

General Relativity notes

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1 Special relativity

Definition 1.1. *An inertial frame is one in which Newton's laws hold: a free body moves with acceleration $a^i = 0$.*

Newton's first law establishes the *existence* of inertial frames.

Proposition 1.1. *The frames O and O' are both inertial frames iff O' moves with constant velocity wrt O .*

Proposition 1.2. *Coordinate transformations between inertial frames are Lorentz boosts, which in some coordinate frame can be written as*

$$t' = \gamma_v \left(t - \frac{vx}{c^2} \right) \quad (1a)$$

$$x' = \gamma_v (x - vt) \quad (1b)$$

$$y' = y \quad (1c)$$

$$z' = z, \quad (1d)$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$.

If $v \ll c$, so $v/c \sim 0$, they simplify to the identity for t, y, z and $x' = x - vt$: these are Galilean transformations.

If we have two events, x^μ and y^μ , they occur with some time and space separation $\Delta x^\mu = x^\mu - y^\mu$. We can compute $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$, where

$$\eta_{\mu\nu} = \text{diag}(-c^2, 1, 1, 1). \quad (2)$$

Proposition 1.3. Under Lorentz transformations Δs^2 is invariant.

We can classify separations between events as

- time-like when $\Delta s^2 < 0$;
- null-like when $\Delta s^2 = 0$;
- space-like when $\Delta s^2 > 0$.

We can draw spacetime diagrams. A light cone is the set of points which are null-like separated from a select point. Things can be only causally related to events inside the light-cone (with $\Delta s^2 \geq 0$).

1.1 Time dilation

Take two events which occur at the same location for O' . In the primed frame they will have coordinates $x^\mu = (t_0, x_0)$ and $y^\mu = (t_1, x_0)$.

Definition 1.2. The proper time between these two events is $t_1 - t_0 \stackrel{\text{def}}{=} \Delta\tau$.

We now see that $\Delta s'^2 = -c^2 \Delta\tau^2$. Then, any other observer will see the same $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = \Delta s'^2$.

This directly implies that $\Delta\tau \leq \Delta t$ for any observer, since $\Delta\tau^2 = \Delta t^2 - \Delta x^2/c^2$. This effect is called *time dilation*.

By how much exactly is time dilated? Of course $\Delta x = v\Delta t$, therefore $\Delta t = \gamma_v \Delta\tau$.
-> Muon problem.

Inverse Lorentz transformation have the same expression, but with $v \rightarrow -v$. This can be proved both mathematically by solving the equations and phisically by reasoning about their meaning. There is no preferential inertial frame.

A Lorentz transformation can be written in matrix form in the (ct, x) plane as:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \begin{bmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{bmatrix} \quad (3)$$

since there is an angle θ such that $\gamma = \cosh \theta$ and $\gamma\beta = \sinh \theta$: the angle θ will be $\theta = \tanh^{-1}(v/c)$. This is true because $\gamma^2 - \beta^2\gamma^2 = 1$.

After a boost the ct' and x' axes are respectively the lines $ct = x/\beta$ and $ct = \beta x$.

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Last lecture we saw the fact that the ct' and x' axes are rotated by equal angles from the ct and x axes towards the $ct = x$ axis.

1.2 Relativity of simultaneity

Consider two events which are simultaneous in the O' frame. Their times in this frame are $t'_A = t'_B$.

In the O frame, instead, we have

$$ct_{A,B} = \frac{v}{c}x_{A,B} + \underbrace{\sqrt{1 - \frac{v^2}{c^2}}}_{\text{a constant}} ct'_{A,B}, \quad (4)$$

so the events are not simultaneous in the O frame.

1.3 Length contraction

If in the O frame, A occurs at $t, x = 0$ while B occurs at $t = 0, x = L$, then L is the measured length of their spatial interval by O . We assume that this is the frame in which the object is moving, and we transform into a frame in which it is stationary: O' .

In the primed frame their coordinates will be:

$$x'_A = \gamma_v \left(x_A - \frac{v}{c} ct_A \right) \quad (5a)$$

$$x'_B = \gamma_v \left(x_B - \frac{v}{c} ct_B \right), \quad (5b)$$

therefore $x'_B - x'_A = \gamma_v(x_B - x_A)$: the length is contracted in the O frame, since $\gamma \geq 1$.

1.4 Addition of velocities

Two observers see an object moving with $v' = dx'/dt'$ and $v = dx/dt$ respectively. Their relative velocity is u . Differentiating we get:

$$v' = \frac{\gamma(dx - v dt)}{\gamma\left(dt - \frac{u dx}{c^2}\right)} = \frac{v - u}{1 - \frac{uv}{c^2}}. \quad (6)$$

Two interesting limits of this formula are: $v' = v - u$ if $u \ll c$ or $v \ll c$; and $v' = c$ if $v = c$ for whatever u .

