Theoretical physics notes

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General information

Written & oral exam.

The oral is optional: but is there grade truncation?

The suggested book is D'Auria & Trigiante [DT11]. For the second part it is also useful to have a look at Mandl & Shaw [MS10].

Live question time in Zoom at half past 11 on Mondays.

Things which will be taken for granted: four-vectors, Lorenz and Poincaré groups, basics of QM, basics of linear operators.

Contents

This course will deal with the basics of Relativistic Quantum Field Theory.

We will discuss the Lagrangian formalism for a Classical Field Theory. We will quantize these theories using canonical quantization, specifically for a scalar, a Dirac fermion, and a vector boson.

Then, we will introduce interactions in our Lagrangian: we will use the S-matrix expansion, and Feynman diagrams.

Chapter 1

Relativistic Quantum Field Theory

1.1 The nonrelativistic wave equation

We will review the derivation of the nonrelativistic Schrödinger equation. We find it starting from the correspondence principle: we start from the expression of the energy

$$E = \frac{p^2}{2m} + V(x)\,, \tag{1.1}$$

and substitute the energy with $E \to i\partial_t$, the momentum with $\vec{p} \to -i\vec{\nabla}_x$ and the position with the position operator $\vec{x}$, all acting on the wavefunction. With this we get

$$i\frac{\partial \psi}{\partial t}(\vec{x}, t) = \left(\frac{-\nabla^2}{2m} + V(\vec{x})\right)\psi(\vec{x}, t)\,. \tag{1.2}$$

We still need to assign a meaning to the wavefunction: this is given by the Bohr condition, which tells us that the probability density of finding the particle in a specific region is

$$\rho(\vec{x}, t) = \left|\psi(\vec{x}, t)\right|^2 \geq 0\,. \tag{1.3}$$

This probability density must be normalized as an initial condition:

$$\mathbb{P}(t_0) = \int_{\mathbb{R}^3} \mathrm{d}^3x \, \rho(\vec{x}, t_0) = 1\,, \tag{1.4}$$

and we wish to show that it will also be normalized at later times:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3x \, \frac{\partial}{\partial t}\left|\psi(\vec{x}, t)\right|^2 \tag{1.5}$$

$$= \int_{\mathbb{R}^3} \mathrm{d}^3x \left(\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right). \tag{1.6}$$

Using the Schrödinger equation we can substitute in the expression for the derivative of the wavefunction:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \frac{1}{i} \left(-\frac{\nabla^2}{2m} + V \right) \psi - \frac{1}{i} \psi \left(-\frac{\nabla^2}{2m} + V \right) \psi^* \right\} \tag{1.7}$$

$$= \frac{i}{2m} \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \nabla^2 \psi - 2m\psi^* V \psi - \psi \nabla^2 \psi^* + 2m\psi V \psi^* \right\}, \tag{1.8}$$

and we use the fact that

$$\psi^* V \psi = \psi V \psi^* = \left(\psi^* V \psi \right)^*, \tag{1.9}$$

which is true since V is a symmetric operator: it has real eigenvalues. This allows us to simplify the terms which include V, and we find:

There seem to be some $2m$ factors missing in the formula in the notes.

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \frac{i}{2m} \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right\} \tag{1.10}$$

$$= \frac{i}{2m} \int_{\mathbb{R}^3} \nabla_{\vec{x}} \cdot \left[\psi^* \vec{\nabla} \psi - \psi \left(\vec{\nabla} \psi^* \right) \right], \tag{1.11}$$

where we integrated by parts[1] so we can define

$$\vec{j}(\vec{x}, t) = -\frac{i}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right), \tag{1.14}$$

so that our equation now reads

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = -\int_{\mathbb{R}^3} \mathrm{d}^3x \, \vec{\nabla}_x \cdot \vec{j} = \int_{\partial\mathbb{R}^3} \vec{j} \cdot \hat{n} \, \mathrm{d}^2x = 0, \tag{1.15}$$

