Astrophysics and cosmology notes

Jacopo Tissino, Giorgio Mentasti

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Chapter 1

Cosmology

3 October 2019

Sabino Matarrese ♡. There is a dropbox folder with notes by a student from the previous years: [Pac18]. Also, there are handwritten notes by Sabino.

Textbooks: there are many. A good one is "Cosmology" by Lucchin and Coles.

On the astroparticle side, there is another book: I did not catch its name.

Exam: traditional oral exam, there are fixed dates but they do not matter: on an individual basis we should write an email to set a date and time.

In october the lessons of GR and this course on fridays are swapped.

1.1 Cosmology

The cosmological principle (or Copernican principle): *we do not occupy a special, atypical position in the universe*. We will discuss the validity of this. A more formal statement is:

Proposition 1.1.1 (Cosmological principle)**.** *Every* comoving observer *observes the Universe around them at a* fixed time *as being* homogeneous *and* isotropic.

We are allowed to use this principle only on very large scales. Comoving is a proxy for the absolute reference frame. When we observe the CMB we see that we are surrounded by radiation of temperature $\sim 3\,\mathrm{K}$. There is a *dipole modulation* though: around a milliKelvin of difference. This is due to the Doppler effect: we are *not* comoving with respect to the CMB.

We, however, ignore the anisotropies on the order of a μK.

The velocity needed to expain the Doppler effect is of the order of $630\,\mathrm{km\,s^{-1}}$. This is *after* correcting for the motion of everything up to a galactic scale.

So, we must assume that to get a *comoving* observer we should launch it in a certain direction at an extremely high velocity. So, we *do* have a preferred frame.

Fixed time refers to the proper time of the comoving observer.

This refers to scales on the order of 100 Mpc.

How can we talk about homogeneity if we can only look at the universe from a single point? We assume that any other observer would also see isotropy as we do.

Isotropy around every point is equivalent to homogeneity. We observe isotropy, we assume homogeneity.

What if we are not typical? a different assumption is that our observer status is *random* according to some distribution... This discussion then starts to involve the anthropic principle.

We must however always keep in mind that these assumptions have to be made before any cosmological study starts.

In GR, our line element will generally be $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$. The *preferred* form of the metric is the one written in the comoving frame: the Robertson-Walker line element,

$$\boxed{\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right)\right) \qquad (1.1)}$$

where $a(t)$ is called the *scale factor*. Sometimes it is convenient to write $\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2 = \mathrm{d}\Omega^2$. The angular part is the same (just rescaled) as the Minkowski one.

The parameter k is a constant, which can be ± 1 or 0.

The parameter r is not a length: we choose our variables so that $a(t)$ is a length, while the stuff in the brackets in (1.1) is adimensional (and so is r).

- $k = 1$ are called closed universes;
- $k = -1$ are open universes;
- $k = 0$ are called flat universes.

The set of flat universes with $k = 0$ has zero measure.

In a 4D flat spacetime we have the 10 degrees of freedom of Lorentz transformations: Minkowski spacetime is *maximally symmetric*.

Other geometries are De Sitter and Anti de Sitter spacetimes.

For the actual universe, however, not all of these symmetries hold: the universe seems not to be symmetric under time translation, and we have a preferential velocity. 4 symmetries are broken (time translation and Lorentz boosts), 6 hold (rotations and spatial translations): our assumption is that there is a 3D maximally symmetric space.

Under these assumptions, we can derive the RW line element.

Say we have a cartesian flat 2D plane: the flat metric for this is $\mathrm{d}l^2 = a^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$. The constant a is included since we want r to be adimensional.

The surface of a sphere has the following line element: $\mathrm{d}l^2 = a^2\Big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\Big)$, where $a^2 = R^2$, the square radius of the sphere.

For a hyperboloid, we will have: $\mathrm{d}l^2 = a^2\Big(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\varphi^2\Big)$, therefore the only difference is that trigonometric functions become hyperbolic ones.

For both of these, let us define the variable: $r = \sin\theta$ in the spherical case, and $r = \sinh\theta$ in the hyperbolic case. Then, these line elements become respectively:

$$\mathrm{d}l^2{}_{\text{sphere}} = a^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\,\mathrm{d}\varphi^2\right) \tag{1.2a}$$

$$\mathrm{d}l^2{}_{\text{hyperboloid}} = a^2\left(\frac{\mathrm{d}r^2}{1+r^2} + r^2\,\mathrm{d}\varphi^2\right). \tag{1.2b}$$

We can rewrite the RW element as:

$$\mathrm{d}l^2 = c^2\,\mathrm{d}t^2 - a^2\begin{cases}\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega^2\\ \mathrm{d}\chi^2 + \chi^2\,\mathrm{d}\Omega^2\\ \mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\Omega^2\end{cases} \tag{1.3}$$

where if $k = +1$ then $r = \sin\chi$, if $k = 0$ then $r = \chi$, and if $k = -1$ then $r = \sinh\chi$.

If we wish to use cartesian coordinates we will have:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(1 + \frac{k|x|^2}{4}\right)^{-2}\Big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\Big). \tag{1.4}$$

Universes in which a is a constant are called *Einstein spaces*. We can change time variable, defining $\mathrm{d}t = a(\eta)\,\mathrm{d}\eta$, where $a(\eta) \overset{\text{def}}{=} a(t(\eta))$: so, we will have

$$\mathrm{d}s^2 = a^2(\eta)\left(c^2\,\mathrm{d}\eta^2 - \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega\right)\right). \tag{1.5}$$

The parameter η is called *conformal time*: RW is said to be *conformal* to Minkowski. Conformal geometry is particularly useful for systems which have no characteristic length.

Photons do not have a characteristic length: they do not perceive the expansion of spacetime.

The photons of the CMB look like they are thermal: they were thermal originally, and remained such despite the expansion of the universe.

For now we did not use any dynamics, but we will insert them later.
The Friedmann equations are:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{1.6a}$$

$$\ddot{a} = -\frac{4\pi G}{3}a\left(\rho + \frac{3P}{c^2}\right) \tag{1.6b}$$

$$\dot{\rho} = \frac{-3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \tag{1.6c}$$

where dots denote differentiation with respect to the proper time of a consmological observer, t, which is called *cosmic time*. These imply that the energy density $\rho = \rho(t)$ and the isotropic pressure $P = P(t)$ only depend on t.

An important parameter is $H(t) \stackrel{\text{def}}{=} \dot{a}/a$, the *Hubble parameter*. We can write an equation for it from the first Friedmann one:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{1.7}$$

If $k = 0$, then there we have a critical energy density $\rho_C(t) = 3H^2(t)/(8\pi G)$: we call $\Omega(t) = \rho(t)/\rho_C(t)$. Is Ω larger or smaller than 1? This tells us about the sign of k; while measuring k directly is *very* hard. The former is a "Newtonian" measurement, while the latter is a "GR" measurement.

We define:

$$H_0 = H(t_0) = 100h \times \mathrm{km\,s^{-1}\,Mpc^{-1}} \tag{1.8}$$

where h is a number: around 0.7, while t_0 just means *now*.

1.1.1 Energy density

How do we measure it? We want the energy density *today* of *galaxies*: ρ_{0g}. This is $\mathscr{L}_g \langle M/L \rangle$, where $\mathscr{L}_g$ is the mean (intrinsic, bolometric) luminosity of galaxies per unit volume, while M/L is the mass to light ratio of galaxies. It is measured in units of $M_\odot/L_\odot$. Reference values for these are $M_\odot \sim 1.99 \times 10^{33}\,\mathrm{g}$, while $L_\odot \sim 3.9 \times 10^{33}\,\mathrm{erg\,s^{-1}}$.

How do we measure this? We have a trick:

$$\mathscr{L}_g = \int_0^\infty \mathrm{d}L\, L\Phi(L) \tag{1.9}$$

where $\Phi(L)$ is the number of galaxies per unit volume and unit luminosity: the *luminosity function*. With of our observations we estimate the shape of $\Phi(L)$. We know that the integral must converge, so we can bound the shape of Φ (?).

4 October 2019

From yesterday we recall that $k = 0$ iff $\rho(t) = \rho_C(t)$.

We can write $H(t_0) = H_0 = 100h \times \mathrm{km\,s^{-1}\,Mpc^{-1}}$. Do note that $1\,\mathrm{Mpc} = 3.086 \times 10^{22}\,\mathrm{m}$.

In the American school, the pupils of Hubble thought $h \sim 0.5$, while the French school thought $h \sim 1$. Now, we know that $h \sim 0.7$. Some people find $h \approx 0.67 \div 68$, others find $h \approx 0.62$.

If we have H we can find $\rho_{0C} = h^2 \times 1.88 \times 10^{-28}\,\mathrm{g\,m^{-3}}$. We have defined $\Omega(t) \stackrel{\text{def}}{=} \rho/\rho_C$: recall that $\operatorname{sign}\Omega - 1 = \operatorname{sign}k$.

So we want to measure the energy density in galaxies to figure out what Ω is.

There is a professor called Schechter who introduced a *universal* luminosity function of the universe.

$$\Phi(L) = \frac{\Phi^*}{L^*}\left(\frac{L}{L^*}\right)^{-\alpha} \exp\left(-\frac{L}{L^*}\right) \tag{1.10}$$

These can be fit by observation: we find $\Phi^* \approx 10^{-2}h^3\mathrm{Mpc}^{-3}$, $L^* \approx 10^{10}h^{-2}L_\odot$ and $\alpha \approx 1$.

[Plot of this function: sharp drop-off at $L = L^*$]

The integral for $\mathscr{L}_g$ converges despite the divergence of $\Phi(L)$ as $L \to 0$: so we do not need to really worry about the low-luminosity cutoff.

The result of the integral is $\mathscr{L}_g = \Phi^* L^* \Gamma(2-\alpha)$ where Γ is the Euler gamma function, and $\Gamma(2-1) = 1$. Numerically, we get $(2.0 \pm 0.7) \times 10^{18} h L_\odot \mathrm{Mpc}^{-3}$.

Spiral galaxies are characterized by rotation.

We plot the velocity of rotation of galaxies v against the radius R. This is measured using the Doppler effect.

We'd expect a roughly linear region, and then a region with $v \sim R^{-1/2}$: we apply $GM(R) = v^2(R)R$ (this comes from Kepler's laws or from the virial theorem). This implies

$$v(R) \propto \sqrt{\frac{M(R)}{R}} \tag{1.11}$$

So in the inside of the galaxy, where $M(R) \propto R^3$, $v \propto R$, while outside of it $M(R)$ is roughly constant, so $v \propto R^{-1/2}$.

Instead of this, we see the linear region and then $v(R)$ is approximately constant. Is Newtonian gravity wrong? (GR effects are trivial at these scales).

An option is MOND: they propose that there is somthing like a Yukawa term at Megaparsec distances. They are wrong for some other reasons.

Another option is that what we thought was the galaxy, from our EM observations, is actually smaller than the real galaxy. We'd need mass obeying $M(R) \sim R$:

since $M(R) = 4\pi \int_0^{R_{\max}} \mathrm{d}R\, R^2 \rho(R)$, we need $\rho(R) \propto R^{-2}$. This is a *thermal* distribution (?): we call it the *dark matter halo*.

People tend to believe that this matter is made up of beyond-the-standard-model particles, like a *neutralino*. An alternative is the *axion*.

The total density of DM is ~ 5 times more than that of regular matter.

If galaxies are not spiral, we look at other things: the Doppler broadening of spectral lines gives us a measure of the RMS velocity.

1.1.2 Virial theorem

Later in the course we will obtain the (nonrelativistic) virial theorem:

$$2T + U = 0 \tag{1.12}$$

This holds when the inertia tensor stabilizes.

The kinetic energy is $T = {}^3\!/_2 M \left\langle v_r^2 \right\rangle$: we expect the radial velocity to account for one third of total energy by equipartition. M is the total mass of the energy.

The potential energy is $U = -GM^2/R$. Substituting this in we get $3\left\langle v_r^2 \right\rangle = GM/R$.

If we account for the extra DM mass, we get $\langle M/L \rangle \approx 300 h M_\odot / L_\odot$. In order to have $\Omega = 1$, we'd need 1390.

So measuring the number density of galaxies and their velocities we get a way to measure Ω_0.

So, only 5% of the energy budget is given by baryionic matter (not all of which is visible), while around 27% is dark matter.

In order to comply with observation, it must be:

$$0.013 \leq \Omega_{\mathrm{b}} h^2 \leq 0.025\,, \tag{1.13}$$

where Ω_b corresponds to the baryionic density: so the universe *cannot* be made only of baryons.

Dark matter likes "clumping": we characterize it by this property.

In the end we have $\Omega = 1 = \Omega_b + \Omega_{DM} + \Omega_{DE}$ (do note that the value of 1 is measured, not theoretical!)

The Friedmann equation would imply deceleration if $\rho, P \geq 0$: dark energy seems to have *negative pressure*.

What about radiation? The CMB appears to be Planckian:

$$\rho_{0\gamma} = \frac{\sigma_r T_{0\gamma}^4}{c^2} = 4.8 \times 10^{-34}\,\mathrm{g\,cm^{-3}} \tag{1.14}$$

where $\sigma_r = \pi^2 k_B^4 / (15\hbar^3 c^3)$, while $\sigma_{SB} = \sigma_r c/4$.

We are going to show that if neutrinos were massless, their temperature would be $T_\nu = (4/11)^{1/3} T_\gamma$.

However, we know that for sure $\sum m_\nu \leq 0.12\,\mathrm{eV}$.

We have:

$$\rho_\nu = 3 N_\nu \frac{\langle m_\nu \rangle}{10\,\mathrm{eV}} 10^{-30}\,\mathrm{g\,cm^{-3}} \tag{1.15}$$

(??? to check)

1.1.3 The Hubble law

It is very simply

$$v = H_0 d\,, \tag{1.16}$$

where v is the velocity of objects far from us, and d is their distance from us. Can we derive this from the Robertson-Walker line element? It was actually derived first by Lemaitre.

We drop the angular part in the FLRW line element (for $k = 0$):

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2 \tag{1.17}$$

So the distance $d = a(t) r$: therefore $\dot{d} = \dot{a} r = \frac{\dot{a}}{a} d$. This is Newtonian and rough, but it seems to work.

Definition 1.1.1 (Redshift)**.** *The redshift z is defined by*

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}\,, \tag{1.18}$$

where λ_0 and λ_e are the observed and emission wavelengths.

We can show that $1 + z = a_0/a_e$. Therefore, $\nu_o/\nu_e = a_e/a_0$.

So, we can define (?) the luminosity distance:

$$d_L = \sqrt{\frac{L}{4\pi\ell}}\,, \tag{1.19}$$

where L is the observed luminosity, while ℓ is the apparent luminosity.

How do we relate the luminosity distance and the scale factor? Geometrically we derive:

$$\ell = \frac{L}{4\pi r^2 a^2} \left(\frac{a_e}{a_0} \right)^2 \tag{1.20}$$

since we can just integrate over angles the FLRW element: the value of k does not enter into the equation. The corrective factor comes from the frequency dependence of energy and the fact that power is energy over time, which also changes.

Therefore:

$$d_L = \frac{a_o^2}{a_e} r = a(1+z) r\,. \tag{1.21}$$

Thu Oct 10 2019

Today at 17 there is a colloquium at Aula Rostagni.

Next week, also at 17, there is a meeting at San Gaetano (for the general public).

We just had a Nobel prize in cosmology: he predicted the existence of the CMB, and two in astrophysics, they found the first exoplanet.

Send the professor an e-mail to get access to his Dropbox folder with many relevant texts.

We come back to the RW metric:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right), \tag{1.22}$$

where $k = 0, \pm 1$ and r is the dimensionless comoving radius.

The distance can be calculated as:

$$d_P(t) = a(t)\frac{r}{\sqrt{1-kr^2}}. \tag{1.23}$$

We are allowed to consider radial trajectories, however this does not account for the fact that we cannot actually measure with a space-like measuring stick.

A better definition is

$$d_P = a(t)\int_0^r \frac{\mathrm{d}\widetilde{r}}{\sqrt{1-k\widetilde{r}^2}}. \tag{1.24}$$

This is called the *proper distance* (the subscript P is for "proper"): the integration, at zero $\mathrm{d}\Omega$ and zero $\mathrm{d}t$, of the radial part of the metric.

On the other hand (OTOH), the flux of photons can give us the *luminosity distance*, which is actually measurable.

We want to derive the Hubble law ($v = H_0 d$) mathematically. It can also be restated as $cz = H_0 d$.

why?

For a fixed source at $d = ar$, naïvely we would have:

$$\dot{d} = \dot{a}r = \frac{\dot{a}}{a}ar = H_0 d \tag{1.25}$$

We can "move away from the current epoch by Taylor expanding".

The scale factor $a(t)$ can be written as

$$a(t) \simeq a_0 + \dot{a}_0(t-t_0) + {}^1\!/\!_2\,\ddot{a}(t-t_0)^2 \tag{1.26a}$$

$$= a_0\left(1 + \frac{\dot{a}}{a_0}\Big|_{t_0}(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)\right), \tag{1.26b}$$

where $q_0 \stackrel{\text{def}}{=} -\ddot{a}_0 a_0 / (\dot{a}_0)^2$ is called the *deceleration parameter* by historical reasons: people thought they would see deceleration when first writing this.

(the Hubble *constant* is not constant wrt *time* , but wrt *direction*!)

The deceleration parameter is actually measured to be negative.

Now, $1 + z = a_0/a$ can be expressed as:

$$1 + z \simeq \left(1 + \frac{\dot{a}}{a_0}|_{t_0}(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)\right)^{-1} . \tag{1.27}$$

Do note that this is derived by using at most second order in the distance interval: when expanding we cannot trust terms of order higher than second. Expanding with this in mind we get:

$$1 + z \simeq 1 - H_0 \Delta t + \frac{q_0}{2} H_0^2 \Delta t^2 + H_0^2 \Delta t^2 , \tag{1.28}$$

therefore

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t)^2 . \tag{1.29}$$

Rearranging the equation we get

$$t_0 - t = z \left(H_0 + \left(1 + \frac{q_0}{2}\right) H_0 z\right)^{-1} , \tag{1.30}$$

but as before we must expand to get only the relevant orders:

$$t_0 - t = H_0(t_0 - t) + \left(1 + \frac{q_0}{2} H_0^2 (t_0 - t)^2\right) \tag{1.31}$$

We would like the time interval to disappear: for photons $\mathrm{d}s^2 = 0$, therefore in that case $c^2\,\mathrm{d}t^2 = a^2(t)\,\mathrm{d}r^2\,/(1 - kr^2)$. Taking a square root and integrating:

$$\int_t^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \pm \int_r^0 \frac{\mathrm{d}\tilde{r}}{\sqrt{1 - kr^2}} , \tag{1.32}$$

where the plus or minus sign comes from...

The integral on the RHS can be solved analytically: it is

$$\begin{cases} \arcsin r \qquad k = 1 \\ r \qquad k = 0 \\ \operatorname{arcsinh} r \qquad k = -1 \end{cases} \tag{1.33}$$

in all cases, it is just r up to *second* order (since the next term in the expansion of an arcsine or hyperbolic arcsine etc is of third order).

On the other side, we have:

$$\frac{1}{a_0}\int_{t_0}^{t} c\,\mathrm{d}\widetilde{t}\left(1+H_0(\widetilde{t}-t_0)-\frac{q_0}{2}H_0^2(\widetilde{t}-t_0)^2\right)^{-1} \tag{1.34}$$

therefore, neglecting *third* and higher order terms, we have:

$$\frac{c}{a_0}\left((t_0-t)+\frac{1}{2}H_0(t_0-t)^2+o(|t_0-t|^2)\right)=r\,, \tag{1.35}$$

since the term proportional to q_0 only gives a third order contribution.

The explicit form of the luminosity distance was

$$d_L=a_0^2\frac{r}{a}=a_0(1+z)r\,. \tag{1.36}$$

The term a_0 should disappear at the end of every calculation: it is a bookkeeping parameter. Moving on:

$$d_L=a_0(1+z)\frac{c}{a_0H_0}\left(z-\frac{1}{2}(1+q_0)z^2\right), \tag{1.37}$$

but this also contains cubic terms:

$$d_L\simeq\frac{c}{H_0}\left(z+\frac{1}{2}(1-q_0)z^2+o(z^2)\right). \tag{1.38}$$

Therefore:

$$cz=H_0\left(d_L+\frac{1}{2}(q_0-1)\frac{H_0}{c}d_L^2\right), \tag{1.39}$$

and we can notice that the relation is approximately linear and independent of acceleration for low redshift, but we can detect the acceleration at higher redshift. Typically we need to go at around 10 Mpc.

The parameter q_0 appears to be negative now.

We can do better than that if we go from *cosmography* to *cosmology*, by understanding what causes the acceleration of the expansion of the universe.

Let us expand on the concept of reshift: photons are emitted with a certain wavelength λ_e, at a comoving radius r from us, and detected at λ_o. The line element for the photon is $\mathrm{d}s^2=0$, therefore $c\,\mathrm{d}t\,/a(t)=\pm\,\mathrm{d}r\,\sqrt{1-kr^2}$.

As before, we can integrate this relation from the emission to the absorption: we call it $f(r)$ (it can be any of the functions shown before).

$$\int_{t}^{t_0}=\frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}=f(r) \tag{1.40}$$

If we map $t\to t+\delta t$ and $t_0\to\delta t_0$ in the integration limits, the integral must be constant since it only depends on r.

When equating these two we can simplify the original integral, rearrange the integration limits and get:

$$\int_t^{t+\delta t} \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}, \tag{1.41}$$

which can we cast into

$$\frac{c\delta t}{a(t)} = \frac{c\delta t_0}{a(t_0)}. \tag{1.42}$$

Since the frequency must be proportional to the inverse of the time intervals δt or δt_0, we have

$$\nu_e a(t_e) = \nu_o a(t_o), \tag{1.43}$$

therefore

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a_0}{a}. \tag{1.44}$$

1.1.4 A Newtonian derivation of the Friedmann equations

It is useful to do it first this way, pedagogically.

Let us take a uniform spacetime with density ρ. We consider a sphere, and take all the mass inside the sphere away.

Consider Birkoff's theorem: the gravitational field of a spherically symmetric body is always described by the Schwarzschild metric. This can also be applied to a hole: in this case, it tells us that the metric inside the cavity is the Minkowski metric.

The mass taken away will be $M(\ell) = {}^{4\pi}/_{3}\rho\ell^3$, where $\vec{l} = a(t)\vec{r}$ is the radius of the sphere.

We suppose that the gravitational field is *weak*:

$$\frac{GM(\ell)}{\ell c^2} \ll 1. \tag{1.45}$$

We put a test mass on the surface of the sphere. What is the motion of the mass due to the gravitational field from the center? It will surely be radial, therefore

$$\ddot{\ell} = -\frac{GM(\ell)}{l^2} = -\frac{4\pi G}{3}\rho\ell. \tag{1.46}$$

This seems to give us a net force even though by our hypotheses there should be none, this is actually not an issue since the unit vectors in our equations will go away in the end.

Then, we have

$$\ddot{a}r = -\frac{4\pi G}{3}\rho a r \to \ddot{a} = -\frac{4\pi G}{3}\rho a. \tag{1.47}$$

This is part of a Friedmann equation: the isotropic pressure term cannot be recovered, it's like the speed of light is infinite now.

Now consider

$$\dot{\ell}\ddot{\ell} = -\frac{GM}{\ell^2}\dot{\ell}\,, \tag{1.48}$$

therefore

$$\frac{1}{2}\dot{\ell}^2 = \frac{4\pi}{3}G\rho\ell^3 + \text{const}\,, \tag{1.49}$$

and integrating we get

$$\dot{a}^2 r^2 = \frac{8\pi G}{3}\rho a^2 r^2 + \text{const} \tag{1.50}$$

or, removing the r^2 term, which is a constant,

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \text{const}\,. \tag{1.51}$$

We know that this new constant is related to the energy per particle.

A universe with negative k expands forever and so on.

The number k was badly defined to be only ± 1 or 0: its newtonian version is much better represented by $k_N = kc^2$.

Fri Oct 11 2019

On Oct 31 Marco Peloso will not give his lecture, so Sabino will do both lectures.

Recalling last lecture: we consider a universe with constant density ρ...

We can also recover the third Friedmann equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right), \tag{1.52}$$

without the last term, which yet again will come from the relativistic consideration.

We will consider *ideal fluids*. From thermodynamics we have:

$$\mathrm{d}E + p\,\mathrm{d}V = 0\,. \tag{1.53}$$

We can write $E = \rho c^2 a^3$. Then, this becomes $\mathrm{d}(\rho c^2 a^3) + p\,\mathrm{d}a^3 = 0$; expanding:

$$c^2 \mathrm{d}\rho a^3 + c^2\rho\,\mathrm{d}a^3 + p\,\mathrm{d}a^3 = 0\,. \tag{1.54}$$

This amounts to the third Friedmann equation, exactly.

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{1.55a}$$

$$\ddot{a} = -\frac{4\pi G}{3} a \left(\rho + \frac{3P}{c^2} \right) \tag{1.55b}$$

$$\dot{\rho} = -\frac{3\dot{a}}{a} \left(\rho + \frac{P}{c^2} \right) \tag{1.55c}$$

The equations are in terms of the three parameters a, ρ and p, which are all functions of time.

The third equation comes from the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$.

Rewriting the first equation:

$$\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - kc^2 \tag{1.56}$$

and

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3} \dot{\rho} a^2 + \frac{16\pi G}{3} \rho \dot{a} a \tag{1.57}$$

We then substitute in the expression we have for $\dot{\rho}$ from the third FE. Everything is multiplied by $\dot{a}$: if it is not zero we have

$$\frac{\ddot{a}}{a} = -4\pi G \rho - \frac{4\pi G p}{c^2} + \frac{8\pi G}{3} \rho = -4\pi G \left(\frac{\rho}{3} + \frac{P}{c^2} \right). \tag{1.58}$$

The equation system is underdetermined. We have to make an assumption: we will assume our fluid is a *barotropic* perfect fluid: $p \overset{!}{=} p(\rho)$.

Very often this equation of state will look like $p = w\rho c^2$, for a constant w (not ω!). This is related to the adiabatic constant γ by $\gamma = w + 1$.

We are going to assume homogeneity and isotropicity.

Some possible equations of state are $p \equiv 0$, or $w = 0$: this means $\gamma = 1$. This is a *dust pressureless fluid*.

What is the speed of sound of our fluid? We only have the adiabatic speed of sound $c_s^2 = \partial p / \partial \rho$, where the derivative is to be taken at constant entropy and is just a total derivative in the barotropic case.

Also in the baryionic case $p = 0$ is a good approximation.

If $p = 0$ we can simplify:

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \rho \implies \rho \propto a^{-3}. \tag{1.59}$$

More generally, not assuming $w = 0$ we get:

$$\dot{\rho} = -3(1+w)\rho \frac{\dot{a}}{a} \implies \rho \propto a^{-3(1+w)} = a^{-3-3\gamma} \tag{1.60}$$

Another case is a *gas of photons*: in that case $p = \rho c^2/3$, so $w = 1/3$, $\gamma = 4/3$, $c_s^2 = c^2/3$: the speed of sound is $c/\sqrt{3}$. These photons are thermal: perturbations can propagate (even without interactions with matter. . .).

In this case we get $\rho \propto p \propto a^{-4}$.

Stiff matter is $p = \rho c^2$, $w = 1$, $\gamma = 2$ and $c_s = c$. This is an incompressible fluids: it is so difficult to set this matter in motion that once one does it travels at the speed of light. Now, $\rho \propto p \propto a^{-6}$.

A possible case is $p = -\rho c^2$: $w = -1$ and $\gamma = 0$: we cannot compute a speed of sound. Now ρ and p are constants. This is the case of dark energy (?).

This can be interpreted as an interpretation of the cosmological constant Λ.

Now we relace the last FE with $w = \text{const}$, $\rho(t) = \rho_* \big(a(t)/a_*\big)^{-3(1+w)}$.

Now, if we substitute into the second FE we get that gravity is attractive ($\ddot{a} < 0$) iff $w > -1/3$.

[Plot: ρ vs a: the cosmological constant is constant, matter is decreasing, radiation is decreasing faster].

In this plot, a can be interpreted as the time. We can insert the spatial curvature in the plot: it decreases, but slower than matter. Now, the dark energy in the universe is more important than the curvature.

Let us solve the first FE: inserting the third one we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_* \left(\frac{a}{a_*}\right)^{-3(1+w)} - \frac{kc^2}{a^2}\,. \tag{1.61}$$

We defined the parameter $\Omega = \frac{8\pi G \rho}{3H^2} = \rho/\rho_C$. Experimentally this is very close to 1. The Einstein-de Sitter model is one where we take $\Omega \equiv 1$: negligible spatial curvature. This amounts to making $k = 0$.

