Radiative processes in astrophysics notes

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Contents

Introduction

The professor, Roberto Turolla, will follow the pdf of the book by Rybicki and Lightman [RL79] on his screen. It is available for free.

Understanding radiative processes is fundamental for the analysis of several phenomena: for example, in the Crab nebula the main process is Synchrotron radiation, in the Coma cluster we have Bremsstrahlung, in Cygnus-X1 we have Compton scattering.

Even in the era of multimessenger astrophysics, most of the information we receive still comes from electromagnetic radiation. The required background for this course is classical EM, special relativity and the basics of atomic structure.

The exam is an oral one. The lectures will be recorded and put on the Moodle until the emergency ends, every Wednesday and Thursday. The duration of recorded lectures will be shorter than the duration of the lectures we would have in the classroom.

Chapter 1

Fundamentals of radiative transfer

1.1 Basic properties of the EM spectrum

Thursday 2020-3-12, compiled 2020-08-19

Electromagnetic radiation can be decomposed into a spectrum according to either the frequency ν or the wavelength λ; these are connected by $c = \lambda\nu$, where c is the speed of light.

Sometimes we give the energy of the photons, which can be found using Planck's constant h: $E = h\nu$.

We conventionally divide the spectrum into bands: γ-rays, X-rays, ultraviolet light, visible light, infrared radiation, radio band.

1.1.1 The radiative flux

Let us consider an area element $\mathrm{d}A$, through which radiation passes for a time $\mathrm{d}t$: the energy will be proportional to both $\mathrm{d}A$ and $\mathrm{d}t$, so we define the radiative flux F by saying that the energy is equal to $F\,\mathrm{d}A\,\mathrm{d}t$. Of course, we need to account for orientation: if the surface is not perpendicular to the source the energy is less.

Let us consider a pointlike source, and draw two spherical surfaces of radii r_1 and r_2 along which we compute the flux: if there is no energy loss we must have

$$F(r_1)A_1\,\mathrm{d}t = F(r_2)A_2\,\mathrm{d}t \tag{1.1.1a}$$

$$F(r_1)4\pi r_1^2 = F(r_2)4\pi r_2^2 \tag{1.1.1b}$$

$$F(r_1) = F(r_2)\frac{r_2^2}{r_1^2}\,. \tag{1.1.1c}$$

This tells us that **the flux emitted from a pointlike source decreases like r^{-2}**.

The flux of energy is a measure of all the energy which passes through the surface; however we can get a more detailed description. We cannot consider photons at a specific frequency: the set has measure 0, we must use the language of probability densities. We look at a "pencil" of radiation: all the radiation coming from a solid angle $\mathrm{d}\Omega$ over an area $\mathrm{d}A$ and carried by photons of frequencies between ν and $\nu + \mathrm{d}\nu$.

Note that the solid angle $\mathrm{d}\Omega$ depends on both angles we use to describe a direction (typically θ, φ).

So, we define the *specific intensity of brightness* I_ν by

$$\mathrm{d}E \stackrel{\text{def}}{=} I_\nu \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}\nu \ . \tag{1.1.2}$$

This will depend on position (where we put the detector area with respect to any sources) and on direction (where we look).

We usually neglect the time-dependence. The units of this quantity are those of energy per unit time, area, frequency, solid angle.

How do we account for the direction? The differential flux for radiation coming with an angle θ to the normal is proportional to the dot product of the normal and the incidence direction:

$$\mathrm{d}F_\nu = I_\nu \cos(\theta) \, \mathrm{d}\Omega \ , \tag{1.1.3}$$

so the total net flux is

$$F_\nu = \int I_\nu \cos(\theta) \, \mathrm{d}\Omega \ . \tag{1.1.4}$$

This is about energy, but we can define the momentum flux per unit time per unit area (which is the pressure) with the same procedure; we get an additional factor of $\cos\theta$ since $\vec{p}$ is a vector, and we are interested in its component along the normal of the surface. So, the global formula for this pressure is

$$P_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \, \mathrm{d}\Omega \ . \tag{1.1.5}$$

These are *moments*: in general, a moment for a direction-dependent quantity like the intensity is something in the form

$$n\text{-th moment} = \int I_\nu \cos^n\theta \, \mathrm{d}\Omega \ . \tag{1.1.6}$$

These are frequency dependent; the corresponding *grey* (that is, frequency-integrated) quantities are in the form

$$F = \int F_\nu \, \mathrm{d}\nu \ . \tag{1.1.7}$$

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1.2 Radiative energy density

We define the energy density per unit solid angle, $u_\nu(\Omega)$ by: $\mathrm{d}E = u_\nu(\Omega) \, \mathrm{d}V \, \mathrm{d}\Omega \, \mathrm{d}\nu$. This is the differential amount of energy in the volume $\mathrm{d}V$, carried by radiation coming from the solid angle $\mathrm{d}\Omega$ which has energies between ν and $\nu + \mathrm{d}\nu$.

We consider a cylinder for our volume, its axis being aligned with the direction the radiation is coming from. Its volume can be expressed as $\mathrm{d}V = \mathrm{d}A \, c \, \mathrm{d}t$, where $\mathrm{d}t$ is the time taken by light to cross the height of the cylinder.

We can also express the differential energy using the definition of the specific intensity: then, we can compare the following two equations:

$$\mathrm{d}E = u_\nu(\Omega) c \,\mathrm{d}A \,\mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}\nu \tag{1.2.1a}$$

$$= I_\nu \,\mathrm{d}A \,\mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}\nu \,, \tag{1.2.1b}$$

which allows us to conclude that $u_\nu = I_\nu / c$. Also, if we want to get the total energy density we just need to integrate over the volume of the whole sphere:

$$u_\nu = \frac{1}{c} \int I_\nu(\Omega) \,\mathrm{d}\Omega \overset{\text{def}}{=} \frac{4\pi}{c} J_\nu \,, \tag{1.2.2}$$

where J_ν is the *mean intensity*: $J_\nu = \langle I_\nu \rangle_\Omega$.

We can also integrate over frequencies to get the total energy density:

$$u = \int u_\nu \,\mathrm{d}\nu = \frac{4\pi}{c} \int J_\nu \,\mathrm{d}\nu \,. \tag{1.2.3}$$

1.2.1 Isotropic radiation field

An isotropic radiation field is one for which the specific intensity does not depend on angles. Let us start from the definitions of u_ν and P_ν:

$$u_\nu = \int \frac{I_\nu}{c} \,\mathrm{d}\Omega = \frac{4\pi J_\nu}{c} \tag{1.2.4a}$$

$$P_\nu = \int \frac{I_\nu}{c} \cos^2\theta \,\mathrm{d}\Omega = \int \frac{I_\nu}{c} \cos^2\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi \,, \tag{1.2.4b}$$

and let us use the assumption that I_ν does not depend on Ω: so we can bring it out of the integrals, to find

$$u_\nu = 4\pi \frac{I_\nu}{c} \tag{1.2.5a}$$

$$P_\nu = -2\pi \frac{I_\nu}{c} \int \cos^2\theta \,\mathrm{d}\cos\theta = 2\pi \frac{I_\nu}{c} \int_{-1}^{1} x^2 \,\mathrm{d}x \tag{1.2.5b}$$

The differential is negative, but we swap the integration bounds.

$$= \frac{4\pi}{3} \frac{I_\nu}{c} \,, \tag{1.2.5c}$$

which gives us the result we sought:

$$P_\nu = \frac{u_\nu}{3} \,. \tag{1.2.6}$$

1.2.2 Specific intensity along a ray

We wish to see how the specific intensity I_ν changes along a beam of light rays. Let us consider two positions $1, 2$ along the beam, separated by a distance R. Then, by definition we will have, for $i = 1, 2$:

$$\mathrm{d}E_i = I_{\nu,i} \,\mathrm{d}A_i \,\mathrm{d}t_i \,\mathrm{d}\Omega_i \,\mathrm{d}\nu_i \,. \tag{1.2.7}$$

First, we make the assumption of the gravitational field being weak: therefore time dilation is negligible, so $\mathrm{d}t_1 = \mathrm{d}t_2$ and $\mathrm{d}\nu_1 = \mathrm{d}\nu_2$. Now, we ask these two expressions to describe the same beam: the same photons will pass through $\mathrm{d}A_1$ and $\mathrm{d}A_2$. Therefore, by conservation of energy, $\mathrm{d}E_1 = \mathrm{d}E_2$.

This means that

$$I_{\nu,1}\,\mathrm{d}A_1\,\mathrm{d}\Omega_1 = I_{\nu,2}\,\mathrm{d}A_2\,\mathrm{d}\Omega_2\ . \tag{1.2.8}$$

We can treat the photons' motion as time-reversal symmetric: so, whether they pass through $\mathrm{d}A_1$ or $\mathrm{d}A_2$ first is irrelevant. The linear scale of the differential area element is infinitesimal, while the separation between the two points, R, is macroscopic: so, we can consider all the photons which come through $\mathrm{d}A_1$ to be coming from a point source from position 2 and vice versa. This will define the angular size as seen from the "pointlike" position 2, $\mathrm{d}\Omega_2$.

