General Relativity notes

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1 Special relativity

Definition 1.1. *An inertial frame is one in which Newton's laws hold: a free body moves with acceleration* $a^i = 0$*.*

Newton's first law establishes the *existence* of inertial frames.

Proposition 1.1. *The frames* O *and* O' *are both inertial frames iff* O' *moves with constant velocity wrt* O*.*

Proposition 1.2. *Coordinate transformations between inertial frames are Lorentz boosts, which in some coordinate frame can be written as*

$$t' = \gamma_v \left(t - \frac{vx}{c^2} \right) \tag{1a}$$

$$x' = \gamma_v (x - vt) \tag{1b}$$

$$y' = y \tag{1c}$$

$$z' = z\,, \tag{1d}$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$*.*

If $v \ll c$, so $v/c \sim 0$, they simplify to the identity for t, y, z and $x' = x - vt$: these are Galilean transformations.

If we have two events, x^μ and y^μ, they occur with some time and space separation $\Delta x^\mu = x^\mu - y^\mu$. We can compute $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$, where

$$\eta_{\mu\nu} = \operatorname{diag}\left(-c^2, 1, 1, 1 \right) . \tag{2}$$

Proposition 1.3. *Under Lorentz transformations Δs^2 is invariant.*

We can classify separations between events as

- time-like when $\Delta s^2 < 0$;
- null-like when $\Delta s^2 = 0$;
- space-like when $\Delta s^2 > 0$.

We can draw spacetime diagrams. A light cone is the set of points which are null-like separated from a select point. Things can be only causally related to events inside the light-cone (with $\Delta s^2 \geq 0$).

1.1 Time dilation

Take two events which occur at the same location for O'. In the primed frame they will have coordinates $x^\mu = (t_0, x_0)$ and $y^\mu = (t_1, x_0)$.

Definition 1.2. *The* proper time *between these two events is* $t_1 - t_0 \overset{def}{=} \Delta\tau$.

We now see that $\Delta s'^{\,2} = -c^2 \Delta\tau^2$. Then, any other observer will see the same $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 = \Delta s'^{\,2}$.

This directly implies that $\Delta\tau \leq \Delta t$ for any observer, since $\Delta\tau^2 = \Delta t^2 - \Delta x^2/c^2$. This effect is called *time dilation*.

By how much exacly is time dilated? Of course $\Delta x = v\Delta t$, therefore $\Delta t = \gamma_v \Delta\tau$.

-> Muon problem.

Inverse Lorentz transformation have the same expression, but with $v \to -v$. This can be proved both mathematically by solving the equations and phisically by reasoning about their meaning. There is no preferential inertial frame.

A Lorentz transformation can be written in matrix form in the (ct, x) plane as:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \tag{3}$$

since there is an angle θ such that $\gamma = \cosh\theta$ and $\gamma\beta = \sinh\theta$: the angle θ will be $\theta = \tanh^{-1}(v/c)$. This is true because $\gamma^2 - \beta^2\gamma^2 = 1$.

After a boost the ct' and x' axes are respectively the lines $ct = x/\beta$ and $ct = \beta x$.

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Last lecture we saw the fact that the ct' and x' axes are rotated by equal angles from the ct and x axes towards the $ct = x$ axis.

1.2 Relativity of simultaneity

Consider two events which are simultaneous in the O' frame. Their times in this frame are $t'_A = t'_B$.

In the O frame, instead, we have

$$ct_{A,B} = \frac{v}{c} x_{A,B} + \underbrace{\sqrt{1 - \frac{v^2}{c^2}} c t'_{A,B}}_{\text{a constant}}, \tag{4}$$

so the events are not simultaneous in the O frame.

1.3 Length contraction

If in the O frame, A occurs at $t, x = 0$ while B occurs at $t = 0$, $x = L$, then L is the measured length of their spatial interval by O. We assume that this is the frame in which the object is moving, and we transform into a frame in which it is stationary: O'.

In the primed frame their coordinates will be:

$$x'_A = \gamma_v \left(x_A - \frac{v}{c} c t_A \right) \tag{5a}$$

$$x'_B = \gamma_v \left(x_B - \frac{v}{c} c t_B \right), \tag{5b}$$

therefore $x'_B - x'_A = \gamma_v (x_B - x_A)$: the length is contracted in the O frame, since $\gamma \geq 1$.

1.4 Addition of velocities

Two observers see an object moving with $v' = \mathrm{d}x'/\mathrm{d}t'$ and $v = \mathrm{d}x/\mathrm{d}t$ respectively. Their relative velocity is u. Differentiating we get:

$$v' = \frac{\gamma(\mathrm{d}x - v\,\mathrm{d}t)}{\gamma\left(\mathrm{d}t - \frac{u \mathrm{d}x}{c^2}\right)} = \frac{v - u}{1 - \frac{uv}{c^2}}. \tag{6}$$

Two interesting limits of this formula are: $v' = v - u$ if $u \ll c$ or $v \ll c$; and $v' = c$ if $v = c$ for whatever u.

1.5 Tensor notation

The position four-vector is $x^\mu = (ct, x, y, z)$. The Euclidean scalar product is given by $x \cdot y = \delta_{\mu\nu} x^\mu x^\nu$. If we substitute the identity $\delta_{\mu\nu}$ with another metric we can find a more general metric space.

The Minkowski metric is $\eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, 1)$. The separation 4-vector is $\mathrm{d}x^\mu = (c\,\mathrm{d}t\,, \mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z)$.

Using Einstein summation notation, we can write the spacetime interval as $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$.

Specifically for the Minkowski metric we have the relation $\eta_{\mu\nu} = \eta^{\mu\nu}$: it is its own inverse. For a general metric $g_{\mu\nu}$ this will not hold.

How do we express the Lorentz boosts? They preserve $\mathrm{d}s^2$, therefore they look like $x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu$, with the $(1, 1)$ tensors $\Lambda_\mu{}^\nu$ satisfying $\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$. This is called the *pseudo-orthogonality* relation.

The metric allows us to raise and lower indices. Raising an index in the pseudo-orthogonality relation gives us: $\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta \eta^{\beta\sigma} = \delta_\alpha{}^\sigma$, therefore $\eta_{\mu\nu}\Lambda^\nu{}_\beta \eta^{\beta\sigma}$ is the inverse of a Lorentz transformation.

Four-vectors can also have their indices down, and they will transform according to the inverse of Lorentz transformations:

$$
\begin{aligned}
(\eta_{\alpha\mu} x^\mu)' &= \eta_{\alpha\mu}\Lambda^\mu{}_\nu x^\nu && \text{(7a)}\\
&= \Lambda_{\alpha\sigma}\delta^\sigma{}_\nu x^\nu && \text{(7b)}\\
&= \Lambda_{\alpha\sigma}\eta^{\sigma\beta}\eta_{\beta\nu} x^\nu && \text{(7c)}\\
&= \Lambda_\alpha{}^\beta x_\beta\,. && \text{(7d)}
\end{aligned}
$$

We will write our laws as tensorial equations, which are covariant.

By pseudo-orthogonality, the scalar product $A_\mu B^\mu$ is a covariant (that is, invariant) scalar. Of course it is equal to $A^\mu B_\mu$.

Definition 1.3 (Tensor). *A* (p, q) tensor *is an object* $M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p}$ *with many components indexed by* $p + q$ *indices, which transforms as:*

$$
M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p} \to \Lambda_{\mu_1}{}^{\mu_1'} \dots \Lambda_{\mu_p}{}^{\mu_p'} \Lambda^{\nu_1}{}_{\nu_1'} \dots \Lambda^{\nu_q}{}_{\nu_q'} M^{\nu_1' \dots \nu_q'}_{\mu_1' \dots \mu_p'} \tag{8}
$$

under Lorentz transformations $\Lambda_\mu{}^\nu$.

Thu Oct 10 2019

Last lecture we introduced tensors.

An example of those is the EM tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_x & B_y \\ -E_y/c & B_x & 0 & -B_z \\ -E_z/c & -B_y & B_z & 0 \end{bmatrix}, \tag{9}$$

which, it can be checked, transforms as a $(0,2)$ tensor. Also, we can define the current vector $j^\mu = (c\rho, j^i)$. Then, the Maxwell equations read:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \qquad \text{and} \qquad \partial_{[\mu} F_{\nu\rho]} \,. \tag{10}$$

They are covariant!

1.6 Particles in motion

In Newtonian mechanics, the motion of a particle is described by a function of time $x^i = x^i(t)$.

In special relativity, we introduce the concept of *worldline*. It must be parametrized with respect to some parameter λ, such that $x^\mu = x^\mu(\lambda)$. A preferred choice for λ is the proper time of the particle, $\lambda = \tau$.

We then define the 4-velocity:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \,. \tag{11}$$

It is a tensor since it is the product of a scalar and a tensor.

Multiplying $u^\mu u_\mu$ we always get $-c^2$, since:

$$u^\mu u_\mu = \frac{\mathrm{d}x^\mu \, \mathrm{d}x_\mu}{\mathrm{d}\tau^2} = -c^2 \frac{\mathrm{d}s^2}{\mathrm{d}s^2} \tag{12}$$

We can make the expression explicit using $\mathrm{d}\tau = \gamma \, \mathrm{d}t$, which gives us $u^\mu = (\gamma c, \gamma v^i)$. In the frame of the particle, $u^\mu = (c, 0)$.

The *four-momentum* of a particle is defined as:

$$p^\mu = m u^\mu = (m\gamma c, m\gamma v^i) \,. \tag{13}$$

The component p^0 is mc at $v = 0$. What does it mean? we can expand it for small v:

$$\frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}} \sim mc\left(1 + \frac{v^2}{2c^2}\right) = mc + \frac{1}{c}\frac{mv^2}{2} \,. \tag{14}$$

We get the mass, plus a kinetic energy term: more explicitly, $cp^0 = mc^2 + {}^1\!/_2 mv^2$.

We can rewrite Newton's first law in SR:

Proposition 1.4 (Newton I)**.** *A free particle moves with constant u^μ, or*

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \tag{15}$$

To express this in an easier way we introduce the 4-acceleration:

$$a^\mu \stackrel{\text{def}}{=} \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} \tag{16}$$

We now with to introduce the concept of a path minimizing proper time. Recall Snell's law, which allows us to relate the angles of incidence of light when it passes between one medium to another, if they have different indices of refraction:

$$\frac{\sin(\theta_2)}{\sin(\theta_1)} = \frac{n_1}{n_2} = \frac{v_2}{v_1}\,. \tag{17}$$

This can be shown to be equivalent to light minimizing the time it takes to move from a point in one medium to a point in the other.

Analogously, saying that a massive particle travels along the worldline which minimizes τ is equivalent to Newton's first principle.

We want to perturb a generic worldline x^μ with some $\mathrm{d}x^\mu$, and consider the proper time functional τ which gives the proper time of a generic trajectory: we impose

$$\frac{\tau[x^\mu + \varepsilon^\mu] - \tau[x^\mu]}{|\varepsilon^\mu|} = \frac{\delta\tau}{\delta x^\mu} \stackrel{!}{=} 0\,, \tag{18}$$

where a limit $|\varepsilon^\mu| \to 0$ is implied, and only the linear terms are considered.

The proper time functional for paths between A and B is given $\tau = \int_A^B \mathrm{d}\tau$. We can rewrite it as:

$$\tau = \int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = \int_A^B \mathrm{d}\tau \, \frac{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}{\mathrm{d}\tau^2}\,. \tag{19}$$

We now consider a perturbation $\varepsilon^\mu = \delta_1^\mu \delta x$:

$$\tau_{AB}[x+\varepsilon] = \int_A^B \mathrm{d}\tau \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\mathrm{d}\delta x}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \right]. \tag{20}$$

We can discard a second order term $\left(\mathrm{d}\delta x/\mathrm{d}\tau\right)^2$, and subtract off $\tau_{AB}[x]$: we are left with

$$\delta\tau = -\frac{2}{c^2}\int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\delta x}{\mathrm{d}\tau} \tag{21}$$

Now, we integrate by parts, disregard the boundary terms since the endpoints of the path cannot be deformed, and get:

$$\frac{\delta\tau_{AB}}{\delta x} = +\frac{2}{c^2}\int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\,, \tag{22}$$

which proves the equivalence for this type of perturbation, the others are analogous.

The generalization of Newton's second law, which at low speeds is $F^i = ma^i$, can be similarly restated as $\delta S = 0$, for the action $S = \int \mathrm{d}\tau$.

1.7 Motion of light rays

For light we cannot compute u^μ with the previous definition, since its proper time is always zero.

Instead, we *define* u^μ to be a normalized null-like vector, such that $x^\mu = \lambda u^\mu$ for some λ.

We know from quantum mechanics that $E = \hbar\omega$, where $\hbar = h/(2\pi)$ and $\omega = 2\pi/T = 2\pi f$.

