# Advanced astrophysics notes

Jacopo Tissino, Giorgio Mentasti

October 8, 2019

## 30 september 2019

## 1 Introduction

Teachers: Paola Marigo, Michele Trabucchi.
paola.marigo@unipd.it and michele.trabucchi@unipd.it

### 1.1 Topics

They are selected topics in stellar physics.

1. Stellar pulsations and Astroseismiology (dr. Michele Trabucchi);
2. stellar winds (dr. Paola Marigo);
3. final fates of massive & very massive stars (dr. Paola Marigo).

Basics in Stellar Physics: "astrophysics II" inside the bachelor's degree in astronomy (second semester). It can be taken as an optional course.

Material:

1. *Introduction to stellar winds* by Lamers, Cassinelli.
2. *Stellar Atmospheres: Theory and observations* (lecture notes from 1996).

and more on Paola Marigo's site.

### 1.1.1 Stellar Winds

Moving flows of materials ejected by stars. 20 to $2 \times 10^3\,\mathrm{km/s}$.

See *Bubble Nebula* in Cassiopea, there is a $45 M_\odot$ star ejecting stellar wind at $1700\,\mathrm{km/s}$.

Diagram: luminosity vs effective temperature. We see the *main sequence*. We can also plot the *mass loss rate*, $\dot{M} > 0$ in solar masses/year. An other important parameter is $v_\infty$.

Diagram: mass loss (or gain) rate vs age of star.

Stellar winds affect stellar evolution, the dynamics of the interstellar medium, the chemical evolution of galaxies.

Momentum is approximately injected with $\dot{M}v$, kinetic energy with $1/2 \dot{M} v^2$. Within $1 \times 10^8\,\mathrm{yr}$ around half of the infalling matter is reemitted.

### 1.1.2 Contents

We will start with the basic theory of stellar winds, and then: *coronal*, *line-driven* and *dust-driven* winds.

## 1.2 Final fates of massive & very massive stars

Masses over $10 M_\odot$.

## 1.3 Stellar oscillations

... see slides.

Material: slides on moodle or Marigo's page.

1. *Pulsating stars* by Catelan & Smith (introductory);
2. *Theory of stellar pulsation* by Cox (harder).

Written exam, partial exam on stellar pulsation.

# 2 Variability in Astronomy

First observations of variable stars: $\sim 1600$, omicron-Ceti. It changes in magnitude by 6 orders of magnitude.

Others are found from the 1600 onwards, but since the XX century the reason is unknown. Is it *rotation*, *eclipses*?

For some it sure are eclipses, but the Cepheids are different. See $\delta$-Cephei, asymmetric continuous curve. What if stars *pulsate*?

The *light curve* is the luminosity curve over time.

We can also look at the *phased* light curve. Of course we need the period: the phase is

$$\varphi = \frac{(t - t_0) \bmod \Pi}{\Pi} \tag{1}$$

where $\Pi$ is the period. $E(t) = \lfloor (t - t_0)/\Pi \rfloor$ is the epoch.

We can then measure the period, but if the light curve is multiperiodic we can subtract the model from the curve to see if there are additional periods: this is *prewhitening*.

We can also look at the luminosity in Fourier space.

Of course there are issues with observational gaps (day-night, full moon): aliases; accuracy, duration of observations...

Also, the period can change in time.

Things have improved a great deal with large-scale surveys and space suveys.

## 2.1 Classification

By variability type: regular, semi-regular or irregular.

By intrinsic variability: extrinsic, external to the star: eclipses, transits, microlensing, rotation; intrinsic: rotation, eclipses (self-occultation), eruptive and explosive variables, oscillations, secular variations (?).

Whether rotation is to be considered intrinsic or extrinsic is a matter of taste.

Oscillations can be classified by several criteria.

The geometry can be radial (classical pulsators) or non-radial.

The restoring force can be the pressure gradient or the gravitational force (bouyancy, not gravitational waves).

The excitation mechanisms can be different.

The evolutionary phase and mass of the star can also be different.

