Early Universe Cosmology

Jacopo Tissino

2020-11-02

Contents

Wednesday 2020-9-30, compiled 2020-11-02

Introduction

Professor Nicola Bartolo, `bartolo@pd.infn.it`. Office 236 in the DFA department.

Live lectures will be in room P1A, usually at the blackboard, sometimes with slides. There will be notes uploaded to Moodle.

Course content An up-to-date overview of the physics of the Early Universe. The goal is to be able to understand and analyze the problems from both a theoretical and observational point of view.

There are three main parts:

1. inflationary models: the issue of the inital conditions;
2. cosmological perturbations, GW of cosmological origin;
3. baryogenesis, production of DM particles.

All of these will be connected with observations: nowadays cosmology is data-driven.

There are connections with: cosmology, astroparticle physics, astrophysics, GR, theoretical physics, field theory, multimessenger astrophysics, GW.

We will do "blended learning", with shifts in the classroom. We will do 48 hours of lectures.

Textbooks: Liddle and Lyth, "The primordial Density Perturbation", "Cosmological inflation and "

Complete

There will be 2 exam dates for each session. The exam is an oral one. Office hours can be arranged anytime.

We should limit questions in the break, prefer asking them during the lecture itself.

Chapter 1

Inflationary models

1.1 A general introduction

The basic issue is to find what initial conditions would produce the universe as we currently observe it.

Observational probes of the Hot Big Bang model: the Hubble diagram, Big Bang nucleosynthesis, the CMB.

On large scales we observe a smooth universe. However, that is a "zeroth-order" approximation: there are structures and anisotropies. All the structures need initial conditions to start from and then grow through gravitational instability.

We have several observables to probe the anisotropies: CMB, LSS, clusters of galaxies, weak gravitational lensing. There are initial fluctuations on the order of

$$\frac{\delta\rho}{\rho} \sim \frac{\delta T}{T} \sim 10^{-5} . \tag{1.1.1}$$

What is the initial time and temperature at which these perturbations start? Is there a dynamical mechanism which produces the perturbations? How do the perturbations evolve exactly? How do they relate to baryogenesis?

Under a Newtonian treatment, relative density perturbations grow like $\delta_m \propto a(t)$. The problem we will address here is how the initial value of δ_m comes about.

The CMB is a very good blackbody, without spectral distortions except for the Sunyaev-Zel'dovich effect (inverse Compton scattering from high-energy electrons in galaxies up-scattering the CMB photons).

We recall some basic concepts about the smooth model of the universe: critical density, Hubble parameter and so on.

The standard ΛCDM model does predict a small deviation, $\mu/T \sim 1.9 \times 10^{-8}$, from the Planckian, whose phase space distribution is:

$$f = \left[\exp\left(\frac{h\nu - \mu}{k_B T} \right) - 1 \right]^{-1} . \tag{1.1.2}$$

Currently we have upper bounds: $\mu/T < 9 \times 10^{-5}$ at 95 % CL.

The CMB radiation is also highly, but not perfectly, isotropic. the scale of the temperature angular anisotropies are of the order $\Delta T/T \sim 10^{-5}$ (the quoted value for $\Delta T/T$ is a root-mean-square, since the average of ΔT is zero). This is to say: in each direction we observe a very good blackbody, whose characteristic temperature changes slightly depending on the direction.

Planck 2018 had an angular resolution of 5 arcminutes, and it also measured the polarization of the CMB.

We also have redshift galaxy surveys like the Sloan Digital Sky Survey. We map galaxies in redshift space. There is a statistical pattern of the galaxies, which is connected to the origin of the inhomogeneities.

The idea is that the seeds of the perturbations are quantum mechanical, coming from the inflaton scalar field, which are made into galaxies and galaxy clusters from gravitational instabilities.

At $z \sim 20$ the DM distribution was quite smooth, it then clustered.[1]

The components of the ΛCDM model are:

1. dark energy 68 %;
2. dark matter 26 %;
3. hydrogen and helium gas 4 %;
4. stars 0.5 %;
5. neutrinos 0.26 %;
6. metals 0.025 %;
7. radiation 0.005 %.

We also need seed perturbations and baryo-leptogenesis. We will see phases in which the universe is not in thermal equilibrium.

We want to find information about energies up to 10^{16} GeV: we will see that the inflationary phase corresponds to this epoch.

GW from inflation travel basically unimpeded from inflation to us.

Today, we have radiation with $w = 1/3$, $\rho \propto a^{-4} \propto T^4$, so, Tolman's law $Ta = \text{const}$.

Baryonic matter has $\Omega_b h^2 = 0.0224 \pm 0.0001$. Its equation of state is $P = nT \ll nm$. So, $\rho \propto a^{-3}$. Dark matter is also nonrelativistic, with $P \approx 0$, and $\Omega_{DM} h^2 0.120 \pm 0.001$.

The cosmological constant has $P = -\rho$, and $\Omega_\Lambda = 0.6847 \pm 0.0073$.

Neutrinos have $\sum m_\nu < 0.12\,\text{eV}$, and $\Omega_\nu h^2 < 0.0012$. Both values are at 95 % CL.

Spatial curvature has $\Omega_k = 1 - \Omega_0 = 0.001 \pm 0.002$, from Planck, Baryon Acoustic Oscillations, local measurements.

The presence of discordance can surely signal systematics, but also new physics. There are certain discordances.

[1] https://www.youtube.com/watch?v=FBkYIqtYb0I.

Beyond galactic rotation curves, we also have evidence for DM from the power spectrum of inhomogeneities. We Fourier-transform the density perturbation field δ to get $\delta_{\vec{k}}$; then we can calculate the power spectrum

$$\Delta^2(k) = \frac{\partial \sigma^2}{\partial \log k} \propto k^3 \left| \delta_{\vec{k}} \right|^2 \propto k^{3+n} T^2(k) . \tag{1.1.3}$$

The Poisson equation reads

$$4\pi G \bar{\rho} \delta = \nabla^2 \Phi \implies \delta_{\vec{k}} \propto k^2 \Phi_{\vec{k}} . \tag{1.1.4}$$

If the gravitational perturbation is written as

$$\Phi_{\vec{k}} = \Phi_{\vec{k}}^{\text{primordial}} T(k) \times \text{growth function} , \tag{1.1.5}$$

and the primordial field perturbation squared is $\left| \Phi_k^{\text{primordial}} \right|^2 \propto k^{n-4}$, where $n = 0.9600 \pm 0.0042$ is a spectral index. This explains the last proportionality sign we wrote earlier, the density power spectrum includes information about the power-spectral index of the initial conditions. This index measures the amplitude of the inhomogeneities in DM density.

Also, if we only had baryons without dark matter the power spectrum of the matter density perturbations would look very different from what they do.

Baryon Acoustic Oscillations are an oscillatory imprint in the power spectrum, they have been measured today.

Inflation is an early epoch in the history of the universe during which expansion is accelerated. The basic predictions of inflation are so far confirmed, however we have not detected the SGWB from it, which would be a "smoking gun".

Monday 2020-10-5, compiled 2020-11-02

These next few lectures, we will consider the motivations for inflationary models. The problems they solved were the **shortcomings of the Hot Big Bang** model.

1. The horizon problem;
2. the flatness problem;
3. unwanted relics / magnetic monopole problem.

We start by recalling some basic elements in cosmology. In order to describe a homogeneous and isotropic universe we use the FLRW metric:

$$\mathrm{d}s^2 = -c^2 \,\mathrm{d}t^2 + a^2(t) \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \,\mathrm{d}\Omega^2 \right] , \tag{1.1.6}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\varphi^2$. The quantity $a(t)$ is called the *scale factor*. The coordinates r, θ and φ are called *comoving coordinates*.

Physical distances and comoving distances are related by

$$\lambda_{\text{phys}} = a(t) \lambda_{\text{comoving}} . \tag{1.1.7}$$

The constant k is the spatial curvature of the universe, which can always be rescaled so that it is equal to

1. $+1$ for a spatially closed universe;
2. 0 for a spatially flat universe;
3. -1 for a spatially open universe.

In terms of the scale factor we define the **Hubble parameter**

$$H = \frac{\dot{a}}{a}, \tag{1.1.8}$$

which describes the rate at which the universe expands.

The dynamics of gravity are described by the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1.9}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the particle species filling the universe, while $G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu}/2$ is the Einstein tensor, describing curvature.

These can be derived from an action principle through the action

$$S = \underbrace{\frac{1}{16\pi G} \int R \sqrt{-g} \, \mathrm{d}^4 x}_{S_{EH}} + S_{\text{matter}}. \tag{1.1.10}$$

Often we use an ideal fluid energy-momentum tensor:

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}, \tag{1.1.11}$$

where $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$ is a projector onto the space orthogonal to the four-velocity. This does not account for any anisotropy, it is the most symmetric energy-momentum tensor. This is a diagonal

In order to solve the Einstein equations we can proceed with some assumptions, without needing to know the action for all the fundamental fields. The perfect fluid S-E tensor has all the FLRW symmetries, as long as ρ and P are only functions of time.

We are *not* saying that this S-E tensor is only allowed if we are in a FLRW universe.

Clarify…

Requiring the FLRW symmetries means that the S-E tensor must be diagonal, however we can have viscosity as long as it is not *shear* but *bulk* viscosity, which adds onto the diagonal terms.

Inserting the FLRW metric into the Einstein equation yields the Friedmann equations:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.1.12}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{1.1.13}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P). \tag{1.1.14}$$

The first two can be derived from the Einstein equations directly, the third comes from the "conservation law" $T_{\mu\nu}{}^{;\nu} = 0$.

They are not independent, only two are. We have too many parameters: a, ρ and P, but only two independent equations, so we "close" the system of equations with an equation of state, commonly $P = P(\rho) = w\rho$.

These equations of state describe many kinds of fluids (approximately): dust with $w = 0$, which means $\rho \propto a^{-3}$; radiation with $w = 1/3$, which means $\rho \propto a^{-4}$; a cosmological constant with $w = -1$, which means $\rho = \text{const}$.

In general as long as $w \neq 1$ we have $\rho \propto a^{-3(1+w)}$ and $a \propto t^{2/3(1+w)}$.

1.1.1 The horizon problem

The **particle horizon**, denoted as $d_H(t)$, is given by

$$d_H(t) = a(t) \int_0^t \frac{c\, \mathrm{d}\tau}{a(\tau)}\,, \tag{1.1.15}$$

and it sets the radius of a sphere centered at an observer O. The points inside this sphere have been able to have causal interactions with observer O in the time from the Big Bang to t.

It is the proper distance (as measured today) which could have been travelled by light starting at the beginning and moving in a geodesic. It can be derived from the FLRW metric by assuming radial light-like motion

$$\mathrm{d}s^2 = -c^2\, \mathrm{d}t^2 + a(t) \frac{\mathrm{d}r^2}{1 - kr^2} = 0\,, \tag{1.1.16}$$

and setting $k = 0$:[2]

$$c\, \mathrm{d}t = \pm a(t)\, \mathrm{d}r\ , \tag{1.1.17}$$

which we can use to calculate the *comoving distance* from the point of emission to today, which we then multiply by the scale factor calculated at a chosen point.