The momentum is proportional to the wavevector k^i: $p^i = \hbar k^i/c$. The relativistic generalization of this fact is

$$p^\mu = \left(\frac{\hbar\omega}{c}, \frac{\hbar k^i}{c}\right) = \frac{\hbar k^\mu}{c}\,. \tag{23}$$

Since the momentum of light must be null we have that necessarily $\omega = |k|$.

1.8 Doppler effect

We take a special case: radiation goes in the same direction as the observer. In the O frame we have $k^\mu = (\omega, \omega, 0, 0)$.

The observer, moving with velocity v, measures $k^{\prime,\mu}$. This can be easily computed with a Lorentz transformation: $k^{\prime,\mu} = \Lambda^\mu{}_\nu k^\nu$.

We are mostly interested in $k^{\prime,0} = \omega'$: it comes out to be $\omega' = \gamma\omega + (-\gamma\beta)\omega = (1 - v/c)\gamma\omega$.

Some notes: at slow speeds $\omega' \approx (1 - v/c)\omega$; we have $f' < f$ when source and observer are moving away from each other.

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1.9 Bases

In Euclidean 2D geometry we can choose, for example, the basis $e_1 = (1, 0)^\top$ and $e_2 = (0, 1)^\top$. This basis is orthonormal with respect to the scalar product $g_{\mu\nu} = \delta_{\mu\nu}$: $e_{(\alpha)} \cdot e_{(\beta)} = e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} g_{\mu\nu} = g_{(\alpha)(\beta)}$.

I use parentheses around indices to denote the fact that they are not tensorial indices, but instead denote which basis vector we are considering. We express our vectors in components with respect to this basis.

In SR, we can do the same: our coordinate basis can be given by $e^{\mu}_{(\alpha)} = \delta^{\mu}_{(\alpha)}$. Now, the orthonormality $e_{(\alpha)} \cdot e_{(\beta)} = g_{(\alpha)(\beta)}$ holds with respect to $g_{\mu\nu} = \eta_{\mu\nu}$.

1.10 Observers & observations

Every observer will be characterized by their trajectory $x^{\mu}(\tau)$. We can associate a coordinate system with the observer: the one in which the observer's own 4-velocity u^{μ} is the time-like unit vector (rescaled by a factor of c: $u^{\mu} = c e_{(0)}$).

When the observer sees a particle with $p^{\mu} = (E_p / c, p^i)$ they measure the energy of the particle to be $p^0 c$: in this frame this is $E_{\text{measured}} = -e^{\mu}_{(0)} p_{\mu} c = -u^{\mu} p_{\mu} c$. Do note that this is a covariant expression, while p^0 is not: the energy of a particle with 4-momentum p^{μ} measured by an observer with 4-velocity u^{μ} is an invariant.

In the rest frame of the observer, their own 4-velocity is $(c, \vec{0}) = c(1, \vec{0})$. In the rest frame of the particle, its own energy is measured to be mc^2. The measured energy by an observer such that the product of the 4-velocities of the particle and of the observer is $-\gamma c^2$ is $m\gamma c^2$.

The Earth moves with speed $10^{-4} c$ around the Sun.

Now we can start using $c = 1$. We can put the c back whenever we want with dimensional analysis.

2 Gravity

2.1 The Equivalence Principle

Just like Newton supposedly thought about universal gravity when, while looking at the sky, an apple fell on his head; Einstein supposedly thought up the equivalence principle when he saw a man falling from a rooftop.

Proposition 2.1 (Equivalence principle). *Experiments in a small free falling system over a short amount of time give the same result as experiments in an inertial frame in empty space.*

Why “small”? The gravitational field is not really homogeneous. The idea is that gravity can only be removed *locally*.

If we were to see that objects fall differently even in the same neighbouhood then we would lose the EP.

Definition 2.1. *The* inertial *mass is an object's resistance to motion:* $m_{inertial} = F^i / a^i$.

Definition 2.2. *The* gravitational *mass is the one which defines the gravitational force on an object:* $m_{gravitational} = |F| r^2 / (GM)$.

These are *a priori* different, but experimentally equal: in general the gravitational acceleration is given by

$$a^i = \frac{GMr^i}{r^3} \frac{m_{\text{gravitational}}}{m_{\text{inertial}}} \tag{24}$$

If the ratio of masses depended on the material, this could vary.

We can do a torsion pendulum experiment: the torsion applied by the Coriolis effect on a pendulum depends on the inertial mass, while its restoring force depends on the gravitational mass. Experimentally we have measured them to be equal with an accuracy of 10^{-12}.

A person on a rocket accelarating at g experiences the same acceleration as a person standing on Earth.

2.2 Gravitational redshift

We treat it now in a weak field approximation.

Alice sends radiation to Bob from a higher altitude on Earth. Alice sends it with frequency f, Bob receives it with f'. They are at rest with respect to one another: there is no kinematic Doppler effect here.

We do this by applying the equivalence principle! We imagine A and B to be standing in a rocket which is accelerating at g: there is no more gravity.

Bob will receive a greater frequency: $f' > f$. This can be seen by imagining two consecutive wavefronts as two particles. Alice sends them $\Delta t_A = 1/f$ apart, Bob receives them as $\Delta t_B = 1/f'$ apart.

If the rocket is at rest, the time for the radiation to reach B is h/c; if the rocket is moving then the time is $< h/c$.

When the second wavefront starts moving the rocket is already going: the second wave starts later but it has less distance to travel. Therefore $\Delta t_A > \Delta t_B$, which implies $f' > f$.

Claim 2.1. *The first terms in the expansion are:*

$$f' = f\left(1 + \frac{gh}{c^2} + O\left(\left(\frac{gh}{c^2}\right)^2\right)\right). \tag{25}$$

2.3 Potentials

In electromagnetism, the potential energy between a charge Q and a test charge q is $U = kQq/r$: then we define the electromagnetic potential $V = U/q$ which has the advantage of being test-charge independent.

Similarly, we define the gravitational potential $\Phi = U/m = gh$. Then, the second order term in the formula for the redshift becomes $\Delta\Phi c^{-2}$: now we can properly say that this *weak field* means $\Delta\Phi c^{-2} \ll 1$.

This is surely the case for the cases we can treat concretely. If two people are separated by 1 km of difference in altitude, they have $\Delta\Phi c^{-2} \approx 10^{-13}$: the difference they will experience is one second in a million years.

Our expression from Φ in the newtonian approximation is $\Phi = GM/r$.

We can say even now by dimensional analysis that $GM/(rc^2)$ is the parameter which tells us how relevant the gravitational effects are: if it is similar to 1 we must consider GR.

This is very close to the expression for the Schwarzschild radius: it is $r = M$ in natural units $c = G = 1$, while the correct expression is $r = 2M$: that one can actually be recovered exactly if we calculate the radius at which the escape velocity is equal to c.

Thu Oct 17 2019

3 The mathematical description of curved spacetime

The metric describes the spacetime. It depends on the coordinates.

Euclidean metric In 2D, it is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. We can express it as

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \, \mathrm{d}x_\mu = \mathrm{d}x^\mu \, \mathrm{d}x^\mu \, g_{\mu\nu} \,, \tag{26}$$

computed with $g_{\mu\nu} = \delta_{\mu\nu}$.

Polar coordinates Let us see how this changes when we change coordinates: for example, we can go to polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\bigl(y/x\bigr) \end{cases} \,. \tag{27}$$

Let us compute the differentials $\mathrm{d}x$ and $\mathrm{d}y$:

$$\mathrm{d}x = \mathrm{d}r\cos(\theta) - r\sin(\theta)\,\mathrm{d}\theta \,, \tag{28}$$

and

$$\mathrm{d}y = \mathrm{d}r \sin(\theta) + r\cos(\theta)\,\mathrm{d}\theta \ . \tag{29}$$

Plugging these into the metric and simplifying we get

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \ . \tag{30}$$

It is not clear to see that these are equivalent. We then want to compute scalar quantities to characterize them.

Another issue is the fact that the polar metric is singular at the origin: we'd have to invert

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{31}$$

at $r = 0$: we cannot do it! This is a *coordinate singularity*. There is nothing wrong with the space; $\mathbb{R}^2$ is perfectly regular at $(0,0)$, but our coordinate description fails: what value of θ should we assign to that point?

Spherical coordinates In $\mathbb{R}^3$ we have the same issue. We use:

$$\begin{cases} x = \cos(\theta)\cos(\varphi) \\ y = \cos(\theta)\sin(\varphi) \\ z = \sin(\theta) \end{cases} , \tag{32}$$

and it is a simple computation as before to see that

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin(\theta)^2 \end{bmatrix} , \tag{33}$$

therefore these coordinates are not defined on the *whole z axis*!

General coordinate transformations Now we consider a general transformation of space, which we denote by $x'^{\mu}(x^{\mu})$: we see how the metric should change in order for the spacetime distance to be invariant: we define $g'_{\mu\nu}$ by

$$g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} \overset{!}{=} g'_{\mu\nu}\,\mathrm{d}x'^{\mu}\,\mathrm{d}x'^{\nu} \ . \tag{34}$$

This means that the metric changes as a tensor:

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \, , \tag{35}$$

and we can see that it transforms as a $(0, 2)$ tensor since the primes are in the denominator: it transforms with the *inverse* of the Jacobian matrix.

The inverse metric transforms as:

$$g^{\mu\nu} \to g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta} \,, \tag{36}$$

which we can check by proving that $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$ is conserved when we transform both the metric and the inverse metric. We get:

$$g'^{\mu\nu} g'_{\sigma\tau} \delta^{\sigma}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta} \delta^{\sigma}_{\nu} \frac{\partial x^{\pi}}{\partial x'^{\sigma}} \frac{\partial x^{\lambda}}{\partial x'^{\tau}} g_{\pi\lambda} \,, \tag{37}$$

which can be simplified using the relations between the partial derivatives:

$$\frac{\partial x'^{\nu}}{\partial x^{\beta}} \delta^{\sigma}_{\nu} \frac{\partial x^{\pi}}{\partial x'^{\sigma}} = \frac{\partial x^{\pi}}{\partial x^{\beta}} = \delta^{\pi}_{\beta} \,, \tag{38}$$

since we are multiplying the Jacobian with its inverse, thus we get the identity. One can make this reasoning more explicit by seeing it as an application of the chain rule: x^{μ} is a function of x'^{μ} which is a function of x^{μ}. Plugging this in we get

$$g'^{\mu\nu} g'_{\sigma\tau} \delta^{\sigma}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} g^{\alpha\beta} \delta^{\pi}_{\beta} \frac{\partial x^{\lambda}}{\partial x'^{\tau}} g_{\pi\lambda} \,, \tag{39}$$

and now we can apply our hypothesis that $g^{\alpha\beta} g_{\beta\lambda} = \delta^{\alpha}_{\lambda}$ to contract some more indices, get one more multiplication of a Jacobian with its inverse, which we can simplify in the same way as before to finally get the identity. This proves that the inverse metric is a $(2, 0)$ tensor.

3.1 Lengths, areas, volumes and so on

In 4D space, if we fix x^0 and x^3, we can consider an area element like $\mathrm{d}A = \sqrt{g_{11}}\, \mathrm{d}x \, \sqrt{g_{22}}\, \mathrm{d}x^2$.

Similarly, the 4-volume is just

$$\mathrm{d}v = \sqrt{-g_{00} g_{11} g_{22} g_{33}}\, \mathrm{d}x^0 \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3 = \sqrt{-g}\, \mathrm{d}^4 x \,, \tag{40}$$

where $g = \det g_{\mu\nu}$. We have shown it for a diagonal metric, but it can be shown that it holds for any general metric. The minus sign comes from the fact that our metric has signature -1 (and, as we will see, this holds in general): the determinant is then negative, and to take the square root we need a positive number.

Claim 3.1 (Unproven)**.** *A metric can be always diagonalized, at least locally.*

When it is diagonal, then $\mathrm{d}v = \sqrt{-g}\,\mathrm{d}^4x$. Then the quantity $\sqrt{-g}\,\mathrm{d}^4x$ is a scalar.

Intuitively, the claim follows from the equivalence principle: we put a free-falling observer in our space. They will perceive spacetime as being flat.

Under a diffeomorphism with the determinant of the jacobian equal to J we have the following transformation law for the determinant:

$$g' = J^{-2}g \qquad \Longrightarrow \qquad \sqrt{-g'} = J^{-1}\sqrt{-g}\,. \tag{41}$$

This can be expressed by saying that $\sqrt{-g}$ is a *tensor density* of weight -1.