# 3 Summary of stellar structure & evolution

Eulerian: properties of a gas are fields, the position is the position of an observer. To differentiate position with respect to time is meaningless: position is an independent variable. $f = f(r^i, t)$.

Lagrangian: we follow an element of fluid: $\mathrm{d}r^i \big/ \mathrm{d}t = v^i$. We can identify univocally these fluid elements.

When treating stellar structure & evolution, we look at mass layers $\mathrm{d}m$. $f = f(m, t)$. Do note that $m$ is the mass of the whole full sphere under a certain layer, not the mass of the shell.

In the lagrangian case, we use the convective derivative $\mathrm{d}/\mathrm{d}t = \partial_t + v^i \partial_i$ where $v^i$ is the velocity defined before.

## 3.1 Equations of stellar structure

We write these in the spherically symmetric case.

The continuity equation is:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \,. \tag{2}$$

Momentum conservation is given by:

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \,, \tag{3}$$

where

Energy conservation is given by:

$$\frac{\mathrm{d}L}{\mathrm{d}m} = \varepsilon - \varepsilon_\nu - \varepsilon_g \,, \tag{4}$$

where $L$ is the luminosity, $\varepsilon$ is the rate of nuclear energy generation per unit mass, while $\varepsilon_\nu$ is the rate of energy loss due to neutrino emission per unit mass, and $\varepsilon_g$ is the work done by the gas per unit mass & time.

The energy transfer equation is:

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P} \nabla \,, \tag{5}$$

where $\nabla = \partial \log T / \partial \log P$ is the temperature gradient, which has contributions from radiation, conduction, convection...

With the diffusion approximation, we can write the gradient as

$$\nabla = \nabla_{\mathrm{rad}} = \frac{3}{16\pi acG} \frac{\kappa_R LP}{mT^4} \,, \tag{6}$$

where $a$ is a constant depending on the Stefan-Boltzmann constant and the speed of light.

where $\kappa_R$ is the Rosseland mean opacity, given by

$$\frac{1}{\kappa_R} = \frac{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T} \frac{1}{\kappa_\nu} \,\mathrm{d}\nu}{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T} \,\mathrm{d}\nu} \tag{7}$$

Substituting in the result in (6) we get:

$$L = -\frac{64\pi^2 ac}{3} r^4 \frac{T^3}{\kappa} \frac{\partial T}{\partial m} \tag{8}$$

where $\kappa$ is a generalized opacity, the harmonic mean of the Rosseland opacity $\kappa_R$ and the convective opacity $\kappa_c = 4acT^3/(3\rho\lambda_c)$.

Where $\lambda_c$ is...?

If we need to deal with convection, this defies any simple modeling. There are instability criteria: where is it relevant? This is given by Ledoux's criterion,

$$\nabla_{\text{rad}} > \nabla_{\text{ad}} - \frac{\chi_\mu}{\chi_T} \nabla_\mu \,, \tag{9}$$

where:

$$\nabla_\mu = \frac{\mathrm{d}\log \mu}{\mathrm{d}\log P} \tag{10a}$$

$$\nabla_{\text{ad}} = \left( \frac{\partial \log T}{\partial \log P} \right)_{\text{ad}} \tag{10b}$$

$$\chi_\mu = \left( \frac{\partial \log P}{\partial \log \mu} \right)_{\rho,T} \tag{10c}$$

$$\chi_T = \left( \frac{\partial \log P}{\partial \log T} \right)_{\rho,\mu} \tag{10d}$$

which are thermodynamic parameters.

how are these called? What do they mean?

In the convective core, $\nabla \approx \nabla_{\text{ad}}$, but outside of it we need something else.

Mixed-length theory model convection with "bubbles":

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Figure 22.8 in some PDF: run of adiabatic, radiation gradients vs $\log T$.

We compute $\nabla_{\text{ad}}$ and $\nabla_{\text{rad}}$ and see whether the region is convective or radiative.

