Path integrals and non-Gaussianity in cosmology

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Abstract

We introduce the main statistical methods needed to study cosmological non-Gaussianities: correlation functions, the power spectrum, spatial filtering. Then, we introduce the Euclidean path integral formalism and discuss how it can be applied in cosmology. Finally, we draw a parallel to the path integral used in Quantum theory.

1 Introduction and motivation

The Large Scale Structure of the cosmos formed as a result of gravitational clustering, starting from "seed" inhomogeneities in the density field in the early universe. The generation of these perturbations is commonly thought to have been the result of quantum

fluctuations in the early universe, which have reached galactic scales in the inflationary stage of the expansion.

These perturbations may be considered as pertaining to the gravitational field Φ or to the density field ρ: the former approach has shown itself to be more fruitful in recent years.

A model which so far has not been contradicted by observation is to consider perturbations as being a realization of a Gaussian random field. Even if the field were not truly Gaussian, we would expect the Gaussian approximation to be a rather good one as it happens generally for processes which arise from the combined effects of many independent random variables [CM18, pag. 2].

However, the study of non-Gaussianities in the primordial universe is crucial. Inflationary models commonly predict a nearly-Gaussian perturbation field, and the deviations from Gaussianity can aid us in the effort to determine which one best fits the data.

It is common to model an almost-Gaussian field with something similar to a Taylor expansion [MVJ00, eq. 1], [CM18, eq. 1] in the Gaussian field φ_L:

$$\Phi = \varphi_L + f_{NL}\left(\varphi_L^2 + \left\langle \varphi_L^2 \right\rangle\right) + g_{NL}\left(\varphi_L^3 - \left\langle \varphi_L^2 \right\rangle \varphi_L\right) + \mathcal{O}\left(\varphi_L^4\right). \tag{1.1}$$

From the most recent Planck mission data, which measured the non-Gaussianity by looking at the Cosmic Microwave Background temperature and polarization anisotropies' correlations, the parameters f_{NL} and g_{NL} are both compatible with zero [Col+19].[1]

Non-Gaussianities may have a large impact on structure formation, since the overdensities which cluster into cosmic structures are heavily dependent on the *tails* of the density perturbation distribution: for example, if φ is Gaussian then its tails decay like $\exp\left(-\varphi^2\right)$, while φ^2 is chi-square distributed (with one degree of freedom), so its tails decay like $\exp\left(-\varphi\right)$.

Therefore, even a small amount of non-Gaussianity may "boost" structure formation significantly.

Modelling the initial conditions for structure formation for a general non-Gaussian field is tricky, and the formalism which is best suited for the task is that of functional integration (the Path Integral): an integral over an infinite-dimensional space

2 Statistical methods in cosmology

2.1 The statistics of fluctuations

In order to study the statistical properties of fluctuations we need to start from an assumption: *ergodicity*. The probabilities we will discuss are defined formally as corresponding to the likelihood of drawing a certain "realization of the universe" from a statistical ensemble of possible universes. We cannot, of course, analyze more than one of these realizations, but we can consider its *spatial* statistics.

[1] The parameter f_{NL} is divided into different contributions, corresponding to different geometric configurations in Fourier space. We shall not go into details here, but they are independently compatible with zero.

All statistical properties can be derived as averages of appropriate functions; ergodicity is precisely the statement that spatial averages correspond to ensemble averages.

With this established, let us discuss how we can characterize the fluctuations of, say, the density field ρ (although this approach can be applied to other fields). We start by defining $\delta = (\rho - \rho_b)/\rho_b$: this is the dimensionless fluctuation, which can range from -1 to $+\infty$. By definition, its average will be 0: $\langle\delta\rangle = 0$, so we need to average at least a product of two δ's in order to have a nonzero result. We evaluate them at two arbitrary point, x and $x + r$: this would correspond to six degrees of freedom, but we lose three from homogeneity and two more from isotropy, therefore we only need to account for the modulus of r. This motivates the definition

$$\xi(r) = \left\langle \delta(x)\delta(x+r) \right\rangle , \tag{2.1}$$

which is called the *two-point correlation function*. By isotropy, we expect its Fourier transform to be a function of the modulus of the wavevector. With this in mind, we write the following ansatz for the correlation function in Fourier space, in terms of the **power spectral density** P:

$$\left\langle \delta(k_1)\delta(k_2) \right\rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2) P(|k_1|) , \tag{2.2}$$

where $\delta(k)$ is the Fourier transform of $\delta(x)$ (they are denoted by the same letter despite being different functions, we distinguish them by the argument):

$$\delta(x) = \int \frac{\mathrm{d}^3 x}{(2\pi)^3} e^{ik\cdot x} \delta(k) . \tag{2.3}$$

Let us then check that the ansatz is indeed correct:

$$\begin{aligned}
\xi(r) &= \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \frac{\mathrm{d}^3 k_2}{(2\pi)^3} \exp(ik_1 x + ik_2 x + ik_2 r) \left\langle \delta(k_1)\delta(k_2) \right\rangle && (2.4)\\
&= \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \frac{\mathrm{d}^3 k_2}{(2\pi)^3} \exp(ik_1 x + ik_2 x + ik_2 r)(2\pi)^3 \delta^{(3)}(k_1 + k_2) P(k_1) && (2.5)\\
&= \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \exp(-ik_1 r) P(k_1) && (2.6)\\
&= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \exp(i|k||r|\cos\theta) P(|k|) . && (2.7)
\end{aligned}$$

This shows that $P(|k|)$ is the Fourier transform of the two-point correlation function $\xi(r)$. We can simplify this expression making use of the fact that there is no angular dependence in the integral in $\mathrm{d}^3 k_1$: the computation yields

$$\xi(r) = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k\, k^2 P(|k|) j_0(kr) , \tag{2.8}$$

where j_0 is a Bessel function defined by

$$j_0(x) = \frac{1}{2} \int_0^\pi e^{ix\cos\theta} \sin\theta \, \mathrm{d}\theta . \tag{2.9}$$

If we evaluate the two-point correlation function ξ at 0 we find the variance of the density fluctuation field at a generic point: $\xi(0) = \langle \delta(x)\delta(x) \rangle = \sigma^2$, which can be expressed as

$$\sigma^2 = \xi(0) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P(|k|)\,. \tag{2.10}$$

The physical meaning of the power spectrum $P(|k|)$ is to describe the distribution of the power of the perturbations into the various spatial frequencies; the previous expression shows that the variance at each point can be recovered by integrating it.

The shape of this power spectrum in the early universe has been measured through the correlations in the CMB by the Planck satellite: the spectrum is well described by a powerlaw, $P(k) = Ak^{n_s}$, with $n_s = 0.9665 \pm 0.0038$ [Col+19, eq. 38].

A value of $n_s = 0$ would correspond to frequency-independent *white noise*; $n_s = 1$ would instead indicate exact *scale invariance*, and this is called a Harrison-Zel'dovich spectrum. The measurement of n_s being slightly smaller than 1 indicates the presence of more energy at longer wavelengths, which is called a *red tilt*.