Therefore, the differential solid angle will look like

$$\mathrm{d}\Omega_2 = \frac{\mathrm{d}A_1}{R^2}\,, \tag{1.2.9}$$

and we can apply the same reasoning reversing the photons' motion to find the same, alternate relation with $(1 \leftrightarrow 2)$. We can use this to write

$$I_{\nu,1}\frac{\mathrm{d}A_1}{\mathrm{d}\Omega_2} = I_{\nu,2}\frac{\mathrm{d}A_2}{\mathrm{d}\Omega_1} \tag{1.2.10a}$$

$$I_{\nu,1}R^2 = I_{\nu,2}R^2 \tag{1.2.10b}$$

$$I_{\nu,1} = I_{\nu,2}\,. \tag{1.2.10c}$$

This means that, under our assumptions, the specific intensity is conserved:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0\,, \tag{1.2.11}$$

where s is a parameter describing the light ray's trajectory. This is useful since, if the variation of the specific intensity is zero in a vacuum, then its variation in the presence of matter will only be due to transfer phenomena, and the sign of the variation will describe whether energy is being added or removed.

1.3 Radiative transfer

In general, as radiation passes through matter, its specific intensity changes. This is due to emission and absorption, but also to scattering, which preserves the total number of photons: even in the low-energy limit it can change the angular distribution of the radiation, and in general it also changes the energy of the photon.

1.3.1 Emission

Emission is a process through which photons are created. We can define the grey emission coefficient j and the monochromatic emission coefficient j_ν as:

$$\mathrm{d}E = j\,\mathrm{d}V\,\mathrm{d}\Omega\,\mathrm{d}t \tag{1.3.1a}$$

$$\mathrm{d}E = j_\nu\,\mathrm{d}V\,\mathrm{d}\Omega\,\mathrm{d}t\,\mathrm{d}\nu\ , \tag{1.3.1b}$$

they quantify the energy added to the radiation field per unit volume, solid angle (in order to account for the direction of emission) and unit time. For the monochromatic coefficient, we restrict ourselves to radiation emitted in the range from ν to $\nu + \mathrm{d}\nu$.

In the case of an isotropic emission we can integrate over the solid angle to find

$$P_\nu = 4\pi j_\nu\,, \tag{1.3.2}$$

the radiated power per unit volume and frequency.

Another useful concept is the emissivity ϵ_ν: it is the energy added to the radiation field per unit time, frequency and mass in the directions described by the solid angle $\mathrm{d}\Omega$. We express the infinitesimal mass as $\mathrm{d}m = \rho\,\mathrm{d}V$, so that in the case of isotropic emission we have

$$\mathrm{d}E = \epsilon_\nu \rho\,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu\,\frac{\mathrm{d}\Omega}{4\pi}\,, \tag{1.3.3}$$

so the emissivity ϵ_ν and the emission coefficient j_ν are connected by

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}\,. \tag{1.3.4}$$

We wish to describe the variation in specific intensity due to this emission. Let us consider a beam of cross section $\mathrm{d}A$ going through a length $\mathrm{d}s$, so that the volume it occupies is $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}s$.

Now, if we compare the definitions of j_ν and I_ν we find that they differ by a factor $\mathrm{d}V\,/\,\mathrm{d}A = \mathrm{d}s$, the length of the beam cylinder we defined.

The difference between the specific intensities at the start and end of the cylinder would be zero without emission, now instead their difference can be calculated from the energy added; as we said most of the differentials simplify and we get that the variation of specific intensity is

$$\mathrm{d}I_\nu = j_\nu\,\mathrm{d}s\ . \tag{1.3.5}$$

1.3.2 Absorption

Absorption is described by a coefficient $\alpha_\nu > 0$, which is dimensionally an inverse length. The absorption law which defines the coefficient gives the decrease in radiative intensity for radiation of intensity I_ν crossing an absorbing medium of length $\mathrm{d}s$:

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu\,\mathrm{d}s\ . \tag{1.3.6}$$

Why should the variation in intensity be proportional to the intensity itself? We give a simple argument: let us assume that absorption is due to randomly absorbers with number density n and (frequency dependent) cross section σ_ν.

Let us consider our usual cylinder with cross sectional area $\mathrm{d}A$ and length $\mathrm{d}s$: the number of absorber in it will be $\mathrm{d}N = n\,\mathrm{d}A\,\mathrm{d}s$. The total effective cross section area presented for absorption will be $\sigma_\nu\,\mathrm{d}N$. The energy contained in photons in this cross sectional area will be lost: the energy lost $-\,\mathrm{d}I_\nu$ can be calculated as

$$-\,\mathrm{d}I_\nu\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu = I_\nu(\sigma_\nu n\,\mathrm{d}A\,\mathrm{d}s)\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \tag{1.3.7a}$$

$$\mathrm{d}I_\nu = -n\sigma_\nu I_\nu\,\mathrm{d}s\ , \tag{1.3.7b}$$

which is the relation written above, with $n\sigma_\nu = \alpha_\nu$. The number density is proportional to the mass density: $n\overline{m} = \rho$ where $\overline{m}$ is the average mass of a particle. Therefore, we can express α_ν as

$$\alpha_\nu = \rho\kappa_\nu = n\overline{m}\frac{\sigma_\nu}{\overline{m}}\,, \tag{1.3.8}$$

so we can see that κ_ν is a cross sectional area per unit mass. It is called the *mass absorption coefficient* or the *opacity*.

Conditions for validity This line of reasoning holds as long as the inter-absorber distances $d \sim n^{-1/3}$ are large compared to the linear scale of the cross section $\sigma_\nu^{1/2}$: we ask

$$\sigma_\nu^{1/2} \ll n^{-1/3}\,, \tag{1.3.9}$$

and also we must assume that the absorbers are independent and randomly distributed (at least locally). These assumptions are usually met in astrophysical systems.

1.3.3 The radiative transfer equation

We can account for both absorption and emission in a combined equation for the derivative with respect to the beam length travelled s of the specific intensity I_ν:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu\,, \tag{1.3.10}$$

and we can see that in the absence of emission and absorption I_ν is unchanged, as we have shown before. If j_ν and α_ν are known we can integrate this differential equation to find the specific intensity.

This will *not* be the case when we will include scattering: the scattering term will not just depend on I_ν but in its integral on the sphere, making this an integro-differential equation.

Solutions to the transfer equation in simple cases

If there is only emission, that is, only j_ν is nonzero, the intensity increases (linearly in s for constant j_ν):

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu \implies I_\nu = I_\nu(0) + \int_{s_0}^{s} j_\nu(\widetilde{s})\,\mathrm{d}\widetilde{s}\,. \tag{1.3.11}$$

If there is only absorption, that is, only α_ν is nonzero, then the intensity decreases (exponentially in s for constant j_ν):

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu \implies I_\nu = I_\nu(s_0)\exp\left(-\int_{s_0}^{s}\alpha_\nu(\widetilde{s})\,\mathrm{d}\widetilde{s}\right). \tag{1.3.12}$$

1.3.4 Optical depth and source function

The optical depth τ_ν is defined so that it increases by 1 when the intensity of light at frequency ν decreases e-fold: its differential is

$$\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}s\,, \tag{1.3.13}$$

so that the solution in the absorption-only case reads $I_\nu \propto e^{-\int \mathrm{d}\tau} = e^{-\tau}$.

So, a useful distinction to make is based on the magnitude of τ, since it quantifies how much light can shine through a medium:

1. if $\tau \gg 1$ the medium is said to be *opaque* or *optically thick*;
2. if $\tau \ll 1$ the medium is said to be *transparent* or *optically thin*;
3. if $\tau \approx 1$ the medium is said to be *translucent*.

If we define the source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu}\,, \tag{1.3.14}$$

we can write the radiative transfer equation as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu\,. \tag{1.3.15}$$

Divided through by α_ν, used definition of τ_ν.

1.3.5 A formal solution of the radiative transfer equation

We can solve this equation by defining $Y_\nu = I_\nu e^{\tau_\nu}$, which obeys

$$\frac{\mathrm{d}Y_\nu}{\mathrm{d}\tau_\nu} = \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}e^{\tau_\nu} + I_\nu e^{\tau_\nu}\,, \tag{1.3.16}$$

so we can multiply the radiative transport equation by e^{τ_ν} to get

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}e^{\tau_\nu} = -I_\nu e^{\tau_\nu} + S_\nu e^{\tau_\nu} \tag{1.3.17a}$$

$$\frac{\mathrm{d}Y_\nu}{\mathrm{d}\tau_\nu} = S_\nu e^{\tau_\nu} \tag{1.3.17b}$$

$$Y_\nu(\tau_\nu) = Y_\nu(0) + \int_0^{\tau_\nu} S_\nu(\widetilde{\tau}_\nu)e^{\widetilde{\tau}_\nu}\,\mathrm{d}\widetilde{\tau}_\nu \tag{1.3.17c}$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\widetilde{\tau}_\nu)e^{\widetilde{\tau}_\nu-\tau_\nu}\,\mathrm{d}\widetilde{\tau}_\nu\,, \tag{1.3.17d}$$

Divided through by e^{τ_ν}.

which has a direct intuitive meaning: the intensity at a certain point must be computed accounting for the initial one and emission all through the beam before the point we are considering, and each of these contributions to the emission is weighted by an exponential factor: the relevance of a term decreases if the optical distance increases.