We can move from the Eulerian and Lagrangian formalisms using the continuity equation. In the Eulerian formalism:

$$m(r) = \int_0^\infty 4\pi r^2 \rho(x)\, \mathrm{d}x \,. \tag{11}$$

We need several *constitutive equations* for the parameters $\rho$, $c_P$ (heat capacity of stellar matter), the opacity $\kappa$, the nuclear transformation rate $r_{ij}$ and the rate of generation of nuclear energy $\varepsilon$. These can all be considered as functions of $P$, $T$, $\mu$.

We define:

$$\mu^{-1} = \sum_i (1 + \nu_i) \frac{X_i}{A_i} \tag{12}$$

(CHECK)

The variables $X$, $Y$ and $Z$ represent the abundances of H, He and metals, and satisfy $X + Y + Z = 1$.

We may need to know the metal mixture inside $Z$, but often we can approximate it as the Sun's distribution.

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \sum_j \left( r_{ji} - r_{ij} \right) \tag{13}$$

## 3.2 Classification of stars

Low mass stars have between 0.8 and 2 solar masses. Intermediate mass stars have masses between 2 and 8 $M_\odot$. Massive stars have masses of over $8M_\odot$. (Add characteristics of these).

### 3.2.1 Low-mass star evolution

See slides for figures. What are Hayashi lines?

### 3.2.2 Intermediate-mass star evolution

Idem

# 4 Stellar oscillations

Today we will look at typical time-scales, the period-mean density relation, the energy equation and perturbation theory w/ linearization.

No derivations of the equations in the exam.

## 4.1 Time-scales

The *free fall* time scale is:

$$\tau_{\text{dyn}} \sim \left( \frac{R}{g} \right)^{1/2} = \left( \frac{R^3}{GM} \right)^{1/2}, \tag{14}$$

this is associated with pulsation. It is calculated using the travel time of a mass in free fall across the stellar radius accelerated by constant acceleration equal to surface acceleration.

We note that $\tau_{\text{dyn}} \propto \overline{\rho}^{-1/2}$. For the Sun it is about 25 min.

The *thermal* time scale is the relaxation time of deviations from thermal equilibrium:

$$\tau_{\mathrm{th}} \sim E_{\mathrm{th}}/L\,. \tag{15}$$

It is calculated as the time required for a star to irradiate all its energy.

*Proof.* Roughly, $3\int_V P\,\mathrm{d}V = -\Omega$ with $\Omega = -\int_M \cdots$ $\square$

Typically, $\tau_{\mathrm{th}} \sim GM^2/(LR) \sim 10^7$ ... It is much larger than the dynamic time scale.

The *nuclear* time scale is even longer.

This allows us to say that oscillations will not be heavily affected by thermal conduction, and even less by nuclear processes: the pulsations will be almost *adiabatic*.

The best candidate for these oscillations are *sound waves*: is the adiabatic speed of sound roughly right?

The speed of sound is:

$$v_s^2 = \Gamma_1 \frac{P}{\rho} \tag{16}$$

If the gas follows the perfect equation of state

$$\frac{P}{\rho} = \frac{k_B}{m_H}\frac{T}{\mu}\,, \tag{17}$$

we get

$$v_s^2 = \frac{\Gamma_1 k_B T}{m_H \mu}\,. \tag{18}$$

Typical values are $\mu \sim (2X + 3Y/4 + Z/2)^{-2} \sim 0.6$, $\Gamma_1 = 5/3$, and $T_{\mathrm{He}} \sim 4.5 \times 10^4\,\mathrm{K}$.

So we get $v_s \sim 32.2\,\mathrm{km/s}$.

The timescale is $\Pi \sim 2R/v_s \sim 22\,\mathrm{d}$, while $\Pi_{\mathrm{obs}} = 5.336\,\mathrm{d}$, in terms of orders of magnitude it works.

We can use the equation for the sound speed in the virial theorem, and get:

$$\Omega = -3\frac{\int_M v_s^2/\Gamma_1\,\mathrm{d}m}{\int_M \mathrm{d}m} M = -3\left\langle \frac{v_s^2}{\Gamma_1} \right\rangle M \tag{19}$$

If $\Gamma_1$ and $v_s$ are independent, we can compute their averages separately.