With this value, we can see that the integral giving σ^2 diverges in the ultraviolet ($k \to \infty$). This divergence is not physical, since cosmic structure does not extend to arbitrarily small scales. Therefore, we fix the problem by introducing a *spatial filter* $W(r, R)$, where r is the spatial radius of interest while R is a fixed characteristic radius below which we do not consider structures. This filter is better characterized through its Fourier transform, $\widetilde{W}(k, R)$: we use a filtered field like

$$\delta_R(k) = \widetilde{W}(k, R)\delta(k) \tag{2.11}$$

$$\delta_R(r) = \int \mathrm{d}^3 y\, W(|x - y|, R)\delta(x)\,. \tag{2.12}$$

The convolution in real space can be visually interpreted as a smoothing over scales of R (of course, the exact workings of this depend on the precise shape of W); in Fourier space we have a simpler multiplication.

There are different ways of choosing $\widetilde{W}(k, R)$, which have in common the fact that they attain high values for $k \ll R^{-1}$ and they go to zero for $k \gg R^{-1}$. One must be careful when choosing a filter: a sharp filter in real space corresponds to a very broad filter in Fourier space, and vice versa. The power spectrum is transformed like $P(k) \to \widetilde{W}^2(k, R)P(k)$ by the application of the filter.

2.2 Peak functions

We have two main observational channels to study perturbations: the CMB and the positions of galaxies. With the latter we are not probing the whole perturbation spectrum: instead, we can describe the *peaks* of the perturbation, which undergo gravitational collapse to form galaxies.

The approach by Matarrese, Lucchin, and Bonometto [MLB86] is to start from the fact that at each point we have a distribution with a variance σ_R^2 (if we are using a smoothing

scale of R), so we can define the density

$$\rho_{\nu,R} = \Theta(\delta(x) - \nu\sigma_R)\,, \tag{2.13}$$

where ν is a real number of order 1, and Θ is the Heaviside function: this density characterizes the points which are ν standard deviations above the background. We can study the N-point correlation functions of this "boolean density":

$$\Pi^{(N)}_{\nu,R}(x_1 \dots x_N) = \left\langle \prod_{r=1}^{N} \rho_{\nu,R}(x_r) \right\rangle\,, \tag{2.14}$$

which, roughly, speaking, can quantify the probability that we will find galaxies in the positions $x_1, \dots, x_N$ if we choose ν appropriately.

A seminal paper by Bardeen et al. [Bar+86] systematically characterized the statistics of these peaks in the case of a Gaussian density perturbation field.

Bertschinger [Ber87] pioneered the use of path integrals in order to generate *constrained realizations* of perturbation fields: these are realizations of the field — points in the statistical ensemble — which satisfy some specific characteristic, such as the presence of a galaxy in a specific location.

This allows for the statistical study of the structure *around* the peak of the density perturbation, as opposed to only considering the statistics of the peak distribution.

2.3 Press-Schechter theory

Press-Schechter theory is a way to calculate the distribution of virialized objects (i. e. galaxies) in a perturbative fashion. Virialization is a deeply nonlinear process; however it can be shown that when the linearized density perturbation reaches a critical value of $\delta_c \approx 1.686$ the corresponding true nonlinear halo has virialized.

Matarrese, Verde, and Jimenez [MVJ00] applied this approach to a mildly non-Gaussian density perturbation field (with only a quadratic term) in order to study the abundance of high-redshift galaxies, finding analytic results with the aid of the path integral formalism.

In order to account for the scale of the growing mode of the linear perturbations they give a redshift-dependence to the critical density: fixing $\Delta_c = 1.686$ they write $\delta_c = \Delta_c / D(z_c|\Omega_{0,m}, \Omega_{0,\Lambda})$, where D is the linear growth factor: D^{-1} increases with redshift.

This means that at higher z it is harder to form virialized objects, but simultaneously the density PDF is more sensitive to non-Gaussianity. This can be seen graphically in figures 4 and 5 of the same paper.[2]

2.4 The bispectrum

As we will explore in more detail later, the two-point function and its associated power spectrum completely characterize a Gaussian field: all the higher-order n-point correlation functions can be derived from the two-point one. Since we want to characterize non-Gaussianity, then, we define the simplest object which allows us to do so: the 3-point

[2] They refer to two different models, denoted as A and B: in the first the density field is perturbed, in the second the gravitational field is perturbed. Nowadays, model B has been found to be the most useful one.

correlation function, and its associated *bispectrum*, defined by the relation

$$\left\langle \delta(\vec{k}_1)\delta(\vec{k}_2)\delta(\vec{k}_3) \right\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B(k_1, k_2, k_3)\,. \tag{2.15}$$

We stress that this kind of statistical tool can be applied to any field; in modern analyses the bispectrum associated with the gravitational field [CM18, eq. 2] B_Φ is used.

This kind of correlation function then can be used to estimate the nongaussianity in the gravitational field distribution: it can be shown that if we consider a linear + quadratic model like that in equation (1.1) (with $g_{NL} = 0$) we will have

$$B_\Phi(k_1, k_2, k_3) = 2 f_{NL} \big(B_\Phi(k_1) B_\Phi(k_2) + B_\Phi(k_2) B_\Phi(k_3) + B_\Phi(k_1) B_\Phi(k_3) \big)\,, \tag{2.16}$$

where $B_\Phi(k)$ is the power spectrum. This allows us to observationally compute f_{NL}, [Pla+20].

3 Path integral

The formalism of the path integral is about assigning a probability density to the space of possible field configurations. This space is infinite-dimensional: in order to treat it as a vector space we need to find some kind of basis, such as the values of the field on a grid of points.

The way to formally treat these problems starts from the theory of linear functionals; for this first section we will follow Zaidi [Zai83] quite closely.

3.1 Linear Functionals

We start from the space of square-integrable functions $q(x)$, endowed with a product and an orthonormal basis ϕ_n. We consider (multi-)linear *functionals*, which are maps from the space of square-integrable functions (or from tuples of them) to $\mathbb{R}$ or $\mathbb{C}$. These can be represented as functions of infinitely many variables, countably so if we use the basis ϕ_n, uncountably so if we use the continuous basis x.

A functional $F[q]$ can be represented as a power series

$$F[q] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int \mathrm{d}x_i \, q(x_i) f(x_1, \ldots, x_n)\,. \tag{3.1}$$

Examples of this are the exponential series corresponding to the function $f(x)$, mapping $q(x)$ to $e^{(f,q)}$ where the brackets denote the scalar product in the space, and the Gaussian series corresponding to the kernel $K(x, y)$, mapping $q(x)$ to $e^{(q,K,q)}$, where

$$(q, K, q) = \int \mathrm{d}x \, \mathrm{d}y \, q(x) q(y) K(x, y)\,. \tag{3.2}$$

Functional derivatives describe how the output of the functional changes as the argument goes from $q(x)$ to $q(x) + \eta(x)$, where $\eta(x)$ is small. This will be a linear functional of η to first order, so we define the functional derivative with the expression

$$F[q + \eta] - F[q]\Big|_{\text{linear order}} = \int \eta(y) \frac{\delta F}{\delta q(y)} \, \mathrm{d}y\,. \tag{3.3}$$

The analogy to finite-dimensional spaces is as follows: the functional derivative $\delta F/\delta q(y)$ corresponds to the *gradient* $\nabla^i F$, while the integral in the previous expression corresponds to the *directional derivative* $(\nabla^i F)\eta^j g_{ij}$. The metric is present since the gradient is conventionally defined with a vector-like upper index; in our infinite-dimensional space the scalar product is given by the integral.

