# Astrophysics and cosmology notes

Jacopo Tissino, Giorgio Mentasti

October 17, 2019

## 3 October 2019

Sabino Matarrese ♡. There is a dropbox folder with notes by a student from the previous years. Also, there are handwritten notes by Sabino.

Textbooks: there are many. A good one is "Cosmology" by Lucchin and Coles.

On the astroparticle side, there is another book: I did not catch its name.

Exam: traditional oral exam, there are fixed dates but they do not matter: on an individual basis we should write an email to set a date and time.

In october the lessons of GR and this course on fridays are swapped.

# 1 Cosmology

The cosmological principle (or Copernican principle): *we do not occupy a special, atypical position in the universe*. We will discuss the validity of this. A more formal statement is:

**Proposition 1.1** (Cosmological principle)**.** *Every* comoving observer *observes the Universe around them at a* fixed time *as being* homogeneous *and* isotropic*.*

We are allowed to use this principle only on very large scales. Comoving is a proxy for the absolute reference frame. When we observe the CMB we see that we are surrounded by radiation of temperature $\sim 3\,\mathrm{K}$. There is a *dipole modulation* though: around a milliKelvin of difference. This is due to the Doppler effect: we are *not* comoving with respect to the CMB.

We, however, ignore the anisotropies on the order of a $\mu\mathrm{K}$.

The velocity needed to expain the Doppler effect is of the order of $630\,\mathrm{km\,s^{-1}}$. This is *after* correcting for the motion of everything up to a galactic scale.

So, we must assume that to get a *comoving* observer we should launch it in a certain direction at an extremely high velocity. So, we *do* have a preferred frame.

*Fixed time* refers to the proper time of the comoving observer.

This refers to scales on the order of 100 Mpc.

How can we talk about homogeneity if we can only look at the universe from a single point? We assume that any other observer would also see isotropy as we do.

Isotropy around every point is equivalent to homogeneity. We observe isotropy, we assume homogeneity.

What if we are not typical? a different assumption is that our observer status is *random* according to some distribution… This discussion then starts to involve the anthropic principle.

We must however always keep in mind that these assumptions have to be made before any cosmological study starts.

In GR, our line element will generally be $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$. The *preferred* form of the metric is the one written in the comoving frame: the Robertson-Walker line element,

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right)\right) \tag{1}$$

where $a(t)$ is called the *scale factor*. Sometimes it is convenient to write $\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2 = \mathrm{d}\Omega^2$. The angular part is the same (just rescaled) as the Minkowski one.

The parameter $k$ is a constant, which can be $\pm 1$ or 0.

The parameter $r$ is not a length: we choose our variables so that $a(t)$ is a length, while the stuff in the brackets in (1) is adimensional (and so is $r$).

- $k = 1$ are called closed universes;
- $k = -1$ are open universes;
- $k = 0$ are called flat universes.

The set of flat universes with $k = 0$ has zero measure.

In a 4D flat spacetime we have the 10 degrees of freedom of Lorentz transformations: Minkowski spacetime is *maximally symmetric*.

Other geometries are De Sitter and Anti de Sitter spacetimes.

For the actual universe, however, not all of these symmetries hold: the universe seems not to be symmetric under time translation, and we have a preferential velocity. 4 symmetries are broken (time translation and Lorentz boosts), 6 hold (rotations and spatial translations): our assumption is that there is a 3D maximally symmetric space.

Under these assumptions, we can derive the RW line element.

Say we have a cartesian flat 2D plane: the flat metric for this is $\mathrm{d}l^2 = a^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$. The constant $a$ is included since we want $r$ to be adimensional.

The surface of a sphere has the following line element: $\mathrm{d}l^2 = a^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$, where $a^2 = R^2$, the square radius of the sphere.

For a hyperboloid, we will have: $\mathrm{d}l^2 = a^2 \left(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\varphi^2\right)$, therefore the only difference is that trigonometric functions become hyperbolic ones.

For both of these, let us define the variable: $r = \sin\theta$ in the spherical case, and $r = \sinh\theta$ in the hyperbolic case. Then, these line elements become respectively:

$$\mathrm{d}l^2{}_{\text{sphere}} = a^2 \left( \frac{\mathrm{d}r^2}{1 - r^2} + r^2\,\mathrm{d}\varphi^2 \right) \tag{2a}$$

$$\mathrm{d}l^2{}_{\text{hyperboloid}} = a^2 \left( \frac{\mathrm{d}r^2}{1 + r^2} + r^2\,\mathrm{d}\varphi^2 \right). \tag{2b}$$

