Advanced astrophysics notes

Jacopo Tissino, Giorgio Mentasti

February 17, 2020

Contents

Chapter 1

Stellar oscillations

30 september 2019

1.1 Introduction

The course is held by Paola Marigo, Michele Trabucchi.
Their adresses are: paola.marigo@unipd.it and michele.trabucchi@unipd.it

1.1.1 Topics

They are selected topics in stellar physics.

1. Stellar pulsations and Astroseismiology (dr. Michele Trabucchi);
2. stellar winds (dr. Paola Marigo);
3. final fates of massive & very massive stars (dr. Paola Marigo).

For information about the basics in stellar physics refer to the course "astrophysics II" inside the bachelor's degree in astronomy (second semester). It can be taken as an optional course.

Material:

1. *Introduction to stellar winds* by Lamers, Cassinelli.
2. *Stellar Atmospheres: Theory and observations* (lecture notes from 1996).

and more on Paola Marigo's site.

Stellar oscillations

...see slides.

Material: slides on moodle or Marigo's page.

1. *Pulsating stars* by Catelan & Smith (introductory);
2. *Theory of stellar pulsation* by Cox (harder).

Written exam, partial exam on stellar pulsation.

Reference books can be found in the Moodle: they are ordered by difficulty, Catelan to Aerts to Salaris.

Stellar Winds

They are moving flows of materials ejected by stars, with speeds generally between 20 to 2×10^3 km/s.

See, for example, the *Bubble Nebula* in Cassiopea, there is a $45M_\odot$ star ejecting stellar wind at 1700 km/s.

Diagram: luminosity vs effective temperature. We see the *main sequence*. We can also plot the *mass loss rate*, $\dot{M} > 0$ in solar masses/year. Another important parameter is v_∞, the asymptotic terminal velocity of the wind.

Diagram: mass loss (or gain) rate vs age of star.

Stellar winds affect stellar evolution, the dynamics of the interstellar medium, the chemical evolution of galaxies.

Momentum is approximately injected with $\dot{M}v$, kinetic energy with $1/2\dot{M}v^2$. Within 1×10^8 yr around half of the infalling matter is reemitted.

We will start with the basic theory of stellar winds, and then discuss *coronal*, *line-driven* and *dust-driven* winds.

Final fates of massive & very massive stars

Masses over $10M_\odot$.

1.2 Stellar oscillations

1.2.1 Variability in Astronomy

The first observations of variable stars happended around the year 1600: Fabritius observed the star omicron-Ceti, in the constellation of Cetus. It changes in magnitude by 6 orders of magnitude: several authors report it as a "new star" in the

16-hundreds, before finally in 1667 Bullialdus puts the pieces together and figures out that the star is periodic, with a period of 333 days.

The star o-Ceti was also called Mira, and it is considered a prototype for these long-period variables: they are called *Miras*.

Others are found from the 1600 onwards, but up to the XX century the reason is still unknown. Is it *rotation*, *eclipses*?

For some the cause was discovered to be indeed eclipses, but the Cepheids are different. See for example the δ-Cephei type: we have an asymmetric continuous curve, with no clearly recognizable *dip*, which we would expect to see if there was an eclipsing system. What if stars *pulsate*?

In order to investigate these phenomena, we need to define the *light curve*: it is the luminosity curve over time.

We can also look at the *phased* light curve: in order to plot it however we need the period. The phase is defined as

$$\varphi = \frac{(t - t_0) \bmod \Pi}{\Pi} \in [0, 1) \tag{1.1}$$

where Π is the period. $E(t) = \lfloor (t - t_0)/\Pi \rfloor$ is the epoch.

So, we can plot the magnitude against φ: we will get several curves in the same plot.

We can then measure the period, but if the light curve is multiperiodic we can subtract the model from the curve to see if there are additional periods: this is *prewhitening*.

We can also look at the luminosity in Fourier space, or more generally use other period measuring techniques, such as the Lomb-Scargle periodogram or *phase dispersion minimization*.

A curiosity: how is phase dispersion minimization actually implemented? Minimizing the area of the convex hull of the data seems error-prone, and it would be nice to have an algorithm which did not rely on the residuals from a *model*. Maybe: for each point compute the distance to the k nearest neighbours, add all of these together and minimize this?

Of course there are issues with observational gaps (day-night, full moon): aliases; accuracy, duration of observations...

Also, the period can change in time.

Things have improved a great deal with large-scale surveys and space suveys.

We also have to account for the Nyquist frequency: if we have n observations spaced with a constant interval Δt we will only be able to measure the frequency with a resolution of $\Delta f = (n \Delta t)^{-1}$.

A useful technique for the assessment of a true period is to plot the observed luminosity at a fixed phase with varying (integer) epoch: if the period was assessed exactly, we expect this to be constant. If we see a straight line, then we know we are under or overestimating the period. If we see some other curve, with this diagram we can start to figure out how the period is changing.

Classification of variable stars

By variability type: regular, semi-regular or irregular.

By variability class: *extrinsic*, external to the star: eclipses, transits, microlensing, rotation; *intrinsic*: rotation, eclipses (self-occultation), eruptive and explosive variables, oscillations, secular variations.

Whether rotation is to be considered intrinsic or extrinsic is a matter of taste.

Oscillations can be classified by several criteria.

The geometry can be *radial* (classical pulsators, such as cepheids, RR Lyrae, Miras) or *non-radial*.

The restoring force can be the pressure gradient (p-modes) or the gravitational force (so, bouyancy) (g-modes).

The excitation mechanisms can be different.

The evolutionary phase and mass of the oscillating star can also be different. We distinguish these populations by the sky region in which we see them.

1.2.2 Summary of stellar structure & evolution

In the *Eulerian* view, properties of a gas are fields, the position is the position of an observer. To differentiate position with respect to time is meaningless: position is an independent variable. Any function is a function of position and time: $f = f(r^i, t)$.

In this case, then, the mass underneath a certain layer is

In the *Lagrangian* view, we follow an element of fluid, which has a velocity $\mathrm{d}r^i \big/ \mathrm{d}t = v^i$. We can identify univocally these fluid elements (since the time-evolution is deterministic).

When treating stellar structure & evolution, we identify the fluid elements as mass layers $\mathrm{d}m$. Any function is then a function of mass and time: $f = f(m, t)$. Do note that m is the mass of the whole full sphere under a certain layer, not the mass of the shell.

In the lagrangian case, the expression for the total derivative with respect to time is given by the convective derivative $\mathrm{d} / \mathrm{d}t = \partial_t + v^i \partial_i$ where v^i is the velocity defined before.

Equations of stellar structure

We write these in the spherically symmetric case, using the Lagrangian formalism.

The *continuity equation* is:

$$\boxed{\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}\,. \qquad (1.2)}$$

In order to switch between the Lagrangian view, in which the derivatives are done with respect to the mass m, and the Eulerian one, in which we differentiate with respect to the radius r, we use the continuity relation.

Momentum conservation is given by:

$$\boxed{\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}\,, \qquad (1.3)}$$

which is the equation for hydrostatic equilibrium: P is the pressure. In the absence of hydrostatic equilibrium, the equation reads:

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2}\,. \tag{1.4}$$

Energy conservation is given by:

$$\boxed{\frac{\mathrm{d}L}{\mathrm{d}m} = \varepsilon - \varepsilon_\nu - \varepsilon_g\,, \qquad (1.5)}$$

where L is the luminosity, ε is the rate of nuclear energy generation per unit mass, while ε_ν is the rate of energy loss due to neutrino emission per unit mass, and ε_g is the work done by the gas per unit mass & time, which can be written as

$$\epsilon_g = \frac{\partial u}{\partial t} - \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\,, \tag{1.6}$$

The *energy transfer* equation is:

$$\boxed{\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla\,, \qquad (1.7)}$$

where $\nabla = \partial \log T / \partial \log P$ is the temperature gradient, which has contributions from radiation, conduction, and convection.

With the diffusion approximation, we can write the gradient as

$$\nabla = \nabla_{\text{rad}} = \frac{3}{16\pi acG}\frac{\kappa_R LP}{mT^4}\,, \tag{1.8}$$

where a is a constant depending on the Stefan-Boltzmann constant and the speed of light.

where κ_R is the Rosseland mean opacity, given by

$$\frac{1}{\kappa_R} = \frac{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{1}{\kappa_\nu}\,\mathrm{d}\nu}{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu}\,, \tag{1.9}$$

where B_ν is the Planck function:

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right)^{-1}\,, \tag{1.10}$$

which can be written like this or multiplied by 4π, if we wish to integrate over all solid angles. It does not matter here, since any constant factor simplifies.

Substituting in the result in (1.7) we get:

$$L = -\frac{64\pi^2 ac}{3} r^4 \frac{T^3}{\kappa_R}\frac{\partial T}{\partial m}\,. \tag{1.11}$$

This can be improved by substituting κ_R with κ, a generalized opacity, which is given by the harmonic mean of the Rosseland opacity κ_R and the convective opacity $\kappa_c = 4acT^3/(3\rho\lambda_c)$.

The term λ_c here is the proportionality factor in the equation for the conductive energy flux in terms of the temperature gradient: $\vec{F}_c = -\lambda_c \vec{\nabla} T$.

If we need to deal with convection, this defies any simple modeling. There are instability criteria: where is conduction relevant? This is given by Ledoux's criterion,

$$\nabla_{\text{rad}} > \nabla_{\text{ad}} - \frac{\chi_\mu}{\chi_T}\nabla_\mu\,, \tag{1.12}$$

where:

$$\nabla_\mu = \frac{\mathrm{d}\log\mu}{\mathrm{d}\log P} \tag{1.13a}$$

$$\nabla_{\text{ad}} = \left(\frac{\partial\log T}{\partial\log P}\right)_{\text{ad}} \tag{1.13b}$$

$$\chi_\mu = \left(\frac{\mathrm{d}\log P}{\mathrm{d}\log\mu}\right)_{\rho,T} \tag{1.13c}$$

$$\chi_T = \left(\frac{\mathrm{d} \log P}{\mathrm{d} \log T} \right)_{\rho,\mu} \tag{1.13d}$$

which are thermodynamic parameters.

TODO: Add commentary about this stuff

In the convective core, $\nabla \approx \nabla_{\mathrm{ad}}$, but outside of it we need something else. Mixed-length theory models convection with "bubbles".

Beyond these equations, we need the constitutive equations for:

1. the density ρ;
2. the heat capacity of stellar matter c_P;
3. the opacities (radiative and conductive) κ;
4. the transformation rates between nuclear species i and j: r_{ij};
5. the generation rate of nuclear energy ϵ

in terms of the pressure P, temperature T and chemical potential μ.

1 October 2019

Figure 22.8 in some PDF: run of adiabatic, radiation gradients vs $\log T$.

We compute ∇_{ad} and ∇_{rad} and see whether the region is convective or radiative.

We can move from the Eulerian and Lagrangian formalisms using the continuity equation. In the Eulerian formalism:

$$m(r) = \int_0^\infty 4\pi r^2 \rho(x) \,\mathrm{d}x \,. \tag{1.14}$$

We define the mean molecular weight μ with:

$$\mu^{-1} = \sum_i (1 + \nu_e(i)) \frac{X_i}{A_i} \,, \tag{1.15}$$

where $\nu_e(i)$ is the number of free electrons coming from element i, X_i is the abundance by mass fraction of the element i, and A_i is its mass number.

The variables X, Y and Z represent the abundances of H, He and metals, and satisfy $X + Y + Z = 1$.

We may need to know the metal mixture inside Z, but often we can approximate it as the Sun's distribution.

The time evolution of the various elements' fractions is given by

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \sum_j \left(r_{ji} - r_{ij} \right) \tag{1.16}$$

Classification of stars

1. Low mass stars have between 0.8 and 2 solar masses. They develop an electron-degenerate core after their time on the Main Sequence.
2. Intermediate mass stars have masses between 2 and 8 $M_\odot$. They start burning helium in a non-degenerate core, then they develop a degenerate C−O core.
3. Massive stars have masses of over $8 M_\odot$. They start burning carbon in a non-degenerate core.

What does *degenerate* mean in this context?

What are Hayashi lines?

1.2.3 Time-scales

The *free fall* time scale is:

$$\tau_{\text{dyn}} \sim \left(\frac{R}{g} \right)^{1/2} = \left(\frac{R^3}{GM} \right)^{1/2} . \tag{1.17}$$

It is associated with pulsation. It is calculated using the travel time of a mass in free fall across the stellar radius accelerated by constant acceleration equal to surface acceleration.

We note that $\tau_{\text{dyn}} \propto \overline{\rho}^{-1/2}$, since $\overline{\rho} \propto M/R^3$. For the Sun it is about 1.6×10^3 s.

The *thermal* time scale is the relaxation time of deviations from thermal equilibrium:

$$\tau_{\text{th}} \sim E_{\text{th}}/L . \tag{1.18}$$

It is calculated as the time required for a star to irradiate all its energy.

Proof. We use the virial theorem to estimate the thermal (so, local kinetic) energy T: we know that $T = -V/2$ where V is the total potential energy of the star. V can be computed as

$$V = -\int \frac{Gm}{r(m)} \, \mathrm{d}m , \tag{1.19}$$

and since $r(m) \approx \sqrt[3]{3m/4\pi\rho}$ if the density is constant, we get

$$\begin{aligned} V &= -\int Gm \sqrt{\frac{4\pi\rho}{3}} m^{-1/3} \, \mathrm{d}m && \text{(1.20a)} \\ &= -\frac{3}{5} G \sqrt[3]{\frac{4\pi\rho}{3}} M^{5/3} = -\frac{3}{5} \frac{GM^2}{R} \sim -\frac{GM^2}{R} . && \text{(1.20b)} \end{aligned}$$

Therefore, $T \sim -GM^2/2R$. □

Typically, $\tau_{\rm th} \sim GM^2/(LR) \sim 10^7 M^2/(LR)$ years. For the Sun we have $\tau_{\rm th} \sim 5 \times 10^{14}$ s. It is much larger than the dynamic time scale.

The *nuclear* time scale is even longer: it is calculated using the efficiency of nuclear fusion of H to He, which is about $\epsilon \sim 0.7\%$, and the fraction of hydrogen in the star, which is about $M_H = 10\% \times M_\odot$. Using these numbers, we get for the Sun:

$$\tau_{\rm nuc} = \frac{\epsilon c^2 M_H}{L_\odot} \approx 3 \times 10^{17}\,{\rm s}\,, \tag{1.21}$$

This allows us to say that oscillations will not be heavily affected by thermal conduction, and even less by nuclear processes: the pulsations will be almost *adiabatic*.

The best candidate for these oscillations are *sound waves*: is the adiabatic speed of sound roughly right?

The speed of sound is given by:

$$v_s^2 = \frac{\partial P}{\partial \rho} = \frac{P}{\rho}\frac{\partial \log P}{\partial \log \rho} = \Gamma_1 \frac{P}{\rho}\,, \tag{1.22}$$

where Γ_1 is the adiabatic exponent, such that $P = \rho^{\Gamma_1}$.

If the gas follows the perfect equation of state

$$\frac{P}{\rho} = \frac{k_B}{m_H}\frac{T}{\mu}\,, \tag{1.23}$$

where μ is the average mass of a gas particle in atomic mass units (units of m_H), so we have $\rho = \mu N m_H / V$. Substituting this in, we get

$$v_s^2 = \frac{\Gamma_1 k_B T}{m_H \mu}\,. \tag{1.24}$$

The mean molecular weight is calculated taking account of the fact that the electrons appear in the count of particles, but we can neglect their mass: therefore, we get a $Z+1$ in the formula, since there are Z electrons and one nucleus. So the value becomes

$$\mu = \left(\sum_j \frac{X_j}{A_j}(1 + Z_j) \right)^{-1} \approx (2X + 3Y/4 + Z/2)^{-1} \sim 0.6\,, \tag{1.25}$$

where we approximated that for each metal $(1+Z)/A \sim 1/2$ and we used the values of the mass fractions of the Sun: $X = 0.7381$, $Y = 0.2485$ and $Z = 0.0134$.[1]

Typical values for the other parameters are $\Gamma_1 = 5/3$, and $T_{\rm He} \sim 4.5 \times 10^4$ K.

[1] See https://arxiv.org/abs/0909.0948

So we get $v_s \sim 32.2\,\mathrm{km/s}$.

The timescale for a perturbation to go from one side of the star to the other is $\Pi \sim 2R/v_s \sim 22\,\mathrm{d}$, while the observed value is $\Pi_{\mathrm{obs}} = 5.336\,\mathrm{d}$, so in terms of orders of magnitude it works. We could say that from a more thorough analysis we would see that the vibration does not actually go all the way from an edge of the star to the other.

We can use the equation for the sound speed in the virial theorem, which for a star relates the gravitational potential energy Ω to the integral of the pressure:

$$\Omega = -3 \int P \, \mathrm{d}V = -3 \int \frac{P}{\rho} \, \mathrm{d}m \,, \tag{1.26}$$

and substitute in $P/\rho = v_s^2/\Gamma_1$:

$$\Omega = -3 \frac{\int_M v_s^2/\Gamma_1 \, \mathrm{d}m}{\int_M \mathrm{d}m} M = -3 \left\langle \frac{v_s^2}{\Gamma_1} \right\rangle M \,, \tag{1.27}$$

We multiply and divide by $M = \int \mathrm{d}m$

where the brackets denote an average weighted by the mass distribution.

If Γ_1 and v_s are independent, we can compute their averages separately: we approximate and do this. Then, we can substitute in our expression for the average $\left\langle v_s^2 \right\rangle \sim \langle v_s \rangle^2$ into $\Pi \sim 2R/v_s$: we get

$$\langle v_s \rangle^2 = -\frac{\Omega \Gamma_1}{3M} \,, \tag{1.28}$$

so

$$\Pi \sim \frac{2R}{\langle v_s \rangle} = \sqrt{\frac{-4R^2 \times 3M}{\Omega \Gamma_1}} \,, \tag{1.29}$$

This means we are writing the period Π with respect to something resembling the moment of inertia, $I = \int_r r^2 \, \mathrm{d}m\,(r) \sim R^2 M$:

$$\Pi \sim \left(\frac{I_{\mathrm{osc}}}{-\Omega} \right)^{1/2} \tag{1.30}$$

This is further evidence that we are dealing with a dynamical phenomenon.

We can refine our model: the speed of sound changes throughout the interior of the star. We compute the period as the travel time of sound waves throughout the diameter:

$$\Pi = 2 \int_0^R \mathrm{d}t(r) = 2 \int_0^R \frac{\mathrm{d}r}{\sqrt{\Gamma_1(r) P(r) / \rho(r)}} \,, \tag{1.31}$$

since $\mathrm{d}t = \mathrm{d}r \,/ v_s$; the factor of 2 comes from the fact that the sound wave must go from one side of the star to the other.

We also integrate the hydrostatic balance equation, which reads

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2} = -\frac{G\rho^2 4\pi r}{3}\,, \tag{1.32}$$

with respect to r: then we get

$$P(r) - \underbrace{P(R)}_{=0} = \int_R^r \frac{\mathrm{d}P}{\mathrm{d}r}\,\mathrm{d}r = \frac{2\pi\rho^2 G}{3}\left(R^2 - r^2\right), \tag{1.33}$$

which we can plug into our formula for the period to find:

$$\Pi = 2\int \mathrm{d}r \left(\Gamma_1 \frac{P}{\rho}\right)^{-1/2} \tag{1.34a}$$

$$= 2\int \mathrm{d}r \left(\Gamma_1 \frac{2\pi\rho G}{3}\left(R^2 - r^2\right)\right)^{-1/2} \tag{1.34b}$$

$$= \sqrt{\frac{6}{\Gamma_1 \pi G \overline{\rho}}} \underbrace{\int \frac{\mathrm{d}r}{\sqrt{R^2 - r^2}}}_{\pi/2} \tag{1.34c}$$

$$= \sqrt{\frac{3\pi}{2\Gamma_1 G\overline{\rho}}}\,, \tag{1.34d}$$

which confirms Ritter's relation $\Pi \propto \overline{\rho}^{-1/2}$. Since the product $\Pi\overline{\rho}^{1/2}$ is approximately constant, we give it the name

$$\mathcal{Q} = \Pi\sqrt{\overline{\rho}} \approx \sqrt{\frac{3\pi}{2\Gamma_1 G}}\,. \tag{1.35}$$

Ritter's relation is consistent with the statement that the period of the oscillations is of the order of the dynamical characteristic time of the star, since

$$\tau_{\mathrm{dyn}} = \sqrt{\frac{R^3}{GM}} \qquad \text{while} \qquad \overline{\rho}^{-1/2} \approx \left(\frac{M}{4\pi R^3/3}\right)^{-1/2} = \sqrt{\frac{4\pi R^3}{3M}}\,. \tag{1.36}$$

This gives an estimate of $\Pi \sim 8.5\,\mathrm{d}$ for a δ-Cephei star, which we have to compare to the observed period of $\Pi_{\mathrm{obs}} = 5.466\,\mathrm{d}$. We have definitely improved our estimate. The prediction of this model is that *dense stars pulstate faster*.

This works for acoustic modes, such as those found in δ-Cephei, o-Ceti and SX-Phe stars, but if we consider non-radial g-modes such as those found in variables of type ZZ Ceti it stops working. Then, the estimate given by this model can be off by three orders of magnitude.

1.2.4 The energy equations

7 October 2019

We introduce the *mirror principle*: when the core contracts or expands, the envelope does the opposite.

The shell must remain at around the same temperature to maintain equilibrium: contracting the core would increase the temperature, therefore the envelope exapands. This heuristic argument is actually derived from simulations.

The relevant time scale for oscillations is the free-fall, dynamical time scale.

We come back to the energy equation

$$\frac{\partial L}{\partial m} = \varepsilon - \varepsilon_\nu - \varepsilon_g \tag{1.37}$$

we incorporate the nuclear energy generation rate and the energy lost as neutrino production into an effective energy generation rate per unit mass $\varepsilon - \varepsilon_\nu = \varepsilon_{\text{eff}}$ and express the energy absorbed by the stellar layer as

$$\varepsilon_g = \frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon_{\text{eff}} - \frac{\partial L}{\partial m} \,. \tag{1.38}$$

This makes the meaning of this transfer equation clearer. Using the first and second laws of thermodynamics, and recalling some thermodynamical values: the specific heat at constant volume $c_V = \left(\frac{\partial Q}{\partial T}\right)_V$, the equation of state exponents χ_T and χ_ρ which satisfy: $P = T^{\chi_T}$ and $P = \rho^{\chi_\rho}$ and the adiabatic exponents $\Gamma_{1,2,3}$, which are defined by

$$\Gamma_1 = \gamma_{\text{ad}} = \left(\frac{\partial \log P}{\partial \log \rho}\right)_s \tag{1.39a}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{1}{\nabla_{\text{ad}}} = \left(\frac{\partial \log P}{\partial \log T}\right)_s \tag{1.39b}$$

$$\Gamma_3 - 1 = \left(\frac{\partial \log T}{\partial \log \rho}\right)_s \tag{1.39c}$$

$$\tag{1.39d}$$

and satisfy

$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1} \,. \tag{1.40}$$

These are all *exponents* in some power law. We use log values since our variables change by orders of magnitude.

