

Radiative processes in astrophysics notes

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Introduction

The professor, Roberto Turolla, will follow the pdf of the book by Rybicki and Lightman [RL79] on his screen. It is available for free.

Understanding radiative processes is fundamental for the analysis of several phenomena: for example, in the Crab nebula the main process is Synchrotron radiation, in the Coma cluster we have Bremsstrahlung, in Cygnus-X1 we have Compton scattering.

Even in the era of multimessenger astrophysics, most of the information we receive still comes from electromagnetic radiation. The required background for this course is classical EM, special relativity and the basics of atomic structure.

The exam is an oral one. The lectures will be recorded and put on the Moodle until the emergency ends, every Wednesday and Thursday. The duration of recorded lectures will be shorter than the duration of the lectures we would have in the classroom.

Chapter 1

Fundamentals of radiative transfer

1.1 Basic properties of the EM spectrum

Electromagnetic radiation can be decomposed into a spectrum according to either the frequency ν or the wavelength λ ; these are connected by $c = \lambda\nu$, where c is the speed of light.

Sometimes we give the energy of the photons, which can be found using Planck's constant h : $E = h\nu$.

We conventionally divide the spectrum into bands: γ -rays, X-rays, ultraviolet light, visible light, infrared radiation, radio band.

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1.1.1 The radiative flux

Let us consider an area element dA , through which radiation passes for a time dt : the energy will be proportional to both dA and dt , so we define the radiative flux F by saying that the energy is equal to $F dA dt$. Of course, we need to account for orientation: if the surface is not perpendicular to the source the energy is less.

Let us consider a pointlike source, and draw two spherical surfaces of radii r_1 and r_2 along which we compute the flux: if there is no energy loss we must have

$$F(r_1)A_1 dt = F(r_2)A_2 dt \quad (1.1.1a)$$

$$F(r_1)4\pi r_1^2 = F(r_2)4\pi r_2^2 \quad (1.1.1b)$$

$$F(r_1) = F(r_2) \frac{r_2^2}{r_1^2}. \quad (1.1.1c)$$

This tells us that **the flux emitted from a pointlike source decreases like r^{-2}** .

The flux of energy is a measure of all the energy which passes through the surface; however we can get a more detailed description. We cannot consider photons at a specific frequency: the set has measure 0, we must use the language of probability densities. We look at a “pencil” of radiation: all the radiation coming from a solid angle $d\Omega$ over an area dA and carried by photons of frequencies between ν and $\nu + d\nu$.

Note that the solid angle $d\Omega$ depends on both angles we use to describe a direction (typically θ, φ).

So, we define the *specific intensity of brightness* I_ν by

$$dE \stackrel{\text{def}}{=} I_\nu dA dt d\Omega d\nu . \quad (1.1.2)$$

This will depend on position (where we put the detector area with respect to any sources) and on direction (where we look).

We usually neglect the time-dependence. The units of this quantity are those of energy per unit time, area, frequency, solid angle.

How do we account for the direction? The differential flux for radiation coming with an angle θ to the normal is proportional to the dot product of the normal and the incidence direction:

$$dF_\nu = I_\nu \cos(\theta) d\Omega , \quad (1.1.3)$$

so the total net flux is

$$F_\nu = \int I_\nu \cos(\theta) d\Omega . \quad (1.1.4)$$

This is about energy, but we can define the momentum flux per unit time per unit area (which is the pressure) with the same procedure; we get an additional factor of $\cos \theta$ since \vec{p} is a vector, and we are interested in its component along the normal of the surface. So, the global formula for this pressure is

$$P_\nu = \frac{1}{c} \int I_\nu \cos^2 \theta d\Omega . \quad (1.1.5)$$

These are *moments*: in general, a moment for a direction-dependent quantity like the intensity is something in the form

$$n\text{-th moment} = \int I_\nu \cos^n \theta d\Omega . \quad (1.1.6)$$

These are frequency dependent; the corresponding *grey* (that is, frequency-integrated) quantities are in the form

$$F = \int F_\nu d\nu . \quad (1.1.7)$$

1.2 Radiative energy density

We define the energy density per unit solid angle, $u_\nu(\Omega)$ by: $dE = u_\nu(\Omega) dV d\Omega d\nu$. This is the differential amount of energy in the volume dV , carried by radiation coming from the solid angle $d\Omega$ which has energies between ν and $\nu + d\nu$.

We consider a cylinder for our volume, its axis being aligned with the direction the radiation is coming from. Its volume can be expressed as $dV = dA c dt$, where dt is the time taken by light to cross the height of the cylinder.

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We can also express the differential energy using the definition of the specific intensity: then, we can compare the following two equations:

$$dE = u_\nu(\Omega) c dA dt d\Omega d\nu \quad (1.2.1a)$$

$$= I_\nu dA dt d\Omega d\nu , \quad (1.2.1b)$$

which allows us to conclude that $u_\nu = I_\nu / c$. Also, if we want to get the total energy density we just need to integrate over the volume of the whole sphere:

$$u_\nu = \frac{1}{c} \int I_\nu(\Omega) d\Omega \stackrel{\text{def}}{=} \frac{4\pi}{c} J_\nu , \quad (1.2.2)$$

where J_ν is the *mean intensity*: $J_\nu = \langle I_\nu \rangle_\Omega$.

We can also integrate over frequencies to get the total energy density:

$$u = \int u_\nu d\nu = \frac{4\pi}{c} \int J_\nu d\nu . \quad (1.2.3)$$

1.2.1 Isotropic radiation field

An isotropic radiation field is one for which the specific intensity does not depend on angles. Let us start from the definitions of u_ν and P_ν :

$$u_\nu = \int \frac{I_\nu}{c} d\Omega = \frac{4\pi J_\nu}{c} \quad (1.2.4a)$$

$$P_\nu = \int \frac{I_\nu}{c} \cos^2 \theta d\Omega = \int \frac{I_\nu}{c} \cos^2 \theta \sin \theta d\theta d\varphi , \quad (1.2.4b)$$

and let us use the assumption that I_ν does not depend on Ω : so we can bring it out of the integrals, to find

$$u_\nu = 4\pi \frac{I_\nu}{c} \quad (1.2.5a)$$

$$P_\nu = -2\pi \frac{I_\nu}{c} \int \cos^2 \theta d \cos \theta = 2\pi \frac{I_\nu}{c} \int_{-1}^1 x^2 dx \quad (1.2.5b)$$

$$= \frac{4\pi}{3} \frac{I_\nu}{c} , \quad (1.2.5c)$$

The differential is negative, but we swap the integration bounds.

which gives us the result we sought:

$$P_\nu = \frac{u_\nu}{3} . \quad (1.2.6)$$

1.2.2 Specific intensity along a ray

We wish to see how the specific intensity I_ν changes along a beam of light rays. Let us consider two positions 1,2 along the beam, separated by a distance R . Then, by definition we will have, for $i = 1, 2$:

$$dE_i = I_{\nu,i} dA_i dt_i d\Omega_i d\nu_i . \quad (1.2.7)$$

First, we make the assumption of the gravitational field being weak: therefore time dilation is negligible, so $dt_1 = dt_2$ and $d\nu_1 = d\nu_2$. Now, we ask these two expressions to describe the same beam: the same photons will pass through dA_1 and dA_2 . Therefore, by conservation of energy, $dE_1 = dE_2$.

This means that

$$I_{\nu,1} dA_1 d\Omega_1 = I_{\nu,2} dA_2 d\Omega_2 . \quad (1.2.8)$$

We can treat the photons' motion as time-reversal symmetric: so, whether they pass through dA_1 or dA_2 first is irrelevant. The linear scale of the differential area element is infinitesimal, while the separation between the two points, R , is macroscopic: so, we can consider all the photons which come through dA_1 to be coming from a point source from position 2 and vice versa. This will define the angular size as seen from the "pointlike" position 2, $d\Omega_2$.

Therefore, the differential solid angle will look like

$$d\Omega_2 = \frac{dA_1}{R^2} , \quad (1.2.9)$$

and we can apply the same reasoning reversing the photons' motion to find the same, alternate relation with $(1 \leftrightarrow 2)$. We can use this to write

$$I_{\nu,1} \frac{dA_1}{d\Omega_2} = I_{\nu,2} \frac{dA_2}{d\Omega_1} \quad (1.2.10a)$$

$$I_{\nu,1} R^2 = I_{\nu,2} R^2 \quad (1.2.10b)$$

$$I_{\nu,1} = I_{\nu,2} . \quad (1.2.10c)$$

This means that, under our assumptions, the specific intensity is conserved:

$$\frac{dI_\nu}{ds} = 0 , \quad (1.2.11)$$

where s is a parameter describing the light ray's trajectory. This is useful since, if the variation of the specific intensity is zero in a vacuum, then its variation in the presence of matter will only be due to transfer phenomena, and the sign of the variation will describe whether energy is being added or removed.

1.3 Radiative transfer

In general, as radiation passes through matter, its specific intensity changes. This is due to emission and absorption, but also to scattering, which preserves the total number of photons: even in the low-energy limit it can change the angular distribution of the radiation, and in general it also changes the energy of the photon.

1.3.1 Emission

Emission is a process through which photons are created. We can define the grey emission coefficient j and the monochromatic emission coefficient j_ν as:

$$dE = j dV d\Omega dt \quad (1.3.1a)$$

$$dE = j_\nu dV d\Omega dt d\nu, \quad (1.3.1b)$$

they quantify the energy added to the radiation field per unit volume, solid angle (in order to account for the direction of emission) and unit time. For the monochromatic coefficient, we restrict ourselves to radiation emitted in the range from ν to $\nu + d\nu$.

In the case of an isotropic emission we can integrate over the solid angle to find

$$P_\nu = 4\pi j_\nu, \quad (1.3.2)$$

the radiated power per unit volume and frequency.

Another useful concept is the emissivity ϵ_ν : it is the energy added to the radiation field per unit time, frequency and mass in the directions described by the solid angle $d\Omega$. We express the infinitesimal mass as $dm = \rho dV$, so that in the case of isotropic emission we have

$$dE = \epsilon_\nu \rho dV dt d\nu \frac{d\Omega}{4\pi}, \quad (1.3.3)$$

so the emissivity ϵ_ν and the emission coefficient j_ν are connected by

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}. \quad (1.3.4)$$

We wish to describe the variation in specific intensity due to this emission. Let us consider a beam of cross section dA going through a length ds , so that the volume it occupies is $dV = dA ds$.

Now, if we compare the definitions of j_ν and I_ν we find that they differ by a factor $dV / dA = ds$, the length of the beam cylinder we defined.

The difference between the specific intensities at the start and end of the cylinder would be zero without emission, now instead their difference can be calculated from the energy added; as we said most of the differentials simplify and we get that the variation of specific intensity is

$$dI_\nu = j_\nu ds. \quad (1.3.5)$$

1.3.2 Absorption

Absorption is described by a coefficient $\alpha_\nu > 0$, which is dimensionally an inverse length. The absorption law which defines the coefficient gives the decrease in radiative intensity for radiation of intensity I_ν crossing an absorbing medium of length ds :

$$dI_\nu = -\alpha_\nu I_\nu ds. \quad (1.3.6)$$

Why should the variation in intensity be proportional to the intensity itself? We give a simple argument: let us assume that absorption is due to randomly absorbers with number density n and (frequency dependent) cross section σ_ν .

Let us consider our usual cylinder with cross sectional area dA and length ds : the number of absorber in it will be $dN = n dA ds$. The total effective cross section area presented for absorption will be $\sigma_\nu dN$. The energy contained in photons in this cross sectional area will be lost: the energy lost $-dI_\nu$ can be calculated as

$$-dI_\nu dA dt d\Omega d\nu = I_\nu (\sigma_\nu n dA ds) dt d\Omega d\nu \quad (1.3.7a)$$

$$dI_\nu = -n\sigma_\nu I_\nu ds, \quad (1.3.7b)$$

which is the relation written above, with $n\sigma_\nu = \alpha_\nu$. The number density is proportional to the mass density: $n\bar{m} = \rho$ where \bar{m} is the average mass of a particle. Therefore, we can express α_ν as

$$\alpha_\nu = \rho\kappa_\nu = n\bar{m}\frac{\sigma_\nu}{\bar{m}}, \quad (1.3.8)$$

so we can see that κ_ν is a cross sectional area per unit mass. It is called the *mass absorption coefficient* or the *opacity*.

Conditions for validity This line of reasoning holds as long as the inter-absorber distances $d \sim n^{-1/3}$ are large compared to the linear scale of the cross section $\sigma_\nu^{1/2}$: we ask

$$\sigma_\nu^{1/2} \ll n^{-1/3}, \quad (1.3.9)$$

and also we must assume that the absorbers are independent and randomly distributed (at least locally). These assumptions are usually met in astrophysical systems.

1.3.3 The radiative transfer equation

We can account for both absorption and emission in a combined equation for the derivative with respect to the beam length travelled s of the specific intensity I_ν :

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu, \quad (1.3.10)$$

and we can see that in the absence of emission and absorption I_ν is unchanged, as we have shown before. If j_ν and α_ν are known we can integrate this differential equation to find the specific intensity.

This will *not* be the case when we will include scattering: the scattering term will not just depend on I_ν but in its integral on the sphere, making this an integro-differential equation.

Solutions to the transfer equation in simple cases

If there is only emission, that is, only j_ν is nonzero, the intensity increases (linearly in s for constant j_ν):

$$\frac{dI_\nu}{ds} = j_\nu \implies I_\nu = I_\nu(0) + \int_{s_0}^s j_\nu(\tilde{s}) d\tilde{s}. \quad (1.3.11)$$

If there is only absorption, that is, only α_ν is nonzero, then the intensity decreases (exponentially in s for constant j_ν):

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \implies I_\nu = I_\nu(s_0) \exp\left(-\int_{s_0}^s \alpha_\nu(\tilde{s}) d\tilde{s}\right). \quad (1.3.12)$$

1.3.4 Optical depth and source function

The optical depth τ_ν is defined so that it increases by 1 when the intensity of light at frequency ν decreases e -fold: its differential is

$$d\tau_\nu = \alpha_\nu ds, \quad (1.3.13)$$

so that the solution in the absorption-only case reads $I_\nu \propto e^{-\int d\tau} = e^{-\tau}$.