We can rewrite the RW element as:

$$\mathrm{d}l^2 = c^2\,\mathrm{d}t^2 - a^2 \begin{cases} \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega^2 \\ \mathrm{d}\chi^2 + \chi^2\,\mathrm{d}\Omega^2 \\ \mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\Omega^2 \end{cases} \tag{3}$$

where if $k = +1$ then $r = \sin\chi$, if $k = 0$ then $r = \chi$, and if $k = -1$ then $r = \sinh\chi$. If we wish to use cartesian coordinates we will have:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t) \left( 1 + \frac{k|x|^2}{4} \right)^{-2} \left( \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \right). \tag{4}$$

Universes in which $a$ is a constant are called *Einstein spaces*. We can change time variable, defining $\mathrm{d}t = a(\eta)\,\mathrm{d}\eta$, where $a(\eta) \overset{\text{def}}{=} a(t(\eta))$: so, we will have

$$\mathrm{d}s^2 = a^2(\eta) \left( c^2\,\mathrm{d}\eta^2 - \left( \frac{\mathrm{d}r^2}{1 - kr^2} + r^2\,\mathrm{d}\Omega \right) \right). \tag{5}$$

The parameter $\eta$ is called *conformal time*: RW is said to be *conformal* to Minkowski. Conformal geometry is particularly useful for systems which have no characteristic length.

Photons do not have a characteristic length: they do not perceive the expansion of spacetime.

The photons of the CMB look like they are thermal: they were thermal originally, and remained such despite the expansion of the universe.

For now we did not use any dynamics, but we will insert them later.

The Friedmann equations are:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{6a}$$

$$\ddot{a} = -\frac{4\pi G}{3}a\left(\rho + \frac{3P}{c^2}\right) \tag{6b}$$

$$\dot{\rho} = \frac{-3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \tag{6c}$$

where dots denote differentiation with respect to the proper time of a consmological observer, $t$, which is called *cosmic time*. These imply that the energy density $\rho = \rho(t)$ and the isotropic pressure $P = P(t)$ only depend on $t$.

An important parameter is $H(t) \stackrel{\text{def}}{=} \dot{a}/a$, the *Hubble parameter*. We can write an equation for it from the first Friedmann one:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{7}$$

If $k = 0$, then there we have a critical energy density $\rho_C(t) = 3H^2(t)/(8\pi G)$: we call $\Omega(t) = \rho(t)/\rho_C(t)$. Is $\Omega$ larger or smaller than 1? This tells us about the sign of $k$; while measuring $k$ directly is *very* hard. The former is a "Newtonian" measurement, while the latter is a "GR" measurement.

We define:

$$H_0 = H(t_0) = 100h \times \mathrm{km\,s^{-1}\,Mpc^{-1}} \tag{8}$$

where $h$ is a number: around 0.7, while $t_0$ just means *now*.

## 1.1 Energy density

How do we measure it? We want the energy density *today* of *galaxies*: $\rho_{0g}$. This is $\mathscr{L}_g \langle M/L \rangle$, where $\mathscr{L}_g$ is the mean (intrinsic, bolometric) luminosity of galaxies per unit volume, while $M/L$ is the mass to light ratio of galaxies. It is measured in units of $M_\odot/L_\odot$. Reference values for these are $M_\odot \sim 1.99 \times 10^{33}\,\mathrm{g}$, while $L_\odot \sim 3.9 \times 10^{33}\,\mathrm{erg\,s^{-1}}$.

How do we measure this? We have a trick:

$$\mathscr{L}_g = \int_0^\infty \mathrm{d}L\, L\Phi(L) \tag{9}$$

where $\Phi(L)$ is the number of galaxies per unit volume and unit luminosity: the *luminosity function*. With of our observations we estimate the shape of $\Phi(L)$. We know that the integral must converge, so we can bound the shape of $\Phi$ (?).

# 4 October 2019

From yesterday we recall that $k = 0$ iff $\rho(t) = \rho_C(t)$.

We can write $H(t_0) = H_0 = 100h \times \mathrm{km\,s^{-1}\,Mpc^{-1}}$. Do note that $1\,\mathrm{Mpc} = 3.086 \times 10^{22}\,\mathrm{m}$.

In the American school, the pupils of Hubble thought $h \sim 0.5$, while the French school thought $h \sim 1$. Now, we know that $h \sim 0.7$. Some people find $h \approx 0.67 \div 68$, others find $h \approx 0.62$.

If we have $H$ we can find $\rho_{0C} = h^2 \times 1.88 \times 10^{-28}\,\mathrm{g\,m^{-3}}$. We have defined $\Omega(t) \overset{\text{def}}{=} \rho/\rho_C$: recall that $\operatorname{sign}(\Omega - 1) = \operatorname{sign}(k)$.