We start with the definition of the entropy differential: $\mathrm{d}Q$ is not an exact differential but $\mathrm{d}Q\,/T = \mathrm{d}S$ is. So, we express it using the first law of thermodynamics: $\mathrm{d}Q = \mathrm{d}E + P\,\mathrm{d}V$. We assume the internal energy E to be a function of the volume V and of the temperature T, so we will have:

$$\mathrm{d}E = \frac{\partial E}{\partial V}\,\mathrm{d}V + \frac{\partial E}{\partial T}\,\mathrm{d}T\ , \tag{1.41}$$

which we can substitute into the expression for the entropy differential:

$$\begin{aligned}
\mathrm{d}S &= \frac{1}{T}\left(\frac{\partial E}{\partial V}\,\mathrm{d}V + \frac{\partial E}{\partial T}\,\mathrm{d}T + P\,\mathrm{d}V\right) && \text{(1.42a)}\\
&= \left(\frac{1}{T}\frac{\partial E}{\partial V} + \frac{P}{T}\right)\mathrm{d}V + \frac{1}{T}\frac{\partial E}{\partial T}\,\mathrm{d}T && \text{(1.42b)}\\
&= \frac{\partial S}{\partial V}\,\mathrm{d}V + \frac{\partial S}{\partial T}\,\mathrm{d}T\ , && \text{(1.42c)}
\end{aligned}$$

so we have identified the partial derivatives of the entropy. By Schwarz's lemma, we then have the equality:

$$\begin{aligned}
\frac{\partial^2 S}{\partial T \partial V} &= \frac{\partial^2 S}{\partial V \partial T} && \text{(1.43a)}\\
\frac{\partial}{\partial T}\left(\frac{1}{T}\frac{\partial E}{\partial V} + \frac{P}{T}\right) &= \frac{\partial}{\partial V}\left(\frac{1}{T}\frac{\partial E}{\partial T}\right) && \text{(1.43b)}\\
\frac{1}{T}\left(\cancel{\frac{\partial^2 E}{\partial T \partial V}} + \frac{\partial P}{\partial T}\right) - \frac{1}{T^2}\left(\frac{\partial E}{\partial V} + P\right) &= \cancel{\frac{1}{T}\frac{\partial^2 E}{\partial V \partial T}} && \text{(1.43c)}\\
\frac{\partial E}{\partial V} &= T\frac{\partial P}{\partial T} - P\,, && \text{(1.43d)}
\end{aligned}$$

but we can turn the derivatives with respect to V with ones with respect to $\rho \propto V^{-1}$, by

$$\frac{\partial}{\partial V} = \frac{\partial \rho}{\partial V}\frac{\partial}{\partial \rho} = -\rho^2\frac{\partial}{\partial \rho}\,, \tag{1.44}$$

so after dividing through by ρ we find:

$$\begin{aligned}
\rho\frac{\partial E}{\partial \rho} &= -\frac{T}{\rho}\frac{\partial P}{\partial T} + \frac{P}{\rho} && \text{(1.45a)}\\
&= -\frac{P}{\rho}\frac{\partial \log P}{\partial \log T} + \frac{P}{\rho} && \text{(1.45b)}\\
\rho\frac{\partial E}{\partial \rho} &= -\frac{P}{\rho}\left(\frac{\partial \log P}{\partial \log T} - 1\right) && \text{(1.45c)}
\end{aligned}$$

$$
\frac{\partial E}{\partial \log \rho} = -\frac{P}{\rho}(\chi_T - 1) \,. \tag{1.45d}
$$

We used the fact that

$$
\frac{\partial}{\partial \log x} = \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x} \,. \tag{1.46}
$$

Now, we can write the first law of thermodynamics for the specific energy density $E(\rho, T)$:

$$
\begin{aligned}
\mathrm{d}Q &= \mathrm{d}E - \frac{P}{\rho^2}\,\mathrm{d}\rho & (1.47\mathrm{a})\\
&= \frac{\partial E}{\partial \log \rho}\,\mathrm{d}\log\rho + \frac{\partial E}{\partial \log T}\,\mathrm{d}\log T - \frac{P}{\rho}\,\mathrm{d}\log\rho & (1.47\mathrm{b})\\
&= \left(-\frac{P}{\rho}(\chi_T - 1) - \frac{P}{\rho}\right)\mathrm{d}\log\rho + \frac{\partial E}{\partial \log T}\,\mathrm{d}\log T & (1.47\mathrm{c})\\
&= -\frac{P\chi_T}{\rho}\,\mathrm{d}\log\rho + \frac{\partial E}{\partial \log T}\,\mathrm{d}\log T \,, & (1.47\mathrm{d})
\end{aligned}
$$

which in the adiabatic ($\mathrm{d}Q = 0$) case reduces to

$$
\begin{aligned}
\Gamma_3 - 1 \overset{\mathrm{def}}{=} \frac{\partial \log T}{\partial \log \rho} &= -\frac{P\chi_T}{\rho}\frac{\partial \log T}{\partial E} = -\frac{P\chi_T}{\rho T c_V} & (1.48\mathrm{a})\\
&= -\frac{1}{\rho} \times \underbrace{\frac{P\chi_T}{T}}_{\frac{\partial P}{\partial T}} \times \underbrace{\frac{1}{c_V}}_{\frac{\partial T}{\partial E}} & (1.48\mathrm{b})\\
&= -\frac{1}{\rho}\frac{\partial P}{\partial E}\,, & (1.48\mathrm{c})
\end{aligned}
$$

where we used the fact that $c_V = \partial E / \partial T$. This can also be written as

$$
\frac{\partial E}{\partial P} = \frac{1}{\rho(\Gamma_3 - 1)} \,. \tag{1.49}
$$

If we drop the hypothesis of adiabaticity, we can study the variation with respect to time of Q, both when writing $E = E(\rho, T)$ and when writing $E = E(\rho, P)$. In the first case we can use equation (1.47a), "dividing through by $\mathrm{d}t$" (more formally, applying the differential covector equation to the vector ∂_t), after some manipulation we can bring out a factor $T\,\partial E / \partial T$ to get:

$$
\frac{\mathrm{d}Q}{\mathrm{d}t} = T\frac{\partial E}{\partial T}\left(\frac{\mathrm{d}\log T}{\mathrm{d}t} + \frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{T\frac{\partial E}{\partial T}}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right), \tag{1.50}
$$

and similarly if we express $E = E(\rho, P)$ we find

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = P\frac{\partial E}{\partial P}\left(\frac{\mathrm{d}\log P}{\mathrm{d}t} + \frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{P\frac{\partial E}{\partial P}}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right). \tag{1.51}$$

We can simplify these two expressions by recalling some results from before, plus two more expression we now derive for the coefficients Γ_1 and $\Gamma_3 - 1$: we assume adiabaticity, and get

$$0 = \mathrm{d}S = \frac{1}{T}\left(\frac{\partial E}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial E}{\partial P}\,\mathrm{d}P - \frac{P}{\rho^2}\,\mathrm{d}\rho\right) \tag{1.52a}$$

$$0 = \left(\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}\right)\mathrm{d}\log\rho + P\frac{\partial E}{\partial P}\,\mathrm{d}\log P \tag{1.52b}$$

$$\Gamma_1 = \left.\frac{\mathrm{d}\log P}{\mathrm{d}\log\rho}\right|_{\mathrm{ad}} = \frac{\frac{P}{\rho} - \rho\frac{\partial E}{\partial \rho}}{P\frac{\partial E}{\partial P}} \tag{1.52c}$$

and

$$0 = \mathrm{d}S = \frac{1}{T}\left(\frac{\partial E}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial E}{\partial T}\,\mathrm{d}T - \frac{P}{\rho^2}\,\mathrm{d}\rho\right) \tag{1.53a}$$

$$0 = \left(\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}\right)\mathrm{d}\log\rho + T\frac{\partial E}{\partial T}\,\mathrm{d}\log T \tag{1.53b}$$

$$\Gamma_3 - 1 = \left.\frac{\mathrm{d}\log T}{\mathrm{d}\log\rho}\right|_{\mathrm{ad}} = \frac{\frac{P}{\rho} - \rho\frac{\partial E}{\partial \rho}}{T\frac{\partial E}{\partial T}}, \tag{1.53c}$$

The identifications are:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = T\underbrace{\frac{\partial E}{\partial T}}_{c_V}\left(\frac{\mathrm{d}\log T}{\mathrm{d}t} + \underbrace{\frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{T\frac{\partial E}{\partial T}}}_{-(\Gamma_3-1)}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right), \tag{1.54}$$

and

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = P\underbrace{\frac{\partial E}{\partial P}}_{1/\rho(\Gamma_3-1)}\left(\frac{\mathrm{d}\log P}{\mathrm{d}t} + \underbrace{\frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{P\frac{\partial E}{\partial P}}}_{-\Gamma_1}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right). \tag{1.55}$$

Finally, we substitute in the equation of energy conservation:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m}, \tag{1.56}$$

to find the equations for the evolution of the energy and pressure:

$$\boxed{\begin{aligned} \frac{\partial \log P}{\partial t} &= \Gamma_1 \frac{\partial \log \rho}{\partial t} + \frac{\rho}{P}(\Gamma_3 - 1)\left(\varepsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m}\right) && \text{(1.57a)} \\ \frac{\partial \log T}{\partial t} &= (\Gamma_3 - 1)\frac{\partial \log \rho}{\partial t} + \frac{1}{c_V T}\left(\varepsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m}\right) && \text{(1.57b)} \end{aligned}}$$

1.2.5 Linear perturbation theory

Say we have a solution for these equations, we look at linear perturbations of them. This makes sense: the main solution is basically static on the pulsation time-scales.

The perturbed model is $f = f(m)$, the unperturbed one is $f_0(m)$. The Lagrangian perturbation is $\delta f(m,t) = f(m,t) - f_0(m,t)$.

Let us consider specific cases for f: the radial displacement is $\delta r(m,t)$. The position of the layer at time t is $r = r_0 + \delta r$.

We can write:

$$r = r_0\left(1 + \frac{\delta r}{r_0}\right) = r_0(1+\zeta), \tag{1.58}$$

where we define $\zeta = \delta r / r_0$.

In general the fractional perturbation $\delta f / f_0$ is assumed to be $\ll 1$. So, $\delta f / f_0 \sim \delta_f / f$. Formally, we only consider terms which are of first order in either perturbed function. We will insert expressions which are functions of perturbations of all our variables, and thus get linear differential equations.

Properties of Lagrangian perturbations

In general for a Lagrangian perturbation we have the following useful properties:

1. we can use the properties of derivatives: $\delta(f^n) = n f_0^{n-1} \delta f$;
2. we can use the properties of logarithmic derivatives:

$$\frac{\delta(\prod f_i)}{\prod f_i} = \sum \frac{\delta f_i}{f_i}; \tag{1.59}$$

3. δ commutes with partial derivation.

Continuity equation

Let us try the continuity equation, substituting in $r = r_0(1+\zeta)$ and $\rho = \rho_0(1+\delta\rho/\rho_0)$.

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{1.60a}$$

$$\frac{\partial}{\partial m}\big(r_0(1+\zeta)\big) = \frac{1}{4\pi r_0^2}(1+\zeta)^{-2}\left(1+\frac{\delta\rho}{\rho_0}\right)^{-1} \tag{1.60b}$$

and we use $(1+x)^n \approx 1+nx$ plus the zeroth order equation: $\partial r_0/\partial m = 1/4\pi r_0^2\rho_0$. With these, we find:

$$4\pi\rho_0^2\left(\frac{\partial r_0}{\partial m}(1+\zeta) + r_0\frac{\partial\zeta}{\partial m}\right) = (1-2\zeta) - \frac{\delta\rho}{\rho_0} \tag{1.61}$$

We can collapse the equation into:

$$\boxed{\frac{\delta\rho}{\rho_0} = -3\zeta - 4\pi r_0^3\rho_0\frac{\partial\zeta}{\partial m}} \tag{1.62}$$

or, the density perturbation is proportional with a negative constant to the radial perturbation, plus a term proportional to $\partial\zeta/\partial m$. If there is a positive gradient of radial perturbation, the corresponding layer expands.

Momentum conservation

Let us also perturb the momentum conservation equation; the unperturbed solution will be at hydrostatic equilibrium, so $\partial^2 r_0 \big/ \partial t^2 = \frac{\partial r_0}{\partial t} = 0$, which means

$$\frac{\partial P_0}{\partial m} = -\frac{Gm}{4\pi r_0^4}\,. \tag{1.63}$$

Substituting in we find that, to linear order:

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2\frac{\partial P}{\partial m} - \frac{Gm}{r^2} \tag{1.64a}$$

$$\frac{\partial^2}{\partial t^2}\big(r_0(1+\zeta)\big) = -4\pi\big(r_0(1+z)\big)^2\frac{\partial}{\partial m}\left(P_0\left(1+\frac{\delta P}{P_0}\right)\right) - \frac{Gm}{r_0^2(1+\zeta^2)} \tag{1.64b}$$

$$r_0 \frac{\partial^2 \zeta}{\partial t^2} = -4\pi r_0^2 (1+2\zeta) \left(\frac{\partial P_0}{\partial m} \left(1 + \frac{\delta P}{P_0} \right) + P_0 \frac{\partial}{\partial m} \left(\frac{\delta P}{P_0} \right) \right) - \frac{GM}{r_0^2} (1-2z) \tag{1.64c}$$

$$\frac{r_0}{4\pi r_0^2} \frac{\partial^2 \zeta}{\partial t^2} = -(1+2\zeta) \left(\frac{\partial P_0}{\partial m} \left(1 + \frac{\delta P}{P_0} \right) + P_0 \frac{\partial}{\partial m} \left(\frac{\delta P}{P_0} \right) \right) + (1-2z) \frac{\partial P_0}{\partial m} \tag{1.64d}$$

Used equation (1.63).

$$\begin{aligned} \frac{1}{4\pi r_0} \frac{\partial^2 \zeta}{\partial t^2} = &- \left(\frac{\partial P_0}{\partial m} \left(1 + \frac{\delta P}{P_0} \right) + P_0 \frac{\partial}{\partial m} \left(\frac{\delta P}{P_0} \right) \right) + \frac{\partial P_0}{\partial m} + \\ &- 2\zeta \left(\frac{\partial P_0}{\partial m} \left(1 + \frac{\delta P}{P_0} \right) + P_0 \frac{\partial}{\partial m} \left(\frac{\delta P}{P_0} \right) \right) - 2\zeta \frac{\partial P_0}{\partial m} \end{aligned} \tag{1.64e}$$

Split the terms into different orders of ζ

$$\begin{aligned} \frac{1}{4\pi r_0} \frac{\partial^2 \zeta}{\partial t^2} = &- \frac{\partial}{\partial m} (\delta P) + \\ &- 4\zeta \frac{\partial P_0}{\partial m} - 2\zeta \frac{\partial}{\partial m} (\delta P) \end{aligned} \tag{1.64f}$$

$$r_0 \frac{\mathrm{d}^2 \zeta}{\mathrm{d} t^2} = -4\pi r_0^2 \left(\frac{\partial}{\partial m} (\delta P) + 4\zeta \frac{\partial P_0}{\partial m} \right) . \tag{1.64g}$$

Neglected a second order term

The final equation then looks like

$$\boxed{r_0 \frac{\partial^2 \zeta}{\partial t^2} = -4\pi r_0^2 \left(\underbrace{P_0 \frac{\partial}{\partial m} \left(\frac{\delta P}{P_0} \right) + \frac{\delta P}{P_0} \frac{\partial P_0}{\partial m}}_{\frac{\partial \delta P}{\partial m}} + 4\zeta \frac{\partial P_0}{\partial m} \right) ,} \tag{1.65}$$

and we can now give a physical interpretation of the various terms.

We have a term $-16\pi r_0^2 \zeta \, \partial P_0 / \partial m = 4\zeta Gm / r_0^2$. This, by itself, is a force moving the system away from equilibrium: the equation with only that term on the RHS is precisely like a harmonic repulsor, $\ddot{\zeta} = \omega^2 z$ with

$$\omega^2 = \frac{4Gm}{r_0^3} . \tag{1.66}$$

This term is of geometric origin: as the layer moves outwards it expands, and the expansion is favoured by the decrease in the gravitational potential and the corresponding increase in pressure due to the increase of the area of the layer.

The equation with only the other term looks like

$$\ddot{\zeta} = -4\pi r_0 \frac{\partial \delta P}{\partial m} = -4\pi r_0 \left(P_0 \frac{\partial}{\partial m} \left(\frac{\delta P}{P_0} \right) + \frac{\delta P}{P_0} \frac{\partial P_0}{\partial m} \right) . \tag{1.67}$$

According to the slides, the action of this term is to be split in two: for the second, a restoring force towards equilibrium, since if the pressure decreases as the layer expands then the force is inward, and a more vague interpretation for the first bit, stating that "a non uniform variation of δP from a layer to the next generates a change in the pressure gradient": but it *is* a change in the pressure field…

Energy conservation

Let us also consider the expression for the time derivative of $\log P$ coming from the energy conservation equation: equation (1.57a); the perturbed equation for the time derivative of $\log T$ (equation (1.57b)) is analogously derived. Besides ρ and P, we also perturb the adiabatic exponents and the $\mathrm{d}Q/\mathrm{d}t = \epsilon_{\text{eff}} - \partial L/\partial m$ term.

After difficult manipulations we get back an equation which relates the changes in density and pressure to the change in energy: we manipulate until we get something which is similar to the original equation:

$$\frac{\partial}{\partial t}\left(\frac{\delta P}{P_0}\right) = \Gamma_{1,0}\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_0}\right) + \frac{\rho_0}{P_0}(\Gamma_{3,0}-1)\delta\left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m}\right) \tag{1.68a}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta T}{T_0}\right) = (\Gamma_{3,0}-1)\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_0}\right) + \frac{1}{c_V T}\delta\left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m}\right), \tag{1.68b}$$

where we note that the first index of the adiabatic exponents denotes *which exponent it is*, while the second indicates that it is the unperturbed value.

The last T in the temperature equation is not unperturbed nor perturbed...?

Luminosity equation

In the radiative case with the diffusion approximation we can perturb the luminosity equation, (1.11). It is much more convenient not to calculate δL but $\delta L/L_0$ instead: this allows us to use the logarithmic derivative properties of the perturbation; also, since

$$T^3\frac{\partial T}{\partial m} = T^4\frac{\partial \log T}{\partial m} \tag{1.69}$$

we use the latter expression, which is more convenient. So we have $L = \prod f_i$, with

$$f_i = \left\{-\frac{64\pi^2 ac}{3}, r^4, \kappa_R^{-1}, T^4, \frac{\partial \log T}{\partial m}\right\}. \tag{1.70}$$

Now we can apply the rule given in equation (1.59): we find

$$\frac{\delta L}{L_0} = 4\zeta + 4\frac{\delta T}{T_0} - \frac{\delta \kappa_R}{\kappa_{R,0}} + \left(\frac{\partial \log T}{\partial m}\right)_0^{-1} \frac{\partial}{\partial m}\left(\frac{\delta T}{T_0}\right), \tag{1.71}$$

where we used the simplification

$$\delta\left(\frac{\partial \log T}{\partial m}\right) = \delta\left(\frac{1}{T}\frac{\partial T}{\partial m}\right) \tag{1.72a}$$

$$= \delta\left(\frac{1}{T}\right)\frac{\partial T_0}{\partial m} + \frac{1}{T_0}\delta\left(\frac{\partial T}{\partial m}\right) \tag{1.72b}$$

$$= -\frac{\delta T}{T_0}\frac{\partial T_0}{\partial m} + \frac{1}{T_0}\frac{\partial \delta T}{\partial m} \tag{1.72c}$$

$$= \frac{\partial}{\partial m}\left(\frac{\delta T}{T_0}\right). \tag{1.72d}$$

Inverse application of the product rule

The last step is to assume a certain dependence of the Rosseland mean opacity on the temperature and density: specifically, it is a "Kramers-like" expression, given by

$$\kappa_R \propto \rho^n T^{-s}, \tag{1.73}$$

which can be substituted into our expression: the proportionality factor does not matter, and we get additional temperature and density terms:

$$\boxed{\frac{\delta L}{L_0} = 4\zeta + (4+s)\frac{\delta T}{T_0} - n\frac{\delta \rho}{\rho_0} + \left(\frac{\partial \log T}{\partial m}\right)_0^{-1} \frac{\partial}{\partial m}\left(\frac{\delta T}{T_0}\right),} \tag{1.74}$$

In the end, we have a set of four linear PDE equations (written as a 5-equation system).

These describe implicitly how the properties of the star change over time.

Pulsation usually affects mostly the outer layers of a star.

1.3 Adiabatic oscillations

1.3.1 Derivation of the LAWE

Exploiting the adiabatic approximation we will get the Linear Adiabatic Wave Equation (LAWE): a single equation which summarizes the 4 and can be solved explicitly.

Recall the heat transfer equation (1.38): if we suppose that each layer does not lose nor gain heat, $\mathrm{d}Q/\mathrm{d}t = 0$, this implies that $\delta(\varepsilon_{\text{eff}} - \partial L/\partial m) = 0$.

Is this approximation justified? The term multiplying $\delta(\varepsilon_{\text{eff}} - \partial L/\partial m)$ in the perturbed energy equation (1.68a) is $\rho(\Gamma_3 - 1)/P = \chi_T/(c_V T)$. Usually $\chi_T \sim 1$, while the density perturbation term is multiplied by $\Gamma_1 \sim 1$.

This term, $\chi_T/(c_V T)\delta(\varepsilon_{\text{eff}} - \partial L/\partial m)$, is of the order $1/\tau_{\text{th}}$, the thermal time scale of this layer, while the term before, $\Gamma_1\, \partial/\partial t\, (\delta\rho/\rho)$, is of the order $1/\tau_{\text{dyn}}$, the dynamical time scale.

Is this just because the first term contains a time derivative while the second one does not?

Therefore, we neglect the second part. This only works for the star as a whole, not for single layers. There are stellar layers which are *strongly* non-adiabatic (driving layers). We will need some non-adiabatic theory to explain how pulsations *start*.

So, the energy conservation equations become

$$\frac{\delta P}{P} = \Gamma_1 \frac{\delta\rho}{\rho} \qquad \text{and} \qquad \frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta\rho}{\rho}, \tag{1.75}$$

which we can substitute into the momentum conservation equation (1.65) to find

$$r_0 \frac{\partial^2 \zeta}{\partial t^2} = -4\pi r_0^2 \left(P_0 \frac{\partial}{\partial m}\left(\Gamma_1 \frac{\delta\rho}{\rho}\right) + \Gamma_1 \frac{\delta\rho}{\rho}\frac{\partial P_0}{\partial m} + 4\zeta \frac{\partial P_0}{\partial m}\right), \tag{1.76}$$

and now we can use the continuity equation (1.62) which gives us an expression for $\delta\rho/\rho$: inserting it we get

$$\begin{aligned} r_0 \frac{\partial^2 \zeta}{\partial t^2} = -4\pi r_0^2 \bigg[& P_0 \frac{\partial}{\partial m}\left(\Gamma_1\left(-3\zeta - 4\pi r_0^3 \rho_0 \frac{\partial \zeta}{\partial m}\right)\right) + \\ & + \Gamma_1\left(-3\zeta - 4\pi r_0^3\rho_0\frac{\partial\zeta}{\partial m}\right)\frac{\partial P_0}{\partial m} + 4\zeta\frac{\partial P_0}{\partial m}\bigg] \end{aligned} \tag{1.77a}$$

$$\begin{aligned} r_0 \frac{\partial^2 \zeta}{\partial t^2} = 4\pi r_0^2 \Bigg(& \underbrace{(3\Gamma_1 - 4)\zeta\frac{\partial P}{\partial m}}_{①} + \underbrace{4\pi r_0^3 \Gamma_1 \rho \frac{\partial\zeta}{\partial m}\frac{\partial P}{\partial m}}_{②} + \\ & + \underbrace{3P\frac{\partial}{\partial m}(\Gamma_1\zeta)}_{③} + \underbrace{4\pi P \frac{\partial}{\partial m}\left(\Gamma_1 r^3 \rho \frac{\partial\zeta}{\partial m}\right)}_{④}\Bigg), \end{aligned} \tag{1.77b}$$

and now we can can manipulate the terms inside the parentheses; we start to drop the zero indices, any quantity not being δ'd is meant to be unperturbed. The first and third terms are respectively given by

$$
① = \frac{\partial P}{\partial m}\zeta(3\Gamma_1 - 4) = \zeta\frac{\partial}{\partial m}\big((3\Gamma_1 - 4)P\big) - 3\zeta P\frac{\partial \Gamma_1}{\partial m} \tag{1.78}
$$

Backwards derivative of a product

and

$$
③ = 3P\frac{\partial}{\partial m}(\Gamma_1\zeta) = 3P\Gamma_1\frac{\partial \zeta}{\partial m} + 3P\zeta\frac{\partial \Gamma_1}{\partial m}, \tag{1.79}
$$

so we can see that the highlighted terms cancel. Also, the other term in equation (1.79) can be rewritten using the continuity equation (1.2):

$$
3P\Gamma_1\frac{\partial \zeta}{\partial m} = 12\pi r^2\rho\Gamma_1 P\frac{\partial \zeta}{\partial m}\frac{\partial r}{\partial m}, \tag{1.80}
$$

so we can see that if we expand the fourth term in (1.77b) we find a thing that is equal to it:

$$
④ = 4\pi P\frac{\partial}{\partial m}\left(r^3 \times \Gamma_1\rho\frac{\partial \zeta}{\partial m}\right) = 4\pi P \times 3r^2\frac{\partial r}{\partial m}\Gamma_1\rho\frac{\partial \zeta}{\partial m} + 4\pi P r^3\frac{\partial}{\partial m}\left(\Gamma_1\rho\frac{\partial \zeta}{\partial m}\right), \tag{1.81}
$$

so we will have twice that contribution in the final result.

For now then, we have shown that

$$
①+③+④ = \zeta\frac{\partial}{\partial m}\big((3\Gamma_1 - 4)P\big) + 2 \times 12\pi P r^2\frac{\partial r}{\partial m}\Gamma_1\rho\frac{\partial \zeta}{\partial m} + 4\pi P r^3\frac{\partial}{\partial m}\left(\Gamma_1\rho\frac{\partial \zeta}{\partial m}\right). \tag{1.82}
$$

Now then, the full equation reads

$$
\begin{aligned}
r_0\frac{\partial^2 \zeta}{\partial t^2} = 4\pi r^2\zeta\frac{\partial}{\partial m}\big((3\Gamma_1 - 4)P\big) + 16\pi^2 r_0^5\Gamma_1\rho\frac{\partial \zeta}{\partial m}\frac{\partial P}{\partial m} + \\
+96\pi^2 P r^4\frac{\partial r}{\partial m}\Gamma_1\rho\frac{\partial \zeta}{\partial m} + 16\pi^2 P r^5\frac{\partial}{\partial m}\left(\Gamma_1\rho\frac{\partial \zeta}{\partial m}\right),
\end{aligned} \tag{1.83a}
$$

and we can notice a certain similarity between the highlighted terms: consider the expression

$$
\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6\frac{\partial \zeta}{\partial m}\right), \tag{1.84}
$$

to which we can apply the general expression, which holds for nonzero differentiable functions of a certain variable x (and if it is interpreted as a limit, even if the functions go to 0):

$$\frac{\partial}{\partial x}\left(\prod_i f_i\right) = \left(\prod_i f_i\right)\sum_i \frac{1}{f_i}\frac{\partial f_i}{\partial x}, \tag{1.85}$$

so

$$\frac{\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6 \frac{\partial\zeta}{\partial m}\right)}{16\pi^2\Gamma_1 P\rho r^6 \frac{\partial\zeta}{\partial m}} = \frac{1}{P}\frac{\partial P}{\partial m} + \frac{1}{\Gamma_1\rho\, \partial\zeta/\partial m}\frac{\partial}{\partial m}\left(\Gamma_1\rho\frac{\partial\zeta}{\partial m}\right) + \frac{1}{r^6}\frac{\partial r^6}{\partial m} \tag{1.86a}$$

$$= \frac{1}{P}\frac{\partial P}{\partial m} + \frac{1}{\Gamma_1\rho\, \partial\zeta/\partial m}\frac{\partial}{\partial m}\left(\Gamma_1\rho\frac{\partial\zeta}{\partial m}\right) + \frac{6}{r}\frac{\partial r}{\partial m}, \tag{1.86b}$$

so

$$\frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6\frac{\partial\zeta}{\partial m}\right) = 16\pi^2 r^5\Gamma_1\rho\frac{\partial\zeta}{\partial m}\frac{\partial P}{\partial m} + \\ +16\pi^2 r^5 P\frac{\partial}{\partial m}\left(\Gamma_1\rho\frac{\partial\zeta}{\partial m}\right) + 96\pi^2 r^4\Gamma_1\rho\frac{\partial\zeta}{\partial m}P\frac{\partial r}{\partial m}. \tag{1.87}$$

Now, we can finally write the simplest form of the LAWE:

$$\boxed{r\frac{\partial^2\zeta}{\partial t^2} = 4\pi r^2\zeta\frac{\partial}{\partial m}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6\frac{\partial\zeta}{\partial m}\right),} \tag{1.88}$$

Next time, we will decompose: $\zeta(m,t) = \eta(m)e^{i\sigma t}$ with a constant σ: putting this into the LAWE we simplify the exponentials and get the space dependent form of the LAWE.

The LAWE is a Storm-Liouville equation.

8 October 2019

An ansatz for the LAWE

Last lecture we started talking about linearization & perturbation theory.

We use a complex spinning ansatz, which is by itself not physical, however its conjugate is also a solution so after we have found a solution we just need to take the real part.

Substituting $\zeta(m,t) = \eta(m)\exp(i\sigma t)$ into the LAWE we find that the exponentials simplify:

$$-r\sigma^2\eta = 4\pi r^2\eta \frac{\partial}{\partial m}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6 \frac{\partial \eta}{\partial m}\right). \tag{1.89}$$

Since we have an explicit r, it is convenient to rewrite this in the Eulerial formalism, using the continuity equation: we get

$$-r\sigma^2\eta = \frac{\eta}{\rho}\frac{\partial}{\partial r}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r^3\rho}\frac{\partial}{\partial r}\left(\Gamma_1 P r^4 \frac{\partial \eta}{\partial r}\right) \tag{1.90a}$$

$$\sigma^2\eta = -\frac{\eta}{\rho r}\frac{\partial}{\partial r}(3\Gamma_1 P - 4P) - \frac{1}{r^4\rho}\frac{\partial}{\partial r}\left(\Gamma_1 P r^4 \frac{\partial \eta}{\partial r}\right), \tag{1.90b}$$

Slide 4.37: there is a minus sign missing, right?

We will have analytic solutions for the adiabatic case, with the additional hypotheses of either $\Gamma_1 = 4/3$ or $\Gamma_1 > 4/3$ and homogeneity.