[1]The calculation, expressed using index notation (and the Einstein summation convention) for clarity, is as follows:

$$\psi^* \partial_i \partial^i \psi = \partial_i \left(\psi^* \partial^i \psi \right) - (\partial_i \psi^*)(\partial^i \psi) \tag{1.12}$$

and similarly for the other term. The terms which come out as the products of two gradients, $(\partial_i \psi^*)(\partial^i \psi)$, are equal for both the terms, so they simplify. Then, we are left with

$$\psi^* \partial_i \partial^i \psi - \psi \partial_i \partial^i \psi^* = \partial_i \left(\psi^* \partial^i \psi - \psi \partial^i \psi^* \right). \tag{1.13}$$

since the wavefunction is integrable: that is, it goes to zero *quickly* as $|\vec{x}| \to \infty$. Therefore, $\left|\vec{j}\right| \to 0$ as $|\vec{x}| \to \infty$. For a more detailed explanation, see the Quantum Mechanics notes by Manzali [Man19, page 147].

So, if the probability is equal to one at a certain time than it keeps being equal to one.

We can express this as a differential equation for the integrand: the *continuity equation*,

$$\frac{\partial}{\partial t}\left|\psi(\vec{x},t)\right|^2 + \vec{\nabla}\cdot\vec{j} = 0\,. \tag{1.16}$$

Let us now consider the way to solve the free Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = -\frac{\nabla^2\psi}{2m}\,. \tag{1.17}$$

We start from an ansatz of the equation being factorizable: $\psi(\vec{x},t) = \chi(t)\varphi(\vec{x})$. So, we get

$$i\frac{\partial\psi_0}{\partial t} = \varphi(\vec{x})i\frac{\partial\chi}{\partial t} \tag{1.18}$$

on the LHS, and

$$H_0(\psi) = -\chi(t)\frac{\vec{\nabla}^2}{2m}\varphi(\vec{x}) \tag{1.19}$$

on the RHS. Dividing both by $\psi = \chi\varphi$ we get

$$i\frac{1}{\chi}\frac{\partial\chi}{\partial t} = -\frac{1}{\varphi}\frac{\vec{\nabla}^2}{2m}\varphi\,, \tag{1.20}$$

and since these are dependent only on time (for the LHS) and only on position (for the RHS) they must be separately constant: let us call their value E. Therefore, we can integrate them to get

$$\frac{\partial\chi}{\partial t} = -iE\chi \implies \chi(t) = \chi(0)\exp(-iEt) \tag{1.21}$$

and

$$\nabla^2\varphi = -2mE\varphi \implies \varphi(\vec{x}) = \varphi(0)\exp\left(i\vec{k}\cdot\vec{x}\right)\,. \tag{1.22}$$

Here , $\vec{k}$ is a 3D vector such that $\left|\vec{k}\right|^2 = 2mE$.

This is called the *dispersion relation*. So, the full solution, which is called a *monochromatic* solution, is

$$\psi(\vec{x}, t) = \exp\Big(-i\Big(Et - \vec{k}\cdot\vec{x}\Big)\Big)\,, \tag{1.23}$$

where $\left|\vec{k}\right|^2 = 2mE$.

The general solution will be a continuous superposition of solutions of this form:

$$\psi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, \widetilde{\varphi}(\vec{k}) \exp\Big(-i\Big(\omega_k - \vec{k}\cdot\vec{x}\Big)\Big)\Big|_{\omega_k = \frac{\left|\vec{k}\right|^2}{2m}}\,. \tag{1.24}$$

Our conventions for the Fourier transform are:

$$\varphi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, \widetilde{\varphi}(\vec{k}) \exp\Big(-i\vec{k}\cdot\vec{x}\Big) \widetilde{\varphi}(\vec{x}) \quad = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x\, \varphi(\vec{k}) \exp\Big(i\vec{k}\cdot\vec{x}\Big)\,, \tag{1.25}$$

so we use the symmetric definition. Other conventions have factors $(2\pi)^{-3}$ on one side and nothing on the other; it is the same but the we must be consistent.

It is a theorem that $\left|\varphi\right|^2 = \left|\widetilde{\varphi}\right|^2$, where the square norm of φ, $\left|\varphi\right|^2$, is just the integral of $\varphi^*\varphi$ over all 3D space.

The 3D dirac delta function is defined as

$$\delta^3(\vec{x} - \vec{y}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k \exp\Big(-i\vec{k}\cdot(\vec{x} - \vec{y})\Big)\,, \tag{1.26}$$

and the 3D delta in the momentum space is perfectly analogous.