So we want to measure the energy density in galaxies to figure out what $\Omega$ is.

There is a professor called Schechter who introduced a *universal* luminosity function of the universe.

$$\Phi(L) = \frac{\Phi^*}{L^*}\left(\frac{L}{L^*}\right)^{-\alpha} \exp\left(-\frac{L}{L^*}\right) \tag{10}$$

These can be fit by observation: we find $\Phi^* \approx 10^{-2}h^3\mathrm{Mpc}^{-3}$, $L^* \approx 10^{10}h^{-2}L_\odot$ and $\alpha \approx 1$.

[Plot of this function: sharp drop-off at $L = L^*$ ]

The integral for $\mathscr{L}_g$ converges despite the divergence of $\Phi(L)$ as $L \to 0$: so we do not need to really worry about the low-luminosity cutoff.

The result of the integral is $\mathscr{L}_g = \Phi^* L^* \Gamma(2-\alpha)$ where $\Gamma$ is the Euler gamma function, and $\Gamma(2-1) = 1$. Numerically, we get $(2.0 \pm 0.7) \times 10^{18} h L_\odot \mathrm{Mpc}^{-3}$.

Spiral galaxies are characterized by rotation.

We plot the velocity of rotation of galaxies $v$ against the radius $R$. This is measured using the Doppler effect.

We'd expect a roughly linear region, and then a region with $v \sim R^{-1/2}$: we apply $GM(R) = v^2(R)R$ (this comes from Kepler's laws or from the virial theorem). This implies

$$v(R) \propto \sqrt{\frac{M(R)}{R}} \tag{11}$$

So in the inside of the galaxy, where $M(R) \propto R^3$, $v \propto R$, while outside of it $M(R)$ is roughly constant, so $v \propto R^{-1/2}$.

Instead of this, we see the linear region and then $v(R)$ is approximately constant. Is Newtonian gravity wrong? (GR effects are trivial at these scales).

An option is MOND: they propose that there is somthing like a Yukawa term at Megaparsec distances. They are wrong for some other reasons.

Another option is that what we thought was the galaxy, from our EM observations, is actually smaller than the real galaxy. We'd need mass obeying $M(R) \sim R$:

since $M(R) = 4\pi \int_0^{R_{\max}} \mathrm{d}R\, R^2 \rho(R)$, we need $\rho(R) \propto R^{-2}$. This is a *thermal* distribution (?): we call it the *dark matter halo*.

People tend to believe that this matter is made up of beyond-the-standard-model particles, like a *neutralino*. An alternative is the *axion*.

The total density of DM is $\sim 5$ times more than that of regular matter.

If galaxies are not spiral, we look at other things: the Doppler broadening of spectral lines gives us a measure of the RMS velocity.

## 1.2 Virial theorem

Later in the course we will obtain the (nonrelativistic) virial theorem:

$$2T + U = 0 \tag{12}$$

This holds when the inertia tensor stabilizes.

The kinetic energy is $T = {}^3\!/_2 M \left\langle v_r^2 \right\rangle$: we expect the radial velocity to account for one third of total energy by equipartition. $M$ is the total mass of the energy.

The potential energy is $U = -GM^2/R$. Substituting this in we get $3 \left\langle v_r^2 \right\rangle = GM/R$.

If we account for the extra DM mass, we get $\langle M/L \rangle \approx 300 h M_\odot / L_\odot$. In order to have $\Omega = 1$, we'd need 1390.

So measuring the number density of galaxies and their velocities we get a way to measure $\Omega_0$.

So, only 5% of the energy budget is given by baryionic matter (not all of which is visible), while around 27% is dark matter.

In order to comply with observation, it must be:

$$0.013 \leq \Omega_\mathrm{b} h^2 \leq 0.025\,, \tag{13}$$

where $\Omega_b$ corresponds to the baryionic density: so the universe *cannot* be made only of baryons.

Dark matter likes "clumping": we characterize it by this property.

In the end we have $\Omega = 1 = \Omega_b + \Omega_{DM} + \Omega_{DE}$ (do note that the value of 1 is measured, not theoretical!)

The Friedmann equation would imply deceleration if $\rho, P \geq 0$: dark energy seems to have *negative pressure*.

What about radiation? The CMB appears to be Planckian:

$$\rho_{0\gamma} = \frac{\sigma_r T_{0\gamma}^4}{c^2} = 4.8 \times 10^{-34}\,\mathrm{g\,cm^{-3}} \tag{14}$$

where $\sigma_r = \pi^2 k_B^4 / (15 \hbar^3 c^3)$, while $\sigma_{SB} = \sigma_r c/4$.