[The justification of the adiabatic approx might be asked at the exam.]

Boundary conditions for the LAWE

What are the boundary conditions we should set? The final result is:

1. $\delta r = 0$ at $r = 0$;
2. $\partial\eta/\partial r = 0$ at $r = 0$;
3. $(4 + R^3\sigma^2/(GM))\eta + \delta P/P = 0$ at $r = R$;
4. $\eta = \delta r/r = 1$ at $r = R$.

Let us show why. The first condition has an immediate physical basis: there cannot be radial displacement at $r = 0$ since there is nowhere to go at $r < 0$, and if there was positive radial displacement δr it would mean there is a vacuum in the region $r < \delta r$, which is unphysical because of the pressure there.

So, by the first condition, we need the term

$$\frac{1}{r^4\rho}\frac{\partial}{\partial r}\left(\Gamma_1 P r^4 \frac{\partial \eta}{\partial r}\right) = \frac{\Gamma_1 P}{\rho}\frac{\partial^2 \eta}{\partial r^2} + 4\frac{\Gamma_1 P}{\rho r}\frac{\partial \eta}{\partial r} + \frac{\Gamma_1}{\rho}\frac{\partial \eta}{\partial r}\frac{\partial P}{\partial r} + \frac{P}{\rho}\frac{\partial \eta}{\partial r}\frac{\partial \Gamma_1}{\partial r} \tag{1.91}$$

to be well behaved (nonsingular) at $r = 0$.

We know that $\partial P/\partial r$ and $\partial\Gamma_1/\partial r$ are zero at $r = 0$, so we can neglect the terms containing them. This is because any physical quantity must approach $r = 0$ with zero $\mathrm{d}/\mathrm{d}r$ slope, in order for it to be differentiable at the center.

In the second term we have a division by r: in order for it not to diverge we must ask that $\partial\eta/\partial r = 0$ at $r = 0$.

Since η is ζ times a phase, this also means that $\partial\zeta/\partial r = 0$ at $r = 0$. We can plug this into the linearized continuity equation, the Eulerian form of (1.62):

$$\frac{\delta\rho}{\rho} = -3\zeta - r\frac{\partial\zeta}{\partial r} \tag{1.92}$$

$$0 = \frac{\partial\zeta}{\partial r} = -\frac{1}{r}\left(3\zeta + \frac{\delta\rho}{\rho}\right) \tag{1.93}$$

$$\implies 0 = 3\zeta + \frac{\delta\rho}{\rho}\,, \tag{1.94}$$

into which we can substitute the linearized adiabatic expressions for the temperature and pressure perturbations (1.75): so we find

$$3\zeta = -\frac{1}{\Gamma_1}\frac{\delta P}{P} = -\frac{1}{\Gamma_3 - 1}\frac{\delta T}{T}\,, \tag{1.95}$$

We use the Eulerian form of the momentum conservation (1.65) to figure out the surface boundary conditions: it reads

$$r\sigma^2\eta = \frac{1}{\rho}\left(P\frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) + \frac{\delta P}{P}\frac{\partial P}{\partial r} + 4\eta\frac{\partial P}{\partial r}\right) \tag{1.96}$$

$$\frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) = \frac{\rho}{P}\left(r\sigma^2\eta\right) - \frac{1}{P}\left(\frac{\delta P}{P}\frac{\partial P}{\partial r} + 4\eta\frac{\partial P}{\partial r}\right) \tag{1.97}$$

$$= \frac{\rho r\sigma^2\eta}{P} - \frac{\delta P}{P}\frac{\partial \log P}{\partial r} - 4\eta\frac{\partial \log P}{\partial r} \tag{1.98}$$

$$= -\frac{\partial \log P}{\partial r}\left(-\frac{\rho r\sigma^2\eta}{\partial P/\partial r} + 4\eta + \frac{\delta P}{P}\right)\,, \tag{1.99}$$

since we assumed that all perturbations are *in phase* to the radial one, and thus wrote them all as proportional to $\exp(i\sigma t)$, which then simplified. We define the *pressure scale height* as:

$$H_P = -\left(\frac{\partial \log P}{\partial r}\right)^{-1}\,, \tag{1.100}$$

it represents "how far we should move in the star for the pressure to change e-fold".

We insert this into the equation; also we use the Eulerian momentum conservation equation

$$\frac{\partial P}{\partial r} = -\rho\frac{Gm}{r^2}\,, \tag{1.101}$$

so that we find

$$\frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) = \frac{1}{H_P}\left(\left(4 + \frac{r^3\sigma^2}{Gm}\right)\eta + \frac{\delta P}{P}\right), \tag{1.102}$$

Since outside the star the pressure is zero, we must have that $H_P \to 0$ when $r \to R$: the pressure must change by an infinite number of e-folds to reach zero, so the displacement needed for a single e-fold change must go to zero. Therefore, the term multiplying $1/H_P$ must also go to zero to avoid divergences: so we ask

$$\left(4 + \frac{r^3\sigma^2}{Gm}\right)\eta + \frac{\delta P}{P} \to 0 \tag{1.103}$$

as $r \to R$. Since we are considering the mass variable at the edge of the star, we also must have $m = M$.

The last condition, $\eta = \delta r / r = 1$ at $r = R$ comes from the fact that we want our study to give us *periods*, not *amplitudes*: we cannot find those out, so we normalize in a way that is convenient. The LAWE is 1-homogeneous!

Solving the LAWE

The LAWE can be written compactly with a linear operator $\mathcal{L}$:

$$\mathcal{L}(\eta) = \sigma^2\eta\,, \tag{1.104}$$

where the expression for $\mathcal{L}$ can be read off equation (1.90b). The equation is in the form of a Sturm-Liouville differential equation; in general those have the form

$$\frac{\partial}{\partial r}\left(p(r)\frac{\partial\eta}{\partial r}\right) + \lambda t(r)\eta - s(r)\eta = 0\,. \tag{1.105}$$

Figure out signs! Why does nothing make sense?

So, the eigenvalue σ^2 is the square of the pulsation. There are infinitely many solutions to the LAWE, only some (finitely many?) fulfill the boundary condition. The eigenvalues are real since $\mathcal{L}$ is Hermitian, and they have a wavefunction associated: $\eta_m(r)$ corresponding to σ_m^2.

If $\sigma^2 > 0$ we have an oscillating solution, if $\sigma^2 < 0$ we have an exponential collapse or explosion since the solution is proportional to $\exp(i\sigma t)$.

We label solutions by *radial order* $m \in \mathbb{N}$: $m = 0$ has the lowest frequency, and then we have overtones. We choose the labels so that $\sigma_{m_1} < \sigma_{m_2} \iff m_1 < m_2$.

The radial order m is also the number of nodes.

The eigenfunctions are orthogonal wrt the scalar product

$$\langle \eta_m | \eta_n \rangle = \int_0^R \eta_m \eta_n \rho r^4 \,\mathrm{d}r = f(n)\delta_{nm} \tag{1.106}$$

Possibly there is a 4π missing in order for this to be consistent with the following?

The functions ζ_m are orthogonal wrt the same product. The system is linear: we can write a general solution as a superposition.

We can define the moment of inertia:

$$J_m = \int_0^M |\zeta_m|^2 r^2 \,\mathrm{d}m \tag{1.107}$$

and then we can recover the eigenvalue by:

$$\sigma_m^2 = \frac{1}{J_m} \int_0^M \zeta_m^* \mathcal{L} \zeta_m r^2 \,\mathrm{d}m = \frac{\langle \zeta_m | \mathcal{L} | \zeta_m \rangle}{\langle \zeta_m | \zeta_m \rangle} \tag{1.108}$$

1.3.2 Simplifications

Period-mean density relation

If $\eta = \text{const}$, and ρ and Γ_1 are also constanst, we can remove the $\partial\eta/\partial r$ terms: so we have

$$\sigma^2 \eta + \frac{\eta}{\rho r} \frac{\partial}{\partial r} \big(P(3\Gamma_1 - 4) \big) = 0 \tag{1.109}$$

$$\sigma^2 = -\frac{3\Gamma_1 - 4}{\rho r} \frac{\partial P}{\partial r} \tag{1.110}$$

$$\sigma^2 = (3\Gamma_1 - 4) \frac{Gm}{r^3}, \tag{1.111}$$

and by inserting the mean density formula $\overline{\rho} = M / \left(4\pi R^3/3 \right)$ we get the period-mean density relation: $\sigma = 2\pi/T$, so we have $T^{-2} \propto \overline{\rho}$, the period-mean density relation.

Actually, this simplified model gives us $m/r^3 = \text{const}$: the density in this case must be constant and equal to $\overline{\rho}$ (even if we did not use the assumption before). The precise relation is given by

$$T^2 \overline{\rho} = \frac{3\pi}{(3\Gamma_1 - 4)G}, \tag{1.112}$$

Polytropic model

It is a gas sphere with the following constitutive equaition:

$$P = K_n \rho^{1+1/n} = K_n \rho^{\frac{n+1}{n}} \tag{1.113}$$

with varying *polytropic* index n.

This can be derived as a solution for the Lane-Emden equation, which comes from Poisson's equation for the gravitational potential: so we do the following manipulation

Laplacian in spherical coordinates under spherical symmetry

Used the hydrostatic equilibrium equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{1.114}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho \tag{1.115}$$

$$-\frac{1}{r^2}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = 4\pi G\rho \tag{1.116}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho r^2 \,, \tag{1.117}$$

so if we assume $\rho = \rho_c \theta^n$ the polytropic equation of state will read

$$P = K_n \rho^{1+1/n} = K_n \rho_c^{1+1/n} \theta^{n(1+1/n)} = K_n \rho_c^{1+1/n} \theta^{n+1} \,, \tag{1.118}$$

where ρ_c is the central density, so we have $\theta(r=0) = 1$ and $\theta'(r=0) = 0$. Inserting this into Poisson's equation gives us

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho_c \theta^n}\frac{\mathrm{d}}{\mathrm{d}r}\left(K_n \rho_c^{1+1/n}\theta^{n+1}\right)\right) = -4\pi G\rho r^2 \tag{1.119}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 (n+1) K_n \rho_c^{1/n} \frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -4\pi G\rho r^2 \,, \tag{1.120}$$

so we can adimensionalize the radial coordinate by putting inside of it all the numbers: $r = \alpha\xi$, with α chosen so that the equation turns into

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) + \theta^n = 0 \,. \tag{1.121}$$

This is always solvable, and it is solvable analytically for certain values of n.

It models spheres with different mass distributions: $n = 0$ is constant density, $n = 5$ is infinite central density, $n = 3$ is the Eddington standard model, which is reasonable for the Sun and stars on the main sequence.

With these assumptions we can explicitly solve for the wavefunctions, and make predictions of the fractional modulus of the oscillations at a certain radius wrt the modulus at the surface (which can be found only experimentally).

The overtones die out toward the center even faster than the fundamental: these oscillations are very much a *surface phenomenon*.

Beyond the ηs we can also plot the pressure perturbations: these will not be normalized.

Slide 4.108: What is ω? Should we not have $\sigma = \omega$ of the mode? It seems like $\mathcal{P}$ is the period, and the numbers are consistent with $\mathcal{P} = 2\pi/\sigma$

Answer, from the first equation in Schwarzschild 1941[a]: ω and σ are related by

$$\omega = \frac{\sigma}{\sqrt{\pi G \Gamma_1 \rho_{0c}}}, \tag{1.122}$$

so ω is adimensional: it is, in a certain sense, an adimensionalized frequency. From the data in slide 4.108 we can derive the value of $\Gamma_1 \rho_{0c} = (\sigma/\omega)^2/(\pi G) \approx 5.184 \times 10^{14}\,\mathrm{kg/m^3}$.

[a] https://ui.adsabs.harvard.edu/abs/1941ApJ....94..245S/abstract

Numerical RR Lyrae models

We integrate the stellar structure equations numerically, so instead of a simple polytropic equilibrium model our equilibrium model is numerical. This is done looking at an RR Lyrae variable stars.

What are their characteristics?

We find that the ratio between the period of the fundamental and of the first overtone is around $P_1/P_0 \approx 0.743 \div 0.745$ in observed RR Lyrae, while the LAWE model predicts 0.731. The observed period is around $P_0 \approx 0.5668\,\mathrm{d}$, while the model predicts $P_0 \sim 0.6454\,\mathrm{d}$.

We can see that $\sigma_m - \sigma_{m-1} \approx$ const when m gets large, see figure 1.1.

The wavefunctions die out faster than the polytropic model when $r \to 0$.

There are "bumps" in the pressure perturbation plot: these are the partial ionization regions of H and He.

These appear because we start from a solution of the stellar structure equations, where all the properties of stellar matter were used, to start off with the LAWE.

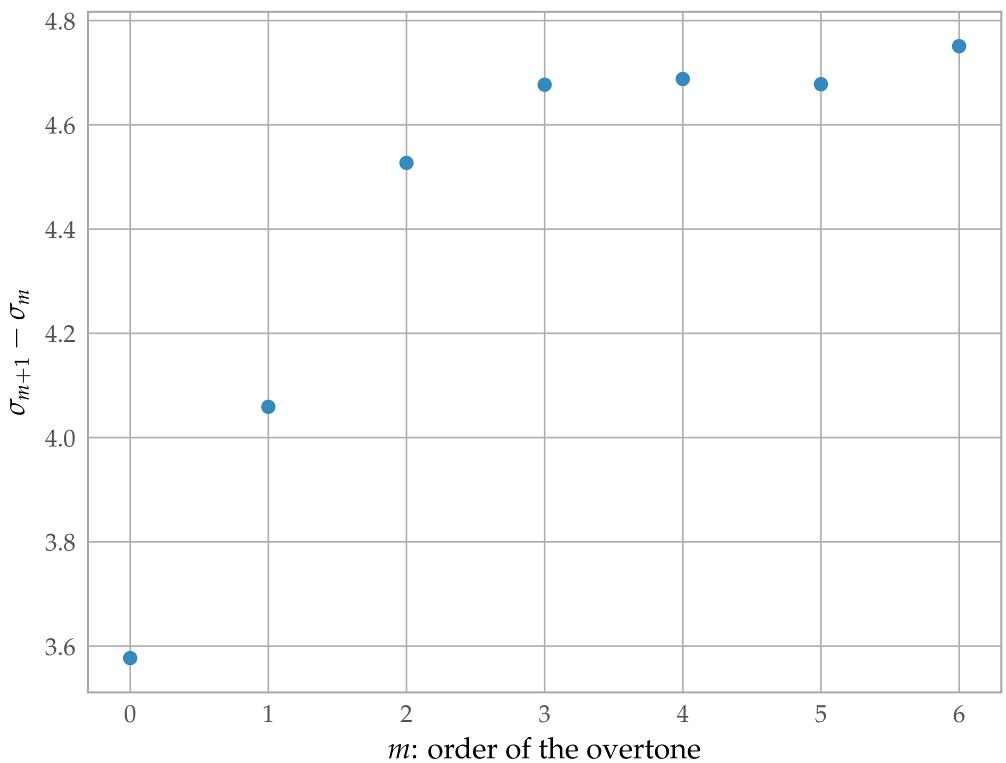

Figure 1.1: RR Lyrae frequency differences. The frequencies are measured in rad/d.

1.4 Non-adiabatic oscillations

1.4.1 Driving and damping layers

How can we tell, theoretically, how stable and how wide the various modes are? We expect to see the stable modes, and not to see the unstable ones.

Let us start from the Lagrangian momentum conservation:

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2} \tag{1.123}$$

and apply to it the identity: $1/2 \frac{\partial}{\partial t} v^2 = \frac{\partial r}{\partial t} \frac{\partial^2 r}{\partial t^2}$, by multiplying everything by $\partial r / \partial t$, and then we integrate everything with respect to m: we find

$$\int_M \frac{1}{2} \frac{\partial}{\partial t} \left(v^2 \right) \mathrm{d}m = - \int_M 4\pi r^2 \frac{\partial P}{\partial m} \frac{\partial r}{\partial t} \mathrm{d}m - \int_M \frac{Gm}{r^2} \frac{\partial r}{\partial t} \mathrm{d}m \,. \tag{1.124}$$

Now we apply some manipulations: first of all, the following:

$$\int_M \left(-\frac{Gm}{r^2}\frac{\partial r}{\partial t} \right) \mathrm{d}m = \int_M \frac{\partial}{\partial t}\left(\frac{Gm}{r} \right) \mathrm{d}m \tag{1.125}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{Gm}{r}\,\mathrm{d}m \tag{1.126}$$

$$= -\frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{1.127}$$

where Ω is the total gravitational potential energy, while

$$-\int_M 4\pi r^2 \frac{\partial P}{\partial m}\frac{\partial r}{\partial t}\,\mathrm{d}m = -\int_M \frac{\partial}{\partial m}\left(4\pi r^2 P \frac{\partial r}{\partial t} \right) \mathrm{d}m + \int_M P \frac{\partial}{\partial m}\left(4\pi r^2 \frac{\partial r}{\partial t} \right) \mathrm{d}m \tag{1.128}$$

$$= \underbrace{- \left. 4\pi r^2 P \frac{\partial r}{\partial t} \right|_{m=0}^{m=M}}_{=0} + \int_M \frac{4\pi}{3} P \frac{\partial}{\partial m}\frac{\partial}{\partial t}\left(r^3 \right) \mathrm{d}m \tag{1.129}$$

$$= \int_M P \frac{\partial}{\partial t} \underbrace{\left(4\pi r^2 \frac{\partial r}{\partial m} \right)}_{1/\rho} \mathrm{d}m \tag{1.130}$$

Commuted m and r derivatives

$$= \int_M P \frac{\partial}{\partial t}\left(\rho^{-1} \right) \mathrm{d}m\,, \tag{1.131}$$

so in the end we get

$$\frac{\partial}{\partial t}\int_M \frac{v^2}{2}\,\mathrm{d}m = -\frac{\mathrm{d}\Omega}{\mathrm{d}t} + \int_M P \frac{\partial}{\partial t}\frac{1}{\rho}\,\mathrm{d}m \tag{1.132}$$

We integrate in time over a pulsation period, which cancels out the gravitational potential term which is conservative:

$$-\int_\Pi \frac{\mathrm{d}\Omega}{\mathrm{d}t}\,\mathrm{d}t = 0\,. \tag{1.133}$$

If we also divide by the period, we are computing an average over a period Π. The term

$$\left\langle \frac{\partial}{\partial t}\int_M \frac{v^2}{2}\,\mathrm{d}m \right\rangle_\Pi = \frac{1}{\Pi}\int_\Pi \frac{\partial}{\partial t}\int_M \frac{v^2}{2}\,\mathrm{d}m\,\mathrm{d}t = \frac{\Delta E_{\mathrm{kin}}}{\Pi} \tag{1.134}$$

is the average power converted to kinetic energy; the variation in kinetic energy is also the work done on the system: $\Delta E_{\mathrm{kin}} = W$. So, taking the average on both sides of the equation we get

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle_\Pi = \frac{1}{\Pi}\int_\Pi \int_M P \frac{\partial}{\partial t}\left(\frac{1}{\rho} \right) \mathrm{d}m\,\mathrm{d}t\,. \tag{1.135}$$

Some layers will provide energy to the oscillation motion (*drive* it), some others will *damp* it. These are characterized by the sign of their contribution the RHS of this equation.

If it is positive, we have instability since more and more work is being done one the star every period; if it is negative the pulsations will tend to die out, giving stability, since they lose kinetic energy every pulse.

The average time scale of change of the perturbations is

$$\kappa \stackrel{\text{def}}{=} \frac{1}{\tau} = -\frac{1}{2}\frac{\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle_\Pi}{\langle \delta\psi \rangle_\Pi}, \tag{1.136}$$

where $\langle \delta\psi \rangle$ is the "average pulsational energy per pulsation cycle":

so, the integral of the absolute value of $\mathrm{d}W/\mathrm{d}t$? This ψ was not defined... Also, this way τ can be negative: maybe the proper definition is $\tau = 1/|\kappa|$?

So, if $\kappa < 0$ we are looking at a driving layer, since then $\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle > 0$, while $\kappa > 0$ means we are in a damping layer.

The term $\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle_\Pi$ can also be interpreted as the net heat gain fed into mechanical work during a pulsation cycle: this can be derived from the first law of thermodynamics (written in its per-unit-mass form, and with the differentials being applied to time derivatives):

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + P\frac{\mathrm{d}(\rho^{-1})}{\mathrm{d}t}, \tag{1.137}$$

so if this is averaged over a period the internal energy change is approximately zero (thermal (i. e. internal) energy changes on the thermal timescale, which is much longer than the dynamical timescale), so we have

$$\left\langle \frac{\mathrm{d}Q}{\mathrm{d}t} \right\rangle_\Pi = \left\langle P\frac{\mathrm{d}}{\mathrm{d}t}(\rho^{-1}) \right\rangle, \tag{1.138}$$

and therefore we can rewrite equation (1.135) as

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle_\Pi = \frac{1}{\Pi}\int_\Pi \int_M \frac{\mathrm{d}Q}{\mathrm{d}t}\,\mathrm{d}m\,\mathrm{d}t\,, \tag{1.139}$$

where Q is a heat per unit mass, so its integral in $\mathrm{d}m$ is the total heat change across all stellar layers.

In the adiabatic case, we had

$$\frac{\partial}{\partial t}\left(\frac{\delta P}{P} - \Gamma_1 \frac{\delta\rho}{\rho} \right) = 0, \tag{1.140}$$

(see equation (1.68a), in which we neglected the heat variation terms in the adiabatic case) so the perturbations were in phase.

Now we add a term to the time derivatives: the pressure and density perturbations will stop being in phase. The sign of the heat variation term gives us the difference between *driving* heat transfer and *damping* heat transfer. Equation (1.68a) will now look like

$$\frac{\partial}{\partial t}\frac{\delta P}{P} = \Gamma_1 \frac{\partial}{\partial t}\frac{\delta \rho}{\rho} + \frac{P}{\rho}(\Gamma_3 - 1)\delta \frac{\mathrm{d}Q}{\mathrm{d}t}\,, \tag{1.141}$$

so the instants of minimum pressure and minimum density will not be synchronized anymore, since there is that extra $\mathrm{d}Q/\mathrm{d}t$ term.

In a PV diagram, we can see that these correspond to right and left oriented loops (as opposed to the loops with zero total signed area we had in the adiabatic case).

The total work performed on the star is given by the area of the loops: the area is positive for a loop going clockwise, and negative for a loop going counter-clockwise.

If the term $\delta\Big(\mathrm{d}Q/\mathrm{d}t\Big) > 0$ we are looking at a *driving layer*, since at the time of maximum compression ($\partial/\partial t\ (\delta\rho/\rho) = 0$) we will have $\partial/\partial t\ (\delta P/P) > 0$: the maximum pressure perturbation will happen as the layer is expanding, so the expansion will be amplified. The symmetric reasoning gives us the fact that if $\delta\Big(\mathrm{d}Q/\mathrm{d}t\Big) < 0$ we have a *damping layer*.

There tend to be more driving layers in the outer parts of the star, especially peaking in power generation in the partial Helium and Hydrogen ionization regions.

What is going on in these regions? Why do they act this way?

The star is effectively a themal engine converting heat into work; this will result in an increased overall entropy of the star, and a smoothing of its temperature gradient, however:

1. the time scales on which this process occurs are much larger than the time scales on which oscillating motions are created and destroyed (specifically, the entropy changes on the thermal time scale);

2. then energies of the oscillations are much smaller than the global thermal energy of the star.

therefore this process is typically not relevant.

Mon Oct 14 2019

1.4.2 The LNAWE

First of all, let us recall the linearized equations of stellar structure:

$$\frac{\delta\rho}{\rho} = -3\zeta - 4\pi r^3 \rho \frac{\partial \zeta}{\partial m} \tag{1.142}$$

$$r\ddot{\zeta} = -4\pi r^2 \left[\left(4\zeta + \frac{\delta P}{P} \right) \frac{\partial P}{\partial m} + P \frac{\partial}{\partial m} \left(\frac{\delta P}{P} \right) \right] \tag{1.143}$$

$$\frac{\partial}{\partial t} \left(\frac{\delta P}{P} \right) = \Gamma_1 \frac{\partial}{\partial t} \left(\frac{\delta \rho}{\rho} \right) + \frac{\rho}{P} (\Gamma_3 - 1) \delta \left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m} \right) \tag{1.144}$$

$$\frac{\partial}{\partial t} \left(\frac{\delta T}{T} \right) = (\Gamma_3 - 1) \frac{\partial}{\partial t} \left(\frac{\delta \rho}{\rho} \right) + \frac{1}{c_V T} \delta \left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m} \right) \tag{1.145}$$

$$\frac{\delta L}{L} = 4\zeta - n \frac{\delta \rho}{\rho} + (s+4) \frac{\delta T}{T} + \left(\frac{\partial \log T}{\partial m} \right)^{-1} \frac{\partial}{\partial m} \left(\frac{\delta T}{T} \right), \tag{1.146}$$

using which we can derive after long manipulations the Linear Non-Adiabatic Wave Equation, which in its Lagrangian formulation reads:

$$\dddot{\zeta} = 4\pi r \left(\dot{z} \frac{\partial}{\partial m} ((3\Gamma_1 - 4)P) - \frac{\partial}{\partial m} \left(\rho (\Gamma_3 - 1) \delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right) \right) \right) + \frac{1}{r^2} \frac{\partial}{\partial m} \left(16\pi^2 \Gamma_1 P \rho r^6 \frac{\partial \dot{\zeta}}{\partial m} \right), \tag{1.147}$$

where we used the fact that $\mathrm{d}Q/\mathrm{d}t = \epsilon_{\text{eff}} - \partial L/\partial m$. On the other hand, the Eulerian formulation is

$$\dddot{\zeta} = \frac{1}{r\rho} \left(\dot{\zeta} \frac{\partial}{\partial r} ((3\Gamma_1 - 4)P) - \frac{\partial}{\partial r} \left(\rho (\Gamma_3 - 1) \delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right) \right) \right) + \frac{1}{r^4 \rho} \frac{\partial}{\partial r} \left(r^4 \Gamma_1 P \frac{\partial \dot{\zeta}}{\partial r} \right). \tag{1.148}$$

The procedure to derive the LNAWE is as follows:

1. substitute the continuity equation (1.142) into the P, ρ form of the energy conservation equation (1.144);
2. substitute the P, ρ form of the energy conservation equation (1.144) into the time derivative of the Eulerian form of the momentum conservation equation

(1.143);

3. simplify.

1.4.3 Solving the LNAWE

Our ansatz for the LNAWE will still be of the form $\zeta(r,t) = \eta(r)e^{i\sigma t}$, but now we insert $\sigma = \omega + i\kappa$: this means we also consider *damped* exponential solutions and *diverging* exponential solutions. We want to simplify the exponentials, so we must assume that

$$\delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right) = \delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\mathrm{sp}} e^{i\sigma t}. \tag{1.149}$$

Why do we assume that the heat derivative perturbation is *in phase* with the displacement? Maybe we do not, and the $_{\mathrm{sp}}$ heat variation is complex?