This allows us to write $\Pi$ wrt the moment of inertia, $I = \int_r r^2\,\mathrm{d}m\,(r)$:

$$\Pi \sim \left(\frac{I_{\mathrm{osc}}}{-\Omega}\right)^{1/2} \tag{20}$$

This is further evidence that we are dealing with a dynamical phenomenon.

Of course the speed of sound changes throughout the interior of the star. We compute the period as the travel time of sound waves throughout the diameter:

$$\Pi = 2\int_0^R \mathrm{d}t(r) = 2\int_0^R \frac{\mathrm{d}r}{\sqrt{\Gamma_1(r)P(r)/\rho(r)}}\,, \tag{21}$$

since $\mathrm{d}t = \mathrm{d}r\,/v_s$.

We also rewrite the differential equation for $P$ substituting $m = \rho r$. Doing this we get

$$\Pi\overline{\rho}^{1/2} = \sqrt{\frac{3\pi}{2\Gamma_1 G}} \tag{22}$$

Which confirms Ritter's relation $\Pi \propto \overline{\rho}^{-1/2}$ .

This works for acoustic modes, but if we consider non-radial g-modes it stops working, such as variables of type ZZ Ceti.

### 4.2 The energy equations

Just an overview for now: we will consider the star as a thermodynamic engine.

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Reference books can be found in the Moodle: they are ordered by difficulty, Catelan to Aerts to Salaris.

The exam for this part of the course: it might be around the end of october.

The subscripts on last lecture *were* inverted after all.

The *mirror principle*: when the core contracts or expands, the envelope does the opposite.

The shell must remain at around the same temperature to maintain equilibrium: contracting the core would increase the temperature, therefore the envelope exapands. This heuristic argument is actually derived from simulations.

The relevant time scale for oscillations is the free-fall, dynamical time scale.

We come back to the energy equation

$$\frac{\partial L}{\partial m} = \varepsilon - \varepsilon_\nu - \varepsilon_g \tag{23}$$

we incorporate $\varepsilon - \varepsilon_\nu = \varepsilon_{\text{eff}}$ and call $\varepsilon_g = \frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon_{\text{eff}} - \frac{\partial L}{\partial m}$.

This makes the meaning of this transfer equation clearer. Using the first and second laws of thermodynamics, and recalling some thermodynamical values $c_V =$

$\left(\frac{\partial Q}{\partial t}\right)_V$, $\chi_T$, $\chi_\rho$, $\Gamma_{1,2,3}$. These are all *exponents* in some power law. We use log values since our variables change by orders of magnitude.

The final result we get from manipulations is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{P}{\rho(\Gamma_3 - 1)}\left(\frac{\partial \log P}{\partial t} - \Gamma_1 \frac{\partial \log \rho}{\partial t}\right) \tag{24a}$$

$$\frac{\partial \log P}{\partial t} = \Gamma_1 \frac{\partial \log \rho}{\partial t} + \frac{\rho}{P}(\Gamma_3 - 1)\left(\varepsilon_{\text{eff}} - \frac{\partial L}{\partial m}\right) \tag{24b}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = c_V T\left(\frac{\partial \log T}{\partial t} - (\Gamma_3 - 1)\frac{\partial \log \rho}{\partial t}\right) \tag{24c}$$

$$\frac{\partial \log T}{\partial t} = (\Gamma_3 - 1)\frac{\partial \log \rho}{\partial r} + \frac{1}{c_V T}\left(\varepsilon_{\text{eff}} - \frac{\partial L}{\partial m}\right) \tag{24d}$$

The second and fourth of these equations substitute our equation of energy conservation.

Say we have a solution for these equations, we look at linear perturbations of them. This makes sense: the main solution is basically static on the pulsation time-scales.

The perturbed model is $f = f(m)$, the unperturbed one is $f_0(m)$. The Lagrangian perturbation is $\delta f(m,t) = f(m,t) - f_0(m,t)$.

Let us consider specific cases for $r$: the radial displacement is $\delta r(m,t)$. The position of the layer at time $t$ is $r = r_0 + \delta r$.