We are going to show that if neutrinos were massless, their temperature would be $T_\nu = (4/11)^{1/3} T_\gamma$.

However, we know that for sure $\sum m_\nu \leq 0.12\,\mathrm{eV}$.

We have:

$$\rho_\nu = 3N_\nu \frac{\langle m_\nu \rangle}{10\,\mathrm{eV}} 10^{-30}\,\mathrm{g\,cm^{-3}} \tag{15}$$

(??? to check)

### 1.3 The Hubble law

It is very simply

$$v = H_0 d\,, \tag{16}$$

where $v$ is the velocity of objects far from us, and $d$ is their distance from us. Can we derive this from the Robertson-Walker line element? It was actually derived first by Lemaitre.

We drop the angular part in the FLRW line element (for $k = 0$):

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2 \tag{17}$$

So the distance $d = a(t)r$: therefore $\dot{d} = \dot{a}r = \frac{\dot{a}}{a}d$. This is Newtonian and rough, but it seems to work.

**Definition 1.1** (Redshift)**.** *The redshift $z$ is defined by*

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}\,, \tag{18}$$

*where $\lambda_0$ and $\lambda_e$ are the observed and emission wavelengths.*

We can show that $1 + z = a_0/a_e$. Therefore, $\nu_o/\nu_e = a_e/a_0$.

So, we can define (?) the luminosity distance:

$$d_L = \sqrt{\frac{L}{4\pi\ell}}\,, \tag{19}$$

where $L$ is the observed luminosity, while $\ell$ is the apparent luminosity.

How do we relate the luminosity distance and the scale factor? Geometrically we derive:

$$\ell = \frac{L}{4\pi r^2 a^2}\left(\frac{a_e}{a_0}\right)^2 \tag{20}$$

since we can just integrate over angles the FLRW element: the value of $k$ does not enter into the equation. The corrective factor comes from the frequency dependence of energy and the fact that power is energy over time, which also changes.

Therefore:

$$d_L = \frac{a_o^2}{a_e} r = a(1+z)r\,. \tag{21}$$

# Thu Oct 10 2019

Today at 17 there is a colloquium at Aula Rostagni.

Next week, also at 17, there is a meeting at San Gaetano (for the general public).

We just had a Nobel prize in cosmology: he predicted the existence of the CMB, and two in astrophysics, they found the first exoplanet.

Send the professor an e-mail to get access to his Dropbox folder with many relevant texts.

We come back to the RW metric:

$$ds^2 = c^2\,dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\,d\Omega^2\right), \tag{22}$$

where $k = 0, \pm 1$ and $r$ is the dimensionless comoving radius.

The distance can be calculated as:

$$d_P(t) = a(t)\frac{r}{\sqrt{1-kr^2}}. \tag{23}$$

We are allowed to consider radial trajectories, however this does not account for the fact that we cannot actually measure with a space-like measuring stick.

A better definition is

$$d_P = a(t)\int_0^r \frac{d\widetilde{r}}{\sqrt{1-k\widetilde{r}^2}}. \tag{24}$$

This is called the *proper distance* (the subscript $P$ is for "proper"): the integration, at zero $d\Omega$ and zero $dt$, of the radial part of the metric.

On the other hand (OTOH), the flux of photons can give us the *luminosity distance*, which is actually measurable.

We want to derive the Hubble law ($v = H_0 d$) mathematically. It can also be restated as $cz = H_0 d$.

why?

For a fixed source at $d = ar$, naïvely we would have:

$$\dot{d} = \dot{a}r = \frac{\dot{a}}{a}ar = H_0 d \tag{25}$$

We can "move away from the current epoch by Taylor expanding".

The scale factor $a(t)$ can be written as

$$a(t) \simeq a_0 + \dot{a}_0(t-t_0) + {}^1\!/_2\,\ddot{a}(t-t_0)^2 \tag{26a}$$

$$= a_0\left(1 + \frac{\dot{a}}{a_0}\Big|_{t_0}(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)\right), \tag{26b}$$

where $q_0 \stackrel{\text{def}}{=} -\ddot{a}_0 a_0/(\dot{a}_0)^2$ is called the *deceleration parameter* by historical reasons: people thought they would see deceleration when first writing this.

(the Hubble *constant* is not constant wrt *time* , but wrt *direction*!)

The deceleration parameter is actually measured to be negative.