Practically speaking, the most convenient way to calculate a functional derivative is by taking $\eta(x)$ to be such that it only differs from zero in a small region near y, and let us define

$$\delta\omega = \int \eta(x)\,\mathrm{d}x\ . \tag{3.4}$$

Then, we define

$$\frac{\delta F}{\delta q(y)} = \lim_{\delta\omega\to 0} \frac{F[q+\eta]-F[q]}{\delta\omega}\,. \tag{3.5}$$

In order for the limit to be computed easily, it is convenient for $\eta(x)$ to be in the form $\delta\omega \times$ fixed function, so that we are only changing the normalization as we shrink $\delta\omega$. A common choice is then

$$\eta(x) = \delta\omega\delta(x-y)\,. \tag{3.6}$$

If we apply this procedure to the identity functional $q \to q$, we find

$$\frac{\delta q(x)}{\delta q(y)} = \lim_{\delta\omega\to 0} \frac{q(x)+\delta\omega\delta(x-y)-q(x)}{\delta\omega} = \delta(x-y)\,. \tag{3.7}$$

The variable q is one-dimensional, if instead we wanted to consider a multi-dimensional coordinate system q_α by the same reasoning we would find

$$\frac{\delta q_\alpha(x)}{\delta q_\beta(y)} = \delta_{\alpha\beta}\delta(x-y)\,. \tag{3.8}$$

An example: the functional derivative of a functional F_n defined by

$$F_n[q] = \int f(x_1,\ldots,x_n)q(x_1)\ldots q(x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\ , \tag{3.9}$$

where f is a symmetric function of its arguments, is given by

$$\frac{\delta F_n}{\delta q(y)} = n\int f(x_1,\ldots,x_{n-1},y)q(x_1)\ldots q(x_{n-1})\,\mathrm{d}x_1\ldots\mathrm{d}x_{n-1}\ , \tag{3.10}$$

a function of y.

A **linear transformation** is in the form

$$q(x) = \int K(x,y)q'(y)\,\mathrm{d}y\ . \tag{3.11}$$

If this transformation has an inverse, which is characterized by the kernel K^{-1}, then we must have the orthonormality relation

$$\int K(x,y)K^{-1}(y,z)\,\mathrm{d}y = \int K^{-1}(x,y)K(y,z)\,\mathrm{d}y = \delta(x-z)\,. \tag{3.12}$$

We can do **Legendre transforms**: if we have a functional F we can differentiate with respect to the coordinate q to find

$$\frac{\delta F[q]}{\delta q(x)} = p(x)\,, \tag{3.13}$$

in analogy to the momentum in Lagrangian mechanics. Then, we can map $F[q]$ to a new functional $G[p]$ which will only depend on the momentum:

$$G[p] = F[p] - \int q(x)p(x)\,\mathrm{d}x\ . \tag{3.14}$$

We can also define functional integration, by

$$\int F[q]\big[\mathrm{d}q\big] = \int \hat{F}(\{q_i\})\prod_i \mathrm{d}q_i\ . \tag{3.15}$$

On the right-hand side we are using the expression of the functional as a function of infinitely many variables which we discussed above; we are then integrating over each of the coordinates in this infinite dimensional function space. The infinite-dimensional measure is also often denoted as $\mathcal{D}q$.

This integral will not always exist, however in the cases in which it does we can change variables. Let us consider a linear change of variable, whose kernel is $K(x,y)$, such that (compactly written) $q = Kq'$.

Then, we want to compute the integral

$$\int F[Kq']\big[\mathrm{d}Kq'\big] \tag{3.16}$$

as an integral in $\big[\mathrm{d}q\big]$: in order to do so, we need to relate the two functional measures. We start by expressing both q and q' in terms of an orthonormal basis ϕ_i: inserting this into the linear transformation law we get

$$q(x) = \int K(x,y)q'(y)\,\mathrm{d}y \tag{3.17}$$

$$\sum_i q_i\phi_i(x) = \int K(x,y)\sum_j q'_j\phi_j(y)\,\mathrm{d}y \tag{3.18}$$

$$\sum_i q_i \underbrace{\int \phi_i(x)\phi_k(x)\,\mathrm{d}x}_{\delta_{ik}} = \sum_j q'_j \underbrace{\int K(x,y)\phi_j(y)\phi_k(x)\,\mathrm{d}y\,\mathrm{d}x}_{k_{jk}} \tag{3.19}$$

$$q_k = \sum_j q'_j k_{jk}\,. \tag{3.20}$$

Then, the measure will transform with the determinant $\det K = \det k$, which we can now express as an infinite product of the eigenvalues of k:

$$[\mathrm{d}q] = \left|\frac{\partial q}{\partial q'}\right| [\mathrm{d}q'] = \det K [\mathrm{d}q'] \,. \tag{3.21}$$

Usually functional integrals cannot be computed analytically; the exception is given by **Gaussian integrals**, which generalize the finite-dimensional result

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}A_{ij}x_i x_j + ib_j x_j\right) \mathrm{d}x_1 \dots \mathrm{d}x_j = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right). \tag{3.22}$$

Here A_{ij} is an n-dimensional real matrix (which WLOG can be taken to be symmetric) while b_i is an n-dimensional vector. The result comes from a transformation of the coordinates according to the finite-dimensional

This can be interpreted as a "functional" (still finite-dimensional, so just a function, but we will generalize soon) of b_i; we write it with an additional normalization N for convenience:

$$Z[b] = N \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}A_{ij}x_i x_j + b_j x_j\right) \mathrm{d}x_1 \dots \mathrm{d}x_j = N\sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right), \tag{3.23}$$

and if we rescale the normalization N so that $Z[\vec{0}] = 1$ we get

$$Z[b] = \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right). \tag{3.24}$$

The infinite-dimensional generalization of this result amounts to replacing all the sums (expressed implicitly with Einstein notation here) with integrals; also conventionally we change the names of the variables to $x \to q$, $A \to K$, $b \to J$:

$$Z[J] = N \int \mathcal{D}q \exp\left(-\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, K(x,y) q(x) q(y) + i \int \mathrm{d}x\, q(x) J(x)\right) \tag{3.25}$$

$$= \exp\left(-\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, J(x) J(y) K^{-1}(x,y)\right). \tag{3.26}$$

Let us now give some examples of applications of this result: $K(x,y) = \sigma^{-2}\delta(x-y)$ means $K^{-1}(x,y) = \sigma^2\delta(x-y)$, so

$$Z[J] = \exp\left(-\frac{\sigma^2}{2}\int \mathrm{d}x\, J^2(x)\right). \tag{3.27}$$

This, as we shall see, can be used to give us a description of white noise, which is uncorrelated in momentum space.

3.2 The probability density functional

We can interpret the quantity

$$\exp\left(-\frac{1}{2}(q,K,q)\right)\mathcal{D}q \tag{3.28}$$

as a Gaussian *probability density functional* $\mathrm{d}\mathcal{P}\,[q]$, since

1. it is positive definite;
2. it is normalized, as long as we set its integral, $Z[0]$, to 1;
3. it goes to zero as $q \to \pm\infty$.

If this is the case, then we ought to be able to compute the average value of a functional $F[q]$ as

$$\left\langle F[q]\right\rangle = \int F[q]\,\mathrm{d}P\,[q] = \int \mathcal{D}q \exp\left(-\frac{1}{2}(q,K,q)\right)F[q]\,, \tag{3.29}$$

which we can generalize to any non-gaussian probability density functional by replacing the exponential $\exp\left(-\frac{1}{2}(q,K,q)\right)$ with a generic $\mathcal{P}[q]$.