We can write:

$$r = r_0\left(1 + \frac{\delta r}{r_0}\right) = r_0(1 + \zeta)\,. \tag{25}$$

In general the fractional perturbation $\delta f / f_0$ is assumed to be $\ll 1$. So, $\delta f / f_0 \sim \delta_f / f$. We will insert expressions which are functions of perturbations of all our variables, and thus get linear differential equations.

Let us try the continuity equation: $r = r_0(1 + \zeta)$ and $\rho = \rho_0(1 + \delta\rho/\rho_0)$.

$$\frac{\partial}{\partial m}\big(r_0(1+\zeta)\big) = \frac{1}{4\pi r_0^2}(1+\zeta)^{-2}\left(1 + \frac{\delta\rho}{\rho_0}\right)^{-1} \tag{26}$$

and we use $(1+x)^n \approx 1 + nx$:

$$4\pi\rho_0^2\left(\frac{\partial r_0}{\partial m}(1+\zeta) + r_0\frac{\partial \zeta}{\partial m}\right) = (1 - 2\zeta) - \frac{\delta\rho}{\rho_0} \tag{27}$$

We can collapse the equation into:

$$\frac{\delta\rho}{\rho_0} = -3\zeta - 4\pi r_0^3 \rho_0 \frac{\partial \zeta}{\partial m} \tag{28}$$

or, the density perturbation is proportional with a negative constant to the radial perturbation, plus a term proportional to $\partial\zeta/\partial m$. If there is a positive gradient of radial perturbation, the corresponding layer expands.

Let us also perturb the momentum conservation equation, assuming hydrostatic equilibrium $\partial^2 r_0 \Big/ \partial t^2$.

We have a purely geometric term $-16\pi r_0^2 \zeta\, \partial P_0/\partial m = 4\zeta G m/r_0^2$: when a layer moves outwards it expands. There is a restoring force toward equilibrium.

Let us also consider the $\log P$ form of the energy conservation equation. We also perturb the adiabatic exponents and the $\mathrm{d}Q/\mathrm{d}t$ term.

After difficult manipulations we get back an equation which relates the changes in density and pressure to the change in energy: we manipulate until we get something which is similar to the original equation.

In general for a Lagrangian perturbation $\delta(f^n) = n f_0^{n-1}\delta f$. We have rules for the perturbation of product which work just like logarithmic derivatives. $\delta$ commutes with partial derivatives.

In the radiative case with the diffusion approximation we can perturb the luminosity: $L \propto r^4 T^4 \kappa_R^{-1} \frac{\partial \log T}{\partial m}$.

$$\frac{\delta L}{L_0} = 4\zeta + 4\frac{\delta T}{T_0} - \frac{\delta \kappa_R}{\kappa_{R,0}} + \frac{\frac{\partial}{\partial m}\frac{\delta T}{T}}{\frac{\partial \log T}{\partial m}} \tag{29}$$

In the end, we have a set of four linear PDE equations (written as a 5-equation system).

These describe implicitly how the properties of the star change over time.

Pulsation usually affects mostly the outer layers of a star.

Moving on to

# 5 Adiabatic oscillations

Explioting the adiabatic approximation we will get the LAWE: Linear Adiabatic Wave Equation: a single equation which summarizes the 4.

This can be solved explicitly.

We suppose that each layer does not lose nor gain heat: $\delta(\varepsilon_{\mathrm{eff}} - \partial L/\partial m) = 0$.

Is this approximation justified? The term multiplying $\delta(\varepsilon_{\mathrm{eff}} - \partial L/\partial m)$ in the perturbed energy equation is $\rho/P(\Gamma_3 - 1) = \chi_T/(c_V T)$. Usually $\chi_T \sim 1$, $\Gamma_1 \sim 1$.

This term, $\chi_T/(c_V T)\delta(\varepsilon_{\mathrm{eff}} - \partial L/\partial m)$, is of the order $1/\tau_{\mathrm{th}}$, the thermal time scale of this layer, while the term before, $\Gamma_1\, \partial/\partial t\ (\delta\rho/\rho)$, is of the order $1/\tau_{\mathrm{dyn}}$, the dynamical time scale.