Now, $1+z = a_0/a$ can be expressed as:

$$1+z \simeq \left(1 + \frac{\dot{a}}{a_0}|_{t_0}(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)\right)^{-1}. \tag{27}$$

Do note that this is derived by using at most second order in the distance interval: when expanding we cannot trust terms of order higher than second. Expanding with this in mind we get:

$$1+z \simeq 1 - H_0 \Delta t + \frac{q_0}{2} H_0^2 \Delta t^2 + H_0^2 \Delta t^2\,, \tag{28}$$

therefore

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t)^2\,. \tag{29}$$

Rearranging the equation we get

$$t_0 - t = z\left(H_0 + \left(1 + \frac{q_0}{2}\right) H_0 z\right)^{-1}, \tag{30}$$

but as before we must expand to get only the relevant orders:

$$t_0 - t = H_0(t_0 - t) + \left(1 + \frac{q_0}{2} H_0^2 (t_0 - t)^2\right) \tag{31}$$

We would like the time interval to disappear: for photons $\mathrm{d}s^2 = 0$, therefore in that case $c^2\,\mathrm{d}t^2 = a^2(t)\,\mathrm{d}r^2\,/(1-kr^2)$. Taking a square root and integrating:

$$\int_t^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \pm \int_r^0 \frac{\mathrm{d}\widetilde{r}}{\sqrt{1-kr^2}}\,, \tag{32}$$

where the plus or minus sign comes from…

The integral on the RHS can be solved analytically: it is

$$\begin{cases} \arcsin r \qquad k=1 \\ r \qquad k=0 \\ \operatorname{arcsinh} r \qquad k=-1 \end{cases} \tag{33}$$

in all cases, it is just $r$ up to *second* order (since the next term in the expansion of an arcsine or hyperbolic arcsine etc is of third order).

On the other side, we have:

$$\frac{1}{a_0}\int_{t_0}^{t} c\,\mathrm{d}\widetilde{t}\left(1+H_0(\widetilde{t}-t_0)-\frac{q_0}{2}H_0^2(\widetilde{t}-t_0)^2\right)^{-1} \tag{34}$$

therefore, neglecting *third* and higher order terms, we have:

$$\frac{c}{a_0}\left((t_0-t)+\frac{1}{2}H_0(t_0-t)^2+o(|t_0-t|^2)\right)=r\,, \tag{35}$$

since the term proportional to $q_0$ only gives a third order contribution.

The explicit form of the luminosity distance was

$$d_L=a_0^2\frac{r}{a}=a_0(1+z)r\,. \tag{36}$$

The term $a_0$ should disappear at the end of every calculation: it is a bookkeeping parameter. Moving on:

$$d_L=a_0(1+z)\frac{c}{a_0H_0}\left(z-\frac{1}{2}(1+q_0)z^2\right), \tag{37}$$

but this also contains cubic terms:

$$d_L\simeq\frac{c}{H_0}\left(z+\frac{1}{2}(1-q_0)z^2+o(z^2)\right). \tag{38}$$

Therefore:

$$cz=H_0\left(d_L+\frac{1}{2}(q_0-1)\frac{H_0}{c}d_L^2\right), \tag{39}$$

and we can notice that the relation is approximately linear and independent of acceleration for low redshift, but we can detect the acceleration at higher redshift. Typically we need to go at around 10 Mpc.

The parameter $q_0$ appears to be negative now.

We can do better than that if we go from *cosmography* to *cosmology*, by understanding what causes the acceleration of the expansion of the universe.

Let us expand on the concept of reshift: photons are emitted with a certain wavelength $\lambda_e$, at a comoving radius $r$ from us, and detected at $\lambda_o$. The line element for the photon is $\mathrm{d}s^2=0$, therefore $c\,\mathrm{d}t\,/a(t)=\pm\,\mathrm{d}r\,\sqrt{1-kr^2}$.

As before, we can integrate this relation from the emission to the absorption: we call it $f(r)$ (it can be any of the functions shown before).

$$\int_t^{t_0}=\frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}=f(r) \tag{40}$$

If we map $t\to t+\delta t$ and $t_0\to\delta t_0$ in the integration limits, the integral must be constant since it only depends on $r$.

When equating these two we can simplify the original integral, rearrange the integration limits and get:

$$\int_{t}^{t+\delta t} \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}\,, \tag{41}$$

which can we cast into

$$\frac{c\delta t}{a(t)} = \frac{c\delta t_0}{a(t_0)}\,. \tag{42}$$

Since the frequency must be proportional to the inverse of the time intervals $\delta t$ or $\delta t_0$, we have

$$\nu_e a(t_e) = \nu_o a(t_o)\,, \tag{43}$$

therefore

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a_0}{a}\,. \tag{44}$$

## 1.4 A Newtonian derivation of the Friedmann equations

It is useful to do it first this way, pedagogically.

Let us take a uniform spacetime with density $\rho$. We consider a sphere, and take all the mass inside the sphere away.

