# Compact Object Astrophysics

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# Contents



Tuesday 2020-9-29, compiled 2020-10-20

### Introduction

Tuesdays and Wednesday at 14.30 PM in room P1A, Paolotti building. 22 people.

This course overlaps with "Computational Astrophysics" by professor Mapelli.

The examination is an oral one, done either online or live.

We start with a brief overview of the final fates of massive stars. We have white dwarfs, neutron stars and black holes under the category of "compact objects", but white dwarfs are not really that compact.

We then discuss accretion onto compact objects, and neutron stars. An open question: what is the EOS of ultradense neutron matter?

"Accretion power in astrophysics", "The physics of Compact Objects", "Astrofisica Relativistica I & II", "Astrofisica delle Alte Energie".

## 0.1 A journey into the life of a massive star

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Stars whose mass is $M \gtrsim 8M_\odot$ go supernova at the end of their life. During their lifetime, hydrogen fuses through two channels: the p-p chain and the CNO cycle.

In the CNO cycle, four protons turn into a $^4$He nuclide, two positrons, two electron neutrinos using heavier nuclides as catalysts.

The critical temperature above which the CNO cycle dominates is around $T_c \sim 2 \times 10^7\,\mathrm{K}$. For the Sun, less than 8 % of energy production is through the CNO cycle.

When the temperature of the core reaches a value around $1 \div 2 \times 10^8\,\mathrm{K}$ and the density is around $10^8 \div 10^9\,\mathrm{g/cm^3}$, helium starts to burn in the $3\alpha$ process, becoming $^{12}$C. The $Q$-value here is around 7.27 MeV.

As soon as we have carbon, this can fuse with an $\alpha$ particle giving rise to a nucleus of oxygen with $Q \approx 7.16\,\mathrm{MeV}$. This oxygen can further catch an $\alpha$ particle, making a $^{20}$Ne nuclide.

Then, we have a temperature around $5 \times 10^8\,\mathrm{K}$ and a density around $3 \times 10^6\,\mathrm{g/cm^3}$. Carbon starts to fuse with itself, making sodium, magnesium, and more neon (plus an $\alpha$ particle).

If you want carbon burning to proceed in a steady way, it must occur in a nondegenerate electron gas. This occurs only if the star is quite massive, more than $8M_\odot$. Otherwise, it is an explosive process.

Now the core temperature reaches $10^9\,\mathrm{K}$. The energy of a typical photon is quite high, $h\nu \sim k_B T \sim 100\,\mathrm{keV}$. Suppose there are neon nuclei in the core (this will be the case since they are a product of fusion).

It is not hard for a Neon to lose an $\alpha$ particle through photodissociation, this produces an Oxygen. The energy required for this is of the order $4.7\,\mathrm{MeV}$, at the high energy tail we have a few photons at this energy.

This is the "neon burning phase", after which we have oxygen and magnesium. Oxygen is the next candidate for nuclear burning, and after a further contraction the star starts burning it. It fuses with itself to produce $^{28}\mathrm{Si}$ plus an $\alpha$, or $^{32}\mathrm{S}$.

Sulfur cannot fuse with itself, the potential barrier is too high. Through successive $\alpha$ captures, the star synthesizes elements in the "iron peak": iron, nickel, cobalt.

The core tries to contract in the attempt to get them to burn, but they have the maximum possible binding energy per nucleon. So, the contraction continues.

If the mass of the contracting core exceeds the Chandrasekhar limit, it cannot become an electron-degenerate object. The mass of the core is always in excess of this limit mass for the stars which are massive enough to reach this stage of stellar burning.

Iron is photodissociated to make helium nuclei first, then bare protons, electrons and neutrons. Protons and electrons can combine into neutrons. The core becomes more and more neutrons rich, but the reaction also produces neutrinos, which can fly away.

The *neutron* degeneracy pressure can stop the collapse in certain cases: this is how a neutron star is formed. The threshold between neutron stars and black holes is hard to determine, but generally speaking with $8M_\odot < M < 25M_\odot$ a neutron star is formed, while for larger masses the core collapses further to form a black hole.

The freefall velocity is a significant fraction of the speed of light. What are the statistics? how many NS and BH are there in our galaxy?

We can model the distribution of star masses in our galaxy with the distribution, the IMF, as a Salpeter IMF,[1] which is given by

$$N(m) \propto m^{-\alpha}\,, \tag{0.1.1}$$

where $\alpha \approx 2.35$. Then, we can calculate the number of stars which have more than $8M_\odot$ by integrating: we find something proportional to $8^{-1.35}$, while the number of stars which have more than $25M_\odot$ we get something proportional $25^{-1.35}$. These will give us the amount of compact objects. The proportionality constant depend on the minimum and maximum mass of stars, but we can calculate the ratio of the two without concern for it. We find

$$\frac{N_{BH}}{N_{NS} + N_{BH}} = \left(\frac{8}{25}\right)^{-1.35} \approx 0.2\,. \tag{0.1.2}$$

[1] See the evil organization in Mission Impossible.

The present rate of supernova explosions in the galaxy is around 1 per century, or $10^{-2}\,\mathrm{yr}^{-1}$. In the age of the galaxy (around $10^{10}\,\mathrm{yr}$), we will then have had around $10^8$ compact objects.

How much can we trust this figure? Kind of, the true number is closer to $10^9$, about $1\,\%$ of the number of stars in the galaxy.

The galaxy roughly looks like a cylinder with radius $R \sim 60\,\mathrm{kpc}$ and height $H \sim 1\,\mathrm{kpc}$. Its volume will then be $V = 2\pi R^2 H \approx 10^{13}\,\mathrm{pc}^3$. Then, the number density of compact objects is around $0.1\,\mathrm{pc}^{-3}$.

The typical separation between them will be something like $20\,\mathrm{pc}$. Compact objects are close, common! Beware!

The closest compact object we know of is a neutron star $60\,\mathrm{pc}$ away: this is on the same order of magnitude.

**Compactness** The gravitational radius characterizing an object is

$$R_g = \frac{GM}{c^2}\,, \tag{0.1.3}$$

while the Schwarzschild radius is $2R_g$. For the Sun, this is approximately $1.5\,\mathrm{km}$. It's small.

The value $R_g/R$ is 0.5 for black holes, 0.15 for neutron stars, $10^{-4}$ for white dwarfs.

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As we said earlier, massive stars go type-2 supernova: this corresponds to Core-Collapse.

Compact objects are quite common in the galaxy.

A compact object is one for which the ratio of the gravitational radius $R_g = GM/c^2$ is comparable to the radius of the true object. For a white dwarf, the ratio is of the order of $10^3$.

We then need GR in order to deal with them. Let us quickly go over exact solutions of the Einstein Field Equations.