We know that the scale factor goes to 0 as t goes to 0, so the integral giving us r_H could diverge. We can show that d_H is finite as long as $\alpha = 2/3(1 + w)$ is smaller than one, meaning that $w > -1/3$, which is equivalent to $\ddot{a} < 0$. In a decelerating universe, the particle horizon is finite.

In general, the calculation yields

$$d_H(t) = \frac{3(1 + w)}{1 + 3w} ct\,. \tag{1.1.18}$$

With $w = 0$, a spatially flat matter-dominated universe, $d_H = 3ct$. This is called an Einstein-De Sitter universe. With $w = 1/3$, a spatially flat radiation-dominated universe, we have $d_H = 2ct$.

[2] This is a good approximation for early times, even if the universe is not flat.

Another way to characterize causality is the Hubble radius:

$$r_C(t) = \frac{c}{H(t)}\,. \tag{1.1.19}$$

The characteristic time of expansion is $\tau(H) = H^{-1}$. We can show that in a FLRW universe, typically after a Hubble time the scale factor doubles.

Since

$$H(t) = \frac{2}{3(1+w)}\frac{1}{t}\,, \tag{1.1.20}$$

we can write

$$R_H \approx \frac{1+3w}{2} d_H(t) \approx d_H(t)\,. \tag{1.1.21}$$

check! exercise

This will *not* happen in an inflationary universe. The two are similar in a regular FLRW universe, while they differ a lot if there is inflation. The particle horizon takes into account all the past history of an observer, the Hubble radius does not care about it: it only described causal connections taking place in a time interval taking place in a Hubble time.

Let us introduce the *comoving Hubble radius*: $r_H(t)$, given by

$$r_H(t) = \frac{r_C(t)}{a(t)}\,. \tag{1.1.22}$$

Let us plot this for a matter or radiation-dominated FLRW universe.

In radiation domination, $r_H \propto \sqrt{t}$, while in matter domination $a \sim t^{2/3}$ so $r_H \propto t^{1/3} \sim a^{1/2}$.

This comoving radius is then always increasing, initially faster and then slower.

Instead, consider the comoving particle horizon: $d_H(t)/a(t)$, so just the integral in the definition of $d_H(t)$:

$$\frac{d_H(t)}{a(t)} = \int \frac{c\,\mathrm{d}t}{a} = \int \frac{\mathrm{d}a}{a}\underbrace{\frac{c}{aH}}_{r_H}\,, \tag{1.1.23}$$

so we can see that the comoving *particle horizon* is the logarithmic integral over the scale factor of the comoving *Hubble radius*: as we mentioned before, this takes into account the whole past history.

In a matter dominated universe, $d_H = 2r_C \approx 5h^{-1}$Gpc.

Now we discuss the horizon problem, which is best understood in a comoving plot. We neglect dark energy for simplicity.

If we choose a fixed comoving size λ, we get in our model that in early times λ is super-horizon, then at a certain point it crosses the horizon, becoming smaller than r_H. The time at which $r_H = \lambda$ is called the *horizon crossing* time, $t_H(\lambda)$.

For times earlier than $t_H(\lambda)$, by definition it is impossible for points at a distance λ to be causally connected. This happens for every scale, and it means that for many regions we are interested in there cannot have been causal connection in the early universe. But, today we observe the universe to exhibit the same properties across the whole sky, even though the regions were causally disconnected earlier.

This is most directly expressed in terms of CMB photons. They would have become causally connected at the quadrupole scale (separations of 90°) almost *today*.

We can compute the size of the horizon at the last scattering epoch: this subtends an angle in the sky of around 1°; however we observe photons with the same temperature on much larger scales, this was already seen by COBE with an angular resolution of 7°.

re-do calculation!

Photons which could not have been in causal contact in the HBB model are observed to have the same temperature.

The inflationary solution to this issue is to think that, before the radiation-dominated epoch, the comoving Hubble radius decreased for a certain period of time.

This allows the parts of the sky to have been in causal contact in the early universe.

This means that $\ddot{a} > 0$ in the inflationary phase, or $w < -1/3$. These are only the *kinematics* of inflation, we are not yet discussing how it might come about.

Wednesday 2020-10-7, compiled 2020-11-02

We come back to the horizon problem. [Plot of the comoving Hubble radius r_H as a function of time]

The problem is solved if there is an early epoch in which r_H decreases in time, due to accelerated expansion.

After the end of this inflation, the regular FLRW universe's history starts, with the radiation, then matter, then cosmological constant dominated phases. An accelerated expansion, however, is not enough to solve the horizon problem: what we need is for *every* observable scale, up to the largest ones, was causally connected in the early universe. In other words, the inflation phase must last *long enough*.

More specifically, our constraint on inflation is that it must start when the Hubble radius was at least as large as it is today. This can be expressed in terms of the *number of e-folds*:

$$N = \log\left(\frac{a_f}{a_{\text{in}}}\right) = \int_{t_{\text{in}}}^{t_f} H(t)\,\mathrm{d}t\ , \tag{1.1.24}$$

the ratio of the scale factor at the beginning and at the end of inflation. The number of elapsed e-folds is a natural measure of time in the epoch of inflation.

We can give the bound $N \gtrsim 60 \div 70$ in order to solve the horizon problem. This is a *huge* expansion! Typical atomic scales of $10^{-15}\,\mathrm{m}$ get stretched to the typical scales of the Solar System, $10^{11}\,\mathrm{m}$.

The condition is $r_H(t_{\text{in}}) \gtrsim r_H(t_0)$. We can express this as

$$\frac{1}{a_{\text{in}} H_{\text{in}}} =\gtrsim \frac{1}{a_0 H_0} \tag{1.1.25}$$

$$\frac{a_f}{a_{\text{in}}} = e^N \gtrsim \frac{H_i}{H_0}\frac{a_f}{a_0}\,. \tag{1.1.26}$$

See on Moodle: paper with the exact computation.

We want to bring on the left all the quantities in the inflationary epoch. Recall that $H^2 \propto \rho \propto a^{-3(1+w_i)/2}$, which we will apply to the inflationary epoch with an equation of state $w_i < -1/3$. This means that

$$\frac{H_i}{H_f} H_f = \left(\frac{a_i}{a_f}\right)^{-3(1+w_i)/2} H_f \,, \tag{1.1.27}$$

so

$$\frac{a_f}{a_i}\left(\frac{a_f}{a_i}\right)^{-3(1+w_i)/2} = \gtrsim \frac{H_f}{H_0}\frac{a_f}{a_0} \tag{1.1.28}$$

$$\left(\frac{a_f}{a_i}\right)^{\frac{-(1+3w_i)}{2}} \gtrsim \frac{T_0}{H_0}\frac{H_f}{T_f} \,, \tag{1.1.29}$$

where we applied Tolman's law, $T \sim 1/a$, neglecting the matter dominated phase — this is a reasonable approximation, we find a similar result to the complete calculation. This yields

$$N \gtrsim -\frac{2}{1+3w_i}\left[\log\frac{T_0}{H_0} + \log\frac{H_f}{T_f}\right] , \tag{1.1.30}$$

where $T_0 = 2.7\,\mathrm{K} \approx 10^{-13}\,\mathrm{GeV}$, while $H_0 \sim 10^{-42}\,\mathrm{GeV}$ in natural units. Therefore, the first logarithm is of the order ~ 67. We also need the *pre-heating* temperature and Hubble parameter: H_f and T_f. This is model-dependent: it is what gives the theoretical uncertainty. The dependence, however, is weak: only logarithmic.

With current measurements, we are starting to be able to measure this term as well. Let us give an estimate for it:

$$H_f^2 \approx \frac{8\pi G}{3}\rho_{\mathrm{rad}} \,, \tag{1.1.31}$$

where $\rho_{\mathrm{rad}} \frac{\pi^2}{30} g_* T^4$. This then yields

$$H_f^2 = \frac{8\pi G}{3}\frac{\pi^2}{30} g_* T^4 \sim \frac{T_f^4}{M_P^2} \,. \tag{1.1.32}$$

There is model dependence here, in g_*! If we go BDSM (beyond de standard model) it could change. We are giving a very rough estimate with the Planck mass. This then means $H_f \sim T_f^2/M_p$. So,

$$\log\left(\frac{H_f}{T_f}\right) \approx \log\frac{T_f}{M_p} \,. \tag{1.1.33}$$

Typically, models predict

$$10^{-5} < \frac{T_f}{M_p} < 1 \,, \tag{1.1.34}$$

but this is not set in stone, we could have different predictions as well.

Now, let us assume that $w_i \sim -1$, something like a cosmological constant. Then, the prefactor is of the order 1, so the bound is $N \gtrsim 60 \div 70$ as was mentioned before.

We now discuss the causal structure of the FLRW metric:

$$ds^2 = -\,dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,d\Omega^2\right]. \tag{1.1.35}$$

Let us express this in different coordinates: we introduce χ, so that

$$r = S_k(\chi) = \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1 \end{cases}. \tag{1.1.36}$$

This allows the term $dr^2/(1-kr^2)$ to become simply $d\chi^2$.

Also, we introduce conformal time: $d\eta = dt/a(t)$, so that the metric becomes

$$ds^2 = a^2(\eta)\left[-\,d\eta^2 + d\chi^2 + S_k^2(\chi)\,d\Omega^2\right]. \tag{1.1.37}$$

The meaning of χ is still a comoving distance. However, the interesting thing is that this metric is conformally related to ("is a time-dependent rescaling of") the Minkowski metric (if we consider radial motion, at least), we say that it is *conformally flat*.

In these coordinates, light propagates at 45° in the (η, χ) plane.

Then, we can draw a diagram for the horizon problem in these coordinates: the Big Bang singularity looks like a straight line at constant η. The last-scattering surface is also a straight line at constant η. We can then draw a past light-cone from a point in the last-scattering surface. Inflation pushes the BB surface back in conformal time, so that light has more time to propagate.

We can show (exercise) that the conformal time at the end of inflation looks like $\eta \propto 2/(1+3w_i)a^{2/(1+3w)}H_*^{-1}$.

So, if $w < -1/3$, we are good.

We can only detect correlations in the CMB up to the quadrupole, since the dipole is correlated with the Earth's motion… Roughly, this means that we can only see correlations on the scale of $\sim r_H$, corresponding to 90° separation, instead of being able to see them on the scale of 180°.

Now, we move to the flatness problem. The HBB model is not intrinsically flawed, however the shortcomings we are discussing tell us that the initial conditions which would be required in order to yield the current universe would be very specific.

We should set initial conditions which are homogeneous and isotropic, with very specific small fluctuations. Inflation provides a dynamical solution to these problems, which is an attractor towards these initial conditions.[3]

[3] See Hossenfelder [Hos19] for a critical discussion of this fine-tuning problem.

The first Friedmann equation reads

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{1.1.38}$$

which we can express through $\Omega = \rho/\rho_c$, where $\rho_c = 3H^2/(8\pi G)$:

$$1 - \Omega = \frac{k}{a^2 H^2} = k r_H^2(t)\,, \tag{1.1.39}$$

so if Ω differs from unity even by a small amount, this difference increases with time.

At 95 % CL, we know that $|\Omega - 1| = |\Omega_k| < 0.4\,\%$, so the universe we observe is consistent with flatness.

