Theoretical cosmology notes

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Contents

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Contents of the course

We start with a derivation of the Friedmann eqs. from the Einstein equations.

We will then discuss the properties of the CMB, deriving the spectrum, and then CMB anisotropies.

Then we will discuss star and structure formation, about the nonlinear evolution of perturbations. We will use the path-integral approach to classical field theory. We will also discuss weak gravitational lensing in the universe.

We will use some smart nonlinear approximations: the Zel'dovich approximation and the adhesion approximation.

We will use an "effective Planck constant", a parameter which can be fit in our model.

References: notes by a student from the previous years, in Italian [Nat17].

Chapter 1

Friedmann equations

In the previous course we used the approximate symmetries of the universe to write the FLRW line element:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\sigma^2 \ , \tag{1.1}$$

do note that we switch signature from the previous course: now we use the mostly plus one. The spatial part is defined by

$$\mathrm{d}\sigma^2 = \widetilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \ , \tag{1.2}$$

where $\widetilde{g}_{ij}$ is the maximally symmetric metric tensor in a 3D space. There are only 3 maximally symmetric 4D spacetimes: Minkowski, dS and AdS.

Since we have maximal symmetry, the Riemann tensor is

$$R_{ijkl} = k\Big(\widetilde{g}_{ik}\widetilde{g}_{jl} - \widetilde{g}_{il}\widetilde{g}_{jk}\Big)\,. \tag{1.3}$$

We can use spherical coordinates:

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2 \ , \tag{1.4}$$

and we can define the coordinate χ by

$$\mathrm{d}\chi = \frac{\mathrm{d}r^2}{\sqrt{1-kr^2}}\,. \tag{1.5}$$

The Einstein equations read

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \tag{1.6}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is its trace, the scalar curvature, while $T_{\mu\nu}$ is the stress energy momentum tensor.

In cosmology we assume to have the SEMT of a perfect fluid. Really, we have particles, between which there is vacuum.

We need to use the Weyl tensor, which describes the parts of the Riemann tensor which are not in the traces. "The real world" is only described by the Weyl tensor, but in cosmology we make a great approximation in ignoring it.

What we do is to insert an ansatz for the metric tensor, which we use to derive the Christoffel symbols, and from these we write the Riemann tensor. Doing it the other way around, starting from the source SEMT, is very difficult.

The Christoffel symbols for the FLRW metric are:

$$\Gamma^{t}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\dot{a}a}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 a\dot{a} & 0 \\ 0 & 0 & 0 & r^2 a\dot{a}\sin^2\theta \end{bmatrix} \tag{1.7}$$

$$\Gamma^{r}_{\mu\nu} = \begin{bmatrix} 0 & \dot{a}/a & 0 & 0 \\ \dot{a}/a & \frac{kr}{(1-kr^2)} & 0 & 0 \\ 0 & 0 & (kr^2-1)r & 0 \\ 0 & 0 & 0 & (kr^2-1)r\sin^2\theta \end{bmatrix} \tag{1.8}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & \dot{a}/a & 0 \\ 0 & 0 & 1/r & 0 \\ \dot{a}/a & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{1.9}$$

$$\Gamma^{\varphi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & \dot{a}/a \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & \cos\theta/\sin\theta \\ \dot{a}/a & 1/r & \cos\theta/\sin\theta & 0 \end{bmatrix}. \tag{1.10}$$

In order to calculate these, we can make use of certain simplifications: the FLRW metric is diagonal, and it does not depend on φ.

Notice that the spatial Christoffel symbols are nonzero even in Minkowski ($k = 0$, $\dot{a} = \ddot{a} = 0$): why is this? This is because we are using curvilinear coordinates, the Christoffel symbols express the *extrinsic* curvature, not the *intrinsic* curvature; they are not tensors, so they can be zero in a reference and nonzero in another.