So, a useful distinction to make is based on the magnitude of τ , since it quantifies how much light can shine through a medium:

1. if $\tau \gg 1$ the medium is said to be *opaque* or *optically thick*;
2. if $\tau \ll 1$ the medium is said to be *transparent* or *optically thin*;
3. if $\tau \approx 1$ the medium is said to be *translucent*.

If we define the source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu}, \quad (1.3.14)$$

we can write the radiative transfer equation as

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu. \quad (1.3.15)$$

Divided through by α_ν , used definition of τ_ν .

1.3.5 A formal solution of the radiative transfer equation

We can solve this equation by defining $Y_\nu = I_\nu e^{\tau_\nu}$, which obeys

$$\frac{dY_\nu}{d\tau_\nu} = \frac{dI_\nu}{d\tau_\nu} e^{\tau_\nu} + I_\nu e^{\tau_\nu}, \quad (1.3.16)$$

so we can multiply the radiative transport equation by e^{τ_ν} to get

$$\frac{dI_\nu}{d\tau_\nu} e^{\tau_\nu} = -I_\nu e^{\tau_\nu} + S_\nu e^{\tau_\nu} \quad (1.3.17a)$$

$$\frac{dY_\nu}{d\tau_\nu} = S_\nu e^{\tau_\nu} \quad (1.3.17b)$$

$$Y_\nu(\tau_\nu) = Y_\nu(0) + \int_0^{\tau_\nu} S_\nu(\tilde{\tau}_\nu) e^{\tilde{\tau}_\nu} d\tilde{\tau}_\nu \quad (1.3.17c)$$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tilde{\tau}_\nu) e^{\tilde{\tau}_\nu - \tau_\nu} d\tilde{\tau}_\nu, \quad (1.3.17d)$$

Divided through by e^{τ_ν} .

which has a direct intuitive meaning: the intensity at a certain point must be computed accounting for the initial one and emission all through the beam before the point we are considering, and each of these contributions to the emission is weighted by an exponential factor: the relevance of a term decreases if the optical distance increases.

If S_ν is a constant, we have the simplified expression

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}), \quad (1.3.18)$$

and we can see that for large optical depths the intensity is dictated purely by the source at that point, since $1 - e^{-\tau_\nu}$ approaches 1 for large τ_ν .

1.3.6 Thermal radiation

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Consider a blackbody enclosure at equilibrium at a certain temperature T . If we open a small hole in the enclosure and measure the radiation inside it, the intensity I_ν we will see will be isotropic, and only depending on the temperature T : let us call the (still unspecified) way this dependence look $I_\nu = B_\nu(T)$. The function $B_\nu(T)$ will be called the **blackbody function**.

Now, imagine we put a small chunk of material inside the blackbody enclosure. If this material is characterized by a source function $S_\nu = j_\nu/\alpha_\nu$, then the evolution of the intensity inside the cavity will be

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu. \quad (1.3.19)$$

If we wait for equilibrium, though, we must have $I_\nu \equiv B_\nu$, meaning that its derivative must be zero: this implies that $I_\nu = B_\nu = S_\nu$.

This is called **Kirkhoff theorem**: a chunk of material which can emit and absorb and which is placed in a blackbody enclosure then it must satisfy

$$B_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu. \quad (1.3.20)$$

Planck function derivation

What is the explicit expression of the blackbody function B_ν ? From Bose-Einstein statistics we know that the phase space density of photons in thermal equilibrium is given by

$$\frac{dN}{d^3x d^3p} = \frac{2}{h^3} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}, \quad (1.3.21)$$

where the 2 comes from the two polarizations of the photons, h^3 is the size of the phase space cell, while the exponential factor is the occupation number.

Now, we would like to write this as a function of frequency, not of the three components of the momentum. The frequency ν and the momentum $p = |\vec{p}|$ of a photon are connected by $h\nu = E = cp$; and the momentum space element d^3p can be written as $d^3p = d\Omega p^2 dp$.

Bringing this to the other side and integrating over the solid angle (since the frequency of the photon does not depend on its direction we can substitute $\int d\Omega = 4\pi$) we can write

$$\frac{dN}{d^3x} = \frac{8\pi}{c^3} \frac{\nu^2 d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}, \quad (1.3.22)$$

the **number density** of photons. From this we can quickly recover the **energy density** as well, since each photon with frequency ν has energy $h\nu$: therefore

$$\frac{dE}{d^3x} = \frac{8\pi}{c^3} \frac{h\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}. \quad (1.3.23)$$

If we want to recover the energy density per unit frequency and per unit solid angle we can divide by $d\Omega d\nu$: this yields

$$u_\nu(\Omega) = \frac{dE}{d^3x d\Omega d\nu} = \frac{2h}{c^3} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}, \quad (1.3.24)$$

which we have previously shown to be equal to $u_\nu(\Omega) = I_\nu/c$: therefore the specific intensity, which corresponds to the Planck function $B_\nu(T)$, is

$$I_\nu = B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}. \quad (1.3.25)$$

Properties of the Planck function

Let us consider the limits of the Planck function. If the energy of the photon, $h\nu$, is much lower than the average photon energies $k_B T$ then we have

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \approx \frac{2h}{c^2} \frac{\nu^3}{1 + \frac{h\nu}{k_B T} - 1} = \frac{2k_B T}{c^2} \nu^2, \quad (1.3.26)$$

the **Rayleigh-Jeans** law.

On the other hand, if the photon energy $h\nu$ is much larger than $k_B T$ we can ignore the -1 in the denominator and just write

$$B_\nu(T) \approx \frac{2h\nu^2}{c^2} \exp\left(-\frac{h\nu}{k_B T}\right), \quad (1.3.27)$$

the **Wien law**. The exponential term dominates, so asymptotically the Planckian goes to zero.

Since it is a continuously differentiable function which increases at low frequency and decreases at high frequency, it will have a maximum, which we expect to appear somewhere around $h\nu \approx k_B T$. This is precisely what is described by the **Wien displacement law**.

Let us see where this maximum lies if the Planckian is expressed as a function of frequency, as we are doing now. If we define $x = h\nu/k_B T$, the Planck function is proportional to

$$B_\nu(T) \propto \frac{x^3}{e^x - 1}, \quad (1.3.28)$$

so we can find its maximum by seeking the stationary point

$$\frac{\partial B_\nu}{\partial \nu} \propto \frac{\partial B_\nu}{\partial x} = 0. \quad (1.3.29)$$

The derivative reads

$$\frac{\partial B_\nu}{\partial x} \propto \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2} \propto e^x(3 - x) - 3 \stackrel{!}{=} 0. \quad (1.3.30)$$

This can also be expressed as $x = 3(1 - e^{-x})$ as well. It is a transcendental equation: there is no closed-form solution, however we can easily approximate it numerically. The maximum is found to be approximately $x = 2.82$, which means that the maximum is attained at the frequency

$$\nu \approx 2.82 \frac{k_B T}{h}. \quad (1.3.31)$$

The fact that the peak of the curve depends linearly on T is precisely the statement of the Wien displacement law.

Another interesting question to ask is about how B_ν varies with T : its partial derivative reads

$$\frac{\partial B_\nu}{\partial T} = \frac{2h^2 \nu^4}{k_B T^2 c^2} \frac{e^{h\nu/k_B T}}{\left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right)^2} > 0, \quad (1.3.32)$$

so it is positive for any value of ν . It will be clearer with a picture, however we can say that this increase with T is much faster for $h\nu < k_B T$ than it is for $h\nu > k_B T$.

The specific energy density is found from $u_\nu = I_\nu/c$ and integrating over the solid angles (so, multiplying by 4π), so we have

$$u_\nu(T) = \frac{4\pi}{c} B_\nu, \quad (1.3.33)$$

and if we want to compute the total energy density we will have

$$u = \int_0^\infty u_\nu(T) d\nu = \frac{4\pi}{c} \frac{2h}{c^2} \int_0^\infty \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} d\nu \quad (1.3.34)$$

$$= \frac{8\pi h}{c^3} \frac{k_B^4 T^4}{h^4} \int_0^\infty \frac{x^3}{e^x - 1} dx, \quad (1.3.35)$$

where, as before, $x = h\nu/k_B T$. This already shows the main result: $u \propto T^4$, however to get a closed form expression we need to solve the integral as well. This integral is of the type

$$\int_0^\infty \frac{x^n}{e^x - 1} dx = n! \zeta(n+1), \quad (1.3.36)$$

which we need to evaluate for $n = 3$. The Riemann zeta function fortunately has a closed form for even integers, in our case $\zeta(4) = \pi^4/90$, which we multiply by $3! = 6$. This yields

$$u = \frac{8\pi h k_B^4 T^4}{c^3} \frac{\pi^4}{15} = \frac{8\pi^5 k_B^4}{15c^3 h^3} T^4 = a T^4, \quad (1.3.37)$$

where the constant $a \approx 7.57 \times 10^{-16} \text{ kg K}^{-4} \text{ m}^{-1} \text{ s}^{-2}$ is called the **blackbody constant**.

Now, we discuss the **flux** of the radiation field. If we choose a surface and measure the flux across it, our assumption of isotropy means that we will always find zero; however it is interesting to measure the *outgoing* flux from an oriented surface, counting only the photons going through it in one direction.

I understand that this gives a factor 1/2, but why should it correspond to integrating over half the solid angle, except by isotropy? The half-sphere does not seem to me to have physical meaning...

The flux we can compute is then:

$$F = \frac{1}{2} \int d\Omega d\nu B_\nu \cos \theta = \frac{2\pi}{2} \int_0^\pi d\theta \cos \theta \sin \theta \int_0^\infty d\nu B_\nu \quad (1.3.38)$$

$$= \pi \underbrace{\int_{-1}^1 \mu d\mu}_{=1} \underbrace{\int_0^\infty B_\nu d\nu}_{=uc/(4\pi)} = \frac{uc}{4} \quad (1.3.39)$$

$$= \frac{2\pi^5 k_B^4}{15c^2 h^3} T^4 = \sigma T^4, \quad (1.3.40)$$

where σ is called the **Stefan-Boltzmann constant**, which is related to a by $\sigma = ca/4$.

Another useful result to state here is that the integral of the Planck function in $d\nu$ is:

$$\int_0^\infty B_\nu d\nu = \frac{uc}{4\pi} = \frac{\sigma}{\pi} T^4. \quad (1.3.41)$$

1.3.7 Scattering

This is the third process which can affect the radiation field, besides absorption and emission. In the astrophysical context, we will only need to concern ourselves with scattering by *free electrons*.

When modelling absorption, we had the true absorption coefficient $\alpha_\nu = n\sigma_\nu$, where σ_ν was the cross section of the individual absorber. We can model scattering in the same way: we can introduce a “scattering absorption coefficient” $\alpha_\nu^{(s)}$, which differs from the true one in that in absorption photons are not destroyed but merely deflected.

However, from the point of view of the beam in a specific direction they disappear, as if they were absorbed. We will then also have to consider the inverse phenomenon:

photons coming into the beam we are considering from another. This can be modelled as a “scattering emission”.

We express the amount of photons lost from our beam as

$$\int \alpha_\nu^{(s)} I_\nu d\Omega . \quad (1.3.42)$$

Let us make a simplifying assumption: that the scattering is **isotropic**, the scattered photon has an equal probability to go in any direction. Also, let us assume that the frequency ν of the photon does not change upon scattering. We will see that both of these conditions are satisfied for Thompson scattering but not in the general situation.

Then, we are left with $\alpha_\nu^{(s)} \int I_\nu d\Omega = 4\pi\alpha_\nu^{(s)} J_\nu$.

This is in any direction, while the radiation scattered in the unit solid angle will be

$$j_\nu^{(s)} = \alpha_\nu^{(s)} J_\nu . \quad (1.3.43)$$

In general, since we do not have either conservation of photon frequency nor isotropy we will need to write a differential scattering cross section like

$$\frac{d\sigma}{d\nu d\Omega d\nu' d\Omega'} , \quad (1.3.44)$$

while the full expression for the radiation scattered in a specific direction Ω at a frequency ν will read

$$j_\nu^{(s)} = \int d\Omega' n \frac{d\sigma}{d\nu d\Omega d\nu' d\Omega'} I_{\nu'}(\Omega') d\nu' . \quad (1.3.45)$$

This is called the **scattering kernel**.

The transfer equation for scattering

If we only account for true emission and absorption the transfer equation looks like

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu . \quad (1.3.46)$$

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For isotropic and conservative scattering we can modify this equation by adding on the scattering absorption and emission terms:

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \alpha_\nu^{(s)}) I_\nu + j_\nu + \alpha_\nu^{(s)} J_\nu . \quad (1.3.47)$$

Now we cannot give a formal solution anymore: the derivative of the intensity now depends not only on the intensity itself but on its average over all solid angles, $J_\nu = \langle I_\nu \rangle_\Omega$. This will typically pose a problem: scattering is often the dominant phenomenon in radiative transfer for astrophysical systems.

In order to solve the equation we can resort to numerical methods: for example we can use an iterative relaxation procedure in which we start off by computing the solution

without scattering, calculate the mean intensity and plug it as a fixed value into the next iteration, and keep going.

