Radiative processes in astrophysics notes

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Contents

Introduction

The professor, Roberto Turolla, will follow the pdf of the book by Rybicki and Lightman [RL79] on his screen. It is available for free.

Understanding radiative processes is fundamental for the analysis of several phenomena: for example, in the Crab nebula the main process is Synchrotron radiation, in the Coma cluster we have Bremsstrahlung, in Cygnus-X1 we have Compton scattering.

Even in the era of multimessenger astrophysics, most of the information we receive still comes from electromagnetic radiation. The required background for this course is classical EM, special relativity and the basics of atomic structure.

The exam is an oral one. The lectures will be recorded and put on the Moodle until the emergency ends, every Wednesday and Thursday. The duration of recorded lectures will be shorter than the duration of the lectures we would have in the classroom.

Chapter 1

Fundamentals of radiative transfer

1.1 Basic properties of the EM spectrum

Thursday 2020-3-12, compiled 2020-09-06

Electromagnetic radiation can be decomposed into a spectrum according to either the frequency ν or the wavelength λ; these are connected by $c = \lambda\nu$, where c is the speed of light.

Sometimes we give the energy of the photons, which can be found using Planck's constant h: $E = h\nu$.

We conventionally divide the spectrum into bands: γ-rays, X-rays, ultraviolet light, visible light, infrared radiation, radio band.

1.1.1 The radiative flux

Let us consider an area element $\mathrm{d}A$, through which radiation passes for a time $\mathrm{d}t$: the energy will be proportional to both $\mathrm{d}A$ and $\mathrm{d}t$, so we define the radiative flux F by saying that the energy is equal to $F\,\mathrm{d}A\,\mathrm{d}t$. Of course, we need to account for orientation: if the surface is not perpendicular to the source the energy is less.

Let us consider a pointlike source, and draw two spherical surfaces of radii r_1 and r_2 along which we compute the flux: if there is no energy loss we must have

$$F(r_1)A_1\,\mathrm{d}t = F(r_2)A_2\,\mathrm{d}t \tag{1.1.1a}$$

$$F(r_1)4\pi r_1^2 = F(r_2)4\pi r_2^2 \tag{1.1.1b}$$

$$F(r_1) = F(r_2)\frac{r_2^2}{r_1^2}\,. \tag{1.1.1c}$$

This tells us that **the flux emitted from a pointlike source decreases like r^{-2}**.

The flux of energy is a measure of all the energy which passes through the surface; however we can get a more detailed description. We cannot consider photons at a specific frequency: the set has measure 0, we must use the language of probability densities. We look at a "pencil" of radiation: all the radiation coming from a solid angle $\mathrm{d}\Omega$ over an area $\mathrm{d}A$ and carried by photons of frequencies between ν and $\nu + \mathrm{d}\nu$.

Note that the solid angle $\mathrm{d}\Omega$ depends on both angles we use to describe a direction (typically θ, φ).

So, we define the *specific intensity of brightness* I_ν by

$$\mathrm{d}E \stackrel{\text{def}}{=} I_\nu \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}\nu \ . \tag{1.1.2}$$

This will depend on position (where we put the detector area with respect to any sources) and on direction (where we look).

We usually neglect the time-dependence. The units of this quantity are those of energy per unit time, area, frequency, solid angle.

How do we account for the direction? The differential flux for radiation coming with an angle θ to the normal is proportional to the dot product of the normal and the incidence direction:

$$\mathrm{d}F_\nu = I_\nu \cos(\theta) \, \mathrm{d}\Omega \ , \tag{1.1.3}$$

so the total net flux is

$$F_\nu = \int I_\nu \cos(\theta) \, \mathrm{d}\Omega \ . \tag{1.1.4}$$

This is about energy, but we can define the momentum flux per unit time per unit area (which is the pressure) with the same procedure; we get an additional factor of $\cos\theta$ since $\vec{p}$ is a vector, and we are interested in its component along the normal of the surface. So, the global formula for this pressure is

$$P_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \, \mathrm{d}\Omega \ . \tag{1.1.5}$$

These are *moments*: in general, a moment for a direction-dependent quantity like the intensity is something in the form

$$n\text{-th moment} = \int I_\nu \cos^n\theta \, \mathrm{d}\Omega \ . \tag{1.1.6}$$

These are frequency dependent; the corresponding *grey* (that is, frequency-integrated) quantities are in the form

$$F = \int F_\nu \, \mathrm{d}\nu \ . \tag{1.1.7}$$

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1.2 Radiative energy density

We define the energy density per unit solid angle, $u_\nu(\Omega)$ by: $\mathrm{d}E = u_\nu(\Omega) \, \mathrm{d}V \, \mathrm{d}\Omega \, \mathrm{d}\nu$. This is the differential amount of energy in the volume $\mathrm{d}V$, carried by radiation coming from the solid angle $\mathrm{d}\Omega$ which has energies between ν and $\nu + \mathrm{d}\nu$.

We consider a cylinder for our volume, its axis being aligned with the direction the radiation is coming from. Its volume can be expressed as $\mathrm{d}V = \mathrm{d}A \, c \, \mathrm{d}t$, where $\mathrm{d}t$ is the time taken by light to cross the height of the cylinder.

We can also express the differential energy using the definition of the specific intensity: then, we can compare the following two equations:

$$\mathrm{d}E = u_\nu(\Omega) c \,\mathrm{d}A \,\mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}\nu \tag{1.2.1a}$$

$$= I_\nu \,\mathrm{d}A \,\mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}\nu \,, \tag{1.2.1b}$$

which allows us to conclude that $u_\nu = I_\nu / c$. Also, if we want to get the total energy density we just need to integrate over the volume of the whole sphere:

$$u_\nu = \frac{1}{c} \int I_\nu(\Omega) \,\mathrm{d}\Omega \overset{\text{def}}{=} \frac{4\pi}{c} J_\nu \,, \tag{1.2.2}$$

where J_ν is the *mean intensity*: $J_\nu = \langle I_\nu \rangle_\Omega$.

We can also integrate over frequencies to get the total energy density:

$$u = \int u_\nu \,\mathrm{d}\nu = \frac{4\pi}{c} \int J_\nu \,\mathrm{d}\nu \,. \tag{1.2.3}$$

1.2.1 Isotropic radiation field

An isotropic radiation field is one for which the specific intensity does not depend on angles. Let us start from the definitions of u_ν and P_ν:

$$u_\nu = \int \frac{I_\nu}{c} \,\mathrm{d}\Omega = \frac{4\pi J_\nu}{c} \tag{1.2.4a}$$

$$P_\nu = \int \frac{I_\nu}{c} \cos^2\theta \,\mathrm{d}\Omega = \int \frac{I_\nu}{c} \cos^2\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi \,, \tag{1.2.4b}$$

and let us use the assumption that I_ν does not depend on Ω: so we can bring it out of the integrals, to find

$$u_\nu = 4\pi \frac{I_\nu}{c} \tag{1.2.5a}$$

$$P_\nu = -2\pi \frac{I_\nu}{c} \int \cos^2\theta \,\mathrm{d}\cos\theta = 2\pi \frac{I_\nu}{c} \int_{-1}^{1} x^2 \,\mathrm{d}x \tag{1.2.5b}$$

The differential is negative, but we swap the integration bounds.

$$= \frac{4\pi}{3} \frac{I_\nu}{c} \,, \tag{1.2.5c}$$

which gives us the result we sought:

$$P_\nu = \frac{u_\nu}{3} \,. \tag{1.2.6}$$

1.2.2 Specific intensity along a ray

We wish to see how the specific intensity I_ν changes along a beam of light rays. Let us consider two positions $1, 2$ along the beam, separated by a distance R. Then, by definition we will have, for $i = 1, 2$:

$$\mathrm{d}E_i = I_{\nu,i} \,\mathrm{d}A_i \,\mathrm{d}t_i \,\mathrm{d}\Omega_i \,\mathrm{d}\nu_i \,. \tag{1.2.7}$$

First, we make the assumption of the gravitational field being weak: therefore time dilation is negligible, so $\mathrm{d}t_1 = \mathrm{d}t_2$ and $\mathrm{d}\nu_1 = \mathrm{d}\nu_2$. Now, we ask these two expressions to describe the same beam: the same photons will pass through $\mathrm{d}A_1$ and $\mathrm{d}A_2$. Therefore, by conservation of energy, $\mathrm{d}E_1 = \mathrm{d}E_2$.

This means that

$$I_{\nu,1}\,\mathrm{d}A_1\,\mathrm{d}\Omega_1 = I_{\nu,2}\,\mathrm{d}A_2\,\mathrm{d}\Omega_2\ . \tag{1.2.8}$$

We can treat the photons' motion as time-reversal symmetric: so, whether they pass through $\mathrm{d}A_1$ or $\mathrm{d}A_2$ first is irrelevant. The linear scale of the differential area element is infinitesimal, while the separation between the two points, R, is macroscopic: so, we can consider all the photons which come through $\mathrm{d}A_1$ to be coming from a point source from position 2 and vice versa. This will define the angular size as seen from the "pointlike" position 2, $\mathrm{d}\Omega_2$.

Therefore, the differential solid angle will look like

$$\mathrm{d}\Omega_2 = \frac{\mathrm{d}A_1}{R^2}\,, \tag{1.2.9}$$

and we can apply the same reasoning reversing the photons' motion to find the same, alternate relation with $(1 \leftrightarrow 2)$. We can use this to write

$$I_{\nu,1}\frac{\mathrm{d}A_1}{\mathrm{d}\Omega_2} = I_{\nu,2}\frac{\mathrm{d}A_2}{\mathrm{d}\Omega_1} \tag{1.2.10a}$$

$$I_{\nu,1}R^2 = I_{\nu,2}R^2 \tag{1.2.10b}$$

$$I_{\nu,1} = I_{\nu,2}\,. \tag{1.2.10c}$$

This means that, under our assumptions, the specific intensity is conserved:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0\,, \tag{1.2.11}$$

where s is a parameter describing the light ray's trajectory. This is useful since, if the variation of the specific intensity is zero in a vacuum, then its variation in the presence of matter will only be due to transfer phenomena, and the sign of the variation will describe whether energy is being added or removed.

1.3 Radiative transfer

In general, as radiation passes through matter, its specific intensity changes. This is due to emission and absorption, but also to scattering, which preserves the total number of photons: even in the low-energy limit it can change the angular distribution of the radiation, and in general it also changes the energy of the photon.

1.3.1 Emission

Emission is a process through which photons are created. We can define the grey emission coefficient j and the monochromatic emission coefficient j_ν as:

$$\mathrm{d}E = j\,\mathrm{d}V\,\mathrm{d}\Omega\,\mathrm{d}t \tag{1.3.1a}$$

$$\mathrm{d}E = j_\nu\,\mathrm{d}V\,\mathrm{d}\Omega\,\mathrm{d}t\,\mathrm{d}\nu\ , \tag{1.3.1b}$$

they quantify the energy added to the radiation field per unit volume, solid angle (in order to account for the direction of emission) and unit time. For the monochromatic coefficient, we restrict ourselves to radiation emitted in the range from ν to $\nu + \mathrm{d}\nu$.

In the case of an isotropic emission we can integrate over the solid angle to find

$$P_\nu = 4\pi j_\nu\,, \tag{1.3.2}$$

the radiated power per unit volume and frequency.

Another useful concept is the emissivity ϵ_ν: it is the energy added to the radiation field per unit time, frequency and mass in the directions described by the solid angle $\mathrm{d}\Omega$. We express the infinitesimal mass as $\mathrm{d}m = \rho\,\mathrm{d}V$, so that in the case of isotropic emission we have

$$\mathrm{d}E = \epsilon_\nu \rho\,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu\,\frac{\mathrm{d}\Omega}{4\pi}\,, \tag{1.3.3}$$

so the emissivity ϵ_ν and the emission coefficient j_ν are connected by

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}\,. \tag{1.3.4}$$

We wish to describe the variation in specific intensity due to this emission. Let us consider a beam of cross section $\mathrm{d}A$ going through a length $\mathrm{d}s$, so that the volume it occupies is $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}s$.

Now, if we compare the definitions of j_ν and I_ν we find that they differ by a factor $\mathrm{d}V\,/\,\mathrm{d}A = \mathrm{d}s$, the length of the beam cylinder we defined.

The difference between the specific intensities at the start and end of the cylinder would be zero without emission, now instead their difference can be calculated from the energy added; as we said most of the differentials simplify and we get that the variation of specific intensity is

$$\mathrm{d}I_\nu = j_\nu\,\mathrm{d}s\ . \tag{1.3.5}$$

1.3.2 Absorption

Absorption is described by a coefficient $\alpha_\nu > 0$, which is dimensionally an inverse length. The absorption law which defines the coefficient gives the decrease in radiative intensity for radiation of intensity I_ν crossing an absorbing medium of length $\mathrm{d}s$:

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu\,\mathrm{d}s\ . \tag{1.3.6}$$

Why should the variation in intensity be proportional to the intensity itself? We give a simple argument: let us assume that absorption is due to randomly absorbers with number density n and (frequency dependent) cross section σ_ν.

Let us consider our usual cylinder with cross sectional area $\mathrm{d}A$ and length $\mathrm{d}s$: the number of absorbers in it will be $\mathrm{d}N = n\,\mathrm{d}A\,\mathrm{d}s$. The total effective cross section area presented for absorption will be $\sigma_\nu\,\mathrm{d}N$. The energy contained in photons in this cross sectional area will be lost: the energy lost $-\,\mathrm{d}I_\nu$ can be calculated as

$$-\,\mathrm{d}I_\nu\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu = I_\nu(\sigma_\nu n\,\mathrm{d}A\,\mathrm{d}s)\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \tag{1.3.7a}$$

$$\mathrm{d}I_\nu = -n\sigma_\nu I_\nu\,\mathrm{d}s\ , \tag{1.3.7b}$$

which is the relation written above, with $n\sigma_\nu = \alpha_\nu$. The number density is proportional to the mass density: $n\overline{m} = \rho$ where $\overline{m}$ is the average mass of a particle. Therefore, we can express α_ν as

$$\alpha_\nu = \rho\kappa_\nu = n\overline{m}\frac{\sigma_\nu}{\overline{m}}\,, \tag{1.3.8}$$

so we can see that κ_ν is a cross sectional area per unit mass. It is called the *mass absorption coefficient* or the *opacity*.

Conditions for validity This line of reasoning holds as long as the inter-absorber distances $d \sim n^{-1/3}$ are large compared to the linear scale of the cross section $\sigma_\nu^{1/2}$: we ask

$$\sigma_\nu^{1/2} \ll n^{-1/3}\,, \tag{1.3.9}$$

and also we must assume that the absorbers are independent and randomly distributed (at least locally). These assumptions are usually met in astrophysical systems.

1.3.3 The radiative transfer equation

We can account for both absorption and emission in a combined equation for the derivative with respect to the beam length travelled s of the specific intensity I_ν:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu\,, \tag{1.3.10}$$

and we can see that in the absence of emission and absorption I_ν is unchanged, as we have shown before. If j_ν and α_ν are known we can integrate this differential equation to find the specific intensity.

This will *not* be the case when we will include scattering: the scattering term will not just depend on I_ν but in its integral on the sphere, making this an integro-differential equation.

Solutions to the transfer equation in simple cases

If there is only emission, that is, only j_ν is nonzero, the intensity increases (linearly in s for constant j_ν):

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu \implies I_\nu = I_\nu(0) + \int_{s_0}^{s} j_\nu(\widetilde{s})\,\mathrm{d}\widetilde{s}\,. \tag{1.3.11}$$

If there is only absorption, that is, only α_ν is nonzero, then the intensity decreases (exponentially in s for constant j_ν):

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu \implies I_\nu = I_\nu(s_0)\exp\left(-\int_{s_0}^{s}\alpha_\nu(\widetilde{s})\,\mathrm{d}\widetilde{s}\right). \tag{1.3.12}$$

1.3.4 Optical depth and source function

The optical depth τ_ν is defined so that it increases by 1 when the intensity of light at frequency ν decreases e-fold: its differential is

$$\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}s\,, \tag{1.3.13}$$

so that the solution in the absorption-only case reads $I_\nu \propto e^{-\int \mathrm{d}\tau} = e^{-\tau}$.

So, a useful distinction to make is based on the magnitude of τ, since it quantifies how much light can shine through a medium:

1. if $\tau \gg 1$ the medium is said to be *opaque* or *optically thick*;
2. if $\tau \ll 1$ the medium is said to be *transparent* or *optically thin*;
3. if $\tau \approx 1$ the medium is said to be *translucent*.

If we define the source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu}\,, \tag{1.3.14}$$

we can write the radiative transfer equation as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu\,. \tag{1.3.15}$$

Divided through by α_ν, used definition of τ_ν.

1.3.5 A formal solution of the radiative transfer equation

We can solve this equation by defining $Y_\nu = I_\nu e^{\tau_\nu}$, which obeys

$$\frac{\mathrm{d}Y_\nu}{\mathrm{d}\tau_\nu} = \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}e^{\tau_\nu} + I_\nu e^{\tau_\nu}\,, \tag{1.3.16}$$

so we can multiply the radiative transport equation by e^{τ_ν} to get

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}e^{\tau_\nu} = -I_\nu e^{\tau_\nu} + S_\nu e^{\tau_\nu} \tag{1.3.17a}$$

$$\frac{\mathrm{d}Y_\nu}{\mathrm{d}\tau_\nu} = S_\nu e^{\tau_\nu} \tag{1.3.17b}$$

$$Y_\nu(\tau_\nu) = Y_\nu(0) + \int_0^{\tau_\nu} S_\nu(\widetilde{\tau}_\nu)e^{\widetilde{\tau}_\nu}\,\mathrm{d}\widetilde{\tau}_\nu \tag{1.3.17c}$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\widetilde{\tau}_\nu)e^{\widetilde{\tau}_\nu-\tau_\nu}\,\mathrm{d}\widetilde{\tau}_\nu\,, \tag{1.3.17d}$$

Divided through by e^{τ_ν}.

which has a direct intuitive meaning: the intensity at a certain point must be computed accounting for the initial one and emission all through the beam before the point we are considering, and each of these contributions to the emission is weighted by an exponential factor: the relevance of a term decreases if the optical distance increases.

If S_ν is a constant, we have the simplified expression

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right), \tag{1.3.18}$$

and we can see that for large optical depths the intensity is dictated purely by the source at that point, since $1 - e^{-\tau_\nu}$ approaches 0 for large τ_ν.

1.4 Thermal radiation

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Consider a blackbody enclosure at equilibrium at a certain temperature T. If we open a small hole in the enclosure and measure the radiation inside it, the intensity I_ν we will see will be isotropic, and only depending on the temperature T: let us call the (still unspecified) way this dependence look $I_\nu = B_\nu(T)$. The function $B_\nu(T)$ will be called the **blackbody function**.

Now, imagine we put a small chunk of material inside the blackbody enclosure. If this material is characterized by a source function $S_\nu = j_\nu/\alpha_\nu$, then the evolution of the intensity inside the cavity will be

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu\,. \tag{1.4.1}$$

If we wait for equilibrium, though, we must have $I_\nu \equiv B_\nu$, meaning that its derivative must be zero: this implies that $I_\nu = B_\nu = S_\nu$.

This is called **Kirkhoff theorem**: a chunk of material which can emit and absorb and which is placed in a blackbody enclosure then it must satisfy

$$B_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu\,. \tag{1.4.2}$$

Planck function derivation

What is the explicit expression of the blackbody function B_ν? From Bose-Einstein statistics we know that the phase space density of photons in thermal equilibrium is given by

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p} = \frac{2}{h^3}\frac{1}{\exp\left(\frac{h\nu}{k_BT}\right) - 1}\,, \tag{1.4.3}$$

where the 2 comes from the two polarizations of the photons, h^3 is the size of the phase space cell, while the exponential factor is the occupation number.

Now, we would like to write this as a function of frequency, not of the three components of the momentum. The frequency ν and the momentum $p = \left|\vec{p}\right|$ of a photon are connected by $h\nu = E = cp$; and the momentum space element d^3p can be written as $\mathrm{d}^3p = \mathrm{d}\Omega\, p^2\,\mathrm{d}p$.

Bringing this to the other side and integrating over the solid angle (since the frequency of the photon does not depend on its direction we can substitute $\int \mathrm{d}\Omega = 4\pi$) we can write

$$\frac{\mathrm{d}N}{\mathrm{d}^3x} = \frac{8\pi}{c^3}\frac{\nu^2\,\mathrm{d}\nu}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,, \tag{1.4.4}$$

the **number density** of photons. From this we can quickly recover the **energy density** as well, since each photon with frequency ν has energy $h\nu$: therefore

$$\frac{\mathrm{d}E}{\mathrm{d}^3x} = \frac{8\pi}{c^3}\frac{h\nu^3\,\mathrm{d}\nu}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,. \tag{1.4.5}$$

If we want to recover the energy density per unit frequency and per unit solid angle we can divide by $\mathrm{d}\Omega\,\mathrm{d}\nu$: this yields

$$u_\nu(\Omega) = \frac{\mathrm{d}E}{\mathrm{d}^3x\,\mathrm{d}\Omega\,\mathrm{d}\nu} = \frac{2h}{c^3}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,, \tag{1.4.6}$$

which we have previously shown to be equal to $u_\nu(\Omega) = I_\nu/c$: therefore the specific intensity, which corresponds to the Planck function $B_\nu(T)$, is

$$I_\nu = B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_BT}\right)-1}\,. \tag{1.4.7}$$

Properties of the Planck function

Let us consider the limits of the Planck function. If the energy of the photon, $h\nu$, is much lower than the average photon energies k_BT then we have

$$B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_BT}\right)-1} \approx \frac{2h}{c^2}\frac{\nu^3}{1+\frac{h\nu}{k_BT}-1} = \frac{2k_BT}{c^2}\nu^2\,, \tag{1.4.8}$$

the **Rayleigh-Jeans** law.

On the other hand, if the photon energy $h\nu$ is much larger than k_BT we can ignore the -1 in the denominator and just write

$$B_\nu(T) \approx \frac{2h\nu^2}{c^2}\exp\left(-\frac{h\nu}{k_BT}\right), \tag{1.4.9}$$

the **Wien law**. The exponential term dominates, so asymptotically the Planckian goes to zero.

Since it is a continuously differentiable function which increases at low frequency and decreases at high frequency, it will have a maximum, which we expect to appear somewhere around $h\nu \approx k_BT$. This is precisely what is described by the **Wien displacement law**.

Let us see where this maximum lies if the Planckian is expressed as a function of frequency, as we are doing now. If we define $x = h\nu/k_B T$, the Planck function is proportional to

$$B_\nu(T) \propto \frac{x^3}{e^x - 1}\,, \tag{1.4.10}$$

so we can find its maximum by seeking the stationary point

$$\frac{\partial B_\nu}{\partial \nu} \propto \frac{\partial B_\nu}{\partial x} = 0\,. \tag{1.4.11}$$

The derivative reads

$$\frac{\partial B_\nu}{\partial x} \propto \frac{3x^2(e^x - 1) - x^3 e^x}{(e^x - 1)^2} \propto e^x(3 - x) - 3 \stackrel{!}{=} 0\,. \tag{1.4.12}$$

This can also be expressed as $x = 3(1 - e^{-x})$ as well. It is a transcendental equation: there is no closed-form solution, however we can easily approximate it numerically. The maximum is found to be approximately $x = 2.82$, which means that the maximum is attained at the frequency

$$\nu \approx 2.82\frac{k_B T}{h}\,. \tag{1.4.13}$$

The fact that the peak of the curve depends linearly on T is precisely the statement of the Wien displacement law.

Another interesting question to ask is about how B_ν varies with T: its partial derivative reads

$$\frac{\partial B_\nu}{\partial T} = \frac{2h^2\nu^4}{k_B T^2 c^2} \frac{e^{h\nu/k_B T}}{\left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right)^2} > 0\,, \tag{1.4.14}$$

so it is positive for any value of ν. It will be clearer with a picture, however we can say that this increase with T is much faster for $h\nu < k_B T$ than it is for $h\nu > k_B T$.

The specific energy density is found from $u_\nu = I_\nu/c$ and integrating over the solid angles (so, multiplying by 4π), so we have

$$u_\nu(T) = \frac{4\pi}{c} B_\nu\,, \tag{1.4.15}$$

and if we want to compute the total energy density we will have

$$u = \int_0^\infty u_\nu(T)\,\mathrm{d}\nu = \frac{4\pi}{c}\frac{2h}{c^2}\int_0^\infty \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}\,\mathrm{d}\nu \tag{1.4.16}$$

$$= \frac{8\pi h}{c^3}\frac{k_B^4 T^4}{h^4}\int_0^\infty \frac{x^3}{e^x - 1}\,\mathrm{d}x\,, \tag{1.4.17}$$

where, as before, $x = h\nu/k_B T$. This already shows the main result: $u \propto T^4$, however to get a closed form expression we need to solve the integral as well. This integral is of the type

$$\int_0^\infty \frac{x^n}{e^x - 1}\,\mathrm{d}x = n!\zeta(n+1)\,, \tag{1.4.18}$$

which we need to evaluate for $n = 3$. The Riemann zeta function fortunately has a closed form for even integers, in our case $\zeta(4) = \pi^4/90$, which we multiply by $3! = 6$. This yields

$$u = \frac{8\pi h}{c^3}\frac{k_B^4 T^4}{h^4}\frac{\pi^4}{15} = \frac{8\pi^5 k_B^4}{15c^3h^3}T^4 = aT^4\,, \tag{1.4.19}$$

where the constant $a \approx 7.57 \times 10^{-16}\,\mathrm{kgK^{-4}m^{-1}s^{-2}}$ is called the **blackbody constant**.

Now, we discuss the **flux** of the radiation field. If we choose a surface and measure the flux across it, our assumption of isotropy means that we will always find zero; however it is interesting to measure the *outgoing* flux from an oriented surface, counting only the photons going through it in one direction.

I understand that this gives a factor 1/2, but why should it correspond to integrating over half the solid angle, except by isotropy? The half-sphere does not seem to me to have physical meaning...

The flux we can compute is then:

$$F = \frac{1}{2}\int \mathrm{d}\Omega\,\mathrm{d}\nu\, B_\nu \cos\theta = \frac{2\pi}{2}\int_0^\pi \mathrm{d}\theta\cos\theta\sin\theta\int_0^\infty \mathrm{d}\nu\, B_\nu \tag{1.4.20}$$

$$= \pi \underbrace{\int_{-1}^1 \mu\,\mathrm{d}\mu}_{=1}\underbrace{\int_0^\infty B_\nu\,\mathrm{d}\nu}_{=uc/(4\pi)} = \frac{uc}{4} \tag{1.4.21}$$

$$= \frac{2\pi^5 k_B^4}{15c^2h^3}T^4 = \sigma T^4\,, \tag{1.4.22}$$

where σ is called the **Stefan-Boltzmann constant**, which is related to a by $\sigma = ca/4$.

