^2} \cos 2\beta \sin 2\chi = (I_l - I_r) \tan 2\chi \tag{1.13}$$

$$V \equiv 2\xi_l^{(0)}\xi_r^{(0)} \sin(\epsilon_l - \epsilon_r) = \xi^{(0)^2} \sin 2\beta = (I_l - I_r) \tan 2\beta \sec 2\chi \tag{1.14}$$

From the above four relations, we can go ahead and prove that

$$
\begin{aligned}
Q^2 + U^2 + V^2 = & \\
&= {\xi^{(0)}}^4 \cos^2 2\beta \cos^2 2\chi + {\xi^{(0)}}^4 \cos^2 2\beta \sin^2 2\chi + {\xi^{(0)}}^4 \sin^2 2\beta \\
&= {\xi^{(0)}}^4 \cos^2 2\beta \left(\sin^2 2\chi + \cos^2 2\chi\right) + {\xi^{(0)}}^4 \sin^2 2\beta \\
&= {\xi^{(0)}}^4 = I^2
\end{aligned}
$$

It is important to remember that, an arbitrary set of stokes parameters can always be considered as a sum of the stokes parameters of two plane waves : the first completely unpolarized i.e. (Q=U=V=0) and the second being the elliptically unpolarized. In general, $I^2 \geq Q^2 + U^2 + V^2$.

1.1.1 Rotation of reference frame

Before understanding the properties of the CMB polarization, we first ask the question how does Stokes parameters behave under rotation of reference frame of the polarized beam from $\chi \longrightarrow \chi - \alpha$. From equations (1.14) we can write (also using $I = {\xi^{(0)}}^2$) the rotation as an operation on I and the transformation is given as $\mathrm{I}' = \mathrm{LI}$

$$
\begin{aligned}
I' &= I \\
Q' &= I \cos 2\beta \cos 2(\chi - \alpha) \\
U' &= I \cos 2\beta \sin 2(\chi - \alpha) \\
V' &= V
\end{aligned}
$$

We note that rotation of the reference frame affects only the Q and U parameters. It will be also important to learn that Thomson scattering does not produce circular polarization, therefore, we should not be concerned with what happens to V under frame transform.

$$
\begin{aligned}
Q' &= \frac{Q}{\cos 2\chi} \left[\cos 2\chi \cos 2\alpha + \sin 2\chi \sin 2\alpha\right] \\
&= Q \left[\cos 2\alpha + \tan 2\chi \sin 2\alpha\right] \qquad \text{We know } \frac{U}{Q} = \tan 2\chi, \text{ using which} \\
&= \left[Q \cos 2\alpha + U \sin 2\alpha\right]
\end{aligned}
$$

Similarly we get $U' = -Q \sin 2\alpha + U \cos 2\alpha$, Therefore the rotation matrix is given by:

$$
\mathrm{L} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha \\ 0 & -\sin 2\alpha & \cos 2\alpha \end{bmatrix} \tag{1.15}
$$

Finally the frame transformation is given as

$$
\begin{bmatrix} Q' \\ U' \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} Q \\ U \end{bmatrix} \tag{1.16}
$$

Additionally, it is important to note that if a matrix satisfies the following property

$$
P'_{ij} = A_i^m A_j^n P_{mn} \tag{1.17}
$$

then the matrix qualifies to be a second rank tensor.

Therefore, if

$$P_{ab} = \begin{bmatrix} Q & U \\ U & -Q \end{bmatrix} \tag{1.18}$$

satisfies the equation (1.17). Then Q and U are components of a second rank, symmetric and trace-free tensor.

We can verify this as following

$$P'_{ij} \equiv \begin{bmatrix} Q' & U' \\ U' & -Q' \end{bmatrix} = \begin{bmatrix} U\sin 2\alpha + Q\cos 2\alpha & U\cos 2\alpha - Q\sin 2\alpha \\ U\cos 2\alpha - Q\sin 2\alpha & -U\sin 2\alpha - Q\cos 2\alpha \end{bmatrix} \text{ and from eq. (1.16)} = \begin{bmatrix} Q' & U' \\ U' & -Q' \end{bmatrix} \tag{1.19}$$