Definition 3.1. *A tensor density of weight w transforms just like a tensor, except we need to multply by a factor J^w in the transformation, where J is the determinant of the Jacobian of the transformation.*

A (p,q) tensor density *of weight w is an object $M^{\nu_1\dots\nu_q}_{\mu_1\dots\mu_p}$ with many components indexed by $p+q$ indices, which transforms as:*

$$M^{\nu_1\dots\nu_q}_{\mu_1\dots\mu_p} \to J^w \Lambda_{\mu_1}{}^{\mu_1'} \dots \Lambda_{\mu_p}{}^{\mu_p'} \Lambda^{\nu_1}{}_{\nu_1'} \dots \Lambda^{\nu_q}{}_{\nu_q'} M^{\nu_1'\dots\nu_q'}_{\mu_1'\dots\mu_p'} \tag{42}$$

under diffeomorphisms with Jacobian matrix $\Lambda_\mu{}^\nu$, with determinant J.

When we transform the 4-volume element $\mathrm{d}^4x' = J\,\mathrm{d}^4x$, we get a Jacobian: the element d^4x is a *tensor density of weight 1*.

Then we see that the volume element $\sqrt{-g}\,\mathrm{d}^4x$ is a *scalar*, since when multiplying tensor densities their weights are added.

3.2 Vectors in curved spacetime

They are not objects *in spacetime*: instead they belong to the *tangent space*.

For each point x in the manifold we define a basis at that point: $\left\{e^\mu_{(\alpha)}\right\}(x)$, where μ is a vector index while (α) denotes which vector we are considering.

Any vector $a^\mu(x)$ can be then decomposed as

$$a^\mu(x) = a^{(\alpha)}(x)e^\mu_{(\alpha)}(x)\,. \tag{43}$$

Since spacetime is not flat, the dependence on x is not trivial.

The scalar product between the vectors can be expressed with respect to the basis:

$$a(x)\cdot b(x) = a^{(\alpha)}b^{(\beta)}e_{(\alpha)}\cdot e_{(\beta)}\,, \tag{44}$$

and we can select our basis so that at every point it is orthonormal:

$$e_{(\alpha)}\cdot e_{(\beta)} = \eta_{(\alpha)(\beta)}\,. \tag{45}$$

We *cannot sum vectors in different tangent spaces*. But we want to: we need to define derivatives! To solve this issue, we will introduce the notion of *parallel transport*.

Fri Oct 18 2019

4 Covariant differentiation

We want to do derivatives, as in:

$$f(x + \mathrm{d}x) = f(x) + \frac{\partial f}{\partial x} \mathrm{d}x \,, \tag{46}$$

which allow us to "move around". In more dimensions, the rule will be

$$f(x + \mathrm{d}x) - f(x) = \mathrm{d}x^\mu \, \partial_\mu f \,. \tag{47}$$

Now, we want to prove that, in Minkowski spacetime, $\partial_\mu f$ is a rank (0,1) tensor if f is a function.

Under a change of variables $f \to f'$ we have

$$\frac{\partial f(x)}{\partial x^\mu} \to \frac{\partial f'(x')}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial}{\partial x^\alpha} f(x) \,, \tag{48}$$

which is the transformation law of a covariant vector, or (0,1) tensor.

However, for a vector $\mathrm{d}x^\nu \, \partial_\nu A^\mu$ is *not* a tensor!

Under a change of coordinates,

$$\begin{aligned}
\frac{\partial A^\mu(x)}{\partial x^\nu} &= \frac{\partial}{\partial x'^\nu} A^\mu(x') && \text{(49a)} \\
&= \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial x'^\mu}{\partial x^\beta} A^\beta(x) \right) && \text{(49b)} \\
&= \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} + \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial A^\beta}{\partial x^\alpha} && \text{(49c)}
\end{aligned}$$

, and we can see that the second term is the transformation we want, but the first term spoils the transformation.

This is not an issue in SR: there, the second derivative vanishes:

$$\frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} = \frac{\partial}{\partial x^\alpha} \Lambda^\mu_\beta = 0 \,, \tag{50}$$

since Lorentz matrices are constant.

So, we construct a *Covariant Derivative* which transforms as a tensor under diffeomorphisms.

We denote it as $\nabla_\nu A^\mu$. For any tensor T of arbitrary rank (p, q) we request $\nabla_\nu T$ to be a tensor of rank $(p, q+1)$. Also, we request $\nabla_\mu \to \partial_\mu$ for flat spacetime.

We define the Christoffel symbols:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} \right) , \tag{51}$$

where we introduced comma notation for partial non-covariant differentiation. They are symmetric in the two lower indices and they are not tensors. Their transformation law is:

$$\Gamma^{\mu}_{\nu\kappa} \to \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu} \partial x'^{\kappa}} , \tag{52}$$

which we note is *not tensorial*!

We define

$$\nabla_{\nu} V_{\mu} = \partial_{\nu} V_{\mu} - \Gamma^{\alpha}_{\nu\mu} V_{\alpha} , \tag{53}$$

and

$$\nabla_{\nu} V^{\mu} = \partial_{\nu} V_{\mu} + \Gamma^{\mu}_{\nu\alpha} V^{\alpha} . \tag{54}$$

How does it transform? For the covariant derivative, we have:

$$\begin{aligned}
\nabla'_{\nu} V'_{\kappa} &= \frac{\partial}{\partial x'^{\nu}} V'_{\kappa} - \Gamma'^{\mu}_{\nu\kappa} V'_{\kappa} && \text{(55a)} \\
&= \partial_{\nu} \left(\frac{\partial x^{\lambda}}{\partial x'^{\kappa}} V_{\lambda} \right) - \left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu} \partial x'^{\kappa}} \right) \left(\frac{\partial x^{\lambda}}{\partial x'^{\mu}} V_{\lambda} \right) && \text{(55b)} \\
&= \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\mu}} V_{\lambda} + \frac{\partial x^{\lambda}}{\partial x'^{\mu}} \frac{\partial V_{\lambda}}{\partial x'^{\nu}} - \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \left(\frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\kappa}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\kappa}} \right) \frac{\partial x^{\lambda}}{\partial x'^{\mu}} V_{\lambda} && \text{(55c)} \\
&= \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\mu}} V_{\lambda} + \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial V_{\lambda}}{\partial x^{\alpha}} - \left(\frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\kappa}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\kappa}} \right) \delta^{\lambda}_{\alpha} V_{\lambda} && \text{(55d)} \\
&= \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\kappa}} V_{\lambda} + \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial V_{\lambda}}{\partial x^{\alpha}} - \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\kappa}} \Gamma^{\alpha}_{\beta\gamma} V_{\alpha} - \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\kappa}} V_{\lambda} && \text{(55e)} \\
&= \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \left(\frac{\partial V_{\lambda}}{\partial x^{\alpha}} - \Gamma^{\sigma}_{\alpha\lambda} V_{\sigma} \right) && \text{(55f)} \\
&= \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \nabla_{\alpha} V_{\lambda} , && \text{(55g)}
\end{aligned}$$

where we used: relabeling of indices, contraction of the Jacobian matrix with its inverse, the chain rule, the product rule, the transformation law of the Christoffel symbols.

The derivative of a contravariant tensor is a tensor: this can be proven by noticing that $\nabla_\mu(A^\alpha B_\alpha) = \partial_\mu(A^\alpha B_\alpha) = B_\alpha \nabla_\mu A^\alpha + A^\alpha \nabla_\mu B_\alpha$.

$$\nabla'_\nu V'^\mu = \frac{\partial V'^\mu}{\partial x'^\nu} + \Gamma'^\mu_{\nu\kappa} V'^\kappa \tag{56a}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial}{\partial x^\lambda} \left(\frac{\partial x'^\mu}{\partial x^\alpha} V^\alpha \right) + \left(\frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\mu} \Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial^2 x'^\mu}{\partial x'^\nu \partial x'^\kappa} \right) \left(\frac{\partial x'^\kappa}{\partial x^\sigma} V^\sigma \right) \tag{56b}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha} V^\alpha + \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial V^\alpha}{\partial x^\lambda} + \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \delta^\gamma_\sigma \Gamma^\alpha_{\beta\gamma} V^\gamma + \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial^2 x'^\alpha}{\partial x^\nu \partial x^\mu} \frac{\partial x'^\kappa}{\partial x^\sigma} V^\sigma \tag{56c}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha} V^\alpha + \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \left(\frac{\partial V^\alpha}{\partial x^\lambda} + \Gamma^\alpha_{\lambda\gamma} V^\gamma \right) \tag{56d}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \nabla_\lambda V^\alpha + \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha} V^\alpha + \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\kappa} \frac{\partial x'^\kappa}{\partial x^\sigma} V^\sigma \,, \tag{56e}$$

and we would like to see that the two last terms cancel: is

$$\frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha} V^\alpha + \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\kappa} \frac{\partial x'^\kappa}{\partial x^\sigma} V^\sigma \stackrel{?}{=} 0 \,, \tag{57}$$

for all V^μ? Let us factor the vector, changing the indices:

$$\frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha} + \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\alpha \partial x'^\kappa} \frac{\partial x'^\kappa}{\partial x^\alpha} \stackrel{?}{=} 0 \,, \tag{58}$$

we can rewrite it as

$$\frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial}{\partial x^\alpha} \frac{\partial x'^\mu}{\partial x^\lambda} + \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial}{\partial x'^\kappa} \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \stackrel{?}{=} 0 \,, \tag{59}$$

which can be recombined into:

$$\frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial}{\partial x^\alpha} \frac{\partial x'^\mu}{\partial x^\lambda} + \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial}{\partial x^\alpha} \frac{\partial x^\lambda}{\partial x'^\nu} \stackrel{?}{=} 0 \,, \tag{60}$$

and becomes

$$\frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\lambda} \right) = \frac{\partial}{\partial x^\alpha} \delta^\mu_\nu = 0 \,. \tag{61}$$

For any order tensor, we add a Christoffel symbol for every index, such as in:

$$\nabla_\mu V_{\alpha\beta} = \partial_\mu V_{\alpha\beta} - \Gamma^\lambda_{\mu\alpha} V_{\lambda\beta} - \Gamma^\lambda_{\mu\beta} V_{\alpha\lambda} \,, \tag{62}$$

or

$$\nabla_\mu V^\beta_\alpha = \partial_\mu V^\beta_\alpha - \Gamma^\lambda_{\mu\alpha} V^\beta_\lambda + \Gamma^\beta_{\mu\lambda} V^\lambda_\alpha \,. \tag{63}$$

4.1 Properties of the covariant derivative

- The covariant derivative of a tensor is a tensor;
- the covariant derivative obeys the Leibniz rule: $\nabla_\mu (AB) = B\nabla_\mu A + A\nabla_\mu B$;
- the metric is covariantly constant: $\nabla_\mu g_{\alpha\beta} = 0$.

Notice that covariant derivatives do *not* commute!
We can check that $\partial_\mu (A^\alpha B_\alpha) = \nabla_\mu (A^\alpha B_\alpha)$. It is

$$\left(\partial_\mu A_\alpha - \Gamma^\lambda_{\mu\alpha} A_\lambda\right) B^\alpha + A_\alpha \left(\partial_\mu B^\alpha + \Gamma^\alpha_{\mu\lambda} B^\lambda\right) , \tag{64}$$

expanding and relabeling indices we get the desired cancellation. Now, for parallel transport:

4.2 Parallel transport

Take a curve $x^\alpha(\lambda)$ and a vector V^μ defined at a certain point along the curve. For infinitesimal displacement we will have $\mathrm{d}V = \mathrm{d}x \cdot \nabla V$: in components $\mathrm{d}V^\mu = \mathrm{d}x^\alpha \, \nabla_\alpha V^\mu$.

Parallel transport means that the vector does not change when it is transported: $\nabla_t V^\mu = 0$ where the index t indicates derivation along the curve's tangent vector $t^\alpha = \mathrm{d}x^\alpha / \mathrm{d}\lambda$.

Thu Oct 24 2019

Last time we looked at the derivative along a curve.

Now, we are going to talk about the

5 Einstein equations

They look like

$$\text{curvature} \propto \text{energy} , \tag{65}$$

we will make this more formal in this lecture.

We take a vector and parallel transport it along a closed path on a curved manifold, such as a sphere [see homework sheet #3]: the vector does *not* come back to its original position. In a flat manifold this is not the case. Do note that flat manifolds can *look* curved: an infinite cylinder's surface is flat.