Specifically, in the Planck epoch we will have

$$\Omega(t_{\text{Pl}}) - 1 \approx (\Omega_0 - 1) \times 10^{-60}\,, \tag{1.1.40}$$

so $\left|\Omega(t_{\text{Pl}}) - 1\right| < 10^{-62}$.

What can be shown is also that

$$\left(\Omega^{-1} - 1\right)\rho a^2 = \text{const}\,. \tag{1.1.41}$$

For times before the matter-radiation equivalence $\rho \propto a^{-4}$, so $\rho(t) = \rho_{\text{eq}}(a_{\text{eq}}/a)^4$. Also, during matter domination up to now (neglecting the cosmological constant)

$$\rho_0 = \rho_{\text{eq}}\left(\frac{a_{\text{eq}}}{a_0}\right)^3\,, \tag{1.1.42}$$

therefore

$$(\Omega^{-1} - 1)\left(\frac{a_{\text{eq}}}{a}\right)^4 a^2 \rho_0 \left(\frac{a_0}{a_{\text{eq}}}\right)^3 \frac{1}{\rho_0 a_0^2} = (\Omega_0^{-1} - 1) \tag{1.1.43}$$

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)\frac{a^2}{a_{\text{eq}} a_0} \tag{1.1.44}$$

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)(1 + z_{\text{eq}})\frac{a^2}{a_0^2}\,, \tag{1.1.45}$$

since $1 + z_{\text{eq}} = a_0/a_{\text{eq}}$. Also, we can approximate $a/a_0 \sim T_0/T_{\text{Pl}}$ by extending Tolman's law. Then,

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)\underbrace{(1 + z_{\text{eq}})}_{\sim 10^4}\underbrace{\frac{T_0^2}{T_{\text{Pl}}^2}}_{\sim 10^{-64}}\,. \tag{1.1.46}$$

This proves the relation we wrote earlier. It is an extreme extrapolation to go back to the Planck time, but even if we only went back to Big Bang nucleosynthesis ($\sim 1\,$MeV) we would get

$$\left|\Omega(t_{\text{BBN}}) - 1\right| < 10^{-18}\,. \tag{1.1.47}$$

How does inflation solve the problem? Recall that $\Omega - 1 = kr_H^2$, and inflation is by definition a time in which r_H decreases. At the end of inflation, $\Omega - 1$ is very close to 0, meaning that r_H is small, but at the start of inflation it could have been relatively far from 1.

Next week we will discuss the proper mechanism of this process. During an inflationary phase, $a(t) \approx \exp(Ht)$. So, as long as H is approximately constant, we have

$$r_H^2 = \frac{1}{a^2 H^2} \propto \frac{1}{a^2} \,. \tag{1.1.48}$$

Then, we have

$$\frac{|\Omega - 1|_{t_f}}{|\Omega - 1|_{t_i}} \sim \left(\frac{a_i}{a_f} \right)^2 \sim \exp(-2N) \,. \tag{1.1.49}$$

This means that, with very broad possible initial conditions, we find $\Omega - 1$ very close to zero at the end of inflation.

A De-Sitter phase is a reference example of a possible inflationary stage. It would correspond to $\rho = \text{const}$, $w = -1$: in general, since the "curvature energy density" scales like a^{-2}, curvature becomes negligible.

Note, however, that this is an unrealistic example: it does not include any method for the inflation to end.

Monday 2020-10-12, compiled 2020-11-02

As we saw last time, inflation provides an "attractor solution" to the flatness problem.

We must impose

$$\frac{1 - \Omega_i^{-1}}{1 - \Omega_0^{-1}} \gtrsim 1 \tag{1.1.50}$$

in order to solve the flatnesss problem. Since $(1 - \Omega^{-1})\rho a^2$ is a constant, this ratio is equal to

$$\frac{1 - \Omega_i^{-1}}{1 - \Omega_0^{-1}} = \frac{\rho_0 a_0^2}{\rho_i a_i^2} = \frac{\rho_0 a_0^2}{\rho_{\text{eq}} a_{\text{eq}}^2} \frac{\rho_{\text{eq}} a_{\text{eq}}^2}{\rho_f a_f^2} \frac{\rho_f a_f^2}{\rho_i a_i^2} \,, \tag{1.1.51}$$

and since $\rho \propto a^{-3(1+w)}$ the term we are considering scales like $\rho a^2 \propto a^{-(1+3w)}$. Substituting this in, accounting for the fact that the first term is in the matter-dominated epoch while the second is in the radiation-dominated one we have

$$\frac{1 - \Omega_i^{-1}}{1 - \Omega_0^{-1}} = \left(\frac{a_0}{a_{\text{eq}}} \right)^{-1} \left(\frac{a_{\text{eq}}}{a_f} \right)^{-2} \left(\frac{a_f}{a_i} \right)^{-(1+3w_f)} e^{N|1+3w_f|} = \frac{1 - \Omega_i^{-1}}{1 - \Omega_0^{-1}} X \,, \tag{1.1.52}$$

where w_f is the equation of state in the inflationary phase, while

$$X = \frac{a_0}{a_{\text{eq}}} \left(\frac{a_{\text{eq}}}{a_f} \right)^2 \tag{1.1.53}$$

$$\approx 60 \,. \tag{1.1.54}$$

Then,

$$e^{N\left|1+3w_f\right|} \gtrsim \underbrace{\frac{1-\Omega_i^{-1}}{1-\Omega_0}}_{\gtrsim 1} X \tag{1.1.55}$$

$$N_{\min} = \frac{\log X}{\left|1+3w_f\right|} \approx 60 \div 70\,, \tag{1.1.56}$$

where we assumed $w_f \sim -1$. There is some model dependence, specifically regarding the transition from inflation to radiation domination.

It is interesting that this is similar to the number of e-folds needed to solve the horizon problem.

We can also write the expression, defining $N = pN_{\min}$, as

$$1-\Omega_0^{-1} = \frac{1-\Omega_i^{-1}}{\exp\Big((p-1)N_{\min}(1+3w_f)\Big)}\,, \tag{1.1.57}$$

so, taking $p = 2$ and $w_i = -1$ we have

$$1-\Omega_0^{-1} = (1-\Omega_i^{-1})e^{-2N_{\min}} = (1-\Omega_i^{-1})X^{-1}\,. \tag{1.1.58}$$

Constraining Ω_0 allows us to constrain $\Delta N = N - N_{\min}$. Right now we have $|1-\Omega_0| < 0.4\,\%$ (at $95\,\%$ CL) we can say $\Delta N \gtrsim 5$.

1.1.2 The flatness problem as an age problem

Consider the HBB model, at $t < t_{\mathrm{eq}}$: the radiation-dominated epoch.

The Planck time, the only characteristic time of the universe a priori, is $t_{\mathrm{Pl}} \approx 10^{-43}\,\mathrm{s}$. If the universe is spatially closed, then $t_{\mathrm{collapse}} = 2t_m$. We would expect for the time of collapse to be of the order of the Planck time.

If the universe was spatially open, we would expect that for $t > t_* \sim t_{\mathrm{Pl}}$ we would have curvature domination: $a(t)/a(t_*) \sim t/t_*$.

Therefore, we would have $t_0 = t_{\mathrm{Pl}} T_{\mathrm{Pl}}/T_0 \sim 10^{-11}\,\mathrm{s}$.

In order for this to not happen, we need the energy density term to finely balance the curvature term in the first Friedmann equation.

1.1.3 The unwanted relics problem

This is also known as the "magnetic monopoles problem". Consider a massive particle X, such that $\Omega_{0X} \gg 1$.

Typically, $\Omega_{0X} \sim 1/\sigma_A$.

Historical examples are cosmic topological defects, arising from the SSB of some gauge theory. Also, we could have cosmic strings: one-dimensional defects, arising from the SSB of $U(1)$. Magnetic monopoles could come from a Grand Unified Theory.

Domain walls arise from the SSB of discrete symmetries, 3D textures arise from the SSB of $U(2)$.

Other examples of unwanted relics are gravitinos (spin 3/2 superpartners of gravitons, with $m \sim 100\,\text{GeV}$) or spin-0 *moduli* from superstring theory.

We will explore how these can overclose the universe. We start with SSB in a cosmological context. SSB means that the ground state has less symmetry than the full theory. How do we describe it in the context of an expanding universe?

If go back in time to when the temperature was very high, we expect a restoration of the symmetry.

We start with a lot of symmetry, and as the universe expands we lose it. There are *phase transitions* accompanying this change.

Wednesday 2020-10-14, compiled 2020-11-02

We were talking about Spontaneous Symmetry Breaking: how does it work in a cosmological context? Specifically, we ask about its effect on cosmological phase transitions at high temperatures in the early universe.

We start out with a scalar field φ with Lagrangian[4]

$$\mathscr{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi)\,, \tag{1.1.59}$$

where we choose a potential

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \sigma^2\right)^2. \tag{1.1.60}$$

This is the typical example of a potential which exhibits SSB. Its vacuum (minimum) is a pair of points at $\left|\varphi\right| = \sigma$.

The Lagrangian is invariant under $\varphi \to -\varphi$; either of the vacuum states is not.

We need to consider finite-temperature effects on the propagator of the scalar field. The temperature corrections to this potentials yields a temperature-dependent mass term, which looks like

$$m_T^2 = \alpha \lambda T^2\,, \tag{1.1.61}$$

where λ is the coupling of the field, while α is a dimensionless order-1 number.

Then, the potential reads

$$V_T(\varphi) = V_{t=0}(\varphi) + \frac{1}{2} \alpha \lambda \varphi^2 T^2\,. \tag{1.1.62}$$

If the temperature is sufficiently high, the potential will have only one vacuum again. This means that if we go far enough back in time the symmetry is restored.

The moment at which the potential goes from one minimum to two is the one at which we have a **phase transition**. At which temperature does it happen? We can find out by considering the sign of the second derivative of the potential at $\varphi = 0$:

$$\left.\frac{\mathrm{d}^2 V}{\mathrm{d}\varphi^2}\right|_{\varphi=0} = -\lambda\sigma^2 + \lambda\alpha T^2\,, \tag{1.1.63}$$

so we have a critical temperature $T \approx \sigma / \sqrt{\alpha}$ at which the symmetry is broken.

[4] We can write it with partial derivatives instead of covariant ones since for a scalar φ they are equal: $\nabla_\mu \varphi = \partial_\mu \varphi$.

1.1.4 Topological defects

The defects are quite similar to the defects we find in regular phase transitions we know at our scales, like water to ice: the crystal which forms is not perfect.

The minima $\varphi = \pm\sigma$ are the *true vacuum* of the system, while $\varphi = 0$ is the *false vacuum*.

There will be regions in the universe in which the scalar field goes to $+\sigma$, and other regions in which it goes into $-\sigma$. This is because the two minima are equivalent: there are even odds for the field at any point to fall into either. In causally connected regions it will go into the same minimum. There will then be boundaries between the regions in which the field goes into $+\sigma$ and $-\sigma$.