If S_ν is a constant, we have the simplified expression

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right), \tag{1.3.18}$$

and we can see that for large optical depths the intensity is dictated purely by the source at that point, since $1 - e^{-\tau_\nu}$ approaches 0 for large τ_ν.

1.3.6 Thermal radiation

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Consider a blackbody enclosure at equilibrium at a certain temperature T. If we open a small hole in the enclosure and measure the radiation inside it, the intensity I_ν we will see will be isotropic, and only depending on the temperature T: let us call the (still unspecified) way this dependence look $I_\nu = B_\nu(T)$. The function $B_\nu(T)$ will be called the **blackbody function**.

Now, imagine we put a small chunk of material inside the blackbody enclosure. If this material is characterized by a source function $S_\nu = j_\nu/\alpha_\nu$, then the evolution of the intensity inside the cavity will be

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu\,. \tag{1.3.19}$$

If we wait for equilibrium, though, we must have $I_\nu \equiv B_\nu$, meaning that its derivative must be zero: this implies that $I_\nu = B_\nu = S_\nu$.

This is called **Kirkhoff theorem**: a chunk of material which can emit and absorb and which is placed in a blackbody enclosure then it must satisfy

$$B_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu\,. \tag{1.3.20}$$

Planck function derivation

What is the explicit expression of the blackbody function B_ν? From Bose-Einstein statistics we know that the phase space density of photons in thermal equilibrium is given by

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p} = \frac{2}{h^3}\frac{1}{\exp\left(\frac{h\nu}{k_BT}\right) - 1}, \tag{1.3.21}$$

where the 2 comes from the two polarizations of the photons, h^3 is the size of the phase space cell, while the exponential factor is the occupation number.

Now, we would like to write this as a function of frequency, not of the three components of the momentum. The frequency ν and the momentum $p = \left|\vec{p}\right|$ of a photon are connected by $h\nu = E = cp$; and the momentum space element d^3p can be written as $\mathrm{d}^3p = \mathrm{d}\Omega\, p^2\,\mathrm{d}p$.

Bringing this to the other side and integrating over the solid angle (since the frequency of the photon does not depend on its direction we can substitute $\int \mathrm{d}\Omega = 4\pi$) we can write

$$\frac{\mathrm{d}N}{\mathrm{d}^3x} = \frac{8\pi}{c^3}\frac{\nu^2\,\mathrm{d}\nu}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,, \tag{1.3.22}$$

the **number density** of photons. From this we can quickly recover the **energy density** as well, since each photon with frequency ν has energy $h\nu$: therefore

$$\frac{\mathrm{d}E}{\mathrm{d}^3x} = \frac{8\pi}{c^3}\frac{h\nu^3\,\mathrm{d}\nu}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,. \tag{1.3.23}$$

If we want to recover the energy density per unit frequency and per unit solid angle we can divide by $\mathrm{d}\Omega\,\mathrm{d}\nu$: this yields

$$u_\nu(\Omega) = \frac{\mathrm{d}E}{\mathrm{d}^3x\,\mathrm{d}\Omega\,\mathrm{d}\nu} = \frac{2h}{c^3}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,, \tag{1.3.24}$$

which we have previously shown to be equal to $u_\nu(\Omega) = I_\nu/c$: therefore the specific intensity, which corresponds to the Planck function $B_\nu(T)$, is

$$I_\nu = B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,. \tag{1.3.25}$$

Properties of the Planck function

Let us consider the limits of the Planck function. If the energy of the photon, $h\nu$, is much lower than the average photon energies k_BT then we have

$$B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_BT}\right)-1} \approx \frac{2h}{c^2}\frac{\nu^3}{1+\frac{h\nu}{k_BT}-1} = \frac{2k_BT}{c^2}\nu^2\,, \tag{1.3.26}$$

the **Rayleigh-Jeans** law.

On the other hand, if the photon energy $h\nu$ is much larger than k_BT we can ignore the -1 in the denominator and just write

$$B_\nu(T) \approx \frac{2h\nu^2}{c^2}\exp\left(-\frac{h\nu}{k_BT}\right)\,, \tag{1.3.27}$$

the **Wien law**. The exponential term dominates, so asymptotically the Planckian goes to zero.

Since it is a continuously differentiable function which increases at low frequency and decreases at high frequency, it will have a maximum, which we expect to appear somewhere around $h\nu \approx k_BT$. This is precisely what is described by the **Wien displacement law**.

Let us see where this maximum lies if the Planckian is expressed as a function of frequency, as we are doing now. If we define $x = h\nu/k_B T$, the Planck function is proportional to

$$B_\nu(T) \propto \frac{x^3}{e^x - 1}\,, \tag{1.3.28}$$

so we can find its maximum by seeking the stationary point

$$\frac{\partial B_\nu}{\partial \nu} \propto \frac{\partial B_\nu}{\partial x} = 0\,. \tag{1.3.29}$$

The derivative reads

$$\frac{\partial B_\nu}{\partial x} \propto \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2} \propto e^x(3 - x) - 3 \stackrel{!}{=} 0\,. \tag{1.3.30}$$

This can also be expressed as $x = 3(1 - e^{-x})$ as well. It is a transcendental equation: there is no closed-form solution, however we can easily approximate it numerically. The maximum is found to be approximately $x = 2.82$, which means that the maximum is attained at the frequency

$$\nu \approx 2.82 \frac{k_B T}{h}\,. \tag{1.3.31}$$

The fact that the peak of the curve depends linearly on T is precisely the statement of the Wien displacement law.

Another interesting question to ask is about how B_ν varies with T: its partial derivative reads

$$\frac{\partial B_\nu}{\partial T} = \frac{2h^2\nu^4}{k_B T^2 c^2} \frac{e^{h\nu/k_B T}}{\left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right)^2} > 0\,, \tag{1.3.32}$$

so it is positive for any value of ν. It will be clearer with a picture, however we can say that this increase with T is much faster for $h\nu < k_B T$ than it is for $h\nu > k_B T$.

The specific energy density is found from $u_\nu = I_\nu/c$ and integrating over the solid angles (so, multiplying by 4π), so we have

$$u_\nu(T) = \frac{4\pi}{c} B_\nu\,, \tag{1.3.33}$$

and if we want to compute the total energy density we will have

$$u = \int_0^\infty u_\nu(T)\,\mathrm{d}\nu = \frac{4\pi}{c}\frac{2h}{c^2}\int_0^\infty \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}\,\mathrm{d}\nu \tag{1.3.34}$$

$$= \frac{8\pi h}{c^3}\frac{k_B^4 T^4}{h^4}\int_0^\infty \frac{x^3}{e^x - 1}\,\mathrm{d}x\,, \tag{1.3.35}$$

where, as before, $x = h\nu/k_B T$. This already shows the main result: $u \propto T^4$, however to get a closed form expression we need to solve the integral as well. This integral is of the type

$$\int_0^\infty \frac{x^n}{e^x - 1}\,\mathrm{d}x = n!\zeta(n+1)\,, \tag{1.3.36}$$

which we need to evaluate for $n = 3$. The Riemann zeta function fortunately has a closed form for even integers, in our case $\zeta(4) = \pi^4/90$, which we multiply by $3! = 6$. This yields

$$u = \frac{8\pi h}{c^3}\frac{k_B^4 T^4}{h^4}\frac{\pi^4}{15} = \frac{8\pi^5 k_B^4}{15c^3h^3}T^4 = aT^4\,, \tag{1.3.37}$$

where the constant $a \approx 7.57 \times 10^{-16}\,\mathrm{kgK^{-4}m^{-1}s^{-2}}$ is called the **blackbody constant**.

Now, we discuss the **flux** of the radiation field. If we choose a surface and measure the flux across it, our assumption of isotropy means that we will always find zero; however it is interesting to measure the *outgoing* flux from an oriented surface, counting only the photons going through it in one direction.

I understand that this gives a factor 1/2, but why should it correspond to integrating over half the solid angle, except by isotropy? The half-sphere does not seem to me to have physical meaning...

The flux we can compute is then:

$$F = \frac{1}{2}\int \mathrm{d}\Omega\,\mathrm{d}\nu\, B_\nu \cos\theta = \frac{2\pi}{2}\int_0^\pi \mathrm{d}\theta\cos\theta\sin\theta\int_0^\infty \mathrm{d}\nu\, B_\nu \tag{1.3.38}$$

$$= \pi \underbrace{\int_{-1}^1 \mu\,\mathrm{d}\mu}_{=1}\underbrace{\int_0^\infty B_\nu\,\mathrm{d}\nu}_{=uc/(4\pi)} = \frac{uc}{4} \tag{1.3.39}$$

$$= \frac{2\pi^5 k_B^4}{15c^2h^3}T^4 = \sigma T^4\,, \tag{1.3.40}$$

where σ is called the **Stefan-Boltzmann constant**, which is related to a by $\sigma = ca/4$.