A useful kind of average we can compute is given by the N-point correlation function,

$$C^{(N)}(x_1,\ldots,x_n) = \left\langle q(x_1)\ldots q(x_n)\right\rangle\,. \tag{3.30}$$

With the formula we gave earlier, this can be computed as

$$C^{(N)}(x_1,\ldots,x_n) = \int \mathcal{D}q\mathcal{P}[q]\prod_i q(x_i)\,. \tag{3.31}$$

Here we can make use of a trick: consider the functional derivative

$$\frac{1}{i}\frac{\delta Z[J]}{\delta J(x_1)}\bigg|_{J=0} = \frac{1}{i}\,\frac{\delta}{\delta J(x)}\bigg|_{J=0}\int \mathcal{D}q \exp\left(-\frac{1}{2}(q,K,q)+i(J,q)\right) \tag{3.32}$$

$$= \int \mathcal{D}q \exp\left(-\frac{1}{2}(q,K,q)\right)q(x_1) = \left\langle q(x_1)\right\rangle = C^{(1)}(x_1)\,, \tag{3.33}$$

which actually holds for any probability density functional, we did not make use of the gaussianity. In general, each functional derivative "brings down a factor $q(x)$", so we will be able to write the N-point correlation function as

$$C^{(N)}(x_1\ldots x_n) = \frac{1}{i^N}\,\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0}\,. \tag{3.34}$$

The correlation functions, which as we discussed in an earlier section are crucial when discussing structure formation, can be "simply" calculated by functional differentiation as long as we have the generating functional $Z[J]$. This generating functional is very similar

mathematically to a partition function in statistical mechanics, and it serves an analogous role: its derivatives allow us to characterize the dynamics of the system.

Now, any functional $\mathscr{F}[q]$ can be expressed through a functional Taylor series:

$$\mathscr{F}[q] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left. \frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)} \right|_{q=0} q(x_1) \dots q(x_n) \,, \tag{3.35}$$

so if we compute the average value $\langle \mathscr{F}[q] \rangle$ we find

$$\langle \mathscr{F}[q] \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left. \frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)} \right|_{q=0} \underbrace{\langle q(x_1) \dots q(x_n) \rangle}_{= \mathrm{C}^{(N)}(x_1 \dots x_n)} \tag{3.36}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left. \frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)} \right|_{q=0} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \right|_{J=0} \tag{3.37}$$

$$= \mathscr{F}\left[-i \frac{\delta}{\delta J} \right] Z[J] \Big|_{J=0} \,. \tag{3.38}$$

The expression with the functional $\mathscr{F}$ being calculated "at" the functional derivative is purely formal: it allows us to compactly write the previous Taylor expansion, and should be interpreted as a shorthand for it.

3.3 Gaussian fields' correlation functions

Consider a Gaussian field, whose partition function $Z[J]$ is given, as we have shown in (3.25), by

$$Z[J] = \exp\left(-\frac{1}{2} (J, K^{-1}, J) \right) . \tag{3.39}$$

Then, as before we can calculate the correlation functions through functional derivatives: the first ones are

$$\langle q(x) \rangle = \frac{1}{i} \left. \frac{\delta Z[J]}{\delta J(x)} \right|_{J=0} \tag{3.40}$$

$$= -i \int \mathrm{d}y \, K^{-1}(x,y) J(y) \exp\left(-\frac{1}{2} (J, K^{-1}, J) \right) \Big|_{J=0} = 0 \tag{3.41}$$

$$\langle q(y) q(x) \rangle = - \left. \frac{\delta^2 Z[J]}{\delta J(y) \delta J(x)} \right|_{J=0} \tag{3.42}$$

$$= - \left(-K^{-1}(x,y) + \int \mathrm{d}z \, \mathrm{d}w \, K^{-1}(x,z) K^{-1}(y,w) J(w) \right) \exp\left(-\frac{1}{2} (J, K^{-1}, J) \right) \Big|_{J=0} \tag{3.43}$$

$$= K^{-1}(x,y) \,. \tag{3.44}$$

So, we have our result: *for a Gaussian variable, the two-point correlation function is the inverse of the kernel*. This is perfectly analogous to the result we find for a zero-mean n-dimensional multivariate Gaussian with covariance matrix K^{-1}: its probability density function is given by

$$\mathcal{N}(\vec{x}|\vec{0}, K^{-1}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det K^{-1}}} \exp\left(-\frac{1}{2}\vec{x}^{\top} K \vec{x}\right), \tag{3.45}$$

and its one and two point functions read $\langle x_i \rangle = 0$, $\left\langle x_i x_j \right\rangle = K^{-1}_{ij}$ respectively.

Coming back to the infinite-dimensional scenario: a similar (albeit longer) calculation to the one before allows us to compute the N-point correlation function for the same Gaussian variable: we expand the exponential in $Z[J]$ in a power series, and when we differentiate it an even number of times we find for the even-numbered correlation functions

$$C^{2N}(x_1 \dots x_{2N}) = \left[K^{-1}(x_1, x_2) K^{-1}(x_3, x_4) \dots K^{-1}(x_{2N-1}, x_{2N})\right]_{\text{symmetrized}}, \tag{3.46}$$

where "symmetrized" means that we must sum over all the permutations of the variables x_i in the argument of the inverse kernels; on the other hand, the odd correlation functions C^{2N+1} all vanish since they correspond to the integrals of odd functions over all space.

As we mentioned in section 2, in the Gaussian case the two-point function (or, equivalently, the kernel) contains all the information. Starting from it, we can reconstruct the Gaussian probability density functional $\mathrm{d}\mathcal{P}[q]$.

3.3.1 Connected correlation functions

In the Gaussian case the N-point correlation functions are redundant for $N > 2$. We wish to introduce a notion of "irreducible" correlation functions which minimize the amount of redundancy: formally, we require that in the Gaussian case the ones with $N > 2$ vanish.

These irreducible functions are called the connected N-point correlation functions, and as we will check in a moment they can be calculated through the *generating functional of connected correlation functions*:

$$\mathscr{W}[J] = \log Z[J], \tag{3.47}$$

through the expansion

$$\mathscr{W}[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \, C_C^N(x_1 \dots x_N) J(x_1) \dots J(x_n) \tag{3.48}$$

$$C_C^n(x_1 \dots x_n) = \frac{1}{i^n} \left.\frac{\delta^n \mathscr{W}[J]}{\delta J(x_1) \dots \delta J(x_n)}\right|_{J=0}. \tag{3.49}$$

We could also have defined $\mathscr{W}[J] = i \log Z[J]$, this is a matter of convention.

In the Gaussian case we have

$$\mathscr{W}[J] = \log Z[J] = -\frac{1}{2}(J, K^{-1}, J), \tag{3.50}$$

which, by direct comparison with the Taylor expansion, means that

$$C_C^2(x_1, x_2) = K^{-1}(x_1, x_2)\,, \tag{3.51}$$

while $C^N \equiv 0$ for any $N \neq 2$. This is a validation of our ansatz: in the Gaussian case it is the construction we hoped to make.

3.4 Computing probabilities

We now wish to apply the construction of probability density functionals: what we will typically ask about, besides correlation functions, is the probability density that the field will attain certain values at certain points. In order to describe this we will need to introduce some mathematical tools, making use of both Legendre and Fourier transforms. What we will achieve is a general framework, although its application to non-Gaussian cases is complicated, therefore we will only treat Gaussian examples.