In general, the Riemann tensor is given by

$$R^{\mu}_{\nu\rho\sigma} = -2\left(\Gamma^{\mu}_{\nu[\rho,\sigma]} + \Gamma^{\alpha}_{\nu[\rho}\Gamma^{\mu}_{\sigma]\alpha}\right), \tag{1.11}$$

where commas denote coordinate derivation, and square square brackets denote antisymmetrization (for clarification on this notation Wikipedia does a good job [19]).

The Ricci tensor is given by the contraction of the Riemann tensor along its first and third component:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = -2\left(\Gamma^{\alpha}_{\mu[\alpha,\nu]} + \Gamma^{\beta}_{\mu[\alpha}\Gamma^{\alpha}_{\nu]\beta}\right). \tag{1.12}$$

A great simplification comes from the observation that the metric being diagonal implies that the Ricci tensor is diagonal as well.

Not true! See [Win96].

The Ricci scalar then comes out to be

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right]. \tag{1.13}$$

The dimensions of the Ricci scalar are those of a length to the -2.

The stress energy tensor is the functional derivative of everything but the curvature in the action with respect to the metric: if our Lagrangian is

$$L = L_g + L_{\text{fluid}}, \tag{1.14}$$

where the gravitational Lagrangian is $L_g = M_P^2 R/2$ (and $M_P = 1/\sqrt{8\pi G}$ in natural units is the reduced Planck mass, and R is the scalar curvature) then

$$T_{\mu\nu} \overset{\text{def}}{=} -2\frac{\delta L_{\text{fluid}}}{\delta g^{\mu\nu}}. \tag{1.15}$$

Discuss why this is equivalent to "flux of momentum component μ across a surface of constant x^ν".

We use perfect fluids: they have a stress-energy tensor like

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + pg^{\mu\nu}, \tag{1.16}$$

where u^μ is the 4-velocity of the fluid element. It is diagonal *in the comoving frame*, in which $u^\mu = \left(1, \vec{0}\right)$.

If we are not comoving, we have additional heat transfer off diagonal terms (this is discussed in my thesis [Tis19, section 4.2]).

If we take the covariant divergence of the Einstein tensor $G_{\mu\nu}$ we get zero; so the stress energy tensor must also have $\nabla_\mu T^{\mu\nu} = 0$. This is *not* a conservation equation.

In SR we had an equation like $\partial_\mu T^{\mu\nu}$: this *was* a conservation equation, a local one. In GR we also need Killing vectors in order to actually have conserved quantities. In cosmology we do not have symmetry with respect to time translation, so there is no timelike Killing vector ξ_μ such that $\xi_\nu \nabla_\mu T^{\mu\nu}$ represents the conservation of energy.

This equation, $\nabla_\mu T^{\mu\nu}$ follows from the fact that our fluid follows its equations of motion.

Let us explore the meaning of these equations: if, in the equation $0 = \nabla_\mu T^\mu_0$, we find

$$0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \tag{1.17}$$

$$= -\dot\rho - 3H\big(\rho + P\big) \,. \tag{1.18}$$

For example consider radiation: $P = \rho/3$. This means that $\dot\rho = -4H\rho$: so, as the Hubble parameter increases, the radiation density decreases.

The other two Friedmann equations can be derived from the time-time and space-space components on the Einstein equations: we get

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\big(\rho + 3P\big) \tag{1.19}$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \,. \tag{1.20}$$

The space-space equation is not a dynamical equation, since it contains no second time derivatives: it is a *constraint* on the evolution of the system.

However, the three Friedmann equations are not independent: the time-time one can be found from the other two.

Exercise: calculate the Christoffel symbols for the FLRW metric, for any k.

Exercise: calculate the Ricci tensor and curvature scalar.

A useful theorem is the fact that for a maximally symmetric space the Ricci tensor must be given by

$$\widetilde{R}_{\alpha\beta} = 2k \widetilde{g}_{\alpha\beta} \,. \tag{1.21}$$

We can write the stress energy tensor as

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu} \,, \tag{1.22}$$

where $h_{\mu\nu}$ is the projection tensor onto the spacelike subspace $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$.

This is more physically meaningful.

Tomorrow we will start the discussion on the CMB.