This lecture, we consider the vacuum Schwarzschild solution. The most general line element which is spherically symmetric (invariant under spatial rotations) must be made up of elements which are themselves invariant under spatial rotations. We will use spherical coordinates: $r, \theta, \varphi, t$.

In flat spacetime, the line element reads

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{0.1.4}$$

and our Schwarzschild solution will need to reduce to this in some limit.

The spatial line element is given by

$$\mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r} = \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{0.1.5}$$

Then, the most general spherically symmetric line element will read

$$\mathrm{d}s^2 = F(r,t)\,\mathrm{d}t^2 + M(r,t)\,\mathrm{d}r^2 + G(r,t)\,\mathrm{d}r\,\mathrm{d}t + C(r,t)r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{0.1.6}$$

however, we can redefine the radial coordinate in order to remove the function multiplying the angular term, so we get

$$\mathrm{d}s^2 = F\,\mathrm{d}t^2 + M\,\mathrm{d}r^2 + G\,\mathrm{d}r\,\mathrm{d}t + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right). \tag{0.1.7}$$

We can also introduce a new time variable:

$$\mathrm{d}t' = \mathrm{d}t + \psi(r,t)\,\mathrm{d}r\ . \tag{0.1.8}$$

If $\psi = rG/F$, then we remove the mixed term, and then we are left with the expression

$$\mathrm{d}s^2 = -B(r,t)\,\mathrm{d}t^2 + A\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\ . \tag{0.1.9}$$

However, we have not yet determined the two functions, and we have not said anything about the Einstein Field Equations, which are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,. \tag{0.1.10}$$

In vacuo, the stress-energy tensor vanishes. The curvature scalar must vanish (we can show this by contracting the EFE with the inverse metric), so the equations reduce to $R_{\mu\nu} = 0$. We restrict ourselves to the static case.

The linearly independent components of the Ricci tensor read

$$R^0_0 = \frac{B''}{2AB} - \frac{A'B'}{4A''B} - \frac{B'^2}{4AB^2} + \frac{B'}{rAB} = 0 \tag{0.1.11}$$
$$R^1_1 = \frac{B''}{2AB} - \frac{A'B'}{4A''B} - \frac{B'^2}{4AB^2} + \frac{A'}{rA^2} = 0 \tag{0.1.12}$$
$$R^2_2 = \frac{1}{4rA}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{r^2}\left(\frac{1}{A} - 1\right). \tag{0.1.13}$$

Computing $R^0_0 - R^1_1 = 0$ we find

$$\frac{1}{rA}\left(\frac{B'}{B} + \frac{A'}{A}\right) = 0 \tag{0.1.14}$$
$$\frac{\mathrm{d}\log(AB)}{\mathrm{d}r} = 0\,, \tag{0.1.15}$$

so $AB$ is constant. Without losing generality we can take $A = 1/B$, since if this is not the case we can just rescale the radial or temporal coordinate until it is.

Then, we can compute

$$R^2_2 = \frac{B}{2r}\left(\frac{B'}{B} + \frac{B'}{B}\right) + \frac{1}{r^2}(B-1) = 0 \tag{0.1.16}$$
$$B' + \frac{B}{r} - \frac{1}{r} = 0 \tag{0.1.17}$$
$$\frac{\mathrm{d}}{\mathrm{d}r}(rB) = 1\,, \tag{0.1.18}$$

so $rB(r) = r + C$, or equivalently

$$B(r) = \frac{C}{r} + 1\,. \tag{0.1.19}$$

After this, we can already substitute into the metric:

$$\mathrm{d}s^2 = -\left(1 + \frac{C}{r}\right)\mathrm{d}t^2 + \frac{1}{1 + C/r}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{0.1.20}$$

For any value of $C$, $B \to 1$ as $r \to \infty$: the metric reduces to the flat one asymptotically. Right now $C$ is an arbitrary constant, however in the weak field limit it is known that

$$g_{00} = -\left(1 + 2\frac{\phi}{c^2}\right), \tag{0.1.21}$$

where $\phi = -GM/r$ is the Newtonian gravitational field. Equating this expression to the one for $g_{00}$, we find

$$g_{00} = -\left(1 - \frac{2GM}{rc^2}\right) \implies C = -\frac{2GM}{c^2}\,. \tag{0.1.22}$$

The constant $M$ in the classical case is the mass of the source, however we are computing a vacuum solution. This is the mass we would compute if we were to measure the orbits of objects around the compact object.

This is then surely a *gravitational* mass, is it also an *inertial* mass? Can we show this in GR?

Then, we can write the Schwarzschild metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{0.1.23}$$

This is derived by assuming time-independence, however the result is the same even in the time-dependent case by the Jebsen-Birkhoff theorem (which we will prove in a moment). The element $g_{rr}$ diverges as $r \to R_g = 2GM/c^2$, however this does not represent any physical divergence: the scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \propto r^{-6}$ does not diverge there.

There are coordinates which do not diverge near the horizon: one classical choice employs the "tortoise" coordinates, which are the same for $r$, $\theta$, $\varphi$ as the Schwarzschild ones, while the time becomes (setting $G = c = 1$)

$$t = t' - 2M\log\left(1 - \frac{r'}{2M}\right). \tag{0.1.24}$$

Substituting this into the metric yields (dropping the primes for clarity):

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{4M}{r}\,\mathrm{d}r\,\mathrm{d}t + \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{0.1.25}$$

There is no pathology at $r = 2M$ anymore, so it was not a physical divergence. The temporal coefficient $g_{00}$ is the same: it can be shown that it is an invariant under coordinate transformations.

If we take two points which are very close along a particle trajectory, they must be separated by an interval $\mathrm{d}s^2 < 0$.

If we consider a (nongeodesic!) radial path described by $r(t)$, we can compute the corresponding line element by neglecting the angular part:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{4M}{r}\,\mathrm{d}r\,\mathrm{d}t + \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 \tag{0.1.26}$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = -\left(1 - \frac{2M}{r}\right) + \frac{4M}{r}\frac{\mathrm{d}r}{\mathrm{d}t} + \left(1 + \frac{2M}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 . \tag{0.1.27}$$

Now, the question we ask is: is it possible for the particle trajectory to be timelike or lightlike ($\mathrm{d}s^2 \leq 0$) and outgoing ($\mathrm{d}r/\mathrm{d}t > 0$) under these conditions? If this is the case, the signs of the three terms read

$$\underbrace{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}_{<0?} = -\left(1 - \frac{2M}{r}\right) \underbrace{+\frac{4M}{r}\frac{\mathrm{d}r}{\mathrm{d}t}}_{>0} \underbrace{+\left(1 + \frac{2M}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2}_{>0} , \tag{0.1.28}$$

so we can see that the equality can be satisfied (a positive number cannot equal a negative one!) as long as the first term on the right-hand side is negative, which means $r > 2M$. If $r \leq 2M$, on the other hand, this cannot be the case: a radial trajectory below the horizon *cannot* be outward.