We can write the transfer equation as

$$\frac{dI_\nu}{ds} = (\alpha_\nu + \alpha_\nu^{(s)}) \left(-I_\nu + \frac{j_\nu + \alpha_\nu^{(s)} J_\nu}{\alpha_\nu + \alpha_\nu^{(s)}} \right) = (\alpha_\nu + \alpha_\nu^{(s)}) (-I_\nu + S_\nu), \quad (1.3.48)$$

where we define a new form for the source function:

$$S_\nu = \frac{j_\nu + \alpha_\nu^{(s)} J_\nu}{\alpha_\nu + \alpha_\nu^{(s)}}, \quad (1.3.49)$$

where, as long as Kirkhoff's law holds, we can substitute $j_\nu = \alpha_\nu B_\nu$.

The optical depth is derived from the *total* absorption coefficient:

$$d\tau_\nu = (\alpha_\nu + \alpha_\nu^{(s)}) ds. \quad (1.3.50)$$

With these definitions, we can write the transfer equation like before,

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu. \quad (1.3.51)$$

If there is scattering this is only apparently simple, the formulation only hides the complexity.

1.3.8 Mean free path

We defined absorption as $\alpha_\nu = n\sigma_\nu$: this is just the inverse of the mean free path, so we have (for true absorption):

$$\ell_\nu = \frac{1}{\alpha_\nu}, \quad (1.3.52)$$

and we can define a mean free path for scattering in the exact same way, with $\alpha_\nu^{(s)}$. These will be the mean free paths of a photon before it undergoes that specific process, if we want to compute the mean free path of a photon before it undergoes *either one* then we can just take the inverse of $\alpha_\nu + \alpha_\nu^{(s)}$, the total absorption coefficient.

Now, consider a medium with scattering, emission and absorption. Typically, a photon will be emitted, scatter a few times, and then be absorbed. Let us say that between emission and absorption it is scattered N times, and let us call the spatial intervals it travels between these scatterings \vec{r}_i , where i goes from 1 to N . The total distance travelled will look like $\vec{R} = \sum_i \vec{r}_i$.

The average of its square value will be

$$\langle R^2 \rangle = \sum_{ij} \langle \vec{r}_i \vec{r}_j \rangle = \sum_i \langle r_i^2 \rangle + 2 \sum_{i < j} \langle \vec{r}_i \vec{r}_j \rangle. \quad (1.3.53)$$

The mixed averages $\langle r_i r_j \rangle$ evaluate to zero, since each scattering is isotropic and independent; on the other hand each of the $\langle r_i^2 \rangle$ is equal to ℓ_v^2 , the square of the mean free path. This means that we have $\langle R^2 \rangle = N \ell_v^2$, meaning that the average distance occurring between each scattering is given by $\ell_* = \sqrt{N} \ell_v$.

Now, let us suppose that the photon is being scattered in a medium whose characteristic length is L . Then typically a photon will need to be scattered N times before escaping the medium, where $L = \sqrt{N} \ell_v$. This means that

$$\sqrt{N} = \frac{L}{\ell_v} = \alpha_v L \sim \tau_v, \quad (1.3.54)$$

where the last order-of-magnitude relation comes from the fact that the *differential* of the optical path is given by $d\tau_v = \alpha_v ds$.

This means that, at least in terms of order of magnitude, $N \sim \tau_v^2$. This holds as long as the medium is optically thick, that is, τ_v is larger (ideally much larger) than 1.

What happens instead if the medium is optically thin, with $\tau_v < 1$? Then, we can neglect emission and scattering emission, and only consider scattering absorption. Then, the transfer equation will read

$$\frac{dI_v}{d\tau_v} = -I_v, \quad (1.3.55)$$

which is solved by $I_v = I_v(0)e^{-\tau_v}$. This can also be written, by adding $I_v(0)$ to both sides, as

$$I_v(0) - I_v = I_v(0)(1 - e^{-\tau_v}) \quad (1.3.56)$$

$$\frac{I_v(0) - I_v}{I_v} = 1 - e^{-\tau_v} \approx \tau_v. \quad (1.3.57)$$

Now, the average number of scatterings will be less than 1 and will correspond to the average relative intensity lost, $N = \Delta I_v / I_v(0)$. Therefore, in this case we will have $N \sim \tau_v$.

Putting together these two limiting cases, we can roughly say that we will have

$$N \approx \max(\tau_v, \tau_v^2). \quad (1.3.58)$$

After a mean free path the photon can be either scattered or absorbed. The probability that it is absorbed as opposed to being scattered is given by

$$\epsilon_v = \frac{\alpha_v}{\alpha_v + \alpha_v^{(s)}}, \quad (1.3.59)$$

while the probability that it is scattered is $1 - \epsilon_v$, and this last quantity is typically called the **single scattering albedo**.

The average number of mean free paths travelled before absorption will be¹

$$N = \frac{1}{\epsilon_v}, \quad (1.3.62)$$

¹ This can be shown using the identity

$$\sum_{i=1}^{\infty} i p^{i-1} = \frac{1}{(1-p)^2}, \quad (1.3.60)$$

so we can make the following manipulation to find an explicit expression for the mean path between emission and absorption:

$$\ell_* = N\ell_v^2 = \frac{\ell_v^2}{\epsilon_v} = \frac{1}{(\alpha_v + \alpha_v^{(s)})^2} \frac{\alpha_v + \alpha_v^{(s)}}{\alpha_v} = \frac{1}{\alpha_v(\alpha_v + \alpha_v^{(s)})}. \quad (1.3.63)$$

This means that

$$\ell_* = \frac{1}{\sqrt{\alpha_v(\alpha_v + \alpha_v^{(s)})}}. \quad (1.3.64)$$

Now we make use again of the order-of-magnitude relation $\tau \sim \alpha L$ where L is the length scale of the medium; if we multiply L by $1/\ell_*$, the effective absorption coefficient, we get

$$\tau \sim \frac{L}{\ell_*} = \sqrt{L^2 \alpha_v(\alpha_v + \alpha_v^{(s)})} = \sqrt{\tau_v(\tau_v + \tau_v^{(s)})}, \quad (1.3.65)$$

which is larger than τ_v alone would be. This is called the **effective optical depth**; the fact that this is larger than τ_v means that scattering traps the photon in the region for a longer time than it would remain there with absorption alone.

1.4 Radiative diffusion

This is a way to approximately solve the radiative transfer equation under certain assumptions, the main one being that the medium should be very **optically thick**.

Another assumption we will make is the **plane-parallel** approximation: this means that the properties of the medium only vary with respect to one coordinate, which we will call z . We will further assume that the properties of the radiation also only vary along z .

Now, this is about the *spatial* dependence of the quantities, however certain ones like the radiation intensity I_ν are also intrinsically vectors, so they also depend on the direction. However, we will have cylindrical symmetry for rotations around the z axis: therefore, the intensity will be a function of z and of the angle θ between the ray and the z axis.

The differential length travelled by the ray can be expressed in terms of the distance travelled along the z axis as

$$ds = \frac{dz}{\cos \theta} = \frac{dz}{\mu}, \quad (1.4.1)$$

where we define the usual shorthand, $\mu = \cos \theta$.

which can be proven by differentiating the geometric series and bringing the derivative into the sum, which converges absolutely.

Then, we can compute the average number of scatterings as

$$\langle N \rangle = \sum_{i=1}^{\infty} \epsilon_\nu i (1 - \epsilon_\nu)^{i-1} = \frac{\epsilon_\nu}{(1 - (1 - \epsilon_\nu))^2} = \frac{1}{\epsilon_\nu}. \quad (1.3.61)$$

The radiative transfer equation reads

$$\frac{dI_\nu}{ds} = (-I_\nu + S_\nu) \left(\alpha_\nu + \alpha_\nu^{(s)} \right), \quad (1.4.2)$$

where we can substitute

$$\frac{dI_\nu}{ds} = \mu \frac{\partial I_\nu}{\partial z}, \quad (1.4.3)$$

and using this we can express the radiative transfer equation like

$$I_\nu(z, \mu) = S_\nu - \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}} \mu \frac{\partial I_\nu}{\partial z}. \quad (1.4.4)$$

Now we make use of the assumption that the optical depth is large. This means that $\tau \sim (\alpha_\nu + \alpha_\nu^{(s)})\ell$ is large, where ℓ is the characteristic scale of the system.

Should we not say explicitly that, besides being optically thick, the medium's thickness should be "slowly varying"?

This means that the term proportional to $\partial I_\nu / \partial z$ is small compared to the source function, so we can apply perturbation theory.

To zeroth order, assuming thermal equilibrium, we will have

$$I_\nu^{(0)} \approx S_\nu^{(0)} = B_\nu(T). \quad (1.4.5)$$

Then, the first-order approximation can be found by inserting the zeroth-order expression into the equation:

$$I_\nu^{(1)}(z, \mu) = B_\nu - \frac{\mu}{\alpha_\nu + \alpha_\nu^{(s)}} \frac{\partial B_\nu}{\partial z}. \quad (1.4.6)$$

An important thing to note is the linear relation between I_ν and the cosine μ of the angle.

With this first-order result we can compute the flux: if we were to use the zeroth-order one we would get zero, since blackbody radiation is isotropic. The flux is given by the integral over the solid angle: the contribution to the integral due to the B_ν term vanishes, since it is isotropic

$$F_\nu(z) = \int I_\nu^{(1)}(z, \mu) \mu d\Omega = 2\pi \int_0^\pi I_\nu^{(1)}(z, \mu) \cos \theta \sin \theta d\theta \quad (1.4.7)$$

$$= -\frac{2\pi}{\alpha_\nu + \alpha_\nu^{(s)}} \frac{\partial B_\nu}{\partial z} \underbrace{\int_1^{-1} \mu^2 (-1) d\mu}_{=2/3} \quad (1.4.8)$$

$$= -\frac{4\pi}{3(\alpha_\nu + \alpha_\nu^{(s)})} \frac{\partial B_\nu}{\partial z} \quad (1.4.9)$$

$$= -\frac{4\pi}{3(\alpha_\nu + \alpha_\nu^{(s)})} \frac{\partial B_\nu}{\partial T} \frac{dT}{dz}, \quad (1.4.10)$$

where we used the fact that the spatial dependence of the B_ν only comes through the temperature $T(z)$. This is useful since, while the derivative of B_ν is frequency-dependent, the derivative of the temperature is not: therefore it can be factored out.

The total flux is given by

$$F = \int_0^\infty F_\nu d\nu = -\frac{4\pi}{3} \frac{dT}{dz} \int_0^\infty \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}} \frac{\partial B_\nu}{\partial T} d\nu \quad (1.4.11)$$

$$= -\frac{4\pi}{3} \frac{dT}{dz} \frac{\int_0^\infty \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu} \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu, \quad (1.4.12)$$

which is the **average value** of the function $1/(\alpha_\nu + \alpha_\nu^{(s)})$ weighted by the *known* function $\partial B_\nu/\partial T$; the integral at the numerator can be explicitly evaluated:

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial}{\partial T} \left(\frac{\sigma}{\pi} T^4 \right) = \frac{4\sigma}{\pi} T^3, \quad (1.4.13)$$

which means that the final expression is

$$F = -\frac{4\pi}{3} \frac{4\sigma T^3}{\pi} \frac{dT}{dz} \frac{\int_0^\infty \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}, \quad (1.4.14)$$

which we can better understand by defining the **Rosseland mean opacity**

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}, \quad (1.4.15)$$

so that the expression reads

$$F = -\frac{16}{3} \frac{\sigma T^3}{\alpha_R} \frac{dT}{dz}. \quad (1.4.16)$$

This is called the **Rosseland approximation**. The result is that we can find a net photon flux which is driven by the temperature gradient. This has many applications, for example in stellar astrophysics.

This equation is very similar to the heat flux equation in one dimension:

$$q = -k \frac{\partial T}{\partial z}, \quad (1.4.17)$$

which in our case has the constant (thermal conductivity)

$$k = \frac{16}{3} \frac{\sigma T^3}{\alpha_R}. \quad (1.4.18)$$

We have assumed that the medium is optically thick, but we can see that if it becomes very optically thick then the photons have a hard time escaping (k becomes small).

1.5 The Eddington approximation

In the Rosseland approximation we found the result

$$I_\nu = B_\nu(T) + \mu \times \text{stuff}, \quad (1.5.1)$$

where the “stuff” is small. We then expanded in this parameter; without it we are in a perfectly isotropic condition, so we can say that we are “expanding around isotropy”.

The idea behind the Eddington approximation is similar: we assume that the intensity can be written as

$$I_\nu(\tau) = a_\nu(\tau) + b_\nu(\tau)\mu, \quad (1.5.2)$$

so as before the angular dependence is linear.

Let us compute the moments of the specific intensity: the zeroth moment is the mean intensity,

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega = \frac{2\pi}{4\pi} \int_{-1}^1 (a_\nu + b_\nu\mu) d\mu = a_\nu, \quad (1.5.3)$$

while the first moment is equal to

$$H_\nu = \frac{1}{4\pi} \int I_\nu \mu d\Omega = \frac{2\pi}{4\pi} \int_{-1}^1 (a_\nu\mu + b_\nu\mu^2) d\mu = \frac{1}{2} \frac{2}{3} b_\nu = \frac{b_\nu}{3}. \quad (1.5.4)$$

The second moment is

$$K_\nu = \frac{1}{4\pi} \int I_\nu \mu^2 d\Omega = \frac{a_\nu}{3} = \frac{1}{3} J_\nu. \quad (1.5.5)$$

The fact that the second moment of the intensity is related to the zeroth one as $K_\nu = J_\nu/3$ is precisely what the Eddington approximation asks.

The plane-parallel radiative transport equation can be written as

$$\mu \frac{\partial I_\nu}{\partial \tau} = I_\nu - S_\nu, \quad (1.5.6)$$

since $d\tau = -(\alpha_\nu + \alpha_\nu^{(s)}) ds$ and $dz = \mu ds$.

Why is the optical depth defined with a negative sign here? It is the same in RL [RL79, eq. 1.115]. Is this to be interpreted as the optical depth towards a far-away observer, positioned at large z ?