Another useful result to state here is that the integral of the Planck function in $\mathrm{d}\nu$ is:

$$\int_0^\infty B_\nu\,\mathrm{d}\nu = \frac{uc}{4\pi} = \frac{\sigma}{\pi}T^4\,. \tag{1.4.23}$$

1.5 Scattering

This is the third process which can affect the radiation field, besides absorption and emission. In the astrophysical context, we will only need to concern ourselves with scattering by *free electrons*.

When modelling absorption, we had the true absorption coefficient $\alpha_\nu = n\sigma_\nu$, where σ_ν was the cross section of the individual absorber. We can model scattering in the same way: we can introduce a "scattering absorption coefficient" $\alpha_\nu^{(s)}$, which differs from the true one in that in absorption photons are not destroyed but merely deflected.

However, from the point of view of the beam in a specific direction they disappear, as if they were absorbed. We will then also have to consider the inverse phenomenon: photons coming into the beam we are considering from another. This can be modelled as a "scattering emission".

We express the amount of photons lost from our beam as

$$\int \alpha_\nu^{(s)} I_\nu \, \mathrm{d}\Omega \, . \tag{1.5.1}$$

Let us make a simplifying assumption: that the scattering is **isotropic**, the scattered photon has an equal probability to go in any direction. Also, let us assume that the frequency ν of the photon does not change upon scattering. We will see that both of these conditions are satisfied for Thompson scattering but not in the general situation.

Then, we are left with $\alpha_\nu^{(s)} \int I_\nu \, \mathrm{d}\Omega = 4\pi\alpha_\nu^{(s)} J_\nu$.

This is in any direction, while the radiation scattered in the unit solid angle will be

$$j_\nu^{(s)} = \alpha_\nu^{(s)} J_\nu \, . \tag{1.5.2}$$

In general, since we do not have either conservation of photon frequency nor isotropy we will need to write a differential scattering cross section like

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu' \, \mathrm{d}\Omega'} \, , \tag{1.5.3}$$

while the full expression for the radiation scattered in a specific direction Ω at a frequency ν will read

$$j_\nu^{(s)} = \int \mathrm{d}\Omega' \, n \frac{\mathrm{d}\sigma}{\mathrm{d}\nu \, \mathrm{d}\Omega \, \mathrm{d}\nu' \, \mathrm{d}\Omega'} I_{\nu'}(\Omega') \, \mathrm{d}\nu \, . \tag{1.5.4}$$

This is called the **scattering kernel**.

The transfer equation for scattering

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If we only account for true emission and absorption the transfer equation looks like

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu \, . \tag{1.5.5}$$

For isotropic and conservative scattering we can modify this equation by adding on the scattering absorption and emission terms:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -(\alpha_\nu + \alpha_\nu^{(s)}) I_\nu + j_\nu + \alpha_\nu^{(s)} J_\nu \, . \tag{1.5.6}$$

Now we cannot give a formal solution anymore: the derivative of the intensity now depends not only on the intensity itself but on its average over all solid angles, $J_\nu = \langle I_\nu \rangle_\Omega$. This will typically pose a problem: scattering is often the dominant phenomenon in radiative transfer for astrophysical systems.

In order to solve the equation we can resort to numerical methods: for example we can use an iterative relaxation procedure in which we start off by computing the solution without scattering, calculate the mean intensity and plug it as a fixed value into the next iteration, and keep going.

We can write the transfer equation as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = (\alpha_\nu + \alpha_\nu^{(s)}) \left(-I_\nu + \frac{j_\nu + \alpha_\nu^{(s)} J_\nu}{\alpha_\nu + \alpha_\nu^{(s)}} \right) = (\alpha_\nu + \alpha_\nu^{(s)})(-I_\nu + S_\nu) \,, \tag{1.5.7}$$

where we define a new form for the source function:

$$S_\nu = \frac{j_\nu + \alpha_\nu^{(s)} J_\nu}{\alpha_\nu + \alpha_\nu^{(s)}} \,, \tag{1.5.8}$$

where, as long as Kirkhoff's law holds, we can substitute $j_\nu = \alpha_\nu B_\nu$.

The optical depth is derived from the *total* absorption coefficient:

$$\mathrm{d}\tau_\nu = (\alpha_\nu + \alpha_\nu^{(s)})\, \mathrm{d}s \ . \tag{1.5.9}$$

With these definitions, we can write the transfer equation like before,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu \,. \tag{1.5.10}$$

If there is scattering this is only apparently simple, the formulation only hides the complexity.

1.5.1 Mean free path

We defined absorption as $\alpha_\nu = n\sigma_\nu$: this is just the inverse of the mean free path, so we have (for true absorption):

$$\ell_\nu = \frac{1}{\alpha_\nu} \,, \tag{1.5.11}$$

and we can define a mean free path for scattering in the exact same way, with $\alpha_\nu^{(s)}$. These will be the mean free paths of a photon before it undergoes that specific process, if we want to compute the mean free path of a photon before it undergoes *either one* then we can just take the inverse of $\alpha_\nu + \alpha_\nu^{(s)}$, the total absorption coefficient.

Now, consider a medium with scattering, emission and absorption. Typically, a photon will be emitted, scatter a few times, and then be absorbed. Let us say that between emission and absorption it is scattered N times, and let us call the spatial intervals it travels between these scatterings $\vec{r}_i$, where i goes from 1 to N. The total distance travelled will look like $\vec{R} = \sum_i \vec{r}_i$.

The average of its square value will be

$$\left\langle R^2 \right\rangle = \sum_{ij} \left\langle \vec{r}_i \vec{r}_j \right\rangle = \sum_i \left\langle r_i^2 \right\rangle + 2 \sum_{i<j} \left\langle \vec{r}_i \vec{r}_j \right\rangle \ . \tag{1.5.12}$$

The mixed averages $\left\langle r_i r_j \right\rangle$ evaluate to zero, since each scattering is isotropic and independent; on the other hand each of the $\left\langle r_i^2 \right\rangle$ is equal to ℓ_ν^2, the square of the mean free path. This means that we have $\left\langle R^2 \right\rangle = N\ell_\nu^2$, meaning that the average distance occurring between each scattering is given by $\ell_* = \sqrt{N}\ell_\nu$.

Now, let us suppose that the photon is being scattered in a medium whose characteristic length is L. Then typically a photon will need to be scattered N times before escaping the medium, where $L = \sqrt{N}\ell_\nu$. This means that

$$\sqrt{N} = \frac{L}{\ell_\nu} = \alpha_\nu L \sim \tau_\nu \,, \tag{1.5.13}$$

where the last order-of-magnitude relation comes from the fact that the *differential* of the optical path is given by $\mathrm{d}\tau_\nu = \alpha_\nu \,\mathrm{d}s$.

This means that, at least in terms of order of magnitude, $N \sim \tau_\nu^2$. This holds as long as the medium is optically thick, that it, τ_ν is larger (ideally much larger) than 1.

What happens instead if the medium is optically thin, with $\tau_\nu < 1$? Then, we can neglect emission and scattering emission, and only consider scattering absorption. Then, the transfer equation will read

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu \,, \tag{1.5.14}$$

which is solved by $I_\nu = I_\nu(0)e^{-\tau_\nu}$. This can also be written, by adding $I_\nu(0)$ to both sides, as

$$I_\nu(0) - I_\nu = I_\nu(0)\left(1 - e^{-\tau_\nu}\right) \tag{1.5.15}$$

$$\frac{I_\nu(0) - I_\nu}{I_\nu} = 1 - e^{-\tau_\nu} \approx \tau_\nu \,. \tag{1.5.16}$$

Now, the average number of scatterings will be less than 1 and will correspond to the average relative intensity lost, $N = \Delta I_\nu / I_\nu(0)$. Therefore, in this case we will have $N \sim \tau_\nu$.

Putting together these two limiting cases, we can roughly say that we will have

$$N \approx \max\left(\tau_\nu, \tau_\nu^2\right). \tag{1.5.17}$$

After a mean free path the photon can be either scattered or absorbed. The probability that it is absorbed as opposed to being scattered is given by

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \alpha_\nu^{(s)}} \,, \tag{1.5.18}$$

while the probability that it is scattered is $1 - \epsilon_\nu$, and this last quantity is typically called the **single scattering albedo**.

The average number of mean free paths travelled before absorption will be[1]

$$N = \frac{1}{\epsilon_\nu} \,, \tag{1.5.21}$$

[1] This can be shown using the identity

$$\sum_{i=1}^{\infty} i p^{i-1} = \frac{1}{(1-p)^2} \,, \tag{1.5.19}$$

so we can make the following manipulation to find an explicit expression for the mean path between emission and absorption:

$$\ell_*^2 = N\ell_\nu^2 = \frac{\ell_\nu^2}{\epsilon_\nu} = \frac{1}{(\alpha_\nu + \alpha_\nu^{(s)})^2}\frac{\alpha_\nu + \alpha_\nu^{(s)}}{\alpha_\nu} = \frac{1}{\alpha_\nu(\alpha_\nu + \alpha_\nu^{(s)})}\,. \tag{1.5.22}$$

This means that

$$\ell_* = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \alpha_\nu^{(s)})}}\,. \tag{1.5.23}$$

Now we make use again of the order-of-magnitude relation $\tau \sim \alpha L$ where L is the length scale of the medium; if we multiply L by $1/\ell_*$, the effective absorption coefficient, we get

$$\tau \sim \frac{L}{\ell_*} = \sqrt{L^2\alpha_\nu(\alpha_\nu + \alpha_\nu^{(s)})} = \sqrt{\tau_\nu(\tau_\nu + \tau_\nu^{(s)})}\,, \tag{1.5.24}$$

which is larger than τ_ν alone would be. This is called the **effective optical depth**; the fact that this is larger than τ_ν means that scattering traps the photon in the region for a longer time than it would remain there with absorption alone.

1.6 Radiative diffusion

This is a way to approximately solve the radiative transfer equation under certain assumptions, the main one being that the medium should be very **optically thick**.

Another assumption we will make is the **plane-parallel** approximation: this means that the properties of the medium only vary with respect to one coordinate, which we will call z. We will further assume that the properties of the radiation also only vary along z.

Now, this is about the *spatial* dependence of the quantities, however certain ones like the radiation intensity I_ν are also intrinsically vectors, so they also depend on the direction. However, we will have cylindrical symmetry for rotations around the z axis: therefore, the intensity will be a function of z and of the angle θ between the ray and the z axis.

The differential length travelled by the ray can be expressed in terms of the distance travelled along the z axis as

$$\mathrm{d}s = \frac{\mathrm{d}z}{\cos\theta} = \frac{\mathrm{d}z}{\mu}\,, \tag{1.6.1}$$

where we define the usual shorthand, $\mu = \cos\theta$.

---

which can be proven by differentiating the geometric series and bringing the derivative into the sum, which converges absolutely.

Then, we can compute the average number of scatterings as

$$\langle N\rangle = \sum_{i=1}^{\infty}\epsilon_\nu i(1-\epsilon_\nu)^{i-1} = \frac{\epsilon_\nu}{(1-(1-\epsilon_\nu))^2} = \frac{1}{\epsilon_\nu}\,. \tag{1.5.20}$$

The radiative transfer equation reads

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = (-I_\nu + S_\nu)\left(\alpha_\nu + \alpha_\nu^{(s)}\right), \tag{1.6.2}$$

where we can substitute

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \mu\frac{\partial I_\nu}{\partial z}, \tag{1.6.3}$$

and using this we can express the radiative transfer equation like

$$I_\nu(z,\mu) = S_\nu - \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}}\mu\frac{\partial I_\nu}{\partial z}. \tag{1.6.4}$$

Now we make use of the assumption that the optical depth is large. This means that $\tau \sim (\alpha_\nu + \alpha_\nu^{(s)})\ell$ is large, where ℓ is the characteristic scale of the system.

Should we not say explicitly that, besides being optically thick, the medium's thickness should be "slowly varying"?

This means that the term proportional to $\partial I_\nu/\partial z$ is small compared to the source function, so we can apply perturbation theory.

To zeroth order, assuming thermal equilibrium, we will have

$$I_\nu^{(0)} \approx S_\nu^{(0)} = B_\nu(T). \tag{1.6.5}$$

Then, the first-order approximation can be found by inserting the zeroth-order expression into the equation:

$$I_\nu^{(1)}(z,\mu) = B_\nu - \frac{\mu}{\alpha_\nu + \alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial z}. \tag{1.6.6}$$

An important thing to note is the linear relation between I_ν and the cosine μ of the angle.

With this first-order result we can compute the flux: if we were to use the zeroth-order one we would get zero, since blackbody radiation is isotropic. The flux is given by the integral over the solid angle: the contribution to the integral due to the B_ν term vanishes, since it is isotropic

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu)\mu\,\mathrm{d}\Omega = 2\pi\int_0^\pi I_\nu^{(1)}(z,\mu)\cos\theta\sin\theta\,\mathrm{d}\theta \tag{1.6.7}$$

$$= -\frac{2\pi}{\alpha_\nu + \alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial z}\underbrace{\int_1^{-1}\mu^2(-1)\,\mathrm{d}\mu}_{=2/3} \tag{1.6.8}$$

$$= -\frac{4\pi}{3(\alpha_\nu + \alpha_\nu^{(s)})}\frac{\partial B_\nu}{\partial z} \tag{1.6.9}$$

$$= -\frac{4\pi}{3(\alpha_\nu + \alpha_\nu^{(s)})}\frac{\partial B_\nu}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}z}, \tag{1.6.10}$$

where we used the fact that the spatial dependence of the B_ν only comes through the temperature $T(z)$. This is useful since, while the derivative of B_ν is frequency-dependent, the derivative of the temperature is not: therefore it can be factored out.

The total flux is given by

$$F = \int_0^\infty F_\nu \, \mathrm{d}\nu = -\frac{4\pi}{3}\frac{\mathrm{d}T}{\mathrm{d}z}\int_0^\infty \frac{1}{\alpha_\nu + \alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu \tag{1.6.11}$$

$$= -\frac{4\pi}{3}\frac{\mathrm{d}T}{\mathrm{d}z}\frac{\int_0^\infty \frac{1}{\alpha_\nu+\alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu\,, \tag{1.6.12}$$

which is the **average value** of the function $1/(\alpha_\nu + \alpha_\nu^{(s)})$ weighted by the *known* function $\partial B_\nu/\partial T$; the integral at the numerator can be explicitly evaluated:

$$\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu = \frac{\partial}{\partial T}\int_0^\infty B_\nu\,\mathrm{d}\nu = \frac{\partial}{\partial T}\left(\frac{\sigma}{\pi}T^4\right) = \frac{4\sigma}{\pi}T^3\,, \tag{1.6.13}$$

which means that the final expression is

$$F = -\frac{4\pi}{3}\frac{4\sigma T^3}{\pi}\frac{\mathrm{d}T}{\mathrm{d}z}\frac{\int_0^\infty \frac{1}{\alpha_\nu+\alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}\,, \tag{1.6.14}$$

which we can better understand by defining the **Rosseland mean opacity**

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\nu+\alpha_\nu^{(s)}}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}\,, \tag{1.6.15}$$

so that the expression reads

$$F = -\frac{16}{3}\frac{\sigma T^3}{\alpha_R}\frac{\mathrm{d}T}{\mathrm{d}z}\,. \tag{1.6.16}$$

This is called the **Rosseland approximation**. The result is that we can find a net photon flux which is driven by the temperature gradient. This has many applications, for example in stellar astrophysics.

This equation is very similar to the heat flux equation in one dimension:

$$q = -k\frac{\partial T}{\partial z}\,, \tag{1.6.17}$$

which in our case has the constant (thermal conductivity)

$$k = \frac{16}{3}\frac{\sigma T^3}{\alpha_R}\,. \tag{1.6.18}$$

We have assumed that the medium is optically thick, but we can see that if it becomes very optically thick then the photons have a hard time escaping (k becomes small).

1.7 The Eddington approximation

In the Rosseland approximation we found the result

$$I_\nu = B_\nu(T) + \mu \times \text{stuff}\,, \tag{1.7.1}$$

where the "stuff" is small. We then expanded in this parameter; without it we are in a perfectly isotropic condition, so we can say that we are "expanding around isotropy".

The idea behind the Eddington approximation is similar: we assume that the intensity can be written as

$$I_\nu(\tau) = a_\nu(\tau) + b_\nu(\tau)\mu\,, \tag{1.7.2}$$

so as before the angular dependence is linear.

Let us compute the moments of the specific intensity: the zeroth moment is the mean intensity,

$$J_\nu = \frac{1}{4\pi}\int I_\nu \,\mathrm{d}\Omega = \frac{2\pi}{4\pi}\int_{-1}^{1}(a_\nu + b_\nu\mu)\,\mathrm{d}\mu = a_\nu\,, \tag{1.7.3}$$

while the first moment is equal to

$$H_\nu = \frac{1}{4\pi}\int I_\nu\mu \,\mathrm{d}\Omega = \frac{2\pi}{4\pi}\int_{-1}^{1}\left(a_\nu\mu + b_\nu\mu^2\right)\mathrm{d}\mu = \frac{1}{2}\frac{2}{3}b_\nu = \frac{b_\nu}{3}\,. \tag{1.7.4}$$

The second moment is

$$K_\nu = \frac{1}{4\pi}\int I_\nu\mu^2 \,\mathrm{d}\Omega = \frac{a_\nu}{3} = \frac{1}{3}J_\nu\,. \tag{1.7.5}$$

The fact that the second moment of the intensity is related to the zeroth one as $K_\nu = J_\nu/3$ is precisely what the Eddington approximation asks.

The plane-parallel radiative transport equation can be written as

$$\mu\frac{\partial I_\nu}{\partial\tau} = I_\nu - S_\nu\,, \tag{1.7.6}$$

since $\mathrm{d}\tau = -(\alpha_\nu + \alpha_\nu^{(s)})\,\mathrm{d}s$ and $\mathrm{d}z = \mu\,\mathrm{d}s$.

Why is the optical depth defined with a negative sign here? It is the same in RL [RL79, eq. 1.115]. Is this to be interpreted as the optical depth towards a far-away observer, positioned at large z?

In this equation, the source function S_ν is assumed to be isotropic. So, if we average over the solid angle (or equivalently in $\mathrm{d}\mu\,/2$) we get

$$\frac{1}{2}\int_{-1}^{1}\mu\frac{\partial I_\nu}{\partial\tau}\,\mathrm{d}\mu = \frac{1}{2}\int_{-1}^{1} I_\nu\,\mathrm{d}\mu - \frac{1}{2}\int_{-1}^{1} S_\nu\,\mathrm{d}\mu \tag{1.7.7}$$

$$\frac{\partial}{\partial\tau}H_\nu = J_\nu - S_\nu\,. \tag{1.7.8}$$

We can also take the first moment of the same equation by multiplying the equation by μ before integrating: this yields

$$\frac{1}{2}\int_{-1}^{1}\mu^2\frac{\partial I_\nu}{\partial\tau}\,\mathrm{d}\mu = \frac{1}{2}\int_{-1}^{1}I_\nu\mu\,\mathrm{d}\mu - \underbrace{\frac{1}{2}\int_{-1}^{1}S_\nu\mu\,\mathrm{d}\mu}_{=0} \tag{1.7.9}$$

$$\frac{\partial K_\nu}{\partial\tau} = H_\nu\,. \tag{1.7.10}$$

However, recall that in the Eddington approximation we have $K_\nu = J_\nu/3$: this means that we have the system of coupled differential equations

$$\frac{\partial H_\nu}{\partial\tau} = J_\nu - S_\nu \qquad \text{and} \qquad \frac{1}{3}\frac{\partial J_\nu}{\partial\tau} = H_\nu\,, \tag{1.7.11}$$

which we can combine into one second order equation by differentiating the second: this yields

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial\tau^2} = J_\nu - S_\nu\,. \tag{1.7.12}$$

Now, the source function can be decomposed in a part depending on J_ν and a part depending on B_ν according to the single scattering albedo ϵ_ν (see equation (1.5.8)):

$$S_\nu = \epsilon_\nu B_\nu + (1-\epsilon_\nu)J_\nu\,. \tag{1.7.13}$$

Substituting this in a term simplifies and we finally find

$$\boxed{\frac{1}{3}\frac{\partial^2 J_\nu}{\partial\tau^2} = \epsilon_\nu(J_\nu - B_\nu)\,,} \tag{1.7.14}$$

the **radiative diffusion equation**. This can be solved relatively simply; however it does not give us the whole intensity, instead we can only find its mean value J_ν.

If we can calculate S_ν using this equation, then we can plug what we have found into the radiative transfer equation

$$\mu\frac{\partial I_\nu}{\partial\tau} = I_\nu - S_\nu\,. \tag{1.7.15}$$

Now that we have discussed the differential equations at length, let us consider for a moment the **boundary conditions** we need to impose.

A possible way to approach the problem is called the *two stream approximation*: we assume that radiation is only travelling along two angles, whose cosines are $\mu = \pm 1/\sqrt{3}$, so $\theta = \pm 55°$. This choice is less arbitrary than it may seem, there is good reason to suppose that in certain condition this is indeed the angle at which radiation will travel. This will be discussed later.

Let us introduce the quantities

$$I_\nu^\pm(\tau) = I_\nu(\tau, \pm\frac{1}{\sqrt{3}})\,. \tag{1.7.16}$$

Then, the average intensity will be given by their average:

$$J_\nu = \frac{I_\nu^+ + I_\nu^-}{2}\,. \tag{1.7.17}$$

The further moments can also be computed: they will be

$$H_\nu = \frac{1}{2\sqrt{3}}\Big(I_\nu^+ + I_\nu^-\Big) \qquad \text{and} \qquad K_\nu = \frac{I_\nu^+ + I_\nu^-}{6}\,, \tag{1.7.18}$$

consistently with the Eddington approximation. Note that, if we want to use a two-stream approximation, the only way for Eddington to hold is to use the values of $\mu = \pm 1/\sqrt{3}$.

The explicit expressions of the moments allow us to recover the $I_\nu^\pm$ as

$$I_\nu^+ = J_\nu + \sqrt{3}H_\nu = J_\nu + \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau} \tag{1.7.19}$$

$$I_\nu^+ = J_\nu - \sqrt{3}H_\nu = J_\nu - \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau}\,. \tag{1.7.20}$$

Now, let us fix an example for concreteness: a "slab" (like, for instance, a stellar atmosphere) extending from τ_0 to $\tau = 0$. The point at $\tau = 0$ is the top of the atmosphere: so, we want to impose the absence of incoming radiation there (supposing that there are no other nearby stars to influence the process): this is implemented as $I_\nu^-(\tau = 0) = 0$. This is called the *non-illuminated atmosphere* condition.

Also suppose that there is no radiation coming into the bottom of the slab: $I_\nu^+(\tau_0) = 0$.

This means that

$$J_\nu + \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau}\bigg|_{\tau_0} = 0 \qquad \text{and} \qquad J_\nu - \frac{1}{\sqrt{3}}\frac{\partial J_\nu}{\partial \tau}\bigg|_{0} = 0\,. \tag{1.7.21}$$

With these boundary conditions, one can solve the equation numerically.

Chapter 2

Results in classical electromagnetism

2.1 Plane electromagnetic waves

In a vacuum, Maxwell's equations read

$$\nabla \cdot E = \nabla \cdot B = 0 \tag{2.1.1}$$

and

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{2.1.2}$$

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}\,. \tag{2.1.3}$$

Using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ we can combine these into a wave equation,

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0\,, \tag{2.1.4}$$

which is solved by plane monochromatic waves

$$\vec{E} = \vec{\epsilon} E_0 \exp\Big(i(\vec{k}\cdot\vec{r} - \omega t)\Big)\,, \tag{2.1.5}$$

where the angular frequency and wavevector satisfy $\omega = c\left|\vec{k}\right|$. These are transverse waves: the scalar product of $\vec{\epsilon}$ and the wavevector $\vec{k}$ is zero, there is no longitudinal oscillation along the direction of motion.

These solutions form a basis, generally the radiation we find is not monochromatic. The electric field will be time-dependent, $\vec{E} = \vec{E}(t)$.

We want to discuss the frequency decomposition of these waves; we start off by assuming that we are dealing with a pulse, such that $\vec{E}(t) \to 0$ as $t \to \pm\infty$.

The Fourier transform of the electric field in the time domain gives us its frequency decomposition: we choose our conventions so that

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t) e^{i\omega t}\,\mathrm{d}t\,. \tag{2.1.6}$$

Since $E(t)$ is real, we must have $\hat{E}^*(\omega) = \hat{E}(-\omega)$. This means that we do not need to worry about negative frequencies.

The fundamental quantity we need to compute is the **energy spectrum** from the electric field: how is the energy in the EM radiation distributed across the various frequencies? The Poynting vector is useful in this context, as it quantifies the amount of energy carried per unit area and time: its modulus is given by

$$S = \frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}A} = \frac{c}{4\pi}E^2(t)\,, \tag{2.1.7}$$

which we need to integrate if we want to find the total energy carried by the pulse:

$$\frac{\mathrm{d}w}{\mathrm{d}A} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\,\mathrm{d}t\ . \tag{2.1.8}$$

Now, **Parseval's theorem** is a statement about Fourier transforms, telling us that the energy of the signal is conserved if we Fourier transform, meaning that

$$\int_{-\infty}^{\infty} E^2(t)\,\mathrm{d}t = 2\pi\int_{-\infty}^{\infty}\left|\hat{E}(\omega)\right|^2\mathrm{d}\omega\ . \tag{2.1.9}$$

Since this square modulus is the same for the conjugate the integrand is symmetric under $\omega \to -\omega$, so we can compute it from 0 to ∞ and multiply by 2 to get the same result. This yields

$$\frac{\mathrm{d}w}{\mathrm{d}A} = \frac{4\pi c}{4\pi}\int_{0}^{\infty}\left|\hat{E}(\omega)\right|^2\mathrm{d}\omega\ , \tag{2.1.10}$$

which can be differentiated with respect to ω to find the energy density per unit time and frequency:

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega} = c\left|\hat{E}(\omega)\right|^2. \tag{2.1.11}$$

This concerns the *total* energy of the pulse: we might want to calculate the energy per unit area, time and frequency! However, there is an issue in doing this: the uncertainty principle. We cannot sample with arbitrarily small intervals in both t and ω, since the inequality $\Delta\omega\Delta t \geq 1$ must be satisfied.

However, if our pulse repeats over some comparatively long time T we can formally define

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega\,\mathrm{d}t} = \frac{1}{T}c\left|\hat{E}(\omega)\right|^2, \tag{2.1.12}$$

so, as long as T is relatively long we can take the limit:

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega\,\mathrm{d}t} = \lim_{T\to\infty}\frac{1}{T}c\left|\hat{E}(\omega)\right|^2. \tag{2.1.13}$$

2.1.1 Polarization

This is a measurable quantity in astrophysics, it is easier to perform polarization measurements for low frequencies but nowadays we are starting to be able to measure polarizations as far as X-rays.