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_* a_*^{3(1+w)} a^{-(1+3w)} \tag{1.62}$$

therefore $\dot{a} = \pm A a^{\frac{1+3w}{2}}$, or $a^{\frac{1+3w}{2}}\,\mathrm{d}a = A\,\mathrm{d}t$. A solution is:

$$a(t) = a_* \left(1 + \frac{3}{2}(1+w) H_* (t - t_*)\right)^{\frac{2}{3(1+w)}} \tag{1.63}$$

where $H_*^2 = \frac{8\pi G}{3}\rho_*$, coupled to:

$$\rho(t) = \rho_* \left(1 + \frac{3}{2}(1 + w H_*(t - t_*))\right)^{-2} \tag{1.64}$$

There is a time where the bracket in $a(t)$ is zero: we call it as t_{BB} define it by

$$1 + \frac{3}{2}(1+w) H_* (t_{\mathrm{BB}} - t_*) = 0\,. \tag{1.65}$$

Since the curvature scalar is $R \propto H^2$, at t_{BB} the curvature is diverges.

Hakwing & Ellis proved that if $w > -1/3$ we unavoidably must have a Big Bang.

We can define a new time variable by $t_{\text{new}} \equiv (t - t_*) + 2H_*^{-1}/(3(1+w))$. Then, we can just write:

$$a \propto t_{\text{new}}{}^{\frac{2}{3(1+w)}} \tag{1.66}$$

and this allows us to get rid of t_*.

Inserting this new time variable, we get

$$\rho(t) = \frac{1}{6(1+w)^2 \pi 4 t^2} \tag{1.67}$$

and the Hubble parameter is:

$$H(t) = \frac{2}{3(1+w)t} \tag{1.68}$$

Some cases are:

$$\begin{cases} w = 0 \implies a \propto t^{2/3} \\ w = 1/3 \implies a \propto t^{1/2} \\ w = 1 \implies a \propto t^{1/3} \end{cases} \tag{1.69}$$

The *De Sitter* universe is one where $w \to 1$: $a(t) \propto \exp(Ht)$ and $H = \text{const}$. (CHECK)

stuff

Thu Oct 17 2019

If we neglect spatial curvature, which is small, we can write the luminosity distance as an integral which we can compute:

$$d_L \equiv \left(\frac{L}{4\pi\ell} \right)^{1/2} . \tag{1.70}$$

Our metric is the FLRW line element. Then, we can write d_L as:

$$d_L = a_0(1+z)r(z) . \tag{1.71}$$

We now define the *conformal time* τ: we want to impose $a^2(\tau)\,\mathrm{d}\tau^2 = \mathrm{d}t^2$. Then, the RW line element becomes:

$$\mathrm{d}s^2 = a^2(\tau)\left(c^2\,\mathrm{d}\tau^2 - \frac{\mathrm{d}r^2}{1-kr^2} - r^2\,\mathrm{d}\Omega^2 \right) . \tag{1.72}$$

This is very important when we talk about zero-mass particles, with no intrinsic length scale. Using the variable χ, we have:

$$ds^2 = a^2(\tau)\left(c^2\,d\tau^2 - d\chi^2 - f_k^2(\chi)\,d\Omega^2\right), \tag{1.73}$$

where $f_k(\chi) = r$ is equal to $\sin(\chi)$, χ or $\sinh(\chi)$ if k is equal to 1, 0 or -1.

If we look at photons moving radially, we get

$$ds^2 = 0 = a^2(\tau)\left(c^2\,d\tau^2 - d\chi^2\right), \tag{1.74}$$

therefore $c^2\,d\tau^2 = d\chi^2$: setting $c = 1$, we get $\tau(t_0) - \tau(t_e) = \chi(r_e) - \chi(0)$, where a subscript e means "emission".

$$d\tau = \frac{dt}{a} = \frac{da}{a\dot{a}}, \tag{1.75}$$

and now recall $a = a_0/(1+z)$: differentiating this we get

$$da = -\frac{a_0}{(1+z)^2}\,dz\ , \tag{1.76}$$

$$\frac{da}{a^2} = \frac{da\,(1+z)^2}{a^2} = -\frac{dz}{a_0}, \tag{1.77}$$

which means

$$de = -\frac{dz}{a_0 H(z)}. \tag{1.78}$$

??? probably there is wrong stuff here

The Hubble parameter is given by

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{1.79}$$

with density $\rho(t) = \rho_r(t) + \rho_m(t) + \rho_\Lambda$, where the first term scales like a^{-4}, the second a^{-3}, the third is constant. Thus they scale like $(1+z)^4$, $(1+z)^3$ and so on.

Then we can write a law for the evolution of $H(z) = H_0 E(z)$. Recall the definition of $\Omega(t)$: we can look at the $\Omega_i(t)$ for i corresponding to matter, radiation and so on:

$$\Omega_i(z) = \frac{8\pi G\rho_i(t)}{3H^2(z)} = \frac{8\pi G}{3H^2}\frac{\rho_i(z)}{E^2(z)} \stackrel{\text{def}}{=} \Omega_{i,0}\frac{(1+z)^\alpha}{E^2(z)}. \tag{1.80}$$

In the case of radiation, $p = \rho c^2/3$, and then $\alpha = 4$.

For matter $p = 0$: $\alpha = 3$.

For the cosmological constant Λ: $\alpha = 0$.

For spatial curvature we have $\alpha = 2$.

For the Ω) corresponding to the curvature we define: $\Omega_k = -tc^2/(a^2H^2)$. We must have

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k \,. \tag{1.81}$$

Recall the definition of $E^2(z)$:

$$E^2(z) = \frac{H^2}{H_0^2} = \Omega_{k,0} + \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(q+z)^3 \,. \tag{1.82}$$

and to get E we just take the square root. We have $\tau(t_0) - \tau(t_e) = \chi(r_e)$. Integrating:

$$\chi(r) = c\int_{a_t}^{a} \frac{\mathrm{d}a}{a\dot{a}} = \int \frac{\mathrm{d}z'}{E(z')} \,, \tag{1.83}$$

therefore

$$r = f_k\left(\frac{c}{a_0 H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right) . \tag{1.84}$$

Two weeks ago we defined the luminosity distance: now we can compute it.

How do we decide which k to use?

Now, suppose we are looking at a certain far-away object with angular size $\Delta\theta$: we fix r in the RW line element, and look at a constant time: then we get a linear size corresponding to the angular one of

$$\mathrm{d}s = a(t) r \Delta\theta \,, \tag{1.85}$$

which, when divided by $\Delta\theta$, is called angular diameter $D_A = ar = a_0 r(z)/(1+z)$. This changes with distance... (?)

$$d_L = a_0(1+z)r \,, \tag{1.86}$$

then

$$\frac{d_L}{d_A} = (1+z)^2 \,. \tag{1.87}$$

Einstein thought that the universe had to be static.

Recall the Friedmann equations (1.6). Now, if we look at static solutions for matter ($p=0$): the third equation becomes the identity, the derivatives of a are zero: therefore the second equation gives us $\rho \equiv 0$: there cannot be matter.

Now we know that the universe is neither static nor stationary.

Einstein modified his equations in order to get a static non-empty solution.

The Einstein equations read

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{1.88}$$

when $c = 1$, where the Einstein tensor $G_{\mu\nu}$ can be defined with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,. \tag{1.89}$$

Einstein added a term $-\Lambda g_{\mu\nu}$ to the LHS of the Einstein equations. This is allowed since

1. it is symmetric;
2. it has zero covariant divergence, since Λ is constant while $\nabla_\mu g^{\mu\nu} = 0$.

Then, we can rewrite the EE with a modified stress-energy tensor, to which we add $\Lambda g_{\mu\nu}/8\pi G$. Comparing this to an ideal fluid tensor

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} , \tag{1.90}$$

we get $\rho \to \rho + \Lambda/8\pi G$ and $p \to p + \Lambda/8\pi G$.

Inserting this into the Friedmann equations we get:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} , \tag{1.91}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \Lambda , \tag{1.92}$$

while in the third equation, in the contribution $\widetilde{\rho} + \widetilde{p}$ the two Λ terms cancel. Then for the first equation we get

$$\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} = \frac{k}{a^2} , \tag{1.93}$$

and for the second:

$$4\pi G \rho = \Lambda \,. \tag{1.94}$$

Then,

$$\Lambda\left(\frac{1}{3} + \frac{2}{3}\right) = \Lambda = \frac{k}{a^2} , \tag{1.95}$$

and we want a solution with $k = 1$, $\Lambda > 0$.

A candidate for the cosmological constant term is the vacuum energy in QFT: however the estimate given there is around 10^{120} times off.

The next topic is a solution of the Friedmann equations. We will try to do it with $p = 0$: $\rho \propto a^{-3}$.

Fri Oct 18 2019

From yesterday: other consequences of inserting Λ. Now we have a different approach from Einstein's: we insert the constant not in order to get a static universe but just as a measurable parameter of our theory.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}\,, \tag{1.96}$$

with $\rho = \rho_0(a_0/a)^3$. The more the universe expands, the more the cosmological constant term dominates, since the other terms are inversely dependent on a.

The asymptotic state, neglecting spatial curvature, gives us a steady-state solution:

$$a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)\,. \tag{1.97}$$

This is called a *de Sitter* solution. An alternative is found if we *do* consider the spatial curvature, with $k = \pm 1$. The two other solutions are also called *de Sitter* and look like cosines (?): they can be mapped into each other.

How?

These belong to the maximally symmetric solutions: Minkowski, dS and Anti de Sitter: the latter has $\Lambda < 0$.

In AdS, do we have $a \propto \exp(ikt)$?

The *no-hair cosmic theorem* is actually a conjecture: it states that asymptotically only the dark energy contribution is relevant: all the matter and everything else is forgotten.

$$a \propto \big(\sinh(At)\big)^{2/3}\,, \tag{1.98}$$

where we define $2A/3 = \sqrt{\Lambda/3}$ is a solution which *interpolates* between the current — matter dominated — universe and the asymptotic one, since the hyperbolic sine is locally a simple exponential.

We can rewrite the Friedmann equation as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k\,, \tag{1.99}$$

and now we will solve it with $k = \pm 1$. In general, for an ODE like $y = f(y')$ with f' continuous we introduce $y' \equiv p$ with $p \neq 0$: then $y = f(p)$, which implies

$$y' = \frac{\mathrm{d}f}{\mathrm{d}p}p'\,, \tag{1.100}$$

and then

$$p = \frac{\mathrm{d}f}{\mathrm{d}p}p' \implies \frac{\mathrm{d}x}{\mathrm{d}p} = \frac{1}{p}\frac{\mathrm{d}f}{\mathrm{d}p}\,, \tag{1.101}$$

which gives by integration the solution:

$$x = \int \mathrm{d}p \, \frac{1}{p} \frac{\mathrm{d}f}{\mathrm{d}p} \qquad \text{with} \qquad y = f(p) \,. \tag{1.102}$$

Using this for our problem, we get $\dot{a}^2 = Aa^{-1} - k$, where $A \equiv 8\pi G a_0^3 \rho_0 / 3$. We can rewrite this as

$$a = \frac{A}{p^2 + k} \qquad \text{where} \qquad p = \dot{a} \,. \tag{1.103}$$

Then we get:

$$\dot{a} = p = -2Ap\dot{p} \frac{1}{(p^2+k)^2} \,, \tag{1.104}$$

and using our formula

$$t = -2A \int \frac{\mathrm{d}p}{(p^2+k)^2} \,. \tag{1.105}$$

In order to go forward, we distinguish the cases: it $k = +1$, then $p = \tan(\theta)$, therefore $1 + p^2 = \sec^2\theta$ which implies $\mathrm{d}p = \mathrm{d}\theta \sec^2\theta$.

For the time:

$$t = \int -2A \, \mathrm{d}\theta \cos^2(\theta) = -A\big(\theta + \sin(\theta)\cos(\theta)\big) + \text{const} \,, \tag{1.106}$$

and we apply the trigonometric identity $\sin(\theta)\cos(\theta) = \sin(2\theta)$:

$$t = -\frac{A}{2}\big(2\theta + \sin(2\theta)\big) + \text{const} \,, \tag{1.107}$$

now we can define $2\theta = \pi - \alpha$, therefore $\sin(2\theta) = \sin(\alpha)$: this gives us $t = A/2(\alpha - \sin(\alpha))$ and $p = 1/\tan(\alpha/2) = \tan(\pi/2 - \alpha/2)$.

So we almost have our solution:

$$a = \frac{A}{1 + \tan^2(\pi/2 - \alpha/2)} = A\cos^2(\frac{\pi}{2} - \frac{\alpha}{2}) = \frac{A}{2}\big(1 + \cos(\pi - \alpha)\big) = \frac{A}{2}\big(1 - \cos(\alpha)\big) \,, \tag{1.108}$$

which should be complemented with the equation we found for t.

Reinserting the constants we have:

$$a = a_0 \frac{\Omega_0}{2(\Omega_0 - 1)}\big(1 - \cos(\alpha)\big) \,, \tag{1.109}$$

and

$$t = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}\big(\alpha - \sin(\alpha)\big) \,, \tag{1.110}$$

but now we switch from α to θ for historical reasons.

[Plot: $a(\theta)$ vs θ].

We have $\dot{a} > 0$ when $0 \leq \theta \leq \theta_m = \pi$, while $\dot{a} < 0$ when $\theta_m \leq \theta \leq 2\pi$. This is the *turn-around* angle. The angles 0 and 2π correspond to the Big Bang and the Big Crunch.

At θ_m we have:

$$a = a_0 \frac{\Omega_0}{\Omega_0 - 1}, \tag{1.111}$$

and

$$t = \frac{\pi}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}, \tag{1.112}$$

therefore if we set $a = 1$ at the current time we get

$$t_0 = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \left(\arccos\left(\frac{2}{\Omega_0} - 1 \right) - \frac{2}{\Omega_0} (\Omega_0 - 1)^{1/2} \right), \tag{1.113}$$

In a pure matter model we would have $a \propto t^{2/3}$, which would imply $H_0 = 2/(3t_0)$ or $t_0 = 2/(3H_0)$: this does not make sense! It would give us an age of the universe of the order 10^9 yr, while the measured age of the universe is 1.4×10^{10} yr. This is for a flat universe.

For a given H_0, we would have a smaller t_0 with a closed universe.

For $k = -1$ we do exactly the same steps with hyperbolic functions instead of trigonometric ones: we get

$$a(\psi) = a \frac{\Omega_0}{2(1 - \Omega_0)} \left(\cosh \psi - 1 \right), \tag{1.114}$$

with $\cosh \psi = 2/\Omega_0 - 1$ and

$$t(\psi) = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} (\sinh \psi - \psi). \tag{1.115}$$

The same reasoning as before gives us a $t_0 > 2/(3H_0)$! This is then more attractive.

Radiation's energy density $\rho_r(a)$ can be seen as a function of the redshift: $\rho_r(z) = \rho_{0r}(1+z)^4$, since $(1+z) = a_0/a$.

For matter $\rho_m(z) = \rho_{0m}(1+z)^3$, for Λ instead $\rho_\Lambda(z) = \rho_{0\Lambda}$.

Let us define a moment called the *equality redshift* z_{eq}. This is when $\rho_r(z_{\text{eq}}) = \rho_m(z_{\text{eq}})$. This means that

$$(1 + z_{\text{eq}}) = \frac{\rho_{0,m}}{\rho_{0,r}} = \frac{\Omega_{0,m}}{\Omega_{0,r}}, \tag{1.116}$$

where we divided and multiplied by the critical density.

We know that $\Omega_{0,m}$ is around 0.3, while for the radiation it would be easier to measure the density. Neutrinos are matter now but they were radiation at the time of equality (???).

Accounting for everything, we think that

$$1 + z_{\text{eq}} \simeq 2.3 \times 10^4 \Omega_{0,m} h^2 \,. \tag{1.117}$$

This means that the recombination of electrons and protons into Hydrogen happened when the universe was already *matter dominated*.

Another time is z_Λ, defined by: $\rho_m(z_\Lambda) = \rho_\Lambda(z_\Lambda)$.

$$1 + z_\Lambda = \left(\frac{\rho_{0,\Lambda}}{\rho_{0,m}}\right)^{1/3} \simeq \left(\frac{0.7}{0.3}\right)^{1/3} , \tag{1.118}$$

which implies that $z_\Lambda \approx 0.33$.

1.2 The thermal history of the universe

The model which won out is the *hot Big Bang* model.

Consider radiation: we know from Stefan's law that $\rho_r \propto T^4$, while $\rho_r \propto a^{-4}$. Therefore we would expect *Tolman's law* to hold: $T \propto 1/a$.

By *total number of galaxies* we mean the galaxies in our past light cone.

When (in natural units) we get temperature of the order of a certain elementary particle, then statistically that type of particle will usually be ultra-relativistic.

The number density of particles is:

$$n = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, f(\vec{q}) \,, \tag{1.119}$$

in units where $c = \hbar = k_B = 1$. The parameter g is the number of helicity states, q is the three-momentum. The function $f(\vec{q})$ is a pdf in phase space. We will have to integrate over position and momentum: the metric we will get in the end must not depend on anything but time, by the cosmological principle.

There is no general rule for g: for photons, we only have two spin states; the "rule" $g = 2s + 1$ is not actually applied, for photons $\vec{s} = 0$ is unphysical, for gravitons $|\vec{s}| \leq 1$ is unphysical. g accounts for all internal degrees of freedom: for atoms we also have vibration, rotation…

The energy density is

$$\rho = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 \vec{q} \, E(q) f(\vec{q}) \,, \tag{1.120}$$

where $E^2 = q^2 + m^2$. For photons $E = q$, for nonrelativistic particles $E \approx m + q^2/2m$.

The adiabatic pressure is

$$P = \frac{g}{(2\pi)^3} \int \mathrm{d}^3\vec{q}\, \frac{q^2 f(\vec{q})}{3E}\,, \tag{1.121}$$

which comes from a consideration of the diagonal components T_{ii} of the stress energy tensor of particles. Alternatively, we can say that this comes from imposing $\mathrm{d}E = P\,\mathrm{d}V + \mathrm{d}Q$.

This definition gives us $P = \rho/3$ for photons directly.

In full generality the distribution is

$$f(\vec{q}) = \left(\exp\left(\frac{E-\mu}{T}\right) \pm 1\right)^{-1}, \tag{1.122}$$

where we have a plus for fermions, and a minus for bosons. Here, μ is the chemical potential: is becomes relevant when the gas becomes hot and dense. It can be introduced as a Lagrange multiplier for changes in number of particles.

The Planck distribution is given by:

$$f_k(\vec{q}) = \left(\exp\left(\frac{q}{T}\right) - 1\right)^{-1}, \tag{1.123}$$

since they are bosons with no chemical potential:

In general we can say that if for some species we have the reaction $i + j \leftrightarrow k + l$, then $\mu_i + \mu_j = \mu_k + \mu_l$. We can deduce them by the known relations: for example, from the annihilation of electron and positron we can derive $\mu_{e^+} = -\mu_{e^-}$. This rule is not trivial; it is an ansatz of thermodynamical equilibrium to get a solution of the Boltzmann equation which allows us to write the Saha equations.

Fri Oct 25 2019

Do we wish to have a part on stellar astrophysics right now, before going on with cosmology? Let him know.

Are we in thermal equilibrium? In general, no. This must be considered when dealing with CMB anisotropies.

QFT must be dealt with not only at zero temperature, but also at finite temperatures. People started doing this in the seventies.

The time variable is then periodic, with period $2\pi\beta$, where $\beta = 1/(k_B T)$.

Is this connected to imaginary time?

This is "in-in" instead of "in-out": there are no equilibrium states before or after the interaction.

We will simplify: we assume thermal equilibrium at any time in the evolution. A key point in cosmologi is the *absence* of time translation invariance, therefore energy is not conserved.

This part will come from Weinberg's book.

We use units where $c = k_B = \hbar = 1$. The number density of a certain species is

$$n(\dots) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, f(\vec{q}) \,, \tag{1.124}$$

where $f(\vec{q})$ is the phase space distribution of the particles, while g is the number of helicity states of that species. The dots will be explained later: they are about the parametrization of the phase space distribution.

For any species, $E = \sqrt{m^2 + |q|^2}$: this applies *on shell*, for the classical equations of motion: when there are quantum fluctuations it does not hold. The energy density is

$$\rho(\dots) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, E(\vec{q}) f(\vec{q}) \,, \tag{1.125}$$

while the pressure is

$$P(\dots) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, f(\vec{q}) \frac{q^2}{3E} \,. \tag{1.126}$$

Because of isotropy, in the cases we need to consider, the dependence on $\vec{q}$ is actually a dependence on $|\vec{q}| \equiv q$.

What is our ansatz for the phase space distribution?

$$f(q) = \frac{1}{\exp\left(\frac{E-\mu}{T}\right) \mp 1} \,, \tag{1.127}$$

where the sign is $-$ for bosons, $+$ for fermions.

The chemical potential deals with the flux of particles. We recover the Planck distribution when $E = q$ and $\mu = 0$: this tells us that photons do not have any chemical potential. If we measured $\mu \neq 0$ that would be called a *spectral distortion*, but is seems like that is not the case.

The rule for the sum of the chemical potentials only holds at equilibrium.

We can relate some chemical potentials by reactions, and this tells us about the conserved quantities which follow from the symmetry group of our theory.

If there was a global charge, phenomenologically we'd expect global magnetic fields, but we see them only of the order of the 10 μG.

So we have an upper bound on the global charge density (?)

We can estimate the orders of magnitude for the various species in the universe.

In the end: very often, $\mu/T \ll 1$. Therefore, the dependence of the number density, energy density and pressure would be also on μ but we forget it, and keep only the temperature dependence: we write $n(T)$, $\rho(T)$ and $P(T)$.

From the second principle of thermodynamics we know that the entropy in a certain volume V at temperature T, $S(V,T)$ is given by:

$$\mathrm{d}S = \frac{1}{T}\big(\mathrm{d}\rho(T,V) + P(T)\,\mathrm{d}V\big)\,. \tag{1.128}$$

Then we have

$$\frac{\partial S}{\partial V} = \frac{1}{T}\big(\rho(T) + P(T)\big)\,, \tag{1.129}$$

and

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{\mathrm{d}\rho(T)}{\mathrm{d}T}\,. \tag{1.130}$$

Recall the Pfaff relations: in order for the differential to be exact it needs to be closed, which means that the second partial derivatives need to commute:

$$\frac{\partial}{\partial T}\left(\frac{1}{T}\big(\rho(T) + P(T)\big)\right) = \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{\mathrm{d}\rho(T)}{\mathrm{d}T}\right)\,. \tag{1.131}$$

We can proceed with the calculation, to get:

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{1}{T}\big(\rho(T) + P(T)\big)\,. \tag{1.132}$$

Cosmology has not entered into the picture yet, but it can by the third Friedmann equation, which can be rewritten as

$$a^3\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(a^3(P+\rho)\Big)\,, \tag{1.133}$$

and these two, when put together, are equivalent (check!) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{T}(\rho + P)\right) = 0\,, \tag{1.134}$$

therefore this quantity is a constant of motion.

(For the RW line element, the square of the determinant $\sqrt{-g} = a^3$, so we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{-g}\frac{\rho + P}{T}\right) = 0\,, \tag{1.135}$$

)

So the quantity which is differentiated is constant. If we plug this back into the differential expression for the entropy, we get:

$$\mathrm{d}S = \mathrm{d}\left(\frac{(\rho + P)V}{T}\right)\,, \tag{1.136}$$

therefore the differentiated quantities are equal up to a constant, but since $V \propto a^3$ we get that the *entropy is constant*.

$$S \equiv S(a^3, T) = \frac{a^3}{T}(\rho + P) = \text{const}\,. \tag{1.137}$$

Since $\rho \propto T^4$ and $\rho \propto a^{-4}$, we expect Tolman's law: $Ta \sim 1$.

Also, for photons $P = \rho/3$: so we get $S \propto {}^4\!/_3 (a^3/T) T^4 \propto T^3 a^3 = \text{const}$.

We only consider photons since they have a much larger number density.

Is this because the suppression is exponential while the ratio of energies is somehow polynomial?

Now we wish to do the integrals for the thermodynamical quantities:

$$\int \mathrm{d}^3\vec{q} = 4\pi \int_0^\infty \mathrm{d}q\, q^2\,, \tag{1.138}$$

so we get

$$n(T) = \frac{g}{(2\pi^2} \int \mathrm{d}q\, q^2 f(q) \tag{1.139a}$$

$$\rho(T) = \frac{g}{2\pi^2} \int \mathrm{d}q\, q^2 f(q) E(q) \tag{1.139b}$$

$$P(T) = \frac{g}{6\pi^2} \int \mathrm{d}q\, q^2 f(q) \frac{q^2}{E(q)}\,, \tag{1.139c}$$

which can be solved analytically in the ultrarelativistic and nonrelativistic approximations.

Ultrarelativistic means $q \gg m$: actually we compare these to the temperature: we define $x = q/T$, and then the energy is $\sqrt{x^2 + m^2/T^2}$: we suppose $m/T \ll 1$.

The momentum will not always be large, but the cases in which it is large give a much greater contribution if the temperature is large.

Ultrarelativistic We approximate $E(q) \sim q$, so

$$n(T) = \frac{g}{2\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^2 \Big(\exp(q/T) \mp 1\Big)^{-1}\,, \tag{1.140}$$

$$\rho(T) = \frac{g}{2\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^3 \Big(\exp(q/T) \mp 1\Big)^{-1}\,, \tag{1.141}$$

$$P(T) = \frac{g}{6\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^3 \Big(\exp(q/T) \mp 1\Big)^{-1}\,, \tag{1.142}$$

so we can see that in this approximation, which is equivalent to $m \approx 0$, we get matter behaving like radiation: $P = \rho/3$. The result of the integrals depends on the type of the particles, and it is

$$n(T) = \begin{cases} \frac{\zeta(3)}{\pi^2} g T^3 & BE \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3 & FD \end{cases}, \tag{1.143}$$

and we get the proportionality to T^3. For the energy density:

$$\rho(T) = \begin{cases} \frac{\pi^2}{30} g T^4 & BE \\ \frac{7}{8}\frac{\pi^2}{30} g T^4 & FD \end{cases}, \tag{1.144}$$

and for the pressure we just divide by 3. For photons, $g = 2$ and they are bosons, so we get

$$\rho_\gamma = \frac{\pi^2}{15} T^4 \,. \tag{1.145}$$

Nonrelativistic Now $m \gg T$, so we get $E \approx m + \frac{q^2}{2m} + O((q/\sqrt{m})^4)$ by expanding.

So, can we just substitute this? It would seem like this makes the exponential very large, so the difference between bosons and fermions becomes negligible. So, to zeroth order in (q^2/m) we get

$$f \approx \exp\left(-\frac{m-\mu}{T}\right), \tag{1.146}$$

and then

$$n = \frac{g}{2\pi^2} \exp\left(-\frac{m-\mu}{T}\right) \int_{\mathbb{R}^+} \mathrm{d}q\, q^2 \,, \tag{1.147}$$

which diverges. This is the ultraviolet catastrophe: we need the *Planckian*, by considering the first order in q^2/m: then

$$n = \frac{g}{2\pi^2} \exp\left(-\frac{m-\mu}{T}\right) \int_{\mathbb{R}^+} \mathrm{d}q\, q^2 \exp\left(-\frac{q^2}{2mT}\right) = g \left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(\frac{\mu - m}{T}\right), \tag{1.148}$$

where we applied the identity

$$\int_{\mathbb{R}} \mathrm{d}x\, x^2 \exp\left(-\alpha x^2\right) = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \,. \tag{1.149}$$

To recover the energy density and pressure we could do $\rho = mn$ and $P = nT$.

Do these come out of doing the other two integrals?

Therefore, $P = T\rho/m$, so $P \ll \rho$: for nonrelativistic particles, we can deal with them as *noninteracting dust*.

$P = nT$ is just the ideal gas law.

If we compare relativistic particles to nonrelativistic ones, the former dominate the latter in terms of all of these three quantities.

Thu Oct 31 2019

Neutrinos have very low mass: therefore they become relativistic very quickly.

We saw last time the case of either bosons' or fermions' number density, energy density and pressure.

In the nonrelativistic case instead we must consider the second order in $q/\sqrt{m}$, and we get the Boltzmann suppression factor out of the integral: $\exp(-m/T)$.

In the nonrelativistic case there is no distinction between bosons and fermions.

What happens in the part of the history of the universe where all the particles were in thermal equilibrium?

If the timescales of the interactions are much larger than the cosmological events timescales (such as the energy of the universe), then those interactions do not happen. These are called *decoupled*.