In this equation, the source function S_ν is assumed to be isotropic. So, if we average over the solid angle (or equivalently in $d\mu/2$) we get

$$\frac{1}{2} \int_{-1}^1 \mu \frac{\partial I_\nu}{\partial \tau} d\mu = \frac{1}{2} \int_{-1}^1 I_\nu d\mu - \frac{1}{2} \int_{-1}^1 S_\nu d\mu \quad (1.5.7)$$

$$\frac{\partial}{\partial \tau} H_\nu = J_\nu - S_\nu. \quad (1.5.8)$$

We can also take the first moment of the same equation by multiplying the equation by μ before integrating: this yields

$$\frac{1}{2} \int_{-1}^1 \mu^2 \frac{\partial I_\nu}{\partial \tau} d\mu = \frac{1}{2} \int_{-1}^1 I_\nu \mu d\mu - \underbrace{\frac{1}{2} \int_{-1}^1 S_\nu \mu d\mu}_{=0} \quad (1.5.9)$$

$$\frac{\partial K_\nu}{\partial \tau} = H_\nu. \quad (1.5.10)$$

However, recall that in the Eddington approximation we have $K_\nu = J_\nu/3$: this means that we have the system of coupled differential equations

$$\frac{\partial H_\nu}{\partial \tau} = J_\nu - S_\nu \quad \text{and} \quad \frac{1}{3} \frac{\partial J_\nu}{\partial \tau} = H_\nu, \quad (1.5.11)$$

which we can combine into one second order equation by differentiating the second: this yields

$$\frac{1}{3} \frac{\partial^2 J_\nu}{\partial \tau^2} = J_\nu - S_\nu. \quad (1.5.12)$$

Now, the source function can be decomposed in a part depending on J_ν and a part depending on B_ν according to the single scattering albedo ϵ_ν (see equation (1.3.49)):

$$S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu. \quad (1.5.13)$$

Substituting this in a term simplifies and we finally find

$$\frac{1}{3} \frac{\partial^2 J_\nu}{\partial \tau^2} = \epsilon_\nu (J_\nu - B_\nu), \quad (1.5.14)$$

the **radiative diffusion equation**. This can be solved relatively simply; however it does not give us the whole intensity, instead we can only find its mean value J_ν .

If we can calculate S_ν using this equation, then we can plug what we have found into the radiative transfer equation

$$\mu \frac{\partial I_\nu}{\partial \tau} = I_\nu - S_\nu. \quad (1.5.15)$$

Now that we have discussed the differential equations at length, let us consider for a moment the **boundary conditions** we need to impose.

A possible way to approach the problem is called the *two stream approximation*: we assume that radiation is only travelling along two angles, whose cosines are $\mu = \pm 1/\sqrt{3}$, so $\theta = \pm 55^\circ$. This choice is less arbitrary than it may seem, there is good reason to suppose that in certain condition this is indeed the angle at which radiation will travel. This will be discussed later.

Let us introduce the quantities

$$I_\nu^\pm(\tau) = I_\nu(\tau, \pm \frac{1}{\sqrt{3}}). \quad (1.5.16)$$

Then, the average intensity will be given by their average:

$$J_\nu = \frac{I_\nu^+ + I_\nu^-}{2}. \quad (1.5.17)$$

The further moments can also be computed: they will be

$$H_\nu = \frac{1}{2\sqrt{3}}(I_\nu^+ - I_\nu^-) \quad \text{and} \quad K_\nu = \frac{I_\nu^+ + I_\nu^-}{6}, \quad (1.5.18)$$

consistently with the Eddington approximation. Note that, if we want to use a two-stream approximation, the only way for Eddington to hold is to use the values of $\mu = \pm 1/\sqrt{3}$.

The explicit expressions of the moments allow us to recover the I_ν^\pm as

$$I_\nu^+ = J_\nu + \sqrt{3}H_\nu = J_\nu + \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau} \quad (1.5.19)$$

$$I_\nu^- = J_\nu - \sqrt{3}H_\nu = J_\nu - \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau}. \quad (1.5.20)$$

Now, let us fix an example for concreteness: a “slab” (like, for instance, a stellar atmosphere) extending from τ_0 to $\tau = 0$. The point at $\tau = 0$ is the top of the atmosphere: so, we want to impose the absence of incoming radiation there (supposing that there are no other nearby stars to influence the process): this is implemented as $I_\nu^-(\tau = 0) = 0$. This is called the *non-illuminated atmosphere* condition.

Also suppose that there is no radiation coming into the bottom of the slab: $I_\nu^+(\tau_0) = 0$.

This means that

$$J_\nu + \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau} \Big|_{\tau_0} = 0 \quad \text{and} \quad J_\nu - \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau} \Big|_0 = 0. \quad (1.5.21)$$

With these boundary conditions, one can solve the equation numerically.

1.6 Plane electromagnetic waves

In a vacuum, Maxwell’s equations read

$$\nabla \cdot E = \nabla \cdot B = 0 \quad (1.6.1)$$

and

$$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \quad (1.6.2)$$

$$\nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t}. \quad (1.6.3)$$

Using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}$ we can combine these into a wave equation,

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0, \quad (1.6.4)$$

which is solved by plane monochromatic waves

$$\vec{E} = \vec{\epsilon} E_0 \exp\left(i(\vec{k} \cdot \vec{r} - \omega t)\right), \quad (1.6.5)$$

where the angular frequency and wavevector satisfy $\omega = c|\vec{k}|$. These are transverse waves: the scalar product of $\vec{\epsilon}$ and the wavevector \vec{k} is zero, there is no longitudinal oscillation along the direction of motion.

These solutions form a basis, generally the radiation we find is not monochromatic. The electric field will be time-dependent, $\vec{E} = \vec{E}(t)$.

We want to discuss the frequency decomposition of these waves; we start off by assuming that we are dealing with a pulse, such that $\vec{E}(t) \rightarrow 0$ as $t \rightarrow \pm\infty$.

The Fourier transform of the electric field in the time domain gives us its frequency decomposition: we choose our conventions so that

$$\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt. \quad (1.6.6)$$

Since $E(t)$ is real, we must have $\hat{E}^*(\omega) = \hat{E}(-\omega)$. This means that we do not need to worry about negative frequencies.

The fundamental quantity we need to compute is the **energy spectrum** from the electric field: how is the energy in the EM radiation distributed across the various frequencies? The Poynting vector is useful in this context, as it quantifies the amount of energy carried per unit area and time: its modulus is given by

$$S = \frac{dw}{dt dA} = \frac{c}{4\pi} E^2(t), \quad (1.6.7)$$

which we need to integrate if we want to find the total energy carried by the pulse:

$$\frac{dw}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) dt. \quad (1.6.8)$$

Now, **Parseval's theorem** is a statement about Fourier transforms, telling us that the energy of the signal is conserved if we Fourier transform, meaning that

$$\int_{-\infty}^{\infty} E^2(t) dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega. \quad (1.6.9)$$

Since this square modulus is the same for the conjugate the integrand is symmetric under $\omega \rightarrow -\omega$, so we can compute it from 0 to ∞ and multiply by 2 to get the same result. This yields

$$\frac{dw}{dA} = \frac{4\pi c}{4\pi} \int_0^{\infty} |\hat{E}(\omega)|^2 d\omega, \quad (1.6.10)$$

which can be differentiated with respect to ω to find the energy density per unit time and frequency:

$$\frac{dw}{dA d\omega} = c \left| \hat{E}(\omega) \right|^2. \quad (1.6.11)$$

This concerns the *total* energy of the pulse: we might want to calculate the energy per unit area, time and frequency! However, there is an issue in doing this: the uncertainty principle. We cannot sample with arbitrarily small intervals in both t and ω , since the inequality $\Delta\omega\Delta t \geq 1$ must be satisfied.

However, if our pulse repeats over some comparatively long time T we can formally define

$$\frac{dw}{dA d\omega dt} = \frac{1}{T} c \left| \hat{E}(\omega) \right|^2, \quad (1.6.12)$$

so, as long as T is relatively long we can take the limit:

$$\frac{dw}{dA d\omega dt} = \lim_{T \rightarrow \infty} \frac{1}{T} c \left| \hat{E}(\omega) \right|^2. \quad (1.6.13)$$

1.6.1 Polarization

This is a measurable quantity in astrophysics, it is easier to perform polarization measurements for low frequencies but nowadays we are starting to be able to measure polarizations as far as X-rays.

By definition a monochromatic plane wave is *linearly polarized*: it oscillates along a specific axis. Consider a superposition of two electric fields at a specific point in space, for simplicity $\vec{r} = 0$: the total electric field will be given by

$$\vec{E} = \underbrace{(\vec{x}E_1 + \vec{y}E_2)}_{\vec{E}_0} e^{-i\omega t}, \quad (1.6.14)$$

where the amplitudes $E_{1,2}$ are in general complex: $E_{1,2} = \tilde{\zeta}_{1,2} e^{i\phi_{1,2}}$. So, we can write the total electric field

$$\vec{E} = \vec{x}\tilde{\zeta}_1 e^{i(\phi_1 - \omega t)} + \vec{y}\tilde{\zeta}_2 e^{i(\phi_2 - \omega t)}. \quad (1.6.15)$$

The physical field along any direction will be given by the real part of the projection of this complex-valued vector along that component:

$$E_x = \text{Re} \left\{ \tilde{\zeta}_1 e^{i(\phi_1 - \omega t)} \right\} = \tilde{\zeta}_1 \cos(\phi_1 - \omega t) = \tilde{\zeta}_1 \cos \omega t \cos \phi_1 + \tilde{\zeta}_1 \sin \omega t \sin \phi_1, \quad (1.6.16)$$

and similarly with E_y , $\tilde{\zeta}_2$ and ϕ_2 .

Now, a short aside: an ellipse can be described parametrically from its principal axes with

$$x' = A \cos \beta \cos \omega t \quad \text{and} \quad y' = -A \sin \beta \sin \omega t, \quad (1.6.17)$$

since these expressions satisfy

$$\frac{x'^2}{A^2 \cos^2 \beta} + \frac{y'^2}{A^2 \sin^2 \beta} = 1, \quad (1.6.18)$$

so we can parametrize any ellipse with an appropriate choice of $A \in R^+$ and $\beta \in [-\pi/2, \pi/2]$. Tuesday

This parametric equation is expressed with respect to the principal axes of the ellipse; if 2020-8-18,
we want to write in a different coordinate system which is obtained from a rotation of the compiled
principal axes we must use the rotation matrix 2020-08-24

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}; \quad (1.6.19)$$

if we make x' and y' explicit we can find

$$x = A \cos \beta \cos \chi \cos \omega t + A \sin \beta \sin \chi \sin \omega t \quad (1.6.20)$$

$$y = A \cos \beta \sin \chi \cos \omega t - A \sin \beta \cos \chi \sin \omega t. \quad (1.6.21)$$

The expressions for x and y and the ones for E_x and E_y are quite similar: we must identify

$$\xi_1 \cos \phi_1 = A \cos \beta \cos \chi \quad (1.6.22)$$

$$\xi_1 \sin \phi_1 = A \sin \beta \sin \chi \quad (1.6.23)$$

$$\xi_2 \cos \phi_2 = A \cos \beta \sin \chi \quad (1.6.24)$$

$$\xi_2 \sin \phi_2 = -A \sin \beta \cos \chi, \quad (1.6.25)$$

which means that the electric field rotates in the shape of an ellipse, which is tilted about the x, y axes by an angle χ .

Stokes parameters

From these four equations, we find that (after working through the simplifications of the trigonometric functions)

$$\xi_1 + \xi_2 = A^2 \stackrel{\text{def}}{=} I \quad (1.6.26)$$

$$\xi_1 - \xi_2 = A^2 \cos 2\beta \cos 2\chi \stackrel{\text{def}}{=} Q \quad (1.6.27)$$

$$2\xi_1 \xi_2 \cos(\phi_1 - \phi_2) = A^2 \sin 2\chi \cos 2\beta \stackrel{\text{def}}{=} U \quad (1.6.28)$$

$$2\xi_1 \xi_2 \sin(\phi_1 - \phi_2) = A^2 \sin 2\beta \stackrel{\text{def}}{=} V, \quad (1.6.29)$$

where we defined the **Stokes parameters** I, Q, U and V . They completely describe the state of an elliptically polarized monochromatic wave.

We can reverse their definitions as

$$A = \sqrt{I} \quad (1.6.30)$$

$$\sin 2\beta = \frac{V}{I} \quad (1.6.31)$$

$$\tan 2\chi = \frac{U}{Q}. \quad (1.6.32)$$

Now let us discuss the meaning of these parameters. Since $A = \xi_0$, $I = \xi_0^2$, while the energy flux is given by

$$S = \frac{c}{4\pi} \xi_0^2 = \frac{c}{4\pi} I. \quad (1.6.33)$$

So, the parameter I is directly proportional to the energy flux. The major axes of the polarization ellipse are $2A \cos \beta$ and $2A \sin \beta$ respectively; therefore their ratio $\tan \beta$ measures the eccentricity of the ellipse, and the ratio V/I can be used to recover β .

The ratio U/Q describes the orientation of the ellipse with respect to the x, y axes. If the wave is linearly polarized then $V = 0$ (this is a degenerate case for our description); if $V > 0$ the polarization is left-handed, while if $V < 0$ the polarization is right-handed.

If we have $U = Q = 0$, this corresponds to circular polarization.

The ellipse is fully determined by three numbers (β, χ, A): why are there 4 Stokes parameters? They cannot all be independent if this is the case...

In fact, they are connected by the relation

$$I^2 = Q^2 + V^2 + U^2. \quad (1.6.34)$$

The reason we use four parameters instead of just three is that this line of reasoning only holds for perfectly polarized monochromatic waves: in the general case, for partially polarized radiation, we have

$$I^2 \geq Q^2 + V^2 + U^2. \quad (1.6.35)$$

For completely unpolarized light we have $Q = V = U = 0$.