In a mostly-plus metric signature, the equation of motion reads

$$\Box\varphi = \frac{\partial V}{\partial \varphi}\,, \tag{1.1.64}$$

so if we neglect the curvature and consider static solutions we will have

$$-\nabla^2\varphi = \frac{\partial V}{\partial \varphi}\,. \tag{1.1.65}$$

Further, we consider an infinite domain wall in the xy plane, assuming that for $z \to \pm\infty$ we have $\varphi = \pm\sigma$. Also, we assume that the whole field has no x or y dependence. Let us then substitute into the equation:

$$-\frac{\partial^2\varphi}{\partial z^2} = -\frac{\partial V}{\partial \varphi} = -\lambda\varphi(\varphi^2 - \sigma^2)\,. \tag{1.1.66}$$

Solving this yields

$$\varphi(z) = \sigma \tanh\left(\frac{z}{\Delta}\right), \tag{1.1.67}$$

where Δ is the *thickness* of the wall, which we can estimate through energetic configurations: the surface energy will have contributions through the gradient of the field: $\Delta(\partial_z\varphi)^2 \sim \Delta\sigma^2/\Delta^2 = \sigma^2/\Delta$; and a potential term $V(\varphi) \sim \Delta V(\varphi = 0) \sim \Delta\lambda\sigma^4/4$.

Add clarification of the estimates.

These scale oppositely with Δ: the kinetic energy decreases for a wide wall, the potential energy decreases for a narrow wall. Then, we can find an optimum at $\Delta \sim 1/(\sigma\sqrt{\lambda})$.

This domain wall is not removable, the configuration is topologically stable.

Now we discuss the Kibble mechanism, which demonstrates how phase transitions always generate domain walls. Let us denote as ξ the typical size of the domains: it is called the *correlation length* of φ.

We know that in the radiation-dominated epoch there is a finite particle horizon $d_H(t) = 2ct$, so we must have $\xi \lesssim d_H(t)$.

Then, we find a lower bound on the number density of the domain walls, $n_X \sim \xi^{-3} \gtrsim d_H^{-3}(t)$.

Recall from regular HBB cosmology that

$$H^2(t) = \frac{8\pi G}{3}\rho_r = \frac{8\pi G}{3}\frac{\pi^2}{30}g_* T^4\,, \tag{1.1.68}$$

so

$$t = \frac{1}{2H} \approx 0.3\frac{1}{\sqrt{g_*}}\frac{M_{\rm Pl}}{T^2}\,, \tag{1.1.69}$$

therefore

$$n_X \gtrsim \left(\frac{\sqrt{g_*}}{0.6}\frac{T}{M_{\rm Pl}}\right)^3 T^3 \sim \left(\frac{\sqrt{g_*}}{0.6}\frac{T}{M_{\rm Pl}}\right)^3 n_\gamma(T)\,, \tag{1.1.70}$$

since the number density of photons scales like $n_\gamma(T) \sim T^3$.

Let us evaluate this number density, for $T \sim T_{\rm GUT} \sim 10^{15}\,\text{GeV}$. Here, $g_* \sim 100$, meaning that we get

$$n_X(T_{\rm GUT}) > 10^{-9} \div 10^{-10} n_\gamma(T_{\rm GUT})\,. \tag{1.1.71}$$

Therefore, we have the ratio

$$\frac{n_X(T_{\rm GUT})}{n_\gamma(T_{\rm GUT})} > 10^{-9} \div 10^{-10}\,. \tag{1.1.72}$$

If we assume that after production these objects are stable, there are no processes which can modify their number. Then, for lower temperatures we keep the same ratio, both number densities will scale like $n_\gamma \sim T^3 \sim a^{-3}$.

This is a very similar number to the ratio of baryons to photons, $\eta = n_b/n_\gamma$! This means that

$$\Omega_{0x} = \frac{m_x \eta_x(t_0)}{\rho_{\rm crit}} \gtrsim \frac{m_x \eta_{0b}}{\rho_{\rm crit}} = \frac{m_x}{m_p}\Omega_{0b}\,, \tag{1.1.73}$$

which means that, since $m_x \sim T_{\rm GUT} \sim 10^{15}\,\text{GeV}$. Then, we must have $\Omega_{0x} \gtrsim 10^{14}$! This definitely over-closes the universe.

How does inflation solve this problem? These objects are produced in these early stages, but their number density is very diluted. Each $\pm\sigma$ region is inflated to the size of the observable universe.

Now we will give some arguments as to why a scalar field makes sense, a characterization of different inflationary models, and discuss the generation of the first primordial density perturbations.

A De Sitter phase is one with a cosmological constant Λ:

$$H^2 = \frac{8\pi G}{3}\rho_\Lambda - \frac{k}{a^2}\,. \tag{1.1.74}$$

Here $P_\Lambda = -\rho_\Lambda$. Then, $a(t) \propto \exp(Ht)$, with $H = \text{const}$. This ρ_Λ is constant as the universe expands.

A cosmological constant term can be written in terms of a vacuum energy density of the quantum system. It appears in the EFE as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \,. \tag{1.1.75}$$

This can be calculated as $\langle 0| T_{\mu\nu} |0\rangle \propto - \langle 0| \varphi |0\rangle g_{\mu\nu} = - \langle \rho \rangle g_{\mu\nu}$.

Plugging this into the EFE we find

$$\Lambda = -8\pi G \langle \rho \rangle \,. \tag{1.1.76}$$

We cannot get rid of the vacuum energy density of the system, since energy gravitates.

Let us come back to the Lagrangian

$$\mathscr{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \,, \tag{1.1.77}$$

whose energy momentum tensor is

$$T^\varphi_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi + \mathscr{L}_\varphi g_{\mu\nu} \,. \tag{1.1.78}$$

Let us look at the Vacuum Expectation Value of the field: $\langle \varphi \rangle = \langle 0| \varphi |0\rangle$. If this is a constant, it should correspond to the minimum of the classical potential: the ground state. This behaves like a cosmological constant. Since φ is a constant, we have

$$\left\langle T_{\mu\nu} \right\rangle = g_{\mu\nu} V(\langle \phi \rangle) \,, \tag{1.1.79}$$

since the derivatives of a constant vanish. This is an effective Λ.

A phase transition can move us away from this VEV.

The VEV of φ can be a function of time.

Are there other options beyond a scalar field? Say, a vector field, or a spinor? The first reason we do not choose this is because it breaks isotropy.

1.2 Dynamics of inflation

Monday 2020-10-19, compiled 2020-11-02

Our action will be in the form

$$S = S_{EH} + S_\varphi + S_{\text{matter}} \,, \tag{1.2.1}$$

where S_{EH} is the Einstein-Hilbert action for the metric, S_φ is the action for the field φ, while "matter" encompasses all the other fields:

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \, \sqrt{-g} R + \int \mathrm{d}^4 x \, \sqrt{-g} \mathscr{L}_\varphi[\varphi, g_{\mu\nu}] + S_{\text{matter}} \,. \tag{1.2.2}$$

We are using the invariant volume element $\mathrm{d}^4 x \sqrt{-g}$, which represents the physical 4-volume regardless of the coordinates.

The simplest possibility for the Lagrangian of a scalar field which is able to drive inflation reads:

$$\mathscr{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) \,, \tag{1.2.3}$$

for a *real* scalar field φ. Also, note that we did not consider any explicit coupling of φ with gravity or other fields: these kinds of terms, which go by the name "nonminimal coupling", might look like $\xi \varphi^2 R$.

These kinds of nonminimal theories represent one of the simplest extensions of GR: they are *scalar-tensor* theories, and in them the field φ as well as $g_{\mu\nu}$ can mediate gravity. One of the theories which currently fits the cosmological data best is of this kind.

What could we put in the potential? The mass is given by $m_\varphi^2 = \frac{\partial^2 *}{\partial V \partial \varphi}$, so a simple mass term would look like $m^2 \varphi^2 / 2$, but we could also have quartic terms like $\lambda \varphi^4 / 4$: these are self-interaction terms.

The other fields will typically be negligible during inflation since their energy density is quickly "redshifted away". Sometimes some of them are non-negligible: this happens if they are coupled to the scalar field, and we must consider them; we typically do so in an "effective" way, by inserting them into the potential $V(\varphi)$.

We can associate an energy-momentum tensor to the scalar field: in general, it is defined by

$$T_{\mu\nu}^{(\varphi)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} \,. \tag{1.2.4}$$

This comes from the way we write the Einstein equations from a variational principle. For our scalar field we find

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left[-\frac{\partial(\sqrt{-g}\mathscr{L}_\varphi)}{\partial g^{\mu\nu}} + \partial_\alpha \frac{\partial(\sqrt{-g}\mathscr{L}_\varphi)}{\partial {g^{\mu\nu}}_{,\alpha}} + \text{higher order terms} \right] . \tag{1.2.5}$$

Is the derivative covariant?

The reason for the alternating signs is that we must integrate by parts in order to get the expression in this form. We then get

$$T_{\mu\nu}^{(\varphi)} = -2 \frac{\mathscr{L}_\varphi}{g^{\mu\nu}} + \frac{2}{\sqrt{-g}} \mathscr{L}_\varphi \frac{\partial \sqrt{-g}}{\partial g^{\mu\nu}} \,, \tag{1.2.6}$$

since there is no dependence on the derivative of the metric in our case. We need the explicit expression

$$\frac{\partial \sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \,, \tag{1.2.7}$$

which comes from the fact that $\operatorname{Tr} \log M = \log \det M$, applied taking $M = g^{\mu\nu}$, and taken considering a functional variation of the whole expression. This yields

$$T_{\mu\nu}^{(\varphi)} = \partial_\mu \varphi \partial_\nu \varphi + g_{\mu\nu} \left[-\frac{1}{2} g^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} - V(\varphi) \right] . \tag{1.2.8}$$

We start by considering a homogeneous and isotropic case, and then perturb it. This is done by splitting the field into the average classical background motion of φ, called φ_0

$$\varphi = \varphi(\vec{x}, t) = \varphi_0(t) + \delta\varphi(\vec{x}, t)\,, \tag{1.2.9}$$

where φ_0 will be the VEV of the field: $\varphi_0 = \langle 0| \varphi(\vec{x}, t) |0\rangle$, while $\delta\varphi$ encompasses the quantum fluctuations.

Are we allowed to do this kind of split? Formally yes, but we need to guarantee that the perturbations are indeed small compared to the classical trajectory: $\left\langle \delta\varphi^2 \right\rangle \ll \varphi_0^2(t)$. We consider the variance since it is the first nonzero moment, as $\langle \delta\varphi \rangle = 0$.

This will not always be the case, sometimes the fluctuation will be dominating; however usually for an inflationary model to work we expect that the condition is satisfied. The fluctuations are what generates the density fluctuations which create the anisotropies in the CMB photons: we know that the size of these anisotropies is of the order of one part in 10^5, so we can give a qualitative argument for the perturbations of the scalar field to be relatively small compared to the mean value.

If we do the explicit computation for the energy-momentum tensor, we get

$$T^0_0 = -\left(\frac{1}{2}\dot{\varphi}_0(t) + V(\varphi_0)\right) = -\rho_\varphi(t) \tag{1.2.10}$$

$$T^i_j = \left(\frac{1}{2}\varphi_0^2(t) - V(\varphi_0)\right)\delta^i_j = P_\varphi \delta^i_j\,. \tag{1.2.11}$$

This is a perfect-fluid energy-momentum tensor. If we are in a regime for which

$$\frac{1}{2}\dot{\varphi}_0^2(t) \ll V(\varphi_0) \tag{1.2.12}$$

when we have $P_\varphi \approx -\rho_\varphi$: this is a *quasi De Sitter* expansion, with $w_\varphi \approx -1$.