With this substitution we get:

$$-i\sigma^3\eta = \frac{i\sigma\eta}{r\rho}\frac{\partial}{\partial r}((3\Gamma_1 - 4)P) - \frac{1}{r\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3 - 1)\delta\left(\frac{\partial Q}{\partial t}\right)_{\mathrm{sp}}\right) + \frac{i\sigma}{r^4\rho}\frac{\partial}{\partial r}\left(r^4\Gamma_1 P\frac{\partial\eta}{\partial r}\right). \tag{1.150}$$

The time scales for these parameters are $\omega \sim \omega_{\mathrm{ad}} \sim \tau_{\mathrm{dyn}}$, while $\kappa \sim 1/\tau_{\mathrm{th}}$: therefore $\omega \gg |\kappa|$.

Using this result, we can make some useful *quasi-adiabatic* approximations: in the LNAWE we will identify the LAWE operator $\mathcal{L}$, and replace its application to the wavefunction with the corrisponding eigenvalue. Basically, we will consider the thermal contribution to be small and work "to first order" with it.

$$\begin{aligned} -i\sigma^3\eta &= -i\sigma\left(-\frac{1}{r\rho}\frac{\partial}{\partial r}((3\Gamma_1 - 4)P) - \frac{1}{r^4\rho}\frac{\partial}{\partial r}r^4\Gamma_1 P\frac{\partial\eta}{\partial r}\right) + \\ &\qquad - \frac{1}{r\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3 - 1)\delta\left(\frac{\partial Q}{\partial t}\right)_{\mathrm{sp}}\right) \end{aligned} \tag{1.151}$$

$$-i\sigma^3\eta = -i\sigma\mathcal{L}(\eta) - \frac{1}{r\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3 - 1)\delta\left(\frac{\partial Q}{\partial t}\right)_{\mathrm{sp}}\right) \tag{1.152}$$

$$\mathcal{L}(\eta) - \sigma^2\eta = \frac{i}{r\sigma\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3 - 1)\delta\left(\frac{\partial Q}{\partial t}\right)_{\mathrm{sp}}\right), \tag{1.153}$$

so we can see that the eigenvalue of the LAWE operator cannot be σ^2 now. We take this equation, multiply it by ηr^2 and integrate it over the whole star in $\mathrm{d}m$: we get

$$i\sigma^3 \int \eta^2 r^2 \,\mathrm{d}m - i\sigma \int \eta \mathcal{L}(\eta) r^2 \,\mathrm{d}m = \int \frac{r}{\rho} \frac{\partial}{\partial r} \left(\rho (\Gamma_3 - 1) \delta \left(\frac{\partial Q}{\partial r} \right)_{\mathrm{sp}} \right) \eta \,\mathrm{d}m \overset{\mathrm{def}}{=} C \,, \tag{1.154}$$

where we defined the *work integral* C. We only look at the first order terms in κ: so we make the approximation $i\sigma^3 \approx \omega^2(i\omega - 3\kappa)$, while (to first order, but also exactly) $i\sigma = i\omega - \kappa$. We substitute these two, and then make the key manipulation: we substitute $\mathcal{L}(\eta)$ with $\omega^2 \eta$. The only thing missing is the definition: $J \overset{\mathrm{def}}{=} \int \eta^2 r^2 \,\mathrm{d}m$. So we get

$$\omega^2 (i\omega - 3\kappa) J - (i\omega - \kappa)\omega^2 J = C \tag{1.155}$$

$$-3\kappa + \kappa = \frac{C}{J\omega^2} \tag{1.156}$$

$$\kappa = -\frac{C}{2\omega^2 J} \,. \tag{1.157}$$

Now we make some considerations on the expression of C and integrate by parts, getting:

$$C = \int_M \frac{1}{r\rho} \frac{\partial}{\partial r} \left(\rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\mathrm{sp}} \right) \eta r^2 \,\mathrm{d}m \tag{1.158}$$

$$= \int_R \frac{\partial}{\partial r} \left(\rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\mathrm{sp}} \right) 4\pi r^3 \eta \,\mathrm{d}r \tag{1.159}$$

$\mathrm{d}m = 4\pi r^2 \rho \,\mathrm{d}r$

$$= \rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\mathrm{sp}} 4\pi r^3 \eta \Bigg|_{r=0}^{r=R} - \int_R \rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\mathrm{sp}} \frac{\partial}{\partial r} \left(4\pi r^3 \eta \right) \mathrm{d}r \tag{1.160}$$

$$= -\int_R \rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\mathrm{sp}} 4\pi r^2 \left(3\eta + r\frac{\partial \eta}{\partial r} \right) \mathrm{d}r \tag{1.161}$$

$\rho = 0$ if $r = R$

$$= \int_R \rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\mathrm{sp}} 4\pi r^2 \left(\frac{\delta\rho}{\rho} \right)_{\mathrm{sp}} \mathrm{d}r \tag{1.162}$$

From equation (1.142): $\frac{\delta\rho}{\rho} = -3\zeta - r\frac{\partial\zeta}{\partial r}$

$$= \int_R (\Gamma_3 - 1) \left(\frac{\delta\rho}{\rho} \right)_{\mathrm{sp}} \delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\mathrm{sp}} \mathrm{d}m \tag{1.163}$$

$$= \int_M \left(\frac{\delta T}{T} \right)_{\mathrm{sp}} \delta \left(\epsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m} \right) \mathrm{d}m \,, \tag{1.164}$$

From the time-integrated equation (1.145), neglecting the heat transfer term and now this expression allows us to study the mechanisms which create perturbations: the coefficient to calculate is κ, which we now know to be given by

$$\kappa = -\frac{1}{2\omega^2 J} \int_M \left(\frac{\delta T}{T} \right)_{\text{sp}} \delta \left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m} \right) \mathrm{d}m \,, \tag{1.165}$$

which we will study in the next section.

1.4.4 Driving mechanisms

The energy of the vibrations comes from the internal thermal energy of the star, which ultimately comes from thermonuclear reactions. We can rewrite equation (1.165) as

$$\kappa = -\frac{1}{2\omega^2 J} \underbrace{\int_M \left(\frac{\delta T}{T} \right)_{\text{sp}} \delta \epsilon_{\text{eff}} \, \mathrm{d}m}_{\text{energy generation}} + \frac{1}{2\omega^2 J} \underbrace{\int_M \left(\frac{\delta T}{T} \right)_{\text{sp}} \frac{\partial \delta L}{\partial m} \, \mathrm{d}m}_{\text{energy transfer}} \,. \tag{1.166}$$

The ϵ-mechanism is about energy generation, which is assumed to be due to nuclear reactions, without considering neutrino processes: this can happen if the magnitude of the temperature and density perturbations are large enough.

The κ-γ-mechanism is about considering the regions where the luminosity gradient and temperature gradient are discordant. This means that the considered stellar layer is absorbing or emitting; this is usually assumed to be happening through free-free interactions (bremsstrahlung and inverse bremsstrahlung).

Tue Oct 15 2019

ϵ mechanism

We now concern ourselves with the energy generation term of equation (1.166).

In general the effective energy generation per unit mass is given by

$$\epsilon_{\text{eff}} = \epsilon_{\text{nuc}} - \epsilon_{\nu} \,, \tag{1.167}$$

but we assume that the neutrino loss of energy contribution in the perturbation is negligible, so when we perturb we will have $\delta \epsilon_{\text{eff}} = \delta \epsilon_{\text{nuc}}$. We also have the approximate relation

$$\frac{\delta T}{T} = (\Gamma_3 - 1) \frac{\delta \rho}{\rho} \,, \tag{1.168}$$

which implies, since $\Gamma_3 > 1$, that the temperature and density relative perturbations are concordant in sign.

Why can we use this relation? Does it have something to do with working "at first order in $\mathrm{d}Q/\mathrm{d}t$"?

Now, the line of reasoning goes like this: the nuclear energy generation perturbation is concordant in sign with the temperature perturbation. So, the term

$$\left(\frac{\delta T}{T}\right)_{\mathrm{sp}} \delta\epsilon_{\mathrm{eff}} \geq 0\,, \tag{1.169}$$

which means that the contribution to κ will be negative (because of the minus sign in equation (1.166)): so, since the dependence of the radial perturbation looks like $\zeta \propto \exp(-\kappa t)$, the perturbation is *amplified*: this is a driving layer.

The reasoning in slide 6.08 seems way too contorted: plus, it mentions the layers "absorbing energy", but we cannot actually talk about that without considering the luminosity gradient perturbation...

Are the fluctuations in ϵ_{nuc} actually enough to power pulsations? *No*. An illustrative example is given by RR-Lyrae stars. There, we can look at the relative magnitude of the realative temperature variation, $\delta T/T$, at various layers in the star: it is of the order 10^{-1} or even more at the surface, but quickly drops as we move towards the center; in the Hydrogen burning shell and in the Helium burning core it is of the order $10^{-9} \div 10^{-7}$, which implies that the actual temperature fluctuations are of the order $\delta T \sim 1\,\mathrm{K}$: way too little to amplify pulsations.

κ-γ mechanism

Today we look at the κ-γ-mechanism, which is about the term

$$\int_M \left(\frac{\delta T}{T}\right)_{\mathrm{sp}} \frac{\partial \delta L}{\partial m}\,\mathrm{d}m\,, \tag{1.170}$$

and we will have driving layers ($\kappa < 0$) in the regions in which the two terms multiplied in the integrand are discordant.

The mean Rosseland opacity is approximated by a law in the form:

$$\kappa_R \approx \overline{\kappa}_R \rho^n T^{-s}\,, \tag{1.171}$$

with $n \approx 1$, $s \approx 7/2$ in the case of free-free absorption (that is, inverse & direct bremsstrahlung) in a non-degenerate, totally ionized gas. This seems to be a good approximation for $4 < \log_{10} T < 8$.

This allows us to relate the perturbations in κ_R to those in T:

$$\frac{\delta\kappa_R}{\kappa_R} \approx n\frac{\delta\rho}{\rho} - s\frac{\delta T}{T} \tag{1.172}$$

$$\approx \big(n - s(\Gamma_3 - 1)\big)\frac{\delta\rho}{\rho}, \tag{1.173}$$

and if we substitute in $\Gamma_3 \approx 5/3$ with the other terms we get

$$\frac{\delta\kappa_R}{\kappa_R} \approx -\frac{4}{3}\frac{\delta\rho}{\rho}, \tag{1.174}$$

which means that the relative mean opacity perturbation is *discordant* in sign with the pressure one, and thus with the temperature one.

So, up to a positive constant, the integrand looks like

$$-\frac{\delta\kappa_R}{\kappa_R}\frac{\partial\delta L}{\partial m}, \tag{1.175}$$

and we want it to be < 0. However, if under the perturbation the layer is *more* opaque than usual ($\delta\kappa_R > 0$) then it will let through *less* light than usual, so we will have $\delta(\partial L/\partial m) < 0$. This can be rendered more formal by considering the luminosity equation (1.11): the dependence of L on κ_R is inverse.

So, the global contribution to κ will be positive, therefore our layer will be *damping*.

κ-mechanism However, layers for which the contribution of this term is negative are present in certain regions: where there is ionization, we can have regions for which the gradient $\partial\kappa_R/\partial T$ reverses: it is usually negative, but it can become locally positive. This provides an additional channel for energy stocking: it is like a *dam* for energy. The ionization energy is then released through mechanical work. This is called the κ-mechanism.

γ-mechanism The γ-mechanism, on the other hand, involves a decrease of Γ_3 which brings it near to 1: recall,

$$\Gamma_3 - 1 = \left.\frac{\partial \log T}{\partial \log \rho}\right|_{\text{adiabatic}}, \tag{1.176}$$

so if it is small that means that the change in temperature, i.e. internal energy, associated with a compression ($\delta\rho > 0$) is small: this can happen in the outer layers of the star, which have partial ionization regions. The work of the compression is partially absorbed in order to ionize the atoms in these layers: if Γ_3 is close enough to 1, we can have

$$n - s(\Gamma_3 - 1) > 0 \tag{1.177}$$

even with $s > 0$.

κ-γ-mechanism We look at the linearized equations of continuity and radiative transfer for an expression for the gradient of δL, neglecting the temperature perturbation gradient term for simplicity: then, starting from equation (1.146) we have

$$\frac{\delta L}{L} = 4\zeta - n\frac{\delta\rho}{\rho} + (s+4)\frac{\delta T}{T}\,, \tag{1.178}$$

while from equation (1.142) we get

$$\frac{\delta\rho}{\rho} = -3\zeta\,. \tag{1.179}$$

Combining these, and using $\delta T/T = (\Gamma_3 - 1)\delta\rho/\rho$ we find:

$$\frac{\delta L}{L} = \left(-\frac{4/3+n}{\Gamma_3 - 1} + s + 4\right)\frac{\delta T}{T}\,, \tag{1.180}$$

and now we make the following consideration: on average, in the outer layers of the stars we have no nuclear energy generation ($\epsilon_{\mathrm{nuc}} \approx 0$) and the heat variation is negligible ($\mathrm{d}Q/\mathrm{d}t \approx 0$). So, we have $\partial L/\partial m \approx 0$ by the energy conservation equation (1.5): using this, we can freely bring L outside of derivatives with respect to m, in order to write

Why?

$$\frac{\partial \delta L}{\partial m} = L\frac{\partial}{\partial m}\left[\left(s + 4 - \frac{4/3+n}{\Gamma_3 - 1}\right)\frac{\delta T}{T}\right]. \tag{1.181}$$

Outside the regions of partial ionization, typical values for the parameters are $\Gamma_3 \sim 1.6$, $s \sim 7/2$, $n \sim 1$. Therefore, we get

$$\frac{\partial \delta L}{\partial m} \approx 3.61 \times L\frac{\partial}{\partial m}\left(\frac{\delta T}{T}\right), \tag{1.182}$$

and a function and its derivative have the same sign, right? so we approximate the derivative of $\partial/\partial m\ (\delta T/T)$ with $\delta T/T$

which means that

$$\frac{\partial \delta L}{\partial m} \sim 3.6L\frac{\delta T}{T}\,, \tag{1.183}$$

so they are concordant in sign, so the layer is a damping layer.

In partial ionization regions, instead, $\kappa_R \propto \rho^n T^{-s}$ with negative s: so it is much easier for the terms to be discordant. Keeping $n = 1$ and $\Gamma_3 = 1.6$, we need $s < -0.11$ in order to have a driving layer. However, Γ_3 also decreases in these layers, so the threshold for s is higher: for example, with $n = 1$ and $\Gamma_3 = 1.4$ we only need $s < 1.83$.

So, the κ in the κ-γ-mechanism refers to the decrease in s while the γ refers to the decrease in Γ_3, right?

Opacity bump mechanism The κ-γ-mechanism cannot explain the pulsation of hotter stars than RR-Lyr, Cepheids, δ-Scuti or SX-Phe, such as β-Ceph, GW Vir and sdBV stars: in these, the stellar material is much more ionized on average: in the regions which, for cooler stars, are partial ionization regions the gas is fully ionized, while the partial ionization regions have moved further out towards the surface, if they have not disappeared entirely.

One could consider the partial second ionization layers: this has been thoroughly considered, and it has been found that they cannot provide a significant enough contribution.

There is an opacity bump at $5 < \log_{10}(T) < 6.5$, which was found in the eighties by Simon: the old graph for $\kappa(T)$ had spikes around $T \sim 10^4$ and $T \sim 10^{4.7}$, and a new spike was found at $T \sim 10^{5.4}$. Do note that these are temperatures reached when looking a bit inside the star, not at the surface (although close to it).

Convective blocking Convection is slow to adapt, so it cannot move away the heat brought by the pulsations. Then, the heat must be turned into work. This happens when the period of the pulsations is of the order of the thermal timescale?

Not sure about this

Convective driving Convective driving is called the δ-mechanism. It happens when the convective timescale is much shorter than the period of the pulsation, so the convection can mix the gas.

And this drives pulsations?

Stochastic driving There also are *stochastically driven* oscillations: they happen in stars which are intrinsically stable, but the pulsations may be fed by convective turbulence. This can happen in Sun-like Main Sequence stars, or in solar-like red giants, such as ξ-Hydrae stars.

Strange modes There are very luminous *strange modes*, very dim convectively driven modes.

The classical instability strip

The pulsation region has boundaries: for the κ-mechanism:

- if the star is too hot, the regions of partial ionization get too close to the surface;
- if the star is too cold, they are too far in: in their outer region the pulsation is damped by convective heat transfer.

So, there is an optimal region for the partial ionization layers to be.

What is the optimal region? We will only look at the fundamental mode. We define

$$\phi(m) \equiv \frac{1}{L(\Pi/2\pi)} \int_m^M c_V T \, \mathrm{d}m \,, \tag{1.184}$$

which represents the thermal balance of a single oscillation: $L(\Pi/2\pi)$ is the luminosity radiated in a radian of the pulsation cycle ($1/2\pi$ cycles), while $c_V T = T\Big(\partial Q/\partial T\Big) \approx Q$, the heat per unit mass, which when integrated from m to M, the whole mass of the star, gives us the global heat variation of the layers above m.

We are in the helium ionization region: there are no nuclear reactions, so we set energy generation to zero in equation (1.145): we get

$$\frac{\partial}{\partial t}\left(\frac{\delta T}{T}\right) = (\Gamma_3 - 1)\frac{\partial}{\partial t}\left(\frac{\delta \rho}{\rho}\right) - \frac{1}{c_V T}\frac{\partial \delta L}{\partial m} \,. \tag{1.185}$$

Now we assume that the temperature, density and luminosity perturbations are oscillating, so they have a certain starting value, and the temporal dependence is absorbed in a factor $e^{i\sigma t}$. Then, we get

$$i\sigma\frac{\delta T}{T} = (\Gamma_3 - 1) i\sigma \frac{\delta \rho}{\rho} - \frac{1}{c_V T}\frac{\partial \delta L}{\partial m} \tag{1.186}$$

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta \rho}{\rho} + \frac{i}{\sigma c_V T}\frac{\partial \delta L}{\partial m} \,. \tag{1.187}$$

Now, consider the following: with our definition of ϕ, we can write:

$$\frac{\partial}{\partial \phi} = \frac{\partial m}{\partial \phi}\frac{\partial}{\partial m} = -\frac{L(\Pi/2\pi)}{c_V T}\frac{\partial}{\partial m} \,; \tag{1.188}$$

do note that ϕ is adimensional, because of the way we define $L(\Pi/2\pi)$:

$$L(\Pi/2\pi) = L/\sigma \,, \tag{1.189}$$

which is an energy. With these facts in mind, plus the fact that approximately $\partial L/\partial m = 0$, we can do the following manipulation:

$$\frac{i}{\sigma c_V T}\frac{\partial \delta L}{\partial m} = -\frac{i}{\sigma L(\Pi/2\pi)}\frac{\partial \delta L}{\partial \phi} \tag{1.190}$$

$$= -i\frac{\partial}{\partial \phi}\left(\frac{\delta L}{L}\right) . \tag{1.191}$$

Now, we can use the chain rule on the derivative in $\partial \phi$ using any auxiliary variable we like: we choose $x = r/R$, the fractionary radius, so we get

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta \rho}{\rho} - i\left(\frac{\partial \phi}{\partial x}\right)^{-1}\frac{\partial}{\partial x}\left(\frac{\delta L}{L}\right) , \tag{1.192}$$

where δT is actually complex, since the perturbations are out of phase.

We can make some considerations on the variations of the terms in the equation:

$$\phi \approx \frac{1}{L(\Pi/2\pi)} \frac{4}{3} \pi R^3 \left(1 - x^3\right) \rho c_V T \propto (1 - x^3) \rho c_V T \,, \tag{1.193}$$

so

$$\frac{\partial \phi}{\partial x} \propto -x^2 \rho c_V T \,. \tag{1.194}$$

The x^2 dependence, however, is not very important: if the internal energy of a layer does not change, we can write the first law of thermodynamics as

$$\mathrm{d}Q = P \,\mathrm{d}\left(\frac{1}{\rho}\right) \implies P \sim \frac{\mathrm{d}Q}{\mathrm{d}(\rho^{-1})} \sim Q \times \rho \,, \tag{1.195}$$

and $c_V T \sim Q$, so we have:

$$\frac{\partial \phi}{\partial x} \propto -x^2 P \,, \tag{1.196}$$

and while x goes to zero in the center, the pressure increases a lot.

From what I could gather in 5 minutes, the Sun's pressure profile is very roughly given by

$$P(x) \sim 2.5 \times 10^{11} \,\mathrm{Pa} \exp(-\lambda x) \,, \tag{1.197}$$

where $\lambda \sim 9$. Now, if we plot $\exp(-9x)x^2$, it becomes small both at the surface and at the center, and has a maximum around $x \sim 0.2$. So, the reasoning still makes sense, but you have to consider the very center of the star as a special case.

On the surface, the imaginary term is negligible. In the interior, it is relevant.

So, near the surface $\partial x / \partial \phi$ is very large: therefore the term multiplying it, $\partial / \partial x \; (\delta L / L)$, must become small in order for the temperature perturbation not to diverge. This means that the luminosity perturbation cannot change much: it is "frozen in". Anyway, the imaginary part of the RHS of equation (1.192) is much larger than the real part.

Right?

On the other hand, near the center the term $\partial x / \partial \phi$ becomes very small, while the spatial derivative of the perturbation does not become very large, so on balance the term remains small: the oscillations are then quasi-adiabatic.

Between these two regimes, we have a *transition region* in which the two terms in equation (1.192) are similar in terms of order of magnitude: depending on the temperature of the star, this can can occur before or after the ionization region, when going radially outward.

The classical instability strip is the region in the Hertzsprung-Russell diagram in which the partial ionization region roughly coincides with the transition region.

Why is the perturbation "frozen in" if its spatial derivative is small? we could have $\delta L / L$ constant with respect to x and it would still evolve in time!

Why are the boundaries of the classical instability strip tilted? It seems like the region is defined by some relation like $\log L = -(\text{large \#}) \log T$, why is this so?

1.4.5 Star botany

RR Lyræ

These are stars with periods between $0.2 \div 1\,\text{d}$, absolute visual magnitude around 0.6, and mean effective temperature around $6000 \div 7250\,\text{K}$.

It is a stage, which lasts no more than $10^8\,\text{yr}$, and it is observed only in clusters older than $10^{10}\,\text{yr}$.

They are classified by a, b, or c according to the shape of the light curve, its amplitude, its period:

1. RRa: sharp rise, large amplitude: the fundamental;
2. RRb: similar to RRa with smaller amplitude, longer period: the fundamental;
3. RRc: more symmetric light curve, short periods, low amplitudes: they pulsate in the first overtone.

We have also RRd: bimodal, RRe: second overtone.

These characteristics are seen well in a Bailey diagram, in which we scatter plot the stars with magnitude on the y axis and period on the x axis.

[Argument for the different amplitudes at different wavelengths: to understand]

We can do spectral analysis on a set of several different stars: we find that, when applying a spectral filter to the star and doing Fourier analysis on that light curve, we can identify a relation in the form

$$M_\nu = k_\nu \log \Pi \,, \tag{1.198}$$

where M_ν is the magnitude measured in that particular spectral band, while k_ν is a constant depending on the band. k_ν can be either positive or negative, and it is increasing with ν.

[Qualitative part of the lecture.]

Mon Oct 21 2019

We were talking about RR Lyrae varibles, classifying them. In a Bailey diagram we plot variables as points with the coordinates amplitude and period.

We can tell whether a star is a RR Lyr variable, and then we can use it as a standard candle.

We have nonlinear models which accurately predict the light curves. There are HUMPs, BUMPs, JUMPs and LUMPs in the light curves: they tell us about the propagation of shock waves in the star.

We also have the Blazhko Effect: when it is not present, the light curve is the same at every cycle; when it is present the light curve changes, it is modulated every cycle. This implies that the phased light curve scatter plot is very spread, no matter how well we fix the period. This is a poorly understood effect, it might have something to do with magnetic fields.

About half of RRab have this effect, very few RRc have it.

In a given Globular Cluster, we look at the distribution of RR Lyr with respect to period: we can see two distinct clusters, corresponding to the fundamental and first harmonic (RRab and RRc respectively). We can characterize them with respect to the Oosterhoff group (the one of the fundamental): if it is big with respect to the first harmonic it is a type I, if it is comparable it is a type II.

We can distinguish these by making a histogram of the average period of GCs: OoII are clustered around $0.65\,\text{d}$, while OoI around $0.55\,\text{d}$. There is a distinct gap between the two.

Metal-rich Globular Clusters tend to few RR Lyrae, metal poor-ones tend to have more, unless their horizontal branch is very extended to the blue, in which case they may have few RR Lyr.

The Oosterhoff gap can be seen in metallicity as well.

The period, theoretically, should only change on evolutionary (i. e. very long) time scales, however we observe much faster period changes, especially in stars which exhibit the Blazhko effect.

So far, RR Lyrae have been found in local dwarf galaxies, in the Magellanic Clouds, in M31 (Andromeda Galaxy), in M33 (Triangulum Galaxy).

RR Lyr beyond the Local Group have not yet been observed.

1.4.6 Classical Cepheids

They massive stars ($4 \div 9 M_\odot$) are younger than RR Lyr, typically around $10^7 \div 10^8\,\text{yr}$. Their absolute visual magnitude M_V can change in a range of typically -2 to -6. Their periods are typically beteen $0.5 \div 135\,\text{d}$.

They are very important for the *cosmic distance ladder* determination, because of the Period-Luminosity relation, which can be used to measure distances: period is not affected by reddening.

The light curves of Cepheids are characterized by a sharp rize and a shallow fall in magnitude if the star is vibrating on the fundamental, a more symmetric light curve if it is vibrating on the first overtone. A second overtone pulsation is rarely found, and it generally has a much lower amplitude. In classical Cepheids we also find multimodal pulsators.

As with RR Lyrae, the amplitude of the pulsation is larger at higher light frequencies such as the UV, and decreases through visible and IR.

Cepheids with periods between 6 and 16 d we often have a *bump*, which could be caused by resonances between the fundamental and the second overtone or by echoes from the core.

Some Cepheids, like some RR Lyrae, also show the Blazhko effect (very long-lived modulations in the light curve).