A useful property is that these parameters are **additive**: the superposition of waves with different Stokes parameters can be described by the sum of the Stokes parameters, as $I_{\text{total}} = \sum_k I_k$ and so on.

This suggests that we write them in a vector:

$$\vec{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}. \quad (1.6.36)$$

In general, we can decompose radiation which is partially polarized into a completely unpolarized part and a completely polarized part:

$$\vec{S} = \underbrace{\begin{bmatrix} I - \sqrt{Q^2 + V^2 + U^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{unpolarized}} + \underbrace{\begin{bmatrix} \sqrt{Q^2 + V^2 + U^2} \\ Q \\ U \\ V \end{bmatrix}}_{\text{polarized}}. \quad (1.6.37)$$

This allows us to define the **polarization degree**: the fraction of the radiation which is polarized, calculated as

$$\Pi_L = \frac{I_{\text{polarized}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}, \quad (1.6.38)$$

which is measurable with an instrument.

Electromagnetic potentials

We can express the electric and magnetic fields through the scalar and vector potentials ϕ and \vec{A} :

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}. \quad (1.6.39)$$

In terms of these potentials Maxwell's equations read

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho \quad (1.6.40)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{j}. \quad (1.6.41)$$

The potentials are redundant, we can impose some conditions on them while still being able to describe any physical system. One we can impose is the *Lorentz gauge*:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0, \quad (1.6.42)$$

under which we can get a formal solution for the potentials in terms of the charge and current densities:

$$\phi(\vec{r}, t) = \int \frac{[\rho]}{|\vec{r} - \vec{r}'|} d^3r' \quad (1.6.43)$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \frac{[\vec{j}]}{|\vec{r} - \vec{r}'|} d^3r'. \quad (1.6.44)$$

The square brackets are a notation meaning that the densities must be evaluated at the *retarded time* $t - |\vec{r} - \vec{r}'|/c$. This accounts for the finite speed of the propagation of light.

Radiation from moving charges

Consider a particle of mass m and charge q moving along a path described by the vector $\vec{r}_0(t)$. The charge and current densities will then be

$$\rho(\vec{r}, t) = q\delta(\vec{r} - \vec{r}_0(t)) \quad (1.6.45)$$

$$\vec{j}(\vec{r}, t) = q \underbrace{\frac{d\vec{r}_0}{dt}}_{\vec{u}(t)} \delta(\vec{r} - \vec{r}_0(t)). \quad (1.6.46)$$

From these expressions we can calculate the potentials generated by the moving charge. Let us introduce the relative radius $\vec{R} = \vec{r} - \vec{r}_0$ and its corresponding unit vector $\vec{n} = \vec{R}/|\vec{R}|$.

The potentials will then be

$$\phi = \left[\frac{q}{kR} \right] \quad \text{and} \quad \vec{A} = \left[\frac{q\vec{n}}{ckR} \right], \quad (1.6.47)$$

where $k = 1 - \vec{n} \cdot \vec{u}/c$, while $R = |\vec{R}|$. These are called the **Liénard-Wiechert** potentials. From them we can perform the differentiation in order to recover the fields, which will be called the Liénard-Wiechert fields. The final expressions for the fields are

$$\vec{E}(\vec{r}, t) = q \left[\frac{(\vec{n} - \vec{\beta})(1 - \beta^2)}{k^3 R^2} \right] + \frac{q}{c} \left[\frac{\vec{n}}{k^3 R} \times ((\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}) \right] \quad (1.6.48)$$

$$\vec{B}(\vec{r}, t) = [\vec{n} \times \vec{E}], \quad (1.6.49)$$

where $\vec{\beta} = \vec{u}/c$. The important thing to notice here is that there are two contributions to the electric field: one goes as R^{-2} and is called the **Coulomb field** \vec{E}_c and the other goes as R^{-1} and is called the **radiation field** \vec{E}_r .

Now we will discuss the radiation field mainly. Since the magnetic field can be found by taking a product with the electric field, we can make the same decomposition there.

The radiation field \vec{E}_r must be perpendicular to \vec{n} , and so must be \vec{B}_r . So, they form an orthogonal triple.

Let us consider some simple cases.

Nonrelativistic charge

Let us first suppose that the particle is moving slowly, $|u|/c = |\beta| \ll 1$: this means that $|\vec{n} - \beta| \sim 1$ and also $k \sim 1$.

With these simplifications we find that the ratio of the magnitudes of the two components of the electric field is

$$\frac{E_r}{E_c} \sim \frac{R\dot{u}}{c^2}. \quad (1.6.50)$$

If the characteristic time across which the velocity of the particle changes is τ , then we have $\dot{u} \sim u/\tau$. This time τ will be associated with a characteristic frequency $\nu = 1/\tau$ and a characteristic wavelength $\lambda = c/\nu$.

...and this is the wavelength of the emitted radiation? is this always the case? it does not seem that obvious

Then we will get

$$\frac{E_r}{E_c} \sim \frac{R\nu v}{c^2} \sim \frac{Ru}{\lambda c}, \quad (1.6.51)$$

so we can see that if

$$\frac{R}{\lambda} \lesssim 1 \quad (1.6.52)$$

then we have $E_r/E_c \lesssim u/c$. This defines an inner region around the emitter for which the Coulomb field dominates; outside it the radiation field dominates.

Typically, we are far enough from the emitters of radiation so that the radiation field dominates. Then, under the assumption $\beta \ll 1$ we have

$$\vec{E}_r = \frac{q}{Rc^2} \left[\vec{n} \times \left(\vec{n} \times \dot{\vec{u}} \right) \right] \quad \text{and} \quad \vec{B}_r = \left[\vec{n} \times \vec{E}_r \right]. \quad (1.6.53)$$

Let us go into some more details regarding how the radiation field looks. Let us suppose that the angle between the acceleration $\dot{\vec{u}}$ and the observation unit vector \vec{n} is Θ : then the absolute value of the electric field and magnetic fields will be

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$$E_r = \frac{q}{Rc^2} \dot{u} \sin \Theta = B_r. \quad (1.6.54)$$

\vec{E} and \vec{B} are orthogonal, so the magnitude of the Poynting vector will be

$$|\vec{S}| = \frac{c}{4\pi} |\vec{E} \times \vec{B}| = \frac{c}{4\pi} E_r^2 = \frac{q^2}{4\pi c^3 R^2} \dot{u}^2 \sin^2 \Theta, \quad (1.6.55)$$

which, as is expected, decays like R^{-2} , since it is the power radiated per unit area. If we instead want the power radiated per unit solid angle then we must use $dA = R^2 d\Omega$:

$$\frac{dw}{dt d\Omega} = SR^2 = \frac{q^2}{4\pi c^3} \dot{u}^2 \sin^2 \Theta. \quad (1.6.56)$$

Integrating this over the solid angle allows us to calculate the total power emitted:

$$\frac{dw}{dt} = \int \frac{q^2}{4\pi c^3} \dot{u}^2 \sin^2 \Theta d\Omega \quad (1.6.57)$$

$$= \frac{q^2 \dot{u}^2}{2c^3} \underbrace{\int_{-1}^1 (1 - \mu^2) d\mu}_{=2-2/3} \quad (1.6.58)$$

Reparametrized
 $\mu = \cos \Theta$, integrated
over ϕ .

$$= \frac{2q^2 \dot{u}^2}{3c^3}, \quad (1.6.59)$$

which is the well-known **Larmor formula**.

What we want to do now is to generalize this result to a system of *many charges*. Each of these will have a charge q_i , a distance R_i from the observer, an acceleration \dot{u}_i . In principle the generalization is simple: the electromagnetic field obeys the superposition principle, so we can just add the $E_{i,r}$ from all the charges together.

The issue is that the retarded time for each of the particles is slightly different from that of another particle. This is a treatable problem, but it makes the calculation more cumbersome.

What we can do in order to mitigate it is to make the **dipole approximation**. Suppose that the system of N charges is contained within a volume whose characteristic length scale is L . Also suppose that τ is the typical scale across which the system evolves. If the

evolution time τ is slow compared to L/c , then we can ignore the differences between the retarded times.

Now, if the system changes significantly over a time τ then the electric field also changes significantly over that time, so we can estimate the frequency of the emitted radiation as $\nu \sim 1/\tau$.

What we are asking, $\tau \gg L/c$, then becomes $\nu \approx 1/\tau \ll c/L$, or equivalently $\lambda \gg L$.

We can also give an order of magnitude for the typical velocity of the particles: if ℓ is the typical path length of a particle, then the typical velocity will be $u \sim \ell/\tau$. This means that $\ell/u = \tau$ must be larger than L/c : rearranging, we have

$$\frac{\ell}{L} \gg \frac{u}{c}, \quad (1.6.60)$$

but since $\ell \lesssim L$ this means that $u/c \ll 1$.

If these conditions are satisfied, then we can get the total electric field by adding all the electric field contributions from all the various particles:

$$\vec{E}_r = \sum_i \vec{E}_{r,i} = \sum_i \frac{q_i}{R_i c^2} \vec{n}_i \times (\vec{n}_i \times \dot{\vec{u}}_i) \quad (1.6.61)$$

$$= \sum_i \frac{1}{R_i c^2} \vec{n}_i \times (\vec{n}_i \times q_i \dot{\vec{u}}_i). \quad (1.6.62)$$

All of these will still need to be computed at the retarded time, although the retarded time will be the same for all the charges.

If the length scale L of the system is smaller than all the distances R_i , then we can approximate any R_i with a constant R_0 ; also the unit vectors \vec{n}_i will be almost equal, and we can approximate them with a single \vec{n} . R_0 does not need to be the mean of the R_i , the point is that all of them are quite similar so the distance to any point inside the source works. The reasoning is similar for \vec{n} .

If we make both of these approximations we find

$$\vec{E}_r = \sum_i \frac{1}{R_0 c^2} \vec{n} \times (\vec{n} \times q_i \dot{\vec{u}}_i) \quad (1.6.63)$$

$$= \frac{1}{R_0 c^2} \vec{n} \times \left(\vec{n} \times \underbrace{\sum_i q_i \dot{\vec{u}}_i}_{\vec{d}} \right), \quad (1.6.64)$$

where we define the dipole moment

$$\vec{d} = \sum_i q_i \vec{r}_i. \quad (1.6.65)$$

Because the dipole moment appeared, this is called the **dipole approximation**. If Θ is the angle between \vec{n} and \vec{d} , then we have

$$E_r = \frac{d \sin \Theta}{R_0 c^2}, \quad (1.6.66)$$

and with this we can calculate the power radiated per unit area

$$\frac{dw}{dA dt} = \frac{c}{4\pi} E_r^2, \quad (1.6.67)$$

and per unit solid angle:

$$\frac{dw}{dt d\Omega} = \frac{c}{4\pi} E_r^2 R_0^2 = \frac{1}{c^3} \frac{\ddot{d}^2}{4\pi} \sin^2 \Theta, \quad (1.6.68)$$

so if we integrate we can find the total emitted power, which generalizes the **Larmor formula**:

$$\frac{dw}{dt} = \frac{2\ddot{d}^2}{3c^3}. \quad (1.6.69)$$

Now let us discuss the **spectral distribution** of the emitted energy. The energy per unit frequency and area will be

$$\frac{dw}{dA d\omega} = c \left| \hat{E}_r(\omega) \right|^2, \quad (1.6.70)$$

while the energy emitted per unit solid angle as usual is calculated by multiplying by R^2 :

$$\frac{dw}{d\Omega d\omega} = R^2 c \left| \hat{E}_r(\omega) \right|^2 = \frac{1}{c^3} \sin^2 \Theta \left| \hat{d}(\omega) \right|^2, \quad (1.6.71)$$

where \hat{d} is the Fourier transform of \ddot{d} . This assumes that Θ is fixed through time; we are allowing its magnitude but not its direction to change.

Derivatives in the Fourier domain are multiplication by $-i\omega$: so, we can recover the second derivative of the dipole moment in the time domain as

$$\ddot{d}(t) = - \int_{-\infty}^{\infty} \omega^2 e^{-i\omega t} \hat{d}(\omega) d\omega, \quad (1.6.72)$$

which means that we can express the electric field in the Fourier domain as

$$\hat{E}(\omega) = -\frac{\omega^2}{c^2 R} \hat{d}(\omega) \sin \Theta, \quad (1.6.73)$$

so that the energy emitted per unit frequency and solid angle becomes

$$\frac{dw}{d\Omega d\omega} = \frac{\omega^4}{c^3} \left| \hat{d}(\omega) \right|^2 \sin^2 \Theta. \quad (1.6.74)$$

If we integrate over the sphere we find

$$\frac{dw}{d\omega} = \frac{8\pi}{3c^2} \omega^4 \left| \hat{d}(\omega) \right|^2. \quad (1.6.75)$$

Note that the spectrum of the radiation is heavily dependent on the frequency of the radiation.

1.6.2 Thompson scattering

This is scattering of radiation by free electrons. Consider a plane, linearly polarized, electromagnetic wave impacting a free electron. The force onto the electron is the Lorentz force:

$$\vec{F} = q \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right). \quad (1.6.76)$$

We know that $|E| = |B|$ for the wave, so if the electron is nonrelativistic then the magnetic term is negligible:

$$\vec{F} \approx e\vec{E}. \quad (1.6.77)$$

Under our assumptions the electric field can be written as

$$\vec{E} = E_0 \sin \omega_0 t \vec{e}, \quad (1.6.78)$$

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where \vec{e} is a unit vector which is perpendicular to the propagation direction: $\vec{e} \cdot \vec{k} = 0$; while E_0 is the amplitude of the electric field and ω_0 is its frequency.