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Today we discuss the CMB. This is discussed in the book Modern Cosmology, we follow Sabino's notes.

Let us suppose we have some particle species interacting, such as $1 + 2 \leftrightarrow 3 + 4$.

In these lectures a dot will refer to conformal time derivatives only. The derivative of the number density of particle type 1 is:

$$a^{-3}\frac{\mathrm{d}(n_1 a^3)}{\mathrm{d}t} = \left[\prod_{i=1}^{4}\int \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}\right] \times (2\pi)^4 \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\delta(E_1 + E_2 - E_3 - E_4) \times |\mathcal{M}|^2 \Big[f_3 f_4 (1 \pm f_1) \tag{1.23}$$

where the delta functions account for momentum and energy conservation: energy is *not conserved* in general in cosmology, *but* we can use the equivalence principle to describe a instantaneous such as this. $\mathcal{M}$ is the scattering matrix. The f_i are the phase space distributions of the different species: the terms including these account for the statistics, we use $-$ for fermions and $+$ for bosons. If there is no interaction, $n_1 \propto a^{-3}$. Bose statistics enhance the process, Pauli statistics block it.

The 2π account for the normalization of the deltas. We set $\hbar = c = k_B = 1$. The energy is given by $E = \sqrt{p^2 + m^2}$.

Why are there $2E$ factors in the denominators? In principle, we should integrate in $\mathrm{d}^4 p$, however we work *on shell*. A priori, the particle does whatever it wants, however solutions to the EOM are preferred in the path integral. So, we impose this condition: we do

$$\int \mathrm{d}^3 p \int_0^\infty \delta(E^2 - p^2 - m^2) = \int \mathrm{d}^3 p \int_0^\infty \frac{\delta(E - \sqrt{p^2 + m^2})}{2E}, \tag{1.24}$$

so we include the term in the denominator.

The term for particle 1, E_1, has a different origin: the time is related to the proper time by p^0, which is E_1. The factor 2 is included for symmetry, it is indifferent if we include it or not since we can normalize the helicities g_i.

Typically we have kinetic equilibrium, if the scattering time is very short with respect to the Hubble time. So, we use

$$f_{\mathrm{BE/FD}} = \left(\exp\left(\frac{E - \mu}{T}\right) \pm 1\right)^{-1}, \tag{1.25}$$

where the sign is a $-$ for Bose Einstein statistics, while a $+$ for Fermi-Dirac statistics.

For the nonrelativistic particles (all of them, except the photons) we have $E - \mu \gg T$. If f becomes very small, then we can drop the terms $(1 \pm f_i)$. This is the Boltzmann limit.

In theory we could not do this for photons, in practice we do it and the magnitude of the error is the same as the ratio $\zeta(3) \approx 1.2$ to 1.

Then, our distributions will be given by

$$f(E) = e^{\mu/T} e^{-E/T}. \tag{1.26}$$

So the phase space distribution term is

$$\exp\left(-\frac{E_1+E_2}{T}\right)\left(e^{(\mu_3+\mu_4)/T}-e^{(\mu_1+\mu_2)/T}\right), \tag{1.27}$$

where we used the fact that $E_1 + E_2 = E_3 + E_4$ by energy conservation, as we said. If we enforced the Saha condition, chemical equilibrium $\mu_1 + \mu_2 = \mu_3 + \mu_4$, then we get precisely zero: the number densities of the species are constant.

The mean number density of species i is given by

$$n_i = g_i e^{\mu_i/T}\int\frac{\mathrm{d}^3p}{(2\pi)^3}e^{-E_i/T}, \tag{1.28}$$

where g_i is the number of helicity states.

So, we find for the whole expression inside the brackets:

$$\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}}-\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}, \tag{1.29}$$

so we can define the time-averaged cross section

$$\langle\sigma v\rangle = \frac{1}{n_1^{(0)}n_2^{(0)}}\prod_i\int\frac{\mathrm{d}^3p}{2E_i}\dots, \tag{1.30}$$

so the final equation is

$$a^{-3}\frac{\mathrm{d}}{\mathrm{d}t}(n_1a^3) = \langle\sigma v\rangle\, n_1^{(0)}n_2^{(0)}\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}}-\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}, \tag{1.31}$$

and the left hand side is typically $\sim n_1/t \sim n_1H$. So, the combination on the RHS must be "squeezed to zero" eventually, which is equivalent to the Saha equation.