The Period-Luminosity relation

The fundamental relation we start from is the Stefan-Boltzmann law:

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 \,, \tag{1.199}$$

which can be converted into a relation involving the bolometric magnitude: we first of all divide through by the solar values for the parameters, and then take the log:

$$\log\left(\frac{L}{L_\odot}\right) = 2\log\left(\frac{R}{R_\odot}\right) + 4\log\left(\frac{T_{\text{eff}}}{T_{\text{eff},\odot}}\right) , \tag{1.200}$$

and we use the definition of bolometric magnitude:

$$M_{\text{bol}} - M_{\text{bol},\odot} = -2.5\log\frac{L}{L_\odot} \,, \tag{1.201}$$

so we substitute it in:

$$M_{\text{bol},\odot} - M_{\text{bol}} = 2.5\left(2\log\left(\frac{R}{R_\odot}\right) + 4\log\left(\frac{T_{\text{eff}}}{T_{\text{eff},\odot}}\right)\right) \tag{1.202}$$

$$M_{\text{bol}} = -5\log\left(\frac{R}{R_\odot}\right) - 10\log\left(\frac{T_{\text{eff}}}{T_{\text{eff},\odot}}\right) + \text{const} \,, \tag{1.203}$$

and now we can use the period-mean density relation:

$$\Pi^2\overline{\rho} = \mathcal{Q} = \text{const} \implies 2\log\Pi + \log\overline{\rho} = \log\mathcal{Q} \,, \tag{1.204}$$

but $\overline{\rho} \sim M/R^3$, so this becomes

$$2\log\Pi + \log M - 3\log R - \log \mathcal{Q} = \text{const} \tag{1.205}$$

$$\log R = \frac{2}{3}\log\Pi + \frac{1}{3}\log M - \frac{2}{3}\log\mathcal{Q} + \text{const}\,. \tag{1.206}$$

We can now substitute this into the other equation; we get

Multiply by 3/10

$$M_{\text{bol}} + 5\left(\frac{2}{3}\log\Pi + \frac{1}{3}\log M - \frac{2}{3}\log\mathcal{Q} + \text{const}\right) + 10\log T_{\text{eff}} = 0 \tag{1.207}$$

$$\frac{3}{10}M_{\text{bol}} + \log\Pi + \frac{1}{2}\log M - \log\mathcal{Q} + 3\log T_{\text{eff}} = \text{const}\,, \tag{1.208}$$

where we started dropping the adimensionalizing divisions by the solar values of the parameters, since they are "constants".

Now, we make the assumption of a mass-luminosity relation like that of Main Sequence stars: $\log M_{\text{bol}} = -8\log M + \text{const}$, so we substitute $\log M$ with $-M_{\text{bol}}/8$. The number multiplying M_{bol} is then $3/10 - 1/16 \approx 0.24$. So, in the end, we get

$$\log(\Pi) = -0.24 M_{\text{bol}} - 3\log(T_{\text{eff}}) + \log(Q) + \text{const}\,, \tag{1.209}$$

which is really useful because the period can be measured precisely, while the *absolute* bolometric magnitude is really difficult.

Since the stars which satisfy the P-L relations have different temperatures we have a certain spread of the P-L relation scatter plot.

Is T_{eff} not very much correlated with M_{bol}?

The derivation of this might be asked at the exam.

The PL relation is quite well respected experimentally: if we plot classical Cepheids' $\log\Pi$ vs $\log\langle L\rangle$ we see two distincts lines, which correspond to fundamental pulsators and first overtone pulsators.

1.5 Non-radial oscillations and astroseismiology

This is just an introduction, a full course might be given at the PhD level.

We use similar assumptions as before, except we lose the spherical symmetry. Specifically:

1. we consider perturbations to a spherically symmetric equilibrium model;
2. we make the Cowling approximation: we consider the unperturbed gravitational potential.

The equations describing the oscillations are

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\zeta_r\right) - \frac{g}{v_s^2}\zeta_r + \left(1 - \frac{L_\ell^2}{\sigma^2}\right)\frac{P'}{v_s^2\rho} = 0 \tag{1.210}$$

$$\frac{1}{\rho}\frac{\partial P'}{\partial r} + \frac{g}{v_s^2\rho}P' + \left(N^2 - \sigma^2\right)\zeta_r = 0\,, \tag{1.211}$$

where:

1. the displacement vector is separated into radial and horizontal components: $\vec{\zeta} = \zeta_r\hat{e}_r + \vec{\zeta}_h$;
2. any perturbed quantity changes in two ways: we write

$$\delta f(\vec{r_0} + \vec{\delta r}) = f'(\vec{r_0}) + \vec{\delta r}\cdot\vec{\nabla} f_0\,, \tag{1.212}$$

Does the prime denote a derivative? how are these things defined?

3. L_ℓ is the Lamb frequency, or acoustic frequency: $L_\ell^2 = \ell(\ell+1)(v_s/r)^2$
4. N is the Brunt-Väisälä frequency, or bouyancy frequency:

$$N^2 = -g\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right), \tag{1.213}$$

   where the term inside brackets is called the *Schwarzschild discriminant* $A(r)$, and it measures the non-adiabaticity of the system, since it is proportional (with a positive constant) to $\nabla - \nabla_{\mathrm{ad}}$. Recall Schwarzschild's criterion: we have convectively unstable regions if $A(r)$ is positive. So, N^2 is *negative* in regions which are convectively unstable according to Schwarzschild's criterion.

Our ansatz for the solution is similar to what is done in quantum mechanics: we have a radial part with a quantum number n, and spherical harmonics indexed by the angular order ℓ and the azimuthal order m, with $|m| \leq \ell$. The temporal periodicity will still look like an oscillating complex exponential, but its frequency will depend on the quantum numbers: so, in the end we will have

$$\zeta(\vec{r}) = R_n(r)Y_{\ell m}(\Omega)\exp(i\sigma_{n\ell m}t)\,, \tag{1.214}$$

where R and Y are the radial and angular components of the perturbation, $\Omega = (\theta, \varphi)$ and $Y_{\ell m}$ are the spherical harmonics, the solutions $Y\colon S^2 \to \mathbb{C}$ to $\nabla^2 Y = -\ell(\ell+1)Y$, satisfying $\partial_\varphi Y = im$: these are restrictions on the unit sphere of $\nabla^2 Y = 0$.

We can approximate the differential equations locally as a single differential equation (by assuming that all the parameters are constant: we do the following change of variables:

$$\frac{\partial \widetilde{\zeta}}{\partial r} = h\frac{r^2}{v_s^2}\left(\frac{L_\ell^2}{\sigma^2} - 1\right)\widetilde{\eta} \tag{1.215}$$

$$\frac{\partial \widetilde{\eta}}{\partial r} = \frac{1}{r^2 h}\left(\sigma^2 - N^2\right)\widetilde{\zeta}\,, \tag{1.216}$$

which can be combined into

$$\frac{\partial^2 \widetilde{\zeta}}{\partial r^2} = -k_r^2 \widetilde{\zeta}\,, \tag{1.217}$$

with a wavenumber given by

$$k_r^2 = -\frac{1}{v_s^2 \sigma^2}\left(L_\ell^2 - \sigma^2\right)\left(\sigma^2 - N^2\right). \tag{1.218}$$

We have several families of solutions:

1. p-modes, mostly radial, pressure waves, high frequency: $\sigma^2 > L_\ell^2, N^2$;
2. g-modes, mostly horizontal, buoyancy, low frequency, $\sigma^2 < L_\ell^2, N^2$;
3. f-modes; without nodes.

We can derive equations for this case similarly to the LAWE.

This is because k_r^2 is negative iff σ^2 is either smaller or larger than both N^2 and L_ℓ^2; these are respectively g and p-modes. If σ^2 is between them, we get a real exponential as a solution, which decays quickly. This is called the *evanescent regime*.

p-modes are characterized by a *turning point*: below a certain radius, the mode is in the evanescent regime. For g-modes we have the opposite behaviour: they only exist in the core, and are evanescent for large radii.

We can plot N and L_ℓ as a function of radius.

Mixed modes arise when the evanescent region is thin, therefore the mode can tunnel through.

The frequencies are degenerate in m, but the degeneracy is split when the star is rotating.

In the slide for "asymptotic behaviour", the y axis is frequency.

We analyze the power spectra of stars.

Tue Oct 22 2019

For the physics undergraduate students: "Fundamental Astronomy" by Karttunen, Kröger, Öja, Poutanen, Donner.

1.5.1 Red variable stars

They include the ξ Hydrae, SR and Mira stars. These are low temperature stars.

They are evolved stars, and most all of the evolved stars are at least somewhat variable.

Miras, SRVs and OSARGs have very long periods, on the order of a year.

We need good estimates for the radius: because of the period-mean density relation, the fundamental at a certain radius can correspond to the first overtone at a larger radius.

We have different period-luminosity relations corresponding to which overtone we see. Using Weisenheit indices, which compensate for self-reddening, these relations are even more evident.

We can simulate galaxies, and observe the same patterns.

The linear models are not appropriate for the fundamental. Today, we do 3D models.

1.6 Summary

- $\tau_{\text{dyn}} \propto 1/\sqrt{\overline{\rho}}$;
- the inequality between the timescales;
- variability is due to mechanical, acoustic phenomena!
- derivation of the period-mean density relation;
- the balance of heat absorption;
- meaning of perturbation theory;
- the final result: the perturbed structure equations;
- the adiabatic approximation: justification, cases in which it does not hold: driving layers, stability;
- ideas of how the LAWE is solved (not boundary conditions): the Sturm-Liouville problem;
- the meaning of the solutions of the LAWE: nodes, the shape of the eigenfunctions, orthogonality;
- stability conditions: inequalities which tell us whether a star pulsates or not, driving and damping layers, phase lag;

- NOT the derivation of the LNAWE, but qualitative characterization of its solutions; the expression of the coefficient κ;
- the ϵ and κ-γ mechanisms: orders of magnitude;
- the opacity bump mechanism;
- the difference between self-excited and stochastically driven oscillation;
- the classical instability strip: not important to know the derivation;
- RR Lyrae, typical parameters;
- Cepheids: derivation of the mass-luminosity relation;
- NOT nonradial oscillations and astroseismiology;
- red variables.

Chapter 2

Stellar winds

Mon Oct 28 2019

2.1 Introduction

With Paola Marigo, now.

One should be able to follow this part even without a strong background in stellar astrophysics.

Bubble Nebula in Cassiopeia: a 45 $M_\odot$ star is ejecting mass at 1.7×10^6 m/s.

Some important quantities: we introduce

1. $\dot{M}$ is the mass loss rate:
$$\dot{M} = -\frac{\mathrm{d}M}{\mathrm{d}t} > 0\,; \tag{2.1}$$
2. v_∞ is the terminal wind velocity, in the limit of radial infinity.

The gas initially escapes from the star at low (subsonic, $\sim$1 km/s) velocity; then it is accelerated. It is accelerated, and in the far field when no more forces are acting on it it approaches v_∞. We describe it with a *velocity law*: $v(r)$, and physically since the force is always radially outward we have

$$\frac{\mathrm{d}v}{\mathrm{d}r} > 0 \tag{2.2}$$

for any r. A typical law is something like:

$$v(r) = v_0 + (v_\infty - v_0)\left(1 - \frac{R_*}{r}\right)^\beta , \tag{2.3}$$

where $\beta \approx 0.8$, and R_* is the radius of the photosphere (where, from infinity, we have an opacity of $\tau = 1$).

In an H-R diagram we can plot the mass loss rate using color: we see that it increases when going up the main sequence, and is also high in the RGB.

Probably the thing is that the mass loss rate increases through the later stages of stellar evolution…

The relation $\dot{M}(M)$ seems to be something like a power law: what is it?

The momentum input can come either from a force, like radiation pressure (line driven winds and dust driven winds) or from heating.

Stellar winds can be characterized by their temperature, measured with respect to the effective temperature of the star; velocity, measured with respect to the escape velocity, and density:

1. late type supergiant stars have *cold*, *slow* and *dense* winds;

2. luminous hot stars have *cold*, *fast* and *dense* winds;

3. cool dwarfs and giants have *hot*, *fast* and *tenuous* winds.

We can also characterized them by their driving mechanisms:

1. coronal winds: driven by gas pressure;

2. line driven winds: driven by radiation pressure on highly ionized atoms, O and B stars;

3. dust driven winds: due to radiation pressure on dust grains, solid particles, which are very opaque.

For line-driven winds we have a relations between luminosity and wind momentum: it is a powerlaw. Specifically, what is measured is $\dot{M}_\infty \sqrt{R/R_\odot}$ versus the visual magnitude.

We will not consider winds driven by pulsation, sound waves and Alfvèn waves (magnetic winds).

Wind structure equations

We will assume:

1. spherical symmetry;

2. stationarity;

3. no magnetic fields.

The equation of continuity with the hypothesis of stationarity is given by $\dot{M} = 4\pi r^2 \rho(r) v(r) \equiv \text{const}$.

If we differentiate $\dot{M}$ with respect to r we get 0 on the LHS (since, by continuity, the mass loss rate across any layer is constant), and on the RHS:

$$0 = 2\rho v + rv\frac{\mathrm{d}\rho}{\mathrm{d}r} + r\rho\frac{\mathrm{d}v}{\mathrm{d}r}\,, \tag{2.4}$$

where we simplified the 4π. If we divide by $\rho v r$ we get:

$$\frac{2}{r} + \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = 0\,, \tag{2.5}$$

and then this gives us

$$2\log(r)' + \log(v)' + \log(\rho)' = 0\,, \tag{2.6}$$

where the prime denotes derivatives with respect to r. So, the gradients of the velocity and density are related.

The force per unit volume is given by

$$F = \rho\frac{\mathrm{d}v}{\mathrm{d}t}\,, \tag{2.7}$$

and if we divide by ρ we get the force per unit mass:

$$f = \frac{F}{\rho} = \frac{\mathrm{d}v}{\mathrm{d}t}\,. \tag{2.8}$$

Under stationarity ($\partial_t = 0$), we have:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v(r)\frac{\mathrm{d}v}{\mathrm{d}r}\,. \tag{2.9}$$

The conservation of momentum gives us the Euler equation:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = \underbrace{-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}}_{f_P}\underbrace{-\frac{GM}{r^2}}_{f_g} + f(r)\,, \tag{2.10}$$

where $f(r)$ is a generic unspecified external force, which we assume to be *outward* (no dissipative effects!), while f_P and f_g are respectively the force due to the pressure gradient and to gravitation.

Do note that while f_P has a minus sign, $\mathrm{d}P/\mathrm{d}r < 0$ so the force due to the pressure gradient is outward. On the other hand, $f_g < 0$. The first principle of thermodynamics gives us:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} + P\frac{\mathrm{d}\rho^{-1}}{\mathrm{d}t}\,, \tag{2.11}$$

where Q is the specific heat, u is the specific internal energy.

The internal energy, for an ideal gas, scales linearly with the temperature:

$$u = \frac{3}{2}\frac{k_B T}{\mu m_u} = \frac{3}{2}\frac{\mathcal{R}T}{\mu}\,, \tag{2.12}$$

where μ is the mean molecular weight, while $m_u \approx m_H$ is the atomic unit of mass. Using these, we define the specific gas constant $\mathcal{R} = k_B/m_u$.

The mean molecular weight is defined as the average weight of a molecule in atomic mass units: $\mu = \overline{m}/m_H$, and for a neutral gas it can be calculated as

$$\frac{1}{\mu m_H} = \frac{\sum_j N_j}{M_{\text{tot}}} = \sum_j \frac{N_j}{N_j m_j}\frac{N_j m_j}{M_{\text{tot}}} = \sum_j \frac{N_j}{N_j A_j m_H} X_j\,, \tag{2.13}$$

so

$$\frac{1}{\mu} = \sum_j \frac{N_j}{A_j}\,, \tag{2.14}$$

where we have used the facts that $m_j = A_j m_H$, we defined the mass fraction $X_j = N_j m_j / M_{\text{tot}}$: the fraction of the total mass which is of the species j. For an ionized gas the calculation is the same except we need to include a factor $(1 + z_j)$, at the numerator in the sum, where z_j is the atomic number of the element, since for every ionized atom we have an additional particle (the electron).

Using the mass fractions for the Sun we get approximately $\mu \approx 1.3$ for neutral gas and $\mu \approx 0.62$ for fully ionized gas.

We will also assume that the gas pressure follows the ideal gas law:

$$P = \frac{k_B T \rho}{\mu m_u} = \frac{\mathcal{R}T\rho}{\mu}\,, \tag{2.15}$$

where we used the fact that $N = M/\overline{m}$, and $\overline{m} = \mu m_H$.

By stationarity, all time derivatives can be written as $\frac{\mathrm{d}}{\mathrm{d}t} = v\frac{\mathrm{d}}{\mathrm{d}r}$.

We define

$$q(r) = \frac{\mathrm{d}Q}{\mathrm{d}r}\,, \tag{2.16}$$

the heat input or loss per unit mass per unit distance in the wind (since Q is already a specific heat). Inserting this in the previous equation, we get the following expression for the energy equation:

$$\boxed{q = \frac{3}{2}\frac{\mathcal{R}}{\mu}\frac{\mathrm{d}T}{\mathrm{d}r} + P\frac{\mathrm{d}\rho^{-1}}{\mathrm{d}r}\,,} \tag{2.17}$$

This can be incorporated, using the momentum conservation as well, into the global energy equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\underbrace{\frac{v^2}{2}}_{A}+\underbrace{\frac{5}{2}\frac{\mathcal{R}T}{\mu}}_{B}\underbrace{-\frac{GM}{r}}_{C}\right)=f(r)+q(r)\,, \qquad (2.18)}$$

where the expression inside the brackets is the total internal energy.

Here is the derivation of the formula: the momentum conservation equation (2.10) can be written as:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right)=-\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{P}{\rho}\right)+P\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\rho}\right)+\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{GM}{r}\right)+f\,, \qquad (2.19)$$

Inverse Leibniz rule on the P, ρ term.

so we can substitute in our expression from the energy equation for $P\,\mathrm{d}\rho^{-1}\big/\mathrm{d}r$: we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v^2}{2}+\frac{P}{\rho}-\frac{GM}{r}\right)=f+q-\frac{3}{2}\frac{\mathcal{R}}{\mu}\frac{\mathrm{d}T}{\mathrm{d}r}\,, \qquad (2.20)$$

but P/ρ is precisely equal to $\mathcal{R}T/\mu$, and the molecular weight and the gas constant are constants, so we can bring that term inside the derivative on the LHS in order to get the $5/2$ factor in the final result.

The terms A, B and C describe the different types of energy we can have: A is kinetic energy, B is chemical potential energy (enthalpy) while C is gravitational potential energy.

We can change the total energy by either adding heat (increasing q) or by adding momentum (increasing f).

Equation (2.18) can be written in integral form, by integrating from a generic radius r_0: a useful choice is often the photospheric radius. We introduce the notation $e(r) = A + B + C$ for the total energy.

Near the photosphere $\left|C\right| \gg A + B$, so $e(r_0) \approx GM/r_0 < 0$ (this is not obvious theoretically: it is an experimental fact that the gas is slow-moving near the surface).

At large radii $B, C \to 0$: $e(r_\infty) \approx v_\infty^2/2 > 0$.

If we integrate from r_0, the radius of the photosphere, to infinity then we get:

$$\frac{v_\infty^2}{2}=-\frac{GM}{r_0}+\int_{r_0}^{\infty} f(r)+q(r)\,\mathrm{d}r\,, \qquad (2.21)$$

so the sum of the two integrals must be large enough for the gas to escape the gravitational well.

2.1.1 Coronal winds

The corona is very hot: it goes to 10^6 K, but with very low density. We know this experimentally by seeing the emission lines of highly ionized elements like $\mathrm{Fe}^{+5\div 13}$.

Why is this the case? We *do not know*. Magnetism?

If the gas particles in the corona have enough kinetic energy to reach terminal velocity and escape the gravitational well, they form what is known as the coronal wind. This is a good model for the Sun: its wind's asymptotic velocity is around 500 km/s, so it reaches the Earth in around 3 d. For the sun $\dot{M} \sim 1 \times 10^9$ kg/s: this means that it loses 0.016 % of its total mass per 10 Gyr, which is appriximately the lifespan of the Sun.

Not 0.1 %, as the slides say!

2.2 Isothermal winds

Now we will treat isothermal winds. These calculations were first done by Parker.

The assumption of constant temperature already gives us $T(r) \equiv T$, which acts as our energy equation: there must be an energy input exactly equal to

$$q = P \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\rho} : \tag{2.22}$$

this is a type of wind which is driven by gas pressure only.

Is this actually the case? it is not mentioned in the slides, but it seems to be a natural consequence of the hypothesis.

This is a good model for low $\dot{M}$, which does not affect the total mass of the star.

Tomorrow we will discuss full solutions in this case.

Tue Oct 29 2019

Recall the equations from last time: continuity, momentum and energy conservation in the isothermal case:

$$\dot{M} = 4\pi r^2 \rho v \tag{2.23a}$$

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} \tag{2.23b}$$

$$T(r) \equiv T\,. \tag{2.23c}$$

Also, by differentiating the ideal gas law $P\mu = \mathcal{R}T\rho$ we get

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathcal{R}T}{\mu}\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\,, \tag{2.24}$$

Divided through by ρ.

since the temperature is constant.

Using this, we manipulate the momentum equation into

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathcal{R}T}{\mu}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2}\,, \tag{2.25}$$

but we know by the continuity equation that the density gradient must correspond to the velocity gradient:

$$-\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\,, \tag{2.26}$$

which we can substitute into the equation: we get

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\mathcal{R}T}{\mu}\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\right) - \frac{GM}{r^2}\,. \tag{2.27}$$

It is a known fact that the isothermal speed of sound is given by

$$a^2 = \frac{\partial P}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{\mathcal{R}\rho T}{\mu}\right) = \frac{\mathcal{R}T}{\mu} \implies a = \sqrt{\frac{\mathcal{R}T}{\mu}}\,, \tag{2.28}$$

so

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = a^2\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\right) - \frac{GM}{r^2}\,. \tag{2.29}$$

Now we move all the terms which are proportional to the velocity gradient on the LHS:

$$\frac{\mathrm{d}v}{\mathrm{d}r}\left(v - \frac{a^2}{v}\right) = \frac{2a^2}{r} - \frac{GM}{r^2}\,, \tag{2.30}$$

which can be written as

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r}\left(v^2 - a^2\right) = \frac{2a^2}{r} - \frac{GM}{r^2}\,, \tag{2.31}$$

so the Jacobian of the differential equation is zero is singular in $v = a$: if $v = a$ we must have $2a^2 r = GM$, which fixes the radius to the so-called Parker radius: $r_P = GM/2a^2$, which is obtained by setting the LHS of the equation to zero. Close to the star, the speed is subsonic ($v < a$), so the denominator D is negative; also the numerator N is negative in

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2a^2/r - GM/r^2}{v^2 - a^2} \stackrel{\text{def}}{=} \frac{N}{D}\,, \tag{2.32}$$

so on the whole N/D is positive, which is consistent with our assumption $\mathrm{d}v/\mathrm{d}r > 0$, which we make since we are considering winds, as opposed to accretion.

Why is the numerator negative? This is equivalent to saying

$$2\frac{\mathcal{R}T}{\mu} < \frac{GM}{r}\,, \tag{2.33}$$

which means that Ⓑ $\times 4/5 <$ Ⓒ, (using the notation from equation (2.18)) which holds since, as we wrote there, Ⓑ $\ll$ Ⓒ near the stellar radius.

Far from the star the numerator is positive, so the speed must be supersonic, so that $N > 0$ and we can still have $N/D > 0$ everywhere, guaranteeing that the velocity gradient is always positive.

So, the only physical solution is transsonic.

The critical velocity *must* be attained at the Parker radius in order to have a physically meaningful transsonic solution; always-subsonic and always supersonic solutions are mathematically possible but not usually observed.

The velocity gradient at the critical point can be found by de l'Hôpital's rule to be

$$\left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r_P} = \pm\frac{a^3}{GM}\,. \tag{2.34}$$

This is calculated by expanding in r; however note that we must vary both r and v when we differentiate with respect to r. We find:

$$\frac{\mathrm{d}v}{\mathrm{d}r} = v\frac{2a^2/r - GM/r^2}{v^2 - a^2} \tag{2.35}$$

$$= \Bigg(a + \underbrace{\frac{\mathrm{d}v}{\mathrm{d}r}}_{\text{Second order}}\Bigg)\frac{-2a^2/r_P^2 + 2GM/r_P^3}{2a\,\mathrm{d}v/\mathrm{d}r} \tag{2.36}$$

Differentiated above and below at $r = r_P$ and $v = a$

$$\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2 = -\frac{a^2}{r_P^2} + \frac{GM}{r_P^3} \tag{2.37}$$

Simplified the $2a$

$$= -a^2\left(\frac{2a^2}{GM}\right)^2 + GM\left(\frac{2a^2}{GM}\right)^3 \tag{2.38}$$

Substituted $r_P = 2a^2/GM$

$$= -4\frac{a^6}{(GM)^2} + 8\frac{a^6}{(GM)^2} \tag{2.39}$$

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \pm 2\frac{a^3}{GM}\,. \tag{2.40}$$

Claim 2.2.1 (Exercise)**.** *The speed of sound at the critical point equals half of the escape velocity at that radius.*

The escape velocity is given by

$$v_{\mathrm{esc}}^2 = \frac{2GM}{r}\,, \tag{2.41}$$

but we know that $r_P = GM/2a^2$: so

$$a^2 = \frac{GM}{2r_P}\,, \tag{2.42}$$

which means that, if we calculate the escape velocity at the Parker radius, we have $v_{\mathrm{esc}}^2/a^2 = 4$, so $v_{\mathrm{esc}}/a = 2$.

There are exactly two transsonic solutions: if we trace a cross in the r, v plane centered on the critical point and speed of sound we see that all solutions meet it perpendicular to it. The solution with the decreasing velocity gradient is an accretion solution: what is plotted is the *absolute value* of the velocity, the accretion solution has always-negative absolute velocity gradient.

Always-supersonic solutions and always-subsonic ones are also found, but they do not obey the monotonicity of the velocity, so we discard them.

The boundary condition is the velocity at some r_0: the problem is second-order, but if we select the specific transsonic solution we eliminate the necessity of one boundary condition. The choice we make for $v_0 = v(r_0)$ is key, and nontrivial.

If we have the density, velocity and radius of the lower boundary we have *fixed* the accretion rate: $\dot{M} = 4\pi r_0^2 \rho_0 v_0$.

This is actually fixed by only specifying the gravity, temperature and density at the star's edge. It would seem like we also should have v_0, but we do not actually need it, as we will show in a moment.

With these hypotheses, we can solve the momentum equation analytically: we get

$$\frac{v}{v_0}\exp\left(-\frac{v^2}{2a^2}\right) = \left(\frac{r_0}{r}\right)^2 \exp\left(\frac{GM}{a^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right) \tag{2.43}$$

$$= \left(\frac{r_0}{r}\right)^2 \exp\left(\frac{2r_c}{r_0} - \frac{2r_c}{r}\right). \tag{2.44}$$

So, to calculate the Mach number we need to solve

$$Me^{-M^2/2} = \frac{v_0}{a}\left(\frac{r_0}{r}\right)^2 \exp\left(\frac{2r_c}{r_0} - \frac{2r_c}{r}\right), \tag{2.45}$$

but there seems to be an issue: the left hand side is bounded (it attains its maximum value of $e^{-1/2} \approx 0.6$ at $M = 1$) while the RHS is maximized at $r = r_c$, where it is equal to

$$\frac{v_0}{a} \left(\frac{r_0}{r_c} \right)^2 \exp\left(\frac{2r_c}{r_0} - 2 \right), \tag{2.46}$$

which is dependent on v_0! If we plug in solar values for the parameters, $T = 1 \times 10^6\,\mathrm{K}$, $\mu = 0.62$ and select an initial velocity like $v_0 = 1\,\mathrm{km/s}$ we get that the RHS's maximum is around 35!