Another useful result to state here is that the integral of the Planck function in $\mathrm{d}\nu$ is:

$$\int_0^\infty B_\nu\,\mathrm{d}\nu = \frac{uc}{4\pi} = \frac{\sigma}{\pi}T^4\,. \tag{1.3.41}$$

1.3.7 Scattering

This is the third process which can affect the radiation field, besides absorption and emission. In the astrophysical context, we will only need to concern ourselves with scattering by *free electrons*.

When modelling absorption, we had the true absorption coefficient $\alpha_\nu = n\sigma_\nu$, where σ_ν was the cross section of the individual absorber. We can model scattering in the same way: we can introduce a "scattering absorption coefficient" $\alpha_\nu^{(s)}$, which differs from the true one in that in absorption photons are not destroyed but merely deflected.

However, from the point of view of the beam in a specific direction they disappear, as if they were absorbed. We will then also have to consider the inverse phenomenon:

photons coming into the beam we are considering from another. This can be modelled as a "scattering emission".

We express the amount of photons lost from our beam as

$$\int \alpha_\nu^{(s)} I_\nu \, \mathrm{d}\Omega \, . \tag{1.3.42}$$

Let us make a simplifying assumption: that the scattering is **isotropic**, the scattered photon has an equal probability to go in any direction. Also, let us assume that the frequency ν of the photon does not change upon scattering. We will see that both of these conditions are satisfied for Thompson scattering but not in the general situation.

Then, we are left with $\alpha_\nu^{(s)} \int I_\nu \, \mathrm{d}\Omega = 4\pi\alpha_\nu^{(s)} J_\nu$.

This is in any direction, while the radiation scattered in the unit solid angle will be

$$j_\nu^{(s)} = \alpha_\nu^{(s)} J_\nu \, . \tag{1.3.43}$$

In general, since we do not have either conservation of photon frequency nor isotropy we will need to write a differential scattering cross section like

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu' \, \mathrm{d}\Omega'} \, , \tag{1.3.44}$$

while the full expression for the radiation scattered in a specific direction Ω at a frequency ν will read

$$j_\nu^{(s)} = \int \mathrm{d}\Omega' \, n \frac{\mathrm{d}\sigma}{\mathrm{d}\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu' \, \mathrm{d}\Omega'} I_{\nu'}(\Omega') \, \mathrm{d}\nu \, . \tag{1.3.45}$$

This is called the **scattering kernel**.

The transfer equation for scattering

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If we only account for true emission and absorption the transfer equation looks like

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu \, . \tag{1.3.46}$$

For isotropic and conservative scattering we can modify this equation by adding on the scattering absorption and emission terms:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -(\alpha_\nu + \alpha_\nu^{(s)}) I_\nu + j_\nu + \alpha_\nu^{(s)} J_\nu \, . \tag{1.3.47}$$

Now we cannot give a formal solution anymore: the derivative of the intensity now depends not only on the intensity itself but on its average over all solid angles, $J_\nu = \langle I_\nu \rangle_\Omega$. This will typically pose a problem: scattering is often the dominant phenomenon in radiative transfer for astrophysical systems.

In order to solve the equation we can resort to numerical methods: for example we can use an iterative relaxation procedure in which we start off by computing the solution

without scattering, calculate the mean intensity and plug it as a fixed value into the next iteration, and keep going.

We can write the transfer equation as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = (\alpha_\nu + \alpha_\nu^{(s)}) \left(-I_\nu + \frac{j_\nu + \alpha_\nu^{(s)} J_\nu}{\alpha_\nu + \alpha_\nu^{(s)}} \right) = (\alpha_\nu + \alpha_\nu^{(s)})(-I_\nu + S_\nu)\,, \tag{1.3.48}$$

where we define a new form for the source function:

$$S_\nu = \frac{j_\nu + \alpha_\nu^{(s)} J_\nu}{\alpha_\nu + \alpha_\nu^{(s)}}\,, \tag{1.3.49}$$

where, as long as Kirkhoff's law holds, we can substitute $j_\nu = \alpha_\nu B_\nu$.

The optical depth is derived from the *total* absorption coefficient:

$$\mathrm{d}\tau_\nu = (\alpha_\nu + \alpha_\nu^{(s)})\,\mathrm{d}s\ . \tag{1.3.50}$$

With these definitions, we can write the transfer equation like before,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu\,. \tag{1.3.51}$$

If there is scattering this is only apparently simple, the formulation only hides the complexity.

1.3.8 Mean free path

We defined absorption as $\alpha_\nu = n\sigma_\nu$: this is just the inverse of the mean free path, so we have (for true absorption):

$$\ell_\nu = \frac{1}{\alpha_\nu}\,, \tag{1.3.52}$$

and we can define a mean free path for scattering in the exact same way, with $\alpha_\nu^{(s)}$. These will be the mean free paths of a photon before it undergoes that specific process, if we want to compute the mean free path of a photon before it undergoes *either one* then we can just take the inverse of $\alpha_\nu + \alpha_\nu^{(s)}$, the total absorption coefficient.

Now, consider a medium with scattering, emission and absorption. Typically, a photon will be emitted, scatter a few times, and then be absorbed. Let us say that between emission and absorption it is scattered N times, and let us call the spatial intervals it travels between these scatterings $\vec{r}_i$, where i goes from 1 to N. The total distance travelled will look like $\vec{R} = \sum_i \vec{r}_i$.

The average of its square value will be

$$\left\langle R^2 \right\rangle = \sum_{ij} \left\langle \vec{r}_i \vec{r}_j \right\rangle = \sum_i \left\langle r_i^2 \right\rangle + 2 \sum_{i<j} \left\langle \vec{r}_i \vec{r}_j \right\rangle\ . \tag{1.3.53}$$

The mixed averages $\left\langle r_i r_j \right\rangle$ evaluate to zero, since each scattering is isotropic and independent; on the other hand each of the $\left\langle r_i^2 \right\rangle$ is equal to ℓ_ν^2, the square of the mean free path. This means that we have $\left\langle R^2 \right\rangle = N\ell_\nu^2$, meaning that the average distance occurring between each scattering is given by $\ell_* = \sqrt{N}\ell_\nu$.

Now, let us suppose that the photon is being scattered in a medium whose characteristic length is L. Then typically a photon will need to be scattered N times before escaping the medium, where $L = \sqrt{N}\ell_\nu$. This means that

$$\sqrt{N} = \frac{L}{\ell_\nu} = \alpha_\nu L \sim \tau_\nu \,, \tag{1.3.54}$$

where the last order-of-magnitude relation comes from the fact that the *differential* of the optical path is given by $\mathrm{d}\tau_\nu = \alpha_\nu \,\mathrm{d}s$.

This means that, at least in terms of order of magnitude, $N \sim \tau_\nu^2$. This holds as long as the medium is optically thick, that it, τ_ν is larger (ideally much larger) than 1.

What happens instead if the medium is optically thin, with $\tau_\nu < 1$? Then, we can neglect emission and scattering emission, and only consider scattering absorption. Then, the transfer equation will read

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu \,, \tag{1.3.55}$$

which is solved by $I_\nu = I_\nu(0)e^{-\tau_\nu}$. This can also be written, by adding $I_\nu(0)$ to both sides, as

$$I_\nu(0) - I_\nu = I_\nu(0)\left(1 - e^{-\tau_\nu}\right) \tag{1.3.56}$$

$$\frac{I_\nu(0) - I_\nu}{I_\nu} = 1 - e^{-\tau_\nu} \approx \tau_\nu \,. \tag{1.3.57}$$

Now, the average number of scatterings will be less than 1 and will correspond to the average relative intensity lost, $N = \Delta I_\nu / I_\nu(0)$. Therefore, in this case we will have $N \sim \tau_\nu$.

Putting together these two limiting cases, we can roughly say that we will have

$$N \approx \max\left(\tau_\nu, \tau_\nu^2\right). \tag{1.3.58}$$

After a mean free path the photon can be either scattered or absorbed. The probability that it is absorbed as opposed to being scattered is given by

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \alpha_\nu^{(s)}} \,, \tag{1.3.59}$$

while the probability that it is scattered is $1 - \epsilon_\nu$, and this last quantity is typically called the **single scattering albedo**.

The average number of mean free paths travelled before absorption will be[1]

$$N = \frac{1}{\epsilon_\nu} \,, \tag{1.3.62}$$

[1] This can be shown using the identity

$$\sum_{i=1}^{\infty} i p^{i-1} = \frac{1}{(1-p)^2} \,, \tag{1.3.60}$$

so we can make the following manipulation to find an explicit expression for the mean path between emission and absorption:

$$\ell_* = N\ell_\nu^2 = \frac{\ell_\nu^2}{\epsilon_\nu} = \frac{1}{(\alpha_\nu + \alpha_\nu^{(s)})^2}\frac{\alpha_\nu + \alpha_\nu^{(s)}}{\alpha_\nu} = \frac{1}{\alpha_\nu(\alpha_\nu + \alpha_\nu^{(s)})}\,. \tag{1.3.63}$$

This means that

$$\ell_* = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \alpha_\nu^{(s)})}}\,. \tag{1.3.64}$$

Now we make use again of the order-of-magnitude relation $\tau \sim \alpha L$ where L is the length scale of the medium; if we multiply L by $1/\ell_*$, the effective absorption coefficient, we get

$$\tau \sim \frac{L}{\ell_*} = \sqrt{L^2\alpha_\nu(\alpha_\nu + \alpha_\nu^{(s)})} = \sqrt{\tau_\nu(\tau_\nu + \tau_\nu^{(s)})}\,, \tag{1.3.65}$$

which is larger than τ_ν alone would be. This is called the **effective optical depth**; the fact that this is larger than τ_ν means that scattering traps the photon in the region for a longer time than it would remain there with absorption alone.