Therefore we can *define* a "curved manifold" by:

$$\nabla_\mu \nabla_\nu V^\alpha \neq \nabla_\nu \nabla_\mu V^\alpha , \tag{66}$$

for at least some directions. To quantify this noncommutativity, let us then look at the commutator:

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu] V^\alpha &= \nabla_\mu \big(\nabla_\nu V^\alpha\big) - (\mu \leftrightarrow \nu) && \text{(67a)} \\
&= \partial_\mu \big(\nabla_\nu V^\alpha\big) \cancelto{0}{- \Gamma^\lambda_{\mu\nu} \big(\nabla_\lambda V^\alpha\big)} + \Gamma^\alpha_{\mu\lambda} \Big(\nabla_\nu V^\lambda\Big) - (\mu \leftrightarrow \nu) && \text{(67b)} \\
&= \partial_\mu \left(\cancelto{0}{\partial_\nu V^\alpha} + \Gamma^\alpha_{\nu\lambda} V^\lambda\right) + \Gamma^\alpha_{\mu\lambda} \Big(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\sigma} V^\sigma\Big) - (\mu \leftrightarrow \nu) && \text{(67c)} \\
&= \partial_\mu \Gamma^\alpha_{\nu\sigma} V^\sigma + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} V^\sigma - (\mu \leftrightarrow \nu) && \text{(67d)} \\
&= \Big(\partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}\Big) V^\sigma && \text{(67e)} \\
&\overset{\text{def}}{=} R^\alpha_{\mu\nu\sigma} V^\sigma \,. && \text{(67f)}
\end{aligned}
$$

We have cancelled many terms which are symmetric with respect to $(\mu \leftrightarrow \nu)$. The (3, 1) tensor we defined is called the *Riemann tensor*.

5.1 Local Inertial Frame

Claim 5.1. *For any spacetime endowed with a metric and for any point P in it we can choose a ("prime") coordinate system for which $g'_{\mu\nu}(x'_p) = \eta_{\mu\nu}$ and $\partial'_\lambda g'_{\mu\nu}(x'_p) = 0$.*

We cannot, however, set all the second coordinate derivatives to 0.

We want to compute the Riemann tensor in the LIF. In the LIF, the Christoffel symbols are all zero since they are linear combinations of the coordinate derivatives of the metric. The *derivatives* of the CS, however, are not zero!

$$
\partial_\sigma \left(\frac{1}{2} g^{\alpha\lambda} \Big(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\Big)\right) = \frac{g^{\alpha\lambda}}{2} \Big(g_{\lambda\mu,\nu\sigma} + g_{\lambda\nu,\mu\sigma} - g_{\mu\nu,\lambda\sigma}\Big) \,. \tag{68}
$$

Therefore, we have:

$$
\begin{aligned}
R^\alpha_{\sigma\mu\nu} &= \frac{g^{\alpha\lambda}}{2} \Big(g_{\lambda\nu,\sigma\mu} + \cancelto{0}{g_{\sigma\lambda,\mu\nu}} - g_{\sigma\nu,\lambda\mu}\Big) - (\mu \leftrightarrow \nu) && \text{(69a)} \\
&= \frac{g^{\alpha\lambda}}{2} \Big(g_{\lambda\nu,\sigma\mu} - g_{\lambda\mu,\sigma\nu} - g_{\sigma\nu,\lambda\mu} + g_{\sigma\mu,\lambda\nu}\Big) \,. && \text{(69b)}
\end{aligned}
$$

We lower an index of the Riemann tensor with the metric and get:

$$
R_{\gamma\sigma\mu\nu} = \frac{1}{2} \Big(g_{\gamma\nu,\sigma\mu} - g_{\gamma\mu,\sigma\nu} - g_{\sigma\nu,\gamma\mu} + g_{\sigma\mu,\gamma\nu}\Big) \,, \tag{70}
$$

which is a reasonably simple expression, however it is only true at a single point.

We can derive some symmetry properties:

$$
\begin{aligned}
R_{\gamma\sigma\mu\nu} &= -R_{\sigma\gamma\mu\nu} && \text{(71a)}\\
R_{\gamma\sigma\mu\nu} &= -R_{\gamma\sigma\nu\mu} && \text{(71b)}\\
R_{\gamma\sigma\mu\nu} &= R_{\mu\nu\gamma\sigma} && \text{(71c)}\\
R_{\gamma\sigma\mu\nu} + R_{\gamma\mu\nu\sigma} + R_{\gamma\nu\sigma\mu} &= 0\,, && \text{(71d)}
\end{aligned}
$$

these can be checked in the LIF, and since they are tensorial expressions they will hold in any frame.

Definition 5.1 (Ricci tensor and scalar)**.** *The Ricci tensor is the trace of the Riemann tensor:*

$$
R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}\,, \tag{72}
$$

while the Ricci scalar, or scalar curvature, is the trace of the Ricci tensor:

$$
R = g^{\mu\nu} R_{\mu\nu}\,. \tag{73}
$$

5.2 The stress-energy tensor

All of the objects we defined are defined locally in the tangent bundle of the manifold.

To discuss energy, we also need a local object: an energy *density*.

This will not be a scalar: it is frame dependent, since it is energy over volume, but volume changes if we change frame.

The number of particles N is a scalar (not a scalar field!). The number density is *not* a scalar field: a moving observer with velocity v will se the volume as being *smaller*. The average number density as measured in LIF will be

$$
n_* = \frac{N}{\mathrm{Vol}_*}\,, \tag{74}
$$

where Vol_* is the spatial volume of the box in its own rest frame. For another observer moving at $v = 1 - 1/\gamma^2$, we will have

$$
n = \frac{N}{\mathrm{Vol}} = \gamma n_* \geq n_*\,, \tag{75}
$$

since $\mathrm{Vol} = \mathrm{Vol}_*/\gamma$.

The 4-velocity of the box for the observer is $u^\mu = (\gamma, \gamma\vec{v})$. Therefore, if we define $n^\alpha = n_* u^\alpha$ we will have $n = n^0$ for any observer. It is not a scalar because it cannot be, but the whole density vector transforms as a proper vector.[]

What are the spatial components of this vector? n^i is called the *number current density*.

We imagine an area which is fixed with respect to the moving observer. How many particles cross the area $\mathrm{d}A$ in a time $\mathrm{d}t$, given that locally near the area the 3-velocity of the particles is $\vec{v}$?

It will be the density times the volume:

$$\frac{n_*}{\sqrt{1-v^2}}\,\mathrm{d}t\,\vec{v}\cdot\mathrm{d}\vec{A} = n^i\,\mathrm{d}A_i\,. \tag{76}$$

We have a scalar product because the area can be at an angle with respect to the velocity, and we need to compute the flux.

As an example, the electromagnetic current for electrons with number density n^μ is simply $j^\mu = -en^\mu$.

Now, we will discuss the *net flux* in or out of a certain region. If we imagine a certain region, the net flux will equal the variation of the particle number in the region: this gives us a conservation equation

$$\int_{\partial V}\mathrm{d}\vec{A}\cdot\vec{n} + \partial_t\int_V \mathrm{d}^3x\,n = \int_V \mathrm{d}^3x\,(\nabla\cdot\vec{n} + \partial_t n) = 0\,, \tag{77}$$

where V is our volume, and its boundary is ∂V. If T is the time for which we consider the problem, this can be restated by integrating over time as well:

$$\int_{V\times T}\partial_\mu n^\mu\,\mathrm{d}^4x = 0\,. \tag{78}$$

We used the divergence theorem, which states that the flux going out of the boundary of a volume is equal to the integral of the divergence over the volume (for any vector field).

This holds for any volume and for any time: therefore the integrand must be identically null, $\partial_\mu n^\mu \equiv 0$.

Fri Oct 25 2019

We define $\Delta N = N^\mu n_\mu \Delta V$, where N^μ is the current density from last time while n^μ is the normal to the surface of a 3-volume.

Now, we define the energy-momentum-stress tensor $T^{\mu\nu}$.

We define it by $\Delta p^\alpha = T^{\alpha\beta}n_\beta\Delta V$, where Δp^μ is the momentum in the volume.

Let us consider an inertial frame in which ΔV is at rest, and take n_μ to be its 4-velocity.

Then the energy density is given by:

$$\epsilon = \frac{\Delta p^0}{\Delta V} = T^{00}\,, \tag{79}$$

while the momentum density is:

$$\Pi^i = \frac{\Delta p^i}{\Delta V} = T^{i0} . \tag{80}$$

Now let us consider a frame moving with velocity $\vec{v}$, in this frame each particle has energy $m\gamma$ and momentum $m\gamma\vec{v}$, and the energy density is $n_*\gamma$.

The moving observer sees exacly the energy density $\epsilon = m\gamma n_* \gamma = T^{00}$, and the momentum density $T^{i0} = m\gamma \vec{v}^i n_* \gamma$.

In general, for particles which do not interact with each other, we have

$$T^{\alpha\beta} = n_* m u^\alpha u^\beta . \tag{81}$$

The stress energy tensor is in general symmetric.

What is T^{0i}? We have $\Delta p^\alpha = T^{\alpha 1}\Delta y \Delta z \Delta t$ (if we select the normal n_μ parallel to the x axis).

For $\alpha = 0$, this is the flux of energy in time (power) along the x direction.

For $\alpha = i$, we can use the same relation:

$$T^{i1} = \frac{\Delta p^i}{\Delta y \Delta z \Delta t} = \frac{F^i}{\Delta y \Delta z} = \frac{F^i}{\text{Area}} . \tag{82}$$

Do note that i can be either x, y or z: we can consider the *pressure*, which is the force along the x axis across the x axis, but also the *deviatoric stresses* along the y or z axes but across the x axis.

Do also note that the stress tensor from fluid dynamics, σ^{ij}, is not equal to T^{ij}: it is its opposite.

The energy density measured by an observer with 4-velocity u^μ is $T_{\mu\nu}u^\mu u^\nu$.

Claim 5.2. *Energy momentum is conserved.*

Proof. In the LIF $\partial_\beta T^{\alpha\beta} = 0$ (in perfect analogy to the current number density) therefore in any frame $\nabla_\beta T^{\alpha\beta} = 0$. □

Is this true even in the absence of translational symmetry?

Definition 5.2 (Perfect fluid)**.** *A fluid is perfect if it has no dissipative effects: heat conduction, viscosity.*

Claim 5.3. *The stress-energy tensor of a perfect fluid is*

$$T^{\alpha\beta} = diag(\rho, p, p, p) . \tag{83}$$

in its own rest frame; in any frame $T^{\alpha\beta} = \rho u^\alpha u^\beta + P h^{\alpha\beta}$*, where* $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$*.*

Definition 5.3 (Equation of state)**.** *It is* $P = \omega\rho$*.*

5.3 The Einstein equations

We require them to be *mathematically consistent* and *physically correct*.

We have the principle of covariance: this tells us that the law should be a tensorial expression.

We know that energy is described by the tensor $T_{\mu\nu}$, while for the curvature part we have the various contractions of the metric and the Riemann tensor.

Our ansatz is

$$R_{\mu\nu} + c_1 R g_{\mu\nu} = c_2 T_{\mu\nu} \,. \tag{84}$$

Symmetry is all right: $R_{\mu\nu}$ is symmetric, and so is the metric. The stress energy tensor is also conserved: therefore we impose

$$\nabla^{\mu} \Big(R_{\mu\nu} + c_1 R g_{\mu\nu} \Big) = 0 \,. \tag{85}$$

By the contracted Bianchi identies, this implies $c_1 = -1/2$.

To get c_2, we can look at the Newtonian limit: they should simplify to $\Box\phi \propto \rho$. This will imply $c_2 = 8\pi G$.

5.4 The Newtonian limit

We start by tracing the equations: we get $R - 2R = c_2 T$, where $g^{\mu\nu} T_{\mu\nu} = T$. Note that $g^{\mu\nu} g_{\mu\nu} = 4$: it is not the trace of the metric, but its Frobenius norm.

We can then rewrite the EFE as

$$R_{\mu\nu} = c_2 \bigg(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \bigg) \,. \tag{86}$$

Let us consider a slowly moving perfect fluid: $p \ll \rho$, and $v \approx 0$, so $u^{\mu} = (1, \vec{0})$.

We consider a weak gravitational field: $g_{\mu\nu} \approx \eta_{\mu\nu}$.

Our stress energy tensor is approximately $\rho u^{\mu} u^{\nu}$. We have $T = -\rho$.

Then, we will get:

$$R_{00} \approx c_2 \bigg(\rho - \frac{\rho}{2} (-)^2 \bigg) \,. \tag{87}$$

Therefore we get $R_{00} \approx c_2 \rho / 2$.

Thu Nov 07 2019

We are trying to fix the last coefficient in the Einstein equations:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = c_2 T_{\mu\nu} \,. \tag{88}$$

The 1/2 on the LHS is fixed by the conservation of the stress-energy tensor.

We look at a low-energy scenario: $\rho \ll P$, $u^\mu = (1, \vec{0})^\top$. Then the EFE become $R_{00} = c_2 \rho / 2$.

We consider a stationary metric $g_{\mu\nu}(\vec{x})$, which does not depend on time. This is consistent with the stuff we usually see: planetary dynamics are quasi-static with respect to the speed of light.

In a LIF we have shown that the Riemann tensor is:

$$R_{\gamma\sigma\mu\nu} = \frac{1}{2}\Big(g_{\gamma\nu,\sigma\mu} - g_{\sigma\nu,\gamma\mu} - g_{\gamma\mu,\sigma\nu} + g_{\sigma\mu,\gamma\nu} \Big) . \tag{89}$$

To get the Ricci tensor we need $R^\alpha_{\sigma\mu\nu} = \eta^{\alpha\gamma} R_{\gamma\sigma\mu\nu}$ since we are in the LIF.