The equations of motion of the charge read

$$m\ddot{\vec{r}} = e\vec{E} = eE_0 \sin \omega_0 t \vec{e}, \quad (1.6.79)$$

which we can also express through the dipole moment $\vec{d} = e\vec{r}$: the equation for its evolution will then read

$$\ddot{\vec{d}} = \frac{e^2}{m} E_0 \sin \omega_0 t \vec{e}. \quad (1.6.80)$$

If we integrate in dt two times we find

$$\vec{d}(t) = -\frac{e^2 E_0}{m\omega_0^2} \sin \omega_0 t \vec{e}, \quad (1.6.81)$$

so the response to the impinging EM wave is an oscillation of the dipole, with a frequency ω_0 equal to that of the EM wave and an amplitude equal to

$$d_0 = \frac{e^2 E_0}{m\omega_0^2}. \quad (1.6.82)$$

The electron is accelerating, so it will radiate: the power emitted per unit solid angle will be

$$\frac{dw}{dt d\Omega} = \frac{1}{c^3} \frac{\dot{d}^2}{4\pi} \sin^2 \Theta = \frac{1}{4\pi c^3} \frac{e^4}{m^2} E_0^2 \sin^2 \omega_0 t \sin^2 \Theta, \quad (1.6.83)$$

where Θ is the angle between the direction of propagation of the wave and the direction of observation. This oscillates in time; we can compute the average over an oscillation: this means that we substitute the square sine with a factor $1/2$:

$$\left\langle \frac{dw}{dt d\Omega} \right\rangle = \frac{e^4 E_0^2 \sin^2 \Theta}{8\pi c^3 m^2}. \quad (1.6.84)$$

Integrating over the solid angle to get the average power amounts to multiplying by 4π times $2/3$, because of the solid angle in the sphere and because of the integral of $\sin^2 \Theta$. This yields

$$\left\langle \frac{dw}{dt} \right\rangle = \frac{e^4 E_0^2}{3m^2 c^3}. \quad (1.6.85)$$

Now, let us compute the flux of energy which is carried away by the incident EM wave:

$$S = \frac{dw}{dA dt} = \frac{c}{4\pi} E_r^2 = \frac{c}{4\pi R^2} E_0^2 \sin^2 \omega_0 t, \quad (1.6.86)$$

then the power per unit solid angle is

$$\frac{dw}{d\Omega dt} = \frac{c}{4\pi} E_0^2 \sin^2 \omega_0 t, \quad (1.6.87)$$

whose average as before is

$$\left\langle \frac{dw}{d\Omega dt} \right\rangle = \frac{c E_0^2}{8\pi}. \quad (1.6.88)$$

Except this, dimensionally, is a power per unit *area*!

Now, let us define the **differential scattering cross section** as

$$\frac{d\sigma}{d\Omega} = \frac{\left\langle \frac{dw}{dt d\Omega} \right\rangle_{\text{emitted}}}{\left\langle \frac{dw}{dt d\Omega} \right\rangle_{\text{incoming}}}, \quad (1.6.89)$$

is this not dimensionally inconsistent? the differential scattering cross section should have the dimensions of an area / steradian... Maybe the incoming power should be considered per unit *area*, since it is a plane wave whose source is at infinity? This seems indeed to be the case, see [RL79, eq. 3.36]

and if we compute this for our case we will have

$$\frac{d\sigma}{d\Omega} = \frac{e^4 E_0^2 \sin^2 \Theta}{8\pi c^3 m^2} \frac{8\pi}{c E_0^2} = \frac{e^4}{c^4 m^2} \sin^2 \Theta. \quad (1.6.90)$$

This expression can also be written through the classical electron radius:

$$r_0 = \frac{e^2}{mc^2} \approx 2.82 \times 10^{-13} \text{ cm}, \quad (1.6.91)$$

whose expression is found by equating the rest energy of the electron mc^2 with its electromagnetic self-energy e^2/r_0 . Then, the scattering cross section can be expressed as

$$\frac{d\sigma}{d\Omega} = r_0^2 \sin^2 \Theta. \quad (1.6.92)$$

The total cross section is found from the integral of this over all the solid angle:

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = r_0^2 \underbrace{\int (1 - \mu^2) d\mu}_{=8\pi/3} \times 2\pi = \frac{8\pi}{3} r_0^2. \quad (1.6.93)$$

This is the **Thomson cross section** of the electron $\sigma_T \approx 0.665 \times 10^{-24} \text{ cm}^2$.

Now, for some observations. This cross section does not change depending on the frequency of the incoming wave: it is “color blind”. This formula for the scattering is not always valid, but in its regime of validity (which will be discussed later) we expect no frequency dependence.

Moreover, this scattering is **conservative** or coherent: the frequency of the scattered radiation is the same as the frequency of the incoming radiation.

It also is **not isotropic** because of the factor $\sin^2 \Theta$. This scattering “prefers” for the light to go along the same direction it came from, and the probability density for it goes to zero for $\Theta = \pi/2$, meaning for a scattered photon orthogonal to the direction of propagation. Also, there is forward-backward symmetry, which will be useful to simplify certain calculations. The degree of anisotropy is also rather mild, it varies slowly over the solid angle, so in certain situations it will be alright to approximate Thomson scattering as isotropic. This is the closest we can get to completely isotropic and coherent scattering.

We can apply this reasoning to any other particle, it does not need to be an electron: for example, let us apply it to a proton. Their square charge is the same — the sign does not matter. The only different is the mass, which appears with a power m^{-2} in the cross section and in the differential cross section.

So, their ratio will be

$$\frac{\sigma_p}{\sigma_e} = \left(\frac{m_e}{m_p} \right)^2 \approx 3 \times 10^{-7}. \quad (1.6.94)$$

This is important in astrophysical applications: if we have a plasma with protons and electron we can almost completely ignore the protons.

Our assumptions have been to treat the electromagnetic field classically, and to assume the electron moves nonrelativistically. The first of these holds as long as the energy of the individual photons is small: $h\nu \ll m_e c^2$. If this is the case, the **recoil** of the electron due to the scattering with the single photon is negligible. Since $m_e c^2 \approx 511 \text{ keV}$, we are asking that $h\nu \ll 511 \text{ keV}$. A rough boundary for when this works is given by considering $h\nu \approx 50 \text{ keV}$, a tenth of the electron rest energy. This puts us in the medium X-rays.

We also assumed that the electron started out at rest, while in practically all situations charges are moving around, if nothing else because of thermal motion. If we are considering a plasma, it must be hot for the gas to stay ionized.

In the electron’s rest frame our description works, however in order to be in that frame we must perform a Lorentz boost, apply Thomson scattering, and boost back: in this way we can describe how the energy of the photon changes.

In astrophysical settings, radiation is generally either completely unpolarized or partially polarized. Therefore, it is interesting to compute the cross section for unpolarized radiation.

Unpolarized radiation can be described as the superposition of two orthogonal polarized waves. One of them will have an angle Θ between the polarization vector $\vec{\epsilon}_1$ and the observation direction \vec{n} , let us suppose that $\vec{\epsilon}_1$, \vec{k} and \vec{n} are coplanar. This can be done, since all we are doing is choosing a convenient basis for the plane orthogonal to the propagation direction.

If this is the case, then if $\vec{\epsilon}_2 \perp \vec{\epsilon}_1$ we must have $\Theta = \pi/2$ for the second wave. This means that we have the two cross sections

$$\left. \frac{d\sigma}{d\Omega} \right|_1 = r_0^2 \sin^2 \Theta \quad \text{and} \quad \left. \frac{d\sigma}{d\Omega} \right|_2 = r_0^2 \sin^2 \frac{\pi}{2} = r_0^2. \quad (1.6.95)$$

The total differential cross section will be given by their average:

$$\frac{d\sigma}{d\Omega} = r_0^2 \frac{1 + \sin^2 \Theta}{2} = r_0^2 \frac{1 + \cos^2 \theta}{2}, \quad (1.6.96)$$

where we define $\theta = \pi/2 - \Theta$, the angle between the propagation direction \vec{k} and the observation direction \vec{n} .

The total unpolarized cross section is the same as the one we had for polarized light:

$$\sigma_{\text{unpol}} = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} r_0^2 = \sigma_T. \quad (1.6.97)$$

So, for the total cross section of Thomson scattering polarization does not matter, while it does change the differential cross section.

The Eddington limit

The total momentum flux of the radiation field is given by

$$P = \int_0^\infty d\nu \int_{4\pi} d\Omega \cos^2 \theta I_\nu. \quad (1.6.98)$$

Suppose we have a differential area dA , and suppose that photons only cross it in the direction of its normal \vec{n} . If this is the case, then we have

$$P = \int_0^\infty d\nu I_\nu = \frac{F}{c}, \quad (1.6.99)$$

since then the angular distribution function is a delta on $\theta = 0$ on the unit sphere.

Suppose then we have a particle at the center of the area element dA , and that we want to calculate the force upon it. It will be the pressure times the cross section:

$$\mathcal{F} = \frac{F}{c} \sigma = P \sigma. \quad (1.6.100)$$

Now, let us consider a source which emits photons only radially, such as a star with perfect spherical symmetry. The flux at any radius r will be $F = L/4\pi r^2$, where L is the luminosity (power) of the source. The photons will only move radially.

Then, the force onto a test particle will be

$$\mathcal{F} = \frac{L\sigma}{c4\pi r^2}. \quad (1.6.101)$$

The source will have a mass M , and it will attract the particles gravitationally.

Now, a typical composition for the material outside a star is a plasma, made up of dissociated hydrogen: protons and electrons. As we have seen, the cross section of the electrons for Thomson scattering will be much larger than that of the protons, while the gravitational force will be much larger on the protons since they are more massive.

If we consider a single electron-proton pair which is bound by electrostatic forces (although not in a bound *state*, since we still have a plasma), then we can calculate the equilibrium point for the forces on the pair. We can then equate the radiative force on the electron to the gravitational force on the proton, since the other two forces are negligible in comparison. This yields:

$$\frac{L\sigma_T}{c4\pi r^2} = \frac{GMm_p}{r^2} \quad (1.6.102)$$

$$L = \frac{4\pi GMm_p c}{\sigma_T} = L_{\text{Edd}}, \quad (1.6.103)$$

the **Eddington luminosity** corresponding to a mass M . In comparison to the Sun, this is roughly

$$L_{\text{Edd}} \approx 3 \times 10^5 L_{\odot} \frac{M}{M_{\odot}}. \quad (1.6.104)$$

This means that the Sun is well below the Eddington limit. This is not a general limit, since it only applies if we have spherical symmetry; if this is broken the “limit” can be violated. Also, we can define analogous limits for different kinds of processes, which will have different cross sections.

1.7 Bremsstrahlung

The German word comes from the words meaning “braking” and “radiation”. It is also called “free-free emission”.

The name is historical: this kind of radiation was first observed in the lab coming from the deceleration of charges which hit a target. This is the radiation emitted by electrons when a force is exerted upon them, for instance the Coulomb force by an ion. So, this kind of radiation can occur in a plasma since there we have both free protons and electrons.

We could also have electron-electron electromagnetic interactions, but in real astrophysical plasmas, as we shall see, this is less relevant.

Suppose we have an ion with a positive charge Ze , and an electron with a negative charge $-e$. Between them we have the attractive electrostatic force, whose modulus is

$$F = \frac{Ze^2}{r^2}. \quad (1.7.1)$$

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If the ion is a proton, then the acceleration on the electron and on the proton can be calculated by equating

$$m_e a_e = m_p a_p = \frac{e^2}{r^2} \quad (1.7.2)$$

in modulus. So, the acceleration of the proton is about $m_p/m_e \approx 1836$ times *smaller* than that of the electron. This allows us to approximate the ion as a fixed source of force, neglecting its acceleration completely.

We know from the Larmor formula (1.6.59) that the power emitted by an accelerating charge is proportional to its acceleration squared, so the power emitted by the ion is about $(m_p/m_e)^2 \approx 4 \times 10^6$ times smaller than that emitted by the electron.

Now, let us consider the case of the repulsion of two identical particles, such as two electrons, in the nonrelativistic dipole approximation (which is not completely justified, often electrons will be relativistic... however we will make it for simplicity).

If \vec{d} is the dipole moment, then we know from the Larmor formula for the dipole (1.6.69) that the power emitted is proportional to $\ddot{\vec{d}}^2$. The dipole moment is

$$\vec{d} = -e \sum_i \vec{r}_i = -\frac{e}{m_e} \sum_i m_e \vec{r}_i = -\frac{e}{m_e} \vec{r}_{CM}. \quad (1.7.3)$$

We can then see that the dipole moment is proportional to the center of mass of the system. Now, we are always implicitly assuming that the two charges we are treating are isolated, so they are not subject to any external forces: therefore, the acceleration of the center of mass is zero.

This means that $\ddot{\vec{d}} = 0$, so the power emitted is zero in the dipole approximation. This is why *a system of nonrelativistic identical charges does not radiate in the dipole approximation*.

Now, let us move to an actual description of bremsstrahlung radiation. We will treat it approximately; a complete description, even accounting for quantum mechanics, can be given, however it is beyond the scope of this course.

We start off by making the small angle approximation: we assume that the electron moves fast enough, so that its trajectory looks like a straight line and the deviation due to the proton is only a small perturbation.

We define the position vector \vec{r} as the one connecting the ion to the electron, while $\vec{v} = \dot{\vec{r}}$ is the velocity of the electron, and the impact parameter $b = \min |\vec{r}|$ measures the distance of closest approach of the particles.

The dipole moment is given by $\vec{d} = -e\vec{r}$, whose second derivative will be $\ddot{\vec{d}} = -e\ddot{\vec{v}}$. Let us move to frequency space:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \ddot{\vec{d}} e^{i\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-e\ddot{\vec{v}}) e^{i\omega t} dt \quad (1.7.4)$$

$$-\omega^2 \hat{d}(\omega) = -\frac{e}{2\pi} \int_{-\infty}^{\infty} \ddot{\vec{v}} e^{i\omega t} dt. \quad (1.7.5)$$

Then, we can see that if we are able to determine the value of the integral on the right-hand side we can directly calculate the dipole moment, and with it the emitted power.