This means that, in order to always have solutions, we need to fix v_0 to set the maxima to be equal. This gives us

$$v_0 = e^{-1/2} \left(\frac{1}{a} \left(\frac{r_0}{r_c} \right)^2 \exp\left(\frac{2r_c}{r_0} \right) e^{-2} \right)^{-1} \tag{2.47}$$

$$= a \left(\frac{r_c}{r_0} \right)^2 \exp\left(-\frac{2r_c}{r_0} + \frac{3}{2} \right). \tag{2.48}$$

Specifically, we can see that for the solar values mentioned before $1\,\mathrm{km/s}$ was an overestimate, and with the values we fixed the initial velocity is actually more like $17\,\mathrm{m/s}$.

At large distances, we get $v \to 2a\sqrt{\ln(r/r_0)}$.

To derive this, take the log of the equation and take the limit $r \to \infty$.

$$-\frac{M^2}{2} + \log M = \log \frac{v_0}{a} + 2 \log \frac{r_0}{r} + \frac{2r_c}{r_0} - \frac{2r_c}{r}, \tag{2.49}$$

so the RHS diverges as $-\log r$ does for $r \to \infty$, which means that the LHS must also diverge and become very large and negative. The polynomial term $-M^2/2$ dominates the logarithmic one. So, asymptotically we have $-M^2/2 \approx 2 \log r_0/r$, which means $M \approx 2\sqrt{\log(r/r_0)}$.

This is reported incorrectly in the slides as $v \to 2a \log(r/r_0)$.

This is unphysical (the velocity diverges!?) and due to the fact that we assume a constant temperature (and, thus, energy input) even for diverging r.

The density profile can be found from the continuity equation to be

$$\frac{\rho}{\rho_0} \exp\left(\frac{1}{2} \left(\frac{v_0 \rho_0 r_0^2}{a \rho r^2} \right)^2 \right) = \exp\left(2r_c \left(\frac{1}{r} - \frac{1}{r_0} \right) \right), \tag{2.50}$$

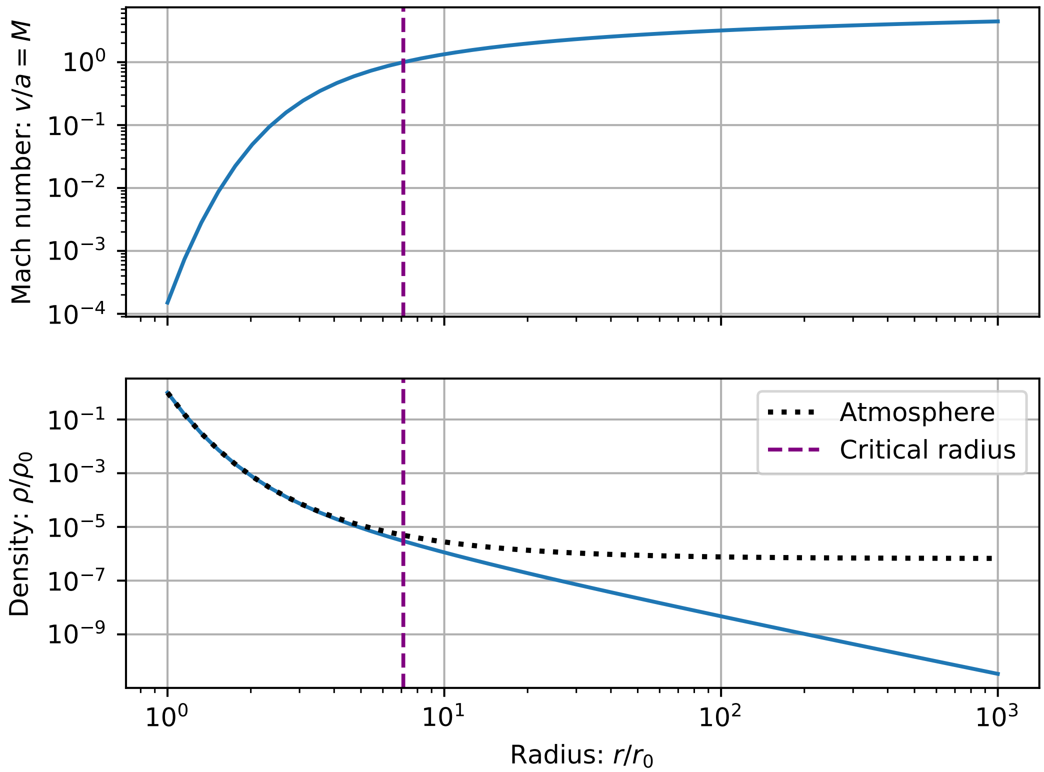

Figure 2.1: Velocity profile (solution to (2.43)) and density wind profile (solution to (2.50)), plus atmosphere (hydrostatic) profile (shown in (2.53)).

Now we look at the structure of the wind in the subcritical region.

In the corresponding slide: the dashed line in the density profile is the density we'd expect in a hydrostatic atmosphere, while the solid one is the solution.

The hydrostatic density structure is given by

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0\,, \tag{2.51a}$$

since we set the (log) velocity gradient to zero. Manipulating this we get

$$\frac{r^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{GM}{a^2}\,, \tag{2.52}$$

So, integrating this we see that the density profile follows a decreasing exponential law:

$$\frac{\rho(r)}{\rho_0} = \exp\left(\frac{-(r-r_0)}{H_0}\frac{r_0}{r}\right)\,, \tag{2.53}$$

where H_0 is the scale height, $H_0 = \mathcal{R}T/(\mu g_0)$ with $g_0 = GM/r_0^2$. The length scale at which this decreases is defined by H_0.

The density profile in the subsonic region is very well approximated by this hydrostatic profile: as we can see in figure 2.1 in the subsonic region the log-velocity gradient is small, so the pressure contribution dominates.

The mass loss rate is our main prediction: we have $\dot{M} = 4\pi r_0^2 \rho_0 v_0 = 4\pi r_c^2 \rho_c a$.

In order to compute it at the critical radius, we can use the density profile equation:

$$\dot{M} = 4\pi r_c^2 a \rho_0 \exp\left(\frac{-(r_c - r_0)}{H_0} \frac{r_0}{r_c} \right) . \tag{2.54}$$

This is an approximation, justified by the fact that in the subsonic region the density profile of the wind is almost the same as in hydrostatic equilibrium. There is a correction factor of $\exp(-1/2)$.

If we consider this numerically, we find that the exponential is the dominant part: the mass loss rate becomes lower when the critical point moves outward. We can specify it by fixing

1. the temperature at the corona, T_C;
2. the radius at the bottom of the corona, r_0;
3. the stellar mass M;
4. the density at the bottom of the corona ρ_0.

In the slides there are numerical estimates. As H_0 increases, the density profile is less steep: the density remains high at larger radii.

Why is the dependence on $(r_c - r_0)/H_0$ so strong? The idea seems to be that in order for the wind to expel a significative quantity of material it needs to give it a large velocity in a small space, otherwise the wind becomes supersonic when its density is very small.

Mon Nov 04 2019

2.2.1 Isothermal winds with an external force

Last time we found that the only physical solution to the stellar wind is the transsonic solution.

Now, let us add an external force:

$$
v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} + f(r)\,, \tag{2.55}
$$

so we get

$$
\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2\frac{a^2}{r} - \frac{GM}{r^2} + f(r)}{v^2 - a^2}\,, \tag{2.56}
$$

with the speed of sound $a = \sqrt{\mathcal{R}T/\mu}$. How does the velocity gradient change? With an outward force, we expect the velocity gradient to be less steep in the subsonic region (the velocity decreases slower: the numerator is *less negative*). Adding this force is formally equivalent to modifying the gravitational field by making it weaker. This increases the pressure scale height, so the density gradient becomes less steep, so the velocity gradient becomes less steep as well.

In the supersonic region, instead the gradient will be larger: the numerator is *more positive* and higher velocities are reached.

The critical radius changes: it is the solution to

$$
r_C = \frac{GM}{2a^2} - \frac{f(r_C) r_C^2}{2a^2}\,, \tag{2.57}
$$

which will shift inward as f goes from 0 to positive. This is shown in figure 2.2.

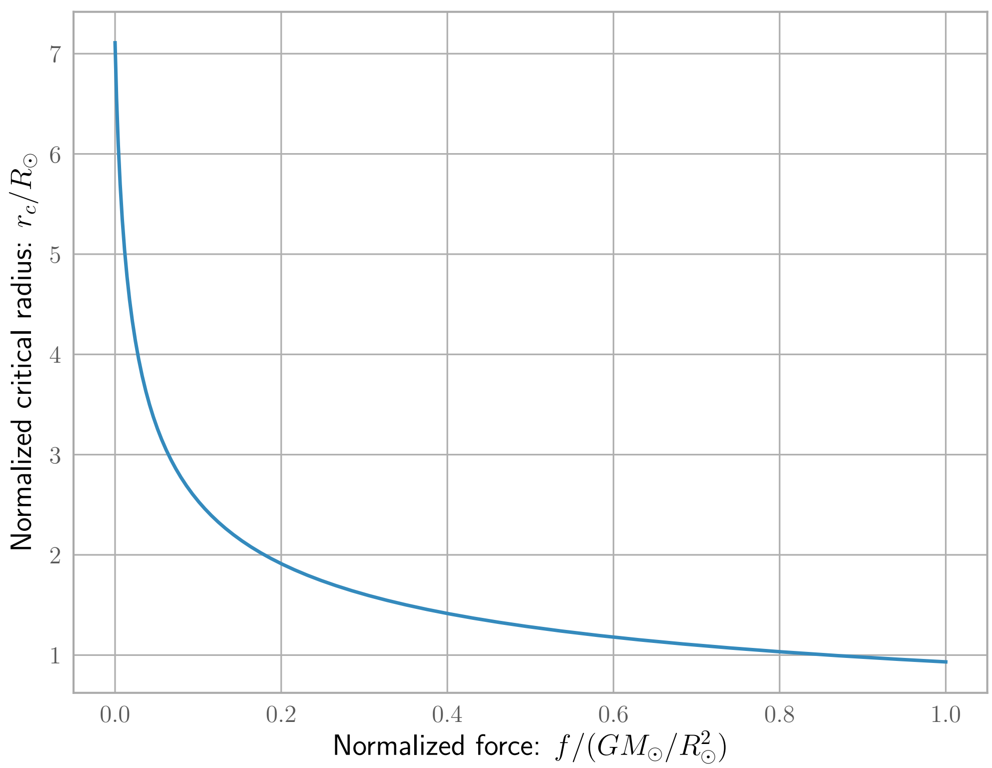

Figure 2.2: Critical radius position in function of a constant force, expressed in units of the gravitational acceleration at the surface. Here, we assume solar parameters and $T = 10^6$ K.

The adiabatic speed of sound is the same, it does not depend on f. Also, the velocity gradient is less steep as we said before. These two facts combined mean that the velocity at the corona, v_0, must be larger.

Since $\rho'/\rho + v'/v + 2/r = 0$, and the critical radius velocity is $v = a$ regardless of the value of f, and the critical radius decreases if we have $f > 0$, we must have that ρ_c, the density at the critical radius, must be smaller than that with $f = 0$.

How do we expect the mass loss rate to change? from the continuity equation at the bottom of the corona, we get $\dot{M} = 4\pi r_0^2 \rho_0 v_0$, and everything on the RHS is fixed but v_0, so when we increase f the LHS must incrase as well.

Let us consider some explicit law scaling as $f \propto r^{-2}$, like a radiative force:

$$g_{\text{rad}} \propto \kappa_F \times \left(\frac{r}{R}\right)^{-2}. \tag{2.58}$$

This is the same as changing the mass of the star, since it scales like the gravitational force. We then take our force to be $f(r) = A/r^2$. Our momentum equation, with the pressure gradient substituted in from the differentiated ideal gas law and the continuity equation, becomes

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \left(\frac{2a^2}{r} - \frac{GM}{r^2} + \frac{A}{r^2}\right) / \left(v^2 - a^2\right), \tag{2.59}$$

which is the same as the equation without force, with a smaller effective mass: $M_{\mathrm{eff}} = M(1 - A/GM)$.

The correction is usually called the Eddington ratio:

$$\Gamma = \frac{A}{GM} = \frac{A/r^2}{GM/r^2}, \tag{2.60}$$

the acceleration of the force divided by the gravitational one.

If A is a consant, the critical radius becomes

$$r_C = \frac{GM}{2a^2}(1 - \Gamma), \tag{2.61}$$

however in general A is not taken as a constant, instead it is modelled with a Heaviside theta function, so that it only activates after a certain radius.

As Γ increases, the critical velocity is reached faster, and the velocity at the corona is greater. The density profile gets less and less steep: the density scale height decreases.

We approximate $A(r) = A[r \geq r_d]$ for some r_d.[1] The justification for this model is that below the dust condensation region there is only gas, above it there is dust which is more opaque to radiation. We are still assuming the wind is isothermal.

The critical point depends on Γ:

$$\frac{r_C}{1 - \Gamma(r_C)} = \frac{GM}{2a^2}. \tag{2.62}$$

If the extra force switches on *outside* the critical region, the mass loss rate is *unchanged*, since it only depends on the subsonic region.

Is there the possibility to have more than one critical radius with only outward force?

There is a maximum value of Γ such that the critical radius r_C is equal to the radius of the star.

As Γ increases towards this value, the mass loss rate increases greatly. If Γ is larger than this value, and v_0 is subsonic, then the RHS of the momentum equation is *negative*: the velocity gradient is negative, the velocity decreases.

[1] Using the Iverson bracket here!

Wed Nov 06 2019

2.3 Non-isothermal winds

Now, we consider the possibility that our winds are *not isothermal*. This will change the structure of the wind, by the introduction of an additional pressure gradient.

It will change the speed of sound and thus the Mach number.

It is useful to define the energy per unit mass e:

$$e(r) = \frac{v^2(r)}{2} - \frac{GM}{r} + \frac{\gamma}{\gamma - 1} \frac{\mathcal{R}T}{\mu}, \tag{2.63}$$

where $\gamma/(\gamma - 1) = 5/2$ for a monoatomic gas, which has $\gamma = 3/2$.

In the lower boundary of the wind the velocity is much less than the escape velocity ($v \ll v_{\text{esc}}$), and also the thermal velocity of the particles is not enough for them to escape the gravitational well: $\mathcal{R}T/\mu \ll v_{\text{esc}}$ at the surface of the star, while far from the star we have $v \gg v_{\text{esc}}$.

If we have an isothemal wind, some energy must be added in order to prevent the adiabatic cooling of the gas, lift it from the potential well, and to increase its kinetic energy.

If a force is applied it increases the momentum, but the heat transmission q also appears in the momentum equation, which is $\Delta e = \int f + q \, dr$. Heat transmission changes the pressure profile, which affects the momentum, even though q does not appear explicitly in the momentum equation.

We define the total heat deposition Q and the total work done by the force W:

$$Q(r) = \int_{r_0}^{r} q(\widetilde{r}) \, d\widetilde{r} \qquad \text{and} \qquad W(r) = \int_{r_0}^{r} f(\widetilde{r}) \, d\widetilde{r}, \tag{2.64}$$

and we will have $e(\infty) - e(r_0) = Q(\infty) + W(\infty)$.

The most general momentum equation is given by:

$$\frac{1}{v} \frac{dv}{dr} = \left(2\frac{c_s^2}{r} - \frac{GM}{r} + f - (\gamma - 1) q \right) / \left(v^2 - c_s^2 \right), \tag{2.65}$$

where we introduce the adiabatic speed of sound $c_s = \sqrt{\gamma a^2}$, a being the *isothermal* speed of sound. In general $-(\gamma - 1) < 0$, therefore if we add heat this is equivalent to pushing *inward*.

If either f or q depend on the velocity gradient dv/dr then the sonic point can *decouple* from the critical point.

There are cases in which we have multiple critical points (specifically, multiple zeros of the denominator).

The momentum equation plus the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v^2}{2}+\frac{5}{2}\frac{RT}{\mu}-\frac{GM}{r}\right)=f(r)+q(r)\,, \tag{2.66}$$

can be solved numerically, and if we impose smooth passage through the critical point this yields the mass loss rate.

Qualitatively, the results are the same as in the isothermal case. Adding either momentum or energy to the subsonic region of the wind increases the bottom-of-the-corona velocity and the mass loss rate. Doing it in the supersonic region has no effect.

This ends our general introduction to stellar winds.

Now we will do a couple of exercises to get familiar with the theory.

Exercise

The wind is isothermal. The solar wind has a mean coronal temperature of $1.5\times10^6\,\mathrm{K}$ and a mass loss rate of 2×10^{-14} solar masses per year. The bottom of the corona is at $r_0\approx1.003R_\odot$, where the density is $\rho(r_0)=10^{-14}\,\mathrm{g/cm^3}$.

Calculate the potential energy, the kinetic energy and the enthalpy of the gas at r_0.

Calculate the same quantities at the critical point. Which of these energies has absorbed the largest fraction of the energy input?

We can use the continuity equation $\dot{M}=4\pi\rho_0r_0^2v_0$ to get

$$v_0=\frac{\dot{M}}{4\pi r_0^2\rho_0}\approx21\,\mathrm{m/s}\,, \tag{2.67}$$

and with this we can calculate

$$e(r_0)=-\frac{GM}{r_0}+\frac{1}{2}v_0^2+\frac{5}{2}\frac{RT}{\mu}\,. \tag{2.68}$$

We get:

$$E_{\mathrm{kin},\,0}=\frac{v_0^2}{2}\approx212\,\mathrm{J/kg}\,, \tag{2.69}$$

while for the gravitational energy we'd need the mass of the star. Assuming it is equal to the solar mass, we find

$$E_{\mathrm{grav},\,0}=-\frac{GM}{r_0}\approx-1.9\times10^{11}\,\mathrm{J/kg}\,. \tag{2.70}$$

The mean molecular weight of the gas for the Sun is something like $\mu = 0.62$ (we count electrons in it). Then we get

$$E_{\text{chem, 0}} = E_{\text{chem, crit}} = \frac{5}{2}\frac{RT}{\mu} \approx 5.0 \times 10^7\,\text{J/kg}\,. \tag{2.71}$$

The enthalpy is the same everywhere in the flow, since the flow is isothermal.

The values at the critical radius are calculated with the same formula. The velocity will be the speed of sound $a = \sqrt{RT/\mu} \approx 4.5 \times 10^3\,\text{m/s}$.

Then we find:

$$E_{\text{kin, crit}} = \frac{a^2}{2} \approx 1.0 \times 10^7\,\text{J/kg} \approx 5 \times 10^4 \times E_{\text{kin, 0}}\,. \tag{2.72}$$

The critical radius is given by $r_c = GM/(2a^2)$: so, we get

$$E_{\text{grav, crit}} = -\frac{GM}{r_c} = -2a^2 \approx -4.0 \times 10^7\,\text{J/kg} \approx 2.1 \times 10^{-4} \times E_{\text{grav, 0}}\,. \tag{2.73}$$

Qualitatively, at the corona we have

$$\left|E_{\text{grav, 0}}\right| \gg E_{\text{chem, 0}} \gg E_{\text{kin, 0}}\,, \tag{2.74}$$

while at the critical point they are similar, and specifically

$$E_{\text{chem, crit}} \gtrsim \left|E_{\text{grav, crit}}\right| \gtrsim E_{\text{kin, crit}}\,. \tag{2.75}$$

Exercise

A star with $T_{\text{eff}} = 3200\,\text{K}$, $R_* = 30R_\odot$, $L_* = 85L_\odot$ and $M_* = 6M_\odot$ has an isothermal corona of $T = 10^6\,\text{K}$ with a density at the lower boundary of $10^{-13}\,\text{g/cm}^3$.

Calculate the energy per unit mass at the bottom of the corona at $r_0 = R_*$.

Calculate the location of the critical point, r_c, and the mass loss rate.

Calculate the energy per gram gained by the wind between r_0 and r. What fraction of the stellar luminosity is used to drive the wind up to the critical point?

The energy per unit mass at the bottom of the corona is given by

$$e(r) = -\frac{GM}{r_0} + \frac{1}{2}v_0^2 + \frac{5}{2}\frac{RT}{\mu}\,, \tag{2.76}$$

but we cannot use this formula since we do not have v_0 nor $\dot{M}$. However, we can approximate the density profile as an exponential, applying the formula

$$\dot{M} = 4\pi r_c^2 a \rho_c \tag{2.77a}$$

$$= 4\pi r_c^2 a \rho_0 \exp\left(-\frac{r_c - r_0}{H_0}\frac{r_0}{r_c}\right), \tag{2.77b}$$

where we have: the length scale $H_0 = RTr_0^2/(GM\mu)$, the critical velocity $a = \sqrt{RT/\mu}$, and the critical radius $r_c = GM/(2a^2)$.

Plugging these in, we find:

$$\dot{M} = 4\pi\left(\frac{GM\mu}{2RT}\right)^2\sqrt{\frac{RT}{\mu}}\rho_0 \exp\left(\frac{\mu GM}{RT}\left(\frac{2RT}{GM\mu} - \frac{1}{r_0}\right)\right) \tag{2.78a}$$

$$= \pi(GM)^2\left(\frac{\mu}{RT}\right)^{3/2}\rho_0 \exp\left(2 - \frac{\mu GM}{RTr_0}\right). \tag{2.78b}$$

One temperature should probably be the effective temperature, but I do not really know what that means.

Then, the mass loss rate can be calculated, since we have all of these quantities. It comes out to be barely anything, since $\mu GM/RTr_0$ is very large and we have a negative exponential of it...

Lamers-Cassinelli: chap 4, section 3: multiple critical points.

Next Tuesday, the 12th, after the lecture, we will likely move to the DFA in Via Marzolo until 14.30.

2.4 Wind types

Coronal winds These are driven by gas pressure due to high temperature. Stars with a convection zone right under the photosphere can have coronas of a few million Kelvin degrees. These are rather well described as isothermal, however the temperature decreases slowly because of conduction.

Dust driven winds For these winds, the driving mechanism is the radiation pressure on the dust grains, which are continuum absorbers. The gas component is dragged along by momentum transfer. This happens for cool stars, whose envelopes have temperatures of less than a thousand degrees Kelvin, in which the dust grains can actually be formed. These can be well modelled as winds with a force like $f = [r \geq r_d] A/r^2$, where the radius r_d is the one after which the temperature is low enough for the grains to form.

Line-driven winds These are winds of hot stars, driven by radiation pressure on spectral lines of abundant ions which have many in the UV and far UV. The pressure depends on the Doppler effect: the ions can absorb different wavelengths of light at different radii because of it. The force depends on the velocity gradient: it cannot be modelled with what we discussed so far.

Pulsation driven winds Stars such as Miras and those in the Asymptotic Giant Branch may pulsate: the atmosphere is tossed up and then falls back. Because of the low gravity, they fall slowly and are hit by an outmoving layer before they have completed their fall. This means that for each pulsation cycle they get a "kick".

This can be a very efficient driving method if we account for dust formation. The shockfront from the pulsation cycle basically travels outwards to infinity.

Sound wave driven winds They are modelled in a way that is similar to coronal winds. The resulting amplitudes are usually small.

Alfvén Wave Driven winds Due to magnetic fields: if the points at which the field lines make contact with the surface move, than a magnetic wave travels forward with a speed $v_A = B/\sqrt{4\pi\rho} \gg a$. This can be very efficient, and result in high wind speeds. It is relevant for stars which do not have strong radiation pressure (ie not more than a thousand times more luminous than the Sun).

Magnetic rotating winds

2.5 Coronal winds

Now, we introduce the next topic. We deal with hot, luminous stars: at the top left of the HR diagram.

[Picture from the slides]

We talk of line-driven winds for hot stars. These are winds driven by *spectral lines*. The blackbody emission for these stars is mainly at high frequencies, like the UV.

One may ask: hydrogen is much more abundant than heavier elements like carbon, nitrogen. . . why do we see the spectral lines for these heavier elements?

This is because hydrogen is completely ionized at these temperatures, and helium is also.

The strongest lines are far in the UV, where the stellar flux is low: only few atoms are hit, the rest of the gas is dragged along.

A peculiar characteristic of the spectrum is the so-called P-Cygni profile.

Now, an overview of the formation of spectral lines: there are 5 processes

1. Line scattering: if it comes from the ground state of the atom, it is called *resonance scattering*, which is the main phenomenon.

2. Emission by recombination: the ion recombines to an excited state.

3. Emissional from collisional or photo-excitation. A photon is absorbed, it excites the atom which then descends to a lower energy level.

4. Pure absorption and then de-excitation of an already excited atom.

5. Masering by stimulated emission: it can happen in a very narrow set of circumstances: an excited atom is hit by a photon which has exactly the same energy as the one between the atom's state and the ground state, so the atom is deexcited and now there are two photons. This happens when there are many excited atoms, and when the velocity gradient is very small — otherwise, the photons are Doppler-shifted out of the right frequency.

Mon Nov 11 2019

Tomorrow we have the meeting at 13.30 in room C.
On the 19th the lecture is in room A (still @ Specola).
Now we deal with line driven winds. The main mechanism is line scattering.

2.5.1 P-Cygni profiles

They are spectra characterized by a blue-shifted absorption and a red-shifted emission.

We consider a source which emits a spherically symmetric wind. The region of the wind which is directed at us corresponds to a blue-shifted absorption. We basically have emission which is symmetric centered at $v = 0$ (since most of the radiation comes from the photosphere), and absorption which is caused by atoms moving towards us at velocities $0 \leq v \leq +v_\infty$: so the absorption is something like $-k[0 \leq v \leq v_\infty]$.

We can gather data from these profiles: for example, the maximum Doppler shift of the absorption corresponds to the maximum velocity.

We can also infer the number densities of the chemical species in the wind:

$$n_i = \frac{X_i \rho}{A_i m_u}, \tag{2.79}$$

assuming some parametric velocity profile $v(r)$ and density profile $\rho(r)$, by this we derive the mass loss rate.

The opacity of the spectral lines is several orders of magnitude larger than the continuum opacity: for example, the opacity of the IV line of Carbon is $\sim 10^6 \kappa_{es}$.

When a photon is absorbed, the momentum increases by $\Delta p = h\nu/c$. The increase in velocity when a typical metal absorbs an UV photon with wavelength 10^{-7} m is of the order $\Delta v \approx 20\,\mathrm{cm/s}$. Typically, due to the redistribution of momentum among atoms, this is something like four orders of magnitude less. All the gas is then accelerated by the radiation.

In order to accelerate the gas to $2000\,\mathrm{km/s}$ we'd need 10^{11} photons. If the distance to accelerate to terminal velocity is about three sun radii, the time to accelerate is of the order $10^4\,\mathrm{s}$. So we need around 10^7 photons per second: the typical lifetime of the transition is of the order $10^{-7}\,\mathrm{s}$,

So only transitions with oscillator strengths $f \gtrsim 0.01$ will contribute significantly.

What is oscillator strength? for $\Delta t \approx 10^{-7}\,\mathrm{s}$ we have $\Delta E = \hbar/\Delta t \approx 6\,\mathrm{neV}$...