We get the Ricci tensor:

$$R_{\sigma\nu} = \eta^{\alpha\gamma}\left(g_{\gamma\nu,\sigma\alpha} - g_{\sigma\nu,\gamma\alpha} - g_{\gamma\alpha,\sigma\nu} + g_{\sigma\alpha,\gamma\nu} \right) , \tag{90}$$

from which we can calculate R_{00}:

$$R_{00} = \frac{1}{2}\eta^{\alpha\gamma}\left(g_{\gamma 0,0\alpha} - g_{00,\gamma\alpha} - g_{\gamma\alpha,00} + g_{0\alpha,\gamma 0} \right) \tag{91}$$

$$= -\frac{1}{2}\eta^{ij} g_{00,ij} = -\frac{1}{2}\sum_i g_{00,ii} = -\frac{1}{2}\nabla^2 g_{00} , \tag{92}$$

where only the second term survives since the metric is time-independent, its time derivatives all vanish.

Therefore, our equation becomes $\nabla^2 g_{00} = c_2 \rho$. We just need to find out what the meaning of g_{00} is, how it is related to the gravitational potential. We do it with gravitational redshift: we found that

$$\Delta\tau_A \approx \Delta\tau_B (1 - \Phi_A + \Phi_B) , \tag{93}$$

if Alice, on the top of a building, is sending photons to Bob who is on the ground. To first order in the field, the expression is equivalent to

$$\Delta\tau_A \approx \Delta\tau_B (1 - \Phi_A)(1 + \Phi_B) \approx \Delta\tau_B \frac{1 + \Phi_B}{1 + \Phi_A} , \tag{94}$$

which is equivalent to the constancy of

$$\frac{\Delta\tau}{1 + \Phi} . \tag{95}$$

The $\Delta\tau$ are proper times measured by the observers A and B at rest. For an observer at rest, the spacetime interval is

$$-\,\mathrm{d}\tau^2 = \mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 , \tag{96}$$

since the $\mathrm{d}x^i$ are null. Therefore, $\mathrm{d}\tau = \sqrt{-g_{00}}\,\mathrm{d}t$.

We have

$$\Delta\tau_A = \sqrt{-g_{00}(A)}\Delta t \qquad \text{and} \qquad \Delta\tau_B = \sqrt{-g_{00}(B)}\Delta t\,, \tag{97}$$

but the Δt are the same, because the metric is constant with respect to time! so we get that $\Delta t = \text{const}$ and specifically it is equal to

$$\frac{\Delta\tau}{\sqrt{-g_{00}}}\,, \tag{98}$$

so we can finally identify $\sqrt{-g_{00}} \approx 1 + \Phi$, or $g_{00} = -(1+\Phi)^2 = -(1+2\Phi)$.

We can now bring the Laplacian inside the g_{00}: we get

$$\nabla^2\Phi = \frac{c_2}{2}\rho\,, \tag{99}$$

Poisson's equation for the gravitational potential.

We know the gravitational potential to be defined by $-\nabla\Phi = \vec{F}_G$.

We can find Gauss' law for the gravitational field just like we did for the electromagnetic field, substituting $Q \to M$ and $1/(4\pi\epsilon_0) \to -G$. I will also denote the electric and gravitational charges with subscripts E and G.

The integral form of this equation is

$$\int_{\partial V} \mathrm{d}\vec{A}\cdot\vec{E} = \int_V \mathrm{d}^3x\,\rho_E\,, \tag{100}$$

but it can be expressed differentially with the divergence theorem as

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_E}{\epsilon_0}\,. \tag{101}$$

For the gravitational field $\vec{F}_G$ then we can just substitute:

$$\vec{\nabla}\cdot\vec{F}_G = -4\pi G\rho_G\,, \tag{102}$$

and then we can substutite the gravitational potential:

$$\vec{\nabla}\cdot(-\vec{\nabla}\Phi) = -\nabla^2\Phi = -4\pi G\rho_G\,, \tag{103}$$

or $\nabla^2\Phi = 4\pi G\rho_G$. So this is what we found before, with $c_2/2 = 4\pi G$. So, our constant is $c_2 = 8\pi G$. So we get Einstein's equations:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{104}$$

The LHS of these is called the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - {}^{R}\!/_{2}g_{\mu\nu}$.

Why do we not write an equation with more derivatives, more indices? We may, but we'd detect these only in the regime of very strong curvature: the relativistic effects are already hard to detect as is! For now the vanilla EFE have always agreed with experiment.

6 Geodesics

In flat spacetime, we know that the curve with $\mathrm{d}^2 x^\mu \Big/ \mathrm{d}\tau^2 = 0$ stationarizes $\tau = \int \mathrm{d}\tau$.

Now we will do the exact same thing, except that $\mathrm{d}\tau$ will be calculated with $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$.

We perturb our path $x^\mu(\lambda)$ as $x^\mu + \delta x^\mu$, and we want to set to zero the first functional derivative $\delta\tau_{AB} / \delta x^\mu$. This means that to first order

$$\delta \left(\int_0^1 \mathrm{d}\sigma \sqrt{-g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}} \right) = 0\,, \tag{105}$$

where we performed a reparametrization of the curve. Computing away, with the notation $u^\mu = \mathrm{d}x^\mu / \mathrm{d}\sigma$:

$$\begin{aligned}
\delta\tau_{AB} &= \int_0^1 \mathrm{d}\sigma \left(\frac{-\delta g_{\alpha\beta} u^\alpha u^\beta}{2\sqrt{\dots}} + \frac{-g_{\alpha\beta} u^\alpha \frac{\mathrm{d}\delta x^\beta}{\mathrm{d}\sigma}}{\sqrt{\dots}} \right) && \text{(106a)} \\
&= -\frac{1}{2} \int_0^1 \mathrm{d}\sigma \left(\delta g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + 2 g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}\delta x^\beta}{\mathrm{d}\sigma} \right) && \text{(106b)} \\
&= -\frac{1}{2} \int_0^1 \mathrm{d}\sigma \left(\partial_\gamma g_{\alpha\beta} \delta x^\gamma \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} - 2 \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \right) \delta x^\beta \right) && \text{(106c)} \\
&= -\frac{1}{2} \int_0^1 \mathrm{d}\tau \left(\partial_\gamma g_{\alpha\beta} \delta x^\gamma \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau} \left(g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \delta x^\beta \right) \right) && \text{(106d)} \\
& \int \mathrm{d}\tau \left(-\frac{1}{2} \partial_\gamma g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau} \left(g_{\alpha\gamma} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \right) \delta x^\gamma \right), && \text{(106e)}
\end{aligned}$$

where we applied the product rule, identified two symmetric terms in the velocity, used the identity

$$\frac{1}{\sqrt{\dots}} \frac{\mathrm{d}}{\mathrm{d}\sigma} = \frac{\mathrm{d}}{\mathrm{d}\tau}\,, \tag{107}$$

expanded the metric using:

$$\delta g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} \delta x^\gamma\,, \tag{108}$$

where we did not need to introduce a covariant derivative since we are just Taylor expanding.

Also, we integrated by parts (without boundary terms since the path variation vanishes at the path boundary), changed variables, and finally gotten an expression which must vanish for any path, therefore we get that

$$-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}+\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\gamma}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\right)=0\,, \tag{109}$$

so we can expand the derivative: we get

$$0=-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}+\partial_\beta g_{\alpha\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}+g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}\,, \tag{110}$$

where the second term can be symmetrized in $\alpha\beta$:

$$0=g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}+\left(\frac{1}{2}\partial_\beta g_{\alpha\gamma}+\frac{1}{2}\partial_\alpha g_{\beta\gamma}\right)-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\,, \tag{111}$$

so we get

$$0=g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}+\frac{1}{2}\Big(g_{\alpha\gamma,\beta}+g_{\beta\gamma,\alpha}-g_{\alpha\beta,\gamma}\Big)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\,, \tag{112}$$

which, raising an index and identifying the Christoffel symbols, gives us

$$0=\frac{\mathrm{d}^2x^\gamma}{\mathrm{d}\tau^2}+\Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\,, \tag{113}$$

the *geodesic equation*.

It can be written also as $u^\mu\nabla_\mu u^\nu=a^\nu=0$.

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It can be shown that the equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}=0\,, \tag{114}$$

in the low-field limit gives the regular acceleration in a gravitational field.

If we define u^μ as $\mathrm{d}x^\mu/\mathrm{d}\tau$, we get that the equation is equivalent to

$$u^\alpha\left(\frac{\partial u^\mu}{\partial x^\alpha}+\Gamma^\mu_{\alpha\beta}u^\beta\right)=u^\alpha\nabla_\alpha u^\mu=a^\mu=0\,. \tag{115}$$

The acceleration we feel corresponds to the difference between our motion and geodesic motion.

The four-velocity has constant square modulus: either 0, or ±1, if we choose an appropriate parametrization. Therefore we can classify geodesics.

Timelike geodesics have $u^\mu u_\mu = -1$, and are related to the motion of a particle. They minimize the proper time $\mathrm{d}\tau = \sqrt{-\,\mathrm{d}s^2}$, which is real in this case. We parametrize them by τ.

Spacelike geodesics have $u^\mu u_\mu = +1$ and can be seen as the shortest path between two points: for them, the integral of $\mathrm{d}s$ is stationary. We parametrize them by s.

Null geodesics have $u^\mu u_\mu = 0$ are characterized by $\mathrm{d}s = 0$. We parametrize them with some parameter of our choosing, λ, which must be independent of proper time or space.

6.1 Solutions of the geodesic equation

We treat the problem in the case of a two-dimensional Euclidean plane, using polar coordinates: we know we should find straight lines, but in these coordinates the problem is nontrivial.

The metric is $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$, and the nonzero Christoffel symbols are (see exercise 3.3):

$$\Gamma^r_{\theta\theta} = -r \qquad \text{and} \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}\,. \tag{116}$$

Our equation for the r coordinate is then:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = 0\,, \tag{117}$$

while for the θ coordinate by symmetry we can identify the two terms:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0\,. \tag{118}$$

In general, in an n dimensional space, motion is defined by n scalar functions, which can be determined by our n differential equations with $2n$ initial conditions.

It is in general useful to find *first integrals*, quantities which are constant along the geodesic.

It can be shown that the second equation can be written as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\right) = 0\,, \tag{119}$$

which gives us the first integral $A = r^2\,\mathrm{d}\theta/\mathrm{d}s$. We can always also use the definition of the differential:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\ , \tag{120}$$

so we can insert our integral:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \frac{A^2}{r^4}\,\mathrm{d}s^2\ , \tag{121}$$

so

$$\mathrm{d}s^2 \left(1 - \frac{A^2}{r^2}\right) = \mathrm{d}r^2\ . \tag{122}$$

This has two solutions, but it can be shown that they give the same solution.

We want the trajectory: the locus of the points the geodesic passes through, $r(\theta)$ or $\theta(r)$.

We do:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{A}{r^2}\frac{1}{\sqrt{1 - \frac{A^2}{r^2}}}\,, \tag{123}$$

so we can integrate this:

$$\theta = \int \mathrm{d}\theta = \int \frac{A}{r^2}\left(1 - \frac{A^2}{r^2}\right)^{-1/2} \mathrm{d}r\ , \tag{124}$$

which comes out to be $\Delta\theta = \arccos(A/r)$: $r\cos(\Delta\theta) = A$: this is equivalent to

$$r\cos(\theta)\cos(\theta_0) + r\sin(\theta)\sin(\theta_0) = A\,, \tag{125}$$

therefore this can be written as $y = \alpha x + \beta$.

7 Euler-Lagrange equations

The time interval can be written as

$$\tau_{AB} = \int \mathrm{d}\tau = \int \sqrt{-\,\mathrm{d}s^2} \tag{126}$$

$$= \int \mathrm{d}\sigma\, \mathscr{L}\left(x^\alpha, \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\right), \tag{127}$$

so under a perturbation we get:

$$0 = \delta\tau \tag{128}$$

$$= \int \mathrm{d}\sigma \left(\frac{\partial \mathscr{L}}{\partial x^\alpha}\delta x^\alpha + \frac{\partial \mathscr{L}}{\partial \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}}\frac{\mathrm{d}\delta x^\alpha}{\mathrm{d}\sigma}\right) \tag{129}$$

$$= \int \mathrm{d}\sigma \left(\frac{\partial \mathscr{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathscr{L}}{\partial \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}}\right)\right)\delta x^\alpha = 0\,, \tag{130}$$

therefore the integrand must vanish identically: this gives us the Euler-Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma} \frac{\partial \mathscr{L}}{\partial \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}} = 0 \tag{131}$$

7.1 Killing vectors

Symmetries of the metric correspond to conserved quantities if our Lagrangian only depends on the metric. If the metric does not depend on a coordinate, then the unit vector in that direction is a Killing vector field.