By definition a monochromatic plane wave is *linearly polarized*: it oscillates along a specific axis. Consider a superposition of two electric fields at a specific point in space, for simplicity $\vec{r} = 0$: the total electric field will be given by

$$\vec{E} = \underbrace{\left(\vec{x}E_1 + \vec{y}E_2\right)}_{\vec{E}_0} e^{-i\omega t}\,, \tag{2.1.14}$$

where the amplitudes $E_{1,2}$ are in general complex: $E_{1,2} = \xi_{1,2}e^{i\phi_{1,2}}$. So, we can write the total electric field

$$\vec{E} = \vec{x}\xi_1 e^{i(\phi_1 - \omega t)} + \vec{y}\xi_2 e^{i(\phi_2 - \omega t)}\,. \tag{2.1.15}$$

The physical field along any direction will be given by the real part of the projection of this complex-valued vector along that component:

$$E_x = \mathrm{Re}\left\{\xi_1 e^{i(\phi_1 - \omega t)}\right\} = \xi_1 \cos\left(\phi_1 - \omega t\right) = \xi_1 \cos\omega t \cos\phi_1 + \xi_1 \sin\omega t \sin\phi_1\,, \tag{2.1.16}$$

and similarly with E_y, ξ_2 and ϕ_2.

Now, a short aside: an ellipse can be described parametrically from its principal axes with

$$x' = A\cos\beta\cos\omega t \qquad \text{and} \qquad y' = -A\sin\beta\sin\omega t\,, \tag{2.1.17}$$

since these expressions satisfy

$$\frac{x'^2}{A^2\cos^2\beta} + \frac{y'^2}{A^2\sin^2\beta} = 1\,, \tag{2.1.18}$$

so we can parametrize any ellipse with an appropriate choice of $A \in R^+$ and $\beta \in [-\pi/2, \pi/2]$.Tuesday

This parametric equation is expressed with respect to the principal axes of the ellipse; if we want to write in a different coordinate system which is obtained from a rotation of the principal axes we must use the rotation matrix 2020-8-18, compiled 2020-09-06

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}\,; \tag{2.1.19}$$

if we make x' and y' explicit we can find

$$x = A\cos\beta\cos\chi\cos\omega t + A\sin\beta\sin\chi\sin\omega t \tag{2.1.20}$$

$$y = A\cos\beta\sin\chi\cos\omega t - A\sin\beta\cos\chi\sin\omega t\,. \tag{2.1.21}$$

The expressions for x and y and the ones for E_x and E_y are quite similar: we must identify

$$\xi_1 \cos\phi_1 = A \cos\beta \cos\chi \tag{2.1.22}$$
$$\xi_1 \sin\phi_1 = A \sin\beta \sin\chi \tag{2.1.23}$$
$$\xi_2 \cos\phi_2 = A \cos\beta \sin\chi \tag{2.1.24}$$
$$\xi_2 \sin\phi_2 = -A \sin\beta \cos\chi\,, \tag{2.1.25}$$

which means that the electric field rotates in the shape of an ellipse, which is tilted about the x, y axes by an angle χ.

Stokes parameters

From these four equations, we find that (after working through the simplifications of the trigonometric functions)

$$\xi_1^2 + \xi_2^2 = A^2 \stackrel{\text{def}}{=} I \tag{2.1.26}$$
$$\xi_1^2 - \xi_2^2 = A^2 \cos 2\beta \cos 2\chi \stackrel{\text{def}}{=} Q \tag{2.1.27}$$
$$2\xi_1\xi_2 \cos(\phi_1 - \phi_2) = A^2 \sin 2\chi \cos 2\beta \stackrel{\text{def}}{=} U \tag{2.1.28}$$
$$2\xi_1\xi_2 \sin(\phi_1 - \phi_2) = A^2 \sin 2\beta \stackrel{\text{def}}{=} V\,, \tag{2.1.29}$$

where we defined the **Stokes parameters** I, Q, U and V.[1] They completely describe the state of an elliptically polarized monochromatic wave.

We can reverse their definitions as

$$A = \sqrt{I} \tag{2.1.30}$$
$$\sin 2\beta = \frac{V}{I} \tag{2.1.31}$$
$$\tan 2\chi = \frac{U}{Q}\,. \tag{2.1.32}$$

Now let us discuss the meaning of these parameters. Since $A = \xi_0$, $I = \xi_0^2$, while the energy flux is given by

$$S = \frac{c}{4\pi}\xi_0^2 = \frac{c}{4\pi} I\,. \tag{2.1.33}$$

So, the parameter I is directly proportional to the energy flux. The major axes of the polarization ellipse are $2A\cos\beta$ and $2A\sin\beta$ respectively; therefore their ratio $\tan\beta$ measures the eccentricity of the ellipse, and the ratio V/I can be used to recover β.

The ratio U/Q describes the orientation of the ellipse with respect to the x, y axes. If the wave is linearly polarized then $V = 0$ (this is a degenerate case for our description); if $V > 0$ the polarization is left-handed, while if $V < 0$ the polarization is right-handed.

[1] They can also be expressed as the difference of the square modulus of the electric field with respect to three bases (corresponding to the three Pauli matrices): one Cartesian (for Q, σ_z) one Cartesian rotated by 45° (for U, σ_x) and one circular (for V, σ_y).

If we have $U = Q = 0$, this corresponds to circular polarization.

The ellipse is fully determined by three numbers (β, χ, A): why are there 4 Stokes parameters? They cannot all be independent if this is the case...

In fact, they are connected by the relation

$$I^2 = Q^2 + V^2 + U^2 \,. \tag{2.1.34}$$

The reason we use four parameters instead of just three is that this line of reasoning only holds for perfectly polarized monochromatic waves: in the general case, for partially polarized radiation, we have

$$I^2 \geq Q^2 + V^2 + U^2 \,. \tag{2.1.35}$$

For completely *un*polarized light we have $Q = V = U = 0$.

A useful property is that these parameters are **additive**: the superposition of waves with different Stokes parameters can be the described by the sum of the Stokes parameters, as $I_{\text{total}} = \sum_k I_k$ and so on.

This suggests that we write them in a vector:

$$\vec{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} . \tag{2.1.36}$$

In general, we can decompose radiation which is partially polarized into a completely unpolarized part and a completely polarized part:

$$\vec{S} = \underbrace{\begin{bmatrix} I - \sqrt{Q^2 + V^2 + U^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{unpolarized}} + \underbrace{\begin{bmatrix} \sqrt{Q^2 + V^2 + U^2} \\ Q \\ U \\ V \end{bmatrix}}_{\text{polarized}} . \tag{2.1.37}$$

This allows us to define the **polarization degree**: the fraction of the radiation which is polarized, calculated as

$$\Pi_L = \frac{I_{\text{polarized}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \,, \tag{2.1.38}$$

which is measurable with an instrument.

Electromagnetic potentials

We can express the electric and magnetic fields through the scalar and vector potentials ϕ and $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{and} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \,. \tag{2.1.39}$$

In terms of these potentials Maxwell's equations read

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho \tag{2.1.40}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j}. \tag{2.1.41}$$

The potentials are redundant, we can impose some conditions on them while still being able to describe any physical system. One we can impose is the *Lorentz gauge*:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0\,, \tag{2.1.42}$$

under which we can get a formal solution for the potentials in terms of the charge and current densities:

$$\phi(\vec{r},t) = \int \frac{[\rho]}{|\vec{r}-\vec{r}'|}\,\mathrm{d}^3r' \tag{2.1.43}$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{[\vec{j}]}{|\vec{r}-\vec{r}'|}\,\mathrm{d}^3r'\,. \tag{2.1.44}$$

The square brackets are a notation meaning that the densities must be evaluated at the *retarded time* $t - \left|\vec{r}-\vec{r}'\right|/c$. This accounts for the finite speed of the propagation of light.

Radiation from moving charges

Consider a particle of mass m and charge q moving along a path described by the vector $\vec{r}_0(t)$. The charge and current densities will then be

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{r}_0(t)) \tag{2.1.45}$$

$$\vec{j}(\vec{r},t) = q\underbrace{\frac{\mathrm{d}\vec{r}_0}{\mathrm{d}t}}_{\vec{u}(t)}\delta(\vec{r}-\vec{r}_0(t))\,. \tag{2.1.46}$$

From these expressions we can calculate the potentials generated by the moving charge. Let us introduce the relative radius $\vec{R} = \vec{r}-\vec{r}_0$ and its corresponding unit vector $\vec{n} = \vec{R}/\left|\vec{R}\right|$.

The potentials will then be

$$\phi = \left[\frac{q}{kR}\right] \qquad \text{and} \qquad \vec{A} = \left[\frac{q\vec{n}}{ckR}\right], \tag{2.1.47}$$

where $k = 1 - \vec{n}\cdot\vec{u}/c$, while $R = \left|\vec{R}\right|$. These are called the **Liénard-Wiechert** potentials. From them we can perform the differentiation in order to recover the fields, which will be called the Liénard-Wiechert fields. The final expressions for the fields are

$$\vec{E}(\vec{r},t) = q\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{k^3R^2}\right] + \frac{q}{c}\left[\frac{\vec{n}}{k^3R}\times\left((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)\right] \tag{2.1.48}$$

$$\vec{B}(\vec{r},t) = \left[\vec{n} \times \vec{E}\right], \tag{2.1.49}$$

where $\vec{\beta} = \vec{u}/c$. The important thing to notice here is that there are two contributions to the electric field: one goes as R^{-2} and is called the **Coulomb field** $\vec{E}_c$ and the other goes as R^{-1} and is called the **radiation field** $\vec{E}_r$.

Now we will discuss the radiation field mainly. Since the magnetic field can be found by taking a product with the electric field, we can make the same decomposition there.

The radiation field $\vec{E}_r$ must be perpendicular to $\vec{n}$, and so must be $\vec{B}_r$. So, they form an orthogonal triple.

Let us consider some simple cases.

Nonrelativistic charge

Let us first suppose that the particle is moving slowly, $|u|/c = |\beta| \ll 1$: this means that $|\vec{n} - \beta| \sim 1$ and also $k \sim 1$.

With these simplifications we find that the ratio of the magnitudes of the two components of the electric field is

$$\frac{E_r}{E_c} \sim \frac{R\dot{u}}{c^2}. \tag{2.1.50}$$

If the characteristic time across which the velocity of the particle changes is τ, then we have $\dot{u} \sim u/\tau$. This time τ will be associated with a characteristic frequency $\nu = 1/\tau$ and a characteristic wavelength $\lambda = c/\nu$.

...and this is the wavelength of the emitted radiation? is this always the case? it does not seem that obvious

Then we will get

$$\frac{E_r}{E_c} \sim \frac{Ru\nu}{c^2} \sim \frac{Ru}{\lambda c}, \tag{2.1.51}$$

so we can see that if

$$\frac{R}{\lambda} \lesssim 1 \tag{2.1.52}$$

then we have $E_r/E_c \lesssim u/c$. This defines an inner region around the emitter for which the Coulomb field dominates; outside it the radiation field dominates.

Typically, we are far enough from the emitters of radiation so that the radiation field dominates. Then, under the assumption $\beta \ll 1$ we have

$$\vec{E}_r = \frac{q}{Rc^2}\left[\vec{n} \times \left(\vec{n} \times \dot{\vec{u}}\right)\right] \qquad \text{and} \qquad \vec{B}_r = \left[\vec{n} \times \vec{E}_r\right]. \tag{2.1.53}$$

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Let us go into some more details regarding how the radiation field looks. Let us suppose that the angle between the acceleration $\dot{\vec{u}}$ and the observation unit vector $\vec{n}$ is Θ: then the absolute value of the electric field and magnetic fields will be

$$E_r = \frac{q}{Rc^2}\dot{u}\sin\Theta = B_r. \tag{2.1.54}$$

$\vec{E}$ and $\vec{B}$ are orthogonal, so the magnitude of the Poynting vector will be

$$\left|\vec{S}\right| = \frac{c}{4\pi}\left|\vec{E}\times\vec{B}\right| = \frac{c}{4\pi}E_r^2 = \frac{q^2}{4\pi c^3 R^2}\dot{u}^2\sin^2\Theta\,, \tag{2.1.55}$$

which, as is expected, decays like R^{-2}, since it is the power radiated per unit area. If we instead want the power radiated per unit solid angle then we must use $\mathrm{d}A = R^2\,\mathrm{d}\Omega$:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} = SR^2 = \frac{q^2}{4\pi c^3}\dot{u}^2\sin^2\Theta\,. \tag{2.1.56}$$

Integrating this over the solid angle allows us to calculate the total power emitted:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \int \frac{q^2}{4\pi c^3}\dot{u}^2\sin^2\Theta\,\mathrm{d}\Omega \tag{2.1.57}$$

$$= \frac{q^2\dot{u}^2}{2c^3}\underbrace{\int_{-1}^{1}(1-\mu^2)\,\mathrm{d}\mu}_{=2-2/3} \tag{2.1.58}$$

Reparametrized $\mu = \cos\Theta$, integrated over ϕ.

$$= \frac{2q^2\dot{u}^2}{3c^3}\,, \tag{2.1.59}$$

which is the well-known **Larmor formula**.

What we want to do now is to generalize this result to a system of *many charges*. Each of these will have a charge q_i, a distance R_i from the observer, an acceleration $\dot{u}_i$. In principle the generalization is simple: the electromagnetic field obeys the superposition principle, so we can just add the $E_{i,r}$ from all the charges together.

The issue is that the retarded time for each of the particles is slightly different from that of another particle. This is a treatable problem, but it makes the calculation more cumbersome.

What we can do in order to mitigate it is to make the **dipole approximation**. Suppose that the system of N charges is contained within a volume whose characteristic length scale is L. Also suppose that τ is the typical scale across which the system evolves. If the evolution time τ is slow compared to L/c, then we can ignore the differences between the retarded times.

Now, if the system changes significantly over a time τ then the electric field also changes significantly over that time, so we can estimate the frequency of the emitted radiation as $\nu \sim 1/\tau$.

What we are asking, $\tau \gg L/c$, then becomes $\nu \approx 1/\tau \ll c/L$, or equivalently $\lambda \gg L$.

We can also give an order of magnitude for the typical velocity of the particles: if ℓ is the typical path length of a particle, then the typical velocity will be $u \sim \ell/\tau$. This means that $\ell/u = \tau$ must be larger than L/c: rearranging, we have

$$\frac{\ell}{L} \gg \frac{u}{c}\,, \tag{2.1.60}$$

but since $\ell \lesssim L$ this means that $u/c \ll 1$.

If these conditions are satisfied, then we can get the total electric field by adding all the electric field contributions from all the various particles:

$$\vec{E}_r = \sum_i \vec{E}_{r,i} = \sum_i \frac{q_i}{R_i c^2} \vec{n}_i \times \left(\vec{n}_i \times \dot{\vec{u}}_i \right) \tag{2.1.61}$$

$$= \sum_i \frac{1}{R_i c^2} \vec{n}_i \times \left(\vec{n}_i \times q_i \dot{\vec{u}}_i \right) . \tag{2.1.62}$$

All of these will still need to be computed at the retarded time, although the retarded time will be the same for all the charges.

If the length scale L of the system is smaller than all the distances R_i, then we can approximate any R_i with a constant R_0; also the unit vectors $\vec{n}_i$ will be almost equal, and we can approximate them with a single $\vec{n}$. R_0 does not need to be the mean of the R_i, the point is that all of them are quite similar so the distance to any point inside the source works. The reasoning is similar for $\vec{n}$.

If we make both of these approximations we find

$$\vec{E}_r = \sum_i \frac{1}{R_0 c^2} \vec{n} \times \left(\vec{n} \times q_i \dot{\vec{u}}_i \right) \tag{2.1.63}$$

$$= \frac{1}{R_0 c^2} \vec{n} \times \left(\vec{n} \times \underbrace{\sum_i q_i \dot{\vec{u}}_i}_{\ddot{\vec{d}}} \right) , \tag{2.1.64}$$

where we define the dipole moment

$$\vec{d} = \sum_i q_i \vec{r}_i . \tag{2.1.65}$$

Because the dipole moment appeared, this is called the **dipole approximation**. If Θ is the angle between $\vec{n}$ and $\ddot{\vec{d}}$, then we have

$$E_r = \frac{\ddot{d} \sin\Theta}{R_0 c^2} , \tag{2.1.66}$$

and with this we can calculate the power radiated per unit area

$$\frac{\mathrm{d}w}{\mathrm{d}A \, \mathrm{d}t} = \frac{c}{4\pi} E_r^2 , \tag{2.1.67}$$

and per unit solid angle:

$$\frac{\mathrm{d}w}{\mathrm{d}t \, \mathrm{d}\Omega} = \frac{c}{4\pi} E_r^2 R_0^2 = \frac{1}{c^3} \frac{\ddot{d}^2}{4\pi} \sin^2\Theta , \tag{2.1.68}$$

so if we integrate we can find the total emitted power, which generalizes the **Larmor formula**:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{2\ddot{d}^2}{3c^3} . \tag{2.1.69}$$

Now let us discuss the **spectral distribution** of the emitted energy. The energy per unit frequency and area will be

$$\frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}\omega} = c\left|\hat{E}_r(\omega)\right|^2, \tag{2.1.70}$$

while the energy emitted per unit solid angle as usual is calculated by multiplying by R^2:

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega\,\mathrm{d}\omega} = R^2 c\left|\hat{E}_r(\omega)\right|^2 = \frac{1}{c^3}\sin^2\Theta\left|\hat{\ddot{d}}(\omega)\right|^2, \tag{2.1.71}$$

where $\hat{\ddot{d}}$ is the Fourier transform of $\ddot{d}$. This assumes that Θ is fixed through time; we are allowing its magnitude but not its direction to change.

Derivatives in the Fourier domain are multiplication by $-i\omega$: so, we can recover the second derivative of the dipole moment in the time domain as

$$\ddot{d}(t) = -\int_{-\infty}^{\infty} \omega^2 e^{-i\omega t}\hat{d}(\omega)\,\mathrm{d}\omega\ , \tag{2.1.72}$$

which means that we can express the electric field in the Fourier domain as

$$\hat{E}(\omega) = -\frac{\omega^2}{c^2 R}\hat{d}(\omega)\sin\Theta\,, \tag{2.1.73}$$

so that the energy emitted per unit frequency and solid angle becomes

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega\,\mathrm{d}\omega} = \frac{\omega^4}{c^3}\left|\hat{d}(\omega)\right|^2\sin^2\Theta\,. \tag{2.1.74}$$

If we integrate over the sphere we find

$$\frac{\mathrm{d}w}{\mathrm{d}\omega} = \frac{8\pi}{3c^2}\omega^4\left|\hat{d}(\omega)\right|^2. \tag{2.1.75}$$

Note that the spectrum of the radiation is heavily dependent on the frequency of the radiation.

2.1.2 Thomson scattering

This is scattering of radiation by free electrons. Consider a plane, linearly polarized, electromagnetic wave impacting a free electron. The force onto the electron is the Lorentz force:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right). \tag{2.1.76}$$

We know that $|E| = |B|$ for the wave, so if the electron is nonrelativistic then the magnetic term is negligible:

$$\vec{F} \approx e\vec{E}\,. \tag{2.1.77}$$

Under our assumptions the electric field can be written as

$$\vec{E} = E_0 \sin \omega_0 t \vec{\epsilon}\,, \tag{2.1.78}$$

where $\vec{\epsilon}$ is a unit vector which is perpendicular to the propagation direction: $\vec{\epsilon} \cdot \vec{k} = 0$; while E_0 is the amplitude of the electric field and ω_0 is its frequency.

The equations of motion of the charge read

$$m\ddot{\vec{r}} = e\vec{E} = eE_0 \sin \omega_0 t \vec{\epsilon}\,, \tag{2.1.79}$$

which we can also express through the dipole moment $\vec{d} = e\vec{r}$: the equation for its evolution will then read

$$\ddot{\vec{d}} = \frac{e^2}{m} E_0 \sin \omega_0 t \vec{\epsilon}\,. \tag{2.1.80}$$

If we integrate in $\mathrm{d}t$ two times we find

$$\vec{d}(t) = -\frac{e^2 E_0}{m\omega_0^2} \sin \omega_0 t \vec{\epsilon}\,, \tag{2.1.81}$$

so the response to the impinging EM wave is an oscillation of the dipole, with a frequency ω_0 equal to that of the EM wave and an amplitude equal to

$$d_0 = \frac{e^2 E_0}{m\omega_0^2}\,. \tag{2.1.82}$$

The electron is accelerating, so it will radiate: the power emitted per unit solid angle will be

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{1}{c^3}\frac{\ddot{\vec{d}}^2}{4\pi} \sin^2\Theta = \frac{1}{4\pi c^3}\frac{e^4}{m^2} E_0^2 \sin^2 \omega_0 t \sin^2\Theta\,, \tag{2.1.83}$$

where Θ is the angle between the direction of propagation of the wave and the direction of observation. This oscillates in time; we can compute the average over an oscillation: this means that we substitute the square sine with a factor $1/2$:

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} \right\rangle = \frac{e^4 E_0^2 \sin^2\Theta}{8\pi c^3 m^2}\,. \tag{2.1.84}$$

Integrating over the solid angle to get the average power amounts to multiplying by 4π times $2/3$, because of the solid angle in the sphere and because of the integral of $\sin^2\Theta$. This yields

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}t} \right\rangle = \frac{e^4 E_0^2}{3m^2 c^3}\,. \tag{2.1.85}$$

Now, let us compute the flux of energy which is carried away by the incident EM wave:

$$S = \frac{\mathrm{d}w}{\mathrm{d}A\,\mathrm{d}t} = \frac{c}{4\pi} E_r^2 = \frac{c}{4\pi R^2} E_0^2 \sin^2 \omega_0 t\,, \tag{2.1.86}$$

then the power per unit solid angle is

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega\,\mathrm{d}t} = \frac{c}{4\pi}E_0^2\sin^2\omega_0 t\,, \tag{2.1.87}$$

whose average as before is

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}\Omega\,\mathrm{d}t} \right\rangle = \frac{cE_0^2}{8\pi}\,. \tag{2.1.88}$$

Except this, dimensionally, is a power per unit *area*!

Now, let us define the **differential scattering cross section** as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\left\langle \frac{\mathrm{d}w}{\mathrm{d}t\mathrm{d}\Omega} \right\rangle_{\text{emitted}}}{\left\langle \frac{\mathrm{d}w}{\mathrm{d}t\mathrm{d}\Omega} \right\rangle_{\text{incoming}}}\,, \tag{2.1.89}$$

is this not dimensionally inconsistent? the differential scattering cross section should have the dimensions of an area / steradian... Maybe the incoming power should be considered per unit *area*, since it is a plane wave whose source is at infinity?
This seems indeed to be the case, see [RL79, eq. 3.36]

and if we compute this for our case we will have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4E_0^2\sin^2\Theta}{8\pi c^3m^2}\frac{8\pi}{cE_0^2} = \frac{e^4}{c^4m^2}\sin^2\Theta\,. \tag{2.1.90}$$

This expression can also be written through the classical electron radius:

$$r_0 = \frac{e^2}{mc^2} \approx 2.82\times10^{-13}\,\mathrm{cm}\,, \tag{2.1.91}$$

whose expression is found by equating the rest energy of the electron mc^2 with its electromagnetic self-energy e^2/r_0. Then, the scattering cross section can be expressed as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = r_0^2\sin^2\Theta\,. \tag{2.1.92}$$

The total cross section is found from the integral of this over all the solid angle:

$$\sigma_T = \int\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = r_0^2\underbrace{\int(1-\mu^2)\,\mathrm{d}\mu\times2\pi}_{=8\pi/3} = \frac{8\pi}{3}r_0^2\,. \tag{2.1.93}$$

This is the **Thomson cross section** of the electron $\sigma_T \approx 0.665\times10^{-24}\,\mathrm{cm}^2$.

Now, for some observations. This cross section does not change depending on the frequency of the incoming wave: it is "color blind". This formula for the scattering is not always valid, but in its regime of validity (which will be discussed later) we expect no frequency dependence.

Moreover, this scattering is **conservative** or coherent: the frequency of the scattered radiation is the same as the frequency of the incoming radiation.

It also is **not isotropic** because of the factor $\sin^2\Theta$. This scattering "prefers" for the light to go along the same direction it came from, and the probability density for it goes to zero for $\Theta = \pi/2$, meaning for a scattered photon orthogonal to the direction of propagation. Also, there is forward-backward symmetry, which will be useful to simplify certain calculations. The degree of anisotropy is also rather mild, it varies slowly over the solid angle, so in certain situations it will be alright to approximate Thomson scattering as isotropic. This is the closest we can get to completely isotropic and coherent scattering.

We can apply this reasoning to any other particle, it does not need to be an electron: for example, let us apply it to a proton. Their square charge is the same — the sign does not matter. The only different is the mass, which appears with a power m^{-2} in the cross section and in the differential cross section.

So, their ratio will be

$$\frac{\sigma_p}{\sigma_e} = \left(\frac{m_e}{m_p}\right)^2 \approx 3 \times 10^{-7}\,. \tag{2.1.94}$$

This is important in astrophysical applications: if we have a plasma with protons and electron we can almost completely ignore the protons.

Our assumptions have been to treat the electromagnetic field classically, and to assume the electron moves nonrelativistically. The first of these holds as long as the energy of the individual photons is small: $h\nu \ll m_e c^2$. If this is the case, the **recoil** of the electron due to the scattering with the single photon is negligible. Since $m_e c^2 \approx 511\,\mathrm{keV}$, we are asking that $h\nu \ll 511\,\mathrm{keV}$. A rough boundary for when this works is given by considering $h\nu \approx 50\,\mathrm{keV}$, a tenth of the electron rest energy. This puts us in the medium X-rays.

We also assumed that the electron started out at rest, while in practically all situations charges are moving around, if nothing else because of thermal motion. If we are considering a plasma, it must be hot for the gas to stay ionized.

In the electron's rest frame our description works, however in order to be in that frame we must perform a Lorentz boost, apply Thomson scattering, and boost back: in this way we can describe how the energy of the photon changes.

In astrophysical settings, radiation is generally either completely unpolarized or partially polarized. Therefore, it is interesting to compute the cross section for unpolarized radiation.

Unpolarized radiation can be described as the superposition of two orthogonal polarized waves. One of them will have an angle Θ between the polarization vector $\vec{\epsilon}_1$ and the observation direction $\vec{n}$, let us suppose that $\vec{\epsilon}_1$, $\vec{k}$ and $\vec{n}$ are coplanar. This can be done, since all we are doing is choosing a convenient basis for the plane orthogonal to the propagation direction.

If this is the case, then if $\vec{\epsilon}_2 \perp \vec{\epsilon}_1$ we must have $\Theta = \pi/2$ for the second wave. This means that we have the two cross sections

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_1 = r_0^2 \sin^2\Theta \qquad \text{and} \qquad \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_2 = r_0^2 \sin^2\frac{\pi}{2} = r_0^2\,. \tag{2.1.95}$$

The total differential cross section will be given by their average:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = r_0^2 \frac{1+\sin^2\Theta}{2} = r_0^2 \frac{1+\cos^2\theta}{2}\,, \tag{2.1.96}$$

where we define $\theta = \pi/2 - \Theta$, the angle between the propagation direction $\vec{k}$ and the observation direction $\vec{n}$.