This is achieved if the potential for the scalar field is "flat enough": then, we reach a *friction-domination* regime, which is commonly called *slow-roll* inflation.

Insert picture of flat potential

If $V(\varphi)$ is approximately a constant, then it mimics a cosmological constant. Inflation is driven by the vacuum energy density associated with the scalar field.

Let us look at the slow-roll dynamics in more detail. What is the equation of motion for this (quantum!) scalar field? The equation reads

$$\Box\varphi = \frac{\partial V}{\partial \varphi}\,, \tag{1.2.13}$$

just like in the flat case, however since we are in curved spacetime we must be careful about the Dalambertian operator, which now reads

$$\Box\varphi = \frac{1}{\sqrt{-g}}\left(g^{\mu\nu}\sqrt{-g}\varphi_{,\mu}\right)_{,\nu}\,, \tag{1.2.14}$$

since we need to use the formula for the quadri-divergence of a vector field.

Let us see what this reduces to in a flat FLRW metric: then $\sqrt{-g} = a^3$, so

$$\Box\varphi = \frac{1}{a^3}\Big(g^{00}a^3\varphi_{,0}\Big)_{,0} + \frac{1}{a^3}\Big(g^{ii}a^3\varphi_{,i}\Big)_{,i} = \frac{\partial V}{\partial\varphi} \tag{1.2.15}$$

$$-\ddot{\varphi} - \dot{\varphi}3\frac{\dot{a}}{a} + \frac{\nabla^2}{a^2}\varphi = \frac{\partial V}{\partial\varphi} \tag{1.2.16}$$

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2\varphi}{a^2} = -\frac{\partial V}{\partial\varphi}\,. \tag{1.2.17}$$

The term $3H\dot{\varphi}$ is a kind of *friction* term: the propagation of the field is "held back" by the expansion. If we consider the background field, it will be constant in space, so it will evolve as

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 = -\frac{\partial V}{\partial\varphi_0}\,. \tag{1.2.18}$$

We then must solve this equation combined with the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\Big(\rho_\varphi + \rho_m + \rho_r\Big) - \frac{k}{a^2}\,. \tag{1.2.19}$$

The matter and radiation densities scale like a^{-3} for ρ_m, a^{-4} for ρ_r; in this early phase the scalar field will dominate the dynamics, so the equation will simplify to

$$H^2 \approx \frac{8\pi G}{3}V(\varphi_0)\,. \tag{1.2.20}$$

Under these slow-roll conditions, we also have $\ddot{\varphi}_0 \ll 3H\dot{\varphi}_0$: therefore

$$3H\dot{\varphi}_0 \approx -\frac{\partial V}{\partial\varphi_0}\,. \tag{1.2.21}$$

We also expect that V and all of its derivatives change very slowly with φ. This means that in this equation we have $\frac{\partial V}{\partial\varphi_0} \approx$ const, as well as $H \approx$ const: this is the same equation which is obeyed by a particle under a constant force and friction: it will then reach the asymptotic "terminal velocity" and move with a constant $\dot{\varphi}_0$. This solution is an attractor.

We typically write this as $\dot{\varphi} = -V'(\varphi_0)/3H$, and $H^2 = \frac{8\pi G}{3}V(\varphi)$.

1.2.1 Slow-roll parameters

These are parameters we need to quantify how much the potential indeed looks like we expected.

The first parameter we define is

$$\epsilon = -\frac{\dot{H}}{H^2} = +4\pi G\frac{\dot{\varphi}^2}{H^2} \approx \frac{3}{2}\frac{\dot{\varphi}^2}{V} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{1.2.22}$$

so requiring $\epsilon \ll 1$ can also be stated as asking that[5]

$$\frac{1}{V}\left(\frac{\partial V}{\partial \varphi}\right)^2 \ll H^2 \,. \tag{1.2.23}$$

So, ϵ gives a bound on the first derivative of the potential; the second derivative is controlled by the parameter

$$\eta = -\frac{\ddot{\varphi}}{H\dot{\varphi}}\,, \tag{1.2.24}$$

and we can also define

$$\eta_V = \frac{1}{3}\frac{V''}{H^2} = \frac{1}{8\pi G}\frac{V''}{V}\,. \tag{1.2.25}$$

Asking that $\eta_V \ll 1$ is equivalent to $V'' \ll H^2$.

These three parameters are related by $\eta = \eta_V - \epsilon$.

Wednesday 2020-10-21, compiled 2020-11-02

We come back to the dynamics of slow-roll inflation. We defined $\epsilon = -\dot{H}/H^2$, which in our model corresponds to $\dot{\varphi}^2/V$. So, the conditions of the potential being flat (V' being small) and the kinetic energy being small compared to the potential are seen to correspond to $\epsilon \ll 1$.

On the other hand, $\eta = -\ddot{\varphi}/(H\dot{\varphi}) = \eta_V - \epsilon$, where

$$\eta_V = \frac{1}{3}\frac{V''}{H^2} = \frac{1}{8\pi G}\frac{V''}{V}\,. \tag{1.2.26}$$

What we ask is that all three of these parameters be small.

This means that H must change slowly over time. We can manipulate the second derivative of the scale factor so that ϵ appears:

$$\ddot{a} = \frac{\mathrm{d}}{\mathrm{d}t}(Ha) = a\dot{H} + \dot{a}H = a\left(\dot{H} + H^2\right) = aH^2\left(1 + \frac{\dot{H}}{H^2}\right) = aH^2(1-\epsilon)\,. \tag{1.2.27}$$

We can see that $\ddot{a} > 0$ only if $\epsilon < 1$. What is then the relevance of $\eta < 1$ then? We must have not only a phase of accelerating expansion, but a phase of accelerating expansion which lasts *sufficiently long*. In order for this to happen, we need $\epsilon \sim$ const. Since $\epsilon \sim \dot{\varphi}^2$ while $\eta \sim \ddot{\varphi}$, requiring $\eta \ll 1$ ensures this.

Also, $\eta \ll 1$ is needed in order to neglect the acceleration term in the Klein-Gordon equation, so that we move towards an attractor solution in the friction-dominated regime.

There could be arbitrarily many more slow-roll parameters, defined in terms of higher-order derivatives [**GravitationalWavesInflation2016**]:

$$\zeta^2 = \frac{1}{8\pi G}\left(\frac{V'V'''}{V''}\right), \tag{1.2.28}$$

[5] A note on the dimensionality: the potential V has the dimensions of the Lagrangian: in natural units, m^{-4}. The field φ has the dimensions of an inverse length (a mass), just like the Hubble rate.

and we will always expand in these parameters; in this course we will usually stop at second order (keeping ϵ and η).

We will then be able to consider both ϵ and η as approximately constant: their time derivatives are of higher order in them, specifically $\dot{\epsilon}, \dot{\eta} = \mathcal{O}\left(\epsilon^2, \eta^2\right)$.

Forgetting about the precise coefficients,

$$\epsilon \sim \frac{1}{\pi G}\left(\frac{V'}{V}\right)^2 \sim \frac{\alpha^2 M_{\mathrm{Pl}}^2}{\varphi^2}, \tag{1.2.29}$$

meaning that $\varphi \gtrsim M_{\mathrm{Pl}}$. Note that we are talking about the *background* solution, dropping the index 0 for simplicity.

These are then called *large-field* models, since the value of φ must be very large. Another kind of potential is in the form

$$V(\varphi) = V_0\left[1 - \left(\frac{\varphi}{\mu}\right)^p + \dots\right], \tag{1.2.30}$$

where $\varphi < \mu < M_{\mathrm{Pl}}$, while $p > 2$. The dots (higher order terms) are important for the end of inflation, not for most of it. This potential has a plateau near $\varphi = 0$.

In this case,

$$\epsilon \sim \frac{1}{\pi G}\left(\frac{V'}{V}\right)^2 \sim \frac{p^2}{\pi G}\frac{\varphi^{2p}}{\varphi^2}[\dots]^{-2} \sim \frac{p^2\varphi^{2p}M_{\mathrm{Pl}}^2}{\varphi^2}, \tag{1.2.31}$$

so $\epsilon \to 0$ as $\varphi \to 0$: these are called *small-field models* of inflation, since we can have small ϵ even with small φ.

Why do we need the condition $\varphi < \mu < M_{\mathrm{Pl}}$?

Claim 1.2.1. *In the case $p = 2$ and in the case $\mu > M_{Pl}$ this model becomes a* large-field *one.*

There is also a third category, but it is mostly excluded by data.

The quantity we are interested in is the *excursion* $\Delta\varphi$ of the field in the *observable window*: the difference between its value when the horizon crosses the largest observable scales (φ_{CMB} — called so since we can measure it from CMB observations) and the value of the field at the end of inflation, φ_{end}: $\Delta\varphi = \varphi_{\mathrm{CMB}} - \varphi_{\mathrm{end}}$. This interval corresponds to the ~ 60 e-folds of inflation; inflation likely lasted much more, however the earliest parts of it, which correspond to scales much larger than the horizon today, are hardly observable.

We can compute $\Delta\varphi$ as

$$\Delta\phi = \int_{\varphi_{\mathrm{CMB}}}^{\varphi_{\mathrm{end}}} \mathrm{d}\varphi = \int_{t_{\mathrm{CMB}}}^{t_{\mathrm{end}}} \dot{\varphi}\,\mathrm{d}t \approx \frac{\dot{\varphi}}{H}\int_{Ht_{\mathrm{CMB}}}^{Ht_{\mathrm{end}}} \mathrm{d}(Ht) = \frac{\dot{\varphi}}{H}N_{\mathrm{CMB}} \sim N_{\mathrm{CMB}}\sqrt{\epsilon}M_{\mathrm{Pl}}, \tag{1.2.32}$$

where $N_{\mathrm{CMB}} \sim 60 \div 70$ is the number of e-folds in the sub-horizon part of inflation.

If ϵ is of the order of $1/N_{\mathrm{CMB}}$, as happens in large-field models, we find $\Delta\varphi \sim \sqrt{N_{\mathrm{CMB}}}M_{\mathrm{Pl}} \gtrsim M_{\mathrm{Pl}}$.

Where does this $\epsilon \sim 1/N$ come from?

On the other hand, if we have $\epsilon \to 0$ we can have smaller $\Delta\varphi \lesssim M_{\text{Pl}}$.

Do we have to account for quantum gravity if $\Delta\varphi \gtrsim M_{\text{Pl}}$? No, since the condition required is actually $V \leq M_{\text{Pl}}^4$. However, a large excursion of the scalar field can constitute a problem, especially if we try to include these models in a "UV-complete" theory, so we must be careful.

We have often taken a De Sitter or quasi-De Sitter phase of expansion, but this is not necessarily the case: recall that

$$\ddot{a} = aH^2 \left(1 + \frac{\dot{H}}{H^2}\right), \tag{1.2.33}$$

so if $\dot{H} = 0$ (which defines De Sitter expansion) we do indeed have accelerated expansion, but $\dot{H} \neq 0$ does not prevent it, as long as $\left|\dot{H}\right| < H^2$ for negative $\dot{H}$, or even more easily with $\dot{H} > 0$.