This is what "horizon" means: it is a *semi-permeable* membrane, particles can surpass it only in one direction.

**Jebsen-Birkhoff** This theorem states that the Schwarzschild solution also describes the spacetime around an object in the spherically-symmetric but time-*dependent* case. Let us give a sketch of its proof, omitting some tedious calculations. If we write out the components of the Ricci tensor, we find something in the form

$$R^0_0 = R^0_0\Big|_{\text{static}} + \dot{A}(\dots) \tag{0.1.29}$$

$$R^1_1 = R^1_1\Big|_{\text{static}} + \dot{A}(\dots) , \tag{0.1.30}$$

while $R^2_2$ and $R^3_3$ are the same. Also, the term $R^1_0$ does not vanish unlike the static case, and is equal to

$$R^1_0 = -\frac{\dot{A}}{rA^2} = 0 , \tag{0.1.31}$$

so $\dot{A} = 0$: the equations then are the same as the static case ones! This, however, is not the end, since now the equation

$$\frac{A'}{A} + \frac{B'}{B} = 0 \tag{0.1.32}$$

is not necessarily solved by $\log A = -\log B$, since a prime denotes a *partial* derivative with respect to $r$, so in general we will have $\log A + \log B = f(t)$, some generic function of time. The metric will then read

$$ds^2 = -\left(1 - \frac{2M}{r}\right) f(t)\, dt^2 + \left(1 - \frac{2M}{r}\right) dr^2 + r^2\, d\Omega^2 \,, \tag{0.1.33}$$

but we can simply rescale the time coordinate to $t \to \sqrt{f} t$ in order for this to reduce to the usual expression. This theorem was originally discovered by the Norwegian physicist Jebsen, and only later popularized in a textbook by Birkoff [JR05].

The source of the geometry can change in time while leaving the outside spacetime unperturbed, however this holds only as long as the variation remains spherically symmetric: collapse, expansion or pulsation. Any asymmetry can lead to the emission of gravitational radiation.

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"Schwarzschild" means "Black Shield": no relation though, it is the name of a German scientist. We have discussed the properties of this metric.

Now, let us consider **geodesic motion** in the vacuum Schwarzschild spacetime.

We cannot choose an arbitrary radius for an orbit around a Black Hole: there is an Innermost Stable Circular Orbit, and this has a direct impact on the accretion efficiency.

We will use Latin letters for spacetime indices. The four-velocity, as usual, is $u^i = dx^i \big/ d\tau$, where $d\tau$ is the proper time. Also, for massive particles we also have the four-momentum $p^i = mu^i$.

Analogously to classical mechanics, the Lagrangian of a free particle is

$$L = -\sqrt{-g_{ij} p^i p^j} = -m\sqrt{-g_{ij} u^i u^j} \,, \tag{0.1.34}$$

and the corresponding Euler-Lagrange equations read

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^i}\right) - \frac{\partial L}{\partial x^i} = 0 \,, \tag{0.1.35}$$

however we will not study these in the general case, it is too complicated. It can be shown that along a geodesic the Lagrangian itself is conserved: $L = \text{const}$. Therefore, $g_{ij} u^i u^j$ is also a constant, a negative one since the velocity is timelike. We can then set it to $-1$ by reparametrizing, so we will have $u^2 = -1$ and $p^2 = -m^2$.

Suppose that the there is a certain coordinate $x^k$ such that

$$\frac{\partial g_{ij}}{\partial x^k} = 0 \,, \tag{0.1.36}$$

that is, the metric does not change along the $x^k$ direction. Therefore,

$$\frac{\partial L}{\partial x^k} = 0 \,, \tag{0.1.37}$$

which tells us that

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^k}\right) = 0 \,, \tag{0.1.38}$$

meaning that this cyclic variable gives us a conserved quantity. If we do the calculation, the constant comes out to be

$$m g_{kj} u^j = p_k = \text{const}\,. \tag{0.1.39}$$

This is the case in Schwarzschild spacetime: the metric only depends on $r$ and $\theta$, while $t$ and $\varphi$ are cyclic. Therefore, we will have two constants of motion: in terms of the momentum, $p_t$ and $p_\varphi$.

We can denote them as $E = -p_t$ and $L = p_\varphi$. Let us start from $L$:

$$L = g_{\varphi k} p^k = g_{\varphi\varphi} m \frac{\mathrm{d}x^\varphi}{\mathrm{d}\tau} \tag{0.1.40}$$

$$= r^2 \sin^2\theta\, m \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \tag{0.1.41}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{L}{m r^2 \sin^2\theta}\,. \tag{0.1.42}$$

Now, geodesic motion is in general planar since the metric is spherically symmetric: we do not lose any generality by setting $\theta = \pi/2$, so we can simplify $\sin^2\theta = 1$.

In general, even in Netwonian mechanics, when discussing orbits we can decompose the velocity into the radial and perpendicular direction. The angular momentum, in classical Newtonian mechanics, has a modulus $\left|\vec{L}\right| = |\vec{r} \wedge m\vec{v}| = m r v_\varphi = m r^2 \dot{\varphi}$. This is the same relation we have found here.

The energy of an object with four-momentum $p_k$ as measured by an observer with four-velocity $u^k$ is given by $E = -p_k u^k$. We choose an observer which is static with respect to the coordinates:[2] then, we have $\mathrm{d}t/\mathrm{d}\tau = \gamma/\sqrt{-g_{tt}}$, so

$$E = -g_{tt} m \frac{\mathrm{d}x^t}{\mathrm{d}\tau} = +\left(1 - \frac{2M}{r}\right) \frac{m}{\sqrt{1-v^2}} \frac{1}{\sqrt{-g_t t}} \tag{0.1.43}$$

$$= \sqrt{1 - \frac{2M}{r}} m\gamma \tag{0.1.44}$$

$$\approx m\left(1 - \frac{M}{r} + \frac{v^2}{2}\right) \tag{0.1.45}$$

so in the Newtonian limit ($M \ll r$, $v \ll 1$) this is regular expression for the conservation of energy, gravitational plus kinetic.