If we have a Killing vector field, then the momentum along the Killing vector is conserved.

If the Killing coordinate is x^1, it can be expressed as

$$\frac{\partial \mathscr{L}}{\partial \frac{\mathrm{d}x^1}{\mathrm{d}\sigma}} = \frac{1}{2\mathscr{L}} \left(-2 g_{1\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma} \right) = -g_{1\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \,, \tag{132}$$

where we performed a change of variable from the derivation with respect to σ to one with respect to τ.

This is usually written as $\xi^\mu u_\mu = \text{const}$, which is actually more general. We can write it with respect to the momentum: $p^\mu \xi_\mu$ since we are considering a constant-mass particle.

We can apply this to our 2D example: the metric does not depend on θ, therefore $\xi = (0, 1)$ is a Killing vector, so $g_{\theta\mu} u^\mu = r^2 \, \mathrm{d}\theta / \mathrm{d}s = \text{const}$.

Thu Nov 14 2019

7.2 Riemann normal coordiates

We want to actually build a LIF, in which $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $g_{\mu\nu,\rho}(P) = 0$.

We can choose a set of vectors $e_{(\mu)}(P)$ which are orthonormal: $e_{(\mu)} \cdot e_{(\nu)} = \eta_{(\mu)(\nu)}$.

If we choose this *tetrad* as a basis, then the metric becomes the Minkowski one at that point.

Take a vector n^μ at P, and consider all possible geodesics which start from P with initial tangent vector n^μ.

Then, the coordinates of a point Q we get by moving for a time τ along this geodesic are $x^\alpha = \tau n^\alpha$ if n^α is timelike, $x^\alpha = s n^\alpha$ if it is spacelike, where

$$\tau = \int_P^Q \mathrm{d}\tau \,. \tag{133}$$

Since these lines are geodesics, they satisfy the geodesic equation:

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0\,, \tag{134}$$

but we can insert the relation $x^\alpha = \tau n^\alpha$ here, to get $\Gamma^\alpha_{\beta\gamma} n^\beta n^\gamma = 0$: but this can be written for any n^α, therefore we immediately get that all the Christoffel symbols vanish: $\Gamma^\alpha_{\beta\gamma} \equiv 0$.

Therefore, we have 4^3 equations of sums of derivatives of the metric which vanish: but the gradient of the metric only has 4^3 independent components, therefore the solution $g_{\mu\nu,\alpha} \equiv 0$ is the only one, as long as $\Gamma^\alpha_{\beta\gamma}(g_{\alpha\beta,\gamma})$ is an invertible system, which is nontrivial to show. This invertibility is equivalent to the system $2\Gamma_{\alpha\beta,\gamma}(g_{\alpha\beta,\gamma})$ being invertible. The following code shows this:

```python
import numpy as np

# linear transformation between
# metric derivatives and Christoffel symbols
M = np.zeros((4**3, 4**3))

# three indices run from 0 to 3:
# we incorporate them into one
# from 0 to 4**3-1
d = lambda i,j,k: 4**2*i+4*j+k

# we populate the matrix with the relevant coefficients,
# starting from the formula for the metric
# in terms of the Christoffel symbols
for i in range(4):
    for j in range(4):
        for k in range(4):
            M[d(i, j, k), d(i, j ,k)] += 1
            M[d(i, j, k), d(i, k, j)] += 1
            M[d(i, j, k), d(k, j, i)] += -1

print(np.linalg.det(M))
```

Can the geodesics cross each other far from the starting point? *Yes*, so the Riemann normal coordinates are only defined in a neighbourhood.

Let us consider an example: Riemann normal coordinates around the north pole of a sphere.

We can map every point on the sphere but the south pole by specifying the meridian and the distance to travel along the meridian.

So in our case, if R is the radius of the sphere, we get $x^\alpha = (R\theta \cos(\phi), R\theta \sin(\phi))$, where then $R\theta = \tau$ and $n^\alpha = (\cos(\theta), \sin(\theta))$. The angles ϕ and θ are the usual spherical coordinate angles: ϕ specifies the meridian, while $R\theta$ gives us the distance travelled away from the north pole (since θ is in radians).

The second order expansion of the metric is

$$g_{ij} = \begin{bmatrix} 1 - \frac{2y^2}{3R^2} & \frac{2xy}{3R^2} \\ \frac{2xy}{3R^2} & 1 - \frac{2x^2}{3R^2} \end{bmatrix}, \tag{135}$$

where $x^\alpha = (x, y)$.

We can see that the metric is δ_{ij} at the north pole, and its derivatives are $g_{ij,k} = 0$ there.

8 The Schwarzschild solution

It descrives the geometry outside a stationary, spherically symmetric object which is not rotating and not electrically charged, such as a star, planet or BH.

In general $\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + C(r)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi\right)$ is our line element.

We can define $\widetilde{r} = C(r)$, and then express A, B with respect to to this, and recalling

$$\mathrm{d}r^2 = \frac{\mathrm{d}\widetilde{r}^2}{(\mathrm{d}C/\mathrm{d}r)^2}, \tag{136}$$

which is what multiplies B, so we redefine $B(\widetilde{r})$ as $B(\widetilde{r})/(\mathrm{d}C/\mathrm{d}r)^2$.

So, we can just relabel B as this: the expression

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi\right) \tag{137}$$

is fully general. So far we have used the hypothesis of stationarity by writing everything only as a function of r.

Recall the inverted Einstein equations:

$$\frac{1}{8\pi G}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right) = R_{\mu\nu}\,. \tag{138}$$

We want to solve these outside of our source: we look for *vacuum solutions*. Then the equations are just $R_{\mu\nu} = 0$. A trivial solution to these is $g_{\mu\nu} = \eta_{\mu\nu}$, but we will show that it is not the only one! As a matter of fact, we know that the solution to a differential equation is determined by the boundary conditions: in our case, the mass of the object which sits at the origin. The Minkowski metric respects these boundary conditions if $M = 0$. In general, it does not.

A homework exercise will be to show that, denoting $\mathrm{d}/\mathrm{d}r$ with a prime:

$$R_{00} = \frac{A''}{2B} - \frac{A'B'}{4B^2} - \frac{A'^2}{4AB} + \frac{A'}{rB} \tag{139a}$$
$$R_{11} = -\frac{A''}{2A} + \frac{A'B'}{4B^2} - \frac{A'^2}{4A^2} + \frac{B'}{rB} \tag{139b}$$
$$R_{22} = 1 - \frac{1}{B} - \frac{rA'}{2AB} + \frac{rB'}{2B^2} \tag{139c}$$
$$R_{33} = \sin^2\theta R_{22} \tag{139d}$$
$$R_{ij} = 0 \qquad \text{if } i \neq j\,. \tag{139e}$$

If we compute $BR_{00} + AR_{11}$ we get many simplifications:

$$0 = A'B + AB' = (AB)'\,, \tag{140}$$

therefore AB is a constant with respect to r.

Also, we can write $A'/A = -B'/B$: we substitute it into R_{22}: we get

$$0 = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{B'}{B}\right) + \frac{rB'}{2B^2}\,, \tag{141}$$

so

$$0 = 1 - \frac{1}{B} + \frac{rB'}{2B^2}\,, \tag{142}$$

which is first order in terms of B. We can solve it by separating the variables.

We get

$$\frac{\mathrm{d}r}{r} = \frac{\mathrm{d}B}{B(1-B)}\,, \tag{143}$$

from which we find that, if at r_* the variable $B = B_*$, then

$$\int_{r_*}^{r} \frac{\mathrm{d}r}{r} = \int_{B_*}^{B} \frac{\mathrm{d}B}{B(1-B)}\,, \tag{144}$$

where we can rewrite the term

$$\frac{1}{B(1-B)} = \frac{1}{B} + \frac{1}{1-B}\,. \tag{145}$$

So,

$$\log\frac{r}{r_*} = \log\frac{B}{B_*} - \log\frac{1-B}{1-B_*}\,, \tag{146}$$

or, exponentiating:

$$\frac{r}{r_*} = \frac{B}{1-B}\frac{1-B_*}{B}\,, \tag{147}$$

from which we can find $B(r)$: first we write

$$\frac{1-B}{B} = \frac{1-B_*}{B}\frac{r_*}{r} = \frac{\gamma}{r}\,, \tag{148}$$

where we collected the integration constants into γ.

So,

$$\frac{1}{B} = \frac{\gamma}{r} + 1 \implies B = \left(1 + \frac{\gamma}{r}\right)^{-1}\,. \tag{149}$$

and because of $AB = k$ we also get

$$A = k\left(1 - \frac{\gamma}{r}\right)\,. \tag{150}$$

We are almost at the full solution: by continuity with the large-r limit, for which we have the Minkowski metric with $A = B = 1$, we have that $k = 1$. Now we have

$$\mathrm{d}s^2 = -\left(1 + \frac{\gamma}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{\gamma}{r}\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,, \tag{151}$$

and γ can only be found by continuity with the weak-field approximation: else, every value for it solves the equations. We know that $g_{00} = -(1 + 2\Phi)$. In the weak-field limit we have $\Phi = -GM/r$, so we identify $\gamma = -2GM$.

8.1 Gravitational redshift

Now we can do the exact calculation for the gravitational redshift: Alice, at r_A, sends photons to Bob at r_B. The motion of the photons need not be radial. Alice and Bob are not following geodesics, we consider them to be stationary in this metric.

We know the metric to be independent of time: $\xi^\mu = (1, \vec{0})$ is a Killing vector field. Light has momentum p^μ and moves along a geodesic. We will use the relation $p^\mu \xi_\mu = \text{const}$ to solve our problem.

Fri Nov 15 2019

We want to study the behaviour of light and particles in a Schwarzschild background, for $r > 2GM$. The metric is:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{152}$$

In general, the motion of light is nonradial. The metric is independent of time: our Killing vector field is $\xi^\alpha = (1, \vec{0})$, so $p \cdot \xi = \text{const}$.

From QM we know that $f_A = E_A/h$, where E_A is the energy measured by A.

The 4-velocity of A has spatial components equal to zero, since A does not move. Its norm must be $u^\mu u_\mu = -1$, therefore $u^\mu_A = (1/\sqrt{1-2GM/r_A}, \vec{0})$, or $u^\alpha_A = (1, \vec{0})/\sqrt{1-2GM/r_A} = \xi^\alpha/\sqrt{1-2GM/r_A}$.

For Bob, we have the exact same thing, except $A \to B$.

So, for $i = A, B$:

$$f_i = -\left(1 - \frac{2GM}{r_A}\right)^{-1/2} \frac{p_{\text{orbit}} \cdot \xi}{h}\,, \tag{153}$$

but the last part is exactly the same since the light moves along a geodesic: so their ratio is

$$\frac{f_B}{f_A} = \sqrt{\frac{1-2GM/r_A}{1-2GM/r_B}}\,, \tag{154}$$

or

$$f_{\text{obs}} = f_{\text{emit}} \frac{\sqrt{1-\frac{2GM}{r_{\text{emit}}}}}{\sqrt{1-\frac{2GM}{r_{\text{obs}}}}}\,. \tag{155}$$

Let us consider the nonrelativistic approximation: $r_A = R + h$, while $r_B = R$, where $R = R_{\text{earth}}$. If we Taylor expand (with $2GM \ll R$), we get:

$$f_{\text{obs}} = f_{\text{emit}}\left(1 - \frac{GM}{r_{\text{emit}}} + \frac{GM}{r_{\text{obs}}}\right)\,, \tag{156}$$

and then we expand in h/R: we get

$$f_{\text{obs}} = f_{\text{emit}}\left(1 - \frac{GM}{R}\left(1 - \frac{h}{R}\right) + \frac{GM}{R}\right) = f_{\text{emit}}\left(1 + \frac{GM}{R^2}h\right) = f_{\text{emit}}(1+gh)\,, \tag{157}$$

if we want more precision then we can keep more orders.

Kepler's laws are *wrong*! We will show this.

In classical circular orbits, for a planet with mass $m = 1$, we have the gravitational force GM/r^2 equalling v^2/r, or with respect to the angular momentum $l = vr$:

$$\frac{GM}{r^2} = \frac{l^2}{r^3}\,, \tag{158}$$

so we get $r = l^2/GM$. The energy is given by

$$\frac{v^2}{2} - \frac{GM}{r} = E\,, \tag{159}$$

and if we want to write these with respect to the velocity vector in polar coordinates, $v_r = \mathrm{d}v/\mathrm{d}t$ and $v_\theta = r\,\mathrm{d}\theta/\mathrm{d}t$ we have

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2, \tag{160}$$

while the angular momentum in general is $\vec{L} = \vec{r} \times \vec{v}$, whose modulus is $l = \left|\vec{L}\right| = rv_\theta = r^2\,\mathrm{d}\theta/\mathrm{d}t$. Therefore, $v_\theta = l/r$. So, the velocity is

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{l^2}{r^2}\,. \tag{161}$$

Then, the equation for the radial motion of the planet is

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} = E\,, \tag{162}$$

which for large r tends to 0 from below, while for small r tends to $+\infty$.