This is basically saying that we eventually reach chemical equilibrium.

1.1 Hydrogen recombination

The process is

$$e^- + p \leftrightarrow H + \gamma, \tag{1.32}$$

so the Saha equation yields

$$\frac{n_en_p}{n_H} = \frac{n_e^{(0)}n_p^{(0)}}{n_H^{(0)}}, \tag{1.33}$$

and charge neutrality implies $n_e = n_p$, *not* $n_e^{(0)} = n_p^{(0)}$.

At this stage in evolution, there are already some Helium nuclei, but we ignore them.

We define the ionization fraction

$$X = \frac{n_e}{n_e + H} \,. \tag{1.34}$$

This then yields

$$\frac{1 - X_e^n}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_e} \right)^{3/2} \exp(\epsilon_0 / T) \,, \tag{1.35}$$

where $\epsilon_0 = m_p + m_e - m_H = 13.6\,\mathrm{eV}$ is the ionization energy of Hydrogen.

Then, we get that the temperature of recombination is $T_{\mathrm{rec}} \approx 0.3\,\mathrm{eV}$.

The evolution of the ionization fraction is

$$\frac{\mathrm{d}X_e}{\mathrm{d}t} = (1 - X_e)\beta(T) - X_e^2 n_b \alpha^{(2)}(T) \,, \tag{1.36}$$

where we defined the ionization rate

$$\beta(T) = \langle \sigma v \rangle \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T} \,, \tag{1.37}$$

and the recombination rate $\alpha^{(2)} = \langle \sigma v \rangle$.

The value of this can be solved numerically: the difference between this and the Saha equation is not great in the prediction in the recombination redshift; however the prediction of the residual ionized hydrogen is different: there is much more than Saha would predict.

The universe gets reionized at $z \gtrsim 6$; this is still under discussion.

There are many ingredients in the interaction of the universe. We are interested in the photons: we want to predict the anisotropies in the CMB. There is a dipole due to the movement of the solar system through the CMB. Now, we want to see what our predictions are if we subtract this.

[Scheme of the interactions.]

The metric interacts with everything, photons interact with electrons through Compton scattering, electrons interact with protons through Coulomb scattering, dark energy, dark matter and neutrinos interact only with the metric.

Instead of Compton scattering, we use its nonrelativistic limit which applies here.

Scattering between electrons and protons is suppressed since protons are very massive. The other terms in the universe affect the geometry and we could see them through this.

There are models which include DM-DE coupling, and quintessence models, and models in which dark energy clusters.

We are not going to consider these.

We go back to first principles:

$$\hat{\mathbb{L}}[f] = \hat{\mathbb{C}}[f]\,, \tag{1.38}$$

where $f = f(x^\alpha, p^\alpha)$, however actually we do not have that much freedom in the phase space distribution. If there are no collisions: $\hat{\mathbb{L}}[f] = 0$, which is equivalent to

$$\frac{\mathcal{D}f}{\mathcal{D}\lambda} = 0\,, \tag{1.39}$$

where λ is the affine parameter.

In the nonrelativistic case,

$$\hat{\mathbb{L}} = \frac{\partial}{\partial t} + \dot{x} \cdot \nabla_x + \dot{v} \cdot \nabla_v = \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_x + \frac{F}{m} \cdot \nabla_v\,, \tag{1.40}$$

while in the GR case we need to account for the geodesic equation: and we write

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma\,, \tag{1.41}$$

where the affine parameter λ has the dimensions of a mass, in order to have dimensional consistency.

Then, the Liouville operator is

$$\hat{\mathbb{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \overset{\text{def}}{=} \frac{\mathcal{D}}{\mathcal{D}\lambda}\,. \tag{1.42}$$

This is a total derivative in phase space with respect to the affine parameter.