The total unpolarized cross section is the same as the one we had for polarized light:

$$\sigma_{\mathrm{unpol}} = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \frac{2\pi r_0^2}{2}\int_{-1}^{1}(1+\mu^2)\,\mathrm{d}\mu = \frac{8\pi}{3}r_0^2 = \sigma_T\,. \tag{2.1.97}$$

So, for the total cross section of Thomson scattering polarization does not matter, while it does change the differential cross section.

The Eddington limit

The total momentum flux of the radiation field is given by

$$P = \int_0^\infty \mathrm{d}\nu \int_{4\pi} \mathrm{d}\Omega \cos^2\theta I_\nu\,. \tag{2.1.98}$$

Suppose we have a differential area $\mathrm{d}A$, and suppose that photons only cross it in the direction of its normal $\vec{n}$. If this is the case, then we have

$$P = \int_0^\infty \mathrm{d}\nu\, I_\nu = \frac{F}{c}\,, \tag{2.1.99}$$

since then the angular distribution function is a delta on $\theta = 0$ on the unit sphere.

Suppose then we have a particle at the center of the area element $\mathrm{d}A$, and that we want to calculate the force upon it. It will be the pressure times the cross section:

$$\mathscr{F} = \frac{F}{c}\sigma = P\sigma\,. \tag{2.1.100}$$

Now, let us consider a source which emits photons only radially, such as a star with perfect spherical symmetry. The flux at any radius r will be $F = L/4\pi r^2$, where L is the luminosity (power) of the source. The photons will only move radially.

Then, the force onto a test particle will be

$$\mathscr{F} = \frac{L\sigma}{c4\pi r^2}\,. \tag{2.1.101}$$

The source will have a mass M, and it will attract the particles gravitationally.

Now, a typical composition for the material outside a star is a plasma, made up of dissociated hydrogen: protons and electrons. As we have seen, the cross section of the electrons for Thomson scattering will be much larger than that of the protons, while the gravitational force will be much larger on the protons since they are more massive.

If we consider a single electron-proton pair which is bound by electrostatic forces (although not in a bound *state*, since we still have a plasma), then we can calculate the equilibrium point for the forces on the pair. We can then equate the radiative force on the

electron to the gravitational force on the proton, since the other two forces are negligible in comparison. This yields:

$$\frac{L\sigma_T}{c4\pi r^2} = \frac{GMm_p}{r^2} \tag{2.1.102}$$

$$L = \frac{4\pi GMm_p c}{\sigma_T} = L_{\text{Edd}}\,, \tag{2.1.103}$$

the **Eddington luminosity** corresponding to a mass M. In comparison to the Sun, this is roughly

$$L_{\text{Edd}} \approx 3 \times 10^5 L_\odot \frac{M}{M_\odot}\,. \tag{2.1.104}$$

This means that the Sun is well below the Eddington limit. This is not a general limit, since it only applies if we have spherical symmetry; if this is broken the "limit" can be violated. Also, we can define analogous limits for different kinds of processes, which will have different cross sections.

2.2 Bremsstrahlung

Saturday 2020-8-22, compiled 2020-09-06

The German word comes from the words meaning "braking" and "radiation". It is also called "free-free emission".

The name is historical: this kind of radiation was first observed in the lab coming from the deceleration of charges which hit a target. This is the radiation emitted by electrons when a force is exerted upon them, for instance the Coulomb force by an ion. So, this kind of radiation can occur in a plasma since there we have both free protons and electrons.

We could also have electron-electron electromagnetic interactions, but in real astrophysical plasmas, as we shall see, this is less relevant.

Suppose we have an ion with a positive charge Ze, and an electron with a negative charge $-e$. Between them we have the attractive electrostatic force, whose modulus is

$$F = \frac{Ze^2}{r^2}\,. \tag{2.2.1}$$

If the ion is a proton, then the acceleration on the electron and on the proton can be calculated by equating

$$m_e a_e = m_p a_p = \frac{e^2}{r^2} \tag{2.2.2}$$

in modulus. So, the acceleration of the proton is about $m_p/m_e \approx 1836$ times *smaller* than that of the electron. This allows us to approximate the ion as a fixed source of force, neglecting its acceleration completely.

We know from the Larmor formula (2.1.59) that the power emitted by an accelerating charge is proportional to its acceleration squared, so the power emitted by the ion is about $(m_p/m_e)^2 \approx 4 \times 10^6$ times smaller than that emitted by the electron.

Now, let us consider the case of the repulsion of two identical particles, such as two electrons, in the nonrelativistic dipole approximation (which is not completely justified, often electrons will be relativistic... however we will make it for simplicity).

If d is the dipole moment, then we know from the Larmor formula for the dipole (2.1.69) that the power emitted is proportional to $\dot{d}^2$. The dipole moment is

$$\vec{d} = -e\sum_i \vec{r}_i = -\frac{e}{m_e}\sum_i m_e\vec{r}_i = -\frac{e}{m_e}\vec{r}_{CM}\,. \tag{2.2.3}$$

We can then see that the dipole moment is proportional to the center of mass of the system. Now, we are always implicitly assuming that the two charges we are treating are isolated, so they are not subject to any external forces: therefore, the acceleration of the center of mass is zero.

This means that $\ddot{d} = 0$, so the power emitted is zero in the dipole approximation. This is why *a system of nonrelativistic identical charges does not radiate in the dipole approximation*.

Now, let us move to an actual description of bremsstrahlung radiation. We will treat it approximately; a complete description, even accounting for quantum mechanics, can be given, however it is beyond the scope of this course.

We start off by making the small angle approximation: we assume that the electron moves fast enough, so that its trajectory looks like a straight line and the deviation due to the proton is only a small perturbation.

We define the position vector $\vec{r}$ as the one connecting the ion to the electron, while $\vec{v} = \dot{\vec{r}}$ is the velocity of the electron, and the impact parameter $b = \min|\vec{r}|$ measures the distance of closest approach of the particles.

The dipole moment is given by $\vec{d} = -e\vec{r}$, whose second derivative will be $\ddot{\vec{d}} = -e\dot{\vec{v}}$. Let us move to frequency space:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ddot{\vec{d}}e^{i\omega t}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-e\dot{\vec{v}})e^{i\omega t}\,\mathrm{d}t \tag{2.2.4}$$

$$-\omega^2\hat{d}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty}\dot{\vec{v}}e^{i\omega t}\,\mathrm{d}t\,. \tag{2.2.5}$$

Then, we can see that if we are able to determine the value of the integral on the right-hand side we can directly calculate the dipole moment, and with it the emitted power. Evaluating it exactly is hard in general, however we can do so in two limiting cases, depending on the "duration of the interaction". The timescale of the process, that is, the rough amount of time over which the electrostatic interaction between the two particles is significant, is of the order $\tau \approx b/v$. Then, we can try to simplify the problem by integrating only over a range of size $\sim\tau$ around zero in the time domain, instead of going from $-\infty$ to $+\infty$, since there we will find the largest contribution.

Now, our limiting cases refer to the frequency ω of the emitted radiation. If this frequency is very high, such that $\omega\tau \gg 1$, then in the integral we will have a slowly-varying term $\dot{v}$ times a quickly oscillating term $\exp(i\omega\tau)$: thus, the value of the integral will be close to zero.

On the other hand, if $\omega\tau \ll 1$, then the exponential will be $\exp(i\omega\tau) \approx 1$: therefore the integral will be

$$\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t}\,\mathrm{d}t \approx \int_{-\infty}^{\infty} \dot{\vec{v}}\,\mathrm{d}t = \Delta\vec{v}\,. \tag{2.2.6}$$

Therefore, the Fourier transform of the dipole moment will be given by

$$\hat{d}(\omega) = \begin{cases} -\frac{e}{2\pi}\frac{\Delta\vec{v}}{\omega^2} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1\,. \end{cases} \tag{2.2.7}$$

With this, we can compute the spectral distribution of the emitted radiation (2.1.75):

$$\frac{\mathrm{d}w}{\mathrm{d}\omega} = \frac{8\pi}{3c^3\omega^4} \begin{cases} \frac{e^2}{4\pi^2}\frac{|\Delta\vec{v}|^2}{\omega^4} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{2.2.8}$$

$$= \begin{cases} \frac{2e^2}{3\pi c^3}|\Delta\vec{v}|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1\,. \end{cases} \tag{2.2.9}$$

Now, then, we need to calculate $\Delta\vec{v}$ in order to find the spectral distribution of the power. However, we already have an interesting result: a **flat power spectrum** in both the high- and low-frequency regimes, at a certain value to be calculated for high frequencies, and at zero for low frequencies. There will need to be some smooth connection between the two regions for $\omega\tau \sim 1$.

Now, as long as the interaction time is short, the electron "flies by" the ion, which then has little time to exert a force on it, and it will do so only in the region in which the electron is close, and in which $\vec{F}$ is approximately perpendicular to $\vec{v}$.

Then, we can compute the variation in velocity as

$$\Delta\vec{v} = \int_{-\infty}^{\infty} \vec{a}\,\mathrm{d}t\ , \tag{2.2.10}$$

and therefore, since the normal (dominant) component of the acceleration is given by $a_N = F_N/m_e$, we have

$$|\Delta\vec{v}| \approx \int_{-\infty}^{\infty} \frac{F_N}{m_e}\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{Ze^2}{m_e r^2}\frac{b}{r}\,\mathrm{d}t\ , \tag{2.2.11}$$

where the factor $b/r = \cos\theta$ accounts for the fraction of the force which is indeed normal: θ is the angle between the radial separation $\vec{r}$ and the velocity $\vec{v}$ (or $\pi-$ this angle), so that $F\cos\theta = F_N$. We can then solve this using the fact that the particle moves linearly, so that $r^2 = b^2 + v^2t^2$: substituting this in we find

$$|\Delta v| = \int_{-\infty}^{\infty} \frac{Ze^2 b}{m_e r^3}\,\mathrm{d}t = \frac{Ze^2 b}{m_e}\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(b^2+v^2t^2)^{3/2}} \tag{2.2.12}$$

$$= \frac{Ze^2}{m_e bv}\underbrace{\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{3/2}}}_{=2} = \frac{2Ze^2}{m_e bv}\,. \tag{2.2.13}$$

We can then insert this into the expression for the spectral distribution of the signal, also using $\tau \sim b/v$:

$$\frac{\mathrm{d}w}{\mathrm{d}\omega} = \begin{cases} \frac{8Z^2e^6}{3\pi c^3 b^2 v^2 m_e^2} & b \ll v/\omega \\ 0 & b \gg v/\omega\,. \end{cases} \tag{2.2.14}$$

Bremsstrahlung in a plasma

We found the spectral density for a single electron: now, we wish to compute it for the whole plasma, whose ion density is n_i, and whose electron density is n_e. To simplify, we will assume that all the electrons have the same speed v, but we will let their impact parameters b vary.

Let us consider this for a single ion, onto which many electrons will impact. The flux of electrons will be given by $n_e v$; so the number of particles crossing an annulus of radii b, $b + \mathrm{d}b$ will be given by $n_e v\,\mathrm{d}A = 2\pi b\,\mathrm{d}b\,n_e v$. Let us then integrate in $\mathrm{d}b$ to find the total emitted power. We will integrate from some minimum impact parameter $b_{\min}$ instead of from zero: this is needed to find a physical result, and it will be explained in more detail later. We find:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega} = \int_{b_{\min}}^{\infty} n_e v \frac{\mathrm{d}w}{\mathrm{d}\omega} 2\pi b\,\mathrm{d}b = 2\pi n_e v \int_{b_{\min}}^{\infty} \frac{\mathrm{d}w}{\mathrm{d}\omega} b\,\mathrm{d}b\,, \tag{2.2.15}$$

where now we need to substitute our expression; however we only have a nonzero contribution in the low-frequency limit, or equivalently the small-b limit. So, our integrand will be zero asymptotically; before that it will go as $1/b$. We account for this by only integrating up to some cutoff $b_{\max} \sim v/\omega$, whose exact value must be determined by a more detailed analysis. We will use the rough estimate $b_{\max} = v/\omega$. If we also account for the ion density n_i to find the power per unit frequency and volume, our integral can be expressed as:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}V} = \frac{16Z^2e^6}{3c^3m_e^2v} n_e n_i \int_{b_{\min}}^{b_{\max}} \frac{b}{b^2}\,\mathrm{d}b \tag{2.2.16}$$

$$= \frac{16Z^2e^6}{3c^3m_e^2v} n_e n_i \log\left(\frac{b_{\max}}{b_{\min}}\right). \tag{2.2.17}$$

Now, what should the value of $b_{\min}$ be? A first approximation we made is the small-angle one, which holds as long as $|\Delta\vec{v}|/|\vec{v}|$ is small, less than unity. Suppose we are at the upper limit of this condition, when $|\Delta\vec{v}| \sim |\vec{v}|$.

Inserting this into our expression for $|\Delta\vec{v}|$ we find the limit where b is so small — the electron comes so close to the ion — that the interaction is too large to be treated perturbatively:

$$|\Delta\vec{v}| \sim v \sim \frac{2Ze^2}{m_e b v} \implies b = b_{\min} = \frac{2Ze^2}{m_e v^2}\,. \tag{2.2.18}$$

A second line of reasoning comes from the quantum-mechanical uncertainty principle: $\Delta x \Delta p \gtrsim \hbar$. If our Δx is $b_{\min}$, this means that we will have

$$b \gtrsim b_{\min} = \frac{\hbar}{m_e v}\,. \tag{2.2.19}$$

Which of these is greater? To make it clearer, in natural units we are comparing ($1/m_e v$ times) the quantities: 1 for the quantum mechanical threshold, and

$$\underbrace{8\pi\alpha}_{\approx 0.18}\frac{Z}{v} \tag{2.2.20}$$

for the small-angle-approximation threshold. Since we are dealing with nonrelativistic particles v will be small, so this can easily become the larger bound of the two, even for a hydrogen ion.

Anyhow, these considerations are all heuristic, and what is typically done is to parametrize the uncertainty in this aspect with a so-called **Gaunt factor**

$$g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi}\log\left(\frac{b_{\max}}{b_{\min}}\right), \tag{2.2.21}$$

where the prefactor is, I think, there for historic reasons. This will in general be a function of both v (inside $b_{\min}$) and of ω (inside of $b_{\max}$). With it, we can write

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}V\,\mathrm{d}\omega} = \frac{16\pi Z^2 e^6 \pi}{3\sqrt{3}c^3 m_e^2 v} n_e n_i g_{ff}(v,\omega)\,. \tag{2.2.22}$$

With a proper quantum-mechanical treatment one can find an exact expression for this factor, and in the literature there are several good approximations which are good in different regimes; also, the values are tabulated. One can then assume that this is a known function.

Sunday 2020-8-23, compiled 2020-09-06

We have treated the bremsstrahlung emission from a single-speed electron distribution, expressing in terms of the free-free Gaunt factor $g_{ff}(\omega, v)$. Now we want to tackle the emission from a distribution with varying velocity.

We will consider a simple case which has broad application in an astrophysical setting: the electrons having a nonrelativistic Maxwellian (thermal) isotropic distribution,

$$\mathrm{d}P \propto \exp\left(-\frac{E}{k_B T}\right)\mathrm{d}^3 v = \exp\left(-\frac{mv^2}{2k_B T}\right)4\pi v^2\,\mathrm{d}v\,. \tag{2.2.23}$$

The last equality is imprecise, what it meant implicitly is that since the distribution is isotropic we can integrate over the solid angle getting a factor 4π; the only true dependence is on the modulus of the velocity.

Then, we will be able to recover the total power emitted by electrons whose velocity distribution looks like this as

$$\frac{\mathrm{d}w}{\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\omega} \propto \int_0^\infty \frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}V}(v,\omega)v^2\exp\left(-\frac{mv^2}{2k_B T}\right)\mathrm{d}v\,. \tag{2.2.24}$$

Why is the total power density also divided by $\mathrm{d}V$, but not the one inside the integral? are they not both densities in terms of space?

Should we really integrate from 0 to $+\infty$? The upper bound is fine, we can have electrons with arbitrarily high energy, however the distribution is exponentially suppressed there so it does not pose an issue. The problem comes from the lower bound: at a velocity v the electron has an energy $mv^2/2$, so if the frequency of the photon is ω and $mv^2/2 < \hbar\omega$ there cannot be emission, since the energy of the photon must come from the kinetic energy.

This means that we must exclude the tail of the distribution at low v, specifically we impose

$$v \geq v_{\min} = \sqrt{\frac{2\hbar\omega}{m}} \,. \tag{2.2.25}$$

So, we will have

$$\frac{\mathrm{d}w}{\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\omega} = \frac{1}{N}\int_{v_{\min}}^{\infty} \frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}V}(v,\omega) v^2 \exp\left(-\frac{mv^2}{2k_BT}\right) \mathrm{d}v \,, \tag{2.2.26}$$

where N is the normalization, which is equal to

$$N = \int_0^{\infty} v^2 \exp\left(-\frac{mv^2}{2k_BT}\right) \mathrm{d}v \,. \tag{2.2.27}$$

Note that in the normalization we still must integrate from 0: the electrons whose energy is lower than the threshold do not matter for our purposes, however they are still there, and so not all the electron density is available for emission.

Now, let us express the single-velocity power distribution (2.2.22) incorporating all the constants into a single constant A:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}V}(v,\omega) = A\frac{g_{ff}(v,\omega)}{v} \,, \tag{2.2.28}$$

where crucially A does not depend on v. Inserting this yields:

$$\frac{\mathrm{d}w}{\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\omega} = \frac{A}{N}\int_{v_{\min}}^{\infty} \frac{g_{ff}(v,\omega)}{v} v^2 \exp\left(-\frac{mv^2}{2k_BT}\,\mathrm{d}v\right) \tag{2.2.29}$$

$$= A\frac{\frac{2k_BT}{m}\int_{x_{\min}}^{\infty} g_{ff}(x,\omega) x e^{-x^2}\,\mathrm{d}x}{(\frac{2k_BT}{m})^{3/2}\int_0^{\infty} x^2 e^{-x^2}\,\mathrm{d}x} \tag{2.2.30}$$

Substituted $x = v\sqrt{2k_BT/m}$.

$$= \frac{A}{\sqrt{2k_BT/m}} \frac{\int_{x_{\min}}^{\infty} g_{ff}(x,\omega) x e^{-x^2}\,\mathrm{d}x}{\int_{x_{\min}}^{\infty} x e^{-x^2}\,\mathrm{d}x} \frac{\int_{x_{\min}}^{\infty} x e^{-x^2}\,\mathrm{d}x}{\int_0^{\infty} x^2 e^{-x^2}\,\mathrm{d}x} \tag{2.2.31}$$

$$= \frac{A}{\sqrt{2k_BT/m}} \overline{g}_{ff}(\omega) \frac{\int_{x_{\min}}^{\infty} x e^{-x^2}\,\mathrm{d}x}{\int_0^{\infty} x^2 e^{-x^2}\,\mathrm{d}x} \,, \tag{2.2.32}$$

where we introduced the **velocity-averaged Gaunt factor** $\overline{g}_{ff}(\omega, T)$ (it depends on the temperature through x), while the ratio of the two integrals equals

$$\frac{\int_{x_{\min}}^{\infty} x e^{-x^2}\,\mathrm{d}x}{\int_0^{\infty} x^2 e^{-x^2}\,\mathrm{d}x} = \frac{\frac{1}{2}e^{-x_{\min}^2}}{\sqrt{\pi}/4} = \frac{4}{\sqrt{\pi}}\frac{1}{2} e^{-\frac{h\nu}{k_BT}} \,, \tag{2.2.33}$$

since $x^2_{\rm min} = m v^2_{\rm min}/2k_BT$.

Therefore, our final expression will be

$$\frac{\mathrm{d}w}{\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\omega} = A\sqrt{\frac{m}{2k_BT}}\overline{g}_{ff}(\omega,T)\frac{1}{2}e^{-\frac{h\nu}{k_BT}}\frac{4}{\sqrt{\pi}} \tag{2.2.34}$$

$$= \frac{16\pi Z^2e^6}{3\sqrt{3}c^3m^2}\frac{1}{2}\frac{4}{\sqrt{\pi}}n_en_ie^{-\frac{h\nu}{k_BT}}\sqrt{\frac{m}{2k_BT}}\overline{g}_{ff}(\omega,T) \tag{2.2.35}$$

$$= \frac{32\sqrt{\pi}Z^2e^6}{3\sqrt{3}c^3m^2}\sqrt{\frac{m}{2k_B}}n_en_iT^{-1/2}e^{-\frac{h\nu}{k_BT}}\overline{g}_{ff}(\omega,T)\,, \tag{2.2.36}$$

which can also be written in terms of frequency $\nu = \omega/2\pi$ as

$$\frac{\mathrm{d}w}{\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu} = \frac{32Z^2e^6}{3mc^3}\sqrt{\frac{2\pi}{3k_Bm}}n_en_i\frac{1}{\sqrt{T}}\exp\left(-\frac{h\nu}{k_BT}\right)\overline{g}_{ff}(\nu,T)\,. \tag{2.2.37}$$

We can then see that the free-free spectrum has an exponential cutoff in frequency, coming from the lower bound on the integral we imposed: if we had not done so the spectrum would be completely flat, except for $\overline{g}_{ff}(\nu,T)$. This is reasonable: that effect is most significant for high-energy photons.

However, the dependence on frequency of the Gaunt factor is quite weak, and it is a reasonable first approximation to assume $\overline{g}_{ff} \approx 1$.

The shape of the spectrum, on a log-log plot, is flat for low frequencies up until $h\nu \sim k_BT$, and then a sharp drop. Now we can calculate the total radiated power per unit volume by integrating in $\mathrm{d}\nu$:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}V} = \int_0^\infty An_en_iT^{-1/2}\overline{g}_{ff}(\nu,T)e^{-h\nu/k_BT}\,\mathrm{d}\nu \tag{2.2.38}$$

$$= An_en_iT^{-1/2}\underbrace{\frac{\int_0^\infty \overline{g}_{ff}(\nu,T)e^{-h\nu/k_BT}}{\int_0^\infty e^{-h\nu/k_BT}\,\mathrm{d}\nu}}_{\overline{\overline{g}}_{ff}}\int_0^\infty e^{-h\nu/k_BT}\,\mathrm{d}\nu \tag{2.2.39}$$

$$= An_en_iT^{-1/2}\frac{k_BT}{h}\overline{\overline{g}}_{ff}(T)\,, \tag{2.2.40}$$

where we have introduced the doubly-averaged (over frequency as well as velocity) Gaunt factor $\overline{\overline{g}}_{ff}$, which now depends only on the temperature. The final result is then

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}V} = \frac{32\pi Z^2e^6}{3mc^3h}\sqrt{\frac{2\pi k_B}{3m}}\overline{\overline{g}}_{ff}(T)n_en_iT^{-1/2}\,. \tag{2.2.41}$$

This frequency-averaged Gaunt factor is very slowly varying, and very close to unity.

Now, let us link this with the **emission** and **absorption coefficients**: the emission coefficient j_ν in the isotropic case is

$$j_\nu = \frac{1}{4\pi}\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}V}\,, \tag{2.2.42}$$

which in our case will then be

$$j_\nu = \frac{8Z^2e^6}{3mc^3}\sqrt{\frac{2\pi}{2mk_B}}\overline{g}_{ff}(\nu,T)n_en_iT^{-1/2}\exp\left(-\frac{h\nu}{k_BT}\right). \tag{2.2.43}$$

The treatment of absorption is more interesting. Our population is in thermal equilibrium, so Kirkhoff's law holds: this means that the ratio of the emission and absorption coefficients is given by the Planck function,

$$B_\nu = \frac{j_\nu}{\alpha_\nu}, \tag{2.2.44}$$

therefore

$$\alpha_\nu = \frac{j_\nu}{B_\nu} = \frac{\frac{8Z^2e^6}{3mc^3}\sqrt{\frac{2\pi}{2mk_B}}\overline{g}_{ff}(\nu,T)n_en_iT^{-1/2}\exp\left(-\frac{h\nu}{k_BT}\right)}{\frac{2h\nu^3}{c^2}\left(\exp\left(\frac{h\nu}{k_BT}\right)-1\right)^{-1}} \tag{2.2.45}$$

$$= \frac{4Z^2e^6}{3mhc}\sqrt{\frac{2\pi}{3mk_B}}\overline{g}_{ff}(\nu,T)n_en_iT^{-1/2}\nu^{-3}\left(1-\exp\left(-\frac{h\nu}{k_BT}\right)\right). \tag{2.2.46}$$

How does the dependence of this absorption coefficient on frequency look like? At low energy, $h\nu \ll k_BT$, we have $1-\exp(-h\nu/k_BT) \approx h\nu/k_BT$, meaning that

$$\alpha_\nu \propto \nu^{-3}\frac{h\nu}{k_BT} \propto \nu^{-2}, \tag{2.2.47}$$

while for high energies $h\nu \gg k_BT$ we have $1-\exp(-h\nu/k_BT) \approx 1$: this means that

$$\alpha_\nu \propto \nu^{-3}. \tag{2.2.48}$$

In the middle, $h\nu \sim k_BT$, we will have some smooth connection of the two powerlaws. In a log-log plot, we have a broken powerlaw.

A quantity we defined earlier is the Rosseland mean absorption coefficient;

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\nu}\frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}\,\mathrm{d}\nu}. \tag{2.2.49}$$

With the explicit expression we now have for the monochromatic absorption coefficient we can compute these integrals: we will need the fact that

$$\frac{\partial B_\nu}{\partial T} \propto \frac{\nu^3}{(e^{h\nu/k_BT}-1)^2}e^{h\nu/k_BT}\frac{\nu}{T^2} \propto \frac{x^3e^xx}{(e^x-1)^2}T^2, \tag{2.2.50}$$

where, as usual, $x = h\nu/k_BT$. We do not need to worry about prefactors since they simplify. The Rosseland mean absorption coefficient is then

$$\frac{1}{\alpha_R} \propto \frac{T^{1/2}}{n_en_iZ^2}\frac{\int_0^\infty \nu^3\left(1-e^{-h\nu/k_BT}\right)^{-1}\overline{g}_{ff}^{-1}\frac{x^4e^x}{(e^x-1)^2T^3\mathrm{d}x}}{\int_0^\infty \frac{x^4e^xT^3}{(e^x-1)^2}\,\mathrm{d}x} \tag{2.2.51}$$

Extra T in the numerator from moving from $\mathrm{d}\nu$ to $\mathrm{d}x$.

$$\propto \frac{T^{1/2}}{n_e n_i Z^2} \frac{T^3 T^3}{T^3} \underbrace{\frac{\int_0^\infty \overline{g}_{ff}^{-1} x^7 (e^x-1)^{-3}\,\mathrm{d}x}{\int_0^\infty x^4 e^x (e^x-1)^{-2}\,\mathrm{d}x}}_{\overline{g}_R^{-1}(T)} \tag{2.2.52}$$

$$\propto \frac{T^{7/2}}{n_e n_i Z^2} \overline{g}_R^{-1}(T)\,, \tag{2.2.53}$$

therefore

$$\alpha_R \propto n_e n_i Z^2 T^{-7/2} \overline{g}_R(T)\,. \tag{2.2.54}$$

The main result is the $T^{-7/2}$ dependence; the constant in front of the expression can be fixed with a more precise calculation.