The circular orbit is the one which corresponds to the bottom of the potential.

Planets *do not* actually orbit in true ellipses, but this is not actually the case even in Newtonian mechanics, since there are other objects in the universe. The orbit of Mercury was expected to precede by $532''$ every 100 yr, but people observed an additional $43''$ every 100 yr. We will compute this.

For semplicity, we will say that in our spherical coordinates Mercury will always have $\theta = \pi/2$. The 4-velocity of the planet will be

$$u^\alpha = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}r}{\mathrm{d}\tau}, 0, \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right), \tag{163}$$

and we have two immediate Killing vectors: $\xi_t^\alpha = (1, \vec{0})$ and $\xi_\varphi^\alpha = (\vec{0}, 1)$ since the metric is independent of t and φ.

We call $e = -\xi_t \cdot u = (1 - 2GM/r)\frac{\mathrm{d}t}{\mathrm{d}\tau}$.

The other Killing vector is $l = \xi_\varphi \cdot u = -r^2 \sin^2\theta\,\mathrm{d}\varphi/\mathrm{d}\tau$, but $\theta = \pi/2$ so $l = r^2\,\mathrm{d}\varphi/\mathrm{d}\tau$.

Now we want to impose the condition $-1 = u \cdot u$:

$$-1 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 \tag{164}$$

$$= -\left(1 - \frac{2GM}{r}\right)\left(\frac{e}{1 - \frac{2GM}{r}}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{l^2}{r^2}\right)^2 \tag{165}$$

$$0 = -e^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{l^2}{r^2} + 1\right)\left(1 - \frac{2GM}{r}\right), \tag{166}$$

so

$$e^2 - 1 = \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{l^2}{r^2} + 1\right)\left(1 - \frac{2GM}{r}\right) - 1 \tag{167}$$

$$E = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}, \tag{168}$$

where

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}, \tag{169}$$

and

$$E = \frac{e^2 - 1}{2}, \tag{170}$$

so the GR effects are exactly contained in that last term GMl^2r^{-3}. This is very small: we consider it as a perturbation.

Are there circular orbits in this case?

This r^{-3} term means that for $r \to 0$ the effective potential goes to $-\infty$.

In general we'd expect two stationary points: one which is closer and unstable, and one which is further and stable.

If we differentiate $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ we get:

$$GMr^2 - l^2 r + 3GMl^2 = 0, \tag{171}$$

and we consider the positive solution:

$$r = \frac{l^2 \pm \sqrt{l^4 - 16G^2M^2l^2}}{2GM}, \tag{172}$$

and we take the plus sign since we want the orbit which is further out.

$$r = \frac{l^2}{2GM}\left(1 + \sqrt{1 - 12\frac{G^2M^2}{l^2}}\right), \tag{173}$$

which is the formula for the stable circular orbit. Expanding for small relativistic corrections we get

$$r_{\text{classical}} = \frac{l^2}{2GM}\left(1 + 1 - 6\frac{G^2M^2}{l^2}\right) = \frac{l^2}{GM} - 3GM\,, \tag{174}$$

which is the Newtonian orbit $r = l^2/GM$ with a correction.

Where is the boundary at which the solution disappears? it is where the square root vanishes:

$$l^2 = 12G^2M^2\,, \tag{175}$$

or $l = GM\sqrt{12}$. Solutions exist as long as the angular momentum is greater than $GM\sqrt{12}$.

For l_{min} we have $r_{\text{min}} = 6GM$. This is called the ISCO: *innermost stable circular orbit*: it is 3 times the Schwazschild radius $2GM$.

Now, we consider elliptical orbits.

The idea is: the angle between two consecutive perihelions is $2\pi + \delta\varphi_{\text{precession}}$.

In order to find the orbits, we want a relation between different coordinates during the orbit: we will use the equations

$$l = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}q} \tag{176a}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} = E\,, \tag{176b}$$

so $\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}$. Then:

$$\frac{l^2}{2r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} = E\,, \tag{177}$$

and it is convenient to solve for $u = r^{-1}$: we get

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}\,, \tag{178}$$

so

$$\frac{l^2}{2}u^4\frac{1}{u^4}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 - GMu + \frac{l^2u^2}{2} - GMl^2u^3 = E \tag{179}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 - \frac{GM}{l}u + \frac{u^2}{2} - GMu^3 = \frac{E}{l^2}\,, \tag{180}$$

and we want to remove E so we differentiate with respect to φ:

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} - \frac{GM}{l^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi} + u\frac{\mathrm{d}u}{\mathrm{d}\varphi} - 3GMu^2\frac{\mathrm{d}u}{\mathrm{d}\varphi} = 0\,, \tag{181}$$

and the orbit is monotonic so $\frac{\mathrm{d}u}{\mathrm{d}\varphi} \neq 0$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{GM}{l^2} + 3GMu^2\,, \tag{182}$$

where the term from GR is precisely $3GMu^2$, the rest is fully Newtonian.

This can be solved exactly with respect to complicated elliptic function, but we do it in a simpler way: a nearly circular orbit: $u = u_c(1 + w(\varphi))$, where $w \ll 1$.

To the order w^0: $u_c = \frac{GM}{l^2} + 3GMu_c^2$, since it is a circular orbit (u_c is a constant!)

To first order in w, instead, we get:

$$u_c\frac{\mathrm{d}^2w}{\mathrm{d}\varphi^2} + u_c(1+w) = \frac{GM}{l^2} + 3GMu_c^2(1+2w)\,, \tag{183}$$

since $w \ll 1$. But the terms without w simplify: they satisfy the zeroth order equation. So, we are left with

$$\frac{\mathrm{d}^2w}{\mathrm{d}\varphi^2} = (6GMu_c - 1)w\,, \tag{184}$$

which is in the form $\ddot{w} + \omega^2 w = 0$, since $u_c < 1/(6GM)$. If we look at unstable orbits with radii smaller than $6GM$, then this is exponentially diverging.

Thu Nov 21 2019

Let's start the show.

The effective potential equation is

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V_{\text{eff}}(r) = E = \frac{e^2-1}{2}\,, \tag{185}$$

where

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}\,, \tag{186}$$

and the timelike Killing vector gives us e, while the azimuthal Killing vector gives us l.

We can satisfy the equation with $V_{\text{eff}} \equiv E$ (eq1), and $\mathrm{d}r/\mathrm{d}t$ (eq2). These are circular orbits; the equations are, explicitly,

$$-\frac{GM}{l} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} = \frac{e^2-1}{2}\text{eq1} \tag{187}$$

and

$$\frac{GM}{r^2} - \frac{l^2}{r^3} + \frac{3GMl^2}{r^4} = 0\text{eq2}\,. \tag{188}$$

Let us compute $\text{eq1} + r(1 - r/2GM)\text{eq2}$: after some algebra, we get

$$\frac{l}{e} = \sqrt{GMr}\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{189}$$

The angular velocity Ω is defined as a derivative with respect to coordinate time:

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\,, \tag{190}$$

so

$$\Omega = \frac{\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}}{\frac{\mathrm{d}t}{\mathrm{d}\tau}} = \frac{l/r^2}{e\left(1 - \frac{2GM}{r}\right)^{-1}} = \frac{\sqrt{GMr}\left(1 - \frac{2GM}{r}\right)^{-1}\frac{1}{r^2}}{\left(1 - \frac{2GM}{r}\right)^{-1}} = \frac{\sqrt{GMr}}{r^2}\,, \tag{191}$$

so $\Omega^2 = GM/r^3$ or $\Omega^2 r = GM/r^2$, a relation which is the same as in Newtonian physics.

Now, review of the equation for $u = r^{-1}$. When we perturbed it, we got a harmonic oscillator. Our equation then is solved by cosinusoidal waves $u = u_c(1 + \cos(\omega\varphi))$, with $\omega = \sqrt{1 - 6GMu_c}$:

$$r(\varphi) = \frac{r_C}{1 + A\cos\left(\sqrt{1 - \frac{6GM}{r_c}}\varphi\right)}\,, \tag{192}$$

so $\Delta\varphi$ in one orbit is

$$2\pi\left(1 + \frac{3GM}{r_c}\right), \tag{193}$$

and then $\delta\varphi = 6\pi GM/r_c$. But $r_c = l^2/GM$: so we get

$$\delta\varphi = 6\pi\left(\frac{GM}{l}\right)^2. \tag{194}$$

We can plug numbers into this: we know that $G = 6.674\,08 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$, and our l has units of $\mathrm{m^2 s^{-1}}$, so we need to insert a c: the calculation then is

$$\delta\varphi = 6\pi \times \frac{180 \times 3600}{\pi} \left(\frac{GM_\odot}{r_{\mathrm{merc}} v_{\mathrm{merc}} c} \right) \times \frac{100\,\mathrm{yr}}{0.241\,\mathrm{yr}} \approx 43'' \,, \tag{195}$$

where the radius and velocity of the orbit of Mercury must be *both* calculated at the perihelion, or at the aphelion, or one can take the average velocity and the semimajor axis of the ellipse. The factor $100\,\mathrm{yr}/0.241\,\mathrm{yr}$ was inserted to account for the number of orbits of Mercury in a century.

```python
from numpy import rad2deg, pi
from scipy.constants import G, c

sun_mass = 2e30
mercury_orbital_velocity = 4.7e4
mercury_semimajor_axis = 57.9e9
mercury_angular_momentum = mercury_orbital_velocity *
    mercury_semimajor_axis
mercury_period_yr = .241
delta_phi = 6*pi*(G*sun_mass/mercury_angular_momentum/c)**2

print(rad2deg(delta_phi) * 3600 * 100/mercury_period_yr)
```

Finish

8.2 Radial orbit

We treat motion with $l = 0$ of an object which is at rest at infinity.

The equation becomes

$$\frac{1}{2} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau} \right)^2 - \frac{GM}{r} = \frac{e^2 - 1}{2} \,, \tag{196}$$

and if the object is at rest at infinity we can calculate e for $r \to \infty$:

$$e = \left(1 - \frac{2GM}{r} \right) \frac{\mathrm{d}t}{\mathrm{d}\tau} = 1 \,. \tag{197}$$

Then, our equation is

$$\frac{1}{2} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau} \right)^2 - \frac{GM}{r} = 0 \,. \tag{198}$$

The time component of the velocity is

$$u^t = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{e}{1 - \frac{2GM}{r}} = \frac{1}{1 - \frac{2GM}{r}} \,, \tag{199}$$

the radial component is given by our equation of motion:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\sqrt{\frac{2GM}{r}} = -\sqrt{\frac{2GM}{r}}\,, \tag{200}$$

where we choose the minus sign since we are falling in. The components v^θ and v^φ are zero since the motion is radial.

Let us compute $\vec{u} \cdot \vec{u}$: we get

$$g_{00}\left(u^0\right)^2 + g_{11}\left(u^1\right)^2 = -\left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-2} + \left(1 - \frac{2GM}{r}\right)^{-1}\frac{2GM}{r} \tag{201}$$

$$= \left(1 - \frac{2GM}{r}\right)^{-1}\left(-1 + \frac{2GM}{r}\right) = -1\,. \tag{202}$$

We can integrate

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2GM}{r}}\,, \tag{203}$$

we get

$$\int_0^r \sqrt{r}\,\mathrm{d}r = -\int_0^\tau \sqrt{2GM}\,\mathrm{d}\tau\,, \tag{204}$$

so

$$r(\tau) = \left(\frac{3}{2}\right)^{2/3}\sqrt[3]{2GM}\sqrt[3]{-\tau}\,. \tag{205}$$

Do keep in mind that we set $\tau = 0$ at $r = 0$. It seems like nothing bad really happens at $r = 2GM$. This solution only makes sense for negative τ.

At $r = 2GM$, a finite $\Delta\tau$ corresponds to an infinite Δt since $\mathrm{d}t/\mathrm{d}\tau$ diverges there.

So, the coordinates t and r seem like a bad choice at the horizon: they vary infinitely for a finite time as measured in the frame of the infalling particle.

The horizon is a *coordinate singularity*.

The 4-velocity for a particle going *outward* is:

$$u^\alpha = \begin{bmatrix} \left(1 - \frac{2GM}{r}\right)^{-1} \\ +\sqrt{\frac{2GM}{r}} \\ 0 \\ 0 \end{bmatrix}\,, \tag{206}$$

if it reaches radial infinity at rest.

What is the escape velocity? What is the three-velocity as measured by an observer at rest at constant r?