Therefore, we neglect the second part. This only works for the star as a whole, not for single layers. There are stellar layers which are *strongly* non-adiabatic (driving layers). We will need some non-adiabatic theory to explain how pulsations *start*.

[Insert derivation of the LAWE, which can be found in the slides].

$$r\frac{\partial^2 \zeta}{\partial t^2} = 4\pi r^2 \zeta \frac{\partial}{\partial m}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2 \Gamma_1 P \rho r^6 \frac{\partial \zeta}{\partial m}\right) \tag{30}$$

We decompose: $\zeta(m,t) = \eta(m)e^{i\sigma t}$ with a constant $\sigma$: putting this into the LAWE we simplify the exponentials and get the space dependent form of the LAWE. It is a Storm-Liouville equation.

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Last lecture we started talking about linearization & perturbation theory.

We will have analytic solutions for the adiabatic case, with the additional hypotheses of either $\Gamma_1 = 4/3$ or $\Gamma_1 > 4/3$ and homogeneity.

The justification of the adiabatic approx might be asked at the exam.

After plugging our ansatz in the LAWE we should take the real part.

What are the boundary conditions we should set?

1. $\delta r = 0$ at $r = 0$;
2. $\partial\eta/\partial r = 0$ at $r = 0$, which allows us to fix many divergences at the center;
3. $(4 + R^3\sigma^2/(GM))\eta + \delta P/P = 0$ at $r = R$;
4. $\eta = \delta r/r = 1$ at $r = R$.

What is the $\frac{\partial^2 \eta}{\partial r^2}$ stuff about?

We use the Eulerian form of the LAWE to figure out the surface boundary conditions. We assume that all perturbation are in phase, and write them all as proportional to $\exp(i\sigma t)$. The pressure scale height is defined as:

$$H_P = -\left(\frac{\partial \log P}{\partial r}\right)^{-1} \tag{31}$$

and represents "how far we should move in the star for the pressure to change $e$-fold". By inserting this in the equation we see that $H_P \to 0$ when $r \to R$: the pressure changes very quickly in the photosphere of the star. Therefore, the term multiplying it must also go to zero.

The last condition comes from the fact that we want our study to give us *periods*, not *amplitudes*: we cannot find those out, so we normalize. The LAWE is 1-homogeneous!

The LAWE can be written compactly with a linear operator $\mathcal{L}$:

$$\mathcal{L}(\eta) = \sigma^2 \eta \,, \tag{32}$$

therefore the eigenvalue is the square of the pulsation. There are infinitely many solutions to the LAWE, only (finitely many?) fulfill the boundary condition. The eigenvalues are real [$\mathcal{L}$ is Hermitian], have a wavefunction associated: $\eta_m(r)$ corresponding to $\sigma_m^2$.

If $\sigma^2 > 0$ we have an oscillating solution, if $\sigma^2 < 0$ we have an exponential collapse or explosion since the solution is proportional to $\exp(i\sigma t)$.

We label solutions by *radial order* $m \in \mathbb{N}$: $m = 0$ has the lowest frequency, and then we have overtones. We choose the labels so that $\sigma_{m_1} < \sigma_{m_2} \iff m_1 < m_2$.

The radial order $m$ is also the number of nodes.

The eigenfunctions are orthogonal wrt the scalar product

$$\langle \eta_m | \eta_n \rangle = \int_0^R \eta_m \eta_n \rho r^4 \, \mathrm{d}r \tag{33}$$

Possibly there is a $4\pi$ missing in order for this to be consistent with the following?

The functions $\zeta_m$ are orthogonal wrt the same product. The system is linear: we can write a general solution as a superposition.

We can define the moment of inertia:

$$J_m = \int_0^M |\zeta_m|^2 r^2 \, \mathrm{d}m \tag{34}$$

and the following holds:

$$\sigma_m^2 = \frac{1}{J_m} \int_0^M \zeta_m^* \mathcal{L} \zeta_m r^2 \, \mathrm{d}m \tag{35}$$

## 5.1 Simplifications

### 5.1.1 Period-mean density relation

If $\eta = \text{const}$, and $\rho$ and $\Gamma_1$ are also constanst, we immediately get:

$$\sigma^2 = (3\Gamma_1 - 4) \frac{GM}{r^3} \tag{36}$$

and by inserting the mean density formula we get the period-mean density relation: this is consistent with our previous assumptions.