1.4 Radiative diffusion

This is a way to approximately solve the radiative transfer equation under certain assumptions, the main one being that the medium should be very **optically thick**.

Another assumption we will make is the **plane-parallel** approximation: this means that the properties of the medium only vary with respect to one coordinate, which we will call z. We will further assume that the properties of the radiation also only vary along z.

Now, this is about the *spatial* dependence of the quantities, however certain ones like the radiation intensity I_ν are also intrinsically vectors, so they also depend on the direction. However, we will have cylindrical symmetry for rotations around the z axis: therefore, the intensity will be a function of z and of the angle θ between the ray and the z axis.

The differential length travelled by the ray can be expressed in terms of the distance travelled along the z axis as

$$\mathrm{d}s = \frac{\mathrm{d}z}{\cos\theta} = \frac{\mathrm{d}z}{\mu}\,, \tag{1.4.1}$$

where we define the usual shorthand, $\mu = \cos\theta$.

which can be proven by differentiating the geometric series and bringing the derivative into the sum, which converges absolutely.

Then, we can compute the average number of scatterings as

$$\langle N\rangle = \sum_{i=1}^{\infty}\epsilon_\nu i(1-\epsilon_\nu)^{i-1} = \frac{\epsilon_\nu}{(1-(1-\epsilon_\nu))^2} = \frac{1}{\epsilon_\nu}\,. \tag{1.3.61}$$

The radiative transfer equation reads

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = (-I_\nu + S_\nu)\left(\alpha_\nu + \alpha_\nu^{(s)}\right), \tag{1.4.2}$$

where we can substitute

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \mu\frac{\partial I_\nu}{\partial z}, \tag{1.4.3}$$

and using this we can express the radiative transfer equation like

$$I_\nu(z,\mu) = S_\nu - \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}}\mu\frac{\partial I_\nu}{\partial z}. \tag{1.4.4}$$

Now we make use of the assumption that the optical depth is large. This means that $\tau \sim (\alpha_\nu + \alpha_\nu^{(s)})\ell$ is large, where ℓ is the characteristic scale of the system.

Should we not say explicitly that, besides being optically thick, the medium's thickness should be "slowly varying"?

This means that the term proportional to $\partial I_\nu/\partial z$ is small compared to the source function, so we can apply perturbation theory.

To zeroth order, assuming thermal equilibrium, we will have

$$I_\nu^{(0)} \approx S_\nu^{(0)} = B_\nu(T). \tag{1.4.5}$$

Then, the first-order approximation can be found by inserting the zeroth-order expression into the equation:

$$I_\nu^{(1)}(z,\mu) = B_\nu - \frac{\mu}{\alpha_\nu + \alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial z}. \tag{1.4.6}$$

An important thing to note is the linear relation between I_ν and the cosine μ of the angle.

With this first-order result we can compute the flux: if we were to use the zeroth-order one we would get zero, since blackbody radiation is isotropic. The flux is given by the integral over the solid angle: the contribution to the integral due to the B_ν term vanishes, since it is isotropic

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu)\mu\,\mathrm{d}\Omega = 2\pi\int_0^\pi I_\nu^{(1)}(z,\mu)\cos\theta\sin\theta\,\mathrm{d}\theta \tag{1.4.7}$$

$$= -\frac{2\pi}{\alpha_\nu + \alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial z}\underbrace{\int_1^{-1}\mu^2(-1)\,\mathrm{d}\mu}_{=2/3} \tag{1.4.8}$$

$$= -\frac{4\pi}{3(\alpha_\nu + \alpha_\nu^{(s)})}\frac{\partial B_\nu}{\partial z} \tag{1.4.9}$$

$$= -\frac{4\pi}{3(\alpha_\nu + \alpha_\nu^{(s)})}\frac{\partial B_\nu}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}z}, \tag{1.4.10}$$

where we used the fact that the spatial dependence of the B_ν only comes through the temperature $T(z)$. This is useful since, while the derivative of B_ν is frequency-dependent, the derivative of the temperature is not: therefore it can be factored out.

The total flux is given by

$$F = \int_0^\infty F_\nu \, \mathrm{d}\nu = -\frac{4\pi}{3}\frac{\mathrm{d}T}{\mathrm{d}z}\int_0^\infty \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu \tag{1.4.11}$$

$$= -\frac{4\pi}{3}\frac{\mathrm{d}T}{\mathrm{d}z}\frac{\int_0^\infty \frac{1}{\alpha_\nu+\alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu\,, \tag{1.4.12}$$

which is the **average value** of the function $1/(\alpha_\nu + \alpha_\nu^{(s)})$ weighted by the *known* function $\partial B_\nu/\partial T$; the integral at the numerator can be explicitly evaluated:

$$\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu = \frac{\partial}{\partial T}\int_0^\infty B_\nu\,\mathrm{d}\nu = \frac{\partial}{\partial T}\left(\frac{\sigma}{\pi}T^4\right) = \frac{4\sigma}{\pi}T^3\,, \tag{1.4.13}$$

which means that the final expression is

$$F = -\frac{4\pi}{3}\frac{4\sigma T^3}{\pi}\frac{\mathrm{d}T}{\mathrm{d}z}\frac{\int_0^\infty \frac{1}{\alpha_\nu+\alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}\,, \tag{1.4.14}$$

which we can better understand by defining the **Rosseland mean opacity**

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\nu+\alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}\,, \tag{1.4.15}$$

so that the expression reads

$$F = -\frac{16}{3}\frac{\sigma T^3}{\alpha_R}\frac{\mathrm{d}T}{\mathrm{d}z}\,. \tag{1.4.16}$$

This is called the **Rosseland approximation**. The result is that we can find a net photon flux which is driven by the temperature gradient. This has many applications, for example in stellar astrophysics.

This equation is very similar to the heat flux equation in one dimension:

$$q = -k\frac{\partial T}{\partial z}\,, \tag{1.4.17}$$

which in our case has the constant (thermal conductivity)

$$k = \frac{16}{3}\frac{\sigma T^3}{\alpha_R}\,. \tag{1.4.18}$$

We have assumed that the medium is optically thick, but we can see that if it becomes very optically thick then the photons have a hard time escaping (k becomes small).

1.5 The Eddington approximation

In the Rosseland approximation we found the result

$$I_\nu = B_\nu(T) + \mu \times \text{stuff}\,, \tag{1.5.1}$$

where the "stuff" is small. We then expanded in this parameter; without it we are in a perfectly isotropic condition, so we can say that we are "expanding around isotropy".

The idea behind the Eddington approximation is similar: we assume that the intensity can be written as

$$I_\nu(\tau) = a_\nu(\tau) + b_\nu(\tau)\mu\,, \tag{1.5.2}$$

so as before the angular dependence is linear.

Let us compute the moments of the specific intensity: the zeroth moment is the mean intensity,

$$J_\nu = \frac{1}{4\pi}\int I_\nu \,\mathrm{d}\Omega = \frac{2\pi}{4\pi}\int_{-1}^{1}(a_\nu + b_\nu\mu)\,\mathrm{d}\mu = a_\nu\,, \tag{1.5.3}$$

while the first moment is equal to

$$H_\nu = \frac{1}{4\pi}\int I_\nu\mu \,\mathrm{d}\Omega = \frac{2\pi}{4\pi}\int_{-1}^{1}\left(a_\nu\mu + b_\nu\mu^2\right)\mathrm{d}\mu = \frac{1}{2}\frac{2}{3}b_\nu = \frac{b_\nu}{3}\,. \tag{1.5.4}$$

The second moment is

$$K_\nu = \frac{1}{4\pi}\int I_\nu\mu^2 \,\mathrm{d}\Omega = \frac{a_\nu}{3} = \frac{1}{3}J_\nu\,. \tag{1.5.5}$$

The fact that the second moment of the intensity is related to the zeroth one as $K_\nu = J_\nu/3$ is precisely what the Eddington approximation asks.

The plane-parallel radiative transport equation can be written as

$$\mu\frac{\partial I_\nu}{\partial\tau} = I_\nu - S_\nu\,, \tag{1.5.6}$$

since $\mathrm{d}\tau = -(\alpha_\nu + \alpha_\nu^{(s)})\,\mathrm{d}s$ and $\mathrm{d}z = \mu\,\mathrm{d}s$.

Why is the optical depth defined with a negative sign here? It is the same in RL [RL79, eq. 1.115]. Is this to be interpreted as the optical depth towards a far-away observer, positioned at large z?