Now we have the tools to study geodesic motion: let us write down $p^2 = -m^2$ explicitly,

$$g_{tt}\left(p^t\right)^2 + g_{rr}\left(p^r\right)^2 + \underbrace{g_{\theta\theta}\left(p^\theta\right)^2}_{=0 \text{ if } \theta \equiv \pi/2} + g_{\varphi\varphi}\left(p^\varphi\right)^2 = -m^2\,, \tag{0.1.46}$$

and if we substitute we must be careful: the constants of motion are related to the *covariant* components of the quantities, so we have

$$g_{tt}\left(g^{tt} p_t\right)^2 + g_{rr}\left(g^{rr} p_r\right)^2 + g_{\varphi\varphi}\left(g^{\varphi\varphi} p_\varphi\right)^2 = -m^2 \tag{0.1.47}$$

[2] Its four velocity will be given by $k^i = (1/\sqrt{-g_{tt}}, \vec{0})$ by normalization.

$$g_{tt}\left(g^{tt}\right)^2E^2 + g_{rr}\left(p^r\right)^2 + g_{\varphi\varphi}\left(g^{\varphi\varphi}\right)^2L^2 = -m^2 \tag{0.1.48}$$

$$g^{tt}E^2 + g^{rr}\left(p_r\right)^2 + g^{\varphi\varphi}L^2 = -m^2 \tag{0.1.49}$$

$$-\left(1-\frac{2M}{r}\right)^{-1}E^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(p^r\right)^2 + \frac{L^2}{r^2} = -m^2 \tag{0.1.50}$$

$$-\left(1-\frac{2M}{r}\right)^{-1}E^2 + \left(1-\frac{2M}{r}\right)^{-1}m^2\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{L^2}{r^2} = -m^2\,, \tag{0.1.51}$$

which is an ODE for the single function $r(\tau)$: we can integrate this to find the shape of the trajectory. We now divide everything by $m^2$, and define the specific energy and angular momentum $\epsilon = E/m$ and $\ell = L/m$:

$$\left(1-\frac{2M}{r}\right)^{-1}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \epsilon^2\right] + \frac{\ell^2}{r^2} = -1\,. \tag{0.1.52}$$

We can then express this as

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -\left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right) + \epsilon^2\,, \tag{0.1.53}$$

which can give us $r(\tau)$ or $r(\varphi)$ in a rather simple way numerically, however the answer is not analytic: it is given by an elliptic integral.

We can study it analytically by looking at the turning points, those with $\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0$. These will tell us about the shape of the orbit. They obey the equation

$$0 = -\left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right) + \epsilon^2\,, \tag{0.1.54}$$

which means

$$\frac{\ell^2}{r^2} = \left(\epsilon^2 - 1 + \frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-1}\,, \tag{0.1.55}$$

which has the solutions

$$l_\pm = \pm r\left[\left(\epsilon^2 - 1 + \frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)\right]^{1/2}\,. \tag{0.1.56}$$

Let us define $\lambda = \ell/2M$ and $x = r/2M$. These are dimensionless in our units. Also, we define $\Gamma = \epsilon^2 - 1$. They satisfy

$$\lambda_\pm = \pm x\sqrt{\left(\Gamma + \frac{1}{x}\right)\left(1-\frac{1}{x}\right)^{-1}} = \pm x\sqrt{\frac{x\Gamma + 1}{x-1}}\,. \tag{0.1.57}$$

We want to draw these as curves in the $x$, $\lambda$ plane. We fix a value of $\ell$, which means we have chosen $\lambda$. This depends on the initial conditions of the motion of the particle. If this $\lambda$ is a $\lambda_+$, then $x$ can move in all the region $x(\lambda_+) < x < \infty$.

This is the relativistic analog of the hyperbolic trajectories in the Keplerian case.

Let us try to compute $\frac{d\lambda_\pm}{dx}$: this yields

$$2x^2\Gamma + x - 3x\Gamma - 2 = 0 \tag{0.1.58}$$

$$\Gamma\left(2x^2 - 3x\right) + x - 2 = 0 \tag{0.1.59}$$

$$\Gamma = \frac{2-x}{2x^2 - 3x}\,. \tag{0.1.60}$$

This means we have a single curve, which will relate $\Gamma$ and $x$. The physical cases are only the ones with $\Gamma > -1$. The minimum is at $x = 3$, corresponding to $r = 6M$, where we have $\Gamma = -1/9$. This corresponds to $\epsilon^2 - 1$: so, $\epsilon = \sqrt{2} \times 2/3$.

For $-1/9 < \Gamma < 0$ we have two choices for $x$, for $\Gamma > 0$ we have only one.

Coming back to the $\lambda_\pm$ curves, we can have $\lambda_\pm$ as two distinct curves, and there is a region around $\lambda = 0$ for which the particle will cross the horizon.

This allows us to compute the cross-section for gravitational capture, which is nonvanishing *even if* we neglect the size of the black hole.

Insert plots

Tuesday 2020-10-13, compiled 2020-10-20

We were discussing geodesics in the external Schwarzschild solution. We wrote down the locus of the inversion point $\dot{r} = 0$: this is

$$\lambda_\pm = \pm x\sqrt{\frac{\Gamma x + 1}{x - 1}}\,. \tag{0.1.61}$$

1. For $\Gamma > 0$ there is one extremum;
2. for $-1/9 < \Gamma < 0$ there are two extrema;
3. for $-1 < \Gamma < -1/9$ there are no extrema.

Let us consider the two-extrema case. Then, there is an intersection with the $x$ axis at $x = -1/\Gamma$ for the $\lambda_\pm$. Then the curve looks like a doorknob: it is a closed curve.

In this case, we are considering *bound states*: the particle can start on the LHS of the curve, and if this is the case it must reach the inversion point and then go back and eventually cross the horizon. If it is in the "doorknob region", then it is trapped there.

These orbits are similar to the Newtonian elliptic ones, but there are differences. These are radially bound, however they are not in general periodic: they will precess.

We can also have circular orbits, both stable and unstable.

In the no-extrema case there are no orbits. The limiting case $\Gamma = -1/9$ is interesting: the minimum becomes an inflection point, with zero second derivative. This happens at $x = 3$. This is the ISCO: the Innermost Stable Circular Orbit.

The corresponding energy is given by $\epsilon^2 - 1 = -1/9$: $\epsilon = 2\sqrt{2}/3$, so

$$E_{\text{ISCO}} = \frac{2\sqrt{2}}{3} mc^2\,. \tag{0.1.62}$$

If we can bring a particle from infinity to the ISCO (which is what happens as an accretion disk is formed) it can release an energy equal to $mc^2 - E_{\text{ISCO}} = (1 - 2\sqrt{2}/3)mc^2$. This is why an accretion disk can produce a large amount of energy.

The efficiency can then be computed as

$$\text{efficiency} = \frac{mc^2 - 2\sqrt{2}mc^2/3}{mc^2} \approx 6\,\%\,. \tag{0.1.63}$$

This is a *very large* efficiency: an order of magnitude more than the efficiency of nuclear burning.

## 0.2 Schwarzschild internal solution

Now we try to solve the EFE under spherical symmetry *with* a source:

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 8\pi T_{ij}\,, \tag{0.2.1}$$

with a metric

$$\mathrm{d}s^2 = -B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{0.2.2}$$

Birkhoff does not apply here: we are *assuming* stationarity. We will use a very simple $T_{ij}$: a perfect fluid, where

$$T_{ij} = (\rho + P) u_i u_j + P g_{ij}\,, \tag{0.2.3}$$

where the energy density is given by $\rho = \rho_0(1+\epsilon)$: $\rho_0$ is the rest energy density of the fluid, while $\epsilon$ accounts for thermal motion and other kinds of internal energy.