Consider Birkoff's theorem: the gravitational field of a spherically symmetric body is always described by the Schwarzschild metric. This can also be applied to a hole: in this case, it tells us that the metric inside the cavity is the Minkowski metric.

The mass taken away will be $M(\ell) = {}^{4\pi}/_{3}\rho\ell^3$, where $\vec{l} = a(t)\vec{r}$ is the radius of the sphere.

We suppose that the gravitational field is *weak*:

$$\frac{GM(\ell)}{\ell c^2} \ll 1\,. \tag{45}$$

We put a test mass on the surface of the sphere. What is the motion of the mass due to the gravitational field from the center? It will surely be radial, therefore

$$\ddot{\ell} = -\frac{GM(\ell)}{l^2} = -\frac{4\pi G}{3}\rho\ell\,. \tag{46}$$

This seems to give us a net force even though by our hypotheses there should be none, this is actually not an issue since the unit vectors in our equations will go away in the end.

Then, we have

$$\ddot{a}r = -\frac{4\pi G}{3}\rho a r \to \ddot{a} = -\frac{4\pi G}{3}\rho a\,. \tag{47}$$

This is part of a Friedmann equation: the isotropic pressure term cannot be recovered, it's like the speed of light is infinite now.

Now consider

$$\dot{\ell}\ddot{\ell} = -\frac{GM}{\ell^2}\dot{\ell}\,, \tag{48}$$

therefore

$$\frac{1}{2}\dot{\ell}^2 = \frac{4\pi}{3}G\rho\ell^3 + \text{const}\,, \tag{49}$$

and integrating we get

$$\dot{a}^2 r^2 = \frac{8\pi G}{3}\rho a^2 r^2 + \text{const} \tag{50}$$

or, removing the $r^2$ term, which is a constant,

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \text{const}\,. \tag{51}$$

We know that this new constant is related to the energy per particle.

A universe with negative $k$ expands forever and so on.

The number $k$ was badly defined to be only $\pm 1$ or 0: its newtonian version is much better represented by $k_N = kc^2$.

# Fri Oct 11 2019

On Oct 31 Marco Peloso will not give his lecture, so Sabino will do both lectures.

Recalling last lecture: we consider a universe with constant density $\rho$...

We can also recover the third Friedmann equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)\,, \tag{52}$$

without the last term, which yet again will come from the relativistic consideration.

We will consider *ideal fluids*. From thermodynamics we have:

$$\mathrm{d}E + p\,\mathrm{d}V = 0\,. \tag{53}$$

We can write $E = \rho c^2 a^3$. Then, this becomes $\mathrm{d}(\rho c^2 a^3) + p\,\mathrm{d}a^3 = 0$; expanding:

$$c^2 \mathrm{d}\rho a^3 + c^2\rho\,\mathrm{d}a^3 + p\,\mathrm{d}a^3 = 0\,. \tag{54}$$

This amounts to the third Friedmann equation, exactly.

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{55a}$$

$$\ddot{a} = -\frac{4\pi G}{3} a \left(\rho + \frac{3P}{c^2}\right) \tag{55b}$$

$$\dot{\rho} = -\frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \tag{55c}$$

The equations are in terms of the three parameters $a$, $\rho$ and $p$, which are all functions of time.

The third equation comes from the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$.

Rewriting the first equation:

$$\ddot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{56}$$

and

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\dot{\rho} a^2 + \frac{16\pi G}{3}\rho \dot{a} a \tag{57}$$

We then substitute in the expression we have for $\dot{\rho}$ from the third FE. Everything is multiplied by $\dot{a}$: if it is not zero we have

$$\frac{\ddot{a}}{a} = -4\pi G\rho - \frac{4\pi G p}{c^2} + \frac{8\pi G}{3}\rho = -4\pi G\left(\frac{\rho}{3} + \frac{P}{c^2}\right). \tag{58}$$

The equation system is underdetermined. We have to make an assumption: we will assume our fluid is a *barotropic* perfect fluid: $p \overset{!}{=} p(\rho)$.

Very often this equation of state will look like $p = w\rho c^2$, for a constant $w$ (not $\omega$!). This is related to the adiabatic constant $\gamma$ by $\gamma = w + 1$.

We are going to assume homogeneity and isotropicity.

Some possible equations of state are $p \equiv 0$, or $w = 0$: this means $\gamma = 1$. This is a *dust pressureless fluid*.

What is the speed of sound of our fluid? We only have the adiabatic speed of sound $c_s^2 = \partial p / \partial \rho$, where the derivative is to be taken at constant entropy and is just a total derivative in the barotropic case.

Also in the baryionic case $p = 0$ is a good approximation.