We define the *classical field*

$$q_{\rm cl}(x) = \frac{\delta \mathscr{W}[J]}{\delta J(x)}\,, \tag{3.52}$$

and the effective action $\Gamma[q_{\rm cl}]$ as the Legendre transform of $\mathscr{W}[J]$:

$$\Gamma[q_{\rm cl}] = \mathscr{W}[J] - \int \mathrm{d}x\, q_{\rm cl}(x) J(x)\,, \tag{3.53}$$

from which we can then recover $J(x)$ as

$$J(x) = -\frac{\delta \Gamma[q_{\rm cl}]}{\delta q_{\rm cl}(x)}\,. \tag{3.54}$$

Also, our expression for $q_{\rm cl}$ yields

$$q_{\rm cl}(x) = -\int \mathrm{d}y\, K^{-1}(x,y) J(y)\,, \tag{3.55}$$

from which we can express $J(x)$ by using the direct kernel $K(x,y)$:

$$\int \mathrm{d}x\, q_{\rm cl}(x) K(x,w) = -\int \mathrm{d}y\, \mathrm{d}w\, K^{-1}(x,y) K(x,w) J(y) = -\int \mathrm{d}w\, \delta(y-w) J(y) = -J(w)\,. \tag{3.56}$$

With an analogous procedure we can show that

$$(J, K^{-1}, J) = (q_{\rm cl}, K, q_{\rm cl})\,. \tag{3.57}$$

The effective action then reads

$$\Gamma[q_{\rm cl}] = -\frac{1}{2}(J, K^{-1}, J) + (J, K^{-1}, J)\Big|_{J=J(q_{\rm cl})} \tag{3.58}$$

$$= \frac{1}{2}(q_{\rm cl}, K, q_{\rm cl})\,. \tag{3.59}$$

Now we have the tools to consider actual probabilities: starting from our classical field q, we want to compute the probability that it takes on a certain value $q \in (\alpha, \alpha + \mathrm{d}\alpha)$ at a point $\overline{x}$: this is expressed with a probability density function in the form

$$\frac{\mathrm{d}P_q}{\mathrm{d}\alpha} = P_{q(x)}(\alpha; \overline{x})\,. \tag{3.60}$$

We want to write this "$P(\alpha)\,\mathrm{d}\alpha$" in terms of the functional integral; in order to do so, we start from the Fourier transform

$$\int \mathrm{d}\beta \exp\bigl(i\beta\varphi\bigr) P_q(\beta; \overline{x}) = \Bigl\langle e^{i\beta\varphi} \Bigr\rangle_\beta\,. \tag{3.61}$$

The integral is a definite one, with bounds corresponding to the region in which P_q is nonzero, typically $\mathbb{R}$. The right-hand side is an average over the possible values taken on by the field q at the point $\overline{x}$, but it can also be computed by averaging over the possible *overall* field configurations, computed at a point $\overline{x}$:

$$\int \mathrm{d}\beta \exp\bigl(i\beta\varphi\bigr) P_q(\beta; \overline{x}) = \Bigl\langle e^{iq(\overline{x})\varphi} \Bigr\rangle_q\,. \tag{3.62}$$

Although the expression is similar we have made a large conceptual leap: $\langle\rangle_\beta$ denotes a one-dimensional integral, the Fourier transform, while $\langle\rangle_q$ denotes an *infinite*-dimensional functional integral, for which we must employ calculation tools such as (3.29). The need for this step is motivated by the fact that the values of the field at different points are not independent: what we want to study are precisely the correlations between them, so in order to treat a single point we must consider the whole field.

If we take the Fourier antitrasform, we find

$$\begin{aligned} P_q(\alpha, \overline{x}) &= \frac{1}{2\pi} \int \mathrm{d}\varphi\, e^{-i\varphi\alpha} \Bigl\langle e^{i\varphi q(\overline{x})} \Bigr\rangle_q & (3.63)\\ &= \frac{1}{2\pi} \int \mathrm{d}\varphi \Bigl\langle e^{i\varphi(q(\overline{x})-\alpha)} \Bigr\rangle_q & (3.64)\\ &= \bigl\langle \delta(q(\overline{x}) - \alpha) \bigr\rangle_q\,. & (3.65) \end{aligned}$$

This equation can be interpreted to mean the following: "the probability that the field q is equal to α at $\overline{x}$ is given by the integral of the probabilities of all the field configurations which satisfy $q(\overline{x}) = \alpha$".

This formalism can be generalized to N-point functions, the notation for the probability that for $j = 1 \ldots N$ the field q takes on the value α_j at position x_j is as follows:

$$\begin{aligned} \mathrm{d}P_q^N &= P_q(\alpha_1 \ldots \alpha_N; x_1 \ldots x_N)\, \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N & (3.66)\\ &= P_q\bigl([\alpha_N]; [x_N]\bigr)\, \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N & (3.67)\\ P_q\bigl([\alpha_N]; [x_N]\bigr) &= \left\langle \prod_{j=1}^{N} \delta(q(x_j) - \alpha_j) \right\rangle_q\,. & (3.68) \end{aligned}$$

Statistical independence corresponds to the statement that the product can be brought out of the average, and we are interested in the general case, in which this does not happen.

The question we generally ask is: what is this probability? We can try to find an expression for it in terms of the partition function $Z[J]$: we use once again the fact that

$$\delta(x) = \frac{1}{2\pi} \int \mathrm{d}\varphi \, e^{i\varphi x} \tag{3.69}$$

to see that

$$P_q([\alpha_N]; [X_N]) = \frac{1}{(2\pi)^N} \int \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_N \exp\left(-i \sum_{j=1}^N \varphi_j \alpha_j \right) \left\langle \exp\left(i \sum_{j=1}^N \varphi_j q(x_j) \right) \right\rangle_q . \tag{3.70}$$

This can be brought back to the partition function by making use of the fact that

$$Z[J] = \left\langle \exp\big(i(J,q)\big) \right\rangle_q \tag{3.71}$$

$$= \int \mathcal{D}q P[q] \exp\left(i \int \mathrm{d}x \, J(x) q(x) \right), \tag{3.72}$$

therefore

$$\left\langle \exp\left(i \sum_{j=1}^N \varphi_j q(x_j) \right) \right\rangle_q = Z\left[\sum_{j=1}^N \varphi_j \delta(x - x_j) \right] = Z[\widetilde{J}_\varphi] \,, \tag{3.73}$$

where we have used the fact that

$$i \int \mathrm{d}x \, q(x) \left(\sum_{j=1}^N \varphi_j \delta(x - x_j) \right) = i \sum_{j=1}^N \varphi_j q(x_j) \,. \tag{3.74}$$

So, we can compute the probability that the field reaches the values α_j at the points x_j as long as we can compute the partition function at $Z[\widetilde{J}_\varphi]$, and then integrate N times:

$$P_q([\alpha_N], [x_N]) = \frac{1}{(2\pi)^N} \int \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_N \exp\left(-i \sum_{j=1}^N \varphi_j \alpha_j \right) Z[\widetilde{J}_\varphi] \,. \tag{3.75}$$

Let us compute this for the case of a Gaussian random field $q(x)$. The partition function evaluated at $\widetilde{J}_\varphi = \sum_j \varphi_j \delta(x - x_j)$ is equal to

$$Z[\widetilde{J}_\varphi] = \exp\left(-\frac{1}{2} (\widetilde{J}_\varphi, K^{-1}, \widetilde{J}_\varphi) \right) \tag{3.76}$$

$$= \exp\left(-\frac{1}{2} \sum_{i,j=1}^N \varphi_i \varphi_j K^{-1}(x_i, x_j) \right) . \tag{3.77}$$