We will assume that the fluid is at rest with respect to the interior: we want to find an analog of the hydrostatic equilibrium equation. So, $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\varphi = 0$, while only $\mathrm{d}t \neq 0$. This means that only $u^0 \neq 0$, and normalization requires $g_{ij}u^i u^j = -1$: this means that $u^0 = 1/\sqrt{B}$, and the whole vector reads $u^i = \delta^i_0 / \sqrt{-g_{00}}$.

Let us then write the nonzero mixed components of the stress-energy tensor:

$$T^0_0 = -\rho \qquad\qquad T^1_1 = T^2_2 = T^3_3 = P\,, \tag{0.2.4}$$

since if we have one upper and one lower index the normalization in $u^i u_j$ simplifies.

Let us also skip the computations: the Einstein tensor reads

$$G^0_0 = \frac{1}{A}\left(\frac{1}{r^2} - \frac{A'}{rA}\right) - \frac{1}{r^2} = -8\pi\rho \tag{0.2.5}$$

$$G^1_1 = \frac{1}{A}\left(\frac{1}{r^2} + \frac{B'}{rB}\right) - \frac{1}{r^2} = 8\pi P \tag{0.2.6}$$

$$G^2_2 = G^3_3 = -\frac{1}{2A}\left[\frac{A'B'}{2AB} + \left(\frac{B'}{B}\right)^2 - \frac{B'}{Br^2} + \frac{A'}{r^2 A} - \frac{B''}{2B}\right] = 8\pi P\,. \tag{0.2.7}$$

We need to solve for $\rho$, $P$, $A$ and $B$: the equations of motion of the particles, $\nabla_i T^{ij} = 0$, are a consequence of the Einstein equations.

We have three equations for four variables: we need an additional one, typically we combine them with an *equation of state* for $P$: a simple case, a *barotropic EOS*, is in the form $P = P(\rho)$.

The first equation can be written as

$$\frac{1}{A} - \frac{rA'}{A^2} = 1 - 8\pi r^2 \rho \,, \tag{0.2.8}$$

which we can integrate to find

$$\frac{r}{A} = r - \int_0^r 8\pi \tilde{r}^2 \rho \, \mathrm{d}\tilde{r} \,. \tag{0.2.9}$$

We introduce the quantity

$$m(r) = \int_0^r 4\pi r^2 \rho \, \mathrm{d}r \,, \tag{0.2.10}$$

so that

$$A(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} \,. \tag{0.2.11}$$

The problem is that $m(r)$ is *not* the total mass-energy enclosed in the region below $r$: we are not integrating with respect to a covariant volume form.

With the second two equations, we end up with

$$\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r} = -\frac{1}{P+\rho}\frac{\mathrm{d}P}{\mathrm{d}r} \,. \tag{0.2.12}$$

We have essentially solved for $B$. We can also manipulate the equations to find:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{m(r)\rho}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi r^3 P}{m(r)}\right)\left(1 - \frac{2m(r)}{r}\right)^{-1} \,, \tag{0.2.13}$$

the **Tolman-Oppenheimer-Volkov** (TOV) equation. This reduces to the hydrostatic equilibrium equation in weak gravity.

The three other terms are basically three relativistic corrections, from velocity and radius: the first are applied based on whether the gas is relativistic *in its own rest frame*; if this is the case, then $P \sim \rho$. The first term is a local correction, the second one is a global one; the third term on the other hand is the usual GR correction, based on $r_{\mathrm{Schw}}/r$.

We can also write

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \,. \tag{0.2.14}$$

This equation, the TOV one and the equation of state form a closed system for $m$, $\rho$ and $P$.

As for boundary conditions, we require that $m(0) = 0$ and $P(R) = 0$. This is hard to do numerically: it is a boundary value problem, not an initial condition one.

Finally, we discuss the meaning of $m(r)$. The total $M$ is given by

$$M = \int_0^R 4\pi r^2 \rho \, \mathrm{d}r \,, \tag{0.2.15}$$

where $R$ is the radius of the star. This is *not* the total mass, since $\mathrm{d}V$ is *not* $4\pi r^2 \, \mathrm{d}r$. The *proper* three-volume element is instead given by $\mathrm{d}V = 4\pi r^2 \sqrt{A(r)} \, \mathrm{d}r$, since the true radial distance is calculated through the metric: $\mathrm{d}s^2 = A(r) \, \mathrm{d}r^2$ if the measured length is radial.

We call $M_*$ the mass which is calculated including the $\sqrt{A}$ term. Their difference is

$$M - M_* = \int_0^R 4\pi r^2 \rho \left(1 - \sqrt{A}\right) \mathrm{d}r \,. \tag{0.2.16}$$

If $r \gg 2m$ (the weak-field limit) then we can approximate $A = 1 - 2m/r$:

$$\begin{aligned} M - M_* &\approx \int_0^R 4\pi r^2 \rho \left(1 - 1 - \frac{2m}{r}\right) \mathrm{d}r && (0.2.17) \\ &\approx - \int_0^R \underbrace{4\pi r^2 \rho \, \mathrm{d}r}_{\mathrm{d}m} \frac{m}{r} && (0.2.18) \\ &\approx - \int_M \frac{Gm}{r} \, \mathrm{d}m = E_g \,. && (0.2.19) \end{aligned}$$

So, in the weak field limit, the mass defect is given by the gravitational potential energy! This is a physical consequence of the *nonlinearity* of the EFE.

The electromagnetic field behaves linearly at low energies, while in QED we can see nonlinearities.

## 0.3 The Kerr solution

Wednesday 2020-10-14, compiled 2020-10-20

Some history: the Schwarzschild was found in 1915 while S. was serving in the army; the second exact solution was found in 1963 by a PhD student in New Zeland [Ker63]. In Cartesian coordinates the metric is extremely ugly; nowadays we use the coordinates defined by Boyer and Lyndquist:

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta \sin\varphi && (0.3.1) \\ y &= \sqrt{r^2 + a^2} \sin\theta \cos\varphi && (0.3.2) \\ z &= r \cos\theta \,. && (0.3.3) \end{aligned}$$

These are then not *spherical* but *spheroidal* coordinates. Constant-$r$ spheroids are oblate ellipsoids in the $z$ direction.