Now, let us discuss what kinds of corrections come up when dealing with **relativistic bremsstrahlung**: the main process which becomes relevant is **electron-electron** bremsstrahlung. Even just treating electron-ion bremsstrahlung in the relativistic regime, however, requires the use of full QED. The final result one gets for the frequency-integrated power is

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}V} \approx 1.4\times 10^{27} T^{1/2} Z^2 n_e n_i \overline{\overline{g}}_{ff} \left(1 + 4.4\times 10^{-10}\frac{T}{\mathrm{K}}\right) \mathrm{erg cm^{-3} s^{-1}}\,, \tag{2.2.55}$$

where the $4.4\times 10^{-10} T$ factor is the relativistic correction. Thermal electrons become relativistic when $k_B T \sim m_e c^2$, meaning that $T \sim 511\,\mathrm{keV} \sim 6\times 10^9\,\mathrm{K}$. The relativistic correction can be written as

$$1 + 2.6\frac{T}{m_e c^2/k_B}\,, \tag{2.2.56}$$

which can become significant quite early: it is of the order of 25 % as early as $6\times 10^8\,\mathrm{K}$, one tenth of the relativistic temperature.

2.3 Electron scattering

2.3.1 Compton scattering onto an electron at rest

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We need to account for the fact that light has a quantum nature, which is not addressed in the classical treatment of scattering. Also, now we will account for the momentum of the photon.

We start of with an electron at rest, and a photon with energy $h\nu$ and momentum $h\nu/c$ impinging on it. After the scattering, the photon will have energy $h\nu'$ and momentum $h\nu'/c$, while the electron will have momentum $mv\gamma$. Let us call the angle between the direction of the incoming photon and the direction of the outgoing one θ.

In terms of four-vectors, we can express the momenta of the photon before and after as

$$k^\mu = \frac{\epsilon}{c}\begin{bmatrix} 1 \\ \vec{\Omega} \end{bmatrix} \qquad \text{and} \qquad k'^\mu = \frac{\epsilon'}{c}\begin{bmatrix} 1 \\ \vec{\Omega}' \end{bmatrix}\,, \tag{2.3.1}$$

where $\vec{\Omega}$ and $\vec{\Omega}'$ are unit vectors defining the propagation directions, such that $\vec{\Omega} \cdot \vec{\Omega}' = \cos\theta$. On the other hand, the momenta of the electron will be

$$p^\mu = \begin{bmatrix} mc \\ \vec{0} \end{bmatrix} \qquad \text{and} \qquad p'^\mu = \gamma \begin{bmatrix} mc \\ m\vec{v} \end{bmatrix} . \tag{2.3.2}$$

Since the particles are unchanged after the scattering, both the incoming and outgoing momenta must satisfy $p^\mu p_\mu = -m^2c^2$ and $k^\mu k_\mu = 0$ (in any frame: they are Lorentz scalars). Because of momentum conservation, we can also impose the four equations

$$p^\mu + k^\mu = p'^\mu + k'^\mu . \tag{2.3.3}$$

Solving the system of equations yields

$$\epsilon' = \frac{\epsilon}{1 + \frac{\epsilon}{mc}\left(1 - \vec{\Omega} \cdot \vec{\Omega}'\right)} , \tag{2.3.4}$$

implying that we must have $\epsilon' \leq \epsilon$: the photon will lose energy in the scattering. The calculation which yields the differential cross section of the scattering is quite complicated and requires the full machinery of QED; here we just give the result:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\vec{\Omega}'} = \frac{r_0^2}{2}\left(\frac{\epsilon'}{\epsilon}\right)^2 \left[\frac{\epsilon}{\epsilon'} + \frac{\epsilon'}{\epsilon} - 1 + \vec{\Omega} \cdot \vec{\Omega}'\right] , \tag{2.3.5}$$

where ϵ is the energy divided by mc^2 and r_0 is the classical electron radius. If we substitute in our formula for the energy, using $\xi = \cos\theta = \vec{\Omega} \cdot \vec{\Omega}'$, we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'\,\mathrm{d}\epsilon'} = \frac{r_0^2}{2} \frac{1 + \xi^2}{\left(1 + \epsilon(1 - \xi)\right)^2} \left[1 + \frac{\epsilon^2(1 - \xi)^2}{(1 + \xi)^2(1 + \epsilon(1 - \xi))}\right] \delta\left(\epsilon' - \frac{\epsilon}{1 + \epsilon(1 - \xi)}\right) . \tag{2.3.6}$$

In the low energy limit, $h\nu \ll m_e c^2$ or $\epsilon \to 0$, we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'\,\mathrm{d}\epsilon'} = \frac{r_0^2}{2}(1 + \xi^2)\delta(\epsilon' - \epsilon) , \tag{2.3.7}$$

which is the Thomson cross section.

Let us now define the **Compton scattering kernel** σ: it is the differential cross section times the electron density,

$$\sigma(\epsilon \to \epsilon', \xi) = n_e \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'\,\mathrm{d}\epsilon'} . \tag{2.3.8}$$

We can integrate this in order to find the total cross section presented by the electrons to photons of an energy ϵ:

$$\sigma(\epsilon) = \int \mathrm{d}\Omega'\,\mathrm{d}\epsilon'\,\sigma(\epsilon \to \epsilon', \xi) \tag{2.3.9}$$

$$= \frac{3}{4} n_e \sigma_T \left[\left(\frac{1+\epsilon}{\epsilon^3} \right) \left(\frac{2\epsilon(1+\epsilon)}{1+2\epsilon} - \log(1+2\epsilon) \right) + \frac{1}{2\epsilon} \log(1+2\epsilon) - \frac{1+3\epsilon}{(1+2\epsilon)^2} \right] . \tag{2.3.10}$$

How does this differ from the Thomson cross section? For $\log \epsilon \lesssim -1$ we have $\sigma \approx \sigma_T$, while as ϵ increases the cross section goes to zero. In the low energy limit we have the expansion

$$\sigma(\epsilon) \approx \sigma_T \left(1 - 2\epsilon + \frac{26}{5} \epsilon^2 \right) . \tag{2.3.11}$$

The introduction of these nonconservative aspects complicates the radiative transfer equation for scattering. The absorption term is

$$-I(\epsilon, \Omega) n_e \int \mathrm{d}\Omega' \, \mathrm{d}\epsilon' \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega' \, \mathrm{d}\epsilon'} = -I(\epsilon, \Omega) \sigma(\epsilon) , \tag{2.3.12}$$

while for the emission term the intensity must go inside the integral, so we have

$$n_e \int \mathrm{d}\Omega' \, \mathrm{d}\epsilon' \, I(\epsilon', \Omega') \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega' \, \mathrm{d}\epsilon'} = \int \mathrm{d}\Omega' \, \mathrm{d}\epsilon' \, \sigma(\epsilon \to \epsilon', \xi) , \tag{2.3.13}$$

which cannot be expressed in terms of the integrated kernel $\sigma(\epsilon)$.

For Thomson scattering the absorption term is similar, with $n_e \sigma_T$ instead of $\sigma(\epsilon)$. The emission term, on the other hand, can now be evaluated to yield

$$\int \mathrm{d}\Omega' \, \mathrm{d}\epsilon' \, \sigma(\epsilon \to \epsilon', \xi) = \frac{r_0^2}{2} n_e \int \mathrm{d}\Omega' \, \mathrm{d}\epsilon' \, I(\epsilon', \Omega') (1 + \xi^2) \delta(\epsilon' - \epsilon) \sim n_e \sigma_T J(\epsilon) , \tag{2.3.14}$$

as long as we neglect the angular part $1 + \xi^2$.

We are always restricting ourselves to electrons which are initially at rest. For them Thomson scattering is a good approximation; photons more energetic than a few tens of keV (hard X-rays) hardly scatter, since the Klein-Nishina cross section drops at high energies.

So, by using the Thomson limit we do not get it wrong by much.

The main conclusions we can draw from this is that we expect no spectral modifications (since Thomson scattering is conservative) but we do expect angular redistribution (since Thomson scattering is essentially isotropic).

2.3.2 Scattering in plasmas

Electrons are not at rest in any realistic astrophysical setting. In order to have free electrons we need a plasma, which is made of (at least partially) ionized gas. So, we need a way to describe the fractional ionization. Let us consider only hydrogen for simplicity: we define the collisional ionization fraction x (ionized atoms divided by total atoms), which can be expressed in terms of the temperature as

$$x = \frac{F}{1+F} \qquad \text{where} \qquad F = 2T \exp\left(-\frac{1.58 \times 10^5 \, \mathrm{K}}{T} \right) . \tag{2.3.15}$$

The way x looks as a function of temperature is a kind of sigmoid: it is close to zero, then at T around $10^4\,\mathrm{K}$ it quickly rises, and becomes close to 1 at a few times $10^4\,\mathrm{K}$. At temperatures larger than $10^5\,\mathrm{K}$ the plasma is basically fully ionized (which makes sense: the first ionization energy of hydrogen is $13.6\,\mathrm{eV} \approx 1.6 \times 10^5\,\mathrm{K}$).

At these temperatures, the electrons will be moving quite a lot because of thermal motion.

2.3.3 Inverse Compton scattering

Scattering onto moving electrons is often called **inverse Compton scattering**, since in this case the photon might gain energy instead of losing it. We start off with a photon with momentum $k^\mu = (\epsilon/c)(1, \vec{\Omega})$ and an electron with momentum $p^\mu = \gamma(mc, m\vec{v})$; the outgoing electron and momentum momenta, energies and velocities will be denoted with a prime.

We know how to deal with scattering off a stationary electron, and stationarity is relative: if we boost to the electron's rest frame we can apply the results from regular Compton scattering, and then we will need to boost back to our frame.

We shall denote quantities calculated in the ERF (electron rest frame) with a pedix e: for example the energy of the photon in the ERF before the scattering will be ϵ_e. This means that in the ERF we will have the equality

$$\epsilon_e' = \frac{\epsilon_e}{1 + \frac{\epsilon_e}{mc}(1 - \cos\Theta)} \approx \epsilon_e\left(1 - \frac{\epsilon_e}{mc}(1 - \cos\Theta)\right) \approx \epsilon_e \qquad \text{if } \epsilon_e \ll mc^2\,. \tag{2.3.16}$$

We are saying that in the rest frame the scattering will basically be conservative, this is for sure an approximation but, because of what we discussed earlier about the Klein-Nishina cross section dropping off at high energies, not a large one.

The Lorentz transform from the LAB frame to the ERF for the energy of the photon is:

$$\epsilon_e = \gamma\epsilon\big(1 - \beta\cos\theta\big)\,, \tag{2.3.17}$$

while the transform back from the ERF to the LAB frame is

$$\epsilon' = \gamma\epsilon_e'\big(1 + \beta\cos\theta_e\big)\,. \tag{2.3.18}$$

Note that the quantities β and γ refer to the velocity of the electron: since we are approximating the scattering as conservative in the ERF calculating them before or after the scattering is the same (the direction of the electron can change: this is accounted for by letting θ and θ' be different). So, we can just insert these two multiplicative factors one after another to find the total change in energy of the photon:

$$\epsilon' \approx \gamma^2\epsilon\big(1 - \beta\cos\theta\big)\big(1 + \beta\cos\theta_e'\big)\,. \tag{2.3.19}$$

Inaccuracy in the slides: the first formula does not really make sense, since the approximation of the nonconservativeness factor being equal to one has been made in part of it (β and γ being the same before and after), so it does not make sense to write it out.

Let us neglect the angular terms for now: generally they are of order unity. Instead, the main factor in the formula is $\gamma^2 > 1$: this means that in general the energy of the scattered photon is of the order $\epsilon' = \gamma^2 \epsilon > \epsilon$. The boost of the energy of the photon depends on how relativistic the electron is; if the electron is very relativistic the boost in energy for the photon can be quite large.

In order to make predictions about the effect of this for a population of electrons and photons, we can Lorentz transform the Compton Scattering Kernel: the final result of the manipulation is

$$\sigma(\epsilon \to \epsilon', \xi) = \frac{D}{D'} \sigma_e(\epsilon_e \to \epsilon'_e, \xi_e) \,, \tag{2.3.20}$$

where $D = 1 - \vec{\Omega} \cdot \vec{v}/c$ and $D' = 1 - \vec{\Omega}' \cdot \vec{v}'/c$ are the factors in the Lorentz transforms, which allow us to write $\epsilon_e = \gamma D \epsilon$ and $\epsilon'_e = \gamma D' \epsilon'$.

The transformation law for $\xi = \vec{\Omega} \cdot \vec{\Omega}$ is

$$1 - \xi = \frac{1 - \xi_e}{\gamma^2 D D'} \,, \tag{2.3.21}$$

while the electron energy density transforms as $n = \gamma n_e$.

This allows us to write down an explicit expression for $\sigma(\epsilon \to \epsilon', \vec{\Omega}, \vec{\Omega})$ in the lab frame for a population of single-speed electrons:

$$\sigma(\epsilon \to \epsilon', \vec{\Omega}, \vec{\Omega}, v) = \frac{n r_0^2}{2 \epsilon \nu \gamma} \left[1 + \left(1 + \frac{1 - \xi}{\gamma^2 D D'} \right)^2 + \frac{\epsilon \epsilon' (1 - \xi)^2}{\gamma^2 D D'} \right] \delta \left(\xi - 1 + \frac{\gamma D}{\epsilon'} - \frac{\gamma D'}{\epsilon} \right) , \tag{2.3.22}$$

but we must also consider the fact that the electrons are distributed with different velocities: if their distribution is isotropic, so that $\mathrm{d}n = n f(v) \, \mathrm{d}^3 v$ then we can integrate across all velocity space, getting an expression like

$$\sigma(\epsilon \to \epsilon', \xi) = \int \mathrm{d}v \sin\theta_v \, \mathrm{d}\theta_v \, \mathrm{d}\phi_v \, v^2 n f(v) \sigma(\epsilon \to \epsilon', \vec{\Omega}, \vec{\Omega}, v) \,. \tag{2.3.23}$$

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We can compute this integral if we know the distribution of the velocity moduli $f(v)$. The angular integral is nontrivial: the single-velocity cross section has an angular dependence.

The calculation is then tricky, the full result is relatively recent, going back only to the 1980s. We will not discuss it here. We are able to reduce the CSK to a single integral in $\mathrm{d}v$.

Now, let us give an explicit expression for the velocity distribution $f(v)$: since the electrons are relativistic we need to use a **relativistic Maxwellian**, also called a Jüttner distribution, which looks like

$$f(v) = \frac{\gamma^5 \exp(-\gamma/\Theta)}{4\pi c^3 \Theta K_2(1/\Theta)} \qquad \text{where} \qquad \Theta = \frac{k_B T}{mc^2} \,, \tag{2.3.24}$$

where γ is the Lorentz gamma factor of the electron, while K_2 is the modified Bessel function of order 2. We will then find a cross section depending on the rescaled temperature Θ. The integral over $\mathrm{d}v$ can then be evaluated numerically.

What we find is that the cross section is strongly peaked, both in energy (temperature, Θ) and in angle ($\xi = \cos\theta$).

We can then compute the total absorption coefficient, by integrating across all possible outgoing photon energies and angles:

$$\sigma(\epsilon;\Theta) = \int \mathrm{d}\Omega'\,\mathrm{d}\epsilon'\,\sigma(\epsilon \to \epsilon', \xi;\Theta) \tag{2.3.25}$$

$$= \frac{n\sigma_T}{2\epsilon K_2(1/\Theta)} \int_0^\infty \mathrm{d}z\,\sigma_0(\epsilon z)\exp\left(-\frac{z+1/z}{2\Theta}\right), \tag{2.3.26}$$

where σ_0 is a known function. This integral is quite easy to compute numerically. What we find is shown in figure

add figure here!

Energy exchange rate

We want to see how much energy is transferred from photons to electrons and vice versa. We start off by considering the energy exchanged by a single electron and many photons.

The electron will be moving with a velocity v, and we consider the photon flux coming from a direction range $\mathrm{d}\Omega$. We also consider the distribution across photon energies, so that we have the number flux of photons $\mathrm{d}f = (I/\epsilon)\,\mathrm{d}\epsilon\,\mathrm{d}\Omega$, where I is the specific intensity. With this, we can compute the number of scatterings per unit time, solid angle of emission $\mathrm{d}\Omega'$ and emission energy $\mathrm{d}\epsilon'$ as

$$\mathrm{d}N_s = (\sigma\,\mathrm{d}\Omega'\,\mathrm{d}\epsilon')\,\mathrm{d}f\ . \tag{2.3.27}$$

Then, the energy change per unit time will be $(\epsilon' - \epsilon)\,\mathrm{d}N_s$. This is all differential: in order to get the full energy change we need to integrate. For a thermal photon distribution we will have

$$\dot{E}_s = \int \mathrm{d}\Omega\,\mathrm{d}\epsilon\,\mathrm{d}\Omega'\,\mathrm{d}\epsilon'\,(\epsilon'-\epsilon)\frac{I}{\epsilon}\sigma(\epsilon\to\epsilon',\xi;\Theta)\,. \tag{2.3.28}$$

Now, we can see that this depends on the number of scatterings, the electron and the radiation temperatures. Let us assume that the distribution of the photons is Planckian as well, with a temperature T_r, and define $\Theta_r = k_B T_R/mc^2$.

Then, we can find that for small Θ_r and small Θ we have

$$\frac{\dot{E}_s}{E_s} = 4(\Theta-\Theta_r)\left(1+\frac{5}{2}\Theta - 21\frac{\zeta(5)}{\zeta(4)}\Theta_r + \dots\right), \tag{2.3.29}$$

where ζ is the Riemann zeta function.

The lowest order contribution is then, for an isotropic photon distribution (not only a blackbody),

$$\frac{\dot{E}_s}{E_s} = 4\Theta - \frac{\left\langle\epsilon^2\right\rangle}{\langle\epsilon\rangle}, \tag{2.3.30}$$

so if the photons are colder than the electrons they gain energy.

What happens if a photon scatters off the electron distribution many times?

Let us assume we have an isotropic electron distribution with number density n, and that the scattering is isotropic. We shall use Thomson scattering, which is not but it has forward-backward symmetry, which is enough for our considerations. So, the cross section will be σ_T.

The photon mean free path is then $\lambda_s = 1/(n\sigma_T)$, while the mean time between scatterings is $t_s = \lambda_s/c$.

The mean number of times the photon scatters in travelling a length L is something we already discussed, and the final result was

$$N_s = \frac{L^2}{\lambda_s^2} = (n\sigma_T L)^2 = \tau_s^2 \,, \tag{2.3.31}$$

where τ_s is the average scattering optical depth, if $\tau_s \gg 1$, and $N_s \sim \tau_s$ if τ_s was $\ll 1$. Then, $N_s \sim \max(\tau_s, \tau_s^2)$.

This is useful in order to define the Compton y parameter:

$$y = \left\langle \frac{\Delta\epsilon}{\epsilon} \right\rangle N_s \approx \frac{4k_B T}{mc^2} \max(\tau_s, \tau_s^2) \,, \tag{2.3.32}$$

where $\langle \Delta\epsilon/\epsilon \rangle$ is the mean fractional energy change per scattering, which as we saw earlier was 4Θ at lowest order for a nonrelativistic thermal distribution in the Thomson limit. This number y tells us how efficient Comptonization is: by how much does the photon's energy change as it travels through the medium?

If this y comes out to be much smaller than 1, then we expect to see little effect on the primary spectrum (of the star, for instance) while if $y \geq 1$ there will be large spectral deformations. We can have small y both if the energy change $\Delta\epsilon/\epsilon$ is small, or if the medium is optically thin: τ_s is small.

Chapter 3

Spectral modification in plasmas

3.1 The Kompaneets equation

This is the tool we use in order to understand what kind of spectral modifications we expect to see for radiation passing through a cloud in which repeated scatterings may occur. It was developed in the 1950s by a Russian physicist working on nuclear weapons.

It is basically a way to rewrite the **Boltzmann equation**, which is the equation describing the evolution of a photon distribution undergoing scatterings with electrons:

$$\frac{\partial n}{\partial t} = c \int \mathrm{d}^3 p \, \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \left[f(\vec{p}\,') n(\epsilon') (1 + n(\epsilon)) - f(\vec{p}) n(\epsilon) (1 + n(\epsilon')) \right] , \tag{3.1.1}$$

where $f(\vec{p})$ is the phase space density of electrons. The reason why there is a $1 + n(\epsilon)$ factor is to account for *stimulated scattering*.

which is...?

We will assume that the energy of the photons is much smaller than mc^2, so $\epsilon \ll 1$, and that the electrons are nonrelativistic. We can then approximate

$$\frac{\epsilon'}{\epsilon} = \frac{1 - \beta \cos\theta}{1 - \beta \cos\theta' + \epsilon(1 - \cos\theta)/\gamma} \approx 1 + \beta \cos\theta' - \beta \cos\theta = 1 + \vec{\beta} \cdot (\vec{\Omega}' - \vec{\Omega}) . \tag{3.1.2}$$

Conservation of 3-momentum and energy gives us

$$\vec{p}\,' = \vec{p} + \epsilon \vec{\Omega} - \epsilon' \vec{\Omega}' \qquad \text{and} \qquad E' = E + \epsilon - \epsilon' , \tag{3.1.3}$$

while if the electrons are thermal as well as nonrelativistic their distribution will be

$$f(E) = n (2\pi m k_B T)^{-3/2} \exp\left(-\frac{E}{\Theta} \right) . \tag{3.1.4}$$

Note that all the energies are in units of mc^2, where m is the electron mass, and the momenta are in units of mc.

We further assume that the energy change of the photon is small compared to the mean electron energy,

$$\Delta = \frac{\epsilon' - \epsilon}{\Theta} = \frac{\epsilon \vec{\beta} \cdot (\vec{\Omega}' - \vec{\Omega})}{\Theta} \ll 1 , \tag{3.1.5}$$

where we used the relation between the photon energies before and after the scattering which follows from the Lorentz transformation.

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We then expand the electron and photon energy distributions *after* the scattering up to second order in Δ: using the fact that $\epsilon' - \epsilon = \Delta\Theta$ (Δ *times* Θ, not the variation of Θ) we find

$$f(E') = f(E - \Delta\Theta) \approx \left(1 + \Delta + \frac{\Delta^2}{2}\right) f(E) \tag{3.1.6}$$

$$n(\epsilon') = n(\epsilon + \Delta\Theta) \approx n(\epsilon) + \frac{\partial n}{\partial \epsilon}\Delta\Theta + \frac{\partial^2 n}{\partial \epsilon^2}\frac{(\Delta\Theta)^2}{2}\,, \tag{3.1.7}$$

where we have used the explicit expression we have for the thermal distribution $f(E) \sim e^{-E/\Theta}$, which gives us a factor $-1/\Theta$ each we differentiate with respect to E.

If we define the usual rescaled energy $x = h\nu/k_B T = \epsilon/\Theta$ we can express the latter expansion as

$$n(x') \approx n(x) + \Delta\frac{\partial n}{\partial x} + \frac{\Delta^2}{2}\frac{\partial^2 n}{\partial x^2}\,. \tag{3.1.8}$$

We will assume that the medium is **homogeneous**, **isotropic** and **infinite**, and that the photon distribution is **isotropic**. The latter point means that n is not a function of Ω. This will be close to true as long as the optical depth is large, since then we will see many interactions which will spread the photons.

We can then take out of the Boltzmann integral all the terms depending only on n, since they have no more angular dependence. This yields the following expression (up to second order in Δ):

$$\frac{\partial n}{\partial t} = \left[n' + n(n+1)\right] \underbrace{\int \mathrm{d}^3 p\, \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} f(E)\Delta}_{I_1} + \left[\frac{n''}{2} + \left(n' + \frac{n}{2}\right)(n+1)\right] \underbrace{\int \mathrm{d}^3 p\, \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} f(E)\Delta^2}_{I_2}\,. \tag{3.1.9}$$

Here we are denoting $n' = \partial n/\partial x$ for simplicity. Let us then try to evaluate the integrals I_1 and I_2, starting from the latter. It depends on Δ^2, which is given by

$$\Delta^2 = x^2\beta^2(\vec{\Omega}' - \vec{\Omega})^2 \cos^2\theta_v\,, \tag{3.1.10}$$

so it evaluates to

$$I_2 = \int \mathrm{d}^3 p\, \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} f(E)\Delta^2 = x^2 \int f(E) p^2 \beta^2 \cos^2\theta_v\, \mathrm{d}p \sin\theta_v\, \mathrm{d}\theta_v\, \mathrm{d}\phi_v \int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}^2\,, \tag{3.1.11}$$

where θ_v and ϕ_v are the two angles defining the unit vector $\hat{p}$, so that $\mathrm{d}^3 p = p^2 \sin\theta_v\, \mathrm{d}p\, \mathrm{d}\theta_v\, \mathrm{d}\phi_v$. We choose them by placing the vector $\vec{\Omega}' - \vec{\Omega}$ on the z axis, so that the angle θ_v is also the one appearing in $(\vec{\Omega}' - \vec{\Omega}) \cdot \vec{p} = p\left|\vec{\Omega}' - \vec{\Omega}\right| \cos\theta_v$.

The angular integrals in $\mathrm{d}\phi_v$ and $\mathrm{d}\theta_v$ are standard, while for the integral in the modulus of p we need to use the actual expression of the Maxwellian: substituting $\beta = p/mc$ and then $z = \beta/\sqrt{\Theta}$ we get

$$I_2 = 2\pi x^2 m^3 c^3 \frac{2}{3}(2\pi m k_B T)^{-3/2} n_e \int_0^\infty \mathrm{d}\beta\, \beta^4 \exp\left(-\frac{\beta^2}{2\Theta}\right) \int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\vec{\Omega}' - \vec{\Omega}\right)^2 \tag{3.1.12}$$

$$= 2\pi x^2 m^3 c^3 \frac{2}{3}\Theta^{5/2}(2\pi m k_B T)^{-3/2} n_e \underbrace{\int_0^\infty \mathrm{d}z\, z^4 \exp\left(-\frac{z^2}{2}\right)}_{=3\sqrt{2\pi}/2} \int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\vec{\Omega}' - \vec{\Omega}\right)^2 \tag{3.1.13}$$

$$= x^2 \Theta n_e \int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left(\vec{\Omega}' - \vec{\Omega}\right)^2, \tag{3.1.14}$$

so we are left with only the integral in the photon space: in order to evaluate it we need to substitute the expression of the modulus of the difference of the unit vectors and the differential cross section:

$$\left(\vec{\Omega}' - \vec{\Omega}\right)^2 = 2(1 - \cos\theta) \qquad \text{and} \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r_0^2}{2}\left(1 + \cos^2\theta\right), \tag{3.1.15}$$

so, also substituting the explicit expression for the radius r_0 we get

$$I_2 = n_e x^2 \Theta \int \sin\theta\, \mathrm{d}\theta \underbrace{\mathrm{d}\phi}_{=2\pi} \frac{3\sigma_T}{16\pi} 2(1 + \cos^2\theta)(1 - \cos\theta) \tag{3.1.16}$$

$$= 2 n_e x^2 \Theta \sigma_T \frac{3}{16\pi} 2\pi \underbrace{\int_{-1}^{1} (1 + \mu^2)(1 - \mu)\, \mathrm{d}\mu}_{=8/3} \tag{3.1.17}$$

$$= 2\sigma_T x^2 n_e \Theta\,. \tag{3.1.18}$$

The integral I_1 looks similar, and one might try to evaluate it similarly. This turns out to be tricky, and there is a better indirect way to do it.