Let us consider ultrarelativistic particles which are not *decoupled*, in the early universe which is radiation dominated (here "radiation" refers to all kinds of ultra-relativistic particles).

In this case, from the "conservation of the stress-energy tensor" we know that $\rho \propto a^{-4}$.

Our equation is

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{1.150}$$

and we want to neglect the last term. The equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_* \left(\frac{a}{a_*}\right)^{-4} - \frac{k}{a^2}\,, \tag{1.151a}$$

but it is not enough to look at the slopes: we can, however, have information about the normalization as well from present-day observations. If the second term is much smaller than the first today, then it was even more so in the far past. We can rewrite the equation as

$$1 = \Omega_{\text{tot}} - \frac{k}{a^2H^2} \equiv \Omega_{\text{tot}} + \Omega_{\text{curvature}}\,, \tag{1.152}$$

where $\Omega_{\text{curvature}} \equiv -k/(a^2H^2)$.

So we neglect the second term: approximately we then have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\text{rad}}\,. \tag{1.153}$$

We know that $a(t) \propto t^{1/2}$, therefore $\dot{a}/a = H = 1/2t$.

Wait, how does this work?

So, we have

$$\frac{1}{4t^2} = \frac{8\pi G}{3} g_* \frac{\pi^2}{30} T^4 \,, \tag{1.154}$$

where

$$g_* = g_*(T) = \sum_{i \in BE} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i \in FD} g_i \left(\frac{T_i}{T}\right)^4 \,, \tag{1.155}$$

where we insert a correction factor in order to not consider particles which are nonrelativistic (?). BE and FD denote the bosonic and fermionic degrees of freedom respectively.

The i spans the different types of particles in the SM. The T_i are the thermalization temperatures of the various species.

The plot of g_* in terms of the temperature goes "down in steps", as more and more species become nonrelativistic.

Then we have a formula for temperature in terms of time:

$$\frac{1}{2t} = \left(\frac{8\pi G}{3}\right)^{1/2} g_*^{1/2} \left(\frac{\pi^2}{30}\right)^{1/2} T^2 \,, \tag{1.156}$$

therefore we get:

$$t \approx 0.301 g_*^{-1/2} \frac{m_P}{T^2} \approx \left(\frac{T}{\mathrm{MeV}}\right)^{-2} s \,, \tag{1.157}$$

where $m_P = G^{-1/2}$ is the Planck mass. Beware: there are different conventions for this! The factor $g_*^{-1/2}$ is of order 1 around $T \approx 1\,\mathrm{MeV}$, so we ignore it.

The number is found by:

$$0.301 \approx \frac{1}{2\sqrt{\frac{8\pi\pi^2}{3\times 30}}} \,. \tag{1.158}$$

The Planck mass is approximately $1.2 \times 10^{19}\,\mathrm{GeV}$. It is the scale at which we need to account for quantum gravitational effects.

When the universe is 1 second old, weak interactions decouple.

Why are we allowing different thermalization temperatures?

A hypothesis is that the particles exit the Planck epoch, $t \approx m_P$, thermalized. When decoupling occurs, they can stop being thermalized.

Another hypothesis is that they stop being thermalized at a certain temperature during inflation.

Some time ago we discussed entropy conservation.

The entropy density is given by

$$s \equiv S/V = (P + \rho)/T = \frac{4}{3}\frac{\rho}{T} = (2\pi^2/45) g_{*s} T^3 \,; \tag{1.159}$$

since $V \propto a^3$ we have $sa^3 = \text{const}$.

We defined a new g_* to account for the different species:

$$g_{*s} \equiv \sum_{i \in BE} g_i \left(\frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{i \in FD} g_i \left(\frac{T_i}{T} \right)^3 , \tag{1.160}$$

so our new Tolman's law is $Tag_{*s}^{1/3} = \text{const}$.

When the neutrinos decouple, their temperature keeps scaling like $1/a$: they keep being thermalized with the photons even though they do not react.

When we reach $T \approx 0.5\,\text{MeV}$ electrons and positrons become thermalized: they decay into photons: the photon temperature T increases with respect to what would happen without this process (it actually just decreases slower).

The neutrinos are not updated when this process happens: their temperature then becomes consistently less than the one of the photons, but they keep scaling the same: one of their temperatures is a constant multiple of the other after this event.

Now we will calculate this multiple.

First, at $1\,\text{MeV}$, neutrinos decouple. Then, at $0.5\,\text{MeV}$, electrons and positrons decouple.

We require continuity at this transition: $T_>$ and $T_<$ must both be T_e.

$$sa^3 = \frac{2\pi^2}{45} g_{*s>} T_>^3 a_>^3 = \frac{2\pi^2}{45} g_{*s<} T_<^3 a_<^3 , \tag{1.161}$$

but we can drop all the factors we know to be continuous on the boundary:

isn't a different on either side of the boundary, if it is not instant?

we are left with

$$T_< = \left(\frac{g_{*s>}}{g_{*s<}} \right)^{1/3} T_< , \tag{1.162}$$

where we can compute

$$g_{*s>} = 2 + \frac{7}{8}(4) = \frac{11}{2} , \tag{1.163}$$

where we considered photons, electrons but not neutrinos (since they are separated). On the other hand,

$$g_{*s<} = 2 , \tag{1.164}$$

since electrons are not thermalized anymore. Therefore

$$T_< = \left(\frac{11}{4} \right)^{1/3} T_> , \tag{1.165}$$

which allows us to compute the neutrino temperature at any time:

$$T_\nu = T_\gamma \left(\frac{4}{11} \right)^{1/3} , \tag{1.166}$$

since they scale the same.

Now let us compute $\rho_r(T = 0.1\,\mathrm{MeV})$. Let us assume that the global temperature is the one of the photons: we get

$$g_* = \sum_{i \in BE} g_i \left(\frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i \in FD} g_i \left(\frac{T_i}{T} \right)^4 \tag{1.167a}$$

$$= 2 + \frac{7}{8} \left(3 \times 2 \left(\frac{T_\nu}{T_\gamma} \right)^4 \right) \tag{1.167b}$$

$$= 2 + \frac{21}{4} \left(\frac{4}{11} \right)^{4/3} \approx 3.36 , \tag{1.167c}$$

since we need to consider neutrinos, which contribute to the total energy density, but not electrons which are not relativistic.

***

Last hour we discussed the Planck mass $m_P \approx 1.2 \times 10^{19}\,\mathrm{GeV}$: it also defines a wavelength, $\lambda_P \approx 10^{-33}\,\mathrm{cm}$ and a timescale $t_P \approx 10^{-43}\,\mathrm{s}$ (both of these are $1/m_P$ in natural units).

When the age of the universe is of this order, our theories are not guaranteed to work.

1.2.1 The cosmological horizon

Let us consider null geodesics in a De Sitter universe with a RW metric: we can take them as radial, and find

$$c^2\,\mathrm{d}t^2 = a^2(t) \frac{1}{1 - kr^2}\,\mathrm{d}r^2 , \tag{1.168}$$

therefore we get:

$$\int^t \frac{c\,\mathrm{d}t}{a(t)} = \int^r \frac{\mathrm{d}\widetilde{r}}{\sqrt{1 - k\widetilde{r}^2}} = f(r) , \tag{1.169}$$

but in order to get something which has the dimensions of a length we need to multiply by a:

$$d_{\mathrm{Hor}}(t) = a(t) \int_0^t \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}\,. \tag{1.170}$$

"If this integral is convergent, we should be worried".

If we integrate from the beginning of time to now, we get the spatial (current) distance elapsed by a photon which started at the start of time. This is the radius of the largest region we could in principle observe. It is of the order of 3 Gpc.

This is not the case, for example, in Minkowski spacetime.

If there is an end of time, there is also a future horizon.

The quantity $d_{\mathrm{Hor}}(t)$ is increasing in time: this causes the Cosmological Horizon problem.

What is the problem?

An important moment is the *recombination of hydrogen*: the formation of the first Hydrogen atoms, so the first moment at which Compton scattering can occur. This causes the decoupling of radiation and baryionic matter. This is the moment at which the radiation in the CMB was emitted.

The CMB is very close to being uniform. It was emitted something like $t = 3.8 \times 10^5$ yr after the BB.

Let us consider the past light cones from two points diametrically opposite with respect to us, at this point in time: they do not overlap, so the CMB cannot be causally correlated. However, it seems like it is!

What is the angular scale at which the light cones overlap?

We have a classification by Bianchi of non-isotropic universes (in 9 classes).

There is the Mixed Master Universe.

Let us consider

$$R_{\mathrm{Hor}} = \int_0^t \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}\,. \tag{1.171}$$

We can hypothesize that there was a period where the comoving radius was decreasing with time. This would solve the problem. We can approximate

$$d_{\mathrm{Hor}}(t) = a(t) \int_0^t \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})} \approx ct \sim \frac{c}{H} \equiv d_{\mathrm{H}}\,, \tag{1.172}$$

where we defined the new *Hubble distance*. We also define the comoving Hubble radius: $r_H = c/(Ha) = c/\dot{a}$.

For it to be decreasing, the condition is

$$\dot{r}_H = -\frac{\ddot{a}}{\dot{a}^2} < 0\,, \tag{1.173}$$

therefore we need $\ddot{a} > 0$ for at least some time. We know that

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right), \tag{1.174}$$

therefore we need to have $\rho + 3P/c^2 < 0$. So, since the energy density is positive, the condition is $P < -\rho/3$.

Let us consider the paramter $\ddot{a}$: it is

$$\ddot{a} = \dot{a}H + a\dot{H} = a(H^2 + \dot{H}) > 0\,, \tag{1.175}$$

so the condition is $H^2 + \dot{H} > 0$. In terms of $P/\rho = w$ we have:

1. $\dot{H} < 0$ while $\dot{H} + H^2 > 0$: this corresponds to $-1 < w < -1/3$;
2. $\dot{H} = 0$: this is De Sitter: $a(t) = \exp(Ht)$, corresponding to $w = -1$;
3. $\dot{H} > 0$: here the solution is

$$a(t) = a_*\left(1 + \frac{3}{2}((1+w)H_*(t-t_*))\right)^{2/(3(1+w))}, \tag{1.176}$$

   which corresponds to $= w < -1$ and means $a(t) \propto t^p$ for some $p > 1$. This is called power-law inflation.

Big Rip singularity: with $a(t) = |t - t_{as}|^{-\alpha}$ with $\alpha > 0$. This is called poli-inflation.

What is this about?

The third condition is very hard to achieve. The boundary at $w = -1$ is called the *phantom divide*.

Now, let us consider the *flatness problem*: this was first proposed bt Dick and Peebles in 1986.

The parameter $\Omega = \frac{8\pi G\rho(z)}{3H^2(z)}$ diverges from 1 as time increases. Measurements of Ω_{tot} gave approximately 0.1.

How is it possible that the universe is still so flat, even when the universe is so old? Oldness and flatness seem incompatible.

This is a type of *fine-tuning* problem. Typically, if there is a fine-tuning problem then it signals that we should improve our theory.

Let us assume that w is constant. Then, $\rho(z) = \rho_0(1+z)^{3(1+w)}$. Recall that

$$H^2(z) = \frac{8\pi G}{3}\rho(z) - \frac{k}{a^2} \tag{1.177a}$$

$$H_0^2 = \frac{8\pi G}{3}\rho_0 - \frac{k}{a_0^2}\,, \tag{1.177b}$$

the latter of which implies $1 = \Omega - k/(a_0^2 H_0^2)$. So,

$$
\begin{aligned}
H^2(z) &= H_0^2\left(\frac{8\pi G}{3H_0^2}\rho(z) - \frac{k}{a^2 H_0^2}\right) && (1.178a)\\
&= H_0^2\left(\frac{\rho(t)}{\rho_0}\Omega_0\frac{a_0^2}{a}(1-\Omega_0)\right) && (1.178b)\\
&= H^2(z) = H_0^2(1+z)^2\Big(\Omega_0(1+z)^{1+3w} + (1-\Omega_0)\Big)\,, && (1.178c)
\end{aligned}
$$

so in the end

$$
\begin{aligned}
\Omega(z) &= \frac{8\pi G\rho_0}{3H_0^2}\frac{\rho(z)}{\rho_0}\frac{H_0^2}{H^2(z)} && (1.179a)\\
&= \Omega_0(1+z)^{1+3w}\Big(1-\Omega_0+\Omega_0(1+z)^{1+3w}\Big)\,, && (1.179b)
\end{aligned}
$$

and since $\rho(z) = \rho_0(1+z)^{3(1+w)}$ we get

$$
\begin{aligned}
\Omega^{-1}(z) - 1 &= \frac{\Omega_0\Big((1-\Omega_0)+\Omega_0(1+z)^{1+3w}\Big) - (1+z)^{1+3w}}{(1+z)^{1+3w}} && (1.180a)\\
&= (\Omega_0^{-1}-1)(1+z)^{-(1+3w)}\,. && (1.180b)
\end{aligned}
$$

If we assume $w = 1/3$ for all times (which is false, but we do it to get a result that is close enough) we get

$$
\Omega^{-1}(z) - 1 = (\Omega_0^{-1}-1)\left(\frac{T_0}{T(z)}\right)^2. \tag{1.181}
$$

If we compute T_{Planck}/T_0 we get approximately 4.5×10^{32}. Then when squaring we get 10^{-64}, without the approximation $w = 1/3$ we get 10^{-60}.

Then, we see that there is something deeply unnatural in the Friedmann model.

Thu Nov 07 2019

We talk about inflation again. The comoving horizon increases with time:

$$
d_H(t) = a(t)\int_0^t \frac{c\,\mathrm{d}\tilde{t}}{a(\tilde{t})} \sim ct \sim \frac{c}{H} \equiv \text{Hubble horizon}\,, \tag{1.182}
$$

it is also called the *past event horizon*. The reason why we can plot the history of the universe in a single spacetime diagram is because there is a well-defined transformation which brings an infinite interval to a finite one, using a hyperbolic arctangent: this gives us a *Penrose diagram*.

The comoving Hubble radius is $r_H = c/aH = c/\dot{a}$. This can grow.

If we have positive pressure $p > 0$, than the scale factor goes like $a \propto t^{2/(3(1+w))}$ with $w > 0$, then the comoving radius increases with time.

We can actually still get increasing comoving radii with a weaker condition: $p > -\frac{1}{3}\rho c^2$. This is the actual boundary (it can be checked looking at the derivative of a).

This is directly connected to the sign of the acceleration in the Friedmann equation.

If these conditions are always met, Hawking and Ellis proved that a Big Bang is inevitable.

The inflation hypothesis is that, even though now we have $\ddot{a} < 0$ now (or, it was so for some time: now the expansion seems to be accelerating), there was a period in which we had $\ddot{a} > 0$.

These are drawn as straight lines, but it is only qualitative: we are looking at the sign of the slope.

Then, there was a time in the past at which the comoving horizon was as large as it is now: now we will see how much inflation there must have been in order to solve the horizon problem up to now, ignoring the fact that now the universe's expansion is accelerating.

The inequality we want to impose is

$$r_H(t_i) \geq r_H(t_0) \,, \tag{1.183}$$

where t_0 is now while t_i is the beginning of inflation.

A sphere with comoving radius $d_H(t_i)$ will expand after inflation up to

$$d_H(t_i) \frac{a(t_f)}{a(t_i)} \,, \tag{1.184}$$

where t_f is the time of the end of inflation.

$$d_H(t_i) \frac{a(t_f)}{a(t_i)} \geq d_H(t_0) \frac{a(t_f)}{a(t_0)} \,. \tag{1.185}$$

We want to see what the limiting condition is.

$$Z_{\text{min}} = \frac{d_H(t_0)}{d_H(t_i)} \frac{a_f}{a_0} = \frac{H_i}{H_0} \frac{a_f}{a_0} \,, \tag{1.186}$$

is Z a redshift?

$$z_{\text{min}} \frac{H_i}{H_0} \frac{a_f}{a_0} = \frac{H_i}{H_f} \frac{H_f}{H_0} \frac{a_f}{a_0}, \tag{1.187}$$

or

$$\frac{H_f}{H_i} z_{\text{min}} = \frac{H_f}{H_0} \frac{a_f}{a_0}, \tag{1.188}$$

in which we can insert our solution to the Friedmann equations, for the scale factor and Hubble parameter in function of time:

$$H(t) = H_* \left(\frac{a(t)}{a_*} \right)^{-\frac{3(1+w)}{2}}. \tag{1.189}$$

This can be found using the results we found some time ago: the expressions for a and H were equal up to a different thing multiplying the parenthesis, and a different exponent.

This is of course an approximation, but it works. A better number can be found by integrating numerically over different more realistic equations of state. We find:

$$z = \frac{a_f}{a_i} =, \tag{1.190}$$

Put earlier

$$\frac{H_f}{H_i} = z_{\text{min}}^{-\frac{3(1+w)}{2}}, \tag{1.191}$$

therefore we get

$$z_{\text{min}} = \left(\frac{H_f}{H_0} \frac{a_f}{a_0} \right)^{-\frac{2}{(1+3w_{\text{inf}})}}, \tag{1.192}$$

where w_{inf} is calculated at the time of matter-radiation equality.

So we get:

$$\frac{H_f}{H_0} = \frac{H_f}{H_{\text{eq}}} \frac{H_{\text{eq}}}{H_0} = \left(\frac{a_f}{a_{\text{eq}}} \right)^{-2} \left(\frac{a_{\text{eq}}}{a_0} \right)^{-3/2} = \left(\frac{a_f}{a_0} \right)^{-2} \left(\frac{a_0}{a_{\text{eq}}} \right)^{-1/2}, \tag{1.193}$$

which means that the minimum inflation redshift must be

$$z_{\text{min}} = \left(\left(\frac{a_f}{a_0} \right)^{-1} \left(\frac{a_0}{a_{\text{eq}}} \right)^{1/2} \right)^{\frac{-2}{1+3w_{\text{inf}}}}, \tag{1.194}$$

so the result can be expressed in terms of temperatures:

$$\frac{a_0}{a_f} = \frac{T_f}{T_0} = \frac{T_f}{T_{\text{pl}}} \frac{T_{\text{pl}}}{T_0} , \tag{1.195}$$

where the T_{pl} is the Planck temperature, and $a_0/a_{\text{eq}} = 1 + z_{\text{eq}}$: in the end our result is

$$z_{\text{min}} = \left(\frac{T_{\text{pl}}}{T_0} (1 + z_{\text{eq}}) \frac{T_f}{T_{\text{pl}}} \right)^{-\frac{2}{1+3w_{\text{inf}}}} . \tag{1.196}$$

Recall that $1\,\text{GeV}$ is equal to $10^{13}\,\text{K}$, and $T_{\text{pl}} = 10^{19}\,\text{GeV}$. Also, $1 + z_{\text{eq}} = 2.3 \times 10^4 \Omega h^2$

What are these units?

We get

$$z_{\text{min}} \approx 10^{30} \frac{T_f}{T_{\text{pl}}} , \tag{1.197}$$

but what is the early universe temperature at the end of inflation? It must allow baryiogenesis, but will still be less than one, but there is an upper bound based on the fact that we have not observed primordial gravitational waves from this time: it must be at most 10^{-3}, so we find that the minimum redshift is of the order $z_{\text{min}} \lesssim 10^{30} \sim e^{60}$, or 60 e-folds.

This is an order of order of magnitude estimate.

From the Friedmann equation we get

$$1 = \Omega(t) - \frac{kc^2}{a^2 H^2} , \tag{1.198}$$

so $\Omega(t) - 1 = k r_H^2$. Now, consider the Ω_i of inflation: we get

$$\frac{\Omega - 1}{\Omega_i - 1} = \left(\frac{r_{H0}}{r_{Hi}} \right)^2 < 1 . \tag{1.199}$$

What we discuss now might be outside of our possibilities of comprehension.

Inflation is equivalent to $\ddot{a} > 0$, which is equivalent to $p < -\frac{1}{3}\rho c^2$:

$$\ddot{a} = -\frac{8\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) a . \tag{1.200}$$

A quantum Hamiltonian for a harmonic oscillator is

$$H = \frac{1}{2} \sum \omega \left(a^\dagger a + a a^\dagger \right) , \tag{1.201}$$

but this might give infinite energy for the ground state. In nonrelativistic QM we know that the energy is defined up to a constant, but in GR this is not the case: this energy gravitates!

Another way this comes up is the Casimir effect: we have virtual particles, which can pop up for times satisfying $\Delta E \Delta t \sim \hbar$. If we put two metallic plates close to each other, we get a force.

In QFT, we either have scalars, vectors or spinors. Can a scalar field have a nonzero expectation value, while respecting the Robertson-Walker symmetries? Yes, we just take a function of time.

For a vector, we cannot have nonzero expectation: a nonzero expectation value gives us a preferred direction. For a spinor, the same holds.

However, an object like $\overline{\psi}\psi$ behaves like a scalar, even though it comes from a vector.

There are almost no scalar particles in nature! The only one is the Higgs field.

The action for GR is given by

$$S = S_\Phi + S_{g_{\mu\nu}} + S_{\text{world}} , \tag{1.202}$$

where

$$S = \int \mathrm{d}^4 x \sqrt{-g} \mathscr{L} , \tag{1.203}$$

and the gravitational Lagrangian is $\mathscr{L}_g = R/16\pi G$. A kinetic Lagrangian is

$$\mathscr{L} = \frac{m}{2} \dot{q}^2 - V(q) , \tag{1.204}$$

for a scalar field its equivalent would be

$$\mathscr{L} = \frac{1}{2} \left(\partial_\mu \Phi \right)^2 - V(\Phi) , \tag{1.205}$$

in Minkowski spacetime. It then becomes:

$$\mathscr{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) , \tag{1.206}$$

but the covariant derivative of a scalar is just its partial derivative.

If we add a massive term, proportional to $R\Phi^2$, we get that adding it to the global action looks like gravity.

Fri Nov 08 2019

We continue the discussion from yesterday on the dynamics of inflation.

The Lagrangian for a scalar field in GR is

$$\mathscr{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) . \tag{1.207}$$

The "contravariant derivative" does not exist.

Why?

We can add a term $\xi R \Phi^2$, which has the right dimensions. Actions are dimensionless since $\hbar = 1$, and since $\mathrm{d}s^2 = g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$ the metric is also dimensionless. Therefore the dimensional analysis of $\int \mathrm{d}^4 x \sqrt{-g} \mathscr{L}$ gives us that $\mathscr{L}$ must have dimensions of m^{-4}.

The field Φ has the dimensions of a mass, which is an inverse length. The coupling constants are conventionally taken to be dimensionless: therefore if we are to add a term to the Lagrangian, it must be $\xi \Phi^2$ times an inverse square length: so we can insert R.

The value of ξ is undetermined: $1/6$ gives us conformal symmetry, while in other cases we get $1/4$. A Weyl transformation allows us to remove the additional term: we move from the Jordan frame (where we *do* have coupling between our scalar field and the curvature) to the Einstein frame, in which we do not.

This is a prototype for modified GR theories.

Varying the Einstein-Hilbert action with respect to the metric gives the LHS of the Einstein equations. If we vary with respect to something else we get the equations of motion of that other thing.

This means that the stress-energy tensor is just the functional variation of everything but the action of the metric in the global acction, with respect to the metric.

We get for our scalar field:

$$T_{\mu\nu}(\Phi) = \Phi_{,\mu} \Phi_{,\nu} - g_{\mu\nu} \left(\frac{1}{2} g^{\rho\sigma} \Phi_{,\rho} \Phi_{,\sigma} - V(\Phi) \right) , \tag{1.208}$$

and then $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.

We can get an explicit solution by using the symmetries of our spacetime: we assume that $\Phi(x^\mu) = \varphi(t)$. This however in QFT is an operator, but we cannot have an operator on the RHS of the EFE: so we do a semiclassical theory, an equivalent of Hartree-Fock: we take an average in the vacuum state, and our equations become

$$G_{\mu\nu} = 8\pi G \left\langle \hat{T}_{\mu\nu} \right\rangle_0 , \tag{1.209}$$

where we define the ground state as that one with the most symmetry allowed.

The symmetries we must consider are only rotations and translations. There are no issues of commutation, since we do not quantize space unlike the quantum loop gravity people.

If we perturb, we get $\Phi = \varphi + \delta\Phi$: so $\left\langle \Phi^2 \right\rangle = \varphi^2 + 2\left\langle \varphi\delta\Phi \right\rangle + \left\langle \delta\Phi^2 \right\rangle$, but the second term is zero since $\left\langle \delta\Phi \right\rangle = 0$ and $\varphi$is constant. The last term in these diverges. We do not know how to deal with it. We therefore assume that it is small.

So: when computing the stress energy tensor we get only diagonal terms: a perfect fluid!

The energy density is the Hamiltonian:

$$\rho = T_{00} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = H\,, \tag{1.210}$$

while the pressure is the Lagrangian:

$$P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) = \mathscr{L}\,. \tag{1.211}$$

We call the stuff in the universe which is not our field "radiation", with energy density ρ_r.

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V + \rho_r\right) \tag{1.212a}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\varphi}^2 - V + \rho_r\right) \tag{1.212b}$$

$$\dot{\rho}_{\text{tot}} = -3\frac{\dot{a}}{a}\left(\rho_{\text{tot}} + P_{\text{tot}}\right), \tag{1.212c}$$

but in the continuity equation we can split the contributions by inserting an unknown factor Γ, the transfer of energy between the field and radiation.

$$\dot{\rho}_\varphi = -3\frac{\dot{a}}{a}\dot{\varphi}^2 + \Gamma \tag{1.213a}$$

$$\dot{\rho}_r = -4\frac{\dot{a}}{a}\rho_r - \Gamma\,, \tag{1.213b}$$

which, denoting $' = \partial_\varphi$:

$$\dot{\rho}_\varphi = \dot{\varphi}\ddot{\varphi} + V'\dot{\varphi} \tag{1.214a}$$

$$\ddot{\varphi}\dot{\varphi} + V'\dot{\varphi} = -3\frac{\dot{a}}{a} + \Gamma\,, \tag{1.214b}$$

but we drop Γ since we assume there is little radiation.

One solution is $\dot{\varphi} = 0$, if not:

$$\ddot{\varphi} + 3\frac{\dot{a}}{a} = -V'\,, \tag{1.215}$$

recall the definition of

$$w = -\frac{1}{3} = \frac{P}{\rho} = \frac{\frac{1}{2}\dot{\varphi}^2 - V}{\frac{1}{2}\dot{\varphi}^2 + V}\,, \tag{1.216}$$

so one possibility we have is

$$\dot{\varphi}^2 \gg 2|V| \implies w = 1\,, \tag{1.217}$$

or else

$$\dot{\varphi}^2 \ll 2|V| \implies w = -1\,. \tag{1.218}$$

The continuity equation gives us the Klein-Gordon equation again: it is tautological.

$\varphi =$ const was one of the first solutions proposed. This model seems so fit the data.

Several proposals were made in the late seventies, early eighties.

A very simple model for a symmetry-breaking potential is the Ginzburg-Landau:

$$V \propto \left(\Phi^2 - \sigma^2\right)^2\,, \tag{1.219}$$

which gives a seeming "mass term" $-2\varphi^2\sigma^2$, which has the wrong sign: it is "tachyonic"!

The configuration at $\Phi = 0$ is unstable. The one at $|\Phi| = \sigma$ is not symmetric under $\Phi \to -\Phi$.

People realized that QFT is a subcase of a condensed matter approach in which we have a thermal bath, an *environment*. This is *finite temperature QFT*.

We consider then an *effective potential* for the temperature: $V_T(\Phi) = V(\Phi) +$ functions of T. This might be $V(\Phi) + \alpha\varphi^2 T^2 + \gamma T^4$, with positive α. The quadratic term then gives us a *positive* mass term: at temperatures larger than some critical temperature we get stability at $\Phi = 0$, but what happens if we lower the temperature?

Then, there is symmetry breaking.

Let us see how our Friedmann equations account for this situation.

The temperature of radiation is $\rho_r = \frac{\pi^2}{30} g_*(r) T^4$. If we start with a universe which is radiation dominate, then it ends up to be De Sitter.

This is a consequence of the *No Hair Cosmic Theorem*.

There is a potential barrier between the metastable "$\Phi = 0$" state, and the symmetry breaking other ones. (even though it does not show in the fourth degree potential model).

This can happen through quantum tunneling, but there is a delay: a *first order phase transition with supercooling* (by "super" what is meant is just that the temperature goes below T_C even though we still are in the center symmetric state).

We get bubbles of symmetry broken by fluctuation, expanding through the universe but never meeting because of the expansion.

This is the "old inflation model".

A new inflation model involves "slow rolling".

The equation $\ddot{\varphi} + 3H\dot{\varphi} = -V'$ looks like a regular equation of motion: after a time $1/H$ the "friction" velocity-dependent term dominates.