The Schrödinger equation is manifestly *non relativistic*: we started from the nonrelativistic expression $E = p^2/2m + V$, so we should expect so. In the differential equation we have a second spatial derivative and a first temporal derivative: there is no way to write such an equation covariantly.

This kind of law of physics is only invariant under *galilean transformations*, which do not change time.

Saturday 2020-3-14, compiled 2020-03-15

1.2 Conventions

1.2.1 Natural units

Two constants which often come up in theoretical physics are Planck's constant $\hbar = h/2\pi \approx 6.582 \times 10^{-22}\,\mathrm{MeV/Hz}$ and the speed of light $c \approx 2.997 \times 10^8\,\mathrm{m/s}$. They are used to convert quantities which are *equivalent*: a length is equivalent to a time interval if light passes through that length in that time interval in a vacuum;

an energy is equivalent to an angular velocity if a photon with that angular velocity has that energy.

So, we can express lengths in "seconds $\times c$", energies in "kilograms times c^2" or "Hertz times $\hbar$", and so on. Since this is very convenient, we go one step further and do not write the c and the $\hbar$. This allows us to not worry about the number of times these should appear in a formula.

Formally, we do this by imposing the conditions $\hbar = c = 1$, where 1 is adimensional. Then, we can select a common unit and use it for everything: a common choice is the electronVolt (or its multiples).

Some examples of physical quantities in natural units:

1. masses and energies are both measured in eV (for masses, we should multiply by c^2);
2. linear momenta $p = mv$ are measured in eV (times c);
3. angular momenta $L = r \wedge p$ are adimensional (they could be expressed in units of $\hbar$);
4. times and lengths are both measured in eV^{-1}.

This shortens formulas, but it obscures their dimensionality. Thankfully we can always reinsert the necessary cs and $\hbar$s by dimensional analysis.

1.2.2 Relativistic notation

A contravariant vector is denoted by writing its components,

$$v^\mu = \begin{bmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{bmatrix}, \tag{1.27}$$

and examples of these include the position 4-vector $x^\mu = (t, \vec{x})$, and the 4-momentum $p^\mu = (E, \vec{p})$.

We shall use the worse convention for the metric, that is, the mostly negative $(+, -, -, -)$ one. This allows us to obtain covariant vectors as

$$x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x}) \,. \tag{1.28}$$

The derivative operator, instead, is naturally covariant:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \vec{\nabla}) \,. \tag{1.29}$$

1.3 The Klein Gordon equation

We shall use the correspondence principle, as we did for the Schrödinger equation; however this time we will apply it to a relativistic particle. Its 4-momentum has a modulus square equal to the square of its mass: M^2, since it is a relativistic invariant and we may compute it in any reference frame we like. In a generic frame, it is

$$M^2 = p^\mu p_\mu = p^\mu \eta_{\mu\nu} p^\nu = E^2 - \left|\vec{p}\right|^2 , \tag{1.30}$$

which allows us to write the *dispersion relation*

$$E^2 = \vec{p}^2 + M^2 . \tag{1.31}$$

This is *quadratic* in the energy, as opposed to the nonrelativistic expression $E = m + mv^2/2$.

Applying the correspondence principle, we find

$$\begin{aligned} -\partial_t^2 \varphi(\vec{x}, t) &= \left(-\nabla^2 + M^2\right) \varphi(\vec{x}, t) && (1.32) \\ 0 &= \left[\left(\partial_t^2 - \nabla^2\right) + M^2\right] \varphi(\vec{x}, t) && (1.33) \\ &= \left[\Box + M^2\right] \varphi(\vec{x}, t) , && (1.34) \end{aligned}$$

where we defined $\Box = \partial^\mu \partial_\mu = \partial_t^2 - \nabla^2$. Another way to see this is that, compactly stated, the correspondence principle is $p_\mu = -i\partial_\mu$, so from $p^2 = M^2$ we get

$$\left[\Box + M^2\right] \varphi(\vec{x}, t) = 0 . \tag{1.35}$$

This is the *free Klein-Gordon relativistic equation*.

Let us check its covariance *a posteriori*, even though it is guaranteed since we started from a covariant relation. Both $M^2 = p^\mu p_\mu$ and $\Box = \partial^\mu \partial_\mu$ are scalars.