### 5.1.2 Polytropic model

It is a gas sphere with the following constitutive equaition:

$$P = K_n \rho^{1+1/n} = K_n \rho^{\frac{n+1}{n}} \tag{37}$$

with varying *polytropic* index $n$. It models spheres with different mass distributions: $n = 0$ is constant density, $n = 5$ is infinite central density, $n = 3$ is the Eddington standard model, which is reasonable for the Sun and stars on the main sequence.

With these assumptions we can explicitly solve for the wavefunctions, and make predictions of the fractional modulus of the oscillations at a certain radius wrt the modulus at the surface (which can be found only experimentally).

The overtones die out toward the center even faster than the fundamental: these oscillations are very much a *surface phenomenon*.

Beyond the $\eta$s we can also plot the pressure perturbations: these will not be normalized.

### 5.1.3 A concrete example

This is done looking at an RR Lyrae variable. We integrate the stellar structure equations numerically. We can see that $\sigma_m - \sigma_{m-1} \approx$ const when $m$ gets large.

The wavefunctions die out faster than the polytropic model when $r \to 0$.

There are "bumps" in the pressure plot: these are the partial ionization regions of H and He.

These appear because we start from a solution of the stellar structure equations, where all the properties of stellar matter were used, to start off with the LAWE.

# 6 Non-adiabatic oscillations

How can we tell, theoretically, how stable and how wide the various modes are? We expect to see the stable modes, and not to see the unstable ones.

Let us start from the Lagrangian momentum conservation:

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r} \tag{38}$$

and apply to it the identity: $1/2 \frac{\partial}{\partial t} v^2 = \frac{\partial r}{\partial t} \frac{\partial^2 r}{\partial t^2}$, by multiplying everything by $\partial r / \partial t$.

We then integrate everything with respect to $m$ and apply some manipulations [see slides].

$$\frac{\partial}{\partial t} \int_M \frac{v^2}{2} \, \mathrm{d}m = -\frac{\mathrm{d}\Omega}{\mathrm{d}t} + \int_M P \frac{\partial}{\partial t} \frac{1}{\rho} \, \mathrm{d}m \tag{39}$$

We integrate a pulsation period, which cancels out the gravitational potential term which is conservative.

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle_{\Pi} = \frac{1}{\Pi} \int_{\Pi} \int_{M} P \frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) \mathrm{d}m \, \mathrm{d}t \tag{40}$$

Some layers will provide energy to the oscillation motion (*drive* it), some others will *damp* it. These are characterized by the sign of the RHS of this equation.

If it is positive, we have instability; if it is negative the pulsations will tend to die out, giving stability.

The average time scale of change of the perturbations is

$$\kappa \stackrel{\text{def}}{=} \frac{1}{\tau} = -\frac{1}{2} \frac{\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle_{\Pi}}{\left\langle \delta\psi \right\rangle_{\Pi}} \tag{41}$$

The term $\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle_{\Pi}$ can also be interpreted as the net heat gain fed into mechanical work during a pulsation cycle.

In the adiabatic case, we had $\frac{\partial}{\partial t}\left( \frac{\delta P}{P} - \Gamma_1 \frac{\delta \rho}{\rho} \right) = 0$: the perturbations were in phase.

Now we add a term to the time derivatives: the pressure and density perturbations will stop being in phase. The sign of the heat variation term gives us the difference between *driving* heat transfer and *damping* heat transfer.

In a PV diagram, we can see that these correspond to right and left oriented loops (as opposed to the loops with zero total signed area we had in the adiabatic case).

The star is effectively a themal engine converting heat into work; this will result in an increased overall entropy of the star, and a smoothing of its temperature gradient, however:

1. the timescales on which this process occurs are much larger than the timescales on which oscillating motions are created and destroyed;
2. then energies of the oscillations are much smaller than the global thermal energy of the star.

therefore this process is typically not relevant.