Then we get a slow-roll regime: $H^2 = \frac{8\pi G}{3} V$ and $\dot{\varphi} \approx -V'/3H$.

We exploit the flatness of the potential. There are quantum fluctuations during inflation.

The solution is *chaotic inflation*, by Linde 1984: Since $\Delta E \Delta t \approx \hbar$, we do not know at which state we are actually. As time passes, the energy uncertainty decreases. The initial condition for the distribution of the universe is then detemined by the uncertainty principle.

An alternative is *eternal* chaotic inflation. If a fluctuation increases the potential universe, then H^2 increases, then the region feels a larger volume. The case where the field goes towards the minimum is unlikely. Why did it happen? This can only be answered with the anthropic principle.

Thu Nov 14 2019

During inflation, the comoving Hubble radius ($r_H = 1/\dot{a}$) decreases.

t_i is the beginning time of inflation, t_f is its end, t_Λ is the time when the cosmological constant became dominant, and then we get to now: t_0.

What happened before inflation?

The cosmic no-hair conjecture is what allows inflation to delete inhomogeneties. So, there might have been perturbations before inflation: we cannot know. Up to which scale? Perturbations on scales larger than the cosmological horizon are not perceivable as perturbations: we only perceive our local mean value.

Below the largest inflation scale, the perturbations are erased by inflation: we see perturbations on these scales which are produced during inflation.

The energy density of radiation scales as $\rho_r \propto a^{-4}$, while the one of matter instead it scales as $\rho_m \propto a^{-3} \propto e^{-3Ht}$ since $a \propto e^{Ht}$.

The maximum temperature of radiation after inflation is given by the one which corresponds to the maximum latent energy released by our scalar field due to its coupling to the rest of the universe, which acts as a sort of viscous force.

So we get that this latent energy ΔV is of the order of T^4_{rad}.

What is the typical temperature needed to produce baryon symmetry?

What is baryon symmetry?

These questions are hard to answer without a grand unification theory.

How much antimatter is there in the universe? A long time ago, it was thought that there might have been regions in the universe which were filled with antimatter by looking for a γ-ray background, but this was not found. We did not find them.

The baryon number is the difference between the number of baryons and antibaryons: $(n_b - n_a)/(n_b + n_a)$. This is actually computed with quark numbers, so it can be computed even when protons and neutrons have not yet formed.

It seems like the only antimatter known is the one which was formed by us, or cosmic rays.

The value $\Omega_{0b} \approx 0.04$ is defined as

$$\rho_{0b} = \Omega_{0b}\rho_c\,, \tag{1.220}$$

where $\rho_c = 3H_0^2/(8\pi G)$.

When we define the number of protons in the universe we also fix the number of neutrons, since the universe is globally neutral. So we can estimate:

$$n_{0b} = \frac{\rho_{0b}}{m_p} \approx 1.12 \times 10^{-5}\,\mathrm{cm}^{-3} \times h^2\Omega_{0b}\,, \tag{1.221}$$

where $h \sim 0.7$, and similarly we can compute

$$n_{0\gamma} \approx 420\,\mathrm{cm}^{-3}\,, \tag{1.222}$$

so we can look at the baryon to photon number ratio:

$$\eta_0 = \frac{n_{0b}}{n_{0\gamma}} \approx 3 \times 10^{-8}\Omega_{0b}h^2\,, \tag{1.223}$$

why does this number have this value?

The denominator in $(n_b - n_a)/(n_b + n_a)$ is approximately $2n_\gamma$ then, and both the difference in the numerator and the numerator scale like a^{-3}, so this value is a constant. We do not see antimatter, therefore $n_a = 0$. So, we get

$$\frac{n_b - n_a}{n_b + n_a} \approx \frac{n_b}{2n_\gamma} = \frac{1}{2}\eta_0\,. \tag{1.224}$$

This must then have been the case also in the matter dominated epoch, during which there was a very slight imbalance in baryons vs antibaryons.

In order to generate a baryon-antibaryon asymmetry we need

1. ?
2. C and CP violation (while CPT symmetry must hold for any well-behaved QFT);
3. out-of-equilibrium processes.

These were proposed by a famous Soviet scientist.

We define the interaction rate Γ: it is the number of interactions per unit time.

The Hubble rate $H = \dot{a}/a$ is also an inverse time: baryons and antibaryons are practically speaking *decoupled* if $\Gamma \leq H$: this is equivalent to saying that the time of interaction is larger than the age of the universe.

When are particles actually coupled or decoupled? Γ can be calculated as $\Gamma = n \langle \sigma v \rangle$, where n is the number density, v is the velocity of the particles, and σ is the cross section of the interaction.

We need to distinguish the types of interactions we are dealing with. In general interactions are carried by gauge bosons. Either they are massless (like the photon) or they are massive (like the the $W^{\pm}$ and Z bosons, as long we are below the scale of electroweak symmetry breaking $\sim 10^2\,\text{GeV}$).

At larger energies than those, the weak interaction also becomes long-range.

In the massless case, the cross-section is $\sigma \sim \alpha^2/T^2$, where $g = \sqrt{4\pi\alpha}$.

In the massive case, for temperatures $T \leq m_x$, the cross section is of the order $\sigma \sim G_x^2 T^2$.

There is an inversion in the T dependence between long and short range interactions.

Typically $G_x = \alpha/m_x^2$. Then, $\sigma \sim \alpha^2 T^2/m_x^4$.

This difference is because in general the formula is like

$$\sigma \sim \frac{T^2}{E^4}\,, \tag{1.225}$$

and we either have $E \sim m$ in the low-speed case, or $E \sim T$ in the high-speed case.

So, we know that $\Gamma = n \langle \sigma v \rangle$. In the massless case this is something like $\Gamma \sim T^3 \sigma \sim T^3 \sim \alpha^2 T$ since $T \sim 1/a$.

Then, we get

$$H = \sqrt{\frac{8\pi G}{3}} g_*^{1/2} \left(\frac{\pi^2}{30}\right)^{1/2} T^2 \sim \frac{T^2}{m_{\text{pl}}}\,, \tag{1.226}$$

since $G \sim 1/m_{\text{pl}}^2$. Then,

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 T m_{\text{pl}}}{T^2} \sim \alpha^2 m_{\text{pl}} \frac{1}{T}\,, \tag{1.227}$$

so we have decoupling when $T > \alpha^2 m_{\text{pl}}$. Essentially, at temperatures larger than $T = 10^{16}\,\text{GeV}$ this massless photon is decoupled.

"Above the Planck epoch, even gravitational interactions are decoupled".

What about the massive interactions? We have $\Gamma \sim T^2 G_x^2 T^2$, to compare with $H \sim T^2/m_{\text{pl}}$: we get

$$\frac{\Gamma}{H} \sim \frac{T^3 G_x^2}{T^2/m_{\text{pl}}} \sim G_x^2 m_{\text{pl}} T^3 \leq 1\,, \tag{1.228}$$

equivalently, $T < m_{\rm pl}^{-1/3} G_x^{-2/3}$.

Suppose that we are considering the gravitational interaction: in that case, we get $T < m_{\rm pl}$ since G_x is related to G_N

What is the relation?

For the weak interaction, we have

$$T < \left(\frac{m_x}{100\,\text{GeV}} \right)^{4/3} \text{MeV}\,, \tag{1.229}$$

which is why below 1 MeV neutrinos are decoupled.

Let us now consider the consequences of these decoupling conditions. First of all we look at the recombination of hydrogen. At very high temperatures, there are free electrons and free protons. Protons first appeared in the universe as non-relativistic, at $T \sim 1\,\text{MeV}$ while $m_p \sim 1\,\text{GeV}$.

At a certain point, it becomes possible to create neutral H atoms from these free particles. We will use a special case of the Boltzmann formula, which governs this and many other phenomena: the Saha equation.

The reaction is $e + p \leftrightarrow H + \gamma$. We want to look at a density in phase space. We'd need all the scattering matrices, and all the phase space densities of the particles. The Saha equation is basically an ansatz at thermal and chemical equilibrium: $\mu_e + \mu_p = \mu_H + \mu_\gamma$. The chemical potential μ entered in the exponent of the FD and BE expressions.

We know that $\mu_\gamma = 0$. At thermal equilibrium the number density of the electrons is:

$$n_e = g_e \left(\frac{m_e T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_e - m_e}{T} \right), \tag{1.230}$$

and an exactly analogous formula holds for n_p and n_H: for protons we have

$$n_p = g_p \left(\frac{m_p T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_p - m_p}{T} \right). \tag{1.231}$$

Degeneracy is not an issue, since we are talking about cosmology.

The number density of photons is given by

$$n_\gamma = \frac{2\zeta(3) T^3}{\pi^2}\,, \tag{1.232}$$

The binding energy of the hydrogen is $B = m_p + m_e - m_H = 13.6\,\text{eV}$. Instead of m_H we write $+m_p + m_e - B$. So the number density of hydrogen atoms is given by:

$$n_H = g_H \left(\frac{m_H T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_p - \mu_e - m_p - m_e + B}{T} \right), \tag{1.233}$$

and we can simplify things since $m_p, m_H \gg m_e, B$, and substitute in the number densities for electrons and protons. We get approximately

$$n_H = \frac{g_H}{g_e g_p} n_e n_p \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp(B/T) \,, \tag{1.234}$$

but the universe is locally and globally neutral: $n_e = n_p$, and $n_b = n_p + n_H$.

We will see that most of the hydrogen we produce will not be when the temperature is of the order of the binding energy, but much later.

Fri Nov 15 2019

We start again from where we left off, with hydrogen recombination.

We want to estimate the moment at which hydrogen first formed, which marks the point at which electrons and photons interact efficently: before, they interacted with Compton scattering which is very efficient; after they interact with hydrogen atoms in a way that is very inefficient.

After this, then, we say that photons and matter are *decoupled*.

The scattering cross section (for Compton?) goes like the inverse square of the mass.

This means that the universe is not only *globally*, but also *locally* neutral.

This decoupling is what allows for star formation. Also, this is when the CMB starts. It is made of microwaves now, but it was higher earlier.

The phase space distribution of photons is scale-invariant since they have zero mass: so we can say that the photons' distribution *looks* thermal, but it is actually not technically since there are no interactions anymore.

However the photons travel freely and are perceived as thermal, and they give an almost perfect blackbody! The errorbars in a plot for it must be magnified by 10^4 in order to be seen.

We have $\Gamma_\gamma = n_e \sigma_T$, where σ_T is the Thompson cross section. Neutrality implies $n_e = n_p$. In principle we should account for Helium: a couple minutes after the BB He-4 nuclei started to form, but they made up only something like 25% of the mass, which means 6% of the number density: so we say that the number density of baryons is

$$n_b = n_p + n_H \,. \tag{1.235}$$

Our ansatz for the Boltzmann equation is $\mu_e + \mu_p = \mu_H$ since photons have no chemical potential.

i denotes a generic one in e, p and H. Then

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T} \right) , \tag{1.236}$$

we need to account for the chemical potential since it is the driver of this process. How much is n_e/n_b? the same as n_p/n_b. We call this quantity X_e, the ionization number. We expect $X_e = 1$ in the early universe, and at the end of the process it will diminish: naively we'd expect it to get to 0, but actually there remains some residual ionization, some free protons and electrons. A proper calculation would account for the non-equilibrium contributions. However, we estimate the process as being in equilibrium: this will underestimate the number of electrons.

Today, most of the hydrogen is ionized (there is a ***-Peterson effect which shows this): this means that matter and radiation interact again.

This is the second important time in the history of the universe.

Can we see the early stars? Not really, we see galaxies only up to $z \sim 10$, these stars would be at something like $z \sim 30$... There might be more to this.

Let us come back to the calculation:

$$n_H = g_H \left(\frac{m_H T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_H - m_H}{T} \right) \tag{1.237a}$$

$$= g_H \left(\frac{m_H T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_e + \mu_p - m_e - m_p + B}{T} \right) \tag{1.237b}$$

$$= \frac{g_H}{g_e g_p} \cancel{\left(\frac{m_H T}{2\pi} \right)^{3/2}} \left(\frac{m_e T}{2\pi} \right)^{-3/2} \cancel{\left(\frac{m_p T}{2\pi} \right)^{-3/2}} n_e n_p \exp\left(\frac{B}{T} \right) \tag{1.237c}$$

$$\frac{n_H}{n_e n_p} = \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) \tag{1.237d}$$

$$\frac{n_b - n_p}{n_p^2} = \left(\frac{m_H T}{2\pi} \right)^{3/2} \exp\left(\frac{B}{T} \right), \tag{1.237e}$$

which we can manipulate, using the following identity:

$$\frac{n_b - n_p}{n_p^2} = \frac{n_b \Big(1 - n_p/n_b \Big)}{n_b^2 X_e^2} = \frac{1}{n_b} \frac{1 - X_e}{X_e}, \tag{1.238}$$

where we use: $n_e = n_p$, and the definition of $X_e = n_p/n_b$. Then, we bring the n_b to the other side of the equation: we get

$$\frac{1 - X_e}{X_e^2} = \underbrace{\frac{n_b}{n_\gamma}}_{\eta_p} \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) n_\gamma \tag{1.239a}$$

$$= \frac{4\sqrt{2}\zeta(3) T^3}{\pi^2} \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) \eta_0 \tag{1.239b}$$

$$= \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta_0 \left(\frac{T}{m_e} \right)^{3/2} \exp\left(\frac{B}{T} \right), \tag{1.239c}$$

which means $T \sim 0.3\,\mathrm{eV}$, much lower than the ionization energy of Hydrogen.

This does depend on the value we assign to Ω_0 and h.

Approximately, it occurred somewhere around $z \sim 1100$ (we conventionally say that recombination happened when $X_e = 0.1$).

After recombination, we have the *last scattering*: the moment at which the CMB was formed.

One Nobel prize this year was awarded to Jim Peebles, a friend of Sabino's: together with his PhD supervisor Dicke, he was the first to calculate this stuff.

Peebles in 1964 (?) did this calculation both in GR and in Brahms-Dicke theory, a modified gravity theory.

Let us describe the early universe, before the first nucleosynthesis. Important papers in this topic are by G. Gamow, and by Alpher, Bethe and Gamow.

Our hypotheses are:

1. the universe passed through a very high temperature phase, with $T > 10^{12}\,\mathrm{K}$;

2. the universe is described by GR and SM;

3. the chemical potentials for the neutrinos μ_ν have certain upper bounds;

4. there is no matter-antimatter separation (as in, "bubbles");

5. there are no strong magnetic fields;

6. the number of exotic particles has a certain upper bound.

There are magnetic fields in the universe, but they are not homogeneous and relatively weak. Exotic particles are predicted by certain unification theories, they are generically defined as ones which we have not observed yet.

We have to explain the fact that we observe an excess of He-4 in the early universe: we define the yield

$$y \equiv \frac{m_{\mathrm{He-4}}}{m_b} > 0.25\,. \tag{1.240}$$

In terms of particle number, the ratio is more like 0.06. We do not produce Carbon or anything higher than it: the process which forms it is inefficient at high temperature, low density like the early universe. Higher Z elements are only produced in stars.

Helium-4 is produced but also destroyed by stars.

The main channels in the early universe are:

1. $n + p \leftrightarrow d + \gamma$ (d denotes deuterium);

2. $d + d \leftrightarrow {}^3\mathrm{He} + n$;

3. $^3\text{He} + d \leftrightarrow {}^4\text{He} + p$.

There is no weak interaction here, unlike what happens in stars. In stars, there are no free neutrons so this process is not possible.

We cannot treat this properly, we only give a story. The slowest process of the three is the first, since it is heavily affected by photons: they destroy deuterium.

The binding energy of deuterium is around 2.2 MeV. After we have produced deuterium, Helium-4 is readily produced.

First of all, we need the neutron to proton ratio. We are working at energies of around 1 MeV, so protons and neutrons are not relativistic anymore. This process takes place around three minutes after the beginning. For $i = n, p$:

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i m_i}{T} \right), \tag{1.241}$$

so their number ratio is around:

$$\frac{n}{p} \sim \exp\left(\frac{m_p - m_n}{T} \right), \tag{1.242}$$

where $m_p - m_n \approx 1.3\,\text{MeV} \approx 1.5 \times 10^{10}\,\text{K}$.

We have the processes

1. $n + \gamma_e \leftrightarrow p + e^-$;
2. $n + e^+ \leftrightarrow p + \overline{\nu_e}$;
3. $n \to p + e^- + \overline{\nu_e}$.

We can replace the temperature in the exponential by $T \to T_{d_\nu}$, the decoupling temperature of the neutrinos, since that is the moment around which this happpens.

So, we get around $\exp(-1.5)$. We define:

$$X_n(t) \equiv \frac{n}{n+p} \sim 0.17\,. \tag{1.243}$$

Later it will change because of β decay, going like:

$$X_n(t) = X_n(t_{d_\nu}) \exp\left(-\frac{t - t_{d_\nu}}{\tau_n} \right), \tag{1.244}$$

where $\tau_n = \log 2 \tau_{1/2}$, and this half-life is $\tau_{1/2} \approx 10.5 \pm 0.2$.

The binding energy of deuterium is around 2.2 MeV. Then, we proceed exactly like we did with hydrogen, and finally get

$$X_d = \frac{3}{4} n_b X_n X_p \left(\frac{m_d}{m_n m_p} \right)^{3/2} \left(\frac{T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) \tag{1.245a}$$

$$= \frac{3}{4}\eta_0 X_n X_p \left(\frac{m_d}{m_n m_p}\right) \frac{2\zeta(3)}{\pi^2} (2\pi)^{3/2} T^{3/2} \exp\left(\frac{B}{T}\right) \tag{1.245b}$$

$$\approx \frac{3}{4}\eta_0 X_n (1 - X_n) \left(\frac{m_d}{m_n m_p}\right)^{3/2} \frac{2}{\pi^2} (2\pi)^{3/2} \zeta(3) T^{3/2} \exp\left(\frac{B_d}{T}\right), \tag{1.245c}$$

Check calculation.

which describes the *deuterium bottleneck*, which is what impedes this process until photons are very diluted. As soon as photons are diluted enough, they stop bottlenecking.

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We exponentiate the equation from before: we get

$$X_d = X_n X_p \exp\left(-29.33 + \frac{25.82}{T_\rho} - \frac{3}{2}\log T_\rho + \log(\Omega_0 h)\right), \tag{1.246}$$

where $T_\rho =$

What is going on? What is T_ρ?

We want to understand why there is so much He-4 in the universe, since it is destroyed in stars!

This model fits observation as long as $0.011h^{-2} \leq \Omega_0 \leq 0.25h^{-2}$. Most people agree that we are around the upper bound.

This is indirect evidence for dark energy: why?

The lifetime of the neutron, $\tau_{1/2}$ is something that is also relevant, since it affects the baryon ratios.

The Gamow factor Γ is proportional to the Fermi coupling constant G_F^2, which is connected to $\tau_{1/2}$.

Let us suppose we increase the lifetime of neutrons, $\tau_{1/2}$. This changes the moment at which we reach equation $\Gamma \sim H$.

Increasing the lifetime of neutrons decreases the amount of He-4 in the universe: less is produced.

We know that

$$H^2 = \frac{8\pi G}{3}\rho_r, \tag{1.247}$$

where $G = 1/m_p^2$ and $\rho = \frac{\pi^2}{30} g_*(T) T^4$.

If we fix the temperature, and change g_* (by adding degrees of freedom), then we get more He-4.

This gives us observational constraints on the additions of exotic particles to our theory, since that would change g_*. This bounds the number of neutrino families by 3.0 something, so there cannot be more than 3 families of light neutrinos.

If gravitons are thermal, then they also contribute to radiation.

There is also another parameter in the Friedmann equation; it is m_P: modified gravity theories often predict variations of the gravitational constant with time.

Dark matter has no relevant electromagnetic interactions: it only interacts gravitationally, and is able to cluster; dark energy, instead, is uniformly distributed.

We divide it into Hot and Cold dark matter: HDM and CDM. There is also something called *Warm* dark matter, which has intermediate properties.

In HDM, particles have very high thermal motion. They move fast, and tend to destroy gravitational potential wells in which they might settle by moving out of them, and thus decreasing the quantity of matter there.

They do this on scales comparable to the maximum distance travelled by them: this is calculated as vt, where v is their average thermal velocity, (and t is the age of the universe?).

The structures formed by these are of scales similar to or larger than $10^{15} M_\odot$, but we observe smaller structures also! They were formed later, by fragmentation: this is the top-down approach.

We also have a bottom-up approach, which is compatible with CDM.

Neutrinos were thought to be Dark Matter, and would have been hot.

The top-down approach, however, is falsified by the observation of high-redshift quasars combined with the scale of the anisotropies of the CMB: in order to account fo high-redshift small-scale structures (we have seen stars at $z \sim 20$!) we would have to increase the amplitude of the anisotropies to a scale which is not compatible to the anisotropies we see in the CMB.

Now, neutrinos are not useful for cosmology.

We have $\Gamma \sim T^5$, and $\Gamma = H$: $T^5/\tau_{1/2} \sim T^2$ implies that $\tau_{1/2} \sim T^3$.

The decouplig temperature for HDM is larger than the temperature at which they become nonrelativistic, which is of the order of the mass.

For CDM, instead, the decoupling temperature is *smaller* than the temperature at which they become relativistic.

Now, we discuss the Boltzmann equation: there is an operator acting in phase space, the Liouville operator, which is equal to the collision operator.

$$\mathbb{L}[f] = \mathbb{C}[f], \tag{1.248}$$

where all the scattering operators live on the RHS.

We start with a Newtonian description: the phase space has position, momentum and time as coordinates, and on it we define a density function $f(\vec{q}, \vec{p}, t)$.

The Einstein equations are blind to the momentum distribution, since $T_{\mu\nu}$ does not depend on the momentum. We can say that all of the momentum has been marginalized.

We define the operator

$$\hat{\mathbb{L}} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \nabla_x + \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \cdot \nabla_p \tag{1.249a}$$

$$= \frac{\partial}{\partial t} + \vec{v} \cdot \nabla_x + \frac{\vec{F}}{m} \cdot \nabla_p \,. \tag{1.249b}$$

In GR, we geometrize the gravitational force: it is included in the inertial motion of the particles.

The relativistic version of this is

$$\hat{\mathbb{L}} = p^\alpha \partial_\alpha - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \,. \tag{1.250}$$

This is because $p^\alpha = \mathrm{d}x^\alpha / \mathrm{d}\lambda$ satisfies the geodesic equation:

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma = 0 \,, \tag{1.251}$$

or $\frac{Dp^\alpha}{Dt} = 0$. We can then write down the Christoffel bit of the geodesic equation instead of the derivative $\mathrm{d}p^\alpha / \mathrm{d}\lambda$. This is because, *on shell*:

$$g^{\alpha\beta} p_\alpha p_\beta = m^2 \,. \tag{1.252}$$

We want to describe the *abundance* of dark matter, in phase space: the number of DM particle per EM-interacting particle.

We then make the Liouville operator explicit, with the Christoffel symbols of the RW metric. Acting on f, we get

$$\hat{\mathbb{L}} = p^0 \partial_t f - \frac{\dot{a}}{a} |\vec{p}|^2 \frac{\partial f}{\partial E} \,, \tag{1.253}$$

which we will not prove.

The derivative with respect to the momentum must be ∂_{p^0},

because of isotropy?

the Christoffel symbols of RW are

$$\Gamma^0_{\beta\gamma} = \frac{\dot{a}}{a} \delta_{\beta\gamma} \,. \tag{1.254}$$

Recall the definition of the number density

$$n(t) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3\vec{p} \, f(|\vec{p}|, t) \,, \tag{1.255}$$

so in the end the equation becomes:

$$\hat{\mathbb{L}} = \dot{n} + 3\frac{\dot{a}}{a}n\,. \tag{1.256}$$

Conventionally we divide by p^0: we get for the Boltzmann equation:

$$\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}\frac{p^2}{E}\frac{\partial f}{\partial E} = \frac{1}{E}\hat{\mathbb{C}}[f]\,. \tag{1.257}$$

Now we integrate in $\mathrm{d}^3\vec{p}$. We get

$$\frac{\partial n}{\partial t} - \frac{\dot{a}}{a}\int \mathrm{d}^3\vec{p}\,\frac{p^2}{E}\frac{\mathrm{d}f}{\mathrm{d}E} = \frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\,\frac{1}{E}\mathbb{C}[f]\,. \tag{1.258}$$

More properly, we are doing $\mathbb{L}\left\langle f \right\rangle$.

We manipulate:

$$\int \mathrm{d}^3 p\,\frac{p^2}{E}\frac{\partial f}{\partial E} = 2\int \mathrm{d}^3 p\, p^2\frac{\partial f}{\partial E^2} \tag{1.259a}$$

$$= 2\int \mathrm{d}^3 p\, p^2\frac{\partial f}{\partial p^2} = \int \mathrm{d}^3 p\, p\frac{\partial f}{\partial p} \tag{1.259b}$$

$$= 2\pi\int_0^\infty \mathrm{d}p\, p^3\frac{\partial f}{\partial p} = -3\int \mathrm{d}^3 p\, f\,. \tag{1.259c}$$

Here, we integrated by parts in the second to last step, and set to zero the boudary term $4\pi p^3 f$, calculated from 0 to infinity, since at 0 we have $p = 0$, and at (momentum) infinity we have $f = 0$.

So, we can see that the LHS is equal to $\dot{n} + 3\dot{a}n/a$: so we find

$$\dot{n} + 3\frac{\dot{a}}{a}n = \frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\,\frac{1}{E}\hat{\mathbb{C}}[f]\,. \tag{1.260}$$

This is the cosmological version of the Boltzmann equation.

We model the RHS as something like

$$\hat{\mathbb{C}}[f] = \Psi - \left\langle \sigma v \right\rangle n^2\,, \tag{1.261}$$

where we have $\Gamma_A = \left\langle \sigma v \right\rangle n$ times n, where Γ is the rate of annihilation.

At equilibrium the LHS is equal to zero: so we can write $\Psi = \left\langle \sigma v \right\rangle n_{\text{eq}}^2$, and the RHS becomes $\left\langle \sigma v \right\rangle \left(n_{\text{eq}}^2 - n^2\right)$.

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Yesterday we arrived at the following equation:

$$\dot{n} + 3\frac{\dot{a}}{a}n = \Psi - \langle \sigma_A v \rangle n^2 \,, \tag{1.262}$$

where the term Ψ is the source of new particles, while the next term accounts for the annihilation of particles.

Under decoupling, the equation reduces to

$$\dot{n} + 3\frac{\dot{a}}{a}n = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(na^3\right) = 0\,. \tag{1.263}$$

Under equilibrium, $\Gamma > H$.

Therefore $\Psi = \langle \sigma_A v \rangle n_{\text{eq}}^2$. Recall that the Γ of annihilation is equal to $\Gamma = \langle \sigma_A v \rangle n$.

If the timescale of the collisions is longer than the age of the universe, $\Gamma < H$, then once again we have $a^3 n = \text{const}$.

We can see thermal *relics*, thermal distributions of objects which are not coupled anymore.

We define

$$n_C = n\left(\frac{a}{a_0}\right)^3 \,, \tag{1.264}$$

so we can simplify the expression:

$$\dot{n} + 3\frac{\dot{a}}{a}n = \dot{n}\left(\frac{a_0}{a}\right)^3 \,. \tag{1.265}$$

We want to factor our a parameter H. We get:

$$\dot{n}_C = -\left(\frac{a}{a_0}\right)^3 \langle \sigma_A v \rangle \left(\frac{a_0}{a}\right)^6 \left(n_C^2 - n_{\text{eq}}^2\right) \,, \tag{1.266}$$

which can be written as

$$\frac{a}{n_{\text{C, eq}}}\frac{\mathrm{d}n_C}{\mathrm{d}a} = -\frac{\langle \sigma_A v \rangle n_{\text{eq}}}{\dot{a}/a}\left(\left(\frac{n}{n_{\text{eq}}}\right)^2 - 1\right) \,, \tag{1.267}$$

so, if we define τ_{coll} as $1/(\langle \sigma_A v \rangle n_{\text{eq}})$ and $\tau_{\text{exp}} = 1/H$, we get

$$n_{\text{C, eq}} a \frac{\mathrm{d}n_C}{\mathrm{d}a} = -\frac{\tau_{\text{exp}}}{\tau_{\text{coll}}}\left(\left(\frac{n}{n_{\text{eq}}}\right)^2 - 1\right) \,, \tag{1.268}$$

so if $\Gamma \gg H$, then $\tau_{\mathrm{exp}}/\tau_{\mathrm{coll}} \gg 1$, therefore $n = n_{\mathrm{eq}}$, which also implies $n_{\mathrm{C}} = n_{\mathrm{C,\,eq}}$. This is the equilibrium case.