In this equation, the source function S_ν is assumed to be isotropic. So, if we average over the solid angle (or equivalently in $\mathrm{d}\mu\,/2$) we get

$$\frac{1}{2}\int_{-1}^{1}\mu\frac{\partial I_\nu}{\partial\tau}\,\mathrm{d}\mu = \frac{1}{2}\int_{-1}^{1} I_\nu\,\mathrm{d}\mu - \frac{1}{2}\int_{-1}^{1} S_\nu\,\mathrm{d}\mu \tag{1.5.7}$$

$$\frac{\partial}{\partial\tau}H_\nu = J_\nu - S_\nu\,. \tag{1.5.8}$$

We can also take the first moment of the same equation by multiplying the equation by μ before integrating: this yields

$$\frac{1}{2}\int_{-1}^{1}\mu^2\frac{\partial I_\nu}{\partial\tau}\,\mathrm{d}\mu = \frac{1}{2}\int_{-1}^{1}I_\nu\mu\,\mathrm{d}\mu - \underbrace{\frac{1}{2}\int_{-1}^{1}S_\nu\mu\,\mathrm{d}\mu}_{=0} \tag{1.5.9}$$

$$\frac{\partial K_\nu}{\partial\tau} = H_\nu\,. \tag{1.5.10}$$

However, recall that in the Eddington approximation we have $K_\nu = J_\nu/3$: this means that we have the system of coupled differential equations

$$\frac{\partial H_\nu}{\partial\tau} = J_\nu - S_\nu \qquad \text{and} \qquad \frac{1}{3}\frac{\partial J_\nu}{\partial\tau} = H_\nu\,, \tag{1.5.11}$$

which we can combine into one second order equation by differentiating the second: this yields

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial\tau^2} = J_\nu - S_\nu\,. \tag{1.5.12}$$

Now, the source function can be decomposed in a part depending on J_ν and a part depending on B_ν according to the single scattering albedo ϵ_ν (see equation (1.3.49)):

$$S_\nu = \epsilon_\nu B_\nu + (1-\epsilon_\nu)J_\nu\,. \tag{1.5.13}$$

Substituting this in a term simplifies and we finally find

$$\boxed{\frac{1}{3}\frac{\partial^2 J_\nu}{\partial\tau^2} = \epsilon_\nu(J_\nu - B_\nu)\,,} \tag{1.5.14}$$

the **radiative diffusion equation**. This can be solved relatively simply; however it does not give us the whole intensity, instead we can only find its mean value J_ν.

If we can calculate S_ν using this equation, then we can plug what we have found into the radiative transfer equation

$$\mu\frac{\partial I_\nu}{\partial\tau} = I_\nu - S_\nu\,. \tag{1.5.15}$$

Now that we have discussed the differential equations at length, let us consider for a moment the **boundary conditions** we need to impose.

A possible way to approach the problem is called the *two stream approximation*: we assume that radiation is only travelling along two angles, whose cosines are $\mu = \pm 1/\sqrt{3}$, so $\theta = \pm 55°$. This choice is less arbitrary than it may seem, there is good reason to suppose that in certain condition this is indeed the angle at which radiation will travel. This will be discussed later.

Let us introduce the quantities

$$I_\nu^\pm(\tau) = I_\nu(\tau, \pm\frac{1}{\sqrt{3}})\,. \tag{1.5.16}$$

Then, the average intensity will be given by their average:

$$J_\nu = \frac{I_\nu^+ + I_\nu^-}{2}\,. \tag{1.5.17}$$

The further moments can also be computed: they will be

$$H_\nu = \frac{1}{2\sqrt{3}}\Big(I_\nu^+ + I_\nu^-\Big) \qquad \text{and} \qquad K_\nu = \frac{I_\nu^+ + I_\nu^-}{6}\,, \tag{1.5.18}$$

consistently with the Eddington approximation. Note that, if we want to use a two-stream approximation, the only way for Eddington to hold is to use the values of $\mu = \pm 1/\sqrt{3}$.

The explicit expressions of the moments allow us to recover the $I_\nu^\pm$ as

$$I_\nu^+ = J_\nu + \sqrt{3}H_\nu = J_\nu + \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau} \tag{1.5.19}$$

$$I_\nu^+ = J_\nu - \sqrt{3}H_\nu = J_\nu - \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau}\,. \tag{1.5.20}$$

Now, let us fix an example for concreteness: a "slab" (like, for instance, a stellar atmosphere) extending from τ_0 to $\tau = 0$. The point at $\tau = 0$ is the top of the atmosphere: so, we want to impose the absence of incoming radiation there (supposing that there are no other nearby stars to influence the process): this is implemented as $I_\nu^-(\tau = 0) = 0$. This is called the *non-illuminated atmosphere* condition.

Also suppose that there is no radiation coming into the bottom of the slab: $I_\nu^+(\tau_0) = 0$.

This means that

$$J_\nu + \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau}\bigg|_{\tau_0} = 0 \qquad \text{and} \qquad J_\nu - \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau}\bigg|_{0} = 0\,. \tag{1.5.21}$$

With these boundary conditions, one can solve the equation numerically.

1.6 Plane electromagnetic waves

In a vacuum, Maxwell's equations read

$$\nabla \cdot E = \nabla \cdot B = 0 \tag{1.6.1}$$

and

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{1.6.2}$$

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}\,. \tag{1.6.3}$$

Using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ we can combine these into a wave equation,

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0\,, \tag{1.6.4}$$

which is solved by plane monochromatic waves

$$\vec{E} = \vec{\epsilon} E_0 \exp\Big(i(\vec{k}\cdot\vec{r} - \omega t)\Big)\,, \tag{1.6.5}$$

where the angular frequency and wavevector satisfy $\omega = c\left|\vec{k}\right|$. These are transverse waves: the scalar product of $\vec{\epsilon}$ and the wavevector $\vec{k}$ is zero, there is no longitudinal oscillation along the direction of motion.

These solutions form a basis, generally the radiation we find is not monochromatic. The electric field will be time-dependent, $\vec{E} = \vec{E}(t)$.

We want to discuss the frequency decomposition of these waves; we start off by assuming that we are dealing with a pulse, such that $\vec{E}(t) \to 0$ as $t \to \pm\infty$.

The Fourier transform of the electric field in the time domain gives us its frequency decomposition: we choose our conventions so that

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t) e^{i\omega t}\,\mathrm{d}t\ . \tag{1.6.6}$$

Since $E(t)$ is real, we must have $\hat{E}^*(\omega) = \hat{E}(-\omega)$. This means that we do not need to worry about negative frequencies.

The fundamental quantity we need to compute is the **energy spectrum** from the electric field: how is the energy in the EM radiation distributed across the various frequencies? The Poynting vector is useful in this context, as it quantifies the amount of energy carried per unit area and time: its modulus is given by

$$S = \frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}A} = \frac{c}{4\pi}E^2(t)\,, \tag{1.6.7}$$

which we need to integrate if we want to find the total energy carried by the pulse:

$$\frac{\mathrm{d}w}{\mathrm{d}A} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\,\mathrm{d}t\ . \tag{1.6.8}$$

Now, **Parseval's theorem** is a statement about Fourier transforms, telling us that the energy of the signal is conserved if we Fourier transform, meaning that

$$\int_{-\infty}^{\infty} E^2(t)\,\mathrm{d}t = 2\pi\int_{-\infty}^{\infty}\left|\hat{E}(\omega)\right|^2\mathrm{d}\omega\ . \tag{1.6.9}$$

Since this square modulus is the same for the conjugate the integrand is symmetric under $\omega \to -\omega$, so we can compute it from 0 to ∞ and multiply by 2 to get the same result. This yields

$$\frac{\mathrm{d}w}{\mathrm{d}A} = \frac{4\pi c}{4\pi}\int_{0}^{\infty}\left|\hat{E}(\omega)\right|^2\mathrm{d}\omega\ , \tag{1.6.10}$$

which can be differentiated with respect to ω to find the energy density per unit time and frequency:

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega} = c\left|\hat{E}(\omega)\right|^2. \tag{1.6.11}$$

This concerns the *total* energy of the pulse: we might want to calculate the energy per unit area, time and frequency! However, there is an issue in doing this: the uncertainty principle. We cannot sample with arbitrarily small intervals in both t and ω, since the inequality $\Delta\omega\Delta t \geq 1$ must be satisfied.

However, if our pulse repeats over some comparatively long time T we can formally define

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega\,\mathrm{d}t} = \frac{1}{T}c\left|\hat{E}(\omega)\right|^2, \tag{1.6.12}$$

so, as long as T is relatively long we can take the limit:

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega\,\mathrm{d}t} = \lim_{T\to\infty}\frac{1}{T}c\left|\hat{E}(\omega)\right|^2. \tag{1.6.13}$$

1.6.1 Polarization

This is a measurable quantity in astrophysics, it is easier to perform polarization measurements for low frequencies but nowadays we are starting to be able to measure polarizations as far as X-rays.