The photon number density in phase space is given by

$$\frac{\mathrm{d}N}{\mathrm{d}V\, \mathrm{d}^3 p} = \frac{2}{h^3} n\,, \tag{3.1.19}$$

so, since the momentum of a photon is proportional to its energy and also to $x = h\nu/k_B T$ we can write

$$\frac{\mathrm{d}N}{\mathrm{d}V} \propto \int_0^\infty \mathrm{d}x\, n x^2\,. \tag{3.1.20}$$

If I'm interpreting this correctly, the slides use x with two different meanings in the same equation: one is position, and the other is the normalized energy. Really confusing if it is the case.

Since the medium is assumed to be homogeneous and isotropic, the photon number *density* is conserved: this means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}N}{\mathrm{d}V}\right) = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \mathrm{d}x\, nx^2 = \int_0^\infty \mathrm{d}x\, \frac{\partial n}{\partial t}x^2 \,. \tag{3.1.21}$$

Now, for reasons which will become clear in a moment, we make an assumption: suppose that there is a function $j(x,t)$ such that the derivative of the photon density can be written as

$$\frac{\partial n}{\partial t} = -\frac{1}{x^2}\frac{\partial[x^2 j(x,t)]}{\partial x} \,. \tag{3.1.22}$$

If this is the case, then the integral which we have just seen shown to equal zero can be written as

$$\int_0^\infty \mathrm{d}x\, \frac{\partial n}{\partial t}x^2 = -\int_0^\infty \frac{\partial[x^2 j(x,t)]}{\partial x} = -x^2 j(x,t)\Big|_0^\infty = 0 \,. \tag{3.1.23}$$

This gives us the boundary conditions for j, which will have to be well-behaved at zero (non diverging) and at infinity (going to zero at least as $1/x^2$). Now, the Boltzmann equation's dependence on n and its derivatives is in the form

$$\frac{\partial n}{\partial t} = C_1(x)n'' + C_2(n,x)n' + C_3(n,x) \,, \tag{3.1.24}$$

and if we compare this expression with the one which defines j[1] we can prove that its dependence on n and x looks like

$$j = g(x)\left[n' + h(n,x)\right] \,, \tag{3.1.26}$$

for some functions g and h. In general h is a function of both n and x, but we will assume it only depends on n.

Why though? Is there a physical reason for it?

Now, we substitute this general expression for j back into the Boltzmann equation: this yields

$$\begin{aligned}\frac{1}{2}I_2 n'' + \left[(1+n)I_2 + I_1\right]n' + \left(I_1 + \frac{I_2}{2}\right)n(1+n) = \\ = -g(x)n'' - \left(\frac{2g(x)}{x} + g(x)\frac{\partial h}{\partial n} + \frac{\partial g}{\partial x}\right)n' - \left(h\frac{2g(x)}{x} + g(x)\frac{\partial h}{\partial x} + h\frac{\partial g}{\partial x}\right)\end{aligned} \,, \tag{3.1.27}$$

[1] Seeing that

$$\frac{\partial n}{\partial t} = -\frac{1}{x^2}\frac{\partial[x^2 j(x,t)]}{\partial x} = -\frac{2j}{x} - \frac{\partial j}{\partial x} \,. \tag{3.1.25}$$

from which, identifying the terms, we can recover the expressions for g, I_1 and h:

$$g = -\frac{I_2}{2} = -x^2 n_e \sigma_T \Theta \tag{3.1.28}$$

$$I_1 = -2\frac{g}{x} - \frac{\partial g}{\partial x} - \frac{I_2}{2} = (4-x) x n_e \sigma_T \Theta \tag{3.1.29}$$

$$h = n(1+n)\,. \tag{3.1.30}$$

This means that the form of j is

$$j = g(x)\left[n' + h(n,x)\right] = -x^2 n_e \sigma_T \Theta (n' + n + n^2)\,, \tag{3.1.31}$$

which we can finally substitute into the Kompaneets equation:

$$\frac{\partial n}{\partial t} = -c\frac{1}{x^2}\frac{\partial [x^2 j]}{\partial x} = n_e \sigma_T c \frac{\Theta}{x^2}\frac{\partial [x^4 (n' + n + n^2)]}{\partial x} \tag{3.1.32}$$

$$\frac{\partial n}{\partial t_s} = \frac{\Theta}{x^2}\frac{\partial [x^4 (n' + n + n^2)]}{\partial x}\,, \tag{3.1.33}$$

where we calculate time in units of the mean time between subsequent scatterings, $T = \frac{1}{n_e \sigma_T c}$, so that $t_s = t/T$.

There is now a c which was not there before...

This is the **Kompaneets equation**. In general this nonlinear equation must be solved numerically, it is an equation of the Fokker-Planck type, whose solution exhibit two main effects:

1. drift of the photon distribution to higher energies, this is called the **secular change**, and is due to the fact that $I_1 \sim 4k_B T - h\nu$ (as long as the photons start out with energies lower than $4k_B T$);

2. broadening of the photon distribution, due to random walk effects: this is related to the integral I_2.

3.1.1 Stationary solutions

We expect that, after the photons have interacted with the medium for a long time, their variation in time will become ever smaller, approaching a stationary solution which will satisfy

$$0 = \frac{\partial [x^4 (n' + n + n^2)]}{\partial x}\,, \tag{3.1.34}$$

meaning that

$$n' + n + n^2 = \frac{A}{x^4}\,. \tag{3.1.35}$$

The constant A must equal zero, since otherwise the solution will not be regular at zero: recall that $x^2 j = x^4(n' + n + n^2)$ must vanish at zero and infinity. Then, we can separate variables in order to integrate $n' + n + n^2 = 0$: this yields

$$\log \frac{n}{1+n} = -x - \mu \tag{3.1.36}$$

for a constant μ, meaning that

$$n = \frac{1}{\exp(x+\mu) - 1}\,, \tag{3.1.37}$$

which is nice! We found a Bose-Einstein distribution with an arbitrary chemical potential μ: photons are at equilibrium with another chemical species.

The reason the photons do not achieve a Planck distribution (which is a Bose-Einstein distribution with $\mu = 0$) is that their number must be conserved under our assumptions.

To understand better: the chemical potential quantifies how much the (energy?) changes if we change the number of particles, right?

The Kompaneets equation is a **parabolic** PDE, like the Fourier heat transport equation. This means that we need to provide boundary conditions not only in the time domain, but also in the frequency domain for $x \to 0$ and $x \to \infty$. We must set the distribution to zero in both regions, in order to avoid photons escaping through the boundary.

If the number of photons we initially insert in the model is smaller than the number of photons which would be contained in a blackbody at that temperature, then the evolution works out smoothly towards the blackbody distribution.

On the other hand, if we insert *more* photons than those which would be contained in the Planckian the numerical solution goes nuts.

3.1.2 The physical effects of Comptonization

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As we have said it is hard to find numerical solutions to the Kompaneets equation, however there is an interesting quantity whose evolution we can study analytically: the photon energy density. The energy density can be calculated as

$$u = \int \mathrm{d}^3 p\, h\nu \frac{\mathrm{d}N}{\mathrm{d}\,\mathrm{d}^3 x} = \int \mathrm{d}^3 p\, h\nu \frac{2n}{h^3} = \frac{8\pi (k_B T)^4}{h^3 c^3} \int_0^\infty \mathrm{d}x\, x^3 n\,, \tag{3.1.38}$$

where, as usual, $x = h\nu/k_B T$, and the last equality comes about by substituting this and $\mathrm{d}^3 p = p^2\, \mathrm{d}p\, \mathrm{d}\Omega$ and assuming isotropy.

In order to see how it changes, let us differentiate this energy density with respect to time (rescaled, as in the Kompaneets equation, by the mean time between two scatterings):

$$\frac{\mathrm{d}u}{\mathrm{d}t_s} = \frac{8\pi (k_B T)^4}{h^3 c^3} \int_0^\infty \mathrm{d}x\, x^3 \frac{\partial n}{\partial t_s} = \frac{8\pi (k_B T)^4}{h^3 c^3} \int_0^\infty \mathrm{d}x\, x^3 \frac{\Theta}{x^2} \frac{\partial}{\partial x} \left[x^4 (n' + n + n^2) \right]. \tag{3.1.39}$$

Used the Kompaneets equation.

Now, let us suppose that n is very small, so that we can neglect n and n^2 inside the derivative, keeping n'

Not super clear why $n \ll n'$ if n is small should hold...

: this yields

$$\frac{\mathrm{d}u}{\mathrm{d}t_s} \approx \frac{8\pi(k_B T)^4}{h^3 c^3}\Theta \int_0^\infty \mathrm{d}x\, x \frac{\partial[x^4 n']}{\partial x}\,. \tag{3.1.40}$$

We integrate by parts twice and assume that $x^k n' \to 0$ for $k > 3$: this give us

$$\frac{\mathrm{d}u}{\mathrm{d}t_s} \approx \frac{8\pi(k_B T)^4}{h^3 c^3} 4\Theta \int_0^\infty \mathrm{d}x\, x^3 n = 4\Theta u\,. \tag{3.1.41}$$

The solution to this is an exponential:

$$u(t_s) = u(0)\exp(4\Theta t_s) = u(0)\exp(t/t_c)\,, \tag{3.1.42}$$

where $t_c = 1/(4\Theta n \sigma_T c)$ is called the Compton time. This means that the radiative energy density increases **exponentially fast**, over a characteristic time of the order t_c.

Wait, is this exponential OK? it diverges as time increases!

3.1.3 Relativistic Kompaneets equation

We made the hypotheses that $\Theta \ll 1$ and $\epsilon \ll 1$ at the start: the electrons and photons were nonrelativistic. If we keep all the other hypotheses (isotropy, homogeneity, small fractional energy change and so on) can we generalize the Kompaneets equation to relativistic particles?

We will need to use a relativistic Maxwellian (Jüttner distribution) instead of a Maxwellian, and the Klein-Nishina cross section instead of the Thomson one. This will allow us to deal with generic Θ and ϵ: the result is

$$\frac{\partial n}{\partial t} = \frac{1}{\epsilon^2}\frac{\partial}{\partial \epsilon}\left[\alpha(\epsilon,\Theta)\left(\Theta\frac{\partial n}{\partial \epsilon} + n + n^2\right)\right], \tag{3.1.43}$$

where α is a certain known function,

$$\alpha(\epsilon,\Theta) = \frac{\epsilon^4}{2K_2(1/\Theta)}\int_0^\infty \mathrm{d}z\, z^2 \alpha_0(\epsilon z)\exp\left(-\frac{z+1/z}{2\Theta}\right), \tag{3.1.44}$$

where α_0 is a known analytical function, whose expression is omitted here.

The form of the relativistic Kompaneets equation is very similar to the nonrelativistic one.

The shape of α is quite similar to ϵ^4, its nonrelativistic counterpart, for low photon energies; while for high photon energies α is lower than ϵ^4.

The take-away, here, is that if we were to need the relativistic form of the equation we can find it in the literature and apply it to our problem.

3.1.4 The emerging spectrum

In astrophysical (non-cosmological) settings we will not find infinite homogeneous media; instead, we need to treat clouds of finite size. However, we cannot simply use the Kompaneets equation for them.

What is the spectrum we expect to observe at infinity? Let us suppose we have a source inside a spherical cloud of radius R; this source will emit a pulse of radiation at a time t, so that the density at a time t_0 is

$$n(\nu, t_0) = n_0(\nu)\delta(t_0 - t)\,. \tag{3.1.45}$$

Photons will propagate and scatter in the cloud, until finally they are emitted and some of them can be observed. What will be the spectral distribution of these?

Roughly speaking, we can express the observed number density as

$$n_{\rm obs}(\epsilon) = \sum_i p_i n(\epsilon, t_i)\,, \tag{3.1.46}$$

where t_i are different times after t_0, $n(\epsilon, t_i)$ are solutions of the Kompaneets equation at these times, and p_i are the probabilities that a photon emitted at t will exit the cloud at t_i. Properly speaking this should be an integral, but we are just giving the rough idea. We will turn it into an integral later.

The escape probability will satisfy

$$p(t)\,{\rm d}t = \frac{{\rm d}n}{n}\,, \tag{3.1.47}$$

so that it describes the fraction of photons (${\rm d}n\,/n$) which escape the medium in a time ${\rm d}t$. The observed intensity will then be

$$n_{\rm obs}(\epsilon) = \int_0^\infty p(\tau) n(\epsilon, \tau)\,{\rm d}\tau\ . \tag{3.1.48}$$

The typical timescale for a photon to leave the cloud will be given by the mean number of scatterings times the typical time per scattering: if the medium is assumed to be optically thick we get

$$t_{\rm esc} \sim N_s t_s \sim \max \frac{\left(\tau, \tau^2\right)}{n\sigma_T c} \sim \frac{\tau^2 R}{nR\sigma_T c} \sim \frac{\tau R}{c}\,, \tag{3.1.49}$$

since $nR\sigma_T = \tau$. This result makes sense: it is the light travel time R/c times the optical depth.

The form of $p(t)$ depends on the shape of the cloud; for example in our spherical cloud we have

$$p(t) = \frac{1}{\tau}\sum_{n=1}^{\infty} \lambda_n \sin\lambda_n \exp\left(-\frac{\lambda_n^2}{3}\frac{t}{t_{\rm esc}}\right), \tag{3.1.50}$$

where the coefficients λ_n are the solutions of $\tan\lambda_n = \lambda_n/(1 - 3\tau/2)$.

It is important to apply this procedure instead of blindly plugging the distribution inside the Kompaneets equation: we know that in that case the photons will eventually thermalize, which is not of physical interest in most situations.

Unsaturated Comptonization and power-law tails

Let us consider a phenomenologically-motivated modified Kompaneets equation, which accounts for the injection and escape of photons in a finite medium:

$$\dot{H} - \dot{S} = \frac{\Theta}{x^2}\frac{\partial[x^4(n' + n + n^2)]}{\partial x}\,, \tag{3.1.51}$$

where $\dot{S} = Q(x)$ quantifies the photons gained per unit time, while $\dot{H} = n/\max(\tau, \tau^2)$ quantifies the photons lost per unit time.

Are the signs of the $\dot{S}$ and $\dot{H}$ not the other way around?

Suppose that the input is *soft*, that is, low-energy. This means that we will have $Q(x)$ different from zero only for x smaller than a certain threshold, $x < x_s \ll 1$. We then look at the high-energy tail in emissions, $x \gg 1$.

We will neglect the n^2 term. The equation will then read

$$0 = \frac{\Theta}{x^2}\frac{\partial[x^4(n' + n)]}{\partial x} + \underbrace{Q(x) - \frac{n}{\max(\tau, \tau^2)}}_{\text{vanish in the high } x \text{ regime}}\,, \tag{3.1.52}$$

so we are left with $n' + n = \text{const}$, meaning that $n \sim \exp(-x)$. This means that at very high energies the decay of the emission is exponential.

Why does the term proportional to n vanish?

Now, instead, we look for mid-energy solutions, in the range between the threshold for the emission x_s and 1. We will suppose that $x_s \ll x \ll 1$. We will then neglect the n and n^2 terms, meaning that we can ignore $Q(x)$ but not the escape term, so we have

$$0 = \frac{\Theta}{x^2}\frac{\partial[x^4(n')]}{\partial x} - \frac{n}{\max(\tau, \tau^2)}\,. \tag{3.1.53}$$

If we take a powerlaw ansatz, $n \propto x^m$, we find that the equation for m becomes

$$m(m+3) - \frac{4}{4\Theta\max(\tau, \tau^2)} = 0\,, \tag{3.1.54}$$

where we recognize the Compton parameter $4\Theta\max(\tau, \tau^2) = y$, which quantifies how much the scatterings affect the spectrum. The second-order equation for m then becomes

$$m = \frac{-3 \pm \sqrt{9 + 16/y^2}}{2}\,. \tag{3.1.55}$$

If we take the positive solution it will be $m_+ > 0$, so the spectrum will increase at high energies, which is unphysical. So, we consider the negative solution: expanding the square root under the assumption that $y < 1$[2] we get

$$m_- \approx -3 - \frac{2}{y}\,. \tag{3.1.56}$$

[2] As a and b get further from one another in magnitude, the approximation $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ gets closer to being true. For small y we have quite a good result, while for y closer to 1 the exponent is slowly-varying anyway.

This is then a powerlaw spectrum: for any soft photon input, then, we have found a qualitative result. It is a powerlaw for medium photon energies and an exponential for high photon energies. This then gives a typical signature for the observational spectrum we observe in Compton scattering.

Dynamical Comptonization

We have so far discussed the effects of repeated scattering of photons onto electrons which are moving due to thermal motion. This is however not the only reason electrons may move!

An interesting situation is one in which the temperature T of the electrons is low, but the bulk velocity of the plasma $\vec{v}$ is high, so that interaction with photons is still capable of causing inverse Compton scattering.

Suppose we have a flow of electrons moving with a uniform $\vec{v}$, constant for all of them. Then, the Electron Rest Frame is the same for all the electrons.

So, in order to deal with the first scattering we must boost to the ERF, but then we are done: we can stay in that frame for all subsequent scatterings. The energy the photon will emerge with will be $\epsilon_2 \approx \gamma^2 \epsilon_1$ regardless of the number of scatterings.

If, on the other hand, the velocity is not uniform we need to insert a relative γ factor for each new electron. This effect will build up for each scattering. So, dynamical Comptonization requires a nonvanishing velocity gradient to work.

He writes $\nabla \cdot \vec{v} \neq 0$, which is not the same... why would the *divergence* not vanishing specifically be the condition? what about the curl, or the other components of $\partial_i v_j$ anyway?

This is similar to *second-order Fermi acceleration*.

what is that?

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An interesting case in which we have dynamical Comptonization is that of spherical infall of material, for example into a black hole: at least close to the event horizon, the particles will be in free fall.

In this case we recover a powerlaw high-energy tail for soft photon input.

3.1.5 Consequences of comptonization

Comptonization in X-ray binaries

Comptonization plays a role in X-ray binaries, which are binary systems in which one of the objects is a compact object (black hole or neutron star).

Because of Roche lobe overflow, matter forms a gas stream flowing from the donor star to the compact object; there it forms an accretion disk and sometimes a jet.

There are Low Mass and High Mass X-ray binaries, the latter have a very complex phenomenology and mostly emit in the X-rays. The spectrum can change dramatically over time, but the typical ingredient is the accretion disk.

The simplest model for this is a Shakura-Sunyaev disk (SSD), which is (geometrically) thin: its height is much smaller than its width. The distribution of the angular velocities of

matter in such a disk will be Keplerian: each element of gas will describe a circular orbit. Also, these disks are optically thick in the direction normal to the disk.

This has the consequence that each annulus will need to be in local thermal equilibrium, and it will need to radiate like a blackbody at the local surface temperature,

$$T(R) = \left[\frac{3GM\dot{M}}{8\pi\sigma R_{\text{in}}^3}\left(\frac{R_{\text{in}}}{T}\right)^3\left(1-\sqrt{\frac{R_{\text{in}}}{R}}\right)\right]^{1/4}, \tag{3.1.57}$$

where T is the temperature, R is the radius of the annulus from the center, R_{in} is the inner radius of the accretion disk, M is the mass of the compact object, $\dot{M}$ is its accretion rate, while σ is the Stefan-Boltzmann constant.

The reference temperature multiplying the dependence on the radius is

$$T_{\text{in}} = \left[\frac{3GM\dot{M}}{8\pi\sigma R_{\text{in}}^3}\right]^{1/4} \approx 1\,\text{keV}\,, \tag{3.1.58}$$

so we can see that the main emission will be in the X-rays.

The total emission we see will be a superposition of the emission curves of all the annuli, which yields three regions: at the low- and high-energy boundaries the emission looks like that of a blackbody, going like ν^2 and $\exp(-h\nu/k_BT)$ respectively, but in the intermediate region there is a peculiar dependence on $\nu^{1/3}$ of the frequency spectrum.

If we look at the X-ray spectrum of a binary like Cyg X-1, we can reproduce it by superimposing a blackbody spectrum with a powerlaw tail at high energies.

In order to understand this we must consider a slightly more complicated model: beyond the disc we need to have a region called the **corona**, which is less dense than the disk itself. It will contain energetic electrons — we are not sure whether they are thermal — which are able to up-scatter the radiation, so they must be hotter than the disc electrons. Then, we will have unsaturated comptonization, which we know to produce a power-law tail.

3.2 Synchrotron and cyclotron radiation

We will describe the radiation emitted by relativistic particles which are accelerated by magnetic fields.

First, we need to generalize some results we found in the dipole approximation for nonrelativistic particles.

The four-momentum is defined as $p^\mu = mu^\mu$, where $u^\mu = \gamma(c, \vec{v})$ and $\gamma = 1/\sqrt{1-v^2/c^2}$.

The four-acceleration is

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}\,, \tag{3.2.1}$$

where τ is the proper time. The equation of motion of a particle is

$$ma^\mu = F^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}\,. \tag{3.2.2}$$

This defines the four-force. If the force is electromagnetic, it can be expressed from the Faraday tensor $F_{\mu\nu}$. The force three-vector is

$$\vec{F} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right), \tag{3.2.3}$$

which can be generalized to

$$F^\mu = \frac{e}{c} F^\mu_\nu u^\nu, \tag{3.2.4}$$

a linear multiple of the four-velocity and the Faraday tensor.

The equation of motion will then read $ma^\mu = (e/c)F^\mu_\nu u^\nu$, whose time component will be

$$\frac{\mathrm{d}p^0}{\mathrm{d}t} = \frac{e}{c} F^0_\nu u^\nu = \frac{e}{c} \vec{E} \cdot \vec{v}, \tag{3.2.5}$$

where we can recall that p^0 is the energy of the particle (divided by c), γmc, so this is telling us that the work on the particle is equal to $e\vec{E} \cdot \vec{v}$.

On the other hand, the space components of the equation are

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = e\left(E^i + \frac{\vec{v}}{c} \times \vec{B}\right). \tag{3.2.6}$$

They give us the variation of the relativistic momentum, $p^i = \gamma m v^i$.

We need to generalize the Larmor formula to relativistic particles. Let us start by defining the instantaneous rest frame of a particle: it is the (non-inertial) frame in which the particle is always stationary. If we are in this frame at a fixed time we can use the nonrelativistic formula locally, since the particle will have nonrelativistic velocities at least for a small while around this point.

The emission from the nonrelativistic Larmor formula depends on $\sin^2\Theta$, so it is not isotropic; however it is (parity) symmetric, so the change of momentum due to the emission of radiation is zero.

He says that the Larmor formula has no angular dependence but this is false!

So, in the (primed) rest frame the momentum emission will be zero, $\mathrm{d}p' = 0$. This means that we can transform to the lab frame to get the energy change as $\mathrm{d}w = \gamma\,\mathrm{d}w'$, where w denotes the energy. Also, the time interval will change like $\mathrm{d}t = \gamma\,\mathrm{d}t'$.

Therefore, the emitted power will be

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\gamma\,\mathrm{d}w'}{\gamma\,\mathrm{d}t'} = \frac{\mathrm{d}w'}{\mathrm{d}t'}, \tag{3.2.7}$$

so the emitted power is the same in both frames, it is a Lorentz invariant (at least as long as emission has forward-backward symmetry).

In the rest frame the emitted power is

$$\frac{\mathrm{d}w'}{\mathrm{d}t'} = \frac{2q^2}{3c^3}\left|\vec{a}'\right|^2, \tag{3.2.8}$$

where $\vec{a}'$ is the 3-velocity in the LRF. In frame the acceleration is nonzero, however the velocity is zero: this means that $\gamma = 1$, and therefore $a_0 \sim \mathrm{d}\gamma/\mathrm{d}t = 0$. So,

$$\left|a'^{\mu}a'_{\mu}\right| = \left|\vec{a}'\right|. \tag{3.2.9}$$

We have then shown that we can calculate the power using the Lorentz-invariant quantity $a^{\mu}a_{\mu}$ in any frame. We will distinguish between the components of the acceleration which are parallel to the velocity, $a_{\parallel}$, and the perpendicular ones, $a_{\perp}$. Under the boost, they transform like

$$a'_{\parallel} = \gamma^3 a_{\parallel} \qquad \text{and} \qquad a'_{\perp} = \gamma^2 a_{\perp}. \tag{3.2.10}$$

Therefore, the power can be calculated using the components of the 3-acceleration in the lab frame as

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{2q^2}{3c^3}\left(a'^2_{\parallel} + a'^2_{\perp}\right) = \frac{2q^2\gamma^4}{3c^3}\left(a^2_{\perp} + \gamma^2 a^2_{\parallel}\right). \tag{3.2.11}$$

One might wonder: since $a_{\mu}a^{\mu}$ is an invariant, why can't we just use the acceleration in the lab frame directly without all the gammas? If we wanted to, we'd also need the a^0 component which is nontrivial in the lab frame. What we did is a way to avoid having to compute it.

Angular distribution of emitted and received energy and power

Friday 2020-8-28, compiled 2020-09-06

We want to discuss how a relativistic particle emits radiation in the lab frame. We start off in the rest frame of the particle: there, it emits an energy $\mathrm{d}w'$ across a solid angle $\mathrm{d}\Omega' = \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi' = \mathrm{d}\mu'\,\mathrm{d}\phi'$; in the lab frame we can say the same thing without the primes. We choose our reference frame for both lab and rest frame so that the particle is moving along the $\hat{x}$ axis: $\vec{v} = (v, 0, 0)$, while our observation direction is positioned at an angle θ from the x axis.

The energies $\mathrm{d}w$ and $\mathrm{d}w'$ are related by a Lorentz transformation, which in general reads

$$\mathrm{d}w = \gamma\,\mathrm{d}w' + \gamma v\,\mathrm{d}p'_x = \gamma\left(\mathrm{d}w' + \frac{v}{c}\cos\theta'\,\mathrm{d}w'\right) = \gamma\,\mathrm{d}w'\left(1 + \beta\mu'\right). \tag{3.2.12}$$

In the second equality we have used the fact that the radiation four-momentum satisfies $p^{\mu}p_{\mu} = 0$ since photons are massless, and therefore $(\mathrm{d}w', \mathrm{d}\vec{p}')$ must be a null vector, meaning that $\mathrm{d}w' = \mu'\,\mathrm{d}w'/c$.