What about the wavefunction φ? It should transform under a Lorentz transformation $x \to x' = \Lambda x$ as

$$\varphi'(x') = \varphi(x) . \tag{1.36}$$

This amounts to saying that it is a *scalar* function. With these constraints, we can say that the Klein-Gordon equation is *covariant*; moreover, since it is a scalar equation it is actually *invariant*. If we apply a Lorentz transformation, we find that indeed

$$\left[\Box' + M'^2\right] \varphi'(x') = \left[\Box + M^2\right] \varphi(x) , \tag{1.37}$$

since $\Box$, M^2 and $\varphi(x)$ are scalars; therefore covariance is proven. If this is zero in an inertial reference frame, it is also zero in any other inertial reference frame.

1.3.1 Continuity equation

Now, let us seek a probability current for the KG equation as we did with the Schrödinger equation. Let us multiply on the left the KG equation by the conjugate of the wavefunction, φ^*. We shall use this equation and its conjugate:

$$\varphi^* \left[\Box + M^2\right] \varphi = 0 \tag{1.38}$$

$$\varphi \left[\Box + M^2\right] \varphi^* = 0\,, \tag{1.39}$$

these both hold since $\Box$ and M^2 are real. If we subtract one of these from the other, the mass terms simplify and we get

$$0 = \varphi^* \Box \varphi - \varphi \Box \varphi^* \tag{1.40}$$

$$= \varphi^* \partial^\mu \partial_\mu \varphi - (\varphi^* \leftrightarrow \varphi)\,, \tag{1.41}$$

where the notation $-(\varphi^* \leftrightarrow \varphi)$ means that we are subtracting the same thing which appears before, but written swapping φ and φ^*. We can expand the derivatives:

$$0 = \partial^\mu \left(\varphi^* \partial_\mu \varphi\right) - \left(\partial^\mu \varphi^*\right) \left(\partial_\mu \varphi\right) - (\varphi^* \leftrightarrow \varphi) \tag{1.42}$$

$$= \partial^\mu \left(\varphi^* \partial_\mu \varphi\right) - (\varphi^* \leftrightarrow \varphi) \tag{1.43}$$

$$= \partial^\mu \left(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*\right) \stackrel{\text{def}}{=} -2i \partial^\mu j_\mu\,, \tag{1.44}$$

The product of the gradients is symmetric under interchange of φ and φ^*, since the metric is constant.

where we defined the 4-current

$$j_\mu = \frac{i}{2} \varphi^* \partial_\mu \varphi - (\varphi^* \leftrightarrow \varphi)\,, \tag{1.45}$$

so we can write the conservation equation $\partial^\mu j_\mu = \partial_\mu j^\mu = 0$ in 3-vector form as:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j}, \tag{1.46}$$

where

$$\rho = \frac{i}{2} \varphi^* \partial_t \varphi - (\varphi^* \leftrightarrow \varphi) \qquad \text{and} \qquad \vec{j} = -\frac{i}{2} \varphi^* \partial^i \varphi - (\varphi^* \leftrightarrow \varphi)\,. \tag{1.47}$$

Minus sign since we want the contravariant components.

We can integrate the continuity equation over all 3D space to obtain a conserved quantity:

$$\int_{\mathbb{R}^3} \mathrm{d}^3 x \, \rho \stackrel{\text{def}}{=} Q\,, \tag{1.48}$$

which is actually constant since

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3 x \, \partial_t \rho = -\int_{\mathbb{R}^3} \mathrm{d}^3 x \, \vec{\nabla} \cdot \vec{j} = -\int_{S^2_\infty} \vec{j} \cdot \hat{n} \, \mathrm{d}A \to 0\,. \tag{1.49}$$

Used the divergence theorem, the surface S^2_∞ is a sphere with diverging radius.

We call this conserved quantity a “charge”. There is an issue: in the Schrödinger case the quantity called ρ was positive by definition; now instead ρ and Q are not necessarily positive.

This can be proven as follows: suppose ρ was positive for a certain wavefunction φ. The conjugate wavefunction φ^* is also a solution to the KG equation, and for it the density will be negative, since permuting φ and φ^* is equivalent to changing the sign of j_μ.

So, we cannot use the Bohr ansatz, interpreting Q as a probability.