By definition a monochromatic plane wave is *linearly polarized*: it oscillates along a specific axis. Consider a superposition of two electric fields at a specific point in space, for simplicity $\vec{r} = 0$: the total electric field will be given by

$$\vec{E} = \underbrace{\left(\vec{x}E_1 + \vec{y}E_2\right)}_{\vec{E}_0} e^{-i\omega t}, \tag{1.6.14}$$

where the amplitudes $E_{1,2}$ are in general complex: $E_{1,2} = \xi_{1,2}e^{i\phi_{1,2}}$. So, we can write the total electric field

$$\vec{E} = \vec{x}\xi_1 e^{i(\phi_1-\omega t)} + \vec{y}\xi_2 e^{i(\phi_2-\omega t)}. \tag{1.6.15}$$

The physical field along any direction will be given by the real part of the projection of this complex-valued vector along that component:

$$E_x = \mathrm{Re}\left\{\xi_1 e^{i(\phi_1-\omega t)}\right\} = \xi_1\cos\left(\phi_1-\omega t\right) = \xi_1\cos\omega t\cos\phi_1 + \xi_1\sin\omega t\sin\phi_1, \tag{1.6.16}$$

and similarly with E_y, ξ_2 and ϕ_2.

Now, a short aside: an ellipse can be described parametrically from its principal axes with

$$x' = A\cos\beta\cos\omega t \qquad \text{and} \qquad y' = -A\sin\beta\sin\omega t, \tag{1.6.17}$$

since these expressions satisfy

$$\frac{x'^2}{A^2\cos^2\beta} + \frac{y'^2}{A^2\sin^2\beta} = 1\,, \tag{1.6.18}$$

so we can parametrize any ellipse with an appropriate choice of $A \in R^+$ and $\beta \in [-\pi/2, \pi/2]$.Tuesday

This parametric equation is expressed with respect to the principal axes of the ellipse; if we want to write in a different coordinate system which is obtained from a rotation of the principal axes we must use the rotation matrix 2020-8-18, compiled 2020-08-19

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}\,; \tag{1.6.19}$$

if we make x' and y' explicit we can find

$$x = A\cos\beta\cos\chi\cos\omega t + A\sin\beta\sin\chi\sin\omega t \tag{1.6.20}$$

$$y = A\cos\beta\sin\chi\cos\omega t - A\sin\beta\cos\chi\sin\omega t\,. \tag{1.6.21}$$

The expressions for x and y and the ones for E_x and E_y are quite similar: we must identify

$$\xi_1\cos\phi_1 = A\cos\beta\cos\chi \tag{1.6.22}$$

$$\xi_1\sin\phi_1 = A\sin\beta\sin\chi \tag{1.6.23}$$

$$\xi_2\cos\phi_2 = A\cos\beta\sin\chi \tag{1.6.24}$$

$$\xi_2\sin\phi_2 = -A\sin\beta\cos\chi\,, \tag{1.6.25}$$

which means that the electric field rotates in the shape of an ellipse, which is tilted about the x, y axes by and angle χ.

Stokes parameters

From these four equations, we find that (after working through the simplifications of the trigonometric functions)

$$\xi_1 + \xi_2 = A^2 \stackrel{\text{def}}{=} I \tag{1.6.26}$$

$$\xi_1 - \xi_2 = A^2\cos 2\beta\cos 2\chi \stackrel{\text{def}}{=} Q \tag{1.6.27}$$

$$2\xi_1\xi_2\cos(\phi_1 - \phi_2) = A^2\sin 2\chi\cos 2\beta \stackrel{\text{def}}{=} U \tag{1.6.28}$$

$$2\xi_1\xi_2\sin(\phi_1 - \phi_2) = A^2\sin 2\beta \stackrel{\text{def}}{=} V\,, \tag{1.6.29}$$

where we defined the **Stokes parameters** I, Q, U and V. They completely describe the state of an elliptically polarized monochromatic wave.

We can reverse their definitions as

$$A = \sqrt{I} \tag{1.6.30}$$

$$\sin 2\beta = \frac{V}{I} \tag{1.6.31}$$

$$\tan 2\chi = \frac{U}{Q}\,. \tag{1.6.32}$$

Now let us discuss the meaning of these parameters. Since $A = \xi_0$, $I = \xi_0^2$, while the energy flux is given by

$$S = \frac{c}{4\pi}\xi_0^2 = \frac{c}{4\pi} I\,. \tag{1.6.33}$$

So, the parameter I is directly proportional to the energy flux. The major axes of the polarization ellipse are $2A\cos\beta$ and $2A\sin\beta$ respectively; therefore their ratio $\tan\beta$ measures the eccentricity of the ellipse, and the ratio V/I can be used to recover β.

The ratio U/Q describes the orientation of the ellipse with respect to the x, y axes. If the wave is linearly polarized then $V = 0$ (this is a degenerate case for our description); if $V > 0$ the polarization is left-handed, while if $V < 0$ the polarization is right-handed.

If we have $U = Q = 0$, this corresponds to circular polarization.

The ellipse is fully determined by three numbers (β, χ, A): why are there 4 Stokes parameters? They cannot all be independent if this is the case...

In fact, they are connected by the relation

$$I^2 = Q^2 + V^2 + U^2\,. \tag{1.6.34}$$

The reason we use four parameters instead of just three is that this line of reasoning only holds for perfectly polarized monochromatic waves: in the general case, for partially polarized radiation, we have

$$I^2 \geq Q^2 + V^2 + U^2\,. \tag{1.6.35}$$

For completely *un*polarized light we have $Q = V = U = 0$.

A useful property is that these parameters are **additive**: the superposition of waves with different Stokes parameters can be the described by the sum of the Stokes parameters, as $I_{\text{total}} = \sum_k I_k$ and so on.

This suggests that we write them in a vector:

$$\vec{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}\,. \tag{1.6.36}$$

In general, we can decompose radiation which is partially polarized into a completely unpolarized part and a completely polarized part:

$$\vec{S} = \underbrace{\begin{bmatrix} I - \sqrt{Q^2+V^2+U^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{unpolarized}} + \underbrace{\begin{bmatrix} \sqrt{Q^2+V^2+U^2} \\ Q \\ U \\ V \end{bmatrix}}_{\text{polarized}}\,. \tag{1.6.37}$$

This allows us to define the **polarization degree**: the fraction of the radiation which is polarized, calculated as

$$\Pi_L = \frac{I_{\text{polarized}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}\,, \tag{1.6.38}$$

which is measurable with an instrument.

Electromagnetic potentials

We can express the electric and magnetic fields through the scalar and vector potentials ϕ and $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{and} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\,. \tag{1.6.39}$$

In terms of these potentials Maxwell's equations read

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \tag{1.6.40}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j}\,. \tag{1.6.41}$$

The potentials are redundant, we can impose some conditions on them while still being able to describe any physical system. One we can impose is the *Lorentz gauge*:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0\,, \tag{1.6.42}$$

under which we can get a formal solution for the potentials in terms of the charge and current densities:

$$\phi(\vec{r},t) = \int \frac{[\rho]}{|\vec{r}-\vec{r}'|}\,\mathrm{d}^3r' \tag{1.6.43}$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{[\vec{j}]}{|\vec{r}-\vec{r}'|}\,\mathrm{d}^3r'\,. \tag{1.6.44}$$

The square brackets are a notation meaning that the densities must be evaluated at the *retarded time* $t - |\vec{r}-\vec{r}'|/c$. This accounts for the finite speed of the propagation of light.

Radiation from moving charges

Consider a particle of mass m and charge q moving along a path described by the vector $\vec{r}_0(t)$. The charge and current densities will then be

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{r}_0(t)) \tag{1.6.45}$$

$$s\vec{j}(\vec{r},t) = q\,\underbrace{\frac{\mathrm{d}\vec{r}_0}{\mathrm{d}t}}_{\vec{u}(t)}\,\delta(\vec{r}-\vec{r}_0(t))\,. \tag{1.6.46}$$

From these expressions we can calculate the potentials generated by the moving charge. Let us introduce the relative radius $\vec{R} = \vec{r} - \vec{r}_0$ and its corresponding unit vector $\vec{n} = \vec{R}/\left|\vec{R}\right|$.

The potentials will then be

$$\phi = \left[\frac{q}{kR}\right] \qquad \text{and} \qquad \vec{A} = \left[\frac{q\vec{n}}{ckR}\right], \tag{1.6.47}$$

where $k = 1 - \vec{n}\cdot\vec{u}/c$, while $R = \left|\vec{R}\right|$. These are called the **Liénard-Wiechert** potentials. From them we can perform the differentiation in order to recover the fields, which will be called the Liénard-Wiechert fields. The final expressions for the fields are

$$\vec{E}(\vec{r},t) = q\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{k^3R^2}\right] + \frac{q}{c}\left[\frac{\vec{n}}{k^3R}\times\left((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)\right] \tag{1.6.48}$$

$$\vec{B}(\vec{r},t) = \left[\vec{n}\times\vec{E}\right], \tag{1.6.49}$$

where $\vec{\beta} = \vec{u}/c$. The important thing to notice here is that there are two contributions to the electric field: one goes as R^{-2} and is called the **Coulomb field** $\vec{E}_c$ and the other goes as R^{-1} and is called the **radiation field** $\vec{E}_r$.

Now we will discuss the radiation field mainly. Since the magnetic field can be found by taking a product with the electric field, we can make the same decomposition there.

The radiation field $\vec{E}_r$ must be perpendicular to $\vec{n}$, and so must be $\vec{B}_r$. So, they form an orthogonal triple.

Let us consider some simple cases.

Nonrelativistic charge

Let us first suppose that the particle is moving slowly, $|u|/c = |\beta| \ll 1$: this means that $|\vec{n} - \beta| \sim 1$ and also $k \sim 1$.