The 4-momentum of the particle is $p^\alpha = mu^\alpha$, while the observer's 4-velocity will be

$$u^\alpha_{\text{obs}} = \begin{bmatrix} \frac{1}{\sqrt{-g_{00}}} \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{207}$$

The energy measured by the observer is $E = -\vec{p} \cdot \vec{u}_{\text{obs}}$. It is

$$E = -g_{00} u^0_{\text{obs}} p^0 = \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-1/2} m \left(1 - \frac{2GM}{r}\right)^{-1} = \frac{m}{\sqrt{1 - \frac{2GM}{r}}} . \tag{208}$$

However, any observer sees $E = m\gamma$, so they identify $\gamma = 1/\sqrt{1 - \frac{2GM}{r}}$.

Then, the escape velocity is $\sqrt{\frac{2GM}{r}}$, just like the Newtonian case.

Now, we will look at the motion of light: by how much is it deflected?

We still have the Killing vector $\xi^\mu = (1, \vec{0})$: so

$$-\xi \cdot u = e = \left(1 - \frac{2GM}{r}\right) \frac{\mathrm{d}t}{\mathrm{d}\lambda} \tag{209}$$

is constant, and from $\xi^\mu = (\vec{0}, 1)$ we have

$$\xi \cdot u = l = r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} , \tag{210}$$

where λ is the parameter of the light trajectory, and we set $\theta = \pi/2$.

The square modulus of the light's velocity is zero, so

$$0 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{e}{1 - \frac{2GM}{r}}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + r^2 \left(\frac{l}{r^2}\right) . \tag{211}$$

This can be rewritten as

$$0 = -\frac{e^2}{l^2} + \frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) , \tag{212}$$

or

$$\frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}}(r) = \frac{1}{b^2} , \tag{213}$$

where $b^2 = l^2/e^2$ and

$$W_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right). \tag{214}$$

A note: λ is a parameter, but we can do affine transformations to it: $\lambda \to a\lambda + b$. Actually we can do any $\lambda' = \lambda'(\lambda)$ which is monotonic. We can check that all physical quantities which appear in the equation are invariant, since all the factors $\mathrm{d}\lambda'/\mathrm{d}\lambda$ cancel.

Also, the equation is even in l, since it must be invariant under $\varphi \to -\varphi$ (which corresponds to P symmetry). Then, we choose l to be positive as our gauge.

At $r = 3GM$ we have a maximum of W_{eff}, which attains the value $1/27G^2M^2$ there. Since it is a maximum, the orbit is unstable.

If a photon approaches the BH with energy $1/b^2$ larger than $W_{\text{eff}}(3GM)$ then it "bounces back": r decreases, there is a minimum r and then r increases.

The critical equation is $b^{-2} > (27G^2M^2)$, which means $l > \sqrt{27}GMe$. It is actually a critical *angular momentum*.

Fri Nov 22 2019

We found the equation

$$\frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}}(r) = \frac{1}{b^2}, \tag{215}$$

where

$$W_{\text{eff}} = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) \tag{216}$$

and

$$b^2 = \frac{l^2}{e^2}, \tag{217}$$

l and e being the integrals corresponding to the Killing vectors of time translations and azimuthal angle rotations.

Now, we want to give the interpretation of b as the impact parameter. We consider a BH at the origin of the x, y axes, and a photon approaching parallel to the x axis with impact parameter d. The impact parameter is the distance between the two lines: the trajectory of the photon far away from the BH and the line parallel to the trajectory and passing through the BH.

The parameter

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \underbrace{\frac{\mathrm{d}r}{\mathrm{d}t}}_{-1} \frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\left(-\frac{d}{r^2}\right), \tag{218}$$

where we used a small angle approximation: $\varphi \approx d/r$, which can be differentiated with respect to r. So

$$b = \frac{l}{e} = \frac{r^2\,\mathrm{d}\varphi/\mathrm{d}\lambda}{\mathrm{d}t/\mathrm{d}\lambda} = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} = d\,, \tag{219}$$

which means that $b = d$: the ratio of l to e is the impact parameter.

So, the photon interacts with the BH and the angle of deflection of the path compared to a straight path is denote as $\delta\varphi$. The total deflection angle is $\Delta\varphi = \pi + \delta\varphi$.

The paramter l is

$$l = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\,, \tag{220}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\,. \tag{221}$$

This allows us to change variables in our 1D equation, and we get

$$\frac{1}{l^2}\frac{l^2}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) = \frac{1}{b^2}\,, \tag{222}$$

so if we change variables to $u = r^{-1}$, with

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}\,, \tag{223}$$

we get

$$u^4u^{-4}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2(1 - 2GMu) = \frac{1}{b^2}\,, \tag{224}$$

and then, differentiating, we find

$$2\frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + 2u\frac{\mathrm{d}u}{\mathrm{d}\varphi} - 6GMu^2\frac{\mathrm{d}u}{\mathrm{d}\varphi} = 0\,, \tag{225}$$

and we can simplify as long as $\mathrm{d}u/\mathrm{d}\varphi \neq 0$, which only fails at one point. So in the end our equation is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = 3GMu^2 \,. \tag{226}$$

We solve it perturbatively. If $GM = 0$, there is no BH and we have a straight line: the impact parameter is constant and equal to $b = r\sin(\varphi)$, therefore

$$u = \frac{1}{b}\sin(\varphi) \,, \tag{227}$$

is the most general solution to the harmonic oscillator which satisfies the boundary conditions. So, we hypothesize that our solution satisfies

$$u(\varphi) = \frac{1}{b}\Big(\sin(\varphi) + W(\varphi)\Big) \,, \tag{228}$$

where W is small.

Then, we insert this:

$$-\frac{1}{b}\sin(\varphi) + \frac{1}{b}\frac{\mathrm{d}^2 W}{\mathrm{d}\varphi^2} + \frac{1}{b}W \approx 3GM\frac{\sin^2(\varphi)}{b} \,, \tag{229}$$

but since the zeroth order equation is satisfied we find:

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\varphi^2} + W \approx \frac{3GM}{b}\sin^2(\varphi) \,. \tag{230}$$

Our ansatz is then:

$$W = A + B\sin^2\varphi \,, \tag{231}$$

which looks like it might solve the equation. Its second derivative is

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\varphi^2} = 2B\Big(\cos^2\varphi - \sin^2\varphi\Big) = 2B - 4B\sin^2\varphi \,. \tag{232}$$

Inserting this we get:

$$2B - 4B\sin^2\varphi + A + B\sin^2\varphi = \frac{3GM}{b}\sin^2\varphi \,, \tag{233}$$

which implies $2B + A = 0$ and $-3B = 3GM/b$. Then,

$$W = \frac{2GM}{b} - \frac{GM}{b}\sin^2\varphi = \frac{2GM}{b}\left(1 - \frac{\sin^2\varphi}{2}\right) \tag{234}$$

solves the perturbed equation. We can see that our condition of $W \ll 1$ is actually the physically meaningful $GM \ll b$, or the impact parameter being much larger than the Schwarzschild radius.

Our complete solution is

$$u(\varphi) = \frac{1}{b}\left(\sin\varphi + \frac{2GM}{b}\left(1 - \frac{\sin^2\varphi}{2}\right)\right), \tag{235}$$

and we are interested in the asymptotic past and future, which correspond to $u = 0$. Now, $\varphi_{\text{in}} = 0$ and $\varphi_{\text{out}} = \pi$ will not be a solution anymore. However, the deflection is small so we write the solution as $\varphi_{\text{in}} = \epsilon_{\text{in}}$ and $\varphi_{\text{out}} = \pi + \epsilon_{\text{out}}$. We substitute these in to the equation $u = 0$:

$$0 = \sin(\epsilon_{\text{in}}) + \frac{2GM}{b}, \tag{236}$$

or $\epsilon_{\text{in}} \approx -2GM/b$, since the deflection is small.

For ϵ_{out} we will have

$$0 = \sin(\pi + \epsilon_{\text{out}}) + \frac{2GM}{b} \approx -\epsilon_{\text{out}} + \frac{2GM}{b}, \tag{237}$$

so $\delta\varphi = 2\varphi_{\text{in}} = 2\varphi_{\text{out}} = 4GM/b$.

This was one of the first tests of GR by Sir Eddington in 1919: during an eclipse he saw a shift in the apparent position of the stars. reinserting c we find that we must divide $4GM/b$ by c^2. Our b is approximately the radius of the Sun: the calculation is

```
from scipy.constants import *
sun_mass = 2e30
sun_radius = 696e6
rad2arcsec = 3600 * 180 / pi
4*G*sun_mass/c**2 /sun_radius * rad2arcsec
```

For the rest of today, we will talk about the Schwarzschild horizon: recall the line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 . \tag{238}$$

It is useful to plot light cones in order to understand the structure of the spacetime. We restrict ourselves to radial motion of light. So, we have

$$0 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 , \tag{239}$$

or

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} . \tag{240}$$

The light cones become slimmer and slimmer as we approach the horizon, they are straight lines at $r = 2GM$: the photon appears to cover less and less dr for a fixed dt as it approaches the horizon. Massive particles are even slower. Let us integrate this relation:

$$\int_0^t \mathrm{d}t = -\int_{r_0}^{r(t)} \frac{\mathrm{d}r}{1 - \frac{2GM}{r}} , \tag{241}$$

which comes out, by a separation of fractions, to be

$$t = r_0 - r(t) + 2GM \log\left(r_0 - 2GM\right) - 2GM \log\big(r(t) - 2GM\big) , \tag{242}$$

which diverges as $r(t)$ approaches $2GM$.

What do we really mean by t? An observer which is far away and at rest has $g_{\mu\nu} = \eta_{\mu\nu}$, and $t = \tau$: t is the proper time, as measured on the clock of an observer who is far away. They will measure the photon as going slower and slower, and becoming redder and redder.

Let us neglect Doppler redshift, which is due to motion. The formula for gravitational redshift is

$$f_{\mathrm{obs}} = f_{\mathrm{emit}}\sqrt{\frac{-g_{00}(\mathrm{emit})}{-g_{00}(\mathrm{obs})}} . \tag{243}$$

In our case we have

$$f_{\mathrm{obs}} = f_{\mathrm{emit}}\sqrt{1 - \frac{2GM}{r}} , \tag{244}$$

which approaches zero as $r \to 2GM$. We see the infalling observer becoming redder and ultimately freezing.

We should use different coordinates to describe the infalling observer who passes through the event horizon.

First of all, we discuss Minkowski spacetime as seen by an accelerating observer: Rindler spacetime and the Riddler horizon.

Recall the exercise in sheet 2, about an accelerating observer: we now consider an observer moving with the position law

$$x(t) = \frac{1}{\kappa}\sqrt{1 + \kappa^2 t^2} , \tag{245}$$

(as opposed to the homework, we remove the constant added to this position). Like in the homework we compute the proper time for the observer:

$$ds^2 = -\,dt^2\left(1-\left(\frac{dx}{dt}\right)^2\right), \tag{246}$$

so

$$d\tau = \frac{dt}{\sqrt{1+\kappa^2 t^2}}, \tag{247}$$

which means

$$t = \frac{1}{\kappa}\sinh(\kappa\tau) \qquad \tau = \frac{1}{\kappa}\operatorname{arcsinh}(\kappa\tau), \tag{248}$$

therefore

$$x = \frac{1}{\kappa}\cosh(\kappa\tau). \tag{249}$$

We want a coordinate system in which

1. the observer is at constant spatial position;
2. where, up to a constant, the time is equal to the proper time.

Our change of variable is

$$\begin{cases} t = \rho\sinh\eta \\ x = \rho\cosh\eta \end{cases} . \tag{250}$$

the observer is at fixed spatial coordinate $\rho_* = 1/\kappa$, and the proper time measured is $\tau = \eta/\kappa = \eta\rho_*$.

Let us consider a family of observers at different spatial locations in the new frame: each has a constant acceleration, this means varying κ or ρ.

If instead we vary η we have:

$$\frac{t}{x} = \tanh\eta \implies \eta = \tanh^{-1}\left(\frac{t}{x}\right), \tag{251}$$

and we can see that since $\tanh 0 = 0$ we have that the $t = 0$ axis has $\eta = 0$, while the lightspeed observers are at $\eta = \pm\infty$.

These coordinates cover one quadrant of Minkowski spacetime. The line element in these new coordinates is

$$\begin{aligned} ds^2 &= -\,dt^2 + dx^2 &(252)\\ &= -\Big(d\rho^2\sinh\eta + \rho\cosh\eta\,d\eta\Big)^2 + \Big(d\rho^2\cosh\eta + \rho\sinh\eta\,d\eta\Big)^2 &(253)\\ &= -\rho^2\,d\eta^2 + d\rho^2\,. &(254) \end{aligned}$$

This is *Rindler geometry*.

If we have an observer staying at $x_0 > 0$, and there is a Rindler observer, then after a time x_0 the observer exits the quarter of the plane covered by the Rindler coordinates: if event A is at $(0, x_0)$, and B is at (x_0, x_0) then the η of event A is zero, while the η of event B is infinite.