1.5 Tensor notation

The position four-vector is $x^\mu = (ct, x, y, z)$. The Euclidean scalar product is given by $x \cdot y = \delta_{\mu\nu} x^\mu x^\nu$. If we substitute the identity $\delta_{\mu\nu}$ with another metric we can find a more general metric space.

The Minkowski metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The separation 4-vector is $dx^\mu = (c dt, dx, dy, dz)$.

Using Einstein summation notation, we can write the spacetime interval as $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

Specifically for the Minkowski metric we have the relation $\eta_{\mu\nu} = \eta^{\mu\nu}$: it is its own inverse. For a general metric $g_{\mu\nu}$ this will not hold.

How do we express the Lorentz boosts? They preserve ds^2 , therefore they look like $x'^\mu = \Lambda^\mu_\nu x^\nu$, with the $(1, 1)$ tensors Λ^μ_ν satisfying $\Lambda^\mu_\nu \Lambda^\sigma_\rho \eta_{\mu\sigma} = \eta_{\nu\rho}$. This is called the *pseudo-orthogonality* relation.

The metric allows us to raise and lower indices. Raising an index in the pseudo-orthogonality relation gives us: $\Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta \eta^{\beta\sigma} = \delta_\alpha^\sigma$, therefore $\eta_{\mu\nu} \Lambda^\nu_\beta \eta^{\beta\sigma}$ is the inverse of a Lorentz transformation.

Four-vectors can also have their indices down, and they will transform according to the inverse of Lorentz transformations:

$$(\eta_{\alpha\mu} x^\mu)' = \eta_{\alpha\mu} \Lambda^\mu_\nu x^\nu \quad (7a)$$

$$= \Lambda_{\alpha\sigma} \delta^\sigma_\nu x^\nu \quad (7b)$$

$$= \Lambda_{\alpha\sigma} \eta^{\sigma\beta} \eta_{\beta\nu} x^\nu \quad (7c)$$

$$= \Lambda_\alpha^\beta x_\beta. \quad (7d)$$

We will write our laws as tensorial equations, which are covariant.

By pseudo-orthogonality, the scalar product $A_\mu B^\mu$ is a covariant (that is, invariant) scalar. Of course it is equal to $A^\mu B_\mu$.

Definition 1.3 (Tensor). A (p, q) tensor is an object $M_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q}$ with many components indexed by $p + q$ indices, which transforms as:

$$M_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_q} \rightarrow \Lambda_{\mu_1}^{\mu'_1} \dots \Lambda_{\mu_p}^{\mu'_p} \Lambda^{\nu_1}_{\nu'_1} \dots \Lambda^{\nu_q}_{\nu'_q} M_{\mu'_1 \dots \mu'_p}^{\nu'_1 \dots \nu'_q} \quad (8)$$

under Lorentz transformations Λ_μ^ν .

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Last lecture we introduced tensors.

An example of those is the EM tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_x & B_y \\ -E_y/c & B_x & 0 & -B_z \\ -E_z/c & -B_y & B_z & 0 \end{bmatrix}, \quad (9)$$

which, it can be checked, transforms as a (0,2) tensor. Also, we can define the current vector $j^\mu = (c\rho, j^i)$. Then, the Maxwell equations read:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad \text{and} \quad \partial_{[\mu} F_{\nu\rho]} = 0. \quad (10)$$

They are covariant!

1.6 The 4-velocity

In Newtonian mechanics, the motion of a particle is described by a function of time $x^i = x^i(t)$.

In special relativity, we introduce the concept of *worldline*. It must be parametrized with respect to some parameter λ , such that $x^\mu = x^\mu(\lambda)$. A preferred choice for λ is the proper time of the particle, $\lambda = \tau$.

We then define the 4-velocity:

$$u^\mu = \frac{dx^\mu}{d\tau}. \quad (11)$$

It is a tensor since it is the product of a scalar and a tensor.

Multiplying $u^\mu u_\mu$ we always get $-c^2$, since:

$$u^\mu u_\mu = \frac{dx^\mu dx_\mu}{d\tau^2} = -c^2 \frac{ds^2}{d\tau^2} \quad (12)$$

We can make the expression explicit using $d\tau = \gamma dt$, which gives us $u^\mu = (\gamma c, \gamma v^i)$. In the frame of the particle, $u^\mu = (c, 0)$.

The *four-momentum* of a particle is defined as:

$$p^\mu = m u^\mu = (m\gamma c, m\gamma v^i). \quad (13)$$

The component p^0 is mc at $v = 0$. What does it mean? we can expand it for small v :

$$\frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}} \sim mc \left(1 + \frac{v^2}{2c^2} \right) = mc + \frac{1}{2} \frac{mv^2}{c}. \quad (14)$$

We get the mass, plus a kinetic energy term: more explicitly, $cp^0 = mc^2 + 1/2mv^2$.