The largest contribution is not the energy of the photons, but their momentum input.

Let us calculate the radiation pressure due to one line. The momentum equation is

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} + f(r)\,, \tag{2.80}$$

and we want to compute the line-driven $f(r)$.

Say we have a line @ wavelength λ_0, which we assume coincides with the peak of the Planckian function for the star. We also assume that the line is so strong that it absorbs or scatters *all* the photons at its wavelength.

So, photons at that wavelength have 0 mean free path?
It seems like that is not necessary actually.

How much mass loss can one optically thick line produce?

The emitted wavelengths which are absorbed are all the ones between λ_0 and $\lambda_0(1 - v_\infty/c)$.

If our velocity range goes from $v = 0$ ar $r = R$ to $v = v_\infty$ at $r = \infty$, then the total energy absorbed is:

$$L_{\mathrm{line}} = \int_{\nu_0}^{\nu_0(1+v_\infty/c)} \underbrace{F_\nu 4\pi R^2}_{L_\nu}\,\mathrm{d}\nu \approx L_{\nu_0}\Delta\nu = L_{\nu_0}\nu_0 v_\infty/c\,, \tag{2.81}$$

where F_ν is the monochromatic flux (specific spectral intensity in $\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}$) at the photosphere, whose radius is R. We approximate the flux to be almost constant with respect to ν.

L_{ν_0} is the specific luminosity per unit frequency at ν_0.

The variation of radiative momentum is

$$\frac{\mathrm{d}p_{\mathrm{rad}}}{\mathrm{d}t} = \frac{L_{\mathrm{line}}}{c} = \frac{L_{\nu_0}\nu_0 v_\infty}{c^2}, \tag{2.82}$$

and the total momentum loss in the wind per second is

$$\frac{\mathrm{d}p_{\mathrm{wind}}}{\mathrm{d}t} = \dot{M} v_\infty, \tag{2.83}$$

so if we approximate the wind to be momentum wind, we can equate these two terms: then we get

$$\frac{L_{\nu_0}\nu_0}{c^2} = \dot{M}. \tag{2.84}$$

Now, the Planck function has the property that $L_{\nu_0}\nu_0 = 0.62L$. Then we see that the mass loss rate of one strong line is linear in the luminosity of the star.

These are additive: if we can approximate the mass loss rate and luminosity with other means, we can find the total number of spectral lines.

Tue Nov 12 2019

Last time we estimated the radiation pressure due to one line.

If we know $\dot{M}$ and the luminosity L, then we can calculate the effective number of lines: $N_{\mathrm{eff}} = \dot{M}c^2/L$.

We define the *efficiency parameter*:

$$\eta = \frac{\dot{M} v_\infty}{L/c}. \tag{2.85}$$

This can be split into various components due to several factors: we always divide by the total luminosity, and have

$$\eta_{\mathrm{pot}} = \frac{\dot{E}_{\mathrm{pot}}}{L} = \frac{\dot{M} G M_*}{R_* L} \tag{2.86a}$$

$$\eta_{\mathrm{kin}} = \frac{\dot{E}_k}{L} = \frac{\dot{M} v_\infty^2}{2L} \tag{2.86b}$$

$$\eta_{\mathrm{th}} = \frac{5\dot{M} R T_w}{2\mu L}, \tag{2.86c}$$

and we will need

$$\eta_{\mathrm{moment}} = \frac{\dot{M} v_\infty c}{L}. \tag{2.87}$$

This is basically the energy emission budget, L, divided into its contributions.

The efficiency for non-momentum components is very low as compared to momentum efficency: what matters, again, is the transfer of momentum, not the transfer of energy.

We'd expect $\eta \leq 1$, but actually sometimes the momentum efficiency can be larger: in one case we have $\eta_{\text{momentum}} \approx 59$.

If across all the spectrum of the star we have completely optically thick absorption, then the momentum of the wind is equal to the momentum of radiation:

$$\dot{M}_{\max} v_\infty = \frac{L}{c} \implies \dot{M}_{\max} = \frac{L}{c v_\infty}\,, \tag{2.88}$$

and typically v_∞ is of the order of 2, 3 times the escape velocity at the photosphere: $v_\infty \approx 3\sqrt{2GM/R}$. For the luminosity we consider $L = 10^5 L_\odot$.

Then, the maximum mass loss rate estimated by (2.88) is similar to the observed one. The ratio $\eta_{\text{momentum}} = \dot{M}/\dot{M}_{\max}$ is typically $0.5 \div 1$, but for some stars it can be of the order 10^1 to 10^2.

This is called the *single scattering upper limit*: we assume that scattering is isotropic, therefore we'd expect that after the first scattering the photon does not contribute anymore. This is not actually the case: *multiple scattering* can contribute, enhancing the maximum mass loss rate by a factor τ_w, equal to

$$\tau_w = \int_{r_c}^{\infty} \kappa \rho \, \mathrm{d}r\ , \tag{2.89}$$

so $\dot{M}_{\text{max, multiple scattering}} = \tau_w \dot{M}_{\text{max, single scattering}}$.

This typically gives us enhancement factors of the order 2 to 6. The quantity τ_w is adimensional, since the units of κ are cm^2/g.

Even though we have this corrective factor, the efficiency is always expressed with respect to the single scattering cross section. Multiple scattering theory accounts for all of the increased mass loss rate, even up to $\tau \sim 10^2$.

Now, we derive the expression for the radiative acceleration privided by one line in a moving atmosphere.

We have a unit volume, of $1\,\mathrm{cm}^3$, it has a velocity gradient inside it from v to $v + \Delta v$, and it will be heated by a monochromatic flux given by $F_\nu = I_\nu/(4\pi r^2)$.

What is the acceleration g_{line}? First we need to know the absorption properties of the medium. The absorption coefficient per cubic centimeter of gas is

$$\kappa_\nu = \frac{\pi e^2}{m_e c} f n_i \phi(\nu)\,, \tag{2.90}$$

where $\pi e^2/(m_e c) \approx 2.654 \times 10^{-2}\,\mathrm{cm}^2/\mathrm{s}$ is the frequency integrated cross section of the classical oscillator (for the electron):

$$\frac{\pi e^2}{m_e c} = \sigma = \int_0^\infty \sigma(\nu) \, \mathrm{d}\nu\ ; \tag{2.91}$$

f is the oscillator strength, which depends on the line (it is a probability); n_i is the number density of atoms which can absorb the line, and $\phi(\nu)$ is the Gaussian profile of the absorption coefficient, centered at the rest frequency ν_0, and normalized so that $\int \phi(\nu)\, d\nu = 1$: therefore the units of ϕ are $1/\mathrm{Hz}$.

The typical profile function is a *Doppler profile*: its width is of an order based on the thermal velocity of the atoms, which can sometimes be approximated as much less than the wind velocity: so, we apply the *Sobolev approximation*, and estimate $\phi(\nu) \sim \delta(\nu - \nu_0)$.

So we need to consider lines which do not overlap.

We get

$$F_{\mathrm{rad}} = \frac{dP_{\mathrm{rad}}}{dt} = \frac{1}{c} \frac{dE_{\mathrm{rad}}}{dt}\,, \tag{2.92}$$

where dE_{rad}/dt is the radiative energy absorbed per unit time and volume by the line.

Then we get

$$g_{\mathrm{line}} = \frac{F_{\mathrm{rad}}}{\rho} = \frac{1}{c\rho} \frac{dE_{\mathrm{rad}}}{dt}\,. \tag{2.93}$$

We still need to calculate the radiative energy absorbed: for a single optically thick line is $dE_{\mathrm{rad}}/dt = F_\nu \kappa_{\mathrm{line}}$.

A more general expression, which however still applies the Sobolev approximation, is

$$g_{\mathrm{rad}} = \frac{F_{\nu_0} \nu_0}{c} \Big(1 - \exp\big(-\tau_{\nu_0}(\mu = 1)\big)\Big) \frac{dv}{dr} \frac{1}{c\rho}\,; \tag{2.94}$$

What is $\mu = 1$ about?

here F_{ν_0} is the monochromatic flux from the star emitted at the line rest frequency ν_0, the quantity

$$\frac{\nu_0}{c} \frac{dv}{dr} \tag{2.95}$$

is the width of the frequency band that can be absorbed, while $1 - \exp(\dots)$ is the probability that the absorption occurs in our cubic centimeter: there τ is the optical depth (integrated up to infinity).

The product of these three terms gives an energy absorption rate dE_{rad}/dt, if we multiply by $1/c$ we get absorbed momentum, if we multiply by $1/\rho$ we get acceleration.

Let us start from optically thin line: they absorb only part of the radiative flux.

Then we get $\exp(-\tau_{\nu_0}) \sim 1 - \tau_{\nu_0}$, therefore $1 - \exp(-\tau_{\nu_0}) = \tau_{\nu_0}$. This is proportional to $\rho\kappa \propto n_i$.

In the end, we get

$$g_{\text{line}} = \frac{F_\nu}{c}\frac{n_i}{\rho}\left(f\frac{\pi e^2}{m_e c}\right) \sim \frac{L_\nu}{r^2}\frac{n_i}{\rho}\,, \tag{2.96}$$

where L_ν is the monochromatic luminosity of the star: we have $F_\nu \sim L_\nu/r^2$. We can approximate n_i/ρ as a constant with respect to the radius: therefore, we have $g_{\text{line}} \propto L_\nu/r^2$.

Mon Nov 18 2019

Let us come back to

$$g_l = \frac{F_\nu}{c}\frac{n_i}{\rho}\frac{\pi e^2}{m_e c}f \sim \frac{L_\nu}{r^2}\frac{n_i}{\rho}\,. \tag{2.97}$$

We can reduce drastically the number of spectral lines we have to account for if we assume that the density of the wind is very low: only lines from the ground state have to be accounted for.

How do we deal with the radiation pressure for an ensemble of lines? We have the CAK formalism: it gives the estimate

$$g_L \sim \left(\rho^{-1}\frac{\mathrm{d}v}{\mathrm{d}r}\right)^\alpha \sim \left(\frac{vr^2}{\dot{M}}\frac{\mathrm{d}v}{\mathrm{d}r}\right)^\alpha\,, \tag{2.98}$$

where α is a parameter quantifying the optical thickness of the line: it goes from 0 for an optically thin line to 1 for an optically thick line. In general we have from the continuity equation $\dot{M} \propto r^2\rho v$, which justifies the expression here.

The expression proposed by CAK was $g_L = g_e M(t)$, where g_e is the radiative acceleration from continuum phenomena (mostly electron scattering), while the *force multiplier* M is in the form:

$$M(t) = kt^{-\alpha}s^\delta\,, \tag{2.99}$$

where κ, α, δ are called *force multiplier parameters*.

The radiative acceleration due to electron scattering is given by

$$g_L(e) = \frac{\kappa_e}{c}\frac{L_*}{4\pi r^2} = \Gamma_e\frac{GM_*}{r^2}\,, \tag{2.100}$$

where Γ_e is the so-called *Eddington factor*:

$$\Gamma_e = \frac{\kappa_e}{4\pi c G} \frac{L_*}{M_*}, \tag{2.101}$$

which is the luminosity divided by the Eddington luminosity. The Compton opacity κ_e is given by

$$\kappa_e = \sigma_e \frac{m_e}{\rho}, \tag{2.102}$$

which is measured in $\mathrm{cm^2 g^{-1}}$. The value $\sigma_e = 6.65 \times 10^{-25}\,\mathrm{cm}^2$ is the Thomson scattering cross section for electrons.

The t in $M(t)$ is defined by

$$t \equiv \kappa_e v_{\text{thermal}} \rho \frac{\mathrm{d}r}{\mathrm{d}v} = \kappa_e \sqrt{\frac{2k_B T}{m_H}} \rho \frac{\mathrm{d}r}{\mathrm{d}v}, \tag{2.103}$$

and is inversely proportional to the velocity gradient.

The s in the definition of $M(t)$ contains information about the degree of ionization: it is

$$s \equiv \frac{10^{-11} \rho}{m_H W}, \tag{2.104}$$

where W is the *geometrical dilution factor*:

$$W(r) = 0.5 \left(1 - \sqrt{1 - \left(\frac{R_*}{r} \right)^2} \right) \sim \left(\frac{R_*}{2r} \right)^2, \tag{2.105}$$

and now we present a proof for the fact that this encodes the radial dependence of the observed intensity. It is a ratio of solid angles:

$$W(r) = \frac{\int_0^\Omega \mathrm{d}\Omega}{4\pi}, \tag{2.106}$$

where the integration limit Ω encodes the solid angle subtended by the star. Assuming simmetry with respect to the azimuthal angle, we can rewrite it as:

$$W(r) = \frac{2\pi}{4\pi} \int_0^{\theta_1} \sin\theta \,\mathrm{d}\theta = \frac{1}{2} \int_1^{\cos(\theta_1)} (-\,\mathrm{d}x) = \frac{1}{2} \left(1 - \sqrt{1 - (R/r)^2} \right), \tag{2.107}$$

so $W(r)$ is the solid angle fraction subtended by a star with radius R at a distance r.

The complete expression for an ensemble of lines is given by

$$g_L = \frac{\kappa_e}{c} \frac{L_*}{4\pi r^2} k t^{-\alpha} s^{\delta}, \tag{2.108}$$

Simulations show that $M(t)$ decreases with t with some kind of power law: the approximation $\log M \sim -\alpha \log t$ is justified.

Numerical simulations show that $\alpha \sim 0.5$, independent of temperature. The δ parameter is almost always of the order $\delta \sim 0.1$.

We also expect a dependence on the metallicity: $M_n(t_n) = M_n(t_n)_\odot (Z/Z_\odot)$.

A typical velocity gradient is something like

$$v(r) = v_\infty \left(1 - \frac{r_0}{r}\right)^\beta , \tag{2.109}$$

with $\beta \sim 0.7$. Then, the profile of g_L can be computed (using the continuity equation): it is in the form

$$g_L \sim r^{-2} \left(\rho \frac{\mathrm{d}r}{\mathrm{d}v}\right)^{-\alpha} \sim r^{2(\alpha-1)} \left(v \frac{\mathrm{d}v}{\mathrm{d}r}\right)^{\alpha} , \tag{2.110}$$

where $\alpha \sim 0.6$. The r dependence is something like

$$g_L \sim r^{-0.8} \left(1 - \frac{r_0}{r}\right)^{0.21} , \tag{2.111}$$

which we can plot.

Tue Nov 19 2019

Yesterday we introduced the CAK formalism.

We consider a spherically symmetric, stationary problem, and we assume that the star looks pointlike to us. Also, we assume that the process is isothermal: $\forall r : T = \text{const}$.

The momentum conservation equation is

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{GM}{r^2} - \frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} + g_c + g_L , \tag{2.112}$$

where we distinguished the continuum acceleration g_c and the line acceleration g_L.

We assume that the gas follows the ideal gas law:

$$P = \frac{R \rho T}{\mu} , \tag{2.113}$$

which implies

$$-\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{RT\rho}{\mu}\right) = -\frac{1}{\rho} \frac{RT}{\mu} \frac{\mathrm{d}\rho}{\mathrm{d}r} , \tag{2.114}$$

where we can recognize the square speed of sound $a^2 = RT/\mu$. Using the continuity equation then we get

$$-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = a^2\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\right). \tag{2.115}$$

For the continuum and line acceleration we use the CAK formalism:

$$g_c = \frac{GM}{r^2}\Gamma_e\,, \tag{2.116}$$

and

$$g_L = \frac{\kappa_e}{c}\frac{L}{4\pi r^2}kt^{-\alpha}s^{\delta}\,, \tag{2.117}$$

so in the end we get:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{GM}{r^2} + a^2\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\right) + \frac{GM}{r^2}\Gamma_c + \frac{\kappa_e}{c}\frac{L}{4\pi r^2}kt^{-\alpha}s^{\delta}\,. \tag{2.118}$$

Now recall

$$t = C'\rho\frac{\mathrm{d}r}{\mathrm{d}v} = C'\left(r^2 v\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{-1}, \tag{2.119}$$

where the constant $C' = \kappa_e\sqrt{2k_B T_{\mathrm{eff}}/m_H}$ encodes all the physical constants.

We plug this into the momentum conservation equation and multiply by r^2:

$$vr^2\frac{\mathrm{d}v}{\mathrm{d}r} = -GM(1-\Gamma_e) + a^2r^2\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\right) + C\left(r^2v\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{\alpha}, \tag{2.120}$$

where

$$C = \frac{\kappa_e}{c}\frac{L}{4\pi}k\left(\kappa_e\sqrt{\frac{2k_BT_{\mathrm{eff}}}{m_H}}\frac{\dot{M}}{4\pi}\right)^{-\alpha}\left(\frac{10^{11}\rho}{m_HW}\right)^{\delta}. \tag{2.121}$$

Also, we can bring a term to the RHS: we get

$$\left(1-\frac{a^2}{v^2}\right)vr^2\frac{\mathrm{d}v}{\mathrm{d}r} = -GM(1-\Gamma_e) + 2a^2r + C\left(r^2v\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{\alpha}. \tag{2.122}$$

Now, we make a simplifying assumption: we neglect the gas pressure. This is reasonable at large distances from the star. It also can be shown that the physical main points we will find are the same as we would find if we did consider the gas pressure. Since the speed of sound is $a^2 = \partial P / \partial \rho$ neglecting the pressure gradient means $a = 0$: removing the terms with a we get

$$r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} - C \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha} = -GM(1 - \Gamma_e) = \text{const}\,, \tag{2.123}$$

and if we denote $r^2 v \, \mathrm{d}v / \mathrm{d}r = D$, the equation can be written as

$$D - CD^{\alpha} = -GM(1 - \Gamma_e)\,. \tag{2.124}$$

Our parameters here are D and $C \propto \dot{M}^{-\alpha}$.

The system is equivalent to $CD^{\alpha} = D + GM(1 - \Gamma_e)$: so the solutions, when plotted with respect to D, are the intersections between a slope-1 straight line and a powerlaw.

There are values of C such that there is no solution.

We select the value of C such that the system has a unique solution — we do this because, inspecting the equation more closely, we find that it is also a critical-point equation, and if we want a monotonic velocity gradient we must have the solution passing through the critical point.

So we differentiate $D - CD^{\alpha} + GM(1 - \Gamma_e)$ with respect to D, set it equal to 0 and find $C = D^{1-\alpha} / \alpha$.

Plugging the solution in we find a differential equation:

$$r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} = D = \frac{\alpha}{1 - \alpha} G M_* (1 - \Gamma_e)\,, \tag{2.125}$$

which can be solved by separation of variables.

The solution is

$$v(r) = \left(\frac{\alpha}{1 - \alpha} 2 G M_* (1 - \Gamma_e) \left(\frac{1}{R_*} - \frac{1}{r} \right) \right)^{1/2} = v_\infty \sqrt{1 - \frac{R_*}{r}}\,. \tag{2.126}$$

this is a β-type law, with $\beta = 0.5$ and

$$v_\infty = \sqrt{\frac{\alpha}{1 - \alpha} 2 G M_* \left(\frac{1 - \Gamma_e}{R_*} \right)} = v_{\text{esc}} \sqrt{\frac{\alpha}{1 - \alpha}}\,, \tag{2.127}$$

since the escape velocity (at the photosphere) is

$$v_{\text{esc}} = \sqrt{\frac{2 G M_* (1 - \Gamma_e)}{R_*}}\,, \tag{2.128}$$

and we include Γ_e in the escape velocity since there is electron scattering there.

This gives us a rather complicated but well-defined expression for the mass loss rate.

The main dependences are $\dot{M} \propto L_*^{1/\alpha} M_*^{(\alpha-1)/\alpha}$.

Copy formula for $\dot{M}$.

For Main Sequence stars $L_* \sim M_*^{2.5}$, so we can turn the luminosity dependence into a mass dependence: $\dot{M} \sim M^{(3.5\alpha-1)/\alpha}$. For $\alpha = 0.52$ we get approximately $\dot{M} \sim M^{1.58}$.

Now, we talk about the corrections due to the finite size of a star. Near the star, we cannot approximate it as a point source. This is in general hard to do, the result is: we get a lower mass loss rate, and a higher terminal velocity. The line acceleration formula picks up a correction factor D_f, which is derived geometrically.

Copy formula for D_f.

The correction factor can be both below and above unity, depending on the distance from the star and on β. As we'd expect in the $r \to \infty$ limit we have $\beta \to 1$.

Typically, the mass loss rate variation due to this correction is similar to multiplying it by $1/2$, while the escape velocity is approximately multiplied by 2.

Phenomena which are not accounted for by this model are:

1. X-ray emission;
2. superionization (beyond the sixth line);
3. discrete absorption components & variability.

In general these winds are unstable: they will feel shocks. These can be simulated: the time-averaged v_∞ and $\dot{M}$ are similar to those found in stationary models.

Mon Nov 25 2019

Now, let us treat the terminal vs escape velocities for O-B stars: we seem to have a linear relation between these, for each group of stars, where we group them by effective temperature.

If we plot $v_\infty / v_{\text{esc}}$ in terms of T_{eff}, we get three horizontal regions. We cannot reproduce all the data with a single α value, instead we need at least three.

We have the maximum mass loss rate when the equation

$$\dot{M} v_\infty = \tau_w \frac{L}{c} \tag{2.129}$$

with $\tau_w = 1$ is satisfied.

We can plot both sides of this equation for O-stars[2] and Wolf-Rayet stars of type WNL. O-stars' observations agree with the model, WNL observations seem not to.

We can make a plot of initial mass vs metallicity and distinghish regions fo the final fates of stars. Metallicity characterizes the amount of heavier-than-Helium elements in the star.

The definition is

$$\text{Metallicity} = \frac{m_{\text{metals}}}{m_{\text{tot}}} = Z\,, \tag{2.130}$$

where m denotes the mass of element of a certain species. $1 = X + Y + Z$ are the fractional masses of hydrogen, helium and metals respectively, so $X = m_{\text{H}}/m_{\text{tot}}$ and so on.

2.6 Dust driven winds

Now, we will speak of dust driven winds: we move to the cool and luminous part of the H-R diagram, which is populated by red giants and so on.

Qualitatively, the driving mechanism is the radiation pressure on dust grains; dust can exist in low temperature regions of the atmosphere, with $T \sim 10^3\,\text{K}$. Dust absorbs momentum and is accelerated outwards: it is very opaque.

The bulk of the outward travelling matter is gas: the dust is a small part, but it is dynamically coupled to the gas (it "drags it along", they share momentum).

A good approximation is a wind with a force $f \sim r^{-2}$. This mechanism critically depends on the dust formation radius: the distance at which the temperature becomes low enough for dust to form.

Dust grains can absorb photons in the continuum, they are then said to be "continuum driven" as opposed to "line driven". Then, this very much diminishes the role of Doppler shift in making it possible for the radiation to pass through much of the gas uninterrupted.

The typical photons in this process are infrared, on the order of $\lambda \sim 1\,\mu\text{m}$.

The dust causes significant reddening. We have low v_∞ and high $\dot{M}$.

Most of the momentum is then transferred in the subsonic region for the high $\dot{M}$, while little energy must be transferred in the supersonic region in order for the exit velocity to be low.

The formation of dust grains must be described with molecular chemistry, and it heavily depends on the ratios of chemical species.

An example: the Helix Nebula.

[2]Hey now, you're an O star, get your game on, go play.

We define the Eddington factor: the ratio of radiative acceleration to gravity. In the case of radiation forces on dust, it is

$$\Gamma_d = \frac{\kappa_{\text{rp}} L_*}{4\pi c G M_*}, \tag{2.131}$$

where the κ_{rp} is called the "radiation pressure mean opacity": it measures the capacity of the dust to absorb photons.

Sometimes, we can have Γ_d greater than unity.

To a good approximation, the transsonic transition corresponds to the point at which Γ_d becomes greater than one.

The plot of Γ_d looks something like $\Gamma_d(r) = \Theta(r - r_{\text{sonic}}) \times 1.4$, where $r_{\text{sonic}} \sim (3 \div 4) r_*$, where Θ is the Heaviside theta.

What are the conditions in order to have a dust driven wind? We first need to form the dust grains: they need to survive against sublimation. Then, the dust must be coupled enough to the gas in order to drive it.

There are two interesting temperatures. The first is the radiative equilibrium temperature T_{rad}, which is the temperature of the grains: it depends on the radiative heating and cooling of the grains.

The second one is the condensations temperature T_{cond}, below it the grain becomes a solid. Above T_{cond} we have sublimation.

$T_{\text{cond}} \sim$ const is not a bad approximation; the radiative equilibrium temperature instead decreases. There is a radius at which they are equal, and for larger radii we have $T_{\text{rad}} < T_{\text{cond}}$. This critical radius is called the dust condensation radius r_d.

Let us suppose that we are at $r > r_d$. The grain then can form, and it can then gain momentum and energy from the Sun's photons, and transfer it to the gas molecules.

We can get a lower limit to the mass loss rate because of the gas-dust coupling condition. Deriving this is complicated, but qualitatively it is since if there is less mass loss rate there is also less gas density, so there is less transfer.

Also, we can have an upper limit on the speed of the dust driven wind. We have the maximal drift speed of a grain through the gas: the drift speed $u_{\text{grain}} - v_{\text{gas}} \propto \rho_{\text{gas}}^{-1/2}$, so there is a radius beyond which the collisions exceed the limiting energy at which they are able to destroy the grains, so there is no further increase in the wind speed, since the dust and gas get progressively more decoupled as the density decreases.

The properties of the grains are very important, but studying the processes in grain formation and growth is hard.

There are both processes of accretion of the gas onto the dust particle and erosion by sputtering, collisions of the gas onto the grain.

There will be a distribution of grain sizes, typically from $0.05 \div 0.1\,\mu\text{m}$: we can sometimes make the small grain approximation.

The composition of the grains depends on the materials in the gas, particularly on the C/O ratio (defined as a ratio of number of atoms).

If $C/O < 1$ typically we form silicate grains, while if it is larger than one we typically form carbonaceous grains.