1.3.2 Solutions to the free KG equation

Let us forget about the physical interpretation for a while, and discuss the general solutions of the KG equation. We can decompose the wavefunction $\varphi(x)$ in terms of its Fourier transform $\widetilde{\varphi}(k)$:

$$\varphi(x) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4k \, e^{-ik^\mu x_\mu} \widetilde{\varphi}(k) \,. \tag{1.50}$$

In order for this to be covariant, if φ is a scalar then $\widetilde{\varphi}$ also must be. The argument of the exponential is also a scalar.

The volume form in the momentum space d^4k is a scalar: under a Lorentz transformation it transforms as

$$\mathrm{d}^4k' = |\det \Lambda| \, \mathrm{d}^4k \,, \tag{1.51}$$

so it does not change since $|\det \Lambda| = 1$ for Lorentz transformations.

Claim 1.3.1. *The inverse Fourier transform reads*

$$\widetilde{\varphi}(k) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4x \, e^{ik^\mu x_\mu} \varphi(x) \,. \tag{1.52}$$

Proof. We take the transform of the antitransform:

$$\begin{aligned}
\widetilde{\varphi}(k) &= \frac{1}{(2\pi)^2} \int \mathrm{d}^4x \, \varphi(x) e^{ik^\mu x_\mu} && (1.53)\\
&= \frac{1}{(2\pi)^4} \int \mathrm{d}^4x \, e^{ik^\mu x_\mu} \int \mathrm{d}^4k' \, e^{-ik'^\mu x_\mu} \varphi(k') && (1.54)\\
&= \int \mathrm{d}^4k' \left[\frac{1}{(2\pi)^4} \int \mathrm{d}^4x \, e^{-i(k'^\mu - k^\mu)x_\mu} \right] \varphi(k') && (1.55)\\
&= \int \mathrm{d}^4k' \, \delta^{(4)}(k-k') \varphi(k') = \varphi(k) \,, && (1.56)
\end{aligned}$$

where we used the definition of the 4D Dirac δ function (here in position space, the definition in momentum space is perfectly analogous):

$$\delta^{(4)}(x^\mu) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k\, e^{-ik_\mu x^\mu}\,, \tag{1.57}$$

and its main property:

$$\int \mathrm{d}^4x\, \delta^{(4)}(x) f(x) = f(0)\,. \tag{1.58}$$

$\square$

Now, to solve the KG equation we insert the Fourier expression of the wavefunction into it:

$$0 = \left[\Box + M^2\right] \varphi(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^2} \left[-k^2 + M^2\right] e^{-ik^\mu x_\mu} \widetilde{\varphi}(k)\,, \tag{1.59}$$

The $\Box$ operator acts in position space, so it has no effect on $\widetilde{\varphi}(k)$: it applies only to the exponential, yielding $-k^2 = (-ik^\mu)(-ik_\mu)$.

and since the integral must be zero the integrand must be zero as well.

What? this is not true in general!

We can use the ansatz $\widetilde{\varphi}(k) = \delta(k^2 - M^2)\widetilde{f}(k)$, where $\widetilde{f}(k)$ is a generic scalar function. If $\widetilde{\varphi}(k)$ is written in this way, it automatically satisfies the KG equation.

Now, recall that for a Dirac delta function applied to a generic function $f(x)$, whose zeroes are enumerated by the index i (that is, $f(x_i) = 0$ for all i between 1 and N) the following property holds:

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{\delta(x - x_i)}{\left|f'(x_i)\right|}\,. \tag{1.60}$$

We can apply this property to $\delta(k^2 - M^2)$. First of all, since $k^2 = k_0^2 - \left|\vec{k}\right|^2$, we can write this expression as $\delta(k_0^2 - \omega_k^2)$, by defining ω_k as the positive root:

$$\omega_k = +\sqrt{|k|^2 + M^2}\,. \tag{1.61}$$

Now, we apply the δ function property:

$$\delta(k^2 - M^2) = \delta(k_0^2 - \omega_k^2) = \frac{\delta(k_0 - \omega_k)}{|2k_0|} + \frac{\delta(k_0 + \omega_k)}{|2k_0|} \tag{1.62}$$

$$= \frac{\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)}{2\omega_k}\,, \tag{1.63}$$

where we substituted $|k_0| = \omega_k$, which holds both if $k_0 = \omega_k$ and if $k_0 = -\omega_k$. This finally gives us