Since it appears in the expression for the partition function, let us define the *covariance matrix*

$$M_{ij} = K^{-1}(x_i, x_j) = C^{(2)}(x_i, x_j) \,, \tag{3.78}$$

so that the probability reads, using the usual result about Gaussian integrals,

$$P_q([\alpha_N],[x_N]) = \frac{1}{(2\pi)^N}\int \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_N \exp\left(-i\sum_{j=1}^{N}\varphi_j\alpha_j\right)\exp\left(-\frac{1}{2}\varphi_i M_{ij}\varphi_j\right) \tag{3.79}$$

$$= \frac{1}{\sqrt{(2\pi)^N \det M}}\exp\left(-\frac{1}{2}\alpha_i (M^{-1})_{ij}\alpha_j\right). \tag{3.80}$$

This is the standard expression for an N-variate Gaussian distribution whose covariance matrix is M_{ij}.

3.5 Avoiding divergences: high- and low-pass filters

We will now discuss the application of the filter functions introduced in section 2 to the path integral formulation. If $W_R(x)$ is our filter function, then the general correlation function can be calculated by substituting the value of the field at a point with its convolution with W_R:

$$C_R^{(N)}(x_1 \ldots x_N) = \int \mathcal{D}q\mathcal{P}[q]\prod_{r=1}^{N}\int \mathrm{d}y_r\, q(y_r) W_R(|y_r - x_r|) \tag{3.81}$$

$$= \int \prod_{r=1}^{N}\mathrm{d}y_r\, W_R(|y_r - x_r|) C^{(N)}(y_1 \ldots y_N)\,. \tag{3.82}$$

The same procedure can be applied to the *connected* correlation functions, so we can calculate the $C_{R,C}^{(N)}$: *smooth, connected* N-point correlation functions.

We can calculate these "smoothed" correlation functions through the generating functional $Z[J]$ by choosing $J(x)$ in the form

$$J(x) = \int \mathrm{d}y\, \varphi(x+y) W_R(y)\,, \tag{3.83}$$

where φ is a generic function.

Since these regularized correlation functions do not diverge at vanishing distances, we can define the N-th order **moment**:

$$\left\langle q_R^N \right\rangle = C_R^{(N)}(x \ldots x) \tag{3.84}$$

and the N-th order **cumulant**:

$$\left\langle q_R^{(N)} \right\rangle_C = C_{R,C}^{(N)}(x \ldots x)\,. \tag{3.85}$$

Due to homogeneity and isotropy, neither of these depends on x. We expect these to diverge if we perform no filtering, which is equivalent to taking the filtering scale $R \to 0$. On the opposite limit, taking $R \to \infty$ amounts to averaging over all space, so we expect $\left\langle q_R^N \right\rangle \sim \langle q \rangle^N$, while $\left\langle q_R^N \right\rangle_C \sim 0$.

The moments of q_R can be obtained through the moment generating function

$$Z(\varphi) = Z[J_\varphi(x)] = Z[\varphi W_R(|x|)]\,, \tag{3.86}$$

and we can define the *cumulant* generating function $W(\varphi)$ analogously. We can then define an effective action through a Legendre transform:

$$\Gamma(q_{R,\mathrm{cl}}) = W(\varphi) - q_{R,\mathrm{cl}}\varphi\,, \tag{3.87}$$

where the classical field $q_{R,\mathrm{cl}}$ is given by

$$q_{R,\mathrm{cl}} = \frac{\mathrm{d}W(\varphi)}{\mathrm{d}\varphi}\,. \tag{3.88}$$

The function $Z(\varphi)$ is just the Fourier transform of $P_R(\alpha, \overline{x})$:

$$Z(\varphi) = \left\langle e^{i\varphi q_R} \right\rangle_q = \int \mathrm{d}\alpha\, e^{i\varphi\alpha} P_{q_R}(\alpha; \overline{x})\,, \tag{3.89}$$

so we can insert the explicit expression for the probability inside the inverse of this transform:

$$P_R(\alpha; \overline{x}) = \frac{1}{2\pi}\int \mathrm{d}\varphi\, e^{-i\alpha\varphi} Z(\varphi) \tag{3.90}$$

$$= \frac{1}{2\pi}\int \mathrm{d}\varphi\, e^{-i\alpha\varphi} \int \mathcal{D}q\mathcal{P}[q] \exp\left(i\varphi \int \mathrm{d}y\, W_R(|y - \overline{x}|) q(y) \right). \tag{3.91}$$

We can expand $Z(\varphi)$ and $W(\varphi)$ as

$$Z(\varphi) = 1 + \sum_{N=1}^{\infty} \frac{i^N}{N!} \varphi^N \left\langle q_R^N \right\rangle \tag{3.92}$$

$$W(\varphi) = \sum_{N=1}^{\infty} \frac{i^N}{N!} \varphi^N \left\langle q_R^N \right\rangle_C\,, \tag{3.93}$$

therefore

$$\left\langle q_R^N \right\rangle = \int \mathrm{d}\alpha\, \alpha^N P_{q_R}(\alpha; \overline{x})\,, \tag{3.94}$$

which clarifies what was meant by saying that these are *moments*. P_{q_R} can be reconstructed starting from these moments.

We can take a different path, by transforming the probability measure: we start from the probability density functional evaluated at q, and calculate the expectation value of a generic functional $F[q]$ as

$$\int \mathcal{D}q P[q] F[q] \propto \int \mathcal{D}q_R P[(W^{-1}, q_R)] F[(W^{-1}, q_R)]\,. \tag{3.95}$$

The proportionality is because we need to include a Jacobian determinant, however probability densities must always be normalized to 1 so any constant is inessential.

Applying this to a Gaussian field amounts to mapping the kernel K to a "smoothed" kernel K_R, which corresponds to the inverse of the smoothed two-point correlation function: $\langle q_R(x) q_R(y) \rangle$.

As we mentioned in section 2, the smoothed two-point function is given by

$$\langle q_R(x) q_R(y) \rangle = G_R(|x-y|) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k \, P(k) \widetilde{W}_R^2(k) e^{ik\cdot(x-y)} \tag{3.96}$$

$$= \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k \, k^2 P(k) \widetilde{W}_R^2(k) j_0(k|x-y|) \,, \tag{3.97}$$

where j_0 is a Bessel function of the first kind. Recall that the two point function is the inverse of the kernel: therefore, from the first expression we can read off the Fourier transform of K_R^{-1}, or, inverting,

$$\widetilde{K}_R(k) = \frac{1}{P(k)\widetilde{W}_R^2(k)} \,. \tag{3.98}$$

Now, let us consider a typical power-spectrum: $P(k) = Ak^n$. It can be shown that the smoothed two-point function will be asymptotically (in the $r \ll R$ or $r \gg R$ limits) proportional to

$$G_R(r) \propto \left[\max(R, r)\right]^{-(n+3)} \,. \tag{3.99}$$

In the $n > 0$ case the smoothed two-point function $G_R(r)$ must be equal to zero somewhere, since we have the constraint[3]

$$\int_0^\infty \mathrm{d}r \, r^2 G_R(r) = 0 \,. \tag{3.101}$$

The first zero-crossing of the correlation function, r_0 such that $G_R(r_0) = 0$, is called the **coherence length**.