With these simplifications we find that the ratio of the magnitudes of the two components of the electric field is

$$\frac{E_r}{E_c} \sim \frac{R\dot{u}}{c^2}. \tag{1.6.50}$$

If the characteristic time across which the velocity of the particle changes is τ, then we have $\dot{u} \sim u/\tau$. This time τ will be associated with a characteristic frequency $\nu = 1/\tau$ and a characteristic wavelength $\lambda = c/\nu$.

...and this is the wavelength of the emitted radiation? is this always the case? it does not seem that obvious

Then we will get

$$\frac{E_r}{E_c} \sim \frac{Ru\nu}{c^2} \sim \frac{Ru}{\lambda c}, \tag{1.6.51}$$

so we can see that if

$$\frac{R}{\lambda} \lesssim 1 \tag{1.6.52}$$

then we have $E_r/E_c \lesssim u/c$. This defines an inner region around the emitter for which the Coulomb field dominates; outside it the radiation field dominates.

Typically, we are far enough from the emitters of radiation so that the radiation field dominates. Then, under the assumption $\beta \ll 1$ we have

$$\vec{E}_r = \frac{q}{Rc^2}\left[\vec{n} \times \left(\vec{n} \times \dot{\vec{u}}\right)\right] \qquad \text{and} \qquad \vec{B}_r = \left[\vec{n} \times \vec{E}_r\right]. \tag{1.6.53}$$

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Let us go into some more details regarding how the radiation field looks. Let us suppose that the angle between the acceleration $\dot{\vec{u}}$ and the observation unit vector $\vec{n}$ is Θ: then the absolute value of the electric field and magnetic fields will be

$$E_r = \frac{q}{Rc^2}\dot{u}\sin\Theta = B_r\,. \tag{1.6.54}$$

$\vec{E}$ and $\vec{B}$ are orthogonal, so the magnitude of the Poynting vector will be

$$\left|\vec{S}\right| = \frac{c}{4\pi}\left|\vec{E} \times \vec{B}\right| = \frac{c}{4\pi}E_r^2 = \frac{q^2}{4\pi c^3 R^2}\dot{u}^2\sin^2\Theta\,, \tag{1.6.55}$$

which, as is expected, decays like R^{-2}, since it is the power radiated per unit area. If we instead want the power radiated per unit solid angle then we must use $\mathrm{d}A = R^2\,\mathrm{d}\Omega$:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} = SR^2 = \frac{q^2}{4\pi c^3}\dot{u}^2\sin^2\Theta\,. \tag{1.6.56}$$

Integrating this over the solid angle allows us to calculate the total power emitted:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \int \frac{q^2}{4\pi c^3}\dot{u}^2\sin^2\Theta\,\mathrm{d}\Omega \tag{1.6.57}$$

$$= \frac{q^2\dot{u}^2}{2c^3}\underbrace{\int_{-1}^{1}(1-\mu^2)\,\mathrm{d}\mu}_{=2-2/3} \tag{1.6.58}$$

Reparametrized $\mu = \cos\Theta$, integrated over ϕ.

$$= \frac{2q^2\dot{u}^2}{3c^3}\,, \tag{1.6.59}$$

which is the well-known **Larmor formula**.

What we want to do now is to generalize this result to a system of *many charges*. Each of these will have a charge q_i, a distance R_i from the observer, an acceleration $\dot{u}_i$. In principle the generalization is simple: the electromagnetic field obeys the superposition principle, so we can just add the $E_{i,r}$ from all the charges together.

The issue is that the retarded time for each of the particles is slightly different from that of another particle. This is a treatable problem, but it makes the calculation more cumbersome.

What we can do in order to mitigate it is to make the **dipole approximation**. Suppose that the system of N charges is contained within a volume whose characteristic length scale is L. Also suppose that τ is the typical scale across which the system evolves. If the

evolution time τ is slow compared to L/c, then we can ignore the differences between the retarded times.

Now, if the system changes significantly over a time τ then the electric field also changes significantly over that time, so we can estimate the frequency of the emitted radiation as $\nu \sim 1/\tau$.

What we are asking, $\tau \gg L/c$, then becomes $\nu \approx 1/\tau \ll c/L$, or equivalently $\lambda \gg L$.

We can also give an order of magnitude for the typical velocity of the particles: if ℓ is the typical path length of a particle, then the typical velocity will be $u \sim \ell/\tau$. This means that $\ell/u = \tau$ must be larger than L/c: rearranging, we have

$$\frac{\ell}{L} \gg \frac{u}{c}\,, \tag{1.6.60}$$

but since $\ell \lesssim L$ this means that $u/c \ll 1$.

If these conditions are satisfied, then we can get the total electric field by adding all the electric field contributions from all the various particles:

$$\vec{E}_r = \sum_i \vec{E}_{r,i} = \sum_i \frac{q_i}{R_i c^2} \vec{n}_i \times \left(\vec{n}_i \times \dot{\vec{u}}_i\right) \tag{1.6.61}$$

$$= \sum_i \frac{1}{R_i c^2} \vec{n}_i \times \left(\vec{n}_i \times q_i \dot{\vec{u}}_i\right). \tag{1.6.62}$$

All of these will still need to be computed at the retarded time, although the retarded time will be the same for all the charges.

If the length scale L of the system is smaller than all the distances R_i, then we can approximate any R_i with a constant R_0; also the unit vectors $\vec{n}_i$ will be almost equal, and we can approximate them with a single $\vec{n}$. R_0 does not need to be the mean of the R_i, the point is that all of them are quite similar so the distance to any point inside the source works. The reasoning is similar for $\vec{n}$.

If we make both of these approximations we find

$$\vec{E}_r = \sum_i \frac{1}{R_0 c^2} \vec{n} \times \left(\vec{n} \times q_i \dot{\vec{u}}_i\right) \tag{1.6.63}$$

$$= \frac{1}{R_0 c^2} \vec{n} \times \left(\vec{n} \times \underbrace{\sum_i q_i \dot{\vec{u}}_i}_{\ddot{\vec{d}}} \right), \tag{1.6.64}$$

where we define the dipole moment

$$\vec{d} = \sum_i q_i \vec{r}_i\,. \tag{1.6.65}$$

Because the dipole moment appeared, this is called the **dipole approximation**. If Θ is the angle between $\vec{n}$ and $\ddot{\vec{d}}$, then we have

$$E_r = \frac{\ddot{d} \sin\Theta}{R_0 c^2}\,, \tag{1.6.66}$$

and with this we can calculate the power radiated per unit area

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}t} = \frac{c}{4\pi}E_r^2\,, \tag{1.6.67}$$

and per unit solid angle:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{c}{4\pi}E_r^2R_0^2 = \frac{1}{c^3}\frac{\ddot{d}^2}{4\pi}\sin^2\Theta\,, \tag{1.6.68}$$

so if we integrate we can find the total emitted power, which generalizes the **Larmor formula**:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{2\ddot{d}^2}{3c^3}\,. \tag{1.6.69}$$

Now let us discuss the **spectral distribution** of the emitted energy. The energy per unit frequency and area will be

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega} = c\left|\hat{E}_r(\omega)\right|^2\,, \tag{1.6.70}$$

while the energy emitted per unit solid angle as usual is calculated by multiplying by R^2:

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega\,\mathrm{d}\omega} = R^2c\left|\hat{E}_r(\omega)\right|^2 = \frac{1}{c^3}\sin^2\Theta\left|\hat{\ddot{d}}(\omega)\right|^2\,, \tag{1.6.71}$$

where $\hat{\ddot{d}}$ is the Fourier transform of $\ddot{d}$. This assumes that Θ is fixed through time; we are allowing its magnitude but not its direction to change.

Derivatives in the Fourier domain are multiplication by $-i\omega$: so, we can recover the second derivative of the dipole moment in the time domain as

$$\ddot{d}(t) = -\int_{-\infty}^{\infty}\omega^2e^{-i\omega t}\hat{d}(\omega)\,\mathrm{d}\omega\,, \tag{1.6.72}$$

which means that we can express the electric field in the Fourier domain as

$$\hat{E}(\omega) = -\frac{\omega^2}{c^2R}\hat{d}(\omega)\sin\Theta\,, \tag{1.6.73}$$

so that the energy emitted per unit frequency and solid angle becomes

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{\omega^4}{c^3}\left|\hat{d}(\omega)\right|^2\sin^2\Theta\,. \tag{1.6.74}$$

If we integrate over the sphere we find

$$\frac{\mathrm{d}w}{\mathrm{d}\omega} = \frac{8\pi}{3c^2}\omega^4\left|\hat{d}(\omega)\right|^2\,. \tag{1.6.75}$$

Note that the spectrum of the radiation is heavily dependent on the frequency of the radiation.

1.6.2 Thompson scattering

This is scattering of radiation by free electrons. Consider a plane, linearly polarized, electromagnetic wave impacting a free electron. The force onto the electron is the Lorentz force:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right). \tag{1.6.76}$$

We know that $|E| = |B|$ for the wave, so if the electron is nonrelativistic then the magnetic term is negligible:

$$\vec{F} \approx e\vec{E}\,, \tag{1.6.77}$$

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