In our models we usually had

$$\dot{H} = -4\pi G \dot{\varphi}^2 < 0\,. \tag{1.2.34}$$

This is to say that inflation is not De Sitter, although it can be close to it in certain phases: for starters, it must end at a certain point. What kind of fluid could correspond to these other two solutions?

In general, for a spatially flat FLRW universe, we have

$$a(t) = a_* \left(1 + \frac{1}{\alpha} H_* (t - t_*)\right)^\alpha \qquad \alpha = \frac{2}{3(1+w)}, \tag{1.2.35}$$

with $w = \text{const}$. This is a good approximation, at least for the subhorizon phase of inflation. Accelerated expansion is achieved for $-1 < w < -1/3$: for these values the scale factor expands like a powerlaw, this is called *powerlaw inflation*.

The De Sitter case is one we know. The case in which $\dot{H} > 0$ is realized with $w < -1$; this goes like

$$a(t) \propto (t - t_{\text{asymptote}})^\alpha\,, \tag{1.2.36}$$

with $\alpha < 0$. This is called *pole inflation*: it is *superexponential*.

In the simple models we are considering we always have $\dot{H} < 0$, so this is not relevant.

We have always worked with equations like $H^2 = \frac{8\pi G}{3} V(\varphi)$, which starts us off with an *unperturbed* FLRW metric.

How can we be sure that the solution we find is indeed an attractor? It seems like we are requiring very specific initial conditions, not general at all!

We need the **cosmic no-hair principle** (sometimes called "theorem", although it is not one). This tells us that starting from very general initial conditions inflation moves us towards a flat FLRW metric. This is discussed in Kolb and Turner [KT94].

We do not have a full GR solution to describe an anisotropic inhomogeneous universe; we consider instead a homogeneous but anisotropic spacetime: a *Bianchi model*. The Bianchi classification distinguishes between different kinds of universes like these.

Bianchi class I universes expand at different rates in different directions:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \sum_i a_i^2(t)\,\mathrm{d}x_i{}^2\ . \tag{1.2.37}$$

We can then define a volume $V = a_1 a_2 a_3$, and an average scale factor $\overline{a}(t) \propto V^{1/3}$: this yields an *averaged* Friedmann equation like

$$\overline{H}^2 = \left(\frac{\dot{\overline{a}}}{\overline{a}}\right)^2 = \frac{1}{9}\left(\frac{\dot{V}}{V}\right)^2 = \frac{8\pi G}{3}\Big[\rho_\varphi + \rho_m + \rho_r\Big] + F(a_1, a_2, a_3)\,, \tag{1.2.38}$$

where the term F represents the dynamical effects of the anisotropic expansion of the universe. It can also include the effects of curvature, with a term like $-k/\overline{a}^2$.

Note that we could also write a full set of equation from the Einstein ones, to describe the evolution of the three anisotropic scale factors a_i: the point is that writing the averaged equation we did we can explicitly see the effects of the anisotropy on the mean scale factor. This allows us

Suppose we have a model of inflation with a scalar field φ and a potential $V(\varphi)$, which drives inflation if it is considered without the anisotropic term F.

The question is: if we account for F, does it "destroy" inflation? This is to say: under these more general anisotropic initial conditions, can inflation move us towards a flat FLRW universe?

The evolution of φ will be governed by the Klein-Gordon equation:

$$\ddot{\varphi} + 3\overline{H}\dot{\varphi} = -\frac{\partial V}{\partial \varphi}\,. \tag{1.2.39}$$

We can then show that usually there is no problem from the anisotropy. Let us start with the simplest model, in which we take $\rho_\varphi = \text{const}$, a cosmological constant.

Typically, $F \propto \overline{a}^\alpha$, where $\alpha \le -2$. This then means that, if ρ_φ is constant, eventually ρ_φ will dominate. Everything else will be exponentially suppressed.

Under certain assumptions this can be proven formally, and is called Wald's Theorem.

There can be exceptions: for example, a closed universe with high Ω_k initially can collapse before inflation starts.

We, however, do not have a cosmological constant but a field: does φ roll to the minimum *before* inflation starts, if the anisotropic term F is present? It does not. The effect of F is to give an additional contribution to H, usually increasing it, which modifies the "friction" term, which allows for the slow-roll inflation to still occur.

What about completely general *inhomogeneous* as well as anisotropic spacetimes? Various analyses have shown (still, not as a mathematical theorem!) that also in this case typically we evolve towards a flat FLRW metric.

Monday 2020-10-26, compiled 2020-11-02

Our scalar field is given by

$$\varphi(\vec{x}, t) = \varphi_0(t) + \delta\varphi(\vec{x}, t)\,, \tag{1.2.40}$$

a background value, which we have studied up to now, plus some quantum fluctuations.

The full field obeys the KG equation:

$$\ddot{\varphi}(\vec{x},t) + 3H\varphi(\vec{x},t) - \frac{\nabla^2\varphi(\vec{x},t)}{a^2} = -\frac{\partial V}{\partial \varphi}\,, \tag{1.2.41}$$

which we can Taylor expand up to linear order around the background value if we assume that the perturbation is indeed small:

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{\nabla^2}{a^2}\delta\varphi = -\frac{\partial^2 V(\varphi)}{\partial\varphi^2}\delta\varphi\,. \tag{1.2.42}$$

We want to see how the dynamics of $\delta\varphi$ affect the super-horizon scales. The first order equation for $\delta\varphi$ is coupled to the background one:

$$\ddot{\varphi}_0(t) + 3H\dot{\varphi}_0(t) = -\left.\frac{\partial V}{\partial\varphi}\right|_{\varphi_0}\,. \tag{1.2.43}$$

Let us differentiate it with respect to time, assuming that we are in a phase of De Sitter expansion with $\dot{H} \approx 0$:

$$\dddot{\varphi}_0 + 3H\ddot{\varphi} = -\left.\frac{\partial^2 V}{\partial\varphi^2}\right|_{\varphi_0}\dot{\varphi}_0\,, \tag{1.2.44}$$

which looks similar to the equation for $\delta\varphi$, except for the Laplacian.

We can neglect the Laplacian on superhorizon scales: for one, on large scales the variations are negligible. More formally, in Fourier space that term is $k^2\delta\varphi/a^2$, to be compared with things like $3H\delta\dot{\varphi} \sim 3H^2\delta\varphi$, since the characteristic time of the variations of the field is $1/H$.

Now, on superhorizon scales $k \gg r_H$, which means that $k \ll aH = r_H^{-1}$, therefore $k^2/a^2 \ll H^2$. We can then neglect the Laplacian term.

What we should do now is to consider the Wronskian of the two-equation system:

$$W(x,y) = \dot{x}y - x\dot{y} \tag{1.2.45}$$

$$W(\dot{\varphi}_0, \delta\varphi) = \ddot{\varphi}_0\delta\varphi - \dot{\varphi}_0\delta\dot{\varphi}\,. \tag{1.2.46}$$

It can be shown that if the Hubble parameter is approximately constant then $\dot{W} = -3HW$, meaning $W \sim \exp(-3Ht)$, which quickly goes to zero.

If $W = 0$, then the two variables $\dot{\varphi}_0$ and $\delta\varphi$ are dependent, then we can decompose $\delta\varphi$ like

$$\delta\varphi(\vec{x},t) = -\delta t(\vec{x})\dot{\varphi}_0(t)\,. \tag{1.2.47}$$

Therefore, the full field is given by reverse-Taylor expanding as

$$\varphi(\vec{x},t) = \varphi_0(t) - \delta t(\vec{x})\dot{\varphi}_0(t) \tag{1.2.48}$$

$$\approx \varphi_0(t - \delta t(\vec{x}))\,. \tag{1.2.49}$$

This expression has a very clear physical interpretation: point by point, the field goes through the same evolution, taking on the same values, only at different times at each point.

This holds as long as we have a single scalar field, in multi-field models of inflation this will not be the case anymore.

What we want to compute is the final effect of these fluctuations on super-horizon scales. The equation for $\delta\varphi$ reads

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{\nabla^2\delta\varphi}{a^2} = V''(\varphi_0)\delta\varphi\,. \tag{1.2.50}$$

We move to Fourier space:

$$\delta\varphi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, e^{i\vec{k}\cdot\vec{x}}\delta\varphi_{\vec{k}}(t)\,. \tag{1.2.51}$$

Since the field is real, $\delta\varphi_{\vec{k}} = \delta\varphi^*_{-\vec{k}}$. These Fourier modes evolve independently of each other up to linear order.

Why do we make a 3D Fourier transform and not a 4D one? We have time-dependent coefficients in the equation, such as H. We Fourier transform to take account of the spatial symmetry: it is not useful to transform in time since there is no time translation symmetry.

By transforming we are implicitly using a stationary wave basis, which is natural in a spatially flat universe: in a non-flat universe instead of $e^{i\vec{k}\cdot\vec{x}}$ we should use generalized solution of the Helmholtz equation

$$\nabla^2 Q_{\vec{k}} + k^2 Q_{\vec{k}} = 0\,, \tag{1.2.52}$$

the eigenvalue equation for the Laplacian operator. These are *time-independent* waves, while propagating waves $\exp\left(-ik_\mu x^\mu\right)$ would not suit our needs. In Fourier space the equation reads

$$\ddot{\delta\varphi}_{\vec{k}} + 3H\dot{\delta\varphi}_{\vec{k}} + \frac{k^2}{a^2}\delta\varphi_{\vec{k}} = -V''(\varphi_0)\delta\varphi_{\vec{k}}\,. \tag{1.2.53}$$

We will need to use the standard tools of second quantization: we start by introducing $\hat{\delta\varphi} = a\delta\varphi$, and we use conformal time, $a\,\mathrm{d}\tau = \mathrm{d}t$. Then, we will have

$$\delta\varphi(\vec{x},\tau) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[u_k(\tau)a_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + u_k^*(\tau)a^\dagger_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\right]. \tag{1.2.54}$$

The functions u are classical functions of time, while a and $a^\dagger$ are the annihilation operators. They are defined so that $a_{\vec{k}}\left|0\right\rangle = 0$ for any $\vec{k}$, and similarly $\left\langle 0\right| a^\dagger_{\vec{k}} = 0$.

Here, $\left|0\right\rangle$ is called the *free vacuum state*.

We impose the normalization condition

$$u_k^* u_k' - u_k u_k^{*,\prime} = -i\,, \tag{1.2.55}$$

which is equivalent to requiring

$$\left[a_{\vec{k}}, a^\dagger_{\vec{k}'}\right] = \hbar\delta^{(3)}(\vec{k}-\vec{k}')\,, \tag{1.2.56}$$

while $[a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0$.

In flat spacetime, once the commutation relations are fixed we are done: the solutions are known to be plane waves, with $u_{\vec{k}} \sim \exp(-i\omega_k t)/\sqrt{2\omega_k}$, and $\omega_k = \sqrt{k^2 + m^2}$.

Now, instead, we need to make some assumptions. Because of the equivalence principle, for small distances and time intervals the solutions should reproduce the flat-spacetime ones. From this guiding principle, we require

$$\frac{k}{aH} \to \infty \implies u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\,, \tag{1.2.57}$$

which is called the *bunch-Davies vacuum choice*. In the denominator we only have k since $k^2 \gg m^2$ for the large scales we are considering.