3.6 Saddle-point expansion

We want to show that in general, our probability measure can be written as

$$\mathcal{P}[q] = \exp\left(-\frac{1}{2}(q, K, q) - V[q]\right) \,, \tag{3.102}$$

where the potential term encompasses all the non-quadratic parts of the probability. Linear terms can be removed with a change of variable, so we can expand it starting from third order:

$$V[q] = \sum_{n=3}^{\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_n \, K^{(n)}(x_1 \ldots x_n) q(x_1) \ldots q(x_n) \,. \tag{3.103}$$

[3] This comes from the following line of reasoning: since the field has zero mean,

$$\int \mathrm{d}^3y \, \langle q_R(x) q_R(y) \rangle = \left\langle q_R(x) \int \mathrm{d}^3y \, q_R(y) \right\rangle = 0 \,. \tag{3.100}$$

We start by ntroducing the "integrating by parts relation":

$$Z[J] = \exp\left(-V\left(-i\frac{\delta}{\delta J}\right)\right)\int \mathcal{D}q \exp\left(-\frac{1}{2}(q,K,q) + i(q,J)\right) \tag{3.104}$$

$$= \exp\left(-V\left(-i\frac{\delta}{\delta J}\right)\right)\underbrace{\exp\left(-\frac{1}{2}(J,K^{-1},J)\right)}_{Z_0[J]} \tag{3.105}$$

$$= \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\left[V\left(-i\frac{\delta}{\delta J}\right)\right]^n Z_0[J]\,. \tag{3.106}$$

With this approach, we can recover the probability of the field attaining a certain value as a function of the cumulants $\left\langle q_R^N\right\rangle_C$, since as we have shown earlier the function $W(\varphi)$ can be expressed in terms of them.[4]

The **saddle point expansion** is the following: we want to compute an integral in the form

$$J = \int \mathrm{d}\tau \exp\big(-f(\tau)\big)\,, \tag{3.107}$$

so we choose a τ_0 such that $f'(\tau_0) = 0$, and then approximate f up to second order:

$$J \approx \int \mathrm{d}\tau \exp\left(-f(\tau_0) - \frac{1}{2}f''(\tau_0)(\tau-\tau_0)^2\right) \tag{3.108}$$

$$\approx \sqrt{\frac{2\pi}{f''(\tau_0)}}\exp\big(-f(\tau_0)\big)\,. \tag{3.109}$$

We are effectively approximating the integrand as a Gaussian, whose mean and variance are computed perturbatively at its stationary point.

We can apply this to the calculation of path integrals: if the probability $P(q)$ can be written in terms of $Z(\varphi) = \exp\big(W(\varphi)\big)$, then we have

$$P(q) = \frac{1}{2\pi}\int \mathrm{d}\varphi \exp\big(-i\varphi q + W(\varphi)\big)\,. \tag{3.110}$$

The classical variable $q_{R,\mathrm{cl}} = \bar{q}$ is defined as $\bar{q} = W'(\varphi)$, and the effective action is written in terms of it: $\Gamma(\bar{q}) = W(\varphi) - \bar{q}\varphi$. Therefore,

$$\frac{\mathrm{d}\bar{q}}{\mathrm{d}\varphi} = W''(\varphi)\,, \tag{3.111}$$

while the derivative of the classical action reads

$$\Gamma''(\bar{q}) = \frac{\mathrm{d}}{\mathrm{d}\bar{q}}(-\varphi) = -\big[W''(\bar{q})\big]^{-1}\,. \tag{3.112}$$

[4] The calculation needed to do so is quite involved, more details can be found in the second section of Matarrese, Lucchin, and Bonometto [MLB86]. Here we merely meant to show that such a procedure is possible, by introducing all the tools needed to perform it.

This means that $\mathrm{d}\varphi = -\Gamma''(\bar{q})\,\mathrm{d}\bar{q}$, which allows us to change variable:

$$P(q) = -\frac{1}{2\pi}\int \Gamma''(\bar{q})\,\mathrm{d}\bar{q}\exp\big(iq\Gamma'(\bar{q}) + \Gamma(\bar{q}) - \bar{q}\Gamma'(\bar{q})\big)\,, \tag{3.113}$$

where we used the fact that

$$-i\varphi q + W(\varphi) = -i\varphi q + \Gamma(\bar{q}) + \bar{q}\varphi \tag{3.114}$$

$$= \Gamma(\bar{q}) + (\bar{q} - iq)\frac{\mathrm{d}\Gamma(\bar{q})}{\mathrm{d}\bar{q}}\,. \tag{3.115}$$

For large q, this oscillating integral is dominated by the points at which the phase does not change much: these are the stationary points of the argument of the exponential,

$$iq\Gamma''(\bar{q}) + \Gamma'(\bar{q}) - \Gamma'(\bar{q}) - \bar{q}\Gamma''(\bar{q}) = \Gamma''(\bar{q})(iq - \bar{q}) = 0\,, \tag{3.116}$$

meaning that $\bar{q} = iq$.

We can then apply the saddle-point approximation:

$$P(q) \approx \frac{e^{i\Gamma(iq)}}{2\pi}\Gamma''(iq)\int \mathrm{d}\bar{q}\exp\left(-\Gamma''(iq)\frac{\big(\bar{q} - iq\big)^2}{2}\right) \tag{3.117}$$

$$\approx \sqrt{\frac{\Gamma''(iq)}{2\pi}}\exp\big(\Gamma(iq)\big)\,. \tag{3.118}$$

In the Gaussian case the probability reads

$$P(q) = \frac{1}{\sqrt{2\pi\sigma_R^2}}\exp\left(-\frac{q^2}{2\sigma_R^2}\right), \tag{3.119}$$

so we can recognize $\Gamma(\bar{q}) = \bar{q}^2/(2\sigma_R^2)$, while $W(\varphi) = -\varphi^2/(2\sigma_R^2)$.

We can apply this approximation in order to prove the claim from the start of this section: suppose we had a generating functional in the form

$$Z[J] = \int \mathcal{D}q\,\underbrace{\exp\left(-\frac{1}{2}(q,K,q) - V[q] + i(q,J)\right)}_{\exp\left(\mathscr{F}[q,J]\right)} \tag{3.120}$$

with arbitrary V.