On the other hand, if $\Gamma \ll H$, then $\tau_{\mathrm{exp}}/\tau_{\mathrm{coll}} \ll 1$ we have decoupling, therefore $n_C = \mathrm{const}$.

We know that at temperatures below 1.5 MeV (after decoupling), the following holds:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma\,, \tag{1.269}$$

and we want to do the same for dark matter.

Neutrinos are nonrelativistic today, but they became so at a relatively low redshift, a short time ago. We can parametrize the number density of neutrinos by the temperature.

The formula for the temperature of neutrinos is a special case of the following formula:

$$T_{0\nu} = \left(\frac{g_{*0}}{g_{*d}}\right)^{1/3} T_{0\gamma}\,, \tag{1.270}$$

where 0 means *now*, while d means *decoupling*. This can be applied to any species.

Let us compute this for a generic species x: we find

$$n_{0x} = B g_* \frac{\zeta(3)}{\pi^2} T_{0x}^3\,, \tag{1.271}$$

where the factor B accounts for the statistics: it is 1 for bosons, $3/4$ for fermions. It is important to note that the temperature is a parameter, but these particles are *not thermal* anymore!

So, we get, using equation (1.232):

$$n_{0x} = \frac{B}{2} n_{0\gamma} g_x \frac{g_{*0}}{g_{*dx}}\,. \tag{1.272}$$

The energy density in general is given by $\rho_{0x} = m_x n_{0x}$. We get:

$$\rho_{0x} = \frac{B}{2} m_x n_{0\gamma} g_x \frac{g_{0x}}{g_{*gx}}\,, \tag{1.273}$$

therefore

$$\Omega_{0x} h^2 = \frac{m_x n_{0x}}{\rho_{0x}} h^2 = 2 B g_x \frac{g_{*0}}{g_{*dx}} \frac{m_x}{10^2\,\mathrm{eV}}\,. \tag{1.274}$$

CDM is made of particles which were already nonrelativistic when they decoupled.

Then, in the formula

$$n_x(T_{dx}) = g_x\left(\frac{m_x T_{dx}}{2\pi}\right)^{3/2} \exp\left(-\frac{m_x}{T_{dx}}\right), \tag{1.275}$$

we can assume that $T_{dx} \ll m_x$. So,

$$n_{0x} = n_x(T_{dx})\left(\frac{a(T_{dx})}{a_0}\right)^3 = n(T_{dx})\frac{g_{*0}}{g_{*x}}\left(\frac{T_{0\gamma}}{T_{dx}}\right)^3, \tag{1.276}$$

but it is difficult to compute T_{dx}, which is the solution to the equation $\Gamma(T_{dx}) = H(T_{dx})$.

We know that

$$H^2(T_{dx}) = \frac{8\pi G}{3} g_{*x}\frac{\pi^2}{30}T_{dx}^4\,, \tag{1.277}$$

or, in terms of the quantity $\tau_{\text{exp}} = 1/H$:

$$\tau_{\text{exp}} = 0.3 g_*^{-1/2}\frac{m_{\text{pl}}}{T_{dx}^2}\,. \tag{1.278}$$

On the other hand, $\Gamma = n\,\langle\sigma_A v\rangle$, and $\tau_{\text{coll}}(T_{dx})$, and in terms of τ_{coll} we get

$$\tau_{\text{coll}}(T_{dx}) = \left(n(T_{dx})\sigma_0\left(\frac{T_{dx}}{m_x}\right)^N\right), \tag{1.279}$$

where $N = 0, 1$: it is a fact from particle physics that the average has this kind of dependence:

$$\langle\sigma_A v\rangle = \sigma_0\left(\frac{T}{m_x}\right)^N. \tag{1.280}$$

So, equaling the two τ we get:

$$\left(n(T_{dx})\sigma_0\left(\frac{T_{dx}}{m_x}\right)^N\right) = 0.3 g_*^{-1/2}\frac{m_{\text{pl}}}{T_{dx}^2}, \tag{1.281}$$

which can be solved iteratively. We solve it in terms of the parameter $x_{dx} = m_x/T_{dx}$, which must be much larger than one: this allows us to select the physical solution to the equation among the nonphysical ones.

The solution is found to be something like:

$$x_{dx} = \log\left(0.038\frac{g_x}{g_{*xd}^{1/2}} m_{\text{pl}} m_x \sigma_0\right) - \left(N - \frac{1}{2}\right)\log\log(\dots)\,. \tag{1.282}$$

Chapter 2

Stellar Astrophysics

Dark matter is collisionless, however it is not *really*: the censorship theorem says that it cannot actually collapse into a naked singularity.

A star is a sphere of matter, usually baryonic matter, characterized by a density $\rho(r)$: the mass enclosed in a radius r is

$$m(r) = \int_0^r 4\pi\widetilde{r}^2\rho(\widetilde{r})\,\mathrm{d}\widetilde{r}\,. \tag{2.1}$$

The gravitational acceleration can be calculated from Gauss' theorem:

$$g(r) = \frac{Gm(r)}{r^2}\,. \tag{2.2}$$

Let us consider a small volume, with its enclosed mass $\Delta M = \rho(r)\Delta A\Delta r$. It can contrast the inward force due to gravity if there is a differential pressure.

Let us denote P as the pressure at the inner surface, and $P+\Delta P$ the pressure at the outer surface. Then,

$$(P+\Delta P)\Delta A - P\Delta A = \left(P(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\Delta r\right)\Delta A - P(r)\Delta A = \frac{\mathrm{d}P}{\mathrm{d}r}\Delta A\Delta r\,. \tag{2.3}$$

The minus sign in what follows is since the force is inward:

$$-\Delta M\ddot{r} = \Delta M g(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\Delta r\Delta A\,. \tag{2.4}$$

The equation of motion is

$$-\ddot{r} = g(r) + \frac{1}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\,, \tag{2.5}$$

so we see that the pressure gradient must have a minus sign.

If the internal energy is used up to do internal (chemical, nuclear) work, then it cannot support the star anymore, and it then collapses.

Usually, stars start from the Jeans phenomenon: dark matter and baryons collapse under their own weight.

Let us start at the decoupling of matter and radiation, something like $z = 1100$.

The clouds of matter collapse freely up to the moment at which their internal pressure starts to slow them down. Then, we get the necessary conditions for the fusion of Hydrogen.

During freefall, we do not have a pressure gradient, so we have

$$-\ddot{r} = g(r)\,. \tag{2.6}$$

It is not generally the case, but let us suppose that the collapse is *orderly*, the interior collapses before the outer layers. We have to account for energy conservation: the total energy when our test shell is at a radius r is conserved as r changes. If the energy (per unit mass) is zero at infinity we have

$$\cancel{E_{\text{tot}}(r_0)} - \frac{Gm_0}{r_0} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{Gm_0}{r}\,, \tag{2.7}$$

therefore

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = Gm_0\left(r^{-1} - r_0^{-1}\right). \tag{2.8}$$

What is the freefall time? we take $\mathrm{d}t/\mathrm{d}r$ from the equation an integrate it from r_0 to 0

$$t_{\text{free fall}} = \int_{r_0}^{0} \mathrm{d}r\,\frac{\mathrm{d}t}{\mathrm{d}r} = -\int_{r_0}^{0} \mathrm{d}r\,\frac{1}{2GM}\left(\frac{1}{r} - \frac{1}{r_0}\right)^{-1/2}\,, \tag{2.9}$$

where we have a minus sign since $\mathrm{d}r/\mathrm{d}t < 0$. We get

$$t_{\text{free fall}} = \sqrt{\frac{r_0^3}{2Gm_0}} \int_0^1 \mathrm{d}x\,\sqrt{\frac{x}{1-x}} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2Gm_0}}\,, \tag{2.10}$$

and if we define the average density $\bar{\rho} = m_0/\left(4\pi r_0^3/3\right)$ we get the simpler expression

$$t_{\text{free fall}} = \sqrt{\frac{3\pi}{32G\bar{\rho}}}\,. \tag{2.11}$$

We might be tempted to ignore the expansion of the universe in these calculations: we know that

$$H^2 = \frac{8\pi G}{3}\bar{\rho}\,, \tag{2.12}$$

and in the purely Newtonian case we know that $a(t) \propto t^{2/3}$ and $H = 2/(3t)$.

Therefore

$$\frac{4}{3t^2} = \frac{8\pi G}{3}\bar{\rho} \implies t^2 = \frac{4}{9}\frac{3}{8\pi G}\bar{\rho}^{-1}, \tag{2.13}$$

so

$$t_{\text{exp}} \propto \bar{\rho}^{-1/2}. \tag{2.14}$$

what is the relation between the universe's density and the star's?
Is the idea: the matter in the universe does not disperse too fast, stars theoretically are allowed to form?

At equilibrium, $\ddot{r} = 0$: so

$$-\ddot{r} = 0 = g(r) + \frac{1}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{2.15}$$

so hydrostatic equilibrium implies that, locally,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G\frac{m(r)\rho(r)}{r^2}. \tag{2.16}$$

We multiply both sides by $4\pi r^3$ and integrate in $\mathrm{d}r$: we get

$$\int_0^R \mathrm{d}r\, 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r} = -G\int_0^R \frac{m(r)\rho(r)4\pi r^2}{r}\,\mathrm{d}r\ , \tag{2.17}$$

and we can change variables: $\rho(r)4\pi r^2\,\mathrm{d}r = \mathrm{d}m$, so we can identify the LHS with the total gravitational energy:

$$E_{\text{grav}} = -G\int \frac{m(G)\,\mathrm{d}m}{r}. \tag{2.18}$$

On the RHS, instead, we can integrate by parts:

$$\Big[P(r)4\pi r^3\Big]_0^R - 3\int_0^R \mathrm{d}r\, 4\pi r^2 P(r), \tag{2.19}$$

where the boundary term is zero: at the origin $r = 0$, at the surface (by definition) $P = 0$.

So, inserting the volume $V(R) \equiv 4\pi R^3/3$, we have

$$E_{\text{grav}} = -3\int_0^R \frac{\mathrm{d}r\, 4\pi r^2 P(r)}{V(R)} V(R), \tag{2.20}$$

which gives us the virial theorem: we can interpret the integral as a weighted average, so we get

$$E_{\text{grav}} = -3\,\langle P\rangle\, V \qquad \text{or} \qquad \langle P\rangle = -\frac{1}{3}\frac{E_{\text{grav}}}{V}. \tag{2.21}$$

This is Newtonian: above a certain limit, the relativistic corrections destabilize the system.

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We derived

$$\langle T \rangle = -\frac{1}{3}\frac{E_{\text{grav}}}{V}\,, \tag{2.22}$$

so now we proceed: we want a relation between kinetic and gravitational energy densities.

We use a statistical mechanics approach.

The rate of momentum transfer in the direction x is given by

$$\frac{N}{L^3} p_x v_x\,, \tag{2.23}$$

and this will be true for either direction by isotropicity: so

$$P = \frac{n}{3}\left\langle \vec{p} \cdot \vec{v} \right\rangle\,, \tag{2.24}$$

in full generality.

Let us consider the two cases of nonrelativistic and fully relativistic. In the nonrelativistic case we have

$$\epsilon_p = mc^2 + \frac{p^2}{2m}\,, \tag{2.25}$$

where $p = mv$. In the ultrarelativistic case we have

$$\epsilon_p = pc\,, \tag{2.26}$$

and the velocity is approximately the speed of light.

Then, for a gas of nonrelativistic particles, we have

$$P = \frac{1}{3} nmv^2 = \frac{2}{3} n \left\langle \frac{1}{2} mv^2 \right\rangle = \frac{2}{3} \times \text{translational KE density}\,, \tag{2.27}$$

while in the relativistic case we have:

$$P = \frac{1}{2} n \left\langle pc \right\rangle = \frac{1}{3} \times \text{translational KE density}\,. \tag{2.28}$$

We will show that, if a star is made of a gas of classical nonrelativistic particles it tends to be stable, if the particles are relativistic then it tends not to be stable.

The virial theorem tells us that

$$2E_{\text{K}} + E_{\text{grav}} = 0\,, \tag{2.29}$$

in the nonrelativistic approximation.

We define: $\Delta E_{\text{tot}} = -\Delta E_{\text{K}} = \frac{1}{2}\Delta E_{\text{grav}}$.

We know that

$$\langle P \rangle = \frac{1}{3}\frac{E_{\text{K}}}{V} = -\frac{1}{3}\frac{E_{\text{grav}}}{V} \tag{2.30}$$

by the virial theorem: so the total binding energy is equal to zero, since this gives us

$$E_{\text{grav}} + E_{\text{K}} = E_{\text{tot}} = 0\,. \tag{2.31}$$

Now we differentiate the law $\mathrm{d}(PV^{\gamma}) = 0$, since it is a constant for an adiabatic transformation: it gives us, using logarithmic derivatives,

$$\gamma\frac{\mathrm{d}V}{V} + \frac{\mathrm{d}P}{P} = 0\,, \tag{2.32}$$

so

$$\mathrm{d}(PV) = -(\gamma - 1)P\,\mathrm{d}V\ , \tag{2.33}$$

and we know that for an adiabatic transformation

$$\mathrm{d}E_{\text{in}} + P\,\mathrm{d}V = 0\,, \tag{2.34}$$

which implies

$$\mathrm{d}E_{\text{in}} = \frac{1}{\gamma - 1}\,\mathrm{d}(PV)\ , \tag{2.35}$$

and let us assume that γ is approximately constant in the transformation: this means

$$E_{\text{in}} = \frac{PV}{\gamma - 1}\,, \tag{2.36}$$

so

$$P = (\gamma - 1)\frac{E_{\text{in}}}{V}\,, \tag{2.37}$$

which justifies the relations we used in cosmology, $P = w\rho$ with $w = \gamma - 1$.

We can rewrite the equation from before as

$$3(\gamma - 1)E_{\text{in}} + E_{\text{gr}} = 0\,, \tag{2.38}$$

which, together with $E_{\text{tot}} = E_{\text{in}} + E_{\text{gr}}$, give us

$$E_{\text{tot}} = -(3\gamma - 4)E_{\text{in}}\,, \tag{2.39}$$

which means that $\gamma > 4/3$ characterizes a bound system, while $\gamma < 4/3$ characterizes a free system.

There are two dangers: one is the fight against the pressure forces, one is the fight against the quantum forces (the Pauli exclusion principle) which do not allow the compression to happen further.

Now we discuss Jeans instability:

$$E_{\text{grav}} = -f \frac{GM^2}{R}, \tag{2.40}$$

where f is a numerical factor of the order 1, depending on the mass distribution. If the distribution is uniform, it is 3/2.

3/2?

The kinetic energy is

$$E_{\text{K}} = \frac{3}{2} N k_B T, \tag{2.41}$$

and we want to impose the condition

$$f \frac{GM^2}{R} > \frac{3}{2} N k_B T, \tag{2.42}$$

and the Jeans criterion is this boundary of the stability region:

$$f \frac{g M_J^2}{R} = \frac{3}{2} \frac{M_J}{\bar{m}} k_B T, \tag{2.43}$$

where $\bar{m} = M/N$. The J denotes the fact that we are considering the specific boundary mass on both sides. Simplifying the formula we find:

$$M_J = \frac{3}{2} \frac{k_B T}{G \bar{m}} R, \tag{2.44}$$

and we can reframe this in terms of the density, which is defined by

$$M_J = \frac{4\pi}{3} \rho_J R^3. \tag{2.45}$$

We cube and multiply on both sides:

$$\rho_J M_J^3 = \left(\frac{3 k_B T}{2 G \bar{m}} \right)^3 R^3 \rho_J, \tag{2.46}$$

so we get

$$\rho_J = \frac{1}{M_J^3} \left(\frac{3 k_B T}{G \bar{m}} \right)^3 \left(\frac{4\pi}{3} \rho_J R^3 \right) \frac{1}{4\pi}, \tag{2.47}$$

so

$$\rho_J = \frac{3}{4\pi M_J^2}\left(\frac{3k_BT}{2G\bar{m}}\right)^3, \tag{2.48}$$

and we will habe stability if the density is larger than this. So, if we want a collapse, we must decrease the mass...

When the last scattering happens, the pions are decoupled from the photons. Dark matter behaves differently from conventional matter.

We have

$$\dot{\rho}_r = -3H\left(\rho_r + P_r\right), \tag{2.49}$$

and

$$\dot{\rho}_m = -3H\left(\rho_m + P_m\right), \tag{2.50}$$

and $P_r = \rho_r/3$, which scale like a^{-4} and also as T^4, which means $T \sim 1/a$.

$$\mathrm{d}\left(\rho_m c^2 a^3\right) + P_m da^3 = 0\,, \tag{2.51}$$

where we usually approximate $\rho_m c^2 = m_p n_b c^2$, but we can include more terms:

$$\rho_m c^2 = m_p n_b c^2\left(1 + (\gamma - 1)^{-1}\frac{k_BT}{m_p c^2}\right), \tag{2.52}$$

while the pressure is given by $P = n_b k_B T$: so in the end we find

$$\mathrm{d}\left(\left(m_p n_b c^2 + \frac{3}{2} m_p n_b \frac{k_BT}{m_p}\right) a^3\right) = -n_b k_B T\,\mathrm{d}a^3\,, \tag{2.53}$$

which after some computation gives us

$$\frac{1}{2}\,\mathrm{d}T = -T\frac{\mathrm{d}a}{a}\,, \tag{2.54}$$

which implies $T_m \propto a^{-2}$ after baryogenesis.

This is for monoatomic baryonic matter, right?

Let us start writing equations for the stellar interior. The continuity equation is

$$\partial_t \rho + \nabla\cdot\left(\rho\vec{v}\right) = 0\,, \tag{2.55}$$

and also we have the Euler equation

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Phi \,, \tag{2.56}$$

which can be written using the convective time derivative:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + \vec{v} \cdot \nabla_x = u^\mu \partial_\mu \,, \tag{2.57}$$

which allows us to write

$$\frac{\mathrm{D}}{\mathrm{D}t} \rho + \rho \nabla \cdot \vec{v} = 0 \,, \tag{2.58}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}t} \vec{v} = -\frac{1}{\rho} - \nabla \Phi \,. \tag{2.59}$$

What happens to the entropy? we define the entropy density s by $S = s\rho$.
An isentropic process is one in which

$$\frac{\mathrm{D}S}{\mathrm{D}t} = 0 \,, \tag{2.60}$$

which in terms of the entropy density is

$$\partial_t s + \vec{v} \cdot \vec{\nabla} s = 0 \,. \tag{2.61}$$

The external force is provided by the gravitational field:

$$\nabla^2 \Phi = 4\pi G \rho \,. \tag{2.62}$$

Jeans looked for a simple solution, an ansatz, called the background solution and then tried to perturb it: if it is stable than it was a good solution.
He started with $\rho = \text{const}$, $\vec{v} = 0$, $s = \text{const}$, $\Phi = \text{const}$.
It is obviously wrong! It cannot satisfy the Poisson equation.
However, we start from it and add some $\delta\rho$, $\delta\vec{v}$ (which we just call $\vec{v}$), δs and $\delta\Phi$; then we only keep the linear terms in these perturbations.
We find:

$$\partial_t \delta\rho + \rho_0 \vec{\nabla} \cdot \vec{v} = 0 \,, \tag{2.63}$$

$$\partial_t \vec{v} = -\frac{1}{\rho_0} \nabla \delta P - \nabla \delta \Phi \,, \tag{2.64}$$

and

$$\nabla^2 \delta\Phi = 4\pi G \delta\rho \,, \tag{2.65}$$

and finally

$$\partial_t \delta s = 0 \,. \tag{2.66}$$

We can expand

$$\delta P = \frac{\partial P}{\partial \rho} \delta\rho + \frac{\partial P}{\partial s} \delta s \,, \tag{2.67}$$

and here we define

$$c_s^2 = \frac{\partial P}{\partial \rho} \,. \tag{2.68}$$

We will then consider an exponential solution:

$$\delta\rho = \delta\rho_0 \exp\left(i \left(\vec{k} \cdot \vec{x} - \omega r \right) \right) \,, \tag{2.69}$$

and similarly for $\vec{v}$, s, Φ.

We will see that we will need to stick to $\delta s = 0$, and find a dispersion relation with ω and $\vec{k}$: it will be

$$\omega^2 = c_s^2 \vec{k}^2 - 4\pi G \rho \,, \tag{2.70}$$

so if the wavenumber is small enough we will have an imaginary ω.

Fri Nov 29 2019

We want to derive the Jeans instability criterion, starting from the structure equations:

1. continuity;
2. momentum conservation;
3. Poisson for the gravitational field;
4. entropy conservation.

Yesterday there was a mistake: the entropy conservation is actually

$$\frac{\mathrm{D}s}{\mathrm{D}t} + s\nabla \cdot \vec{v} = 0\,. \tag{2.71}$$

There is no need to make an ansatz for the pressure, since we can compute $P = P(\rho, s)$: we have the relation seen last time, involving the speed of sound.

Our *ansatz* is

$$\delta x_i = x_{i0} \exp\Big(i\vec{k} \cdot \vec{x} - i\omega t\Big)\,, \tag{2.72}$$

with $x_i = \rho, \vec{v}, \Phi, s$.

Our equations become:

$$\begin{aligned} i\omega\delta\rho_0 + \rho_0 i\vec{k} \cdot \vec{v}_0 &= 0 & (2.73a)\\ i\omega\vec{v}_0 &= \frac{1}{\rho_0} i\vec{k}\left(c_s^2\delta\rho_0 + \frac{\partial P}{\partial s}\delta s_0\right) - i\vec{k}\delta\Phi_0 & (2.73b)\\ k^2\delta\Phi &= 4\pi G\delta\rho_0 & (2.73c)\\ \omega\delta s_0 &= 0\,, & (2.73d) \end{aligned}$$

so one class of solutions will have $\omega = 0$, that is, we consider time-independent ones:

$$\begin{aligned} \rho_0 i\vec{k} \cdot \vec{v}_0 &= 0 & (2.74a)\\ 0 &= \frac{1}{\rho_0}\vec{k}\left(c_s^2\delta\rho_0 + \frac{\partial P}{\partial s}\delta s_0\right) - \vec{k}\delta\Phi_0 & (2.74b)\\ k^2\delta\Phi &= 4\pi G\delta\rho_0\,, & (2.74c) \end{aligned}$$

and we have a result from Helmholtz: the fact that every velocity field can be decomposed into $\vec{v} = \nabla\Psi + \vec{T}$ with $\nabla \cdot \vec{T} = 0$.

As soon as we ask the perturbation to be time independent we only find vortical motions, with $\vec{v} \perp \vec{k}$.

Now we consider the velocity to be irrotational we consider the term $\delta s_0 = 0$, since $\omega \neq 0$ in general.

$$\begin{aligned} \omega\delta\rho_0 + \rho_0\vec{k} \cdot \vec{v}_0 &= 0 & (2.75a)\\ \omega\vec{v}_0 &= \frac{1}{\rho_0}\vec{k}c_s^2\delta\rho_0 - \vec{k}\delta\Phi_0 & (2.75b)\\ k^2\delta\Phi &= 4\pi G\delta\rho_0\,, & (2.75c) \end{aligned}$$

which we can write as a linear system for the vector $\delta\rho_0, v_0, \delta\Phi_0$.

The 5x5 coefficient matrix is:

$$\begin{bmatrix} \omega & \rho_0\vec{k} & 0 \\ \frac{1}{\rho_0}\vec{k}c_s^2 & \omega & \vec{k} \\ 4\pi G & 0 & k^2 \end{bmatrix}, \tag{2.76a}$$

which has determinant

$$\omega k^2 - \rho_0\vec{k}\cdot\left(\frac{1}{\rho_0}\vec{k}c_s^2 - 4\pi G\vec{k}\right) = 0\,, \tag{2.77}$$

which gives the dispersion relation $\omega^2 = c_s^2 k^2 - \rho_0 4\pi G$.

A solution will be a combination of these. If $\omega^2 < 0$, which can happen because of the minus sign. If that happens, instead of a propagating wave we have a standing wave.

We have $\omega = (4\pi G\rho_0)^{1/2}$, so the characteristic time is

$$\tau = \frac{1}{\sqrt{4\pi G\rho_0}}\,, \tag{2.78}$$

The free fall time can be computed to be:

$$\tau_{\text{free fall}} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2}. \tag{2.79}$$

We have a typical Jeans wavenumber:

$$k_J^2 = \frac{4\pi G\rho_0}{c_s^2}\,, \tag{2.80}$$

corresponding to when the frequency becomes imaginary.

We can compare this to a plasma of charged particles, with the electrostatic potential instead of the gravitational field. In that case we find

$$\omega^2 = c_s^2 k^2 + \frac{4\pi n_e e^2}{m_e}\,, \tag{2.81}$$

where n_e is the number density of electrons, m_e is the electron mass.

We have the following analogies:

$$n_e \to \rho_0/m \tag{2.82a}$$
$$m_e \to m \tag{2.82b}$$
$$e^2 \to Gm^2 \tag{2.82c}$$
$$+ \to -\,; \tag{2.82d}$$

the first of these are due to the inertial mass being equal to the gravitational mass; the plus becoming a minus is due to the fact that we do not have negative charge in the gravitational setting, so there cannot be a screening effect.

We now come back to the RW line element:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right), \tag{2.83}$$

in which the most important thing is that a is a function of time.

We will now use a Newtonian approach: we ignore spatial curvature, that is, set $k = 0$: this is good for small enough scales. The thing is: the time scales of gravitational collapse are long, and we cannot ignore the effects of spacetime expansion.

We use a coordinate defined by $\vec{r} = a(t)\vec{x}$, so that the line element becomes

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}|x|^2 - |x|^2\,\mathrm{d}\Omega^2\,. \tag{2.84}$$

right?

What will be done now could have been done by Newton: he did not just because he thought the universe was static. We have (dropping the vector sign, but still impying it):

$$u = \dot{r} = \dot{a}x + a\dot{x} = \frac{\dot{a}}{a}r + v\,, \tag{2.85}$$

where $v = a\dot{x}$ is called the peculiar velocity which galaxies and such can have.

Let us forget about the pressure gradient and just look at how the instability evolves: this will also apply to dark matter which has no pressure. We look at

$$\left[\frac{\partial\rho}{\partial t}\right]_{\vec{r}} + \nabla_{\vec{r}}(\rho\vec{u}) = 0\,, \tag{2.86}$$

the velocity equation is

$$\left[\frac{\partial\vec{u}}{\partial r}\right]_{\vec{r}} + (\vec{u}\cdot\nabla_{\vec{r}})\vec{u} = -\nabla_{\vec{r}}\Phi\,, \tag{2.87}$$

and finally

$$\nabla^2_{\vec{r}} = 4\pi G\rho\,. \tag{2.88}$$

Our expression for $\vec{u}$ is $H\vec{r} + \vec{v}$, also we have

$$\rho(\vec{r}, t) = \rho_b(t) + \delta\rho(\vec{r}, t)\,, \tag{2.89}$$

a background plus a perturbation.

To find the background, we assume that the density of matter is space independent but time dependent: then we find

$$\Phi_b(\vec{r}, t) = \frac{2\pi G}{3}\rho_b(t) r^2 \,, \tag{2.90}$$

which has gradient

$$\nabla_{\vec{r}}\Phi_b = \frac{4\pi G}{3}\rho_b(t)\vec{r} \,, \tag{2.91}$$

and laplacian

$$\nabla^2_{\vec{r}}\Phi_b = 4\pi G\rho_b \,, \tag{2.92}$$

but this diverges at $r \to \infty$! this is not an issue with the solution, but with Newtonian mechanics.

This is an elliptic differential equation: hyperbolic ones look like $\Box\Phi = 0$, elliptic ones like $\nabla^2\Phi = 0$, while parabolic ones (???)

We want to get equations in the comoving coordinates, not the local inertial ones. Take a generic function $f(\vec{r}, t)$, which can also be expressed with respect to $(\vec{x}, t)$.

Then, the difference between the derivatives is:

$$\left[\frac{\partial f}{\partial t}\right]_{\vec{x}} = \left[\frac{\partial f}{\partial t}\right]_{\vec{r}} + H(\vec{r} \cdot \nabla_{\vec{r}}) f \,. \tag{2.93}$$

Take the convective derivative

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t}_{\vec{r}} + \frac{\partial f}{\partial \vec{r}} \cdot \dot{\vec{r}} \,, \tag{2.94}$$

where $\dot{\vec{r}} = \vec{u} = H\vec{r} + \vec{v}$. From what we saw, this can be expressed as

$$\frac{\partial f}{\partial t}_{\vec{r}} + H(\vec{r} \cdot \nabla_{\vec{r}}) + (\vec{v} \cdot \nabla_{\vec{r}}) f \,, \tag{2.95}$$

but we can also do it the other way round, keeping $\vec{x}$ constant: that way, we find

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t}_{\vec{x}} + \frac{\partial f}{\partial \vec{x}} \,, \tag{2.96}$$

and identifying the terms we ge the desired relation, equation (2.93).