The Kerr solution, for who wants to see the computation, is derived in "The mathematical theory of Black Holes" by Chandrasekhar [Cha98]. The metric reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) \mathrm{d}t^2 + \frac{\Sigma}{\Delta} \mathrm{d}r^2 + \Sigma \, \mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{aA}{\Sigma}\right) \sin^2\theta \, \mathrm{d}\varphi^2 - \frac{2A}{\Sigma} \mathrm{d}t \, \mathrm{d}\varphi \,, \tag{0.3.4}$$

where:

$$\Delta = r^2 - 2Mr + a^2 \qquad \Sigma = r^2 + a^2\cos^2\theta \qquad A = 2Mar\sin^2\theta\,, \tag{0.3.5}$$

while $a = J/M$ is the *specific angular momentum* of the black hole.

It describes the spacetime outside an axially symmetric, stationary body. In the Schwarzschild spacetime we have identified an horizon at $r = 2M$ by the properties that $g_{00} \to 0$ and $g_{rr} \to \infty$.

What about Kerr? Are there horizons? In this case, the two conditions do not happen in the same place. The region $g_{00} = 0$ is known as the *limit of staticity*: if it is the case, an observer's worldline cannot be both *timelike* and *stationary* (meaning time-directed).

The condition $g_{00} > 0$ is equivalent to $\Sigma < 2Mr$. Now, consider the $\mathrm{d}r^2$ coefficient: $\Sigma/\Delta$: this is positive always.

If we are in the region $\theta = \pi/2$, at $g_{00} > 0$ we can still have the metric's signature be $-+++$, since we have the $\mathrm{d}t\,\mathrm{d}\varphi$ term.

As long as $a\,\mathrm{d}\varphi > 0$, that term in the metric is negative, allowing the signature of the metric to be preserved.

This means that **the particle must be co-rotating** with the black hole if it is in the region $g_{00} > 0$.

As long as this is the case, however, a particle can remain at fixed $r$ and even escape. This is, then, *not* an horizon.

The horizon, instead, is found when $g_{rr}$ diverges: this is equivalent to $\Delta \to 0$, which means

$$r^2 - 2Mr + a^2 = 0 \implies r = M \pm \sqrt{M^2 - a^2}\,. \tag{0.3.6}$$

Both of these radii correspond to an horizon. Let us denote $r_+ = r_H$, since the inner horizon cannot really affect any observations.

In order for these two solutions to be real, we must have $a < M$. There is an horizon as long as $a/M < 1$.

If $a > M$, we have a **naked singularity**, since the singularity at $\Sigma = 0$ is still there. This singularity, in any case, is not shaped like a point: we only reach it along the equatorial plane.

Penrose proposed the cosmic **censorship hypothesis**: the universe is a "prude", it always hides singularities with horizons. There are good theoretical reasons to believe that this is verified.

Where is the limit of staticity? The equation is

$$r^2 + a^2\cos^2\theta - 2Mr = 0\,, \tag{0.3.7}$$

which is solved by

$$r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{0.3.8}$$

There are two of these surfaces as well, and we consider the outer one as before: $r_E = r_+$. If the horizon exists, then this region also exists.

This region is called the **ergosphere**, and the region between $r_H < r < r_E$ is called the **ergoregion**.

Insert figure for the shape of the region

The name comes from the fact that we can extract rotational energy from the BH. "Ergo" means energy.

There has been a long debate about whether the Penrose process actually occurs in a realistic astrophysical setting: the consensus is that the trajectory a particle must take in order for this to happen is way too peculiar.

Note that in this case we also have the cyclic coordinates $\varphi$ and $t$. We have two constants of motion like in Schwarzschild.

It can be shown that there exists a third constant of motion, beyond $E$ and $L_z$: $Q$, called Carter's constant.

For motion in the equatorial plane we can write the expression

$$E = \frac{r^{3/2} - 2r^{1/2} \pm aM^{1/2}}{r^{3/2}\left(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}\right)^{1/2}}, \tag{0.3.9}$$

where $\pm$ refers to whether the particle moves along a prograde or retrograde trajectory.

The expression for the last stable circular orbit is

$$r_{LS} = M\left[3 + z_2 \mp \left[(3 - z_1)(3 + z_1 + 2z_2)\right]^{1/2}\right], \tag{0.3.10}$$

check equation

where

$$z_1 = 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3}\left[\left(1 + \frac{a}{M}\right)^{-1/3} + \left(1 - \frac{a}{M}\right)^{-1/3}\right] \tag{0.3.11}$$

$$z_2 = \left(3\frac{a^2}{M^2} + z_1^2\right)^{1/2}. \tag{0.3.12}$$

This reduces to $r_{LS} = 6M$ in the $a = 0$ case, since then we have $z_1 = z_2 = 3$.

What happens for an extreme Kerr BH, with $a = M$? Then $z_1 = 1$, $z_2 = 2$: so,

$$r_{LS} = M\left[3 + 2 \mp \sqrt{2 \times 8}\right] = M[5 \mp 4], \tag{0.3.13}$$

which yields $r_{LS} = M$ in the corotating case, and $r_{LS} = 9M$ in the counter-rotating case.

I am writing $LS$ for last-stable, the professor uses $MS$ for marginally-stable

.

We expect that the efficiency of a Kerr BH in the extraction of energy from matter will be higher than the Schwarzschild solution. Let us use the expression we found for the specific energy $E$, with respect to the parameter $x = r/M$:

$$E = \frac{x^{3/2} - 2x^{1/2} \pm a/M}{x^{3/2}\sqrt{x^{3/2} - 3x^{1/2} \pm 2a/M}}, \tag{0.3.14}$$

which in the extreme case becomes

$$E = \frac{x^{3/2} - 2x^{1/2} \pm 1}{x^{3/2}\sqrt{x^{3/2} - 3x^{1/2} \pm 2}}, \tag{0.3.15}$$

so we can take the limit : in the direct case, sending $x \to 1$ we get $E = 1/\sqrt{3} \approx 0.577$.

The efficiency is given by

$$\eta = \frac{E_\infty - E}{E_\infty} = 1 - 1/\sqrt{3} \approx 42\,\%. \tag{0.3.16}$$

This is a huge amount of energy: we do not expect real black holes to be extreme. There are estimates of the spins of the black holes. What is found is that they seem to cover the whole range $0 < a/M < 1$.

An issue regarding a wide-spread misconception: the parameter $M$ in the interior Schwarzschild solution is the same as the $M$ in the corresponding *exterior* solution. Can we apply the same kind of reasoning for Kerr? Is there an interior Kerr solution? We don't know, but most probably not. Many efforts were put into seeking it, and they all failed. The properties of the matter and radiation inside are weird.

The wide-spread misconception is to claim that the Kerr spacetime describes the spacetime around a rotating star. It sounds reasonable, but it's wrong. Numerically we can derive the true form of the spacetime outside something like a neutron star: it is very different from the Kerr spacetime.

A qualitative argument: a star will generally have a quadrupole moment and emit GWs, while Kerr does not. Kerr is a Petrov-type-B spacetime, which is *nonradiating*.

To clarify.

Next time, we will discuss Equations of state and degenerate gasses.