We want to express this in terms of μ, the cosine of the angle in the lab frame. A geometric result we can derive is the **angular aberration formula**,

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'}, \tag{3.2.13}$$

which can be differentiated to yield

$$\mathrm{d}\mu = \frac{\mathrm{d}\mu'}{\gamma^2\left(1 + \beta\mu'\right)^2}. \tag{3.2.14}$$

The Lorentz boost does not affect the azimuthal direction since it is orthogonal to it, so $\mathrm{d}\phi = \mathrm{d}\phi'$. Therefore, we can transform the solid angle differential as

$$\mathrm{d}\Omega = \mathrm{d}\mu\,\mathrm{d}\phi = \frac{\mathrm{d}\mu'\,\mathrm{d}\phi'}{\gamma^2(1+\beta\mu')^2} = \frac{\mathrm{d}\Omega'}{\gamma^2(1+\beta\mu')^2}\,. \tag{3.2.15}$$

The angular distribution of energy in the lab frame then can be expressed as

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega} = \frac{\gamma(1+\beta\mu')\,\mathrm{d}w'}{\mathrm{d}\Omega'}\gamma^2(1+\beta\mu')^2 = \gamma^3(1+\beta\mu')^3\frac{\mathrm{d}w'}{\mathrm{d}\Omega'}\,. \tag{3.2.16}$$

This is the distribution of the *energy* emission: in order to find the *power* emission we need to divide by time. However, we need to choose a time by which to divide: do we choose the one in the lab or rest frame? they are related by

$$\mathrm{d}t = \gamma(1-\beta\mu)\,\mathrm{d}t'\ , \tag{3.2.17}$$

and it can be shown that the power emission in the two frames is related by

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{\gamma^3(1+\beta\mu')^3}{\gamma(1-\beta\mu)\,\mathrm{d}t'}\frac{\mathrm{d}w'}{\mathrm{d}\Omega'} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{\mathrm{d}w'}{\mathrm{d}\Omega'\,\mathrm{d}t'}\,. \tag{3.2.18}$$

Now, if radiation emission is isotropic in the particle frame then $\mathrm{d}w'\,/\,\mathrm{d}\Omega'\,\mathrm{d}t'$ is independent of the angle. Under this assumption, the angular distribution of the received radiation is given by the prefactor.

Let us analyze this for an ultra-relativistic particle, with $\gamma \gg 1$. Then, we can approximate

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}\,. \tag{3.2.19}$$

Inserting this into the prefactor yields

$$\frac{1}{\gamma^4(1-\beta\mu)^4} \approx \frac{1}{\gamma^4\left[1-\left(1-\frac{1}{2\gamma^2}\right)\mu\right]^4}\,, \tag{3.2.20}$$

which in general will be small because of the γ^{-4} suppression. However, if $\mu \sim 1$ then the quantity in square brackets is close to zero. We can expand $\mu \sim 1-\theta^2/2$ since it is the cosine of θ: with this we can manipulate the expression into

$$\frac{1}{\gamma^4(1-\beta\mu)^4} \sim \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^4\,, \tag{3.2.21}$$

for small θ at least. This is sharply peaked around $\theta = 0$, with the FWHM of the peak of the order of $1/\gamma$. This effect is **relativistic beaming**.

So, how does this affect the emission from an accelerated particle? in its rest frame the emission follows Larmor's formula:

$$\frac{\mathrm{d}w'}{\mathrm{d}t'\,\mathrm{d}\Omega'} = \frac{q^2a'^2}{4\pi c^3}\sin^2\Theta'\,, \tag{3.2.22}$$

where Θ' is the angle between the direction of emission and the direction of acceleration (in the rest frame).

In the rest frame of the particle the acceleration $\vec{a}'$ is orthogonal to the velocity $\vec{v}$, since $a'^{\mu}u_{\mu}$ vanishes identically and a'^{0} is zero, therefore $a' \cdot \vec{v} = 0$.

The two polar angles θ' and ϕ' are chosen so that they define $\vec{k}$, the direction of emission, while θ'_a and ϕ'_a define the direction of the acceleration, both with respect to the direction of motion $\vec{v}$ which is chosen as the polar axis.

It can then be shown by fiddling with the vectors that

$$\cos\Theta = \cos\theta' \cos\theta'_a + \sin\theta' \sin\theta'_a \cos\left(\phi' - \phi'_a\right), \tag{3.2.23}$$

however since a' is perpendicular to $\vec{v}$ we have:

He says that a' being perpendicular to $\vec{v}$ is a special case, but it can be proven by a general kinematic argument: is it not always true?

$$\cos\Theta = \sin\theta' \cos\phi' \implies \sin^2\Theta = 1 - \cos^2\Theta = 1 - \sin^2\theta' \cos^2\phi', \tag{3.2.24}$$

and we can transform these into the lab frame as usual:

$$\sin^2\theta' = 1 - \mu'^2 = \frac{1-\mu^2}{\gamma^2(1-\beta\mu)^2} = \frac{\sin^2\theta}{\gamma^2(1-\beta\mu)^2} \tag{3.2.25}$$

$$\cos^2\phi' = \cos^2\phi\,. \tag{3.2.26}$$

Finally, we can substitute into the Larmor formula in order to find the angular distribution of the power which is seen by an external observer:

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{\mathrm{d}w'}{\mathrm{d}t'\,\mathrm{d}\Omega'} \tag{3.2.27}$$

$$= \frac{q^2 a'^2_{\perp}}{4\pi c^3}\frac{1}{\gamma^4(1-\beta\mu)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right] \tag{3.2.28}$$

$$= \frac{q^2 a^2_{\perp}}{4\pi c^3}\frac{1}{(1-\beta\mu)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}\right], \tag{3.2.29}$$

note that we write $a'^2_{\perp}$ just to specify that in that frame the acceleration is indeed perpendicular to the velocity.

3.2.1 Synchrotron radiation

This is the radiation emitted by a particle under the effect of a uniform magnetic field (with no electric field). Recall that the equation of motion will be

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}[m\gamma\vec{v}]}{\mathrm{d}t} = \frac{q}{c}\vec{v}\cdot\vec{B}\,, \tag{3.2.30}$$

while the time component of the equation tells us that

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}[mc^2\gamma]}{\mathrm{d}t} = q\vec{v}\cdot\vec{E} = 0\,. \tag{3.2.31}$$

This has an important consequence: γ is a constant, so the energy of the particle is conserved, the velocity of the particle has a constant modulus. Therefore, we can rewrite the equations of motion as

$$m\gamma\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{c}\vec{v}\times\vec{B}\,. \tag{3.2.32}$$

Let us decompose the velocity $\vec{v}$ into two vectors, one orthogonal and one parallel to $\vec{B}$: $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$.

Now, since the Lorentz force is always perpendicular to $\vec{B}$ it cannot affect $\vec{v}_{\parallel}$, which will then be a constant, while the perpendicular component will evolve as

$$\frac{\mathrm{d}\vec{v}_{\perp}}{\mathrm{d}t} = \frac{q}{mc\gamma}\vec{v}_{\perp}\times\vec{B}\,. \tag{3.2.33}$$

This means that the motion *along the field* is uniform, while the motion *orthogonal to the field* is circular with constant angular velocity: the acceleration corresponding to $\vec{v}_{\perp}$, $\vec{a}_{\perp}$, is orthogonal to $\vec{v}_{\perp}$ as well as to $\vec{B}$. The radius of the circle will then be defined by the equation

$$a_{\perp} = \frac{q}{mc\gamma}v_{\perp}B = \frac{v_{\perp}^2}{R}\,, \tag{3.2.34}$$

and we can introduce the angular velocity as $v_{\perp} = \omega_B R$: this is

$$\omega_B = \frac{v_{\perp}}{R} = \frac{qB}{m\gamma c}\,. \tag{3.2.35}$$

From this we can calculate the radiated power from the relativistic Larmor formula:

$$P = \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{2q^2}{3c^3}\gamma^4 a_{\perp}^2 = \frac{2q^2}{3c^3}\gamma^4\frac{q^2B^2}{m^2c^2\gamma^2}v_{\perp}^2 \tag{3.2.36}$$

$$= \frac{2}{3}\frac{r_0^2}{c}\gamma^2B^2v_{\perp}^2 = \frac{2}{3}r_0^2c\gamma^2B^2\beta_{\perp}^2\,, \tag{3.2.37}$$

where r_0 is the classical particle radius.

3.2.2 Spectrum of synchrotron radiation

Saturday 2020-8-29, compiled 2020-09-06

We will use a rough, heuristic approach. Recall that the spectrum depends on

$$\frac{\mathrm{d}w}{\mathrm{d}\omega\,\mathrm{d}\Omega} \propto \left|\hat{E}(\omega)\right|^2\,, \tag{3.2.38}$$

where $\hat{E}(\omega)$ is the Fourier transform of the electric field $E(t)$.

When we compute the Fourier transform we do so referring to the time as seen by the observer; we will see that the observer in any specific direction will only see the radiation for a very short amount of time each revolution because of relativistic beaming. In order to see this, consider a single charge moving along a helical path.

Let us choose two points 1 and 2 along the trajectory, such that the path length between them is Δs and the angle difference is $\Delta\theta$. Let us also denote the radius of curvature of the trajectory as a: then we will have $\Delta s = a\Delta\theta$.

We know that radiation is emitted in a cone with aperture $\sim 1/\gamma$: let us choose the two points so that they are at the edges of where the radiation starts and then stops being received by the observer. Then, radiation emitted between the two points will always hit the observer.

We know that in this case $\Delta\theta = 2/\gamma$ by the geometry of the setup, since $2/\gamma$ is the angle between two diametrically opposed points in the cone. The equation of motion is

$$m\gamma\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{c}\vec{v}\times\vec{B}\,, \tag{3.2.39}$$

so the modulus of the velocity change, for small enough time differences, is

$$\frac{|\Delta\vec{v}|}{\Delta t} = \frac{q}{mc\gamma}vB\sin\alpha\,, \tag{3.2.40}$$

where α is the angle between the velocity and $\vec{B}$, which is called the *pitch angle*. Since the modulus of $\vec{v}$ is constant, we can write $|\Delta\vec{v}| = v\Delta\theta$, so that

$$v\frac{\Delta\theta}{\Delta t} = \frac{q}{mc\gamma}vB\sin\alpha\,, \tag{3.2.41}$$

but since the path travelled is given by $\Delta s = v\Delta t$ we have

$$v^2\frac{\Delta\theta}{\Delta s} = \frac{q}{mc\gamma}vB\sin\alpha \tag{3.2.42}$$

$$\frac{\Delta\theta}{\Delta s} = \frac{q}{mcv\gamma}B\sin\alpha \tag{3.2.43}$$

$$\frac{2/\gamma}{\Delta s} = \frac{q}{mcv\gamma}B\sin\alpha \tag{3.2.44}$$

$$\Delta s = \frac{2mcv}{qB\sin\alpha}\,, \tag{3.2.45}$$

which is related to the frequency $\omega_B = qB/mc\gamma$ by

$$\Delta s = \frac{2b}{\gamma\omega_B\sin\alpha}\,. \tag{3.2.46}$$

Then, we can derive the time needed to go from point 1 to point 2:

$$\Delta t = \frac{\Delta s}{v} = \frac{2}{\gamma\omega_B\sin\alpha}\,, \tag{3.2.47}$$

however we must be careful: although this is the time as measured in the lab frame, the motion of the particle is highly relativistic and the radiation is emitted in two different places.

Let us call t_1^A and t_2^A the times at which the radiation corresponding to the start and the end of the pulse arrive at the observer. They will be given by

$$t_1^A = t_1 + \frac{L}{c} + \frac{\Delta s}{c} \qquad \text{and} \qquad t_2^A = t_2 + \frac{L}{c}\,. \tag{3.2.48}$$

This means that

$$t_1^A - t_2^A = t_2 - t_1 - \frac{\Delta s}{c} = \frac{2}{\gamma\omega_B \sin\alpha} - 2\frac{v/c}{\gamma\omega_B \sin\alpha} = \frac{2}{\gamma\omega_B \sin\alpha}\left(1 - \frac{v}{c}\right), \tag{3.2.49}$$

but if the motion is highly relativistic we can approximate

$$1 - \frac{v}{c} \approx 1 - \left(1 - \frac{1}{2\gamma^2}\right) = \frac{1}{2\gamma^2}\,, \tag{3.2.50}$$

therefore

$$\Delta t^A = \frac{2}{\gamma\omega_B \sin\alpha}\frac{1}{2\gamma^2} = \frac{1}{\gamma^3\omega_B \sin\alpha}\,, \tag{3.2.51}$$

so the reduction of the pulse duration is shorter by a factor $\sim 1/\gamma^3$ than $1/\omega_B$: it becomes **very short!**

Therefore, the spectrum will be rather broad in frequency, of the order $\sim \gamma^3\omega_B$.

Let us define the **critical pulsation** and the **critical frequency**:

$$\omega_c = \frac{3}{2}\gamma^3\omega_B \sin\alpha \qquad \text{and} \qquad \nu_c = \frac{3}{4\pi}\gamma^3\omega_B \sin\alpha\,, \tag{3.2.52}$$

which will roughly be around the cutoff of the spectrum.

We can see by looking at the old formulas we derived that the dependence of the emitted power per unit solid angle on the angle is always in the form

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\Omega} = F(\gamma\theta)\,, \tag{3.2.53}$$

which is a consequence of relativistic beaming. Therefore, this will also hold for the dependence of the electric field.

Now, we want to find out how θ is related to t: we know that $\theta = s/a$ while $t = (s/v)(1 - v/c)$; so

$$t = \frac{a\theta}{v}\left(1 - \frac{v}{c}\right) \tag{3.2.54}$$

$$\gamma t = \frac{\gamma a\theta}{v}\left(1 - \frac{v}{c}\right) \tag{3.2.55}$$

$$\gamma\theta = \gamma t\frac{v}{a}\left(1 - \frac{v}{c}\right)^{-1} \approx \gamma t\omega_B \sin\alpha 2\gamma^2 \propto \omega_c t\,. \tag{3.2.56}$$

Therefore, the electric field depends on time like $E(t) \propto g(\omega_c t)$ for some function g. Its Fourier transform will then look like

$$\hat{E}(\omega) \propto \int_{-\infty}^{\infty} g(\omega_c t) e^{-i\omega t}\, \mathrm{d}t \tag{3.2.57}$$

$$\propto \int_{-\infty}^{\infty} g(z) e^{-i\frac{\omega}{\omega_c} z}\, \mathrm{d}z\,, \tag{3.2.58}$$

meaning that *the dependence of $\hat{E}(\omega)$ on ω is only through the ratio ω/ω_c.*

Recall that the power per unit solid angle is given by

$$\frac{\mathrm{d}w}{\mathrm{d}\omega\, \mathrm{d}\Omega} \propto \left|\hat{E}(\omega)\right|^2, \tag{3.2.59}$$

so if we integrate over the solid angle and divide by the period we find that the power per unit frequency is

$$\frac{\mathrm{d}w}{\mathrm{d}t\, \mathrm{d}\omega} = c_1 F\left(\frac{\omega}{\omega_c}\right), \tag{3.2.60}$$

where c_1 is some constant and F is some function. We can evaluate c_1 since we know what is the total radiated power:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = P = \frac{2}{3} r_0^2 c \gamma^2 \beta_\perp^2 B^2 = \frac{2}{3} \frac{q^4 \beta^2 \sin^2\alpha B^2 \gamma^2}{m^2 c^3}\,, \tag{3.2.61}$$

where we used the fact that $\beta_\perp^2 = \beta^2 \sin^2\alpha$ and $r_0^2 = q^4/m^2c^4$. In terms of the constant c_1 and the function F this will be the integral over the frequency space:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = c_1 \int_0^\infty F(\frac{\omega}{\omega_c})\, \mathrm{d}\omega = c_1 \omega_c \int_0^\infty F(z)\, \mathrm{d}z \overset{!}{=} \frac{2}{3} \frac{q^4 \beta^2 \sin^2\alpha B^2 \gamma^2}{m^2 c^3}\,, \tag{3.2.62}$$

so we can write the value of the constant c_1 as

$$c_1 = \frac{2}{3} \frac{q^4 \beta^2 \sin^2\alpha B^2 \gamma^2}{m^2 c^3} \frac{2}{3\gamma^3 \omega_B \sin\alpha} \frac{1}{\int_0^\infty F(z)\, \mathrm{d}z} \tag{3.2.63}$$

$$= \frac{q^3 \beta^2 \sin\alpha B}{mc^2} \frac{1}{\int_0^\infty F(z)\, \mathrm{d}z}\,. \tag{3.2.64}$$

As we shall show later the integral will evaluate to $\int_0^\infty F(z)\, \mathrm{d}z = 2\pi/\sqrt{3}$, so, as long as the particle is ultrarelativistic ($\beta \approx 1$) we will have

$$c_1 = \frac{q^3 \sin\alpha B}{mc^2} \frac{\sqrt{3}}{2\pi}\,, \tag{3.2.65}$$

therefore

$$\frac{\mathrm{d}w}{\mathrm{d}t\, \mathrm{d}\omega} = \frac{\sqrt{3}}{2\pi} \frac{q^3 B \sin\alpha}{mc^2} F\left(\frac{\omega}{\omega_c}\right). \tag{3.2.66}$$

The explicit expression of F can be derived [RL79, par. 6.4], but we will not do it here. It is given by the modified Bessel function

$$F(z) = z \int_z^\infty k_{5/3}(x)\,\mathrm{d}x\ . \tag{3.2.67}$$

It is a smooth function of z, going to zero at 0 and $+\infty$, and it has its peak around 0.29, where it reaches a value close to 1 (this means that the peak of synchrotron radiation is around $\omega \sim 0.29\omega_c$). Its decay is then exponential. Asymptotically, we have that for $z \ll 1$

$$F(z) \sim \frac{4}{\sqrt{3}\Gamma(1/3)}\left(\frac{z}{2}\right)^{1/3}, \tag{3.2.68}$$

while for $z \gg 1$:

$$F(z) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-z} z^{1/2}\,. \tag{3.2.69}$$

Synchrotron emission from a nonthermal electron distribution

In many astrphyiscal applications we need to consider the emission from electrons whose energies do not follow a thermal distribution but instead something like a powerlaw; the energy losses and gains caused by synchrotron emission actually cause the spectrum to get closer to a powerlaw, which gives further justification for considering this situation.

So, we will consider a population such that the number density of electrons per unit energy is given by

$$\frac{\mathrm{d}N}{\mathrm{d}E} = CE^{-P}\,. \tag{3.2.70}$$

Since the energy is given by $E = mc^2\gamma$, we can also write

$$\mathrm{d}N = C\gamma^{-P}\,\mathrm{d}\gamma\ . \tag{3.2.71}$$

Then, the emission per unit volume, frequency and time will be

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}V\,\mathrm{d}\omega} = \int_{\gamma_1}^{\gamma_2} \frac{\mathrm{d}N}{\mathrm{d}\gamma}\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega}\,\mathrm{d}\gamma \tag{3.2.72}$$

$$= \int_{\gamma_1}^{\gamma_2} c\gamma^{-P} c_1 F\left(\frac{\omega}{\omega_c}\right)\mathrm{d}\gamma\ . \tag{3.2.73}$$

Remember that ω_c depends on γ, as $\omega_c \propto \gamma^2$, so we cannot bring it out of the integral. We can write this integral, keeping only the variable parts, as

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}V\,\mathrm{d}\omega} \propto \int_{\gamma_1}^{\gamma_2} \gamma^{-P} F\left(\frac{\omega}{\omega_c}\right)\mathrm{d}\gamma \tag{3.2.74}$$

$$\propto \int_{x_1}^{x_2} \omega^{-P/2} x^{P/2} \omega^{1/2}(x)^{3/2} F(x)\,\mathrm{d}x \tag{3.2.75}$$

$$= \int_{x_1}^{x_2} \omega^{-(P-1)/2} x^{(P-3)/2} F(x)\,\mathrm{d}x \tag{3.2.76}$$

$$= \omega^{-(P-1)/2} \int_{x_1}^{x_2} x^{(P-3)/2} F(x)\,\mathrm{d}x\ , \tag{3.2.77}$$

in the second equality, should the integration bounds not change?

where $x = \omega/\omega_c$. The integral bounds x_1 and x_2 will depend on ω in general — depending on what the regime of validity of the powerlaw – but we will brutally approximate $x_1 \sim 0$ and $x_2 \sim +\infty$. In this case, the integral will not depend on ω, therefore

$$\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}V\,\mathrm{d}\omega} \propto \omega^{-(P-1)/2}\,. \tag{3.2.78}$$

So, the spectrum of the synchrotron radiation emitted by an electron population whose energies are distributed according to a powerlaw will **also be a powerlaw**. The new spectral index is given by $S = (P-1)/2$.

3.2.3 Polarization of synchrotron radiation

Saturday 2020-8-29, compiled 2020-09-06

Measuring polarization is in general difficult, but still it is important to be able to provide an estimate of how much radiation is actually polarized.

In order to describe synchrotron radiation we will need some reference unit vectors. Let us define $\vec{\epsilon}_\perp$ and $\vec{\epsilon}_\parallel$ such that, given the observation direction $\vec{n}$ and the magnetic field $\vec{B}$, $\vec{\epsilon}_\perp$ is perpendicular to both, while $\vec{\epsilon}_\parallel$ is perpendicular to $\vec{n}$ only, while being in the plane defined by $\vec{n}$ and $\vec{B}$.

These two unit vectors define a basis for the radiation seen in the direction $\vec{n}$, since as we recall the polarization of electromagnetic radiation is always transverse.

It can be shown that the power spectrum emitted in each polarization is given by

$$\left.\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega}\right|_\perp = \frac{\sqrt{3}q^3 B\sin\alpha}{4\pi mc^2}(F+G) \tag{3.2.79}$$

$$\left.\frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\omega}\right|_\parallel = \frac{\sqrt{3}q^3 B\sin\alpha}{4\pi mc^2}(F-G)\,, \tag{3.2.80}$$

where F is the function of ω/ω_c we defined earlier,

$$F\left(\frac{\omega}{\omega_c}\right) = \frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(z)\,\mathrm{d}z\ , \tag{3.2.81}$$

while the new function G is defined by

$$G\left(\frac{\omega}{\omega_c}\right) = \frac{\omega}{\omega_c} K_{2/3}\left(\frac{\omega}{\omega_c}\right). \tag{3.2.82}$$

The total emitted power per unit frequency is given by the sum of the two contributions in each polarization; so the terms containing G simplify, and we are left with the F term only, the result we found earlier.

The polarization fraction at each frequency for a single electron will be given by

$$\Pi(\omega) = \frac{P_\perp(\omega) - P_\parallel(\omega)}{P_\perp(\omega) + P_\parallel(\omega)} = \frac{G(\omega/\omega_c)}{F(\omega/\omega_c)}\,, \tag{3.2.83}$$

while the polarization fraction for a whole family of electrons whose energies are distributed according to a powerlaw will be given by the ratio of the integrals of G and F weighted by the electron distributions:

$$\Pi = \frac{\int_0^\infty G(\omega/\omega_c)\gamma^{-P}\,\mathrm{d}\gamma}{\int_0^\infty F(\omega/\omega_c)\gamma^{-P}\,\mathrm{d}\gamma} \tag{3.2.84}$$

$$= \frac{\int_0^\infty G(x)\gamma^{-P}\,\mathrm{d}\gamma}{\int_0^\infty F(x)\gamma^{-P}\,\mathrm{d}\gamma}\,, \tag{3.2.85}$$

where we must be careful, since x is a function of γ in the integral. Specifically, $x = \omega/\omega_c \sim 1/\gamma^2$, so $\gamma \sim x^{-1/2}$, therefore $\mathrm{d}\gamma \propto -1/2x^{-3/2}\,\mathrm{d}x$. This yields, up to multiplicative factors which cancel out in the ratio:

$$\Pi = \frac{\int_0^\infty G(x)x^{(P-3)/2}\,\mathrm{d}x}{\int_0^\infty F(x)x^{(P-3)/2}\,\mathrm{d}x}\,. \tag{3.2.86}$$

These integrals of special functions are evaluated in the literature, we have general explicit expressions for expressions like $\int x^\mu F(x)\,\mathrm{d}x$ and similarly for G.

Substituting these in, we find

$$\Pi = \frac{1+P}{P+7/3}\,. \tag{3.2.87}$$

This is always < 1 as it should be, however with very steep powerlaws (high P) it can reach values of Π which are arbitrarily close to 1.

If, on the other hand, we are only considering one electron, we can compute the *frequency-integrated* polarization fraction

$$\Pi = \frac{\int_0^\infty G(x)\,\mathrm{d}\omega}{\int_0^\infty F(x)\,\mathrm{d}\omega} = \frac{\int_0^\infty G(x)\,\mathrm{d}x}{\int_0^\infty F(x)\,\mathrm{d}x}\,, \tag{3.2.88}$$

which coincides with the polarization fraction from a population of electrons if we take $P = 3$. Therefore, we find $\Pi = (1+3)/(3+7/3) = 3/4$.

3.3 Einstein coefficients

In order to understand what the absorption from synchrotron radiation looks like we need to take a step back and consider the Einstein coefficients.

Let us reconsider Kirkhoff's law: if emitters and absorbers are in thermal equilibrium, then the following relation holds:

$$j_\nu = \alpha_\nu B_\nu\,. \tag{3.3.1}$$

This law holds at the macroscopic level: however, the process giving rise to it have a microscopic origin, so we should be able to give a microscopic version of this law.

Let us consider an atom with two energy levels, at E and $E + h\nu_0$ respectively. In the population these will have a different statistical weight: let us call these weights g_1 and g_2.

Einstein identified the three main processes which can occur when such an atom interacts with radiation:

1. spontaneous emission, where the system starts off in the higher energy level and decays to the lower one, emitting a photon of energy $h\nu_0$. This can happen regardless of the presence of an external radiation field. The Einstein A coefficient describes this: A_{21} is the transition probability per unit time connected to this process.

2. Absorption, where the atom starts off in the lower energy state and absorbs a photon of energy $h\nu_0$, going to the higher energy level. It should be stressed that the absorption is not actually monochromatic; because of several effects it is described by a distribution $\phi(\nu)$ which is sharply peaked around ν_0, and which is normalized so that $\int \phi(\nu)\,\mathrm{d}\nu = 1$. Then, the absorption probability per unit time is given by

$$B_{12} \int_0^\infty J_\nu \phi(\nu)\,\mathrm{d}\nu \ , \tag{3.3.2}$$

   where J_ν is the mean intensity of the external radiation field.