Skipping some passages, from the continuity equation we get:

$$\frac{\partial \rho}{\partial t}_{\vec{x}} + 3H\rho + \frac{1}{a}\nabla_{\vec{x}}(\rho\vec{v}) = 0 \,, \tag{2.97}$$

while for the Euler equation we have:

$$\frac{\partial H\vec{r}}{\partial t}_{\vec{r}} + H^2(\vec{r}\cdot\nabla_{\vec{r}})\vec{r} = -\nabla_{\vec{r}}\Phi_b\,. \tag{2.98}$$

The divergence term is just $r^j\partial_j r^i = r^i$. Inserting the expression we know for the gravitational potential, whose gradient is proportional to $\vec{r}$, we get:

$$\vec{r}\left(\dot{H} + H^2 = -\frac{4\pi G}{3}\rho_b\right), \tag{2.99}$$

which must hold for any $\vec{r}$, so we drop it and recover one of the Friedmann equations! Note that we did not use any GR, everything was Newtonian.

We will get another result:

$$\frac{\partial\vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla_{\vec{x}})\vec{v} = -\frac{1}{a}\nabla_{\vec{x}}\delta\Phi\,, \tag{2.100}$$

our new perturbed Euler equation; while the new Poisson equation is

$$\nabla^2\delta\Phi = a^2 4\pi G\delta\rho\,. \tag{2.101}$$

Some cosmologists like to define

$$\delta(\vec{x},t) = \frac{\delta\rho}{\rho_b} = \frac{\rho(\vec{x},t)-\rho_b(t)}{\rho_b(t)}\,, \tag{2.102}$$

which can be anywhere from -1 to $+\infty$.

Then, we can interpret a negative $\delta\rho$ as a negative charge.

Why the Newtonian approach? Nobody knows how to write down the equations for a general relativistic self-gravitating fluid.

Is the Newtonian approximation a good one? In it, we have

$$\nabla^2\delta\Phi = 4\pi G\rho_b a^2\delta\,, \tag{2.103}$$

can we make the weak field approximation?

Typically

$$\delta\Phi \sim 4\pi G\rho_b a^2\delta\lambda^2\,, \tag{2.104}$$

where λ is the variation scale, from the Laplacian.

From the Friedmann equation, $H^2 = \frac{8\pi G}{3}\rho_b$, then we have

$$\delta\Phi \sim \frac{3}{2}H^2\delta a^2\lambda^2 \sim \left(\frac{\lambda^2}{\lambda^2_{\text{hor}}}\right)\delta\,, \tag{2.105}$$

and $\lambda \sim$ Mpc, the galactic perturbation scale, while $\lambda_{\text{hor}} \sim$ Gpc.

As long as the perturbations are only galactic, the Newtonian approximation is good.

Thu Dec 05 2019

There will be a section on gravitational waves in cosmology in January.

We found the relation:

$$\vec{r} = a(\lambda)\vec{x}\,, \tag{2.106}$$

and

$$\dot{\vec{r}} = \vec{u} = \dot{a}\vec{x} + a\dot{\vec{x}} = H\vec{r} + \vec{v}\,, \tag{2.107}$$

since $H = \dot{a}/a$.

An issue we have is the *redshift space distortion*, caused by *peculiar velocities* which cause the light we see to have redshift beyond the one caused by cosmology alone.

In comoving coordinates, we can use the regular Newtonian fluid dynamics equations.

Since we know that the gravitational instability is relevant when the gravitational force overcomes the pressure forces, we set the pressure gradient to be equal to zero. We have

$$\frac{\partial \rho}{\partial t}\Big|_{\vec{r}} + \nabla_{\vec{r}}(\rho\vec{u}) = 0 \tag{2.108a}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \left(\vec{u}\cdot\vec{\nabla}_{\vec{r}}\right)\vec{u} = 0 \tag{2.108b}$$

$$\nabla^2_{\vec{r}}\Phi = 4\pi G\rho\,, \tag{2.108c}$$

and we consider a background $\Phi_b(\vec{r}, t)$, and $\rho = \overline{\rho}(t)$: we have a peculiar gravitational field beyond the Robertson-Walker background: we define

$$\phi = \Phi - \Phi_b\,, \tag{2.109}$$

and analogously we define by

$$\delta\rho = \rho - \overline{\rho} = \overline{\rho}(1+\delta)\,, \tag{2.110}$$

with $-1 \leq \delta \leq \infty$.

The Laplace equation for ϕ is given by

$$\nabla^2_{\vec{x}}\phi(\vec{x}, t) = 4\pi G a^2(t)\overline{\rho}(t)\delta(\vec{x}, t)\,, \tag{2.111}$$

and we actually *can* have negative quantities on the RHS. Not repulsive gravity, but kind of.

The relation between the derivatives is

$$\frac{\partial}{\partial r} = \frac{1}{a}\frac{\partial}{\partial x}\,, \tag{2.112}$$

therefore the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_x\big(\rho\vec{v}\big) = 0\,, \tag{2.113}$$

we get that the density scales like a^{-3} if there is no velocity: then it becomes

$$\frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a} = 0\,. \tag{2.114}$$

For the momentum equation the calculation is longer, but in the end we find:

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla_x)\vec{v} = -\frac{1}{a}\nabla_x\phi\,, \tag{2.115}$$

and once again we remark that we are not using GR because, in the absence of symmetry, that is definitely too difficult.

If we perturb, we will find that plane waves are not solutions anymore, because we will have time-dependent coefficients.

The perturbation of the density is:

$$\rho(\vec{x},t) = \overline{\rho}(t)\big(1+\delta(\vec{x},t)\big)\,, \tag{2.116}$$

and for the field:

$$\phi(\vec{x},t) = \Phi - \Phi_b\,. \tag{2.117}$$

The derivative is:

$$\frac{\partial \rho}{\partial t} = \frac{\partial \overline{\rho}}{\partial t}(1+\delta) + \overline{\rho}\frac{\partial \delta}{\partial t}\,, \tag{2.118}$$

and we can discard higher-order terms: simplifying the zeroth-order equation we find for the continuity equation:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}\cdot\vec{v} = 0\,. \tag{2.119}$$

On the other hand, for the momentum equation we find:

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} = -\frac{1}{a}\nabla\phi\,, \tag{2.120}$$

and in order to solve these we can expand in Fourier space:

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\,\widetilde{\delta}(t)\exp\Big(i\vec{k}\cdot\vec{x}\Big)\,, \tag{2.121}$$

and similarly for $\vec{v}$ and ϕ.

The actual quantities must be real: therefore we know that $\widetilde{\delta}^*_{\vec{k}}(t) = \widetilde{\delta}_{-\vec{k}}(t)$.

Any vector field $\vec{V}$ can be decomposed by the Helmholtz theorem into a gradient and a divergence:

$$\vec{V} = \nabla\Psi + \vec{T}\,, \tag{2.122}$$

where $\nabla \cdot \vec{T} = 0$.

Using the convective time derivative, we have

$$\frac{\mathrm{D}\vec{v}}{\mathrm{d}t} + H\vec{v} = -\frac{1}{a}\nabla\phi\,. \tag{2.123}$$

Inserting here the Helmholtz decomposition applied to the velocity, we find

$$\frac{\mathrm{D}\vec{T}}{\mathrm{D}t} + H\vec{T} = 0\,, \tag{2.124}$$

which means that vorticity is conserved in fluid motion.

This means that any vorticity which might have been there at the beginning would have been diluted out over time.

We project the equations along the versor $\vec{u}_{\vec{k}} = \vec{k}/\left|\vec{k}\right|$.

Then the equation becomes

$$\dot{\delta}_{\vec{k}} + \frac{ik}{a}v_{\vec{k}} = 0\,, \tag{2.125}$$

and

$$\dot{v}_{\vec{k}} + Hv_{\vec{k}} = -\frac{i\vec{k}}{a}\phi_{\vec{k}}\,, \tag{2.126}$$

and finally

$$-k^2\phi_{\vec{k}} = 4\pi G a^2\overline{\rho}\delta_{\vec{k}}\,. \tag{2.127}$$

Now, by differentiating in one single step we will get a linear second order differential equation for δ which is separated from the others.

Differentiating one more time and making the $\vec{k}$ implicit we get:

$$\ddot{\delta} + \frac{ik}{a}\dot{v} - \frac{ik}{a^2}\dot{a} = 0 \tag{2.128a}$$

$$\ddot{\delta} + \frac{ik}{a}\left(-Hv - \frac{ik}{a}\phi\right) - \frac{ik}{a}Hv = 0 \tag{2.128b}$$

$$\ddot{\delta} - \frac{2ik}{a}Hv + \frac{k^2\phi}{a^2} = 0\,, \tag{2.128c}$$

but the last term is $-4\pi G a^2 \overline{\rho}\delta$: then in the end we find

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\overline{\rho}\delta = 0\,, \tag{2.129}$$

and we use the following solutions of the background equations:

$$a(t) \propto t^{2/3} \tag{2.130a}$$

$$H = \frac{2}{3t} \tag{2.130b}$$

$$\overline{\rho} = \left(6\pi G t^2\right)^{-1}\,, \tag{2.130c}$$

and plugging these in we find

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0\,, \tag{2.131}$$

which will be a power of t: plugging t^α we find the equation

$$\alpha(\alpha - 1) + \frac{4}{3}\alpha - \frac{2}{3} = 0\,, \tag{2.132}$$

which gives $\alpha = -1, 2/3$.

The solution with $\alpha = 2/3$ is called the growing mode, while the one with $\alpha = -1$ is called the decaying mode. When the inflaton field becomes classical we lose a degree of freedom: this removes the decaying solutions. Another way to see it is to see that the decaying mode explodes as $t \to 0$.

Thus, we usually remove the decaying solution.

An interesting observation is the fact that the growing mode grows just as fast as the scale factor.

Inserting this into the other equations we find $v \propto t^\beta$ with $\beta = 1/3$ or $-4/3$, while $\phi \propto t^\gamma$ with $\gamma = 0, -5/2$.

understand what the equation

$$\chi_J = c_s\sqrt{\frac{\pi}{4\overline{\rho}}}\,, \tag{2.133}$$

The Jeans density is given by

$$\rho_J = \frac{3}{4\pi M^2}\left(\frac{3k_B T}{2G\overline{m}}\right)^3\,. \tag{2.134}$$

$$\chi_J = c_s\left(\frac{\pi}{4\overline{\rho}}\right)^{1/2}\,, \tag{2.135}$$

$$M_J = \frac{4\pi}{3}\overline{\rho}\left(\frac{\lambda}{2}\right)^3 , \tag{2.136}$$

$$c_s^2 \sim \frac{k_B T}{\overline{m}} , \tag{2.137}$$

and now we are assuming that $\rho > \rho_J$.

Stars are made of baryons.

Molecular hydrogen:

$$\begin{aligned} H + e &\leftrightarrow H^- + \gamma && (2.138a) \\ H^- + H &\leftrightarrow H_2 + e && (2.138b) \\ H + p &\leftrightarrow H_2^+ + \gamma && (2.138c) \\ H_2^+ + H &\leftrightarrow H_2 + p , && (2.138d) \end{aligned}$$

and these processes are generally not very efficient: they start to be efficient at redshifts of about $z \sim 200$.

It is easier to get the Jeans density if the mass is high and the temperature is low.

The formation of stars is somewhat top-down: larger structures form first.

The critical energy density today is $\rho_{0c} \sim 10^{-29}\,\mathrm{gcm^{-3}}$, while for baryons we have $\rho_{0b} \sim 1 \times 10^{-31}\,\mathrm{gcm^{-3}}$.

On the other hand, at $z \sim 200$ we must multiply the density by a factor $(1+z)^3$ we get a density of $\rho_b(z \sim 200) \sim 10^{-22}\,\mathrm{kgm^{-3}}$.

If we take $M = 1000 M_\odot$ and $T \approx 20\,\mathrm{K}$, we have a Jeans critical density of around $10^{-20}\,\mathrm{kgm^{-3}}$.

Is 20 K a typical temperature then?

The matter starts free-falling: then the kinetic energy increases. However, it is used in chemical processes. That way matter can become opaque.

We have the equation

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{Gm_0}{r} - \frac{Gm_0}{r_0} , \tag{2.139}$$

which gives us a typical free-fall time of $2 \times 10^4\,\mathrm{yr}$.

The energy needed to ionize all the hydrogen is

$$E = \frac{M}{2m_H}\epsilon_D + \frac{M}{m_H}\epsilon_I , \tag{2.140}$$

where M is the mass of the star, m_H is the mass of a hydrogen atom, while $\epsilon_D \approx 4.5\,\mathrm{eV}$ and $\epsilon_U \approx 13.6\,\mathrm{eV}$.

We are going from a radius R_1 to a radius R_2: we get $R_1 \sim \sqrt[3]{3M/4\pi\rho_J}$, since that is the radius at which the collapse starts.

However, the R_1 term hardly contributes: the term R_2 is typically 4 orders of magnitude smaller.

Our objects must still become hotter and smaller in order to reach the 10^7 K needed for ignition.

The final equation we get comes from the virial theorem:

$$2E_k + E_{\text{gr}} = 0 \,, \tag{2.141}$$

and

$$E_k = \frac{3}{2} N k_B T = \frac{3}{2} \frac{M}{\overline{m}} k_B T \approx \frac{3M}{m_H} k_B T \,, \tag{2.142}$$

since $\overline{m} = 0.5 m_H$: so in the end, equating the energies, we have

$$2 \times 3 k_B T \frac{M}{m_H} = \frac{M}{m_H} \left(\frac{\epsilon_D}{2} + \epsilon_I \right) , \tag{2.143}$$

which means that $k_B T = \frac{1}{12}(\epsilon_D + 2\epsilon_I) \sim 2.6\,\text{eV}$, which is still very low compared to the temperature we need.

Fri Dec 06 2019

Today we will discuss another possible *caveat* for star formation: the case where the star collapses but does not ignite.

After recombination the baryons' temperature decays more rapidly than the radiation's, so we get a temperature low enough to satisfy the instability criterion.

The temperature stays low during the collapse: the thermal energy is used in order to break the bonds in H_2 and ionize the hydrogen.

Then we have a gas which is opaque to radiation: there is scattering, which means we lose energy through radiation very slowly. Then the Virial theorem is very close to being true.

At the end of the process, even the mass does not matter anymore: we got 2.6 eV as the temperature regardless of the mass. This is equivalent to around 30×10^3 K. We must compare this to the ignition temperature: that is in the order of keV (15×10^6 K is equivalent to around 1.3 keV).

After free-fall the radius is on the order of 10^{10} m for a solar mass star, while the Sun's radius is smaller by two orders of magnitude.

Now we discuss the *conditions for stardom*: we need to account for the fermionic nature of protons and electrons, which will give us a maximum density due to the Pauli exclusion principle.

The way we will treat this today will be quite rough.

We know that the De Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p} \,. \tag{2.144}$$

How do we calculate p? we assume that the particles are nonrelativistic and apply $E_k = p^2/2m_e$.

Objects which will not satisfy the conditions we talk about today become brown dwarves.

The kinetic energy is of the order $k_B T$, therefore

$$p \sim \sqrt{2m_e k_B T} \,, \tag{2.145}$$

and the critical density is defined by

$$\rho_c = \frac{\overline{m}}{\lambda^3} \,, \tag{2.146}$$

where from the formula we found

$$\lambda = \frac{2\pi\hbar}{\sqrt{2m_e k_B T}} \,, \tag{2.147}$$

which gives approximately

$$\rho_c \sim \overline{m}\frac{(m_e k_B T)^{3/2}}{(2\pi\hbar)^3} \,, \tag{2.148}$$

and from the virial theorem $2E_k + E_{\text{gr}} = 0$, with

$$E_k = \frac{3}{2} N k_B T = \frac{3}{2}\frac{M}{\overline{m}} k_B T \,, \tag{2.149}$$

while

$$E_{\text{gr}} = -\frac{GM^2}{R} \,, \tag{2.150}$$

which means

$$3Nk_B T = \frac{GM^2}{R} \,, \tag{2.151}$$

and we can rewrite this as

$$\frac{3k_B T}{\overline{m}} = \frac{GM}{R} \,, \tag{2.152}$$

and we can express the mass as

$$M = \frac{4}{3}\pi\bar{\rho}R^3 \,, \tag{2.153}$$

so

$$\frac{1}{R} = \left(\frac{4\pi}{3}\frac{\bar{\rho}}{M}\right)^{1/3} \,, \tag{2.154}$$

which gives us the result

$$k_B T = \frac{GM\overline{m}}{3}\left(\frac{4\pi}{3}\frac{\bar{\rho}}{M}\right)^{1/3} \,, \tag{2.155}$$

and if we substitute the critical density in for $\bar{\rho}$ we will get the maximum possible temperature allowed at a given mass.

This yields

$$k_B T = \frac{GM\overline{m}}{3}\left(\frac{4\pi}{3M}\right)^{1/3}\frac{\overline{m}^{1/3}}{(2\pi\hbar)}(m_e k_B T)^{1/2} \,, \tag{2.156}$$

which it is convenient to square:

$$(k_B T)^2 = \frac{G^2 M^2 \overline{m}^2}{9}\left(\frac{4\pi}{3M}\right)^{2/3}\frac{\overline{m}^{2/3}}{(2\pi\hbar)^2} m_e k_B T \,, \tag{2.157}$$

so we can simplify, and up to an order-1 constant

$$k_B T = \frac{G^2 \overline{m}^{8/3} M^{4/3}}{(2\pi\hbar)^2} \,, \tag{2.158}$$

and inserting the ignition temperature of around 1 keV we get $M_{\text{min}} \sim 0.08 M_\odot$.

This is confirmed experimentally.

Let us consider the Sun. Its mass is $M_\odot \approx 1.99 \times 10^{30}\,\text{kg}$, the radius is $R_\odot \approx 6.96 \times 10^8\,\text{m}$, the electromagnetic luminosity is $L_\odot = 3.86 \times 10^{26}\,\text{W}$.

The age of the Sun is around $t_\odot \approx 4.55 \times 10^9\,\text{yr}$, which is comparable to the age of the Universe.

The central density is $\rho_c \approx 1.48 \times 10^5\,\text{kgm}^{-3}$, while the central temperature is $T_c = 1.56 \times 10^7\,\text{K}$, and the central pressure is around $P_c = 2.29 \times 10^{16}\,\text{Pa}$.

The effective temperature is around $T_E \approx 5780\,\text{K}$.

Definition?

What is the corresponding free fall time? It is much shorter than the age of the Sun: the Sun is not in free fall.

What is the number?

We know that

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\mathrm{gr}}}{V}\,, \tag{2.159}$$

where $E_{\mathrm{gr}} = -GM^2/R$ while $V = 4\pi R^3/3$: plugging the Sun's numbers we get

$$\langle P \rangle = 10^{14}\,\mathrm{Pa}\,, \tag{2.160}$$

100 times less than the central density.

The density of the Sun is actually very similar to the density of water.

Was it correct to use nonrelativistic equations? (???)

$$\langle P \rangle = \frac{\overline{\rho}}{\overline{m}} k_B T_I\,, \tag{2.161}$$

where T_I is the mean internal temperature of the Sun. The value of $\overline{m} \approx 0.61$ instead of 0.5 when considering the proper composition of the Sun.

We get

$$k_B T_I \approx \frac{GM_\odot \overline{m}}{3R_\odot} \approx 1.5\,\mathrm{keV} \approx 6 \times 10^6\,\mathrm{K}\,. \tag{2.162}$$

We have evidence that the Sun is a blackbody, we can write

$$L_\odot = 4\pi R_\odot \sigma T_E^4\,, \tag{2.163}$$

where σ is Stefan's constant: $\sigma \approx 5.67 \times 10^{-8}\,\mathrm{W m^{-2} K^{-4}}$.

In principle we could define another quantity:

$$L'_\odot = 4\pi R_\odot^2 \sigma T_I^4\,, \tag{2.164}$$

which does not fit the data. Why is this?

We must consider a photon which comes from the interior of the Sun and goes towards the outside: it will follow a random walk scattering many times. However, depending on the density of electron in the outer regions (electron scattering dominates in the outside), the last scattering is in the outermost regions of the star.

We need to deal with the Fourier equation. First of all, we need to write the Langevin equation: this is connected to the work by Einstein on Browinan motion.

These are processes in which there is a "random" force (due to the fact that our description of the microscopic state is probabilistic), and possibly deterministic forces.

What is the stuff about viscosity?

The equation in the end is like:

$$\dot{\vec{x}} = \vec{\eta} \,, \tag{2.165}$$

What?

with $\left\langle \eta_i(t)\eta_j(t') \right\rangle = 2D\delta_{ij}\delta(t-t')$: there is complete uncorrelation. This is the Markov property: the process has no memory.

This gives us a Gaussian distribution of the positions of the particles, and we get the equation:

$$\frac{\partial P}{\partial t} = D\nabla^2 P \,, \tag{2.166}$$

where $P = P(\vec{x}, t)$. This is a *parabolic equation*, the *Fokker-Planck* formula. We need to give it both initial conditions and boundary conditions.

What is P?

There are three kinds of boundary conditions:

1. nothing: free boundary, the solution can diverge;
2. a reflective boundary: particles "bounce back", in order to deal with this we use the images method, as in electromagnetism;
3. an absorbing boundary: particles disappear if they reach the boundary.

This is equivalent to the Fourier transport equation.

The solution without boundary is a Gaussian with variance $\left\langle x^2 \right\rangle = 2Dt$ and centered around zero.

If $\sigma = \sqrt{\langle x^2 \rangle}$ becomes greater than the boundary then most of the particles have escaped.

$\vec{D}$ is the displacement vector of the Sun: it is

$$\vec{D} = \sum_i \vec{l}_i \,, \tag{2.167}$$

where the $\vec{l}_i$ are the displacement vectors of its various steps.

$$\left\langle \vec{D}^2 \right\rangle = \sum_i \left\langle \vec{l}_i^2 \right\rangle + \sum_{i<j} \left\langle \vec{l}_i \cdot \vec{l}_j \right\rangle \,, \tag{2.168}$$

but if we have isotropy then the scalar products have mean zero. This might be unphysical since the steps are larger at the boundary than at the center...

Then we find:

$$\left\langle \vec{D}^2 \right\rangle = Nl^2 = R_\odot^2 \,, \tag{2.169}$$

where $N = R_\odot^2/l^2$.

The time it takes for a photon to cover a distance l is $t = l/c$. Then we have $t_{RW} = Nt = R_\odot^2 l/(l^2 c)$, which means

$$t_{RW} = \frac{R_\odot^2}{cl} \,, \tag{2.170}$$

while in direct flight the photon would take $t_0 = R_\odot/c$: their ratio is

$$\frac{t_{RW}}{t_0} = \frac{R_\odot}{l} \,, \tag{2.171}$$

and then

$$L_\odot = L'_\odot \frac{l}{R_\odot} \,, \tag{2.172}$$

which means

$$T_E = \left(\frac{l}{R_\odot} \right)^{1/4} T_I \,, \tag{2.173}$$

so we can gather l by knowing the other three parameters: we get $l = 1\,\mathrm{mm}$. This is actually an average of the mean free paths.

$$L = L' \frac{l}{R} = 4\pi R_\odot^2 \sigma T_I^4 \frac{l}{R} \,, \tag{2.174}$$

and we know that $k_B T_I = \frac{GM\overline{m}}{3\hbar}$: replacin this we find

$$L = 4\pi R_\odot^2 \sigma \left(\frac{GM\overline{m}}{3Rk_B} \right)^4 \frac{l}{R} = \frac{(4\pi)^2}{3^5} \frac{\sigma}{k_B^4} G^4 \overline{m}^4 \overline{\rho} l M^3 \,. \tag{2.175}$$

What is the typical lifespan of a star? This gives us $L \sim M^3$, which means $\tau \sim M/L \sim M^{-2}$. Observationally, this matches the data pretty well.

Now we discuss thermonuclear fusion. The reactions we need are

$$\mathrm{p} + \mathrm{p} \to \mathrm{d} + \mathrm{e}^+ + \nu_e \tag{2.176a}$$

$$\mathrm{p} + \mathrm{d} \to {}^3\mathrm{He} + \gamma \tag{2.176b}$$

$${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + \mathrm{p} + \mathrm{p} \,, \tag{2.176c}$$

which involves the weak interaction (for the first process: $\tau \sim 5 \times 10^9\,\mathrm{yr}$), EM interaction (for the second process: $\tau \sim 1\,\mathrm{s}$) and strong interaction (for the third process: $\tau \sim 3 \times 10^5\,\mathrm{yr}$). On the other hand, the process

$$
\begin{aligned}
\mathrm{n} + \mathrm{p} &\to \mathrm{d} + \gamma && (2.177\mathrm{a}) \\
\mathrm{d} + \mathrm{d} &\to {}^3\mathrm{He} + \mathrm{n} && (2.177\mathrm{b}) \\
{}^3\mathrm{He} + \mathrm{d} &\to {}^4\mathrm{He} + \mathrm{p}\,, && (2.177\mathrm{c})
\end{aligned}
$$

which does not use the weak interactions. However, later there are no more free neutrons: this means that even if it is slower the first process is the only one which can happen.

The net balance is 4 protons in, 1 ^{4}He out.

Thu Dec 12 2019

Check room availability for the first week of January.

Tomorrow / next week we will discuss some black holes and neutron stars.

Last week we discussed the nuclear processes which occur in the center of the Sun when $T \sim 10^7\,\mathrm{K}$ is reached.

The minimum mass for a star in order to ignite fusion, as we found, is around $0.08 M_\odot$.

The most important difference between the processes outlined last week is in the first equation of each: there are very few free neutrons.

So, the weak interaction process dominates: since it is so slow, it has a very low power density: $P = 4 \times 10^{26}\,\mathrm{W}$, but we need to take its ratio to the volume of the Sun. This gives a density lower than that of a human.

For each ^{4}He nucleus we get $26\,\mathrm{MeV}$, and we need 4 protons to make it. Then, this gives us the number of protons per second the Sun uses in order to produce the power it does. We will use the following relation:

$$
1\,\mathrm{MeV} \approx 1.78 \times 10^{-30}\,\mathrm{kg} \approx 1.6 \times 10^{-13}\,\mathrm{J}\,, \tag{2.178}
$$

and then, in SI units: we get

$$
\frac{4 \times 10^{26}}{2.6 \times 10^{-13}/4} \approx 4 \times 10^{38} \frac{\text{protons}}{\mathrm{s}}\,. \tag{2.179}
$$

For each process we also emit one electron neutrino, and we need to do the first two steps of the process twice for each ^{4}He nucleus, so we are producing 2×10^{38} neutrinos per second.

A proton's mass is around $m_p \approx 1\,\mathrm{GeV} \approx 1.78 \times 10^{-27}$ kg, so in the Sun there are around 10^{56} of them: this means that the typical lifetime of the Sun is around 10^{10} years.

What is the final state of the Sun? After hydrogen runs out, the Sun should start the next process: helium burning. The core contracts and heats. However we also have the degeneracy pressure.

There is a boundary, the Chandrasekar mass $M_C \approx 1.4 M_\odot$, between the final fate of the star being a white dwarf or a neutron star.

From the point of view of the state of matter, brown dwarves and white dwarves are very similar.

So the core becomes denser and hotter and starts burning helium. The external parts, by the virial theorem, must then expand.

So the possibilities, in order of mass, are white dwarf, neutron star, black hole.

The matter which is expelled can form a *planetary nebula*.

Process	Fuel	Products	T_{min}	M_{min}
Hydrogen burning	Hydrogen	Helium	10^7 K	$0.08 M_\odot$
Helium burning	Helium	Carbon, Oxygen	10^8 K	$0.5 M_\odot$
Carbon burning	Carbon	Oxygen, Neon, Sodium	5×10^8 K	$8 M_\odot$
Neon burning	Neon	Magnesium, Oxygen	10^9 K	$9 M_\odot$
Oxygen burning	Oxygen	Magnesium to Sulphur	2×10^9 K	$10 M_\odot$
Silicon burning	Silicon	Iron and nearby elements	3×10^9 K	$11 M_\odot$

Figure 2.1: Solar processes.

A plot of the binding energy per nucleon $B = E - M_{\text{nucleons}}$ shows that it is maximum for iron.

Something about the possibility to have ^{8}Be be stable in order for potassium to be formed.

When the nucleus cannot reach the temperature needed for the next process, the collapse is stopped by electron degeneracy.

The final radius of the red giant phase of the Sun is around 70 times the radius of the Sun, while the white dwarf phase is 70 times smaller.

Is this correct? I couldn't quite hear.