## 0.4 The equation of state and degenerate gasses

Tuesday 2020-10-20, compiled 2020-10-20

We move away from the relativistic realm, and treat the more classical Equation of State (EoS). In general $P$ could be a function of $\rho$, $\mu$, $T$ and other variables. An often-used one is $P = P(\rho)$; also sometimes we use $u = u(\rho)$, where $u$ is the internal energy density.

We will treat the equation of state of a completely degenerate gas.

Let us start for a very simple system: a **hydrogen plasma**. It is a collection of $e^-$ and protons $p$.

We have complete collisional ionization for $T \gtrsim 10^5\,\mathrm{K}$. Under which conditions is this plasma relativistic or nonrelativistic? This is shown as the blue area in figure 1.

The electrons are surely relativistic if $k_B T \gtrsim m_e c^2$, which corresponds to $T \gtrsim 6 \times 10^9\,\mathrm{K}$. For the protons, an analogous equation yields $T \gtrsim 10^{13}\,\mathrm{K}$: these temperatures are basically never reached for realistic astrophysical scenarios.

In the region of $10^5\,\mathrm{K} \lesssim T \lesssim 10^9\,\mathrm{K}$ (for low densities), the plasma will be ionized but not relativistic.

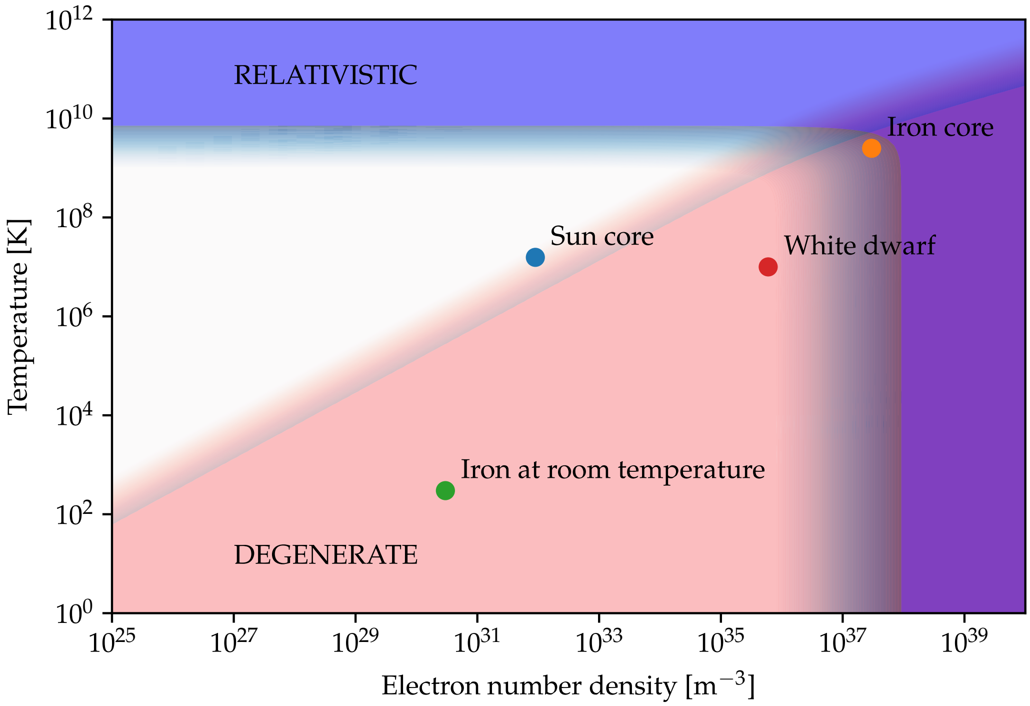


Figure 1:

The ideal gas law for the electrons reads $P_e = n_e k_B T$, for the ions $P_i = n_i k_B T$. However, electrons are much lighter than protons, so for the same forces they will accelerate more, so they will radiate away more energy.

Another complication is the following: consider a collection of $N$ ionic species, then for each of these (labelled by $k$) we will have $P_{i,k} = n_{i,k} k_B T$. The total pressure can be calculated by summing over all of these, plus the electrons:

$$P = n_e k_B T + \sum_k n_{i,k} k_B T = k_B T \left( n_e + \sum_k n_{i,k} \right) \tag{0.4.1}$$

$$= \frac{n k_B T}{\mu}, \tag{0.4.2}$$

where $\mu$ is the *mean molecular weight*, calculated through the total baryonic number density $n$:

$$\mu = \left[ \frac{n_e}{n} + \frac{\sum_k n_{i,k}}{n} \right]^{-1}. \tag{0.4.3}$$

For a pure hydrogen plasma, we have one electron per proton, so we find

$$\mu_{\rm H} = [1+1]^{-1} = \frac{1}{2}. \tag{0.4.4}$$

For a pure helium plasma we have two electrons per Helium nucleus, which contains 4 baryons: so,

$$\mu_{^4\mathrm{He}} = \left[\frac{2}{4} + \frac{4}{4}\right]^{-1} = \frac{4}{3}\,. \tag{0.4.5}$$

This holds under the assumption that the plasma behaves like an ideal gas, and that the electrons and protons are nonrelativistic. There is a crucial reason why the first assumption fails: **degeneracy**.

Electrons obey Dirac statistics, and if the density is high enough they can behave very similarly to the $T = 0$ limit even for high temperatures. The region in which they behave like a fully degenerate gas is shown in pink in figure 1.

The distribution function for a system of fermions reads

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p} = \frac{2}{h^3}\underbrace{\left[\exp\left(\frac{E}{k_BT} - \alpha\right) + 1\right]^{-1}}_{f}\,. \tag{0.4.6}$$

Here, the *degneracy parameter* is $\alpha = \mu/k_BT$, where $\mu$ is the chemical potential. The energy is expressed as $E = \sqrt{m^2c^4 + p^2c^2}$.

If we want to compute $\alpha$, we can just integrate:

$$n = \int_{\mathbb{R}^3} \frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p}\,\mathrm{d}^3p \tag{0.4.7}$$

$$= \frac{2}{h^3}\int_0^\infty \left[\exp\left(\frac{E}{k_BT} - \alpha\right) + 1\right]^{-1} 4\pi p^2\,\mathrm{d}p\,. \tag{0.4.8}$$

This integral can be computed for any value of $\alpha$ and $T$: we then find $n = n(\alpha, T)$. Inverting this relation, we find $\alpha = \alpha(n, T)$.

The logarithm of the absolute value of $\alpha k_BT = \mu$ can be plotted against $\log n$ for different values of the temperature $T$. For each $T$, there is a cusp: $\alpha$ changes sign. It can become very big and positive or very big and negative.

Do plot!