3. Stimulated emission, which is a consequence of the quantum nature of light. Planck's law only holds if this is included. This corresponds to the fact that the emission probability is enhanced by the presence of a radiation field. This is similar to how absorption occurs, in that the transition probability per unit time is

$$B_{21} \int_0^\infty J_\nu \phi(\nu)\,\mathrm{d}\nu \ . \tag{3.3.3}$$

Let us consider a system of many two-level atoms, whose number density is n. Some of them will be in level 1 and some will be in level 2; let us call the corresponding number densities n_1 and n_2.

If we are in thermodynamic equilibrium, then the transition rate from 1 to 2 must equal that from 2 to 1. This can be written as

$$\underbrace{n_1 B_{12} \bar{J}}_{\text{absorption}} = \underbrace{n_2 A_{21}}_{\text{emission}} + \underbrace{n_2 B_{21} \bar{J}}_{\text{stimulated emission}} \tag{3.3.4}$$

$$\bar{J} = \int_0^\infty J_\nu \phi(\nu)\,\mathrm{d}\nu \ . \tag{3.3.5}$$

We can solve this equation for $\bar{J}$: it comes out to be

$$\bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1} \ . \tag{3.3.6}$$

Since we know that the system is in thermodynamic equilibrium, we can also write that the ratio of the number densities of the two populations will be

$$\frac{n_1}{n_2} = \frac{g_1 \exp(-E/k_B T)}{g_2 \exp(-E/k_B T - h\nu_0/k_B T)} = \frac{g_1}{g_2} \exp\left(\frac{h\nu_0}{k_B T}\right). \tag{3.3.7}$$

Inserting this in the previous relation we find

$$\bar{J} = \frac{A_{12}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/k_B T) - 1}. \tag{3.3.8}$$

Also, we know that the mean intensity J_ν will be equal to the Planck function B_ν. The Planckian is quite slowly varying while $\phi(\nu)$ is very peaked around ν_0: therefore, it is safe to say that

$$\bar{J} = \int_0^\infty B_\nu \phi(\nu)\, \mathrm{d}\nu \approx B_{\nu_0}, \tag{3.3.9}$$

since $\phi(\nu)$ is normalized. Equating this to the expression we found above, we get

$$\bar{J} = B_{\nu_0} = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu_0}{k_B T}\right) - 1} = \frac{A_{12}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/k_B T) - 1}. \tag{3.3.10}$$

This must hold for any temperature T, so we can identify the terms in the two similar expressions: this tells us that

$$\frac{g_1 B_{12}}{g_2 B_{21}} = 1 \qquad \text{and} \qquad \frac{2h\nu^3}{c^2} = \frac{A_{12}}{B_{21}}. \tag{3.3.11}$$

These are called the **detailed balance relations**. They must be satisfied even if the system is outside of thermal equilibrium, since the temperature T does not appear in them.

What? How? Why? We used the hypothesis of thermal equilibrium a lot...

The emission and absorption coefficients

Let us assume that radiation which is emitted in the transition $2 \to 1$ has the same frequency as the radiation which is absorbed in the transition $1 \to 2$, so both processes will be characterized by the funciton $\phi(\nu)$.

With this, we can express the energy emitted per unit volume, solid angle, frequency and time as

$$J_\nu \,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu = \frac{h\nu_0}{4\pi} \phi(\nu) n_2 A_{21}\,\mathrm{d}\Omega\,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu \tag{3.3.12}$$

$$J_\nu = \frac{h\nu_0}{4\pi} \phi(\nu) n_2 A_{21}. \tag{3.3.13}$$

For absorption, we can apply a similar line of reasoning: the energy absorbed per unit volume and time is

$$\mathrm{d}w_\nu = \mathrm{d}V\,\mathrm{d}t\, h\nu_0 n_1 B_{12} \frac{1}{4\pi} \int \mathrm{d}\Omega\,\mathrm{d}\nu\, I_\nu \phi(\nu), \tag{3.3.14}$$

which we can specify to the energy taken out of the beam in the specific frequency and solid angle unit range by removing the integral. If we write the volume element as $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}s$, where $\mathrm{d}s$ is the distance travelled along the path of the radiation, and if we remember the radiative absorption law $\mathrm{d}I_\nu = -\alpha_\nu I_\nu\,\mathrm{d}s$, we can write

$$-\mathrm{d}I_\nu = -\frac{\mathrm{d}w_\nu}{\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu} = \underbrace{h\nu_0 n_1 \frac{B_{12}}{4\pi}\phi(\nu)}_{\alpha_\nu} I_\nu\,\mathrm{d}s \tag{3.3.15}$$

$$\alpha_\nu = \frac{h\nu_0}{4\pi} n_1 B_{12}\phi(\nu)\,. \tag{3.3.16}$$

We could also write $h\nu$ instead of $h\nu_0$ if we were to approximate $\phi(\nu)$ as a delta function $\phi(\nu) = \delta(\nu - \nu_0)$.

Now, since stimulated emission depends on $\bar{J}$, it is convenient to treat it as a kind of "negative absorption": with the same steps as before we find that its coefficient is

$$\alpha_\nu^{(s)} = -\frac{h\nu_0}{4\pi} n_2 B_{21}\phi(\nu)\,. \tag{3.3.17}$$

The total absorption coefficient will then be the sum of these:

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)(n_1 B_{12} - n_2 B_{21})\,. \tag{3.3.18}$$

In general, when we talk about absorption we always mean absorption minus stimulated emission.

3.3.1 Sychrotron absorption

Tuesday 2020-9-1, compiled 2020-09-06

We have seen what the total absorption coefficient (accounting for stimulated emission) is for a two-level system.

We want to use this expression in order to evaluate the total absorption due to the synchrotron process: so, we need to generalize it to a system with a continuum of energy levels, the free particle states with arbitrary velocity.

We will approach this by discretizing the space of possible energies of the particle. There is a slight complication, in that for a given energy $h\nu$ there are now *many* pairs of levels having that energy between them. We will need to sum over them, still denoting all the higher-energy states in the pairs by "2" and the lower-energy states in the pairs by "1". In this discretized description, then, we shall have an expression like

$$\alpha_\nu = \frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2}\left[n(E_1)B_{12} - n(E_2)B_{21}\right]\phi_{12}\,, \tag{3.3.19}$$

where the sum is performed across all the possible energy levels E_1 and E_2 such that $E_2 - E_1 = h\nu$, while $n(E_1)$ and $n(E_2)$ are the respective number densities of the two energy states. On the other hand, ϕ_{12} is the transition width.

The result we derived earlier was found by assuming that emission and absorption are isotropic; this is not true anymore since the magnetic field offers a preferential direction.

However, we can work around this problem by assuming that the magnetic field is "tangled", so that the direction of its value at a randomly chosen point in space is essentially uniformly distributed along the sphere.

We can also quickly evaluate the emission coefficient j_ν (which is the power emitted per unit volume, solid angle and frequency): if the power per unit frequency emitted by a single electron is $P(E,\nu)$ then the coefficient is

$$j_\nu = \frac{\mathrm{d}w}{\mathrm{d}t\,\mathrm{d}\nu}\frac{n(E)}{4\pi} = P(E,\nu)\frac{n(E)}{4\pi}\,, \tag{3.3.20}$$

where $n(E)$ is the density of electrons at a specific energy E. Note that this is not a differential quantity since we discretized the energy levels, so that they are countable. We are also assuming isotropicity, by the same reasoning as before.

The coefficient j_ν can also be expressed in terms of the Einstein coefficients, as

$$j_\nu = \frac{h\nu}{4\pi}\phi_{21}(\nu)n_2A_{21}\,, \tag{3.3.21}$$

which like before can be generalized to the situation in which we have many possible energy levels giving rise to the same transition as

$$j_\nu = \sum_{E_1}\frac{h\nu}{4\pi}\phi_{21}n_2A_{21} = \frac{h\nu}{4\pi}n_2\sum_{E_1}\phi_{21}A_{21} = \frac{n_2}{4\pi}P(E_2,\nu)\,, \tag{3.3.22}$$

meaning that now we know

$$P(E_2,\nu) = h\nu\sum_{E_1}\phi_{21}A_{21}\,. \tag{3.3.23}$$

From the *detailed balance* relations we can also relate the Einstein coefficients as

$$A_{21} = \frac{2h\nu^3}{c^2}B_{21}\,, \tag{3.3.24}$$

which we can substitute in, to find

$$P(E_2,\nu) = h\nu\left(\frac{2h\nu^3}{c^2}\right)\sum_{E_1}\phi_{21}B_{21}\,. \tag{3.3.25}$$

The reasoning we went to these great lengths to express the spectral power as a function of the Einstein coefficients is that we hope to invert the relation we found, giving us the coefficients as a function of the spectral power.

We have an expression for α_ν in terms of the coefficients B_{21} and B_{12}; however we know from the detailed balance relations that, as long as the statistical weights of the energy levels are equal (which holds for us since we are considering free states), $B_{21} = B_{12}$, so we can write the expression as

$$\alpha_\nu = \frac{h\nu}{4\pi}\sum_{E_1}\sum_{E_2}B_{21}\big(n(E_1) - n(E_2)\big)\phi_{21} \tag{3.3.26}$$

$$= \frac{h\nu}{4\pi} \sum_{E_2} \big(n(E_2 - h\nu) - n(E_2)\big) \sum_{E_1} B_{21}\phi_{21} \tag{3.3.27}$$

Used $E_2 - E_1 = h\nu$.

$$= \frac{h\nu}{4\pi} \sum_{E_2} \big(n(E_2 - h\nu) - n(E_2)\big) \frac{P(E_2,\nu)}{h\nu \frac{2h\nu^3}{c^2}} \tag{3.3.28}$$

Used equation (3.3.25).

$$= \frac{c^2}{8\pi h\nu^3} \sum_{E_2} \big(n(E_2 - h\nu) - n(E_2)\big) P(E_2,\nu)\,. \tag{3.3.29}$$

This works well for a system which we can discretize: however, if we want to use the language of the continuum of states we will need to turn the sum into an integral. In order to do this, we need to introduce the distribution function of electrons in momentum space, $f(\vec{p})$, which we assume to be isotropic. Using it, we write

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int \mathrm{d}^3 p_2 \left[f(\overline{p}_2) - f(p_2)\right] P(E_2,\nu)\,, \tag{3.3.30}$$

where p_2 is the modulus of the momentum associated with E_2, and similarly $\overline{p}_2$ is associated with $E_1 = E_2 - h\nu$.

Let us apply this for the well-known case in which the electron distribution is thermal, so that

$$f(\vec{p}) = k \exp\left(-\frac{E}{k_B T}\right), \tag{3.3.31}$$

which means that

$$f(\overline{p}_2) - f(p_2) = k\left(\exp\left(-\frac{E - h\nu}{k_B T}\right) - \exp\left(-\frac{E}{k_B T}\right)\right) \tag{3.3.32}$$

$$= k \exp\left(-\frac{E}{k_B T}\right)\left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right) = f(p_2)\left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right). \tag{3.3.33}$$

We can plug this into the expression we have for the absorption coefficient to find

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int \mathrm{d}^3 p\, f(p) \left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right) P(E,\nu) \tag{3.3.34}$$

$$= \frac{c^2}{8\pi h\nu^3} \left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right) \underbrace{\int \mathrm{d}^3 p\, f(p) P(E,\nu)}_{4\pi j_\nu} \tag{3.3.35}$$

$$= \frac{j_\nu}{B_\nu}\,, \tag{3.3.36}$$

since the rest of the expression we have is precisely the inverse of the Planck function. We have **recovered Kirkhoff's law**, which is expected since the electrons are assumed to be in thermal equilibrium.

Now we want to proceed in the general case in which the distribution of the electron energies is not Maxwellian; a specific case of interest is a power-law spectrum, which as we have seen can be the effect of synchrotron radiation.

Are we sure about this? was the powerlaw tail not the effect of Comptonization?

We wish to switch our integral from one over the particle momentum to one over the particle energy; in order to simplify the change of variable we will make the assumption that the particles are ultrarelativistic, therefore $E \approx pc$. The change of variable is then

$$4\pi p^2 \,\mathrm{d}p = 4\pi \frac{E^2}{c^2}\frac{\mathrm{d}E}{c} = \frac{4\pi E^2}{c^3}\,\mathrm{d}E\ . \tag{3.3.37}$$

while the number of particles in this differential element is given by

$$\mathrm{d}N = N(E)\,\mathrm{d}E = 4\pi p^2\,\mathrm{d}p\, f(p) = \frac{4\pi E^2}{c^3}\,\mathrm{d}E\, f(p)\,, \tag{3.3.38}$$

so we can identify

$$N(E) = \frac{4\pi E^2}{c^3} f(p) \qquad \text{or} \qquad f(p) = \frac{N(E)}{E^2}\frac{c^3}{4\pi}\,, \tag{3.3.39}$$

where $f(p)$ is calculated at the momentum $p = E/c$. Plugging this into the expression for the absorption coefficient we find

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int \frac{4\pi}{c^3}E^2\,\mathrm{d}E\,\frac{c^3}{4\pi}\left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right]P(E,\nu)\,. \tag{3.3.40}$$

We will further assume that the particle energy E is much smaller than the photon energy $h\nu$; this is justified by the fact that we are not using the full machinery of QED which would be required if the calculation was nonclassical.

Not clear how $E \ll h\nu$ means classical... Is it not actually $h\nu \ll E$? the photon would be relativistic then

If this is the case, then we can expand the expression in a power series around E, so that it reads

$$\alpha_\nu = -\frac{c^2}{8\pi h\nu^3}\int E^2 P(E,\nu)\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]h\nu\,. \tag{3.3.41}$$

Let us restrict this result to a powerlaw electron distribution, so that $N(E) = CE^{-P}$ for some $P \in \mathbb{R}^+$. If this is the case, then the expression inside the integral reads

$$-E^2\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{N(E)}{E^2}\right) = -E^2\frac{\mathrm{d}}{\mathrm{d}E}\left[CE^{-P-2}\right] = CE^2(P+2)E^{-P-3} = (P+2)\frac{N(E)}{E}\,, \tag{3.3.42}$$

so the absorption coefficient will be approximately

$$\alpha_\nu = (P+2)\frac{c^2}{8\pi\nu^2}\int \frac{N(E)}{E}P(E,\nu)\,\mathrm{d}E\ . \tag{3.3.43}$$

We must insert the known expression for the power radiated by a single charge as $P(E, \nu)$; this can be evaluated to finally find

$$\alpha_\nu \propto \nu^{-(P+4)/2}\,. \tag{3.3.44}$$

Now, recall that in general the definition of the source function S_ν is

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu}\,, \tag{3.3.45}$$

where $P(\nu)$ is the total spectral power, and since we know that its dependence on ν is as $P(\nu) \propto \nu^{-(P-1)/2}$, we can calculate the general source function for a powerlaw distribution:

$$S_\nu \propto \frac{\nu^{-(P-1)/2}}{\nu^{-(P+4)/2}} = \nu^{-\frac{P-1+P-4}{2}} = \nu^{5/2}\,, \tag{3.3.46}$$

which is **independent of** P!

Knowing the source function we can make certain predictions about the emission of a medium which is optically thick at least for the low-energy part of the spectrum.

We have shown at the beginning of the course that if the optical depth in a medium is large then the specific intensity I_ν becomes close to the source function: $I_\nu \sim S_\nu$. In the case of synchrotron emission we have seen that $\alpha_\nu \sim \nu^{-(P+4)/2}$, a *decreasing* function of the frequency, so the absorption will be large at low energies and small at high energies. The same will generally hold for the optical depth, since it is proportional to the absorption coefficient if the length ($\sim$ size of the medium) is fixed.

On the other hand, if the medium is optically thin then $I_\nu \sim j_\nu \sim P(\nu)$.

So, if we make a plot (let us make it log-log as is usually done) of the specific intensity I_ν as a function of frequency ν we will have a "thick" region at low ν where the intensity increases as $I_\nu \sim \nu^{5/2}$, and a "thin" region at high ν where the intensity decreases as $I_\nu \sim \nu^{-(P-1)/2}$.

3.4 Radiative Transitions

Saturday 2020-9-5, compiled 2020-09-06

This topic will be treated more quickly than what was originally planned for lack of time.

The basic question we will ask today is: given the knowledge of the various energy states of a certain atom, how do we characterize the radiation emitted by the various transitions of this atom?

First, we need to find out which transitions are actually permitted and thus can be observed in reality: **selection rules** will help with this. We will then determine what is the strength of the radiation from the transitions which are allowed.

A note: no transition in the cases we will treat is strictly *forbidden*, meaning that it is actually *impossible* for it to happen; rather, we use the word as a shorthand to denote transitions whose probability is quite small, so that the intensity of radiation they emit is vanishingly small (at least as compared to the "allowed" transitions).

We will use a **semi-classical** approach, treating the atom quantum-mechanically and radiation classically. The justification for this (which amounts to not using QED, basically) is that the self-interaction of photons is negligible.

Is the condition we should check not that the number of photons is very large, so that any single-photon effect washes out?

The transition probability per unit time from an initial state i to a final state f is denoted w_{fi}. There is an explicit quantum-mechanical expression for this quantity in terms of the Hamiltonian of the system and the wavefunctions corresponding to the two states of the electrons.

Because of the detailed balance equation

... and time-reversal symmetry?

we have that $w_{fi} = w_{if}$.

The general expression for the transition rate contains integrals in the form

$$\int \phi_f^* e^{i\vec{k}\cdot\vec{r}} \vec{\ell} \cdot \vec{\nabla}_j \phi_i \, \mathrm{d}^3 r \,, \tag{3.4.1}$$

where $\vec{\ell}$ is a unit vector in the direction of the polarization of the wave [RL79, eq. 10.13 onwards], while $\vec{k}$ is the wavevector while $\vec{r}$ is the position vector.

In the **long-wave** approximation, in which the wavelength of the radiation is much larger than the characteristic scale of the atom, we can expand

$$e^{i\vec{k}\cdot\vec{r}} \approx 1 + i\vec{k}\cdot\vec{r} + \frac{1}{2}\left(i\vec{k}\cdot\vec{r}\right)^2 + \dots \,. \tag{3.4.2}$$

Let us make the long-wave condition more explicit: typically we will have $\left|\vec{r}\right| \sim a_0 = \hbar/(m_e \alpha c)$, the Bohr radius, while the wavevector's magnitude will typically be $\left|\vec{k}\right| \sim \omega/c$. So, the quantity we want to be small is

$$\vec{k}\cdot\vec{r} \sim \frac{a_0\omega}{c} = a_0 \frac{\Delta E}{\hbar c} \,, \tag{3.4.3}$$

where ΔE is the transition energy. Since the binding energy is electrostatic in nature, the typical value of ΔE is $\Delta E \sim Ze^2/2a_0$. Then, the Bohr radius simplifies, and we get

$$\vec{k}\cdot\vec{r} \sim \frac{Ze^2}{2}\frac{1}{\hbar c} = \frac{Z\alpha}{2} \ll 1 \,, \tag{3.4.4}$$

so the approximation of taking the scalar product to be small works as long as we consider atoms which are not too far down the periodic table.

Truncating the series at the lowest order is equivalent to the dipole approximation. The link comes from the fact that $Z\alpha \approx v/c$, at least to first order. This means that, as long as the condition is satisfied, the electron is moving non-relativistically around the nucleus.

In the dipole approximation we do indeed have certain transitions with probability exactly equal to zero, but this is an artifact of the approximation.

If we consider an ensemble of atoms whose orientations are random, then the *unpolarized* transition rate reads

$$\left\langle w_{fi} \right\rangle = \frac{4\pi^2}{3\hbar^2 c} \left| d_{fi} \right|^2 \mathcal{I}(\omega_{fi}) \,, \tag{3.4.5}$$

where $\mathcal{I}(\omega_{fi}) = 4\pi J(\omega_{fi})$ is the intensity of radiation integrated over all solid angles and computed at the transition frequency, while d_{fi} is the dipole operator (an operator acting on the initial wavefunction and returning its dipole moment?).

Now, this is related to the Einstein B coefficient as

$$\langle w_{lu} \rangle = B_{lu} J(\nu_{ul}) \,, \tag{3.4.6}$$

where we switched from i and f, meaning initial and final, to l and u, meaning lower and upper energy state.

We have the conversion

$$J(\nu_{ul}) = \frac{\mathcal{I}(\nu_{ul})}{4\pi} = \frac{2\pi}{4\pi} \mathcal{I}(\omega_{ul}) = \frac{\mathcal{I}(\omega_{ul})}{2} \,. \tag{3.4.7}$$

What? a change by a factor 2π in the *argument* results in a change in 2π in the *function*? This only holds if intensity is a *linear* function of the frequency!

Therefore,

$$\langle w_{ul} \rangle = B_{lu} \frac{\mathcal{I}(\omega_{ul})}{2} \,, \tag{3.4.8}$$

which means that the Einstein B coefficient can be expressed in terms of the dipole operator as

$$B_{lu} = \frac{8\pi^2}{4c\hbar^2} |d_{ul}|^2 \,. \tag{3.4.9}$$

It is common to express this in terms of a quantity called the *oscillator strength* for that given transition, however in order to introduce that formalism we need to generalize what we have found to transitions between degenerate energy levels.

If we have several degenerate upper levels, while the final lower state has a statistical weight g_l, then we can write

$$B_{lu} = \frac{32\pi^2}{3c\hbar^2} \frac{1}{g_l} \sum_l |d_{ul}|^2 \,. \tag{3.4.10}$$

We are averaging over the initial states, and summing over the final ones.

why is the statistical weight in the denominator?

why is this expression multiplied by a factor 4 compared to the nondegenerate one?

Why is g_l outside the sum?

The definition of the oscillator strength f_{lu} is then given by expressing the Einstein coefficient as

$$B_{lu} = \frac{4\pi^2 e^2}{h\nu_{ul} mc} f_{lu} \tag{3.4.11}$$

$$f_{lu} = \frac{2m}{3\hbar^2 g_l c^2} (E_u - E_l) \sum |d_{lu}|^2 . \tag{3.4.12}$$

3.5 Line broadening

So far, we have implicitly assumed that the transition can be described by a single frequency, but as was discussed earlier in the course this is not really the case; instead, the line should be described by a certain function $\phi(\nu)$ which is peaked around that frequency.

There are several processes which can broaden a line beyond the most basic time-energy uncertainty relation.

Doppler broadening

We are often dealing with plasmas which contain ions moving at high velocities due to thermal motion; since this is a random motion the distribution of the velocities' components along the observation direction will be quite wide. Since we need to move from the atom rest frame to the observer frame, we will need to consider the Doppler effect.

The velocity component along the observation direction affects the effective frequency of the radiation as measured by us. Let us make this reasoning quantitative.

Suppose that the observation direction is the z axis, and let us denote the velocity component in that direction as v_z. Further, let us assume that the velocity is nonrelativistic so that we can work to first order in v/c. Then, the Doppler shift reads

$$\nu - \nu_0 = \nu_0 \frac{v_z}{c} , \tag{3.5.1}$$

where the (variable) frequency ν is in the observer frame, while the (fixed) frequency ν_0 is in the atom frame.

Since the velocities are assumed to be nonrelativistic and corresponding to thermal equilibrium, their distribution will be Maxwellian, meaning that each of the three components will have a Gaussian distribution: explicitly, the number density of atoms whose velocity component is near v_z in an interval of width $\mathrm{d}v_z$ is

$$\mathrm{d}N \propto \exp\left(-\frac{m v_z^2}{2 k_B T} \right) \mathrm{d}v_z . \tag{3.5.2}$$

We can refer this expression to the frequency change by inverting the Doppler relation: we find

$$v_z = c \frac{\nu - \nu_0}{\nu_0} \implies \mathrm{d}v_z = \frac{c}{\nu_0} \mathrm{d}\nu , \tag{3.5.3}$$

using which we can refer the Gaussian distribution of the velocity components back to the emitted frequency distribution as

$$\mathrm{d}N \propto \phi(\nu) \propto \exp\left(-\frac{mc^2(\nu-\nu_0)^2}{2k_BT\nu_0^2}\right)\mathrm{d}\nu\ . \tag{3.5.4}$$

From this, we can say that the line profile function is given by

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}}\exp\left(-\frac{(\nu-\nu_0)^2}{\Delta\nu_D^2}\right), \tag{3.5.5}$$

where the width of the line is given by

$$\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2k_BT}{m}}, \tag{3.5.6}$$

and the normalization is fixed by imposing $\int_0^\infty \phi(\nu)\,\mathrm{d}\nu = 1$.

Natural broadening

This effect is due to the fact that if a level has a lifetime of Δt, then there must be a spread ΔE in the transition energies of the level so that $\Delta t\Delta E \gtrsim \hbar$. Since no level has an infinite lifetime, not even at $T = 0$ can a transition be truly monochromatic.

The decay rate of a state n is given by

$$\gamma = \sum_{n'} A_{nn'}, \tag{3.5.7}$$

where n' is a state with a lower energy than n into which the atom can decay, while $A_{nn'}$ is the Einstein A coefficient for the transition from n to n'.

It can be shown that the line profile due to natural broadening is given by a Lorentzian or Lorentz profile,

$$\phi(\nu) = \frac{1}{4\pi^2}\frac{\gamma}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}\,. \tag{3.5.8}$$

It can also be shown that certain other effects, such as microturbulence, can be described by a similar profile: this is then accomplished by defining a "generalized transition rate" Γ to replace the transition rate γ. We will use the letter Γ in the next section.

Combined Doppler and Lorentz profile

We can give a description of the combined effects of the two kinds of broadening we discussed so far, which is needed since they will both occur in general.

The Lorentz profile referred to the observed frequency of the transition (which can be shifted due to the Doppler effect) is given, like before, as

$$\phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu-\nu_0^{\mathrm{obs}})^2 + (\Gamma/4\pi)^2}, \tag{3.5.9}$$

where

$$\nu_0^{\rm obs} = \nu_0 + \nu_0 \frac{v_z}{c}\,, \tag{3.5.10}$$

so we can refer the profile to the atom frame like

$$\phi(\nu) = \frac{\Gamma/4\pi^2}{\left[\nu - \nu_0(1+v_z/c)\right]^2 + (\Gamma/4\pi)^2}\,. \tag{3.5.11}$$

In order to find the observed broadening we need to average this over all the possible v_z according to their Gaussian distribution: this yields

$$\phi(\nu) = \frac{\Gamma}{4\pi^2}\int_{-\infty}^{\infty} \frac{\sqrt{m/2\pi k_B T}\exp\left(-mv_z^2/2k_BT\right)}{\left[\nu - \nu_0(1+v_z/c)\right]^2 + (\Gamma/4\pi)^2}\,{\rm d}v_z \tag{3.5.12}$$

$$= \frac{1}{\Delta\nu_D\sqrt{\pi}} H(a,u)\,, \tag{3.5.13}$$

where we introduced the Voigt function, which cannot be computed analytically:

$$H(a,u) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}\,{\rm d}y}{a^2 + (u-y)^2}\,, \tag{3.5.14}$$

whose arguments are $a = \Gamma/4\pi\Delta\nu_D$ and $u = (\nu - \nu_0)/\Delta\nu_D$. This finally yields the Voigt profile.

Bibliography

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