What is the equation for u? In conformal time ($' = \frac{\mathrm{d}}{\mathrm{d}\tau}$) it is

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + a^2 \frac{\partial^2 V}{\partial \varphi^2}\right] u_k(\tau) = 0\,, \tag{1.2.58}$$

which is associated to the rescaled variable $\hat{\delta\varphi} = a\delta\varphi$. We can also see the reason why we have chosen to use the rescaled $\delta\varphi$ instead of the regular one: what we have found with this ansatz is basically a harmonic oscillator with a time-dependent frequency, changing according to the accelerated expansion of the universe. The ansatz yields a canonically-normalized kinetic term in the harmonic oscillator.

In order to solve the equation, we will assume de-Sitter expansion with $H = \text{const}$, a massless scalar field with $m_\varphi^2 = \partial^2 V \Big/ \partial \varphi^2 = 0$. Then, we know that $\mathrm{d}\tau = \mathrm{d}t\,/a = \mathrm{d}t\, e^{-Ht}$, so

$$\tau = -\frac{1}{H} e^{-Ht} = -\frac{1}{aH}\,. \tag{1.2.59}$$

Conventionally we choose the integration bounds so that τ runs from negative infinity to zero. This is just a matter of convention, a constant time shift has no effect on the dynamics. We also have

$$\frac{a''}{a} = \frac{2}{\tau^2}\,, \tag{1.2.60}$$

so if $\lambda_{\text{phys}} = a\lambda \ll H^{-1}$, therefore $\lambda \gg aH$. Therefore, being in the subhorizon scale means that

$$\frac{a''}{a} = 2a^2 H^2 \ll k^2\,, \tag{1.2.61}$$

therefore on subhorizon (microscopic) scales the equation reads

$$u_k'' + k^2 u_k = 0\,, \tag{1.2.62}$$

solved by

$$u_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\,. \tag{1.2.63}$$

As expected, we recover the regular flat-spacetime solution. However, what is really interesting to us are the cosmological, superhorizon solutions: this means $k \ll aH$, so

$$u_k''(\tau) - \frac{a''}{a} u_k(\tau) = 0\,, \tag{1.2.64}$$

which is a second-order equation: it will have two independent solutions, and it can be shown that the generic solution is written as

$$u_k(\tau) = \underbrace{B(k)a(\tau)}_{\text{growing mode}} + \underbrace{C(k)a^{-2}(\tau)}_{\text{decaying mode}}\,. \tag{1.2.65}$$

We neglect the decaying mode, since even if it is excited it will quickly decay. The physical fluctuation $\delta\varphi$ is proportional to $u_k/a \sim B(k)$, therefore we can see that the "growing mode" is actually asymptotically constant on superhorizon scales.

We then want to determine the scale of $B(k)$.

We need to match the subhorizon and superhorizon solutions, at the point $k = aH$.

$$\left|B(k)\right| a = \left|\frac{e^{-ik\tau}}{\sqrt{2k}}\right| \tag{1.2.66}$$

$$\left|\delta\varphi_k\right| = \left|B(k)\right| = \frac{1}{a\sqrt{2k}}\,, \tag{1.2.67}$$

recover last bit

The fluctuation gets "frozen in" at horizon crossing:

$$\left|\delta\varphi_k\right| = \frac{H}{2k^3}\,. \tag{1.2.68}$$

This is called a *gravitational amplification mechanism*. This is analogous to pair production from vacuum under a strong electrostatic field. The electric field separates the newly-formed electron-positron pair.

Wednesday 2020-10-28, compiled 2020-11-02

Today we will see how to explicitly solve the equation

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + a^2 m^2\right] u_k(\tau) = 0\,, \tag{1.2.69}$$

where τ is the conformal time, while $m^2 = \partial^2 V \big/ \partial\varphi^2$. The stage in which $m^2 = 0$ is the quasi de-Sitter stage. Then, $\epsilon = -\dot{H}/H^2 \ll 1$. We have

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon)\,. \tag{1.2.70}$$

Integrating the conformal time definition (we need to also integrate by parts) we can find

$$\tau = -\frac{1}{aH(1-\epsilon)}\,. \tag{1.2.71}$$

Then, the equation becomes

$$u_k''(\tau) + \left[k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right] u_k(\tau) = 0\,, \tag{1.2.72}$$

where $\nu^2 = 9/4 + 3\epsilon$. This is a Bessel equation: these equations are generally in the form

$$z^2 y''(z) + z y'(z) + (z^2 - \nu^2) y(z) = 0\,. \tag{1.2.73}$$

The solutions are called Hankel functions, and the general solution will read

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k) H_\nu^{(1)}(-k\tau) + c_2(k) H_\nu^{(2)}(-k\tau)\right], \tag{1.2.74}$$

where $H_\nu^{(2)} = H_\nu^{(1),*}$. We want to impose the asymptotic behaviour of the solution: we want $k/aH \gg 1$. This means that

$$u_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}\,, \tag{1.2.75}$$

and the asymptotics of the Hankel functions are

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} \exp\left(i(x - \frac{\pi}{2}\nu - \frac{\pi}{4})\right) \tag{1.2.76}$$

for $x \gg 1$. This works well for us: we can set $c_2(k) = 0$ and only use $c_1(k)$.

We must choose

$$c_1(k) = \frac{\sqrt{\pi}}{2} \exp\left(i\left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right) \tag{1.2.77}$$

in order to have the correct normalization. Then, our solution will read

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} \exp\left(i(\nu + \frac{1}{2})\frac{\pi}{2}\right) \sqrt{-\tau} H_\nu^{(1)}(-k\tau)\,, \tag{1.2.78}$$

so in the superhorizon $k/aH \ll 1$ (or $-k\tau \ll 1$) regime we have the asymptotic $x \ll 1$ expansion of the Hankel function:

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi}} e^{-i\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} x^{-\nu}\,, \tag{1.2.79}$$

therefore

$$u_k(\tau) \approx \exp\left(i(\nu - \frac{1}{2})\frac{\pi}{2}\right) 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (-k\tau)^{1/2-\nu} \frac{1}{2k}\,, \tag{1.2.80}$$

so if we want the modulus $\left|\delta\varphi_k\right| = |u_k|/a$ we can make use of the fact that $\tau \sim -1/aH \propto 1/a$, therefore $\left|\delta\varphi_k\right| \approx -H\tau|u_k|$. Inserting the expression we have for u_k yields

$$\left|\delta\varphi_k\right| \approx 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{H}{\sqrt{2k^2}} \left(\frac{k}{aH}\right)^{3/2-\nu} \tag{1.2.81}$$

$$\approx \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3/2-\nu}, \tag{1.2.82}$$

Expanded to first order in ϵ, so that $3/2-\nu \approx -\epsilon$.

which holds at super-horizon scales. In the De Sitter case, the exact result reads

$$u_k(\tau) \propto \sqrt{-\tau} H^{(1)}_{3/2}(-k\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{1.2.83}$$

The next step is to generalize to a scalar field which is not massless anymore: we shall consider a mass $m^2 = \partial^2 V \big/ \partial\varphi^2 \ll H^2$, so a "light" scalar field. We require this because the slow-roll parameter is $\eta_V = m^2/(3H^2) \ll 1$: a massive field (compared to H^2) would not be able to drive slow-roll inflation.

Then, the m^2a^2 term in the equation is nothing but

$$m^2a^2 = 3\eta_V a^2 H^2 = \frac{3\eta_V}{\tau^2}, \tag{1.2.84}$$

but we also know that

$$\frac{a''}{a} = \frac{2}{\tau^2}\left[1 + \frac{3}{2}\epsilon\right], \tag{1.2.85}$$

so we can still recast the equation into a Bessel form: only the coefficient of the τ^{-2} term changes.

The equation will read

$$u_k''(\tau) + \left[k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right] u_k(\tau) = 0, \tag{1.2.86}$$

with $\nu^2 = 9/4 + 3\epsilon - 3\eta_V$. Then, the solution has exactly the same form as before: on super-horizon scales, with $k \ll aH$, we have

$$\left|\delta\varphi_k\right| = \frac{H}{\sqrt{2k}}\left(\frac{k}{aH}\right)^{3/2-\nu} \qquad \nu \approx \frac{3}{2} + \epsilon - \eta_V. \tag{1.2.87}$$

Note that we did not require the field to be anything but light: the computation we have done can apply to an inflaton field but also to any other scalar field evolving in this phase of the expansion of the universe. If it is another scalar field, why should we require it to be light? It can be shown that if m^2 is large compared to H^2 the superhorizon-scale fluctuations of that field have a very hard time being excited: the field basically remains in its vacuum state.

The Klein-Gordon equation reads

$$\ddot{\varphi}(\vec{x}, t) + 3H\dot{\varphi}(\vec{x}, t) - \frac{\nabla^2\varphi(\vec{x}, t)}{a^2} = -\frac{\partial V}{\partial\varphi}, \tag{1.2.88}$$

and we have obtained is starting from the expression

$$\Box\varphi = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\right), \tag{1.2.89}$$

where the metric is taken to be a FLRW one. We have neglected a crucial component: we should also perturb the spacetime, beyond the FLRW metric, if we want to have a consistent discussion.

This is important to do for the inflaton especially since φ dominates the energy density of the universe. The Einstein equations,

$$G_{\mu\nu} = 8\pi G T^{(\varphi)}_{\mu\nu} \,, \tag{1.2.90}$$

are perturbed with a variation in the field, $\delta\varphi$, which perturbs the energy-momentum tensor $\delta T^{(\varphi)}_{\mu\nu}$, which means that we must also perturb the Einstein tensor $\delta G_{\mu\nu}$ and then finally the metric $\delta g_{\mu\nu}$.

So, we find an equation

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + M^2 a^2\right] u_k(\tau) = 0 \,. \tag{1.2.91}$$

If we account for metric perturbations, we get

$$\frac{M^2}{H^2} \approx 3\eta_V - 6\epsilon \,. \tag{1.2.92}$$

Then, we get

$$\nu^2 = \frac{9}{4} + 9\epsilon - 3\eta_V \,. \tag{1.2.93}$$

Just like before we can calculate

$$\left|\delta\varphi_k\right| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{3/2-\nu} \,. \tag{1.2.94}$$

Then we find that in order to calculate Δu_k we can use

$$Q_\varphi = \delta\varphi + \frac{\varphi}{H}\hat{\Phi} \,, \tag{1.2.95}$$

where $\hat{\Phi} = \Phi + (1/6)\nabla^2\chi^{\parallel}$, where Φ and $\chi^{\parallel}$ are scalar perturbations of g_{ij}.

What is Q?

We want to show how primordial GW are generated from the inflaton perturbations. We consider a tensor-perturbed FLRW metric

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + \Big(\delta_{ij} + h_{ij}(\vec{x},\tau)\Big)\,\mathrm{d}x^i\,\mathrm{d}x^j\right] . \tag{1.2.96}$$

We are neglecting all scalar and vector perturbations for simplicity. The symmetric tensor h_{ij} can be chosen to be traceless: $h^i_i = \partial^{\mathrm{BG}}_i h^{ij} = 0$, where the derivative is that constructed from the background unperturbed spacetime. This is basically the TT-gauge, but in a cosmological context the conditions come about naturally.