If $p = 0$ we can simplify:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \implies \rho \propto a^{-3}. \tag{59}$$

More generally, not assuming $w = 0$ we get:

$$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a} \implies \rho \propto a^{-3(1+w)} = a^{-3-3\gamma} \tag{60}$$

Another case is a *gas of photons*: in that case $p = \rho c^2/3$, so $w = 1/3$, $\gamma = 4/3$, $c_s^2 = c^2/3$: the speed of sound is $c/\sqrt{3}$. These photons are thermal: perturbations can propagate (even without interactions with matter…).

In this case we get $\rho \propto p \propto a^{-4}$.

Stiff matter is $p = \rho c^2$, $w = 1$, $\gamma = 2$ and $c_s = c$. This is an incompressible fluids: it is so difficult to set this matter in motion that once one does it travels at the speed of light. Now, $\rho \propto p \propto a^{-6}$.

A possible case is $p = -\rho c^2$: $w = -1$ and $\gamma = 0$: we cannot compute a speed of sound. Now $\rho$ and $p$ are constants. This is the case of dark energy (?).

This can be interpreted as an interpretation of the cosmological constant $\Lambda$.

Now we relace the last FE with $w = \text{const}$, $\rho(t) = \rho_* \big(a(t)/a_*\big)^{-3(1+w)}$.

Now, if we substitute into the second FE we get that gravity is attractive ($\ddot{a} < 0$) iff $w > -1/3$.

[Plot: $\rho$ vs $a$: the cosmological constant is constant, matter is decreasing, radiation is decreasing faster].

In this plot, $a$ can be interpreted as the time. We can insert the spatial curvature in the plot: it decreases, but slower than matter. Now, the dark energy in the universe is more important than the curvature.

Let us solve the first FE: inserting the third one we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_* \left(\frac{a}{a_*}\right)^{-3(1+w)} - \frac{kc^2}{a^2} . \tag{61}$$

We defined the parameter $\Omega = \frac{8\pi G \rho}{3H^2} = \rho/\rho_C$. Experimentally this is very close to 1. The Einstein-de Sitter model is one where we take $\Omega \equiv 1$: negligible spatial curvature. This amounts to making $k = 0$.

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_* a_*^{3(1+w)} a^{-(1+3w)} \tag{62}$$

therefore $\dot{a} = \pm A a^{\frac{1+3w}{2}}$, or $a^{\frac{1+3w}{2}}\,\mathrm{d}a = A\,\mathrm{d}t$. A solution is:

$$a(t) = a_* \left(1 + \frac{3}{2}(1+w)H_*(t - t_*)\right)^{\frac{2}{3(1+w)}} \tag{63}$$

where $H_*^2 = \frac{8\pi G}{3}\rho_*$, coupled to:

$$\rho(t) = \rho_* \left(1 + \frac{3}{2}(1 + wH_*(t - t_*))\right)^{-2} \tag{64}$$

There is a time where the bracket in $a(t)$ is zero: we call it as $t_{\text{BB}}$ define it by

$$1 + \frac{3}{2}(1+w)H_*(t_{\text{BB}} - t_*) = 0 . \tag{65}$$

Since the curvature scalar is $R \propto H^2$, at $t_{\mathrm{BB}}$ the curvature is diverges.

Hakwing & Ellis proved that if $w > -1/3$ we unavoidably must have a Big Bang.

We can define a new time variable by $t_{\mathrm{new}} \equiv (t - t_*) + 2H_*^{-1}/(3(1+w))$. Then, we can just write:

$$a \propto t_{\mathrm{new}}{}^{\frac{2}{3(1+w)}} \tag{66}$$

and this allows us to get rid of $t_*$.

Inserting this new time variable, we get

$$\rho(t) = \frac{1}{6(1+w)^2 \pi 4 t^2} \tag{67}$$

and the Hubble parameter is:

$$H(t) = \frac{2}{3(1+w)t} \tag{68}$$

Some cases are:

$$\begin{cases} w = 0 \implies a \propto t^{2/3} \\ w = 1/3 \implies a \propto t^{1/2} \\ w = 1 \implies a \propto t^{1/3} \end{cases} \tag{69}$$

The *De Sitter* universe is one where $w \to 1$: $a(t) \propto \exp(Ht)$ and $H = \mathrm{const.}$ (CHECK)

stuff

# Thu Oct 17 2019

If we neglect spatial curvature, which is small, we can write the luminosity distance as an integral which we can compute:

$$d_L \equiv \left(\frac{L}{4\pi\ell}\right)^{1/2}. \tag{70}$$

Our metric is the FLRW line element. Then, we can write $d_L$ as:

$$d_L = a_0(1+z)r(z)\,. \tag{71}$$

We now define the *conformal time* $\tau$: we want to impose $a^2(\tau)\,\mathrm{d}\tau^2 = \mathrm{d}t^2$. Then, the RW line element becomes:

$$\mathrm{d}s^2 = a^2(\tau)\left(c^2\,\mathrm{d}\tau^2 - \frac{\mathrm{d}r^2}{1-kr^2} - r^2\,\mathrm{d}\Omega^2\right). \tag{72}$$