In the FLRW background, $f = f(|p|, t)$ and

$$\hat{\mathbb{L}} = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}|p|^2 \frac{\partial f}{\partial E}\,, \tag{1.43}$$

so if we define the number density

$$n(t) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, f(E, t)\,, \tag{1.44}$$

so if we integrate over momenta we get

$$\int \frac{\mathrm{d}^3 p}{E} \hat{\mathbb{L}}[f]\,, \tag{1.45}$$

we find the equation from before:

$$\dot{n} + 3\frac{\dot{a}}{a}n\,, \tag{1.46}$$

??? to check

Now we use a perturbed FLRW metric, in the Poisson gauge.

$$ds^2 = -e^{2\Phi}\,dt^2 + 2a\omega_i\,dx^i\,dt + a^2\left(e^{-2\Psi}\delta_{ij} + \chi_{ij}\,dx^i\,dx^j\right), \tag{1.47}$$

where we neglect spatial curvature (which we will do from now on). This is because we would never be able to see the effect of spatial curvature in the geometry (although we could see it in the dynamics).

Let us describe the quantities we introduced. We have 10 degrees of freedom in the metric. We account for them like this: Φ and Ψ are scalar, ω is a vector, χ_{ij} is a tensor. This is explained in more detail in the class by Nicola Bartolo ("early Universe"). These are GR perturbations.

"Perturbation" means that we compare the physical spacetime and the idealized FLRW metric. We need to do this since we cannot solve the EFE if there is no symmetry. So, we say that spacetime is *close* to the idealized spacetime.

We need a map between the physical and idealized spacetimes: this is called a *gauge*. Perturbations are classified with respect to their effect on FLRW. In euclidean space we know scalars, vectors, tensors. The perturbations will behave as such, under a change of coordinates in the Cartesian space which is the 3D space-like slice of FLRW.

ω_i carries a 3D vector index. χ_{ij} contains the off-diagonal perturbations. Let us start with ω_i. In principle: Helmholtz's theorem says that we can decompose $\omega_i = \omega_{i,\text{transverse}} + \partial_i\omega$. We say that the part we are interested in is the transverse one, but we still have a gradient.

We choose our χ such that $\chi^i_j = \chi^i_{j,i} = 0$. χ could also contain vectors (objects with a vector index), as long as they are divergenceless.

It can also contain tensors: these are GWs.

So, we have 3 scalars (a, Ψ, Φ), 3 components of a transverse vector (ω_i), plus the divergence ω, while in the tensor we have $6-3-1=2$ degrees of freedom.

So we have 10 total degrees of freedom. The reason we do this is that the degrees of freedom obey independent eqs. of motion.

There is gauge ambiguity in our problem: we could change the mapping between physical space and FLRW. We could do a change such as $x^\mu \to x'^\mu = x^\mu +$ a perturbation.

This is explained better in the notes called "GR perturbations".

Gauge freedom allows us to drop 2 of the 4 scalars we have.

We use Poisson or longitudinal gauge, which is sometimes incorrectly called "Newtonian gauge" even though it is not Newtonian.

Our next goal will be to solve the geodesic eqs. for the motion of the particles in this gauge.

Bibliography

[19] *Antisymmetric Tensor*. In: *Wikipedia*. Page Version ID: 926964086. Nov. 19, 2019. URL: https://en.wikipedia.org/w/index.php?title=Antisymmetric_tensor&oldid=926964086 (visited on 03/12/2020).

[Nat17] U. Natale. *Note Del Corso Di Cosmologia*. 2017.

[Tis19] Jacopo Tissino. "Relativistic Non-Ideal Flows". Laurea triennale. Sept. 9, 2019. URL: http://tesi.cab.unipd.it/63377/ (visited on 03/04/2020).

[Win96] K. Z. Win. "Ricci Tensor of Diagonal Metric". In: (Feb. 8, 1996). arXiv: gr-qc/9602015. URL: http://arxiv.org/abs/gr-qc/9602015 (visited on 03/12/2020).