Evaluating it exactly is hard in general, however we can do so in two limiting cases, depending on the “duration of the interaction”. The timescale of the process, that is, the rough amount of time over which the electrostatic interaction between the two particles is significant, is of the order $\tau \approx b/v$. Then, we can try to simplify the problem by integrating only over a range of size $\sim \tau$ around zero in the time domain, instead of going from $-\infty$ to $+\infty$, since there we will find the largest contribution.

Now, our limiting cases refer to the frequency ω of the emitted radiation. If this frequency is very high, such that $\omega\tau \gg 1$, then in the integral we will have a slowly-varying term \dot{v} times a quickly oscillating term $\exp(i\omega t)$: thus, the value of the integral will be close to zero.

On the other hand, if $\omega\tau \ll 1$, then the exponential will be $\exp(i\omega t) \approx 1$: therefore the integral will be

$$\int_{-\infty}^{\infty} \dot{v} e^{i\omega t} dt \approx \int_{-\infty}^{\infty} \dot{v} dt = \Delta \vec{v}. \quad (1.7.6)$$

Therefore, the Fourier transform of the dipole moment will be given by

$$\hat{d}(\omega) = \begin{cases} -\frac{e}{2\pi} \frac{\Delta \vec{v}}{\omega^2} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1. \end{cases} \quad (1.7.7)$$

With this, we can compute the spectral distribution of the emitted radiation (1.6.75):

$$\frac{dw}{d\omega} = \frac{8\pi}{3c^3\omega^4} \begin{cases} \frac{e^2}{4\pi^2} \frac{|\Delta \vec{v}|^2}{\omega^4} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \quad (1.7.8)$$

$$= \begin{cases} \frac{2e^2}{3\pi c^3} |\Delta \vec{v}|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1. \end{cases} \quad (1.7.9)$$

Now, then, we need to calculate $\Delta \vec{v}$ in order to find the spectral distribution of the power. However, we already have an interesting result: a **flat power spectrum** in both the high- and low-frequency regimes, at a certain value to be calculated for high frequencies, and at zero for low frequencies. There will need to be some smooth connection between the two regions for $\omega\tau \sim 1$.

Now, as long as the interaction time is short, the electron “flies by” the ion, which then has little time to exert a force on it, and it will do so only in the region in which the electron is close, and in which \vec{F} is approximately perpendicular to \vec{v} .

Then, we can compute the variation in velocity as

$$\Delta \vec{v} = \int_{-\infty}^{\infty} \vec{a} dt, \quad (1.7.10)$$

and therefore, since the normal (dominant) component of the acceleration is given by $a_N = F_N/m_e$, we have

$$|\Delta \vec{v}| \approx \int_{-\infty}^{\infty} \frac{F_N}{m_e} dt = \int_{-\infty}^{\infty} \frac{Ze^2}{m_e r^2} \frac{b}{r} dt, \quad (1.7.11)$$

where the factor $b/r = \cos \theta$ accounts for the fraction of the force which is indeed normal: θ is the angle between the radial separation \vec{r} and the velocity \vec{v} (or $\pi -$ this angle), so that $F \cos \theta = F_N$. We can then solve this using the fact that the particle moves linearly, so that $r^2 = b^2 + v^2 t^2$: substituting this in we find

$$|\Delta v| = \int_{-\infty}^{\infty} \frac{Ze^2 b}{m_e r^3} dt = \frac{Ze^2 b}{m_e} \int_{-\infty}^{\infty} \frac{dt}{(b^2 + v^2 t^2)^{3/2}} \quad (1.7.12)$$

$$= \frac{Ze^2}{m_e b v} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{3/2}}}_{=2} = \frac{2Ze^2}{m_e b v}. \quad (1.7.13)$$

We can then insert this into the expression for the spectral distribution of the signal, also using $\tau \sim b/v$:

$$\frac{dw}{d\omega} = \begin{cases} \frac{8Z^2 e^6}{3\pi c^3 b^2 v^2 m_e^2} & b \ll v/\omega \\ 0 & b \gg v/\omega. \end{cases} \quad (1.7.14)$$

Bremsstrahlung in a plasma

We found the spectral density for a single electron: now, we wish to compute it for the whole plasma, whose ion density is n_i , and whose electron density is n_e . To simplify, we will assume that all the electrons have the same speed v , but we will let their impact parameters b vary.

Let us consider this for a single ion, onto which many electrons will impact. The flux of electrons will be given by $n_e v$; so the number of particles crossing an annulus of radii b , $b + db$ will be given by $n_e v dA = 2\pi b db n_e v$. Let us then integrate in db to find the total emitted power. We will integrate from some minimum impact parameter b_{\min} instead of from zero: this is needed to find a physical result, and it will be explained in more detail later. We find:

$$\frac{dw}{dt d\omega} = \int_{b_{\min}}^{\infty} n_e v \frac{dw}{d\omega} 2\pi b db = 2\pi n_e v \int_{b_{\min}}^{\infty} \frac{dw}{d\omega} b db, \quad (1.7.15)$$

where now we need to substitute our expression; however we only have a nonzero contribution in the low-frequency limit, or equivalently the small- b limit. So, our integrand will be zero asymptotically; before that it will go as $1/b$. We account for this by only integrating up to some cutoff $b_{\max} \sim v/\omega$, whose exact value must be determined by a more detailed analysis. We will use the rough estimate $b_{\max} = v/\omega$. If we also account for the ion density n_i to find the power per unit frequency and volume, our integral can be expressed as:

$$\frac{dw}{dt d\omega dV} = \frac{16Z^2 e^6}{3c^3 m_e^2 v} n_e n_i \int_{b_{\min}}^{b_{\max}} \frac{b}{b^2} db \quad (1.7.16)$$

$$= \frac{16Z^2 e^6}{3c^3 m_e^2 v} n_e n_i \log \left(\frac{b_{\max}}{b_{\min}} \right). \quad (1.7.17)$$

Now, what should the value of b_{\min} be? A first approximation we made is the small-angle one, which holds as long as $|\Delta\vec{v}|/|\vec{v}|$ is small, less than unity. Suppose we are at the upper limit of this condition, when $|\Delta\vec{v}| \sim |\vec{v}|$.

Inserting this into our expression for $|\Delta\vec{v}|$ we find the limit where b is so small — the electron comes so close to the ion — that the interaction is too large to be treated perturbatively:

$$|\Delta\vec{v}| \sim v \sim \frac{2Ze^2}{m_e b v} \implies b = b_{\min} = \frac{2Ze^2}{m_e v^2}. \quad (1.7.18)$$

A second line of reasoning comes from the quantum-mechanical uncertainty principle: $\Delta x \Delta p \gtrsim \hbar$. If our Δx is b_{\min} , this means that we will have

$$b \gtrsim b_{\min} = \frac{\hbar}{m_e v}. \quad (1.7.19)$$

Which of these is greater? To make it clearer, in natural units we are comparing ($1/m_e v$ times) the quantities: 1 for the quantum mechanical threshold, and

$$\underbrace{8\pi\alpha}_{\approx 0.18} \frac{Z}{v} \quad (1.7.20)$$

for the small-angle-approximation threshold. Since we are dealing with nonrelativistic particles v will be small, so this can easily become the larger bound of the two, even for a hydrogen ion.

Anyhow, these considerations are all heuristic, and what is typically done is to parametrize the uncertainty in this aspect with a so-called **Gaunt factor**

$$g_{ff}(v, \omega) = \frac{\sqrt{3}}{\pi} \log \left(\frac{b_{\max}}{b_{\min}} \right), \quad (1.7.21)$$

where the prefactor is, I think, there for historic reasons. This will in general be a function of both v (inside b_{\min}) and of ω (inside of b_{\max}). With it, we can write

$$\frac{dw}{dt dV d\omega} = \frac{16\pi Z^2 e^6 \pi}{3\sqrt{3} c^3 m_e^2 v} n_e n_i g_{ff}(v, \omega). \quad (1.7.22)$$

With a proper quantum-mechanical treatment one can find an exact expression for this factor, and in the literature there are several good approximations which are good in different regimes; also, the values are tabulated. One can then assume that this is a known function.

We have treated the bremsstrahlung emission from a single-speed electron distribution, expressing in terms of the free-free Gaunt factor $g_{ff}(\omega, v)$. Now we want to tackle the emission from a distribution with varying velocity.

We will consider a simple case which has broad application in an astrophysical setting: the electrons having a nonrelativistic Maxwellian (thermal) isotropic distribution,

$$dP \propto \exp\left(-\frac{E}{k_B T}\right) d^3v = \exp\left(-\frac{mv^2}{2k_B T}\right) 4\pi v^2 dv. \quad (1.7.23)$$

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The last equality is imprecise, what it meant implicitly is that since the distribution is isotropic we can integrate over the solid angle getting a factor 4π ; the only true dependence is on the modulus of the velocity.

Then, we will be able to recover the total power emitted by electrons whose velocity distribution looks like this as

$$\frac{dw}{dV dt d\omega} \propto \int_0^\infty \frac{dw}{dt d\omega dV}(v, \omega) v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) dv. \quad (1.7.24)$$

Why is the total power density also divided by dV , but not the one inside the integral? are they not both densities in terms of space?

Should we really integrate from 0 to $+\infty$? The upper bound is fine, we can have electrons with arbitrarily high energy, however the distribution is exponentially suppressed there so it does not pose an issue. The problem comes from the lower bound: at a velocity v the electron has an energy $mv^2/2$, so if the frequency of the photon is ω and $mv^2/2 < \hbar\omega$ there cannot be emission, since the energy of the photon must come from the kinetic energy.

This means that we must exclude the tail of the distribution at low v , specifically we impose

$$v \geq v_{\min} = \sqrt{\frac{2\hbar\omega}{m}}. \quad (1.7.25)$$

So, we will have

$$\frac{dw}{dV dt d\omega} = \frac{1}{N} \int_{v_{\min}}^\infty \frac{dw}{dt d\omega dV}(v, \omega) v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) dv, \quad (1.7.26)$$

where N is the normalization, which is equal to

$$N = \int_0^\infty v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) dv. \quad (1.7.27)$$

Note that in the normalization we still must integrate from 0: the electrons whose energy is lower than the threshold do not matter for our purposes, however they are still there, and so not all the electron density is available for emission.

Now, let us express the single-velocity power distribution (1.7.22) incorporating all the constants into a single constant A :

$$\frac{dw}{dt d\omega dV}(v, \omega) = A \frac{g_{ff}(v, \omega)}{v}, \quad (1.7.28)$$

where crucially A does not depend on v . Inserting this yields:

$$\frac{dw}{dV dt d\omega} = \frac{A}{N} \int_{v_{\min}}^\infty \frac{g_{ff}(v, \omega)}{v} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) dv \quad (1.7.29)$$

$$= A \frac{\frac{2k_B T}{m} \int_{x_{\min}}^{\infty} g_{ff}(x, \omega) x e^{-x^2} dx}{\left(\frac{2k_B T}{m}\right)^{3/2} \int_0^{\infty} x^2 e^{-x^2} dx} \quad (1.7.30) \quad \text{Substituted } x = v\sqrt{2k_B T/m}.$$

$$= \frac{A}{\sqrt{2k_B T/m}} \frac{\int_{x_{\min}}^{\infty} g_{ff}(x, \omega) x e^{-x^2} dx}{\int_{x_{\min}}^{\infty} x e^{-x^2} dx} \frac{\int_{x_{\min}}^{\infty} x e^{-x^2} dx}{\int_0^{\infty} x^2 e^{-x^2} dx} \quad (1.7.31)$$

$$= \frac{A}{\sqrt{2k_B T/m}} \bar{g}_{ff}(\omega) \frac{\int_{x_{\min}}^{\infty} x e^{-x^2} dx}{\int_0^{\infty} x^2 e^{-x^2} dx}, \quad (1.7.32)$$

where we introduced the **velocity-averaged Gaunt factor** $\bar{g}_{ff}(\omega, T)$ (it depends on the temperature through x), while the ratio of the two integrals equals

$$\frac{\int_{x_{\min}}^{\infty} x e^{-x^2} dx}{\int_0^{\infty} x^2 e^{-x^2} dx} = \frac{\frac{1}{2} e^{-x_{\min}^2}}{\sqrt{\pi}/4} = \frac{4}{\sqrt{\pi}} \frac{1}{2} e^{-\frac{h\nu}{k_B T}}, \quad (1.7.33)$$

since $x_{\min}^2 = mv_{\min}^2/2k_B T$.

Therefore, our final expression will be

$$\frac{dw}{dV dt d\omega} = A \sqrt{\frac{m}{2k_B T}} \bar{g}_{ff}(\omega, T) \frac{1}{2} e^{-\frac{h\nu}{k_B T}} \frac{4}{\sqrt{\pi}} \quad (1.7.34)$$

$$= \frac{16\pi Z^2 e^6}{3\sqrt{3} c^3 m^2} \frac{1}{2} \frac{4}{\sqrt{\pi}} n_e n_i e^{-\frac{h\nu}{k_B T}} \sqrt{\frac{m}{2k_B T}} \bar{g}_{ff}(\omega, T) \quad (1.7.35)$$

$$= \frac{32\sqrt{\pi} Z^2 e^6}{3\sqrt{3} c^3 m^2} \sqrt{\frac{m}{2k_B}} n_e n_i T^{-1/2} e^{-\frac{h\nu}{k_B T}} \bar{g}_{ff}(\omega, T), \quad (1.7.36)$$

which can also be written in terms of frequency $\nu = \omega/2\pi$ as

$$\frac{dw}{dV dt d\nu} = \frac{32Z^2 e^6}{3mc^3} \sqrt{\frac{2\pi}{3k_B m}} n_e n_i \frac{1}{\sqrt{T}} \exp\left(-\frac{h\nu}{k_B T}\right) \bar{g}_{ff}(\nu, T). \quad (1.7.37)$$

We can then see that the free-free spectrum has an exponential cutoff in frequency, coming from the lower bound on the integral we imposed: if we had not done so the spectrum would be completely flat, except for $\bar{g}_{ff}(\nu, T)$. This is reasonable: that effect is most significant for high-energy photons.