Our equation of hydrostatic equilibrium is:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G m(r) \rho(r)}{r^2}\,, \tag{2.180}$$

and the equation giving the variation of the mass is $\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r)$. So we get

$$\frac{r^2}{\rho(r)} \frac{\mathrm{d}P}{\mathrm{d}r} = -G m(r)\,, \tag{2.181}$$

which can be restated as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\frac{\mathrm{d}m}{\mathrm{d}r} = -4\pi G\rho(r)r^2\,, \tag{2.182}$$

more commonly stated as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho(r)\,, \tag{2.183}$$

which holds if we have hydrostatic equilibrium.

Commonly the equation of state used for this is called *polytropic*:

$$P = k\rho^{\frac{n+1}{n}}\,, \tag{2.184}$$

where $k = \text{const}$ and $n = 1/(\gamma - 1)$: so if $\gamma = 5/3$, which holds for a monoatomic gas, then $n = 3/2$ while if $\gamma = 4/3$, which holds for an ultrarelativistic gas, then $n = 3$.

Using this law, we can write this equation in terms of either only the density or only the pressure.

Let us kave only the density:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho^{\frac{n+1}{n}}\right)\right) = -4\pi G\rho(r)\,, \tag{2.185}$$

and we need two boundary conditions: we can fix the central density $\rho(r = 0) = \rho_c$ and the derivative of the density at the center:

$$\frac{\mathrm{d}\rho_c}{\mathrm{d}r}(r = 0) = 0\,. \tag{2.186}$$

This is because:

$$\rho^{1/n}\frac{\mathrm{d}\rho}{\mathrm{d}r} \propto -\frac{Gm(r)}{r^2}\rho(r)\,, \tag{2.187}$$

and the mass is given by $m(r) \sim \rho_c r^3$: therefore $\rho^{1/n}\,\mathrm{d}\rho/\mathrm{d}r \propto r$, so it makes sense to set the derivative to zero.

The radius of the star is defined as the one at which the density goes to 0: $\rho(R) = 0$, and the mass of the star is given by $m(R) = M$.

These equations can be solved numerically. Also, we can use an ansatz.

A model by Claytron 1986: $P_c = 2 \times 10^{16}\,\text{Pa}$, and since

$$\langle P\rangle = -\frac{1}{3}\frac{E_{\text{gr}}}{V_\odot}\,, \tag{2.188}$$

where $E_{\mathrm{gr}} = GM_\odot^2/R_\odot$, and $V_\odot = {}^4\!/\!_3\pi R_\odot^3$: therefore the mean pressure is approximately 1/200 times the pressure at the center.

We can write the equation of hydrostatic equilibrium as

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2} \approx -\frac{4\pi G}{3}\rho_c^2 r\,, \tag{2.189}$$

so the pressure gradient goes to zero linearly in r. So, we know that

$$\frac{\mathrm{d}P}{\mathrm{d}r}(r=0) = 0\,, \tag{2.190}$$

also, as $r \to R$ we get $\mathrm{d}P/\mathrm{d}r \to 0$ as well, since it is proportional to $\rho(r)$ in that region.

The pressure gradient around the center follows the density. Its variation is much larger near the center than on the outside. The ansatz by Clayton is

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4\pi}{3}G\rho_c^2 r \exp\left(-\frac{r^2}{a^2}\right), \tag{2.191}$$

where the parameter a has the dimensions of a length, and we take it to be $a \ll R$. This model is quite accurate near the center, not so much near the surface!

Integrating we find:

$$P(r) = \frac{2\pi}{3}G\rho_c^2 a^2\left(\exp\left(-\frac{r^2}{a^2}\right) - \exp\left(-\frac{R^2}{a^2}\right)\right), \tag{2.192}$$

so that the pressure is exactly zero at the surface: $P(R) = 0$. We will check *a posteriori* that this model makes sense for a typical star.

Do we set the pressure to stay at zero for $r > R$? Maybe does not matter…

We have the relation

$$Gm(r)\,\mathrm{d}m = -4\pi r^4\,\mathrm{d}P\,, \tag{2.193}$$

which can be integrated in order to give the following result:

$$\frac{1}{2}Gm^2(r) = -4\pi\int_0^r \mathrm{d}\widetilde{r}\,\widetilde{r}^4\frac{\mathrm{d}P}{\mathrm{d}\widetilde{r}}\,. \tag{2.194}$$

We recover $m(r)$ by taking the square root:

$$m(r) = \frac{4\pi a^3}{3}\rho_c\Phi(x)\,, \tag{2.195}$$

where $x = r/a$ and $\Phi(x)$ is defined from the integral from before:

$$\Phi^2(x) = 6 \int_0^x \mathrm{d}y\, y^5 e^{-y^2} = 6 - 3\Big(x^4 + 2x^2 + 2\Big)e^{-x^2}\,, \tag{2.196}$$

and the second term is almost zero near the surface if $a \ll R$: the power series starts from the sixth power (? what?).

The density profile then is given by:

$$\rho(r) = \frac{1}{4\pi r^2}\frac{\mathrm{d}m}{\mathrm{d}r} = \rho_c \left(\frac{x^3 e^{-x}}{\Phi(x)}\right)\,, \tag{2.197}$$

and we can also recover the temperature profile from the ideal gas law:

$$T(r) = \frac{\overline{m}}{k_B}\frac{P(r)}{\rho(r)}\,, \tag{2.198}$$

and, when $x \ll 1$ we find

$$\Phi(x) \sim \left(x^6 - \frac{3}{4}x^8 + \frac{3}{10}x^{10} - \frac{1}{12}x^{12} + \dots\right)^{1/2}\,, \tag{2.199}$$

and inserting this we find

$$\rho(r) \approx \rho_c \left(1 - \frac{5}{8}\frac{\overline{r}^2}{a^2} + \dots\right)\,, \tag{2.200}$$

and

$$T(r) \approx T_c \left(1 - \frac{3}{8}\frac{r^2}{a^2} + \dots\right)\,, \tag{2.201}$$

which allows us to get the total mass

$$M = m(R) = \frac{4\pi\rho_c a^3}{3}\Phi(R/a) \approx \frac{4\pi\rho_c a^3\sqrt{6}}{3}\,, \tag{2.202}$$

and lots of it are within a radius a:

$$\rho(a) = 0.53\rho_c \qquad \text{and} \qquad m(a) = 0.28 M_\odot\,. \tag{2.203}$$

We also have the relation $a = R_\odot/5.4$, and

$$\frac{\langle\rho\rangle}{\rho_c} \sim \left(\frac{a}{R}\right)^3 \sim 1\,. \tag{2.204}$$

Where does this come from?

We have the relations

$$m(r) = \frac{4\pi a^3}{3}\rho_c \Phi(x)\,, \tag{2.205}$$

and

$$P_c = \frac{2\pi}{3} G\rho_c^2 a^2\,. \tag{2.206}$$

Then we can combine these:

$$P_c = \left(\frac{\pi}{36}\right)^{1/3} G M^{2/3} \rho_c^{4/3}\,. \tag{2.207}$$

The factor $(\pi/36)^{1/3} \sim 0.44$. Changing γ we get: for $\gamma = 5/4$ the factor is 0.48. We use the ideal gas relation: then we find

$$k_B T_c = \left(\frac{\pi}{36}\right)^{1/3} G\overline{m} M^{1/3} \rho_c^{1/3}\,. \tag{2.208}$$

Then, we can figure out which processes actually can happen.

Fri Dec 13 2019

We discuss the maximum mass of stars.

We were able to give meaning to the parameter a: the pressure is given by

$$P(r) = \frac{2\pi}{3} G\rho_c^2 a^2 \left(\exp\left(-\frac{r^2}{a^2}\right) - \exp\left(-\frac{R^2}{a^2}\right)\right)\,, \tag{2.209}$$

so we can see that

$$a = \left(\frac{3M}{4\pi\rho_c\sqrt{6}}\right)^{1/3}\,, \tag{2.210}$$

and then we got an expression for the central pressure P_c: the parameter multiplying it is approximately 0.44, while more accurate models give: if $\gamma = 5/3$ (ideal gas) we get 0.48 whiile if $\gamma = 4/3$ (ultrarelativistic) we get 0.36.

We have the relation

$$P_c = \frac{\rho_c}{\overline{m}} k_B T_c\,, \tag{2.211}$$

which we use to get the last relation from last time.

This allows us to get some figures for main sequence (hydrogen burning) stars. In the Herzsprung Russel diagram we plot $L/L_\odot$ versus T_{eff}, the latter increasing right to left.

The Main Sequence runs from the upper left to the lower right, we have Red Giants on the upper right and White Dwarves on the lower left. Most of the stars are on the Main Sequence: the hydrogen burning phase lasts a long time.

What is the maximum mass for Main Sequence stars? A star becomes unstabel when most of its material becomes ultrarelativistic: then, its total energy goes from a negative value to 0 and the adiabatic index approaches $4/3$.

Suppose that the central energy is partly given by radiation and partly by matter. We write

$$P_c = P_\rho + P_r = \beta P_c + (1-\beta) P_c \,, \tag{2.212}$$

where the terms of the two sums exactly correspond to each other, and

$$\beta P_c = P_\rho = \frac{\rho_c k_B T_c}{\overline{m}} \tag{2.213}$$

and

$$(1-\beta) P_c = P_r = \frac{1}{3} a \,, \tag{2.214}$$

where

$$a = \frac{\pi^2 k_B^2}{15 \hbar^3 c^3} \,. \tag{2.215}$$

So we have

$$\left(\beta P_c\right)^4 = \frac{\rho_c^4}{\overline{m}^4} (k_B T_c)^4 \,, \tag{2.216}$$

and

$$\frac{1-\beta}{\beta^4} P_c^{-3} = \frac{a}{3} \left(\frac{k_B \rho_c}{\overline{m}} \right)^{-4} \,. \tag{2.217}$$

Inverting this we can eliminate the temperature dependence:

$$P_c = \left(\frac{3}{a} \frac{1-\beta}{\beta^4} \right)^{1/3} \left(\frac{k_B \beta}{\overline{m}} \right)^{4/3} = \left(\frac{\pi}{36} \right)^{1/3} G M^{1/3} \rho_c^{4/3} \,, \tag{2.218}$$

so

$$\left(\frac{\pi}{36} \right)^{1/3} GM^{2/3} = \left(\frac{3}{a} \frac{(1-\beta)}{\beta^4} \right)^{1/3} \left(\frac{k_B}{\overline{m}} \right)^{4/3} , \tag{2.219}$$

so as β decreases, M increases.

[Plot of $1-\beta$ versus $M/M_\odot$, showing this.]

Now we deal with the degenerate electron gas in stars.

The distribution function is approximately given by

$$f(p) \propto \frac{1}{\exp\left(\frac{\epsilon_p - \mu}{k_B T} \right) + 1} , \tag{2.220}$$

where $\epsilon_p = c\left(p^2 + m^2c^2 \right)^{1/2}$. Then, as $T \to 0$ we get: $f(\epsilon_p) = 1$ if $\epsilon_p \leq \epsilon_F$ and $f(\epsilon_p) = 0$ if $\epsilon_p > \epsilon_F$; we can express this energy in terms of the momentum:

$$\epsilon_F^2 = c^2 p_F^2 + m^2 c^4 . \tag{2.221}$$

The number of electrons is given by

$$n_e = 2 \int_0^{p_F} \mathrm{d}p \, p^2 4\pi \frac{1}{h^3} = \frac{8\pi}{3} \left(\frac{p_F}{h} \right)^3 , \tag{2.222}$$

where we have a factor of 2 to account for the spin-1/2 nature of the electrons. This means that

$$p_F = \left(\frac{3n}{8\pi} \right)^{1/3} h . \tag{2.223}$$

The density is given by

$$\rho = \frac{2}{h^3} \int_0^{p_F} \mathrm{d}p \, p^2 4\pi \epsilon_p , \tag{2.224}$$

and we can consider either the nonrelativistic or the ultrarelativistic limit. In the nonrelativistic limit we find

$$\epsilon_p = mc^2 + \frac{p^2}{2m} , \tag{2.225}$$

so the result becomes

$$\rho = n \left(mc^2 + \frac{3}{10} \frac{p_F^2}{m} \right) , \tag{2.226}$$

and we know that for a nonrelativistic gas the pressure is given by

$$P = \frac{2}{3} \frac{E_k}{V}, \tag{2.227}$$

where E/V is the kinetic energy density.

$$P = n \frac{p_F^2}{5m}, \tag{2.228}$$

where does this come from?

$$P = k_{NR} n^{5/3}, \tag{2.229}$$

where

$$k_{NR} = \frac{h^2}{5m} \left(\frac{3}{8\pi} \right)^{2/3}. \tag{2.230}$$

In the relativistic case, on the other hand, we get $\epsilon_p \approx cp$, and $\rho = \frac{3}{4} n \rho_F c$. In this case, we also know that the pressure becomes

$$P = \frac{1}{3} \frac{E_k}{V}. \tag{2.231}$$

So we find

$$P = K_{VR} n^{4/3}, \tag{2.232}$$

where

$$K_{VR} = \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3}. \tag{2.233}$$

We make a plot: on the x axis we have the number density in m^{-3}, on the y axis we have the temperature in K.

We divide the plot into:

1. Classical UR: $P \propto n k_B T$;
2. classical NR (like the Sun);
3. degenerate NR: $P = K_{NR} n^{4/3}$
4. degenerate UR.

Classical vs degenerate is marked by a line similar to $T \sim n$, while we have NR for both T and n lower than certain critical values (since a degenerate gas can become ultrarelativistic even at low temperatures! this is the point).

We have

$$P_c = \frac{\rho_C}{\overline{m}} k_B T_c \,, \tag{2.234}$$

and

$$k_B T_c = \left(\frac{\pi}{36}\right)^{1/3} G \overline{m} M^{2/3} \rho_c^{1/3} \,, \tag{2.235}$$

and it can be (easily?) shown that

$$\overline{m} = 2m_H \times \frac{1}{1 + 3x_1 + 0.5x_4} \,, \tag{2.236}$$

where $x_{1,4}$ are the concentrations of hydrogen and helium respectively.

To estimate the maximum achievable central temperature we do:

$$P_c = k_{NR} n_e^{5/3} + n_i k_B T_c \,, \tag{2.237}$$

where $n_e = n_i = \rho_c / \overline{m}_H$.

$$\left(\frac{\pi}{36}\right)^{1/3} G M^{2/3} \rho_c^{4/3} = k_{NR} \left(\frac{\rho_c}{m_H}\right)^{5/3} + \frac{\rho_c}{m_H} k_B T_c \,, \tag{2.238}$$

which implies

$$k_b T_c = \left(\frac{\pi}{36}\right)^{1/3} G m_H M^{2/3} \rho_c^{1/3} - k_{NR} \left(\frac{\rho_c}{m_H}\right)^{2/3} \,, \tag{2.239}$$

which is in the shape $k_B T_c = A \rho_c^{1/3} - B \rho_c^{2/3}$: we can find a maximum of this function, which comes out to be at $\rho_c = (A/2B)^3$, where we have

$$k_B T_c = \frac{A^2}{2B} = \left(\frac{\pi}{36}\right)^{2/3} \frac{G^2 m_H^{8/3}}{4 k_{NR}} M^{4/3} \,. \tag{2.240}$$

Now, we can set this temperature to be larger than the ignition temperature for any process we want, to see whether it will happen.

The minimum mass is

$$M_{\min} = \left(\frac{36}{\pi}\right)^{1/2} \left(\frac{4 k_{NR}}{G^2 m_H^{8/3}}\right)^{3/4} \left(k_B T_{\text{ign}}\right)^{3/4} \,. \tag{2.241}$$

The potential energy between two hydrogen nuclei separated by a distance equal to their quantum wavelength is

$$E_g = -\frac{Gm_H^2}{r} = -\frac{Gm_H^3 c}{\hbar}, \tag{2.242}$$

where we inserted $r = \hbar/m_H c$. This corresponds to an energy $E = m_H c^2$, and we have a value

$$\alpha_G = \frac{E_g}{E} = \frac{Gm_H^2}{\hbar c} \sim 5.9 \times 10^{-39}. \tag{2.243}$$

For electromagnetic interaction, we get

$$\alpha_{EM} = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}, \tag{2.244}$$

which is *much greater*.

Then we find

$$M_{\min} \approx 16 \left(\frac{k_B T_{\text{ign}}}{m_e c^2} \right)^{3/4} \alpha_G^{-3/2} m_H. \tag{2.245}$$

If $T_{\text{ign}} \sim 1.5 \times 10^6$ K, one tenth of the temperature of the Sun, we find

$$M_{\min} \sim 0.03 \alpha_G^{-3/2} m_H, \tag{2.246}$$

while for the maximum mass, from before with $\beta = 0.5$ and $\overline{m} = 0.61 m_H$ we get

$$M_{\max} \approx 56 \alpha_G^{-3/2} m_H, \tag{2.247}$$

so this hints to the fact that $m_* = \alpha_G^{-3/2} m_H$ is an important characteristic mass. This is around $1.85 M_\odot$.

This corresponds to a number of particles:

$$N_* = \frac{m_*}{m_H} \approx 2 \times 10^{52}. \tag{2.248}$$

Suppose the core of a star is held together by the pressure of degenerate electrons: we discuss *white dwarves*. We define

$$n_e = Y_e \frac{\rho_c}{m_H}, \tag{2.249}$$

where $Y_e = (1 + x_1)/2$. The pressure is

$$P = k_{NR} n_e^{5/3} = k_{NR} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3}, \tag{2.250}$$

which must compared to

$$P_c = \left(\frac{\pi}{36} \right)^{1/3} G M^{2/3} \rho_c^{4/3} \,, \tag{2.251}$$

and we assume that $P = P_c$: then we get

$$\rho_c \approx \frac{3.1}{Y_e^5} \frac{M}{m_*} \frac{m_H}{(h/m_e c^2)^3} \,. \tag{2.252}$$

Missing square on the M/m_*?

The pressure is

$$P = k_{UR} n_e^{4/3} = k_{UR} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3} \,, \tag{2.253}$$

so we get

$$k_{UR} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3} \approx \left(\frac{\pi}{36} \right)^{1/3} G M^{2/3} \rho_c^{4/3} \,, \tag{2.254}$$

so we can see that in this limit the expression becomes independent of ρ_c. This will give us a limit mass: the Chandrasekhar mass.

$$M_{CH} = \left(\frac{36}{\pi} \right)^{1/2} \left(\frac{Y_e}{m_H} \right)^2 \left(\frac{k_{UR}}{G} \right)^{3/2} \approx 2.3 Y_e^2 m_* \approx 4.3 Y_e^2 M_\odot \approx 1.4 M_\odot \,. \tag{2.255}$$

This is the maximum mass of a white dwarf to remain stable, held together by the degeneracy pressure of electrons. Above this, it becomes a neutron star.

Thu Dec 19 2019

The lessons on gravitational waves will be on the 8th and 9th of January, from 14:30 to 16:30, in rooms LUF2 and P2B respectively.

We want to find the Chandrasekhar limit in a more precise manner.

The number density of electrons is given by

$$n_e = Y_e \frac{\rho_c}{m_H} \,, \tag{2.256}$$

in the nonrelativistic case the pressure is given by

$$P_c = k_{\mathrm{NR}} n_e^{5/3} = k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} \,, \tag{2.257}$$

but we can also derive it by

$$P_c = k_{\mathrm{NR}} n_e^{5/3} = k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} , \tag{2.258}$$

and equating these we find:

$$k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} = k_{\mathrm{NR}} n_e^{5/3} = k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} , \tag{2.259}$$

which implies

$$\rho_c = \frac{3.1}{Y_e^5} \left(\frac{M}{M_*} \right)^2 \frac{m_H}{(h/m_e c^2)^3} , \tag{2.260}$$

where $\alpha_G = G m_H^2 / (\hbar c) \approx 5.9 \times 10^{-30}$, while

$$m_* = \alpha_G^{-3/2} m_H = 1.85 M_\odot , \tag{2.261}$$

while for the ultrarelativistic case we get

$$P_C = k_{\mathrm{UR}} n_e^{4/3} = k_{\mathrm{UR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3} , \tag{2.262}$$

so in this particular case the density ρ_c simplifies from the equations: we get a critical mass

$$M_{\mathrm{CHANDRA}} = \left(\frac{36}{\pi} \right)^{1/2} \left(\frac{Y_e}{m_H} \right)^2 \left(\frac{k_{\mathrm{UR}}}{G} \right)^{3/2} \approx 2.3 Y_e^2 m_* \approx 4.3 Y_e^2 M_\odot , \tag{2.263}$$

and we assume that we are in the fully degenerate case: in the integration over momenta of the phase space distribution we insert a cutoff at the Fermi energy.

We find:

$$P = \frac{4\pi}{3h^3} g_* \int_0^{p_F} \mathrm{d}p \, p^2 \frac{p^2 c^2}{\epsilon_p} , \tag{2.264}$$

where $\epsilon_p = \left(p^2 c^2 + m^2 c^4 \right)^{1/2}$. We integrate with the adimensional variable $x = p/(m_e c)$: we get

$$P = \frac{8\pi}{3h^3} m_e^4 c^5 \int_0^{x_F} \frac{x^4}{(1 + x^2)^{1/2}} \, \mathrm{d}x , \tag{2.265}$$

so we get $P = k_{\text{UR}} n_e^{4/3} I(x_F)$, where we incorporated the integral in the term $I(x_F)$: this is given by

$$I_(x) = \frac{3}{2x^4} \left(x(1+x^2)^{1/2} \left(\frac{2x^2}{3} - 1 \right) + \log \left(x + (1+x^2)^{1/2} \right) \right) , \tag{2.266}$$

and

$$x_F = \frac{p_F}{m_e c} = \left(\frac{3 n_e}{8\pi} \right)^{1/3} \frac{h}{m_e c} = \left(\frac{3 Y_e \rho_c}{8 \pi m_H} \right)^{1/3} \frac{h}{m_e c} , \tag{2.267}$$

so if $x_F \gg 1$ we have $I(x_F) \sim 1$, the ultrarelativistic case, while if $x_F \ll 1$ we have $I(x_F) \sim 4 x_F / 5$. This then interpolates between our different cases.

$$k_{\text{UR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3} I(x_F) \approx \left(\frac{\pi}{36} \right)^{1/3} G M^{4/3} \rho_c^{4/3} , \tag{2.268}$$

so we can extract the mass:

$$M = I(x_F)^{3/2} M_{\text{CH}} , \tag{2.269}$$

and the important thing is that $x_F \propto n_e^{1/3} \propto \rho_c^{1/3}$.

[Graph: on the x axis M/M_{CHANDRA}, on the y axis ρ_c.]

If we increase the mass, the star is not able to support itself by the pressure due to being a gas of degenerate electrons.

This gives us

$$M_{\text{CHANDRA}} = 3.1 Y_e^2 m_* = 5.8 Y_e^2 M_\odot = 1.4 M_\odot . \tag{2.270}$$

It can be shown that the mean density is around

$$\langle \rho \rangle = \frac{1}{6} \rho_c = \frac{0.51}{Y_e^2} \left(\frac{M}{m_*} \right)^2 \frac{m_H}{(h/m_e c)^3} , \tag{2.271}$$

so we can estimate the radius as

$$R = \left(\frac{3M}{4\pi \langle \rho \rangle} \right)^{1/3} \approx 0.77 Y_e^{5/3} \left(\frac{M}{m_*} \right)^{1/3} \alpha_G^{-1/2} \frac{h}{m_e c} , \tag{2.272}$$

and the object at the end is

$$\alpha_G^{-1/2} \frac{h}{m_e c} \approx 3 \times 10^7 \, \text{m} , \tag{2.273}$$

so if we take $Y_e = 0.5$ we find:

$$R = \frac{R_\odot}{74}\left(\frac{M_\odot}{M}\right)^{1/3}. \tag{2.274}$$

The luminosity is given by

$$L = 4\pi R^2 \sigma T_E^4 = \frac{1}{74^2}\left(\frac{M_\odot}{M}\right)^{4/3}\left(\frac{T_E}{6000\,\mathrm{K}}\right) L_\odot\,, \tag{2.275}$$

so if we take a typical effective temperature of around $10^4\,\mathrm{K}$ (recall that neutron stars are in the blue part of the HR diagram), $M = 0.4 M_\odot$ we get $L \approx 3 \times 10^{-3} L_\odot$.

We deal with degenerate stars: the Pauli exclusion principle plays a critical role, and the process

$$\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \overline{\nu}_e \tag{2.276}$$

is inhibited, since there is no more room for electrons. On the other hand, the process

$$\mathrm{e}^- + \mathrm{p} \to \mathrm{n} + \nu_e \tag{2.277}$$

is favoured. We can look at the Saha formula to get numerical estimates for this. The chemical potential of neutrinos can be neglected, therefore we find

$$\mu_\mathrm{n} = \mu_\mathrm{p} + \mu_\mathrm{e}\,, \tag{2.278}$$

and the same equation holds for their Fermi energies.

This means that a certain point we will have only neutrons.

We need to exploit the Fermi exclusion principle. Why does the equation

$$\epsilon_{F,\mathrm{n}} = \epsilon_{F,\mathrm{p}} + \epsilon_{F,\mathrm{e}}\,, \tag{2.279}$$

favour neutrons? we have the constraint that the number of protons must equal the number of electrons, but the Fermi energy depends on the number density of these. typical numbers then become

$$n_\mathrm{p} = n_\mathrm{e} = \frac{n_\mathrm{n}}{200}\,. \tag{2.280}$$

We will have

$$n_\mathrm{n} = Y_n \frac{\rho_c}{m_\mathrm{n}} \approx \frac{\rho_c}{m_\mathrm{n}} \qquad \text{since} \qquad Y_n \approx 1\,. \tag{2.281}$$

For a neutron star we have

$$\rho_c \approx 3.1 \left(\frac{M}{M_*}\right)^2 \frac{m_n}{(h/m_n c)^3}\,, \tag{2.282}$$

while for a white dwarf we have

$$\rho_c \approx \frac{3.1}{Y_e^5}\left(\frac{M}{M_*}\right)^2 \frac{m_H}{(h/m_e c^2)^3}\,, \tag{2.283}$$

the mass is given by $M_* = \alpha_G^{-3/5} m_n \approx 1.85 M_\odot$ and for the radius we get

$$R = 0.77\left(\frac{M_*}{M}\right)^{1/3} \alpha_G^{-1/2} \frac{h}{m_n c}\,, \tag{2.284}$$

where the characteristic length is given by

$$L_n = \alpha_G^{-1/2} \frac{h}{m_n c} \approx 17\,\text{km} \approx \frac{1}{1200} L_e\,, \tag{2.285}$$

1200 times smaller than the corresponding length scale for electrons: L_n is the characteristic length scale for a NS, L_e is the characteristic length for a white dwarf. The maximum mass is $M_{\text{max}}^{\text{NS}} = 3.1 M_* = 5.8 M_\odot$. This is the mass of a star *remnant*, which encompasses mass from the core only: the initial star will be much larger.

Can this NS become a BH? we need to compute

$$\frac{GM}{Rc^2} \approx 0.2\left(\frac{M}{M_*}\right)^{4/3}\,, \tag{2.286}$$

so if the mass is large enough we can reach the critical value of $GM/Rc^2 = 2$.

Neutron stars were first detected as rotating objects: *pulsars*. We have

$$\frac{GM}{R^2} \approx R\omega_{\text{max}}^2\,, \tag{2.287}$$

the maximum angular velocity which can be supported gravitationally: it comes out to be

$$\tau_{\text{min}} = \frac{2\pi}{\omega_{\text{max}}} \approx 2\pi\left(\frac{R^3}{GM}\right)^{1/2}\,, \tag{2.288}$$

which is of the order

$$\tau_{\text{min}} \approx 11\left(\frac{M_*}{M}\right) \alpha_G^{-1/2} \frac{h}{m_n c^2} \approx 0.6 \frac{M_*}{M}\text{ms}\,. \tag{2.289}$$

This gives us a bound for gravitational waves of astrophysical origin.

no comment on the GR corrections to this formula: however I'd expect them to be significant

We move on to *the GR issue*.

In a star, the classical equation of hydrostatic balance is:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2}, \tag{2.290}$$

while the GR equations for this are the TOV equation: Tolman-Oppenheimer-Volkov:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}, \tag{2.291}$$

and the first correction is reminiscent of cosmology: *the pressure itself contributes to the inertia of the system*.