This is very important when we talk about zero-mass particles, with no intrinsic length scale. Using the variable $\chi$, we have:

$$\mathrm{d}s^2 = a^2(\tau)\left(c^2\,\mathrm{d}\tau^2 - \mathrm{d}\chi^2 - f_k^2(\chi)\,\mathrm{d}\Omega^2\right), \tag{73}$$

where $f_k(\chi) = r$ is equal to $\sin(\chi)$, $\chi$ or $\sinh(\chi)$ if $k$ is equal to 1, 0 or $-1$.

If we look at photons moving radially, we get

$$\mathrm{d}s^2 = 0 = a^2(\tau)\left(c^2\,\mathrm{d}\tau^2 - \mathrm{d}\chi^2\right), \tag{74}$$

therefore $c^2\,\mathrm{d}\tau^2 = \mathrm{d}\chi^2$: setting $c = 1$, we get $\tau(t_0) - \tau(t_e) = \chi(r_e) - \chi(0)$, where a subscript $e$ means "emission".

$$\mathrm{d}r = \frac{\mathrm{d}t}{a} = \frac{\mathrm{d}a}{a\dot{a}}, \tag{75}$$

therefore:

$$\mathrm{d}a = -\frac{a_0}{(1+z)^2}\,\mathrm{d}z\,, \tag{76}$$

$$\frac{\mathrm{d}a}{a^2} = \frac{\mathrm{d}a\,(1+z)^2}{a^2} = -\frac{\mathrm{d}z}{a_0}, \tag{77}$$

which means

$$\mathrm{d}e = -\frac{\mathrm{d}z}{a_0 H(z)}\,. \tag{78}$$

???

The Hubble parameter is given by

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{79}$$

with density $\rho(t) = \rho_r(t) + \rho_m(t) + \rho_\Lambda$, where the first term scales like $a^{-4}$, the second $a^{-3}$, the third is constant. Thus they scale like $(1+z)^4$, $(1+z)^3$ and so on.

Then we can write a law for the evolution of $H(z) = H_0 E(z)$. Recall the definition of $\Omega(t)$: we can look at the $\Omega_i(t)$ for $i$ corresponding to matter, radiation and so on:

$$\Omega_i(z) = \frac{8\pi G\rho_i(t)}{3H^2(z)} = \frac{8\pi G}{3H^2}\frac{\rho_i(z)}{E^2(z)} \stackrel{\text{def}}{=} \Omega_{i,0}\frac{(1+z)^\alpha}{E^2(z)}\,. \tag{80}$$

In the case of radiation, $p = \rho c^2/3$, and then $\alpha = 4$.

For matter $p = 0$: $\alpha = 3$.

For the cosmological constant $\Lambda$: $\alpha = 0$.

For spatial curvature we have $\alpha = 2$.

For the $\Omega$) corresponding to the curvature we define: $\Omega_k = -tc^2/(a^2H^2)$.

We must have

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k \,. \tag{81}$$

Recall the definition of $E^2(z)$:

$$E^2(z) = \frac{H^2}{H_0^2} = \Omega_{k,0} + \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(q+z)^3 \,. \tag{82}$$

and to get $E$ we just take the square root. We have $\tau(t_0) - \tau(t_e) = \chi(r_e)$. Integrating:

$$\chi(r) = c\int_{a_t}^{a} \frac{\mathrm{d}a}{a\dot{a}} = \int \frac{\mathrm{d}z'}{E(z')} \,, \tag{83}$$

therefore

$$r = f_k\left( \frac{c}{a_0 H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')} \right) . \tag{84}$$

Two weeks ago we defined the luminosity distance: now we can compute it.

How do we decide which $k$ to use?

Now, suppose we are looking at a certain far-away object with angular size $\Delta\theta$: we fix $r$ in the RW line element, and look at a constant time: then we get a linear size corresponding to the angular one of

$$\mathrm{d}s = a(t) r \Delta\theta \,, \tag{85}$$

which, when divided by $\Delta\theta$, is called angular diameter $D_A = ar = a_0 r(z)/(1+z)$. This changes with distance... (?)

$$d_L = a_0(1+z)r \,, \tag{86}$$

then

$$\frac{d_L}{d_A} = (1+z)^2 \,, \tag{87}$$