The equations of motion read

$$h''_{ij} + 2\frac{a'}{a} - \nabla^2 h_{ij} = 0\,, \tag{1.2.97}$$

where a prime denotes a derivative with respect to the conformal time τ. If we wanted to use cosmic time instead, the equation would read

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2 h_{ij}}{a^2} = 0\,. \tag{1.2.98}$$

We can see that this is the same as $\Box^{\mathrm{BG}} h_{ij} = 0$. The equation is the same as a massless, minimally coupled scalar field. We already know the solution for these equations: the mechanism of quantum vacuum amplification still works. We have a 0 on the right-hand side since we are working up to linear order, in full generality there will be other source terms, commonly denoted as π^T_{ij} (T for "source of tensor modes"). This term describes the tensor component of the anisotropic stresses, from the quadrupole up.

Measuring these primordial GWs would be the first probe of the quantum nature of gravity.

GWs start out with 6 degrees of freedom, but 4 are gauged away, so we are left with only 2. We can then expand h_{ij} in a plane wave basis:

$$h_{ij}(\vec{x}, \tau) = \int \sum_{\lambda=+,\times} \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{ik\cdot x} h_\lambda(\vec{k}, \tau)\epsilon^\lambda_{ij}(\vec{k})\,, \tag{1.2.99}$$

so that in Fourier space the equations of motion for the coefficients read

$$h''_\lambda + 2\frac{a'}{a}h'_\lambda + k^2 h_\lambda = 0\,. \tag{1.2.100}$$

This is exactly the same as the equations of motion for the scalar massless minimally coupled field. We can then see that if $k \gg aH$ we get

$$h_\lambda \sim \frac{e^{-ik\tau}}{a}\,, \tag{1.2.101}$$

while for $k \ll aH$ the gravitational perturbation has a constant mode and a decaying mode. We can treat the two coefficients like $h_{+,\times} = \phi_{+,\times}\sqrt{32\pi G}$, so that ϕ has the dimension of a mass while h is dimensionless. Then, like we saw before in the superhorizon regime $k \ll aH$ we get

$$|h_{+,\times}| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{-\epsilon}\sqrt{32\pi G}\,. \tag{1.2.102}$$

We can distinguish GWs from inflation from those produced by single astrophysical events: inflation yields a *stochastic background* of GWs. These GWs are a crucial target of future experiments, both in interferometry and in CMB maps.

We need to introduce the power spectrum of perturbations: the two-point correlation function, in real space, is given in terms of a generic stochastic perturbation field $\delta(\vec{x})$: we can calculate

$$\left\langle \delta(\vec{x}+\vec{r})\delta(\vec{x}) \right\rangle = \xi(r)\,. \tag{1.2.103}$$

In Fourier space, we define the power spectrum as the Fourier transform of the two-point correlation function: the field is written as

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, e^{i\vec{k}\cdot\vec{x}}\delta(\vec{k})\,, \tag{1.2.104}$$

and the power spectrum $P(k)$ is defined by

$$\left\langle \delta(\vec{k})\delta(\vec{k}') \right\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')P(k)\,. \tag{1.2.105}$$

The plus is there since we wrote $\delta(k)\delta(k')$ instead of $\delta(k)\delta^*(k')$: if we included the conjugate than the argument of the δ would be $k-k'$, due to the fact that since $\delta(x)$ is real we have $\delta^*(k) = \delta(-k)$.

We have

$$\sigma^2 = \left\langle \delta^2(x) \right\rangle \frac{1}{2\pi^2}\int \mathrm{d}k\, k^2 P(k) \tag{1.2.106}$$

$$= \frac{1}{2\pi^2}\int \frac{\mathrm{d}k}{k} k^3 P(k) \tag{1.2.107}$$

$$= \int \mathrm{d}\log k\, \Delta^2_\delta(k)\,, \tag{1.2.108}$$

where we define the dimensionless power spectrum:

$$\Delta^2_\delta(k) = \frac{k^3}{2\pi^2}P(k)\,. \tag{1.2.109}$$

We can show that $P(k)$ is indeed the Fourier transform of $\xi(r)$:

$$\left\langle \delta(x+r)\delta(x) \right\rangle = \frac{1}{(2\pi)^6}\int \mathrm{d}^3k_1\, e^{ik_1(x+r)}\int \mathrm{d}^3k_2\, e^{ik_2x}\left\langle \delta(k_1)\delta(k_2) \right\rangle \tag{1.2.110}$$

$$= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k_1\, \mathrm{d}^3k_2\, \delta^{(3)}(k_1+k_2)P(k_1)e^{ik_1(x+r)+ik_2x} \tag{1.2.111}$$

$$= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k_1\, e^{ik_1r}P(k_1)\,, \tag{1.2.112}$$

as we wanted to show. In the $r=0$ case we recover the expression from above for the variance.

Let us continue with our discussion of the power spectrum: the **spectral index** n_s is defined as[6]

$$n_s - 1 = \frac{\mathrm{d}\log\Delta(k)}{\mathrm{d}\log k}\,. \tag{1.2.113}$$

[6] Note that the dimensionless power spectrum is sometimes denoted as Δ^2 and sometimes as Δ: here (in this section) we use the latter definition.

The index s means "scalar". In general this will depend on the wavenumber k; it is a convenient description of the shape of the power spectrum.

If n_s were a constant, then we would have a *powerlaw* spectrum: $\Delta(k) = \Delta(k_0)(k/k_0)^{n_s-1}$. If $n_s = 1$, we have the **Harrison-Zel'dovich** power spectrum, for which Δ does not depend on k. This would be a *scale-invariant* power spectrum.

In a quantum-mechanical formalism, we will calculate the power spectrum as

$$\langle 0| \delta\varphi_{\vec{k}_1} \delta\varphi_{\vec{k}_2} |0\rangle \,, \tag{1.2.114}$$

which will be written in terms of creation and annihilation operators: we have

$$\langle 0| aa |0\rangle = \langle 0| a^\dagger a |0\rangle = \langle 0| a^\dagger a^\dagger |0\rangle = 0\,, \tag{1.2.115}$$

while

$$\langle 0| aa^\dagger |0\rangle = \underbrace{\langle 0| \left[a, a^\dagger\right] |0\rangle}_{\delta^{(3)}(\vec{k}_1 - \vec{k}_2)} - \underbrace{\langle 0| a^\dagger a |0\rangle}_{=0}\,, \tag{1.2.116}$$

so

$$\left\langle \delta\varphi_{\vec{k}_1} \delta\varphi_{\vec{k}_2} \right\rangle = (2\pi)^3 \left|\delta\varphi_{\vec{k}_1}\right|^3 \delta^3(\vec{k}_1 - \vec{k}_2)\,, \tag{1.2.117}$$

where $\delta\varphi_{kl} = u_{kl}/a$. Therefore, the power spectrum is given by

$$P(k) = \left|\delta\varphi_k\right|^2 . \tag{1.2.118}$$

Recall that in the superhorizon case we found

$$\left|\delta\varphi_k\right| = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{3/2-\nu}\,, \tag{1.2.119}$$

where $\nu^2 = 9/4 + 9\epsilon - 3\eta_V$, so $\nu \approx 3/2 + 3\epsilon - \eta_V$, meaning that the index is $3/2 - \nu = \eta_V - 3\epsilon$.

Then, the power spectrum reads

$$\Delta_{\delta\varphi}(k) = \frac{k^3}{2\pi}\left|\delta\varphi_k\right|^2 = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} . \tag{1.2.120}$$

There will be a weak scale dependence proportional to the slow-roll parameters: $3 - 2\nu = 2\eta_V - 6\epsilon$.

1.3 From $\delta\varphi$ to primordial density perturbations

The first Friedmann equation will read

$$H^2 = \frac{8\pi G}{3}\rho_\varphi \approx \frac{8\pi G}{3} V(\varphi)\,, \tag{1.3.1}$$

so the density fluctuation can be written as

$$\delta\rho_\varphi \approx V'(\varphi)\delta\varphi \approx -3H\dot{\varphi}\delta\varphi\,. \tag{1.3.2}$$

Recall that we can define a time shift $\delta t = -\delta\varphi/\dot{\varphi}$. This means that we will have perturbations in the expansion of the universe from place to place.

The number of e-folds is given by

$$N = \log\left(\frac{a(t)}{a(t_*)}\right) = \int_{t_*}^{t} H(\tilde{t})\,\mathrm{d}\tilde{t}\,. \tag{1.3.3}$$

The fluctuations will perturb the number of e-folds by

$$\zeta = \delta N = H\delta t = -H\frac{\delta\varphi}{\dot{\varphi}} \approx -H\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi}\,. \tag{1.3.4}$$

This is called the "δN formalism" for the study of large-scale perturbations. The last equality in (1.3.4) comes from the fact that

$$\dot{\rho}_\varphi = -3H\Big(\rho_\varphi + P_\varphi\Big) = -3H\dot{\varphi}^2\,, \tag{1.3.5}$$

so indeed

$$H\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi} = \frac{-3H^2\varphi\delta\varphi}{-3H\dot{\varphi}^2} = H\frac{\delta\varphi}{\dot{\varphi}}\,. \tag{1.3.6}$$

The quantity $\delta N = \zeta$ is **gauge invariant**. It is written as

$$\zeta = -\hat{\Phi} - H\frac{\delta\rho}{\rho}\,, \tag{1.3.7}$$

where $\hat{\Phi}$ is related to scalar perturbations of the spatial part of the metric, g_{ij}. We shall explore this later.

This ζ is called the **curvature perturbation on uniform energy density hypersurfaces**. Why did we write this with ρ instead of ρ_φ? This definition is completely general; it can be applied at any time. We can then specify it to

$$\zeta_\varphi \approx -H\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi}\,. \tag{1.3.8}$$

What we will show is that on superhorizon scales ζ remains constant (for single-field inflation, at least). Therefore, this keeps a sort of "record" of what happened after horizon crossing.

Let us denote as $t_H^{(1)}(k)$ the time of horizon crossing during inflation for perturbations with wavenumber k, and $t_H^{(2)}(k)$ the time *after inflation* of the second horizon crossing, when the perturbation comes back inside the horizon. The value of ζ at these two times will be the same.

We know that $\delta\varphi \sim H/2\pi$ (statistically), and $H^2 \approx 8\pi G V(\varphi)/3$: then, specifying the potential gives a prediction for the power spectrum.

Suppose that the perturbation re-enters during the radiation-dominated epoch. Then,

$$H\frac{\delta\rho}{\dot\rho} \approx \frac{H\delta\rho_\gamma}{-4H\rho_\gamma} = -\frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}\,, \tag{1.3.9}$$

so the dimensionless power spectrum is

$$\Delta_{\delta\rho/\rho}(k) = \frac{H^2}{\dot\varphi^2}\Delta_{\delta\varphi}(k)\bigg|_{t_H^{(1)}(k)}\,, \tag{1.3.10}$$

and

$$\Delta_{\delta\varphi}(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}\,, \tag{1.3.11}$$

so

Recover a few minutes

Bibliography

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