The functional integral will be dominated by the stationary points of $\mathscr{F}[q,J]$: we define q_0 as the field configuration satisfying

$$\left.\frac{\delta\mathscr{F}[q,J]}{\delta q}\right|_{q_0} = (K,q_0) + \left.\frac{\delta V[q]}{\delta q}\right|_{q_0} - iJ = 0\,. \tag{3.121}$$

This will depend on J in general. We then expand up to second order, as usual:

$$\mathscr{F}[q,J] \approx \mathscr{F}[q_0,J] + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\left.\frac{\delta^2\mathscr{F}[q,J]}{\delta q(x)\delta q(y)}\right|_{q_0}\left.(q(x) - q_0(x))(q(y) - q_0(y))\right|_{\text{symm}}\,. \tag{3.122}$$

Thanks to the functional equation for q_0 (3.121), we can write $\mathscr{F}[q_0, J]$ as

$$\mathscr{F}[q_0, J] = \frac{1}{2}(q_0, K, q_0) + V[q_0] - i(J, q_0) \tag{3.123}$$

$$= (K, q_0) + \left.\frac{\delta V[q]}{\delta q}\right|_{q_0} - \frac{i}{2}(q_0, J)\,. \tag{3.124}$$

This is usually the "bottleneck" in the calculation, practically speaking. It is really hard to solve this functional equation. Also, we define

$$\left.\frac{\delta^2 \mathscr{F}}{\delta q(x_1)\delta q(x_2)}\right|_{q_0} = K(x_1, x_2) + \left.\frac{\delta^2 V[q]}{\delta q(x_1)\delta q(x_2)}\right|_{q_0} = \mathscr{Q}(x_1, x_2)\,. \tag{3.125}$$

We then perform the integration with respect to the field $q - q_0$ (this amounts to adding a constant, so the measure does not change) and find

$$Z[q_0, J] \approx \mathscr{N} \frac{\exp\bigl(-\mathscr{F}[q_0, J]\bigr)}{\sqrt{\det \mathscr{Q}[q_0, J]}}\,. \tag{3.126}$$

Using the identity $\log\det M = \operatorname{Tr}\log M$ we can compute the generating functional for cumulants:

$$W[q_0, J] \approx \mathscr{F}[q_0, J] - \frac{1}{2}\operatorname{Tr}\log \mathscr{Q}[q_0, J] + \text{const}\,. \tag{3.127}$$

3.7 Path integrals in Quantum Mechanics

We draw a parallel to the quantum-mechanical version of the path integral

3.7.1 Nonrelativistic quantum mechanics

The path integral formalism in quantum mechanics was first introduced in the first half of the twentieth century by the works of Dirac and Feynman.

In extreme synthesis, in order to compute the probability amplitude that a particle is first found in a (one dimensional) position x_1 at a time t_1 and then at a position x_2 at a time t_2 we compute a functional integral in the form

$$\langle x_2, t_2 | x_1, t_1\rangle = \int \mathcal{D}q \exp\left(i\int_q L(x, \dot{x}, t)\,\mathrm{d}t\right) = \int \mathcal{D}q \exp\bigl(iS[q]\bigr)\,, \tag{3.128}$$

where L is the Lagrangian of the particle, $q(t)$ is a possible path it can take from x_1 to x_2, and the integral extends over all of them: the integration measure can be understood as the limit

$$\mathcal{D}q = \lim_{N\to\infty} \prod_{i=1}^{N-1} \mathrm{d}x_i\,, \tag{3.129}$$

where we "slice" the time interval $t_2 - t_1$ into $N \to \infty$ equal parts, $\mathrm{d}x_i$ being the integration elements for the i-th slice.

The integrand is oscillatory: if we are far from the classical path, defined by $\delta S[q_{\mathrm{cl}}] = 0$, the contributions to the integral destructively interfere, while constructive interference happens near the classical path. The fully classical approach is to just consider the equations of motion $\delta S = 0$; we can take a *semiclassical* approach with a saddle-point approximation, by expanding the action up to second order. The paths which will contribute will be, roughly speaking, those for which $S[q] - S[q_{\mathrm{cl}}] \sim 1 = \hbar$.

Qualitatively speaking, this oscillatory version of the integral of $\exp\big(iS[q]\big)$ converges just like the Euclidean integral of $\exp\big(-S[q]\big)$ we have seen up to now; however it does so much more slowly. In both cases we expect the greatest contribution to come from the maxima of $S[q]$.

Because of this, a common technique is the **Wick rotation**, in which we map the time to a new "euclidean time": $t \to -it_E$, which allows us to move to a Euclidean path integral in 4D space with a Cartesian signature.

3.7.2 Quantum Field Theory

Let us look at a relatively simple example application of the path integral in QFT, describing the motion of a massive scalar boson with Lagrangian

$$\mathscr{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2}_{\mathscr{L}_0} + \mathscr{L}_I(\phi)\,, \tag{3.130}$$

where the self-interaction term is some non-quadratic function of ϕ, often taken to be proportional to ϕ^3 or ϕ^4.

The Feynman path integral corresponding to this Lagrangian is given by the functional

$$Z[J] = N\int \mathcal{D}\phi \exp\left(i\int \mathscr{L}(\phi) + J\phi\,\mathrm{d}x\right). \tag{3.131}$$

Let us start with the non-interacting case, that is, we compute Z_0 with only the quadratic term in the Lagrangian. This can be expressed, in the formalism from before, using the kernel

$$K(x,y) = (-\Box_x - \mu^2)\delta(x-y)\,. \tag{3.132}$$

Now, the expression the functional is given in terms of K^{-1}: what is the inverse of this kernel? The definition reduces to

$$\int K(x,y)K^{-1}(y,z)\,\mathrm{d}y = \delta(x-z) \tag{3.133}$$

$$-\Big(\Box_x + \mu^2\Big)K^{-1}(x,z) = \delta(x-z)\,, \tag{3.134}$$

which is readily solved in momentum space, with a $+i\epsilon$ prescription in order to avoid the pole in the integration: what we find is called the *Green's function*,

$$K^{-1}(x,z) = G(x-z) = \frac{1}{(2\pi)^4}\int \frac{e^{-ik\cdot(x-z)}}{k^2+\mu^2-i\epsilon}\,\mathrm{d}k\,, \tag{3.135}$$

so the unperturbed functional reads

$$Z_0[J] = \exp\left(-\frac{i}{2}\int \mathrm{d}x\,\mathrm{d}y\, G(x-y)J(x)J(y)\right). \tag{3.136}$$

This by itself might not seem very useful, the motion of a free massive boson can be calculated with easier methods. However, the real power of this path integral is the possibility to write the interacting term perturbatively: the interaction Lagrangian is a function of ϕ, which is what we find if we perform a functional integration of the argument of the exponential in $Z_0[J]$ with respect to J; so we can express the full functinal as

$$Z[J] = \exp\left(i\int \mathrm{d}x\,\mathscr{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right)\underbrace{\int \mathcal{D}\phi \exp\left(i\int \mathrm{d}x\,\left(\mathscr{L}_0 + J\phi\right)\right)}_{=Z_0[J]} \tag{3.137}$$

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}\left[\int \mathrm{d}x\,\mathscr{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]^n Z_0[J]\,. \tag{3.138}$$

We can use this to compute the Green's functions:

$$G(x_1,\ldots,x_n) = \frac{1}{i^n}\frac{\delta Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0}. \tag{3.139}$$

3.7.3 Diagrammatic approach

The calculation of the path integral in the interacting case is generally intractable; we proceed perturbatively, and a very useful calculation tool is given by Feynman diagrams. We will only discuss their application as a way to better understand the correlation functions we defined earlier.

Green's functions, which correspond to N-point correlation functions in the cosmological case, can be graphically represented by drawing lines between N points.

Each term in the interaction Lagrangian corresponds to a way to join these points: for example, if we had a ϕ^3 term in the interaction Lagrangian (or in the potential, if we are considering self-interaction) it would correspond to a vertex with three lines coming from it. On the other hand, the regular ϕ^2 Gaussian term allows us to connect two points. This allows us to understand the result from 3.3: if we want to connect $2N$ points we can do so with lines in several ways, while if we wish to connect $2N+1$ points at least one will be left out.

Further, what is meant by "connected" correlation functions has a direct graphical interpretation: if we connect $N > 2$ points in pairs the graph is composed of disjoint parts. The only possible connected graph in the Gaussian case is the $N = 2$ one.

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