$$\widetilde{\varphi}(k) = \frac{1}{2\omega_k}\big(\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)\big)\widetilde{f}(k)\,, \tag{1.64}$$

which we can insert this into the Fourier transform of $\varphi(x)$:

$$\begin{aligned}
\varphi(x) &= \frac{1}{(2\pi)^2}\int \mathrm{d}^4k\,\widetilde{\varphi}(k)e^{-ik^\mu x_\mu} &(1.65)\\
&= \frac{1}{(2\pi)^2}\int \frac{\mathrm{d}^4k}{2\omega_k}\big(\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)\big)e^{-ik^\mu x_\mu}\widetilde{f}(k) &(1.66)\\
&= \frac{1}{(2\pi)^2}\int \frac{\mathrm{d}^3k}{2\omega_k}\left[e^{-i\omega_k x_0}e^{i\vec{k}\cdot\vec{x}}\widetilde{f}(\omega_k,\vec{k}) + e^{i\omega_k x_0}e^{i\vec{k}\cdot\vec{x}}\widetilde{f}(-\omega_k,\vec{k})\right] &(1.67)\\
&= \frac{1}{(2\pi)^2}\int \frac{\mathrm{d}^3k}{2\omega_k}\left[e^{-ik^\mu x_\mu}\widetilde{f}(k^\mu) + e^{ik^\mu x_\mu}\widetilde{f}(-k^\mu)\right]_{k_0=\omega_k}, &(1.68)
\end{aligned}$$

We integrate in $\mathrm{d}k^0$ to get rid of the δ functions.

where we indicate $k^\mu\big|_{k_0=\omega_k} = (\omega_k, \vec{k})$.

We used the fact that, in the Fourier transform integral, the terms $e^{i\vec{k}\cdot\vec{x}}\widetilde{f}(\vec{x})$ and $e^{-i\vec{k}\cdot\vec{x}}\widetilde{f}(-\vec{x})$ are equivalent: this is because, since we are integrating over all of 3D space, any contributions which are *odd* in $\vec{k}$ will not affect the total integral, therefore we can only consider the even part of the integrand.

Now, in order to simplify the notation we define

$$a(k) = \frac{\widetilde{f}(k)}{\sqrt{2\pi}\sqrt{2\omega_k}} \qquad \text{and} \qquad b(k) = \frac{\widetilde{f}(-k)}{\sqrt{2\pi}\sqrt{2\omega_k}}\,, \tag{1.69}$$

which are arbitrary like the initial function $\widetilde{f}$, however they are connected since $a(k) = b^*(-k)$. So, the final solution reads:

$$\begin{aligned}
\varphi(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\left[a(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\right]_{k_0=\omega_k} &(1.70)\\
&= \varphi_+(x) + \varphi_-(x)\,. &(1.71)
\end{aligned}$$

Recall that all ks appearing in the expression are to be interpreted as $(\omega_k, \vec{k})$. The part dependent on $a(k)$ is conventionally called the *positive energy* solution φ_+, while the part depending on $b^*(k)$ is the *negative energy* solution φ_-.

This is because, as the energy of a wavefunction φ is computed by $E = i\partial_t\varphi$, we have

$$E(\varphi_+) = i\partial_0\left\{\frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}a(k)e^{-ik^\mu x_\mu}\right\} = i(-i)\omega_k\varphi_+ = \omega_k\varphi_+ \tag{1.72}$$

$$E(\varphi_-) = i\partial_0\left\{\frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}b^*(k)e^{ik^\mu x_\mu}\right\} = i(i)\omega_k\varphi_- = -\omega_k\varphi_-\,, \tag{1.73}$$

so φ_+ has a positive energy while φ_- has a negative one.

This is the main difference between the Schrödinger and KG equations.

The solution to the KG equation is not explicitly covariant, but all the steps preserved covariance so the final solution is still covariant.

The KG equation is real, since $\Box$ is a real operator and M^2 is real, so it will admit real solutions. In order to find these we impose $\varphi = \varphi^*$.