Let us start with the case $|\alpha| \gg 1$ and $\alpha < 0$: this corresponds to low $n$, high $T$. Then, the $-\alpha$ appearing in the distribution is large and positive: then, the distribution looks like $f \sim \exp\left(-\frac{E}{k_BT}\right)$, the Maxwell-Boltzmann distribution. The gas is behaving like an ideal gas.

Another option is $|\alpha| \gg 1$, $\alpha > 0$. This is the case in which we have low $T$, high $n$. Even in the $T \to 0$ limit, the product $\alpha k_BT$ stays finite: this is a function of $n$, and is called $E_F$.

The distribution then looks like $f \sim \left[\exp\left((E - E_F)/k_BT\right) + 1\right]^{-1} \to [E \leq E_F]$ in the limit $T \to 0$. (I use the Iverson bracket: [proposition] is 1 if the proposition is true, 0 if it is false).

The higher $n$ is, the higher the $T$ for which the behavior is close to the $T \to 0$ limit.

In the $T \to 0$ limit, we can do the integration analytically: this yields an explicit expression for $n$ in terms of the Fermi momentum $p_F$ corresponding to the Fermi energy $E_F$: inverting it we get

$$p_F = \sqrt[3]{\frac{3n}{8\pi}} h \,. \tag{0.4.9}$$

This momentum is a characteristic of $n$ independently of the temperature: for $T > 0$ it will not be a hard limit anymore, but it is still a good descriptor of the Fermi gas.

We can write

$$E_F = \sqrt{m^2c^4 + p_F^2c^2} = \sqrt{1 + x_F^2} mc^2 \,, \tag{0.4.10}$$

where $x_F = p_F/mc^2$. For a nonrelativistic particle distribution $x_F \ll 1$, so $E_F \approx mc^2 + x_F^2 mc^2/2$. The dependence on the number density of the kinetic part is $\sim x_F^2 \sim n^{2/3}$.

On the other hand, in the ultrarelativistic limit $E_F \approx x_F mc^2 \sim x_F \sim n^{1/3}$.

This is the reason why in figure 1 the pink boundary curves down in the blue (relativistic region).

It is useful to define this quantity in terms of proper density, in $\mathrm{g/cm^3}$, instead of number density.

We have

$$x_F = \frac{p_F}{mc^2} = \left( \frac{3h^3}{8\pi} \right)^{1/3} \frac{1}{mc^2} n^{1/3} \,; \tag{0.4.11}$$

if we multiply above and below by the mean baryon mass, we have

$$n_e = \frac{n m_b}{m_e m_b} = \frac{\rho}{m_e m_b} \,, \tag{0.4.12}$$

which gives us

$$x_F \sim 10^{-2} \left( \frac{\rho}{m_e} \right)^{1/3} \,. \tag{0.4.13}$$

Are we sure about this? I think I missed something.

The internal energy density $u$ is computed in general as

$$u = \frac{2}{h^3} \int E(p) \frac{\mathrm{d}N}{\mathrm{d}^3x \, \mathrm{d}^3p} \, \mathrm{d}^3p \,, \tag{0.4.14}$$

which in our case is

$$u = \frac{2}{h^3} \int_0^{p_F} \sqrt{m^2c^4 + p^2c^2} 4\pi p^2 \, \mathrm{d}p \tag{0.4.15}$$

$$= \frac{8}{h^3} \pi mc^2 (mc)^3 \int_0^{p_F} \sqrt{1 + x^2} x^2 \, \mathrm{d}x \tag{0.4.16}$$

$$= \frac{8\pi m^4 c^6}{h^3} \frac{x_F^4}{4} I(x_F)\,, \tag{0.4.17}$$

where $I(x_F)$ can be computed analytically, but it is of order 1 as can be seen by the asymptotics of the integral.

The integral reads

$$\int_0^{p_F} \sqrt{1+x^2}x^2\,\mathrm{d}x = \frac{1}{8}\left[x_F(1+2x_F)^2\sqrt{1+x_F^2} - \log\left(x_F + \sqrt{1+x_F^2}\right)\right]. \tag{0.4.18}$$

For the pressure we have a similar integral, which however is more complicated from the conceptual point of view.

The first law of thermodynamics states that

$$\mathrm{d}U + p\,\mathrm{d}V = 0\,, \tag{0.4.19}$$

as long as the transformation does not exchange heat with its surroundings. Note that $\mathrm{d}u = \mathrm{d}(U/V) = V^{-1}\,\mathrm{d}U - UV^{-2}\,\mathrm{d}V$, which means that

$$\frac{\mathrm{d}U}{V} = \mathrm{d}u + \frac{U}{V}\,\mathrm{d}V\ . \tag{0.4.20}$$

Substituting into the first law of thermodynamics,

$$\mathrm{d}u + \frac{U}{V}\,\mathrm{d}V + P\,\mathrm{d}V = 0\,, \tag{0.4.21}$$

but since $V \propto 1/n$ we have

$$\frac{\mathrm{d}V}{V} = -\frac{\mathrm{d}n}{n}\,. \tag{0.4.22}$$

Substituting this in, we get

$$\mathrm{d}u - (P+u)\frac{\mathrm{d}n}{n} = 0\,. \tag{0.4.23}$$

This means that

$$n\frac{\mathrm{d}u}{\mathrm{d}n} = P + u\,, \tag{0.4.24}$$

which allows us to compute the pressure! The only step remaining is to replace $\mathrm{d}n\,/n$ with an expression in terms of $x_F$, which is

$$\frac{\mathrm{d}x_F}{x_F} = 3\frac{\mathrm{d}n}{n}\,. \tag{0.4.25}$$

This finally yields

$$P = \frac{x_F}{3}\frac{\mathrm{d}u}{\mathrm{d}x_F} - u\,. \tag{0.4.26}$$

Then we are almost done: we can compute the pressure with $u$, for which we have an analytic expression, and $\mathrm{d}u/\mathrm{d}x_F$, which we can easily find since the original expression for $u$ was an integral in $\mathrm{d}x_F$, from which we can read off the integrand.

This yields

$$P = \frac{8\pi m^4 c^5}{h^3}\left[\frac{1}{3}x_F^3\sqrt{1+x_F^2} - \frac{1}{8}\left(x_F(1+2x_F)^2\sqrt{1+x_F^2} - \log\left(x_F + \sqrt{1+x_F^2}\right)\right)\right] \tag{0.4.27}$$

$$= \frac{m^4 c^5 \pi}{h^3}\left[x_F\sqrt{1+x_F^2}\left(\frac{8}{3}x_F^2 - \left(1+2x_F^2\right)\right) + \log\left(x_F + \sqrt{1+x_F^2}\right)\right] \tag{0.4.28}$$

$$= \frac{m^4 c^5 \pi}{h^3}\left[x_F\sqrt{1+x_F^2}\left(\frac{2}{3}x_F^2 - 1\right) + \log\left(x_F + \sqrt{1+x_F^2}\right)\right]. \tag{0.4.29}$$

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