Wed Nov 27 2019

(Based on few Giorgio notes-BOFGN hereafter) In this lecture the attention was focused on the importance of C/O ratio in the formation of dust grains in dust driven winds. In AGB stars the core, mainly made of heavier elements such as carbon and oxygen, is ourrounded by an helium envelope and, more externally, an hydrogen envelope. Between these areas there exist some convective flows that can bring oxygen and carbon from the core to the more external parts of the star. Reaching thanks to convection these external layers, which lay at a lower temperature, the heavy elements can form moleculary bounds. In particular the strongest bounded molecule that can be formed is carbon monoxyde (CO). This implies that, if the star has a C/O ratio >1 (i.e. it has more abbundance of carbon over oxygen) almost all the oxygen is used to form carbon monoxyde, while the rest of the carbon can form other molecules: for this reason, in these stars we observe carbon grains. Viceversa, in oxygen-rich stars we have with the specular mechanism silicate grains. At this point we started a discussion that will be continued in the following lecture about dust grains opacity, focusing first on cross section. Let a be the radius of the grain, and Q the efficency of the cross section. In this case we have a cross section $c = \pi a^2$ and, considering both absorption and scattering cross sections c_a, c_s, we have

$$c_t ot = c_a + c_s = \pi a^2 (Q_a(a, \lambda) + Q_c(a, \lambda)) \tag{2.132}$$

where $Q_a << Q_c$ for IR wavelengths. Now we can define the opacity

$$k_\lambda = \frac{\int_{a_{min}}^{a_{max}} Q_a \pi a^2 n da}{\rho} \tag{2.133}$$

and the mean opacity

$$k = \int d\lambda k_\lambda \tag{2.134}$$

which is the relevant quantity for the momentum equation

Mon Dec 02 2019

Coming back to concepts from last time: the cross section is given by the product of the geometrical cross section times an adimensional efficiency; we define it both for absorption and scattering.

$$C^{\mathrm{A,\,S}}(a,\lambda) = \pi a^2 Q^{\mathrm{A,\,S}}(a,\lambda)\,, \tag{2.135}$$

while for the total cross section we have

$$C^{\mathrm{TOT}}(a,\lambda) = \pi a^2 \Big(Q^{\mathrm{A}}(a,\lambda) + Q^{\mathrm{S}}(a,\lambda) \Big)\,. \tag{2.136}$$

We are speaking about dust grains. If $\lambda \gg a$, then the following holds:

$$Q^A \propto a\lambda^{-p}\,, \tag{2.137}$$

where p depends on the grain composition, and approaches 2 for increasing λ for any composition.

At infrared wavelengths, we have $Q^S \sim \lambda^{-4}$, so $Q^S \ll Q^A$ there.

The values of p are not the result of any theoretical model, they are observed.

Today we will speak more about mean opacities.

We define

$$Q^A_p(a,T) = \frac{\int_0^\infty Q^A(a,\lambda) B_\lambda(T)\,\mathrm{d}\lambda}{\int_0^\infty B_\lambda(T)\,\mathrm{d}\lambda}\,, \tag{2.138}$$

where we use the Planck function $B_\lambda(T)$; with it we define

$$\kappa_p(T) = \int_{a_{\min}}^{a_{\max}} Q^A_p(a,T)\pi a^2 n(a)\,\mathrm{d}a\,, \tag{2.139}$$

which is called the Planck mean absorption coefficient. This quantity will enter into the balance between cooling and heating of the single grain. We are effectively doing a mean over energy, since

$$B_\lambda\,\mathrm{d}\lambda = \mathrm{d}E \tag{2.140}$$

for a blackbody.

For hydrodynamical calculations instead we use the radiation pressure mean efficiency:

$$Q_{\mathrm{rp}}(a) = \frac{\int_0^\infty \Big(Q^A(a,\lambda) + (1-g_\lambda) Q^S(a,\lambda) \Big) F_\lambda\,\mathrm{d}\lambda}{\int_0^\infty F_\lambda\,\mathrm{d}\lambda}\,. \tag{2.141}$$

Here we are accounting for both the absorption efficiency and the scattering efficiency. Similarly, we can define the radiation pressure mean opacity κ_{rp}:

$$\kappa_{\mathrm{rp}} = \frac{\int_0^\infty (\kappa_\lambda + (1-g_\lambda)\sigma_\lambda) F_\lambda\,\mathrm{d}\lambda}{\int_0^\infty F_\lambda\,\mathrm{d}\lambda}\,, \tag{2.142}$$

where

$$F_\lambda = \frac{L_\lambda}{4\pi r^2} \tag{2.143}$$

is the monochromatic flux.

Also, g_λ is the *mean cosine* of the scattering angle: if it is equal to 1 we have forward scattering, if it is equal to -1 we have backward scattering, while if it is equal to 0 we have isotropic scattering.

We are shown a plot of g_λ in terms of the wavelength: it decreases from 0.8 to 0 as λ goes from $0.1\,\mu\text{m}$ to $10\,\mu\text{m}$.

Do note that the F_λ is not generally known *a priori*: it is a solution to the radiative transfer equation.

The temperaure of the grain is determined by the balance of the heating and cooling rates: heating occurs because of collisions with fast-moving gas particles or because of the absorption of radiation.

Cooling, on the other hand, occurs because of collisional energy transfer or by emission of thermal radiation.

Are we modelling the dust grain as a black body?

We make an approximation: we only consider radiative processes, and estimate the temperature of the dust grain with the radiative equilibrium temperature.

The equilibrium equation is

$$\int_0^\infty \kappa_\lambda B_\lambda(T_d)\,\mathrm{d}\lambda = \int_0^\infty \kappa_\lambda y_\lambda\,\mathrm{d}\lambda\ , \tag{2.144}$$

where

$$y_\lambda = \frac{1}{4\pi}\int_0^{4\pi} I_\lambda\,\mathrm{d}\Omega = W(r)B(T_{\text{eff}})\,, \tag{2.145}$$

where $W(r)$ is the geometrical dilution factor:

$$W(r) = \frac{1}{2}\left(1 - \sqrt{1 - \left(\frac{R}{r}\right)^2}\right) \sim \left(\frac{R}{2r}\right)^2 \qquad \text{as } r \gg R\,, \tag{2.146}$$

where R is the radius of the star, r is our considered radial position.

This allows us to fix r and compute $T_d(r)$.

The LHS of the balance equation corresponds to the radiative cooling, while the RHS corresponds to the radiative heating.

In both terms we make the dependence on Q_p^A explicit:

$$\int_0^\infty \pi a^2 Q_p^A(a, T_d)B_\lambda(T_d)\,\mathrm{d}\lambda = \int_0^\infty \pi a^2 Q_p^A(a, T_{\text{eff}})B_\lambda(T_{\text{eff}})W(r)\,\mathrm{d}\lambda\ . \tag{2.147}$$

We are assuming that diffusion of heat between gas grains is negligible: we simplify some terms.

$$Q_p^A(a, T_d) \int_0^\infty B_\lambda(T_d)\, \mathrm{d}\lambda = W(r) Q_p^A(a, T_{\text{eff}}) \int_0^\infty B_\lambda(T_{\text{eff}})\, \mathrm{d}\lambda \,, \tag{2.148}$$

and we know that the integral of the Planck function is given by $\sigma_{\text{SB}} T^4$, where σ is the Stefan-Boltzmann constant. In the end then our expression is

$$Q_p^A(a, T_d) T_d^4 = T_{\text{eff}}^4 Q^A(a, T_{\text{eff}}) W(r) \,, \tag{2.149}$$

so

$$T_d(r) \sim T_{\text{eff}} \left(\frac{R}{2r} \right)^{1/2} \left(\frac{Q_p^A(a, T_{\text{eff}})}{Q_p^A(a, T_d)} \right)^{1/4} . \tag{2.150}$$

An immediate application is to find the condensation radius: what is the radius at which the temperature becomes low enough so that the grains are not broken up by thermal motion?

Typically the condensation temperature is $T_c \sim 1 \div 1.5 \times 10^3\,\text{K}$.

The equation is

$$r_c \sim \frac{R}{2} \left(\frac{T_{\text{eff}}}{T_c} \right)^2 \sqrt{\frac{Q_p^A(a, T_{\text{eff}})}{Q_p^A(a, T_c)}} \,, \tag{2.151}$$

and this allows us to see that typically the condensation radius is around $2 \div 4$ star radii, for stars at a few thousand Kelvin.

Tomorrow we will speak of the combined dust & wind equation.

Tue Dec 03 2019

(BOFGN) Following the lecture of the day before, now we consider the flow of a mixture of gas and dust. To do this, we have to implement some assumptions:

1. a fixed radius a of dust grains
2. a radial profile of flow velocity $u(r)$
3. the existence of a drag force $f_{drag}(r)$

In this case the equation for the momentum can be written as

$$u \frac{\partial u}{\partial r} = -\frac{GM}{r^2} + Q_{RP} \pi a^2 \frac{L}{c m_d 4\pi r^2} - \frac{f_{drag}}{m_d} \tag{2.152}$$

Now, considering the fact that the thermal speed of sound is defined by

$$\frac{ma_{th}^2}{2} = K_b T \tag{2.153}$$

we want to study flows for which the drift velocity $w_{drift} = u - v$ is much higher than this a_{th}. In this case the drag force, postulated to be

$$f_{drag} = \pi a^2 \rho w \sqrt{w_{drift}^2 + a_{th}^2} \tag{2.154}$$

reduces to $f_{drag} = \pi a^2 \rho w_{drift}^2$. Imposing now that at infinity the velocity gradient is null, we have

$$0 = Q_{RP} \pi a^2 \frac{L}{c m_d 4\pi r^2} - \frac{f_{drag}}{m_d} \tag{2.155}$$

that leads to an expression for the drift velocity

$$w_{drift} = \sqrt{Q_{RP} \frac{L}{4\pi r^2 \rho c}} \tag{2.156}$$

and remembering that ρ scales as $\dot{M}$ in the mass loss equation,

$$w_{drift} = \sqrt{Q_{RP} \frac{Lv}{\dot{M} c}} \tag{2.157}$$

We know from momentum equation

$$v\frac{\partial v}{\partial r} = -\frac{1}{P}\frac{dP}{dr} - \frac{GM}{r^2} + n_d \frac{f_{drag}}{\rho} = -\frac{1}{P}\frac{dP}{dr} - \frac{GM}{r^2} + n_d \frac{Q_{RP}\pi a^2 v}{\rho 4\pi c} = -\frac{1}{P}\frac{dP}{dr} - \frac{GM}{r^2}(1-\Gamma_d) \tag{2.158}$$

where we defined the corrective factor, Γ_d that encodes the pull against gravity driven by the drag force. The physical interpretation of this solution is that radiation transfers momentum to dust, that can transfer it to the gas, reaching a terminal velocity. We know from previous lectures that mass loss rate is lower for static atmosphere, because the scale factor H_r is very small in this cases, implying an higher density than the hydrostatic case. From mass loss equation we have

$$\dot{M} = 4\pi r^2 \rho v \tag{2.159a}$$

$$\dot{M}_d = 4\pi r^2 n_d m_d u_{dust} \tag{2.159b}$$

This implies a lower limit on the luminosity L in the case $\Gamma_d > 1$

Chapter 3

Massive and very massive stars

Mon Dec 09 2019

(BOFGN) Massive stars play a key role in several astrophysical and cosmological phaenomena. In particular, after their collapse into the supernova, the residuals heavy metals and dusts contribute drastically to new stars formation. Massive stars also are the most important candidate for the explanation of the existence of black holes, neutron stars, atomic isotopes and gamma ray bursts: that is the reason why we will focus on these stars in our lecture. Before doing it for massive stars, we saw a graph picturing the evolution of stars for different mass values: as a rough scheme we have massive stars that become supernovas and then black holes-neutron stars, while low-mass stars slowly turn to red dwarfs. The most important equations for stellar evolution are the usual ones:

- mass equation
- momentum equation
- energy conservation equation
- energy transport equation
- chemestry equation

while the most important variables involved are:

- pressure $P(\rho, T, X)$ in the equation of state
- opacity $k(\rho, T, X)$
- energy production $\epsilon(\rho, T, X)$

Hereafter we consider very massive stars (M>80 M-sun), laying in the main sequence of H-R diagram. In the first time of their life, these stars burn hydrogen in the CNO cycle. When H finishes in the core, it starts the burning of helium in the triple alpha reaction, producing carbon. Note that in this reaction thermal (i.e. weak-interactions origined) neutrinos are produced. Then it starts the synthesis of oxygen from carbon, neon from oxygen, magnesium from neon, silicium and iron from magnesium che al mercato mio padre comprò. These elements lay in the star having the heaviest in the core and the others in order of mass. We said before that pressure is one among the most important variables we shall consider. Its dependence on some other thermodynamical variables is strongly affected by the electron characterization:

$$P_R = \frac{aT^4}{3} \qquad \text{in the relativistic case} \tag{3.1a}$$

$$P_{class} = \frac{R\rho T}{\mu} \qquad \text{in the perfect-gas case} \tag{3.1b}$$

$$P_{deg,NR} = \rho^{\frac{5}{3}} \qquad \text{for a degenerate gas of non relativistic electrons} \tag{3.1c}$$

$$P_{deg,R} = \rho^{\frac{4}{3}} \qquad \text{for a degenerate gas of relativistic electrons} \tag{3.1d}$$

Now we consider neutrino producion in this phase of the star, since it has been shown that $\epsilon_{nuc} \approx \epsilon_\nu$, i.e. almost all the power due to nuclear reactions is brought away by neutrinos. For strong-interaction born neutrinos we have that the free path is

$$l_\nu = \frac{1}{\sigma_\nu n} = \frac{\mu m_\mu}{\rho \sigma_\mu} \approx 3000R - sun \tag{3.2}$$

which means that neutrinos in average do not interact with the star and can not give back energy to it. This explains the strong neutrinos luminosity (higher than electromagnetic) of almost all the stars, and the consequent mass loss rates. For weak-interaction (/thermal) neutrinos, the main mechanism that can product them are the following:

- photo-neutrinos production ($\gamma + e^- \rightarrow \nu + \bar{\nu} + e^-$)
- pair-neutrinos production ($e^+ + e^- \rightarrow \nu + \bar{\nu}$)
- plasma-neutrinos production ($\gamma \rightarrow \nu + \bar{\nu}$)
- Bremmstrahlung-neutrinos production (similar, $\gamma \rightarrow \nu + \bar{\nu}$)

In general these processes are very rare in nature, since

$$\frac{P(\nu\bar{\nu}}{\gamma} = 3x10^{-18}\left(\frac{E_\nu}{m_e c^2}\right)^4 \tag{3.3}$$

for this reason neutrinos are important for the core-cooling process and the speed-up of core reactions.

Tue Dec 10 2019

We continue with the evolution of a $15 M_\odot$ star. We use Kippenhahn diagrams. There are four major burning phases:

1. Hydrogen;
2. Helium;
3. Carbon;
4. Silicon.

Check

There can be different burning stages simultaneously, in different shells.

Stellar winds are of general application in the study of stellar evolution.

We can make an H-R diagram using, instead of the luminosity L, the bolometric magnitude

$$M_{\text{bol}} = -2.5 \log L + 4.73 \,, \tag{3.4}$$

where the luminosity is measured in solar luminosities.

The Eddington luminosity is calculated by setting

$$\Gamma_{\text{Edd}} = \frac{a_{\text{rad}}}{g} = \frac{\kappa L}{4\pi R^2 c} \frac{R^2}{GM} = 1 \,, \tag{3.5}$$

which implies

$$L_{\text{Edd}} = \frac{cGM4\pi}{\kappa_e} \,. \tag{3.6}$$

Wolf-Rayet stars have high effective temperatures and high luminosities, they expel a great quantity off mass through stellar winds.

The spectra of WR stars have little of no H, and have an abundance of either He + N or C + O.

We distinguigh them into

1. WNL;
2. WNE;
3. WC;
4. WO.

These are actually evolutionary stages: the outer layers are successively stripped by winds, and the inner ones are exposed. We observe lots of ^{14}N in WNE stars, since the Nitrogen burning phase since that is the slowest process.

We have these stars when $\log T_{\text{eff}} > 4$, and $M > 30 M_{\odot}$.

Depending on the mass of the progenitor red supergiant, we have different behaviours: below $30 M_{\odot}$ the star explodes as a red supergiant; more massive stars will experience more mass loss and explode as blue supergiants, further left on the HR diagram.

We see a simulation, a $15 M_{\odot}$ star starts off on the high part of the Main Sequence, then quickly moves right becoming a red supergiant when its core collapses.

A $3 M_{\odot}$ star instead moves away from the MS more slowly, and then after having moved right it quickly moves far left and cools, as a white dwarf.

Now, let us discuss the engine of the explosion of these massive stars. The explosion is triggered by an implosion: we are at the end of the Silicon-burning phase. The pressure of the core is mantained by the degenerate electrons.

The degenerate iron core starts off with $\rho \approx 10^9\,\text{gcm}^{-3}$, $T \approx 10^{10}\,\text{K}$, $M_{\text{Fe}} \approx 1.5 M_{\odot}$.

What triggers the collapse of the iron core? The adiabatic exponent

$$\gamma_{\text{ad}} = \left(\frac{\partial \log P}{\partial \log \rho} \right)_{\text{ad}} \tag{3.7}$$

falls below the critical value of $4/3$. In general, if it falls below this value we expect a dynamical instability. It is a local quantity, it can be defined for each layer; we can use a global parameter to measure the instability:

$$\int \left(\gamma_{\text{ad}} - \frac{4}{3} \right) \frac{P}{\rho} \, \mathrm{d}m \,, \tag{3.8}$$

which will tell us whether the system as a whole is stable.

Why the factor P/ρ in the average?

There is a very important weak process: electron capture, or *neutronization*: $\text{p}^+ + \text{e}^- \rightarrow \text{n} + \nu_e$. This decreases the electron pressure!

The Chandrasekar mass scales as the square of the mean molecular weight:

$$M_{\text{ch}} = \frac{5.83}{\mu_e^2} \sim 1.26 M_{\odot} \tag{3.9}$$

for the iron core. A huge amount of neutrinos are produced. Now we follow a work by Janka et al (2007) at the Max Planck, which outlines the steps of the collapse.

As the density of the core increases, it becomes opaque to neutrinos: we have the scattering cross section

$$\sigma_\nu \approx 10^{-49} A^2 \left(\frac{\rho}{\mu_e} \right)^{2/3} \text{cm}^2 \,, \tag{3.10}$$

and in the meantime the collapse is proceding, until it reaches values of $\rho \sim 10^{14}\,\text{gcm}^{-3}$, comparable to the nuclear density, at which point the material becomes incompressible.

A proto-neutron star is formed, and a shock front travels back.

What triggers the initial infall and compression in the first place?

It is unclear whether the energy of the shock is enough to blow off the envelope. The energy of the delayed neutrinos which are still trapped in the nucleus will contribute…

Now we start an interesting new part: the advanced burning stages, after the He-burning phases.

As the initial mass increases, the fraction of carbon produced decreases from around 0.35 to 0.15 as M goes from 10 to 120 solar masses.

What about the density structure in the pre-supernova phase? The starting mass of the iron core increases with the stellar mass: as it goes from 10 to 120 solar masses it goes from 1.2 to 1.8 solar masses.

As the total mass increases, at a fixed radius we have more internal mass: the density *increases* with mass.

Some of the material near the core, when the shock occcurs, falls back on it if it is inside a certain critical radius, if it is outside that radius it is thrown out. How much of it depends on the physical properties of the core.

What does the remnant look like? It can either be a black hole or a neutron star. The possibilities

1. Explosion and Neutron Star;
2. Implosion and Black Hole;
3. (rarely) Explosion and Black Hole.

There is no clear-cut law. One thing to look at is the bounce-compactness parameter:

$$\xi_{M^*} = \left| \frac{M^*/M_\odot}{R(M^*)/10^7\,\text{m}} \right|_{t=t_{\text{bounce}}} \,, \tag{3.11}$$

and it seems like this paramter being high is correlated to a black hole being formed. We have also the parameter

$$\mu_4 = \left| \frac{\mathrm{d}m \,/ M_\odot}{\mathrm{d}\tau \,/ 10^7\,\mathrm{m}} \right|_{s=4}, \tag{3.12}$$

the normalized mass inside a dimensionless entripy per nucleon of $s = 4$. We can make a plot of the various final fates of the parts of the initial mass: some of it is expelled by winds, some of it is expelled through the supernova explosion, some of it ends up in the final remnant.

The mass of the iron core is never very much larger than a couple of solar masses, but BHs can become more massive through fallback.

Then it seems that WR stars of more than $30M_\odot$ will not explode: they will not eject material as supernovas.

Simulations show that there is not a clear-cut boundary between neutron star and BH formation, but then we go on as if there were…

The behaviour of LA89 massive stars is quite different, NS are more favoured towards higher masses.

Mon Dec 16 2019

In a binary system things are more complicated.

It seems that stars more massive than $35M_\odot$ have large fallback: they do not contribute to the heavy element population.

There are still many uncertain areas: convection is not well understood, especially when it is heavily coupled to nuclear burning.

Rotation can cause convective overshooting (what is it?).

We do not have a self-consistent hydrodinamical model which includes neutrino transport.

We can plot the mass of the various mass components of the star with respect to $\log \left(t_{\mathrm{collapse}} - t \right)$.

We can also plot the mass fractions of $^{12}\mathrm{C}$ and $^{16}\mathrm{O}$ with respect to the fraction of $X(^4\mathrm{He})$, which decreases with time. At some point the helium burning phase stops, and then we are left with a carbon/oxygen fraction which can be either higher or lower than one.

Depending on whether we are using a LA89 or an LN00 model we can have different behaviours.

The final kinetic energy after the fallback is a "foe": 10^{51} erg.

If we have a higher mass-loss rate then we will have a lower mass of the CO core, which will then be less compact. The fallback will be less efficient.

The state of the art is that at higher initial masses BHs are more favoured. It seems that BHs are more often produced by WR stars.

According to a certain model, BHs are mainly produced by *failed* supernovae: there is a discontinuity around $M \sim 35M_\odot$, above which we get a collapse and a failed supernova. As the metallicity increases, the final masses of the failed supernovae (so, the resulting BHs) decrease.

Now, we discuss the evolution and final fate of very massive stars. We consider stars with $100M_\odot < M_{\text{in}} < 5 \times 10^4 M_\odot$. For these, we do not have a core collapse, but a thermonuclear explosion instead.

There is some evidence for the existence of these stars, for example in the cluster R136a1 and R136a2.

These stars seem to be very young ($\sim 2\,\text{Myr}$).

We have Super Luminous supernovae (SLSNe) which can have luminosities brighter by a factor of 10 than the regular supernovae.

We will need to understand under which metallicity regime we can have these SLSNe: there seems to be a metallicity threshold.

These are "pair creation supernovae" because the radiation they emit is more energetic than $1\,\text{MeV}$?

These might be the first contributors to the presence of metal in the universe, the Black Sabbath of the cosmos.

Pair Creation SNe and Pair Instability SNe are the same thing.

The relevant phase in stellar evolution is when a large amount of thermal energy goes into pair production instead of producing pressure.

An important parameter is the mass of the helium core.

We'd expect more and more massive stars to lose more and more mass loss through winds. We need a Helium core with mass such that $40M_\odot < M_{\text{He}} < 133M_\odot$. If $M_{\text{He}} < 65M_\odot$ then we have "Pulsation pair instability supernovae", above this we have "Pair creation supernovae".

The pair creation triggers the collapse. Then we have a runaway thermonuclear reaction. If this exceeds the binding energy of the star, then we have complete disintegration.

The pair production makes $\gamma_{\text{ad}} < 4/3$: this causes a gravitational collapse.

For $M_{\text{He}} > 133M_\odot$ the infall produces a BH.

This occurs before oxygen burning.

We have the relation $1\,\text{MeV} \sim 10^{10}\,\text{K}$.

The star is able to eject most of the material in a series of pulses even if the nuclear energy is smaller than the binding energy.

The pulses are separated by time intervals on the order of $10^5\,\text{s}$ to $10^9\,\text{s}$.

Tue Dec 17 2019

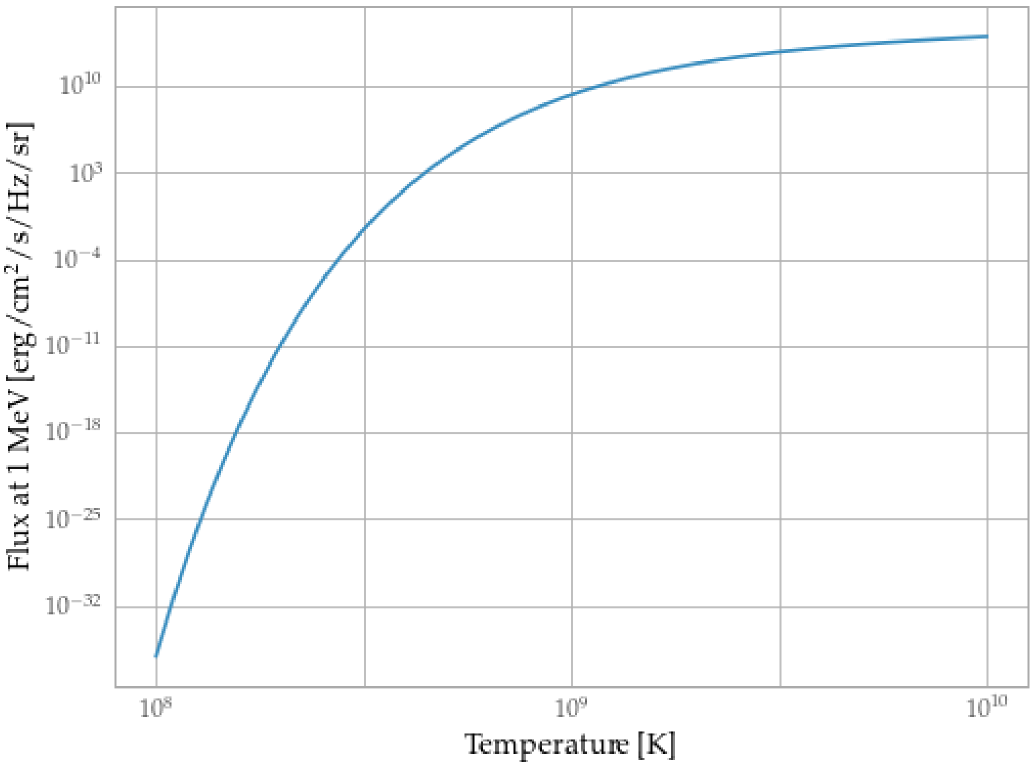

Figure 3.1: Flux at 1 MeV for a blackbody with varying temperature T.

The number of pulsation instabilties decreases as the He core mass increases. After $60M_\odot$ only one pulsation is needed in order to blow off the whole core.

People generally focus on low metallicity, very massive stars. For example, they are found in the Magellanic clouds.

Many factors affect the growth of the oxygen core:

1. ^{12}C$(\alpha, \gamma)^{16}$O rate;
2. convection reduces the minimum mass for a PCSN;
3. rotation-induced chemical mixing: this reduces the minimum stellar mass needed in order to have a PCSN;
4. rotation decreases the binding energy: it increases the maximum mass needed in order to have a PCSN.

Plots of the nuclear, binding and kinetic energies are useful to understand the process.

The nuclear front crosses the various layers of the star. If we increase the initial mass of the star, the amount of Nickel produced increases a lot.

We can plot the production factor of various nuclides relative to the solar rate of production: as the isotope number Z varies: we plot $\log(\text{prod}/\text{solar prod})$. We see a zig-zag pattern: a large difference between even and odd nuclides. Even ones are produced more.

Nickel is important because of the decay chain Ni $\to$ Co $\to$ Fe, which could explain the shape of the decaying light curve of the supernova.

Referring to PCSNe: how much to they contribute to the chemical composition of galaxies? We can plot the metal yield multiplied by the probability to have a star of that mass (which is roughly $\varphi(M) \sim M^{-2.5}$) as the mass of the star varies. We include Core Collapse SNe (with $M < 50M_\odot$), then there is a BH region, and then we have PISNe from $120M_\odot < M < 260M_\odot$, then BHs again.

Partial test on the 14th of January.