Claim 1.3.2. *$\varphi = \varphi^*$ implies $a = b$.*

Proof. We write only the argument of the integrals for simplicity:

$$\varphi \sim ae^{-ikx} + b^*e^{ikx} \tag{1.74}$$

$$\varphi^* \sim a^*e^{ikx} + be^{-ikx}\,, \tag{1.75}$$

so if $\varphi = \varphi^*$ we must identify these component by component, so we must have $a = b$, and $a^* = b^*$. ☐

Then, the most general real solution to the KG equation is

$$\varphi_{\mathbb{R}}(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\left[a(k)e^{-ik\cdot x} + a^*(k)e^{ik\cdot x}\right]_{k_0=\omega_k}. \tag{1.76}$$

Claim 1.3.3. *For a real KG solution, the function $a(k)$ can be written as*

$$a(k) = \frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3x}{\sqrt{2\omega_k}}\left(\omega_k\varphi(x) + i\partial_0\varphi(x)\right)\left.e^{ik\cdot x}\right|_{k_0=\omega_k}. \tag{1.77}$$

Proof. The solution is found by direct substitution of $\varphi_{\mathbb{R}}$ into the expression for a in order to verify it; the operations are all reversible so we can use the derivation backwards or forwards equivalently. We find

$$a \stackrel{?}{=} \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k\,\mathrm{d}^3x}{2\omega_k}\left[\omega_k\left(ae^{-ik\cdot x} + a^*e^{ik\cdot x}\right) + i\partial_0\left(ae^{-ik\cdot x} + a^*e^{ik\cdot x}\right)\right]\left.e^{ik\cdot x}\right|_{k_0=\omega_k} \tag{1.78}$$

$$= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k\,\mathrm{d}^3x}{2\omega_k}\left[\left(\omega_k + i(-ik_0)\right)a^{-ik\cdot x} + \left(\omega_k + i(ik_0)\right)a^*e^{ik\cdot x}\right]\left.e^{ik\cdot x}\right|_{k_0=\omega_k} \tag{1.79}$$

$$= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k\,\mathrm{d}^3x}{2\omega_k}\left[2\omega_k a^{-ik\cdot x}\right]\left.e^{ik\cdot x}\right|_{k_0=\omega_k} \tag{1.80}$$

Used the fact that $\omega_k = k_0$, and $i^2 = -1$.

$$= \frac{1}{(2\pi)^{3/2}}\int\mathrm{d}^3x\,e^{i\vec{k}\cdot\vec{x}}\left[\frac{1}{(2\pi)^{3/2}}\int\mathrm{d}^3k\,e^{-i\vec{k}\cdot\vec{x}}a\right] = a\,. \tag{1.81}$$

3D Fourier inverse and direct transform. We simplified two factors of $e^{ik_0x^0}$ and $e^{-ik_0x^0}$ since they are equal everywhere ($k_0 = \omega_k$).

☐

Claim 1.3.4. *For a complex KG solution, the functions $a(k)$ and $b^*(k)$ can be written as*

$$a(k) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3x}{\sqrt{2\omega_k}} \big(\omega_k \varphi(x) + i\partial_0 \varphi(x)\big)\, e^{ik\cdot x}\Big|_{k_0=\omega_k} \tag{1.82}$$

$$b^*(k) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3x}{\sqrt{2\omega_k}} \big(\omega_k \varphi(x) - i\partial_0 \varphi(x)\big)\, e^{-ik\cdot x}\Big|_{k_0=\omega_k} . \tag{1.83}$$

Proof. The derivation is the same as the real-valued solution case. The b^* terms simplify if there is a plus in front of the $i\partial_0$ term, if instead we have a minus the a terms simplify; everything else is precisely the same. □

Claim 1.3.5. *Given two real solutions to the KG equation, φ_1 and φ_2, one can always write a complex solution $\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$.*

Then, the functions a and b^* for the complex solution can be written in terms of the a_1 and a_2 for the real solution as:

$$a = \frac{a_1 + ia_2}{\sqrt{2}} \tag{1.84}$$

$$b = \frac{a_1 - ia_2}{\sqrt{2}} . \tag{1.85}$$

Proof. We write out the complex function:

$$\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \left[\frac{a_1 + ia_2}{\sqrt{2}} e^{-ik\cdot x} + \frac{a_1^* + ia_2^*}{\sqrt{2}} e^{ik\cdot x} \right] , \tag{1.86}$$

so, since $b^* = \left(a_1^* + ia_2^*\right)/2$, we can get b by conjugating,

$$b = \frac{a_1 - ia_2}{\sqrt{2}} , \tag{1.87}$$

while a can be directly read off the expression. □

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