However, the dependence on frequency of the Gaunt factor is quite weak, and it is a reasonable first approximation to assume $\bar{g}_{ff} \approx 1$.

The shape of the spectrum, on a log-log plot, is flat for low frequencies up until $h\nu \sim k_B T$, and then a sharp drop. Now we can calculate the total radiated power per unit volume by integrating in $d\nu$:

$$\frac{dw}{dt dV} = \int_0^{\infty} A n_e n_i T^{-1/2} \bar{g}_{ff}(\nu, T) e^{-h\nu/k_B T} d\nu \quad (1.7.38)$$

$$= A n_e n_i T^{-1/2} \underbrace{\frac{\int_0^{\infty} \bar{g}_{ff}(\nu, T) e^{-h\nu/k_B T} d\nu}{\int_0^{\infty} e^{-h\nu/k_B T} d\nu}}_{\bar{\bar{g}}_{ff}} \int_0^{\infty} e^{-h\nu/k_B T} d\nu \quad (1.7.39)$$

$$= A n_e n_i T^{-1/2} \frac{k_B T}{h} \bar{\bar{g}}_{ff}(T), \quad (1.7.40)$$

where we have introduced the doubly-averaged (over frequency as well as velocity) Gaunt factor $\bar{\bar{g}}_{ff}$, which now depends only on the temperature. The final result is then

$$\frac{dw}{dt dV} = \frac{32\pi Z^2 e^6}{3mc^3 h} \sqrt{\frac{2\pi k_B}{3m}} \bar{\bar{g}}_{ff}(T) n_e n_i T^{-1/2}. \quad (1.7.41)$$

This frequency-averaged Gaunt factor is very slowly varying, and very close to unity.

Now, let us link this with the **emission** and **absorption coefficients**: the emission coefficient j_ν in the isotropic case is

$$j_\nu = \frac{1}{4\pi} \frac{dw}{dt d\nu dV}, \quad (1.7.42)$$

which in our case will then be

$$j_\nu = \frac{8Z^2 e^6}{3mc^3} \sqrt{\frac{2\pi}{2mk_B}} \bar{\bar{g}}_{ff}(\nu, T) n_e n_i T^{-1/2} \exp\left(-\frac{h\nu}{k_B T}\right). \quad (1.7.43)$$

The treatment of absorption is more interesting. Our population is in thermal equilibrium, so Kirkhoff's law holds: this means that the ratio of the emission and absorption coefficients is given by the Planck function,

$$B_\nu = \frac{j_\nu}{\alpha_\nu}, \quad (1.7.44)$$

therefore

$$\alpha_\nu = \frac{j_\nu}{B_\nu} = \frac{\frac{8Z^2 e^6}{3mc^3} \sqrt{\frac{2\pi}{2mk_B}} \bar{\bar{g}}_{ff}(\nu, T) n_e n_i T^{-1/2} \exp\left(-\frac{h\nu}{k_B T}\right)}{\frac{2h\nu^3}{c^2} \left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right)^{-1}} \quad (1.7.45)$$

$$= \frac{4Z^2 e^6}{3mhc} \sqrt{\frac{2\pi}{3mk_B}} \bar{\bar{g}}_{ff}(\nu, T) n_e n_i T^{-1/2} \nu^{-3} \left(1 - \exp\left(-\frac{h\nu}{k_B T}\right)\right). \quad (1.7.46)$$

How does the dependence of this absorption coefficient on frequency look like? At low energy, $h\nu \ll k_B T$, we have $1 - \exp(-h\nu/k_B T) \approx h\nu/k_B T$, meaning that

$$\alpha_\nu \propto \nu^{-3} \frac{h\nu}{k_B T} \propto \nu^{-2}, \quad (1.7.47)$$

while for high energies $h\nu \gg k_B T$ we have $1 - \exp(-h\nu/k_B T) \approx 1$: this means that

$$\alpha_\nu \propto \nu^{-3}. \quad (1.7.48)$$

In the middle, $h\nu \sim k_B T$, we will have some smooth connection of the two powerlaws. In a log-log plot, we have a broken powerlaw.

A quantity we defined earlier is the Rosseland mean absorption coefficient;

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}. \quad (1.7.49)$$

With the explicit expression we now have for the monochromatic absorption coefficient we can compute these integrals: we will need the fact that

$$\frac{\partial B_\nu}{\partial T} \propto \frac{\nu^3}{(e^{h\nu/k_B T} - 1)^2} e^{h\nu/k_B T} \frac{\nu}{T^2} \propto \frac{x^3 e^x x}{(e^x - 1)^2} T^2, \quad (1.7.50)$$

where, as usual, $x = h\nu/k_B T$. We do not need to worry about prefactors since they simplify. The Rosseland mean absorption coefficient is then

$$\frac{1}{\alpha_R} \propto \frac{T^{1/2}}{n_e n_i Z^2} \frac{\int_0^\infty \nu^3 \left(1 - e^{-h\nu/k_B T}\right)^{-1} \bar{g}_{ff}^{-1} \frac{x^4 e^x}{(e^x - 1)^2 T^3} dx}{\int_0^\infty \frac{x^4 e^x T^3}{(e^x - 1)^2} dx} \quad (1.7.51)$$

Extra T in the numerator from moving from $d\nu$ to dx .

$$\propto \frac{T^{1/2}}{n_e n_i Z^2} \frac{T^3 T^3}{T^3} \frac{\int_0^\infty \bar{g}_{ff}^{-1} x^7 (e^x - 1)^{-3} dx}{\underbrace{\int_0^\infty x^4 e^x (e^x - 1)^{-2} dx}_{\bar{g}_R^{-1}(T)}} \quad (1.7.52)$$

$$\propto \frac{T^{7/2}}{n_e n_i Z^2} \bar{g}_R^{-1}(T), \quad (1.7.53)$$

therefore

$$\alpha_R \propto n_e n_i Z^2 T^{-7/2} \bar{g}_R(T). \quad (1.7.54)$$

The main result is the $T^{-7/2}$ dependence; the constant in front of the expression can be fixed with a more precise calculation.

Now, let us discuss what kinds of corrections come up when dealing with **relativistic bremsstrahlung**: the main process which becomes relevant is **electron-electron** bremsstrahlung. Even just treating electron-ion bremsstrahlung in the relativistic regime, however, requires the use of full QED. The final result one gets for the frequency-integrated power is

$$\frac{dw}{dt dV} \approx 1.4 \times 10^{27} T^{1/2} Z^2 n_e n_i \bar{\bar{g}}_{ff} \left(1 + 4.4 \times 10^{-10} \frac{T}{\text{K}}\right) \text{erg cm}^{-3} \text{s}^{-1}, \quad (1.7.55)$$

where the $4.4 \times 10^{-10} T$ factor is the relativistic correction. Thermal electrons become relativistic when $k_B T \sim m_e c^2$, meaning that $T \sim 511 \text{ keV} \sim 6 \times 10^9 \text{ K}$. The relativistic correction can be written as

$$1 + 2.6 \frac{T}{m_e c^2 / k_B}, \quad (1.7.56)$$

which can become significant quite early: it is of the order of 25 % as early as $6 \times 10^8 \text{ K}$, one tenth of the relativistic temperature.

1.8 Electron scattering

1.8.1 Compton scattering onto an electron at rest

We need to account for the fact that light has a quantum nature, which is not addressed in the classical treatment of scattering. Also, now we will account for the momentum of the photon.

We start of with an electron at rest, and a photon with energy $h\nu$ and momentum $h\nu/c$ impinging on it. After the scattering, the photon will have energy $h\nu'$ and momentum $h\nu'/c$, while the electron will have momentum $m\mathbf{v}$. Let us call the angle between the direction of the incoming photon and the direction of the outgoing one θ .

In terms of four-vectors, we can express the momenta of the photon before and after as

$$k^\mu = \frac{\epsilon}{c} \begin{bmatrix} 1 \\ \vec{\Omega} \end{bmatrix} \quad \text{and} \quad k'^\mu = \frac{\epsilon'}{c} \begin{bmatrix} 1 \\ \vec{\Omega}' \end{bmatrix}, \quad (1.8.1)$$

where $\vec{\Omega}$ and $\vec{\Omega}'$ are unit vectors defining the propagation directions, such that $\vec{\Omega} \cdot \vec{\Omega}' = \cos \theta$. On the other hand, the momenta of the electron will be

$$p^\mu = \begin{bmatrix} mc \\ \vec{0} \end{bmatrix} \quad \text{and} \quad p'^\mu = \gamma \begin{bmatrix} mc \\ m\vec{v} \end{bmatrix}. \quad (1.8.2)$$

Since the particles are unchanged after the scattering, both the incoming and outgoing momenta must satisfy $p^\mu p_\mu = -m^2 c^2$ and $k^\mu k_\mu = 0$ (in any frame: they are Lorentz scalars). Because of momentum conservation, we can also impose the four equations

$$p^\mu + k^\mu = p'^\mu + k'^\mu. \quad (1.8.3)$$

Solving the system of equations yields

$$\epsilon' = \frac{\epsilon}{1 + \frac{\epsilon}{mc} (1 - \vec{\Omega} \cdot \vec{\Omega}')}, \quad (1.8.4)$$

implying that we must have $\epsilon' \leq \epsilon$: the photon will lose energy in the scattering. The calculation which yields the differential cross section of the scattering is quite complicated and requires the full machinery of QED; here we just give the result:

$$\frac{d\sigma}{d\vec{\Omega}'} = \frac{r_0^2}{2} \left(\frac{\epsilon'}{\epsilon} \right)^2 \left[\frac{\epsilon}{\epsilon'} + \frac{\epsilon'}{\epsilon} - 1 + \vec{\Omega} \cdot \vec{\Omega}' \right], \quad (1.8.5)$$

where ϵ is the energy divided by mc^2 and r_0 is the classical electron radius. If we substitute in our formula for the energy, using $\xi = \cos \theta = \vec{\Omega} \cdot \vec{\Omega}'$, we find

$$\frac{d\sigma}{d\Omega' d\epsilon'} = \frac{r_0^2}{2} \frac{1 + \xi^2}{(1 + \epsilon(1 - \xi))^2} \left[1 + \frac{\epsilon^2(1 - \xi)^2}{(1 + \xi)^2(1 + \epsilon(1 - \xi))} \right] \delta \left(\epsilon' - \frac{\epsilon}{1 + \epsilon(1 - \xi)} \right). \quad (1.8.6)$$

In the low energy limit, $h\nu \ll m_e c^2$ or $\epsilon \rightarrow 0$, we find

$$\frac{d\sigma}{d\Omega' d\epsilon'} = \frac{r_0^2}{2} (1 + \xi^2) \delta(\epsilon' - \epsilon), \quad (1.8.7)$$

which is the Thompson cross section.

Let us now define the **Compton scattering kernel** σ : it is the differential cross section times the electron density,

$$\sigma(\epsilon \rightarrow \epsilon', \xi) = n_e \frac{d\sigma}{d\Omega' d\epsilon'}. \quad (1.8.8)$$

We can integrate this in order to find the total cross section presented by the electrons to photons of an energy ϵ :

$$\begin{aligned} \sigma(\epsilon) &= \int d\Omega' d\epsilon' \sigma(\epsilon \rightarrow \epsilon', \xi) \\ &= \frac{3}{4} n_e \sigma_T \left[\left(\frac{1 + \epsilon}{\epsilon^3} \right) \left(\frac{2\epsilon(1 + \epsilon)}{1 + 2\epsilon} - \log(1 + 2\epsilon) \right) + \frac{1}{2\epsilon} \log(1 + 2\epsilon) - \frac{1 + 3\epsilon}{(1 + 2\epsilon)^2} \right]. \end{aligned} \quad (1.8.9)$$

$$(1.8.10)$$

How does this differ from the Thompson cross section? For $\log \epsilon \lesssim -1$ we have $\sigma \approx \sigma_T$, while as ϵ increases the cross section goes to zero. In the low energy limit we have the expansion

$$\sigma(\epsilon) \approx \sigma_T \left(1 - 2\epsilon + \frac{26}{5} \epsilon^2 \right). \quad (1.8.11)$$

The introduction of these nonconservative aspects complicates the radiative transfer equation for scattering. The absorption term is

$$-I(\epsilon, \Omega) n_e \int d\Omega' d\epsilon' \frac{d\sigma}{d\Omega' d\epsilon'} = -I(\epsilon, \Omega) \sigma(\epsilon), \quad (1.8.12)$$

while for the emission term the intensity must go inside the integral, so we have

$$n_e \int d\Omega' d\epsilon' I(\epsilon', \Omega') \frac{d\sigma}{d\Omega' d\epsilon'} = \int d\Omega' d\epsilon' \sigma(\epsilon \rightarrow \epsilon', \xi), \quad (1.8.13)$$

which cannot be expressed in terms of the integrated kernel $\sigma(\epsilon)$.

For Thomson scattering the absorption term is similar, with $n_e \sigma_T$ instead of $\sigma(\epsilon)$. The emission term, on the other hand, can now be evaluated to yield

$$\int d\Omega' d\epsilon' \sigma(\epsilon \rightarrow \epsilon', \xi) = \frac{r_0^2}{2} n_e \int d\Omega' d\epsilon' I(\epsilon', \Omega') (1 + \xi^2) \delta(\epsilon' - \epsilon) \sim n_e \sigma_T J(\epsilon), \quad (1.8.14)$$

as long as we neglect the angular part $1 + \xi^2$.

Bibliography

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