If we have constant density, in the Newtonian case we get

$$m(r) = \frac{4\pi}{3}\rho_0 r^3, \tag{2.292}$$

so than we can integrate and get

$$P(r) = \frac{2\pi G}{3}\rho_0^2\left(R^2 - r_0^2\right), \tag{2.293}$$

where we inserted the boundary condition $P(R) = 0$. In the first-order GR case, this can still be solved analytically! We get

$$P(r) = \rho_0 c^2\left(\frac{(1-2GMr^2/R^3c^2)^{1/2} - (1-2GM/Rc^2)^{1/2}}{3\left(1-2GM/Rc^2\right)^{1/2} - \left(1-2GMr^2/R^3c^2\right)^{1/2}}\right), \tag{2.294}$$

Check exponents

so we get

$$P_c = \frac{2\pi}{3}G\rho_0^2 R^2 = \left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\rho_c^{4/3}, \tag{2.295}$$

so we can look at what happens when we consider $r = 0$: we get

$$P_c = \rho_0 c^2\left(\frac{1-\sqrt{1-2GM/Rc^2}}{3\sqrt{1-2GM/Rc^2}-1}\right), \tag{2.296}$$

so we can see that the central pressure is finite as long as

$$\frac{GM}{Rc^2} < \frac{4}{9}, \tag{2.297}$$

which is not $1/2$ since we have made some approximations.

Then we have a bound

$$M_{\max} \approx \left(\frac{8\pi f}{9}\right)^{3/2} M_* \,, \tag{2.298}$$

with $f \sim 1$. This means than the objects becomes contained inside its Schwarzschild radius, $R = 2GM/c^2$.

Tomorrow we will speak of galaxy formation.

Fri Dec 20 2019

The lectures in january will be about *cosmological* gravitational waves mainly.

Write to him if you do not have access to the dropbox.

On the exam: a traditional oral exam, with questions on the main topics dealt with in class. The days on the calendar do not mean anything. The exams should be agreed upon by email.

Now we talk about the formation of dark matter halos. This is in Sabino's notes in the dropbox.

We say we have a spherical object in the universe, focus on it and apply Birkoff's theorem: we can study it independently of the surroundings.

We can consider halos of different densities to account for over and under densities. In the real world, we will not have spheres, but three-axial ellipsoids: it is known that if we have over-densities the non-sphericity will increase, if we have under-densities it will decrease.

However we treat spherical models because it is simple. Historically the Americans supported the spherical models, the Russians supported the "pancake model". Now we know that we have pancakes with spheres inside (?).

We consider a sphere. We define:

$$\delta(\vec{x}, t) = \frac{\rho(\vec{x}, t) - \overline{\rho}(t)}{\overline{\rho}(t)} \,, \tag{2.299}$$

and we assume $0 < \delta \ll 1$ (a *small over-density*), although in general we could have $-1 < \delta < +\infty$.

We found in previous lessons that we have a growing mode $\delta \propto t^{2/3}$ and a decaying mode $\delta \propto t^{-1}$, for which we had $v \propto t^{1/3}$ and $v \propto t^{-4/3}$ respectively.

So we choose a time t_i such that

$$\delta(t) = \delta_+(t_i)\left(\frac{t}{t_i}\right)^{2/3} + \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1} . \tag{2.300}$$

The linearized continuity equation was:

$$v = i\frac{\dot{\delta}}{k}a \propto \left(\frac{2}{3}\delta_+(t_i)\left(\frac{t}{t_i}\right)^{1/3} - \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-4/3}\right), \tag{2.301}$$

since $a \propto t^{2/3}$. This is in order to write explicitly the fact that we need two initial conditions.

We suppose that at $t = t_i$ we have *unperturbed Hubble flow*: $v(t_i) = 0$, which means

$$\delta_-(t_i) = \frac{2}{3}\delta_+(t_i), \tag{2.302}$$

and we can have a generic initial density, which we can express as:

$$\delta(t_i) = \delta_i = \delta_+ + \delta_- = \frac{5}{3}\delta_+ . \tag{2.303}$$

Our perturbation can be dealt with cosmologically as being a *local FRLW closed universe*. Say our sphere has a radius R: then if we have $\delta = \delta_i$ then

$$\Omega(t_i) = 1 + \delta_i . \tag{2.304}$$

We have then, from the Friedmann equations:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k, \tag{2.305}$$

where $k = +1$ if $\delta_i > 0$, because of the fact that our sphere is locally a closed universe. This then means:

$$-k = (1 - \Omega)a^2 H^2, \tag{2.306}$$

so

$$\frac{\dot{a}^2}{a_i^2} = H_i^2\left(\Omega_p(t_i)\frac{a_i}{a} + (1 - \Omega_p(t_i))\right), \tag{2.307}$$

where the index p denotes the fact that we are talking about the perturbation. We have:

$$\rho_p(t) = \rho_p(t_i)\left(\frac{a_{pi}}{a_p}\right)^3 \tag{2.308a}$$

$$= \rho_c(t_i)\Omega_p(t_i)\left(\frac{a_{pi}}{a}\right)^3 . \tag{2.308b}$$

We want to derive a time for the *turnaround time* t_m:

$$\rho_p(t_m) = \rho_c(t_i)\Omega_p(t_i)\left(\frac{\Omega_p(t_i)-1}{\Omega_p(t_i)}\right)^3 \tag{2.309a}$$

$$= \rho_c(t_i)\frac{\left(\Omega_p(t_i)-1\right)^3}{\Omega_p(t_i)^2}\,, \tag{2.309b}$$

since at $t = t_m$ we have $\frac{\dot{a}^2}{a_i^2} = 0$.

since $\dot{a}(t_m) = 0$?

Some time ago we had found:

$$t(\theta_m = \pi) = t_m = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\,, \tag{2.310}$$

but calculating this *now* is arbitrary: the same formula holds replacing 0 with i.

Then we have

$$t_m = \frac{\pi}{2H_i}\left(\frac{\rho_c(t_i)}{\rho_p(t_m)}\right)^{3/2}\,, \tag{2.311}$$

since we have found that in the relation from a few lessons ago we have exactly the inverse of the relation we just derived connecting Ω and ρ.

It is convenient to consider unperturbed Hubble flow so that the Hubble parameter inside and outside is the same.

Then we have

$$H^2(t_i) = \frac{8\pi G}{3}\rho_c(t_i)\,, \tag{2.312}$$

so we have a cancellation: we find

$$t_m = \frac{\pi}{2H_i}\left(\frac{\rho_c(t_i)}{\rho_p(t_i)}\right)^{1/2} = \left(\frac{3\pi}{32G\rho_p(t_m)}\right)^{1/2}\,, \tag{2.313}$$

so we get

$$\rho_p(t_m) = \frac{3\pi}{32Gt_m^2}\,, \tag{2.314}$$

which holds inside the sphere, while the critical density *outside* is

$$\rho_c(t_m) = \frac{1}{6\pi Gt_m^2}\,. \tag{2.315}$$

We are implicitly using the *synchronous gauge*, and we are finding an exact solution to the EFE: The Lemaître-Tolman-Bondi solution.

We can ask: at a certain given time, how much is the interior density larger than the exterior one? this is given by

$$1+\delta_p(t_m)=\chi(t_m)=\frac{\rho_p(t_m)}{\rho_c(t_m)}=\frac{3\pi}{32G}6\pi G=\left(\frac{3\pi}{4}\right)^2\approx 5.6\,. \tag{2.316}$$

Then we have $\delta_p(t_m)\approx 4.6$. How does this compare to linear theory?

$$\delta_p(t_m)=\delta_t(t_i)\left(\frac{t_m}{t_i}\right)^{2/3}+\delta_-(t_i)\left(\frac{t_m}{t_i}\right)^{-1}\approx\frac{3}{5}\delta_p(t_i)\left(\frac{t_m}{t_i}\right)^{2/3}, \tag{2.317}$$

since the second term vanishes.

We know that $H_i=2/(3t_i)$, so

$$t_m=\frac{3\pi t_i}{4}\times\left(\frac{1+\delta_i}{\delta_i^{3/2}}\right)\approx\frac{3\pi t_i}{4}\delta_i^{-3/2}, \tag{2.318}$$

which comes from the equation from the perturbation density at t_m, in which we substitute $\Omega_p-1=(\delta_p+1)-1=\delta_p$, and we approximate $1+\delta\approx 1$ since δ is small.

So we have

$$\frac{t_m}{t_i}=\frac{3\pi}{4}\delta_i^{-3/2}, \tag{2.319}$$

which gives us a correction

$$\delta_p(t_m)=\frac{3}{5}\delta_i\left(\frac{3\pi}{4}\delta_i^{-3/2}\right)^{2/3}=\frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3}\approx 1.87\,. \tag{2.320}$$

So linear theory would have given us a wrong answer, as we should expect: we are very far from the $\delta\ll 1$ regime in which we expect linearization to hold.

We speak of the virialization time (the time after which the VT starts applying?). It is hard to calculate, different books give different values.

$$t_{\text{vir}}\approx t_c=2t_m\,. \tag{2.321}$$

Then we have

$$E_{\text{tot}}=T+E_{\text{gr}}=\frac{1}{2}E_{\text{gr}}=-T\,, \tag{2.322}$$

since $2T+E_{\text{gr}}=0$.

The total energy at virialization is given by

$$E_{\mathrm{eq}} = -\frac{1}{2}\frac{3}{5}\frac{GM^2}{R_{\mathrm{eq}}}\,, \tag{2.323}$$

where the factor $3/5$ is given by the geometry of the system, we are assuming constant density for our sphere, while the $1/2$ comes from the VT. At the time of collapse instead the energy is given by

$$E_m = -\frac{3}{5}\frac{GM^2}{R_m}\,, \tag{2.324}$$

This then means that the radius at virialization, R_{eq}, is twice the radius at the start of the collapse, R_m:

$$2R_{\mathrm{eq}} = R_m\,. \tag{2.325}$$

This also means, by mass conservation, that $\rho_{\mathrm{vir}} = 8\rho_m$ since the volume shrinks by 2^3.

We want to know the density at collapse::

$$\frac{\rho_p(t_c)}{\rho_c(t_c)} = \underbrace{\frac{\rho_p(t_c)}{\rho_p(t_m)}}_{8}\underbrace{\frac{\rho_p(t_m)}{\rho_c(t_m)}}_{\chi\approx 5.6}\underbrace{\frac{\rho_c(t_m)}{\rho_c(t_c)}}_{2^2} \approx 180\,. \tag{2.326}$$

This object is called $1+\delta(t_c)$, so $\delta(t_c)\approx 179$.

What would happen if we were to use linear theory at this time? Then we would find

$$\delta_+(t_c) = \delta_+(t_m)\left(\frac{t_c}{t_m}\right)^{2/3}\,, \tag{2.327}$$

which gives us

$$\delta_+(t_c) \approx \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} 2^{2/3} \approx 1.686\,, \tag{2.328}$$

which is a kind of "clock": how long can we use linear theory for?

We deal with *** theory: we introduce

$$n(M) = \frac{\mathrm{d}N}{\mathrm{d}M} = \#\text{ of objects per unit volume with mass in } [M, M+\mathrm{d}M]\,. \tag{2.329}$$

Let us consider linear perturbations $\delta(\vec{x},t)$: perturbations in the matter density dealt with using linear theory only.

These tend to fluctuate a lot. We need a *filter*: we use a *low-pass* filter, ignoring the high-frequency modes. It is $W_R(\vec{x})$: R is a spatial radius, we ignore modes with spatial frequency larger than $1/R$.

We use a filter labelled by a mass M, which we find by assuming a certain density, and then using $M \propto R^3$.

It is generally a good idea to assume gaussianity. In the Planck data it was found by Sabino's team that the bounds on non-gaussianity are very low, the data are almost gaussian. So, we have

$$p(\delta_M)\,\mathrm{d}\delta_M = \frac{1}{\sqrt{2\pi\sigma_M^2}} \exp\left(-\frac{\delta_M^2}{2\sigma_M^2}\right) \mathrm{d}\delta_M \;, \tag{2.330}$$

where the variance is typically diverging if we do not apply the filter: we have

$$\sigma_M^2 = \left\langle \delta_M^2 \right\rangle \propto M^{-2\alpha}\,, \tag{2.331}$$

and typically $\alpha \sim 1/2$.

We define a threshold for the value of δ_M, and we want to compute the probability of the value becoming larger than it. We use for the critical value $\rho_c = 1.686$ from before. We have

$$\mathbb{P}_{>\delta_c}(M) = \int_{\delta_c}^{\infty} \mathrm{d}\delta_M \, p(\delta_M)\,. \tag{2.332}$$

so we have

$$\begin{aligned}
n(M)M\,\mathrm{d}M &= \rho_m \big(\mathbb{P}_{>\delta_c}(M) - \mathbb{P}_{>\delta_c}(M + \mathrm{d}M)\big) && \text{(2.333a)}\\
&= \rho_m \left|\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}M}\right| \mathrm{d}M && \text{(2.333b)}\\
&= \rho_m \left|\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}\sigma_M}\right| \left|\frac{\mathrm{d}\sigma_M}{\mathrm{d}M}\right| \mathrm{d}M \;. && \text{(2.333c)}
\end{aligned}$$

Integrating $\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}M}$ we expect to find the matter density again, but we find $1/2$ of it.

This comes from a miscount: as the mass we are considering shrinks, we might me already including smaller objects inside the gravitational influence of larger ones. Properly accounting for this one gets precisely a factor 2.

Integrating we get

$$n(M) = \frac{2}{\sqrt{\pi}} \frac{\rho_m}{M_*^2} \alpha \left(\frac{M}{M_*}\right)^{\alpha} \exp\left(-\left(\frac{M}{M_*}\right)^{2\alpha}\right), \tag{2.334}$$

where $M_* = (2/\delta_c)^{1/2\alpha} M_0$.

Accounting for non spherical collapse we get much better estimates.

Wed Jan 08 2020

2.1 Gravitational waves and interferometry

Guest lecture by Angelo Ricciardone.
An outline:

1. introduction about gravitational waves;
2. frequency bands $\iff$ sources of GW $\iff$ detectors;
3. GWs and observables;
4. stochastic background of GW: characterization, sources, detection (here we discuss *cosmological* sources).

References:

1. Book: "Gravitational Waves - theory and experiments" by Michele Maggiore (Oxford University Press, 2007);
2. Book: "Gravitational Waves - Astrophysical sources" (2018);
3. Paper: "The basics of gravitational wave theory", F. Flannagan (https://arxiv.org/abs/gr-qc/0501041);
4. Review: "Gravitational waves from inflation", (https://arxiv.org/abs/1605.01615);
5. Review: "Cosmological background of gravitational waves", (https://arxiv.org/abs/1801.04268);
6. Kind of unrelated review: "Inflation and the Theory of Cosmological Perturbations" https://arxiv.org/abs/hep-ph/0210162.

We start with some general facts.

Gravitational waves appear naturally in GR: they are propagating oscillations of the gravitational field.

The gravitational interaction is *weak*: this implies that GWs travel freely, but they are also hard to detect.

The frequencies we see for the EM spectrum are in the range $f_{EM} \sim 10^4\,\text{Hz} \div 10^{20}\,\text{Hz}$, radio waves to γ rays.

The typical frequencies of GWs are, instead, in the range $f_{GW} \sim 10^{-16}\,\text{Hz} \div 10^4\,\text{Hz}$, the lower end of this range is the frequency of the CMB GWs while the higher end is given off by astrophysical sources. Ground-based detectors can detect the higher end of this spectrum.

But the CMB is electromagnetic! Is there gravitational radiation corresponding to it?

The wavelength and frequency are typically comparable to the size of the object which is emitting the GWs.

Let us consider astrophysical sources:

$$T = 2\pi \sqrt{\frac{R^3}{GM}} \implies f = \sqrt{\frac{G\rho}{4\pi^2}}, \tag{2.335}$$

and since the mass is related to the density by $M = \rho V = \frac{4}{3}\pi R^3 \rho$, then we have

$$f_{GW} \approx \frac{1}{2\pi} \sqrt{\frac{3GM}{4\pi R^3}}. \tag{2.336}$$

We know that the radius of the object must be greated than the Schwarzschild radius $R_s = 2GM/c^2$: substituting this in we get

$$f_{GW} \approx \frac{1}{4\pi} \frac{c^3}{GM}, \tag{2.337}$$

which means, substituting the numbers, that

$$f_{GW} \approx 10^4 \,\text{Hz} \frac{M_\odot}{M}, \tag{2.338}$$

Is the difference between the formulas coming from different geometries of the problem? Like, a rotating objects versus two inspiralling ones?

We make some estimates for neutron stars. Typically they have $M \sim 1.4 M_\odot$ and $R \sim 10^4$ m. So, we have $f \sim 10^4$ Hz.

For small BHs, we have $M \sim 30 M_\odot$, so $f_{GW} \sim 300$ Hz. This is the band in which LIGO/VIRGO works: these interferometers cannot measure GWs with frequency smaller than 1 Hz.

If we increase the mass, if we use $M \sim 10^7 M_\odot$, we get $f_{GW} \sim 10^{-3}$ Hz: this is the band in which LISA will work.

There are indirect evidences of GW emission.

Pulsars slow down: this is the Hulse-Taylor binary pulsar, two neutron stars which are rotating around each other emit gravitational waves. The GW emission back-reacts on the dynamics of the binary, on a timescale which is observable.

A reference for this is https://arxiv.org/abs/astro-ph/0407149.

We can make a plot: radius of the system vs mass of the system. Since $f \propto M^{1/2} R^{-3/2}$, constant frequency means a powerlaw in this plot, so a straight line in $\log R$ vs $\log M$.

The chirp of the inspiral is *also* a powerlaw. The GR predictions for the period decrease of the Hulse-Taylor binary NS match observations very precisely.

On the 14/09/2015, we had the first direct evidence of GWs with the first detection.

On 17/08/2017 we had the first detection of a NS/NS merger: this was the birth of multimessenger astronomy, since we saw the event with optical telescopes also. This meant that the signal got to us with a speed which is the same as the electromagnetic speed of light.

Now, there is an app: the "GW events app" https://apps.apple.com/us/app/gravitational-wave-events/id1441897107.

In GR, gravitational waves can be polarized in two different ways. These are called "plus" and "cross" polarizations.

Can one of these be rotated into the other? There might be some issue since one is a pseudotensor...

We distinguish: the High Frequency band goes from 10^4 Hz to 1 Hz; the Low Frequency band goes from 1 Hz to 10^{-4} Hz; the Very Low Frequency band goes from 10^{-7} Hz to 10^{-9} Hz; the Extremely Low frequency badn goes from 10^{-15} Hz to 10^{-18} Hz.

There are gaps since in certain frequency regions the methods we have on either side fail for different reasons leaving a gap.

Let us start from HF: it is the domain of Earth-based interferometers: LIGO (Livingstone & Hanford), which has 4 km arms, the two detectors are separated by 3000 km. Also, there is VIRGO near Pisa: an arm is 3 km long.

There is also Geo600 in Hannover, Germany: it has 600 m arms. In Japan there is Kagra: it has 4 km arms.

There are several reasons to have more than one detector: we can identify the position of the source, we can verify signals.

There will be the new Einstein Telescope, in a triangular configuration, in Italy or the Netherlands. There will be the Cosmic Explorer, with 40 km arms.

Now, let us discuss the main sources in the HF band. We have:

1. coalescence of stellar-mass BH binaries and NSs, for these we have an upper bound of $M \lesssim 10^3 M_\odot$;
2. rotation of neutron stars (pulsar);
3. stellar collapse: supernova to BH or NS.

In the LF band, the domain of space-based interferometers, we will have LISA, in a triangular shape with 2.5×10^6 km, and the Japanese DECIGO, with $L \sim 1000$ km.

In the LF band, the sources are:

1. white dwarves merging;

2. NS merging;

3. inspiral and coalescence of SMBH (masses from 100 to $10^8\ M_\odot$).

In the VLF band, we can use Pulsar-Time Array. The sources in this frequency range are:

1. GWs from SMBH with $M > 11M_\odot$, but it seems like there are no black holes this large;

2. GWs from cosmic strings & from phase transitions;

In the ELF band, we have cosmological sources:

1. primordial GWs: here we have $h \sim (E_{\text{infl}}/M_P)^2$.

Typically we have amplitudes increasing as the frequency decreases.
For scalar perturbations we have

$$\frac{\Delta T}{T} \propto \delta\phi\,. \tag{2.339}$$

Vector perturbations decay with the expansion of the universe. We do have tensor perturbations. B mode polarizations correspond to primordial gravitational waves.

What are B modes?

In a simple MM interferometry, $\Delta L \propto h$.

LISA will orbit the Sun at 20° from the Earth. The astrophysical targets for LISA are MBHBs, EMRIs and compact WDs.

Also, there are potential cosmological sources. We have first order phase transitions around the TeV, inflationary GWs, cosmic strings and using MBHBs as standard sirens.

We put powerlaw amplitude spectra in a graph. What is Ω?

We saw powerlaw spectra with Ω increasing with f: do they not have a UV problem since the energy density increased with the frequency? No, since because of the transfer function between the early universe and now at a certain point the powerlaw becomes decreasing.

Cosmological sources are stochastic: they give a background.

If we will have a detector with high sensitivity, we will see many events: they will form a stochastic background.

Can we not correlate the signals in such a way that we only look at GWs from a specific angular region?

A stochastic background of GW comes from a large number of independent uncorrelated sources that are not individually resolvable.

CSGWB is a candidate source for LISA.

For tomorrow, we can either give an overview of cosmological sources for GWs or we can derive the relation

$$\frac{\Delta L}{L} \propto h \,, \tag{2.340}$$

Thu Jan 09 2020

Many of these topics will be discussed in more details in the course by Nicola Bartolo on cosmological perturbations and in the course by Giacomo Ciani on gravitational waves.

Today we will focus on the LISA detector, which works for frequencies $f \sim 10^{-5} \div 10^{-1}\,\text{Hz}$, and on the cosmological stochastic background of GWs (CSGWB).

We can make an analogy with the CMB: we expect to see a background of gravitational radiation coming from all directions.

The stochastic background can have different origins:

1. astrophysical origin: a coherent superposition of a large number of astrophysical sources, which are too weak to be detected separately;
2. cosmological origin: it is generated in the early universe by a variety of mechanisms:
   (a) amplification of primordial tensor fluctuations via inflation;
   (b) GWs from phase transition around the TeV scale (this is not the only energy scale at which they can be emitted, but it is the energy we'd need in order to detect them with LISA);
   (c) GWs from topological defects.

A stochastic background of cosmological origin is expected to be

1. isotropic;
2. stationary;
3. unpolarized, which means that the cross and plus polarizations will have the same amplitude.

Let us discuss the main properties of the frequency spectrum: it is characterized

1. in terms of (normalized) energy density per unit logarithmic interval of frequency: this is called $h_0^2 \Omega_{GW}(f)$;

2. in terms of the spectral density of the ensemble average of the Fourier component of the metric $S_h(f)$;

3. more on the experimental side: in terms of a characteristic amplitude of the stochastic background $h_c(f)$.

Let us define these three quantities. The energy density is given by

$$\Omega_{GW} = \frac{1}{\rho_c} \frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f}\,, \tag{2.341}$$

where ρ_{GW} is the energy density of SGWB, f is the frequency, while ρ_c is the present value of the critical energy density, defined as

$$\rho_c = \frac{3H_0^2}{8\pi G}\,, \tag{2.342}$$

where we usually write $H_0 = h_0 \times 100\,\mathrm{km}/(\mathrm{sMpc})$.

So, usually we plot $h_0^2\Omega_{GW}$ in order to ignore the uncertainties on the measurements of H_0.

We write stochastic GW at a given point $\vec{x} = 0$ in the transverse traceless (TT) gauge: $h_{ii} = 0$ and $\partial_i h^{ij}$: we get

$$h_{ab}(t) = \sum_{A=+,\times} \int_{\mathbb{R}} \mathrm{d}f \int_{S^2} \mathrm{d}\Omega\, \hat{h}_A(f,\Omega) \exp\big(-2\pi i f t\big) e^A_{ab}(\hat{\Omega})\,, \tag{2.343}$$

where we must have the condition $\hat{h}_A(f,\Omega) = \hat{h}^*_A(-f,\Omega)$. Here $\hat{\Omega}$ is a unit vector representing the direction of propagation of the wave, and $\mathrm{d}\hat{\Omega} = \mathrm{d}\cos(\theta)\,\mathrm{d}\phi$, e^A_{ab} are the polarization tensors:

1. $e^+_{ab}(\hat{\Omega}) = 2\hat{m}_{[a}\hat{m}_{b]}$;

2. $e^\times_{ab}(\hat{\Omega}) = 2\hat{m}_{(a}\hat{m}_{b)}$;

where $\hat{m}_{a,b}$ are unit vectors orthogonal to each other and to the propagation direction.

We have the condition $e^A_{ab}e^{Bab} = 2\delta^{AB}$.

Then, assuming a SGWB which is isotropic, unpolarized and stationary we will have

$$\left\langle \hat{h}^*_A(f,\hat{\Omega})\hat{h}_{A'}(f',\hat{\Omega}') \right\rangle = \delta(f-f')\frac{1}{4\pi}\delta^{(2)}(\hat{\Omega},\hat{\Omega}')\delta_{AA'}\frac{1}{2}S_h(f)\,, \tag{2.344}$$

where

$$\delta^{(2)}(\hat{\Omega},\hat{\Omega}') = \delta(\phi-\phi')\delta\big(\cos(\theta)-\cos(\theta')\big)\,. \tag{2.345}$$

The factors of $1/4\pi$ and $1/2$ are for convention, for normalization, so that $\int S_h(f)\,\mathrm{d}f = 1$.

Then the remaining bit, $S_h(f)$, is called the spectral density.

So, using the equaitions we found so far, we get

$$\left\langle h_{ab}(t)h^{ab}(t)\right\rangle = 2\int_{\mathbb{R}} \mathrm{d}f\, S_h(f) \tag{2.346a}$$
$$= 4\int_{f=0}^{f=\infty} \mathrm{d}\log f\, f S_h(f)\,. \tag{2.346b}$$

We define the characteristic amplitude $h_c(f)$ as

$$\left\langle h_{ab}(t)h^{ab}(t)\right\rangle = 2\int_{f=0}^{f=\infty} \mathrm{d}\log f\, h_c^2(f)\,, \tag{2.347}$$

which means that $h_c^2(f) = 2fS_h(f)$.

The last step is to relate $h_c(f)$ and $h_0^2\Omega_{GW}(f)$.

The energy density is defined as:

$$\rho_{GW} = \frac{1}{32\pi G}\left\langle \dot{h}_{ab}\dot{h}^{ab}\right\rangle\,, \tag{2.348}$$

where the average is performed over a wavelength, but by the ergodic theorem it can also be performed over a period. So,

$$\rho_{GW} = \frac{4}{32\pi G}\int_{f=0}^{f=\infty} \mathrm{d}(\log f)\, f(2\pi f)^2 S_h(f)\,, \tag{2.349}$$

whicm means that

$$\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f} = \frac{\pi}{4G}f^2 h_c^2(f) \tag{2.350a}$$
$$= \frac{\pi}{2G}f^3 S_h(f)\,, \tag{2.350b}$$

so in the end we have

$$\Omega_{GW}(f) = \frac{2\pi^2}{3H_0^2}f^2 h_c^2(f)\,, \tag{2.351}$$

therefore

$$\Omega_{GW}(f) = \frac{4\pi^2}{3H_0^2}f^3 S_h(f)\,. \tag{2.352}$$

The *strain* is the quantity S_h.

An advantage of GWs is the fact that they decouple right after emission: they maintain the spectral shape.

If we start from the Einstein-Hilbert action

$$S = \int \mathrm{d}^4x \, Fg \frac{M_P^2}{2} R \,, \tag{2.353}$$

and plug in a tensor-perturbed FRLW metric:

$$\mathrm{d}s^2 = -a^2 \Big(\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij}) \, \mathrm{d}\vec{x}^2 \Big) \,, \tag{2.354}$$

we get the equations of motion for gravitational waves. The scenario of slow-roll inflation gives rise to an energy spectrum which is unobservable with LISA as well as ground-based detectors.

If we add an axion to the inflaton Lagrangian:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) + \underbrace{\frac{\phi}{\Lambda} F_{\mu\nu} \widetilde{F}^{\mu\nu}}_{\text{Additional term}} \,, \tag{2.355}$$

we can see enhanced gravitational waves, which we hope would be detectable with LISA! However the enhancement is very small, we cannot see it.

Chaotic preheating models predict higher amplitudes, however they are at very high frequencies.

We can get observable signals by fine-tuning the parameters, unlikely scenario.

What about phase transition, the collisions of primordial vacuum bubbles? We can get numerical estimates on the spectral shape of this signal, and they seem promising and detectable.

What about cosmic defects? We can have Domain Walls, Cosmic Strings and Cosmic Monopoles.

The prediction here is a flat spectrum at observable frequencies, at possibly observable amplitudes depending on the emission time.

Bibliography

[Pac18] Leonardo Pacciani. *Appunti del corso The Physical Universe*. 2018.