Theoretical cosmology project

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Contents

0.1 Statistical methods in cosmology

From [Nat17, sec. 4].

Two point correlation function:

$$1 + \xi(r_{12}) = \frac{\mathrm{d}P}{\mathrm{d}P_{\text{indep}}} = \frac{\mathrm{d}P}{n^2 \,\mathrm{d}V_1 \,\mathrm{d}V_2} \,. \tag{0.1.1}$$

Fractal dimension! The number of galaxies within a radius R around a given one scales like $R^{3-\gamma}$.

Hierarchical models: N-point correlation functions can be calculated from the two-point one.

Bias model: $\delta_g = b\delta$ with constant b, where δ_g is the density perturbation for galaxies and δ is the one for dark matter.

Power spectrum definition, which by Wiener-Khinchin is the Fourier transform of the two-point correlation function. Expression for ξ in terms of P and Bessel functions as a single integral.

0.2 Path integral basics

Following [Zai83].

We start from the space of square-integrable functions $q(x)$, endowed with a product and an orthonormal basis ϕ_n. We consider (multi-)linear *functionals*, which are maps from the space of square-integrable functions (or from tuples of them) to $\mathbb{R}$ or $\mathbb{C}$. These can be represented as functions of infinitely many variables, countably so if we use the basis ϕ_n, uncountably so if we use the continuous basis x.

A functional $F[q]$ can be represented as a power series

$$F[q] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int \mathrm{d}x_i \, q(x_i) f(x_1, \dots, x_n) \,. \tag{0.2.1}$$

Examples of this are the exponential series corresponding to the function $f(x)$, mapping $q(x)$ to $e^{(f,q)}$ where the brackets denote the scalar product in the space, and the Gaussian series corresponding to the kernel $K(x,y)$, mapping $q(x)$ to $e^{(q,K,q)}$, where

$$(q,K,q) = \int \mathrm{d}x\,\mathrm{d}y\, q(x)q(y)K(x,y)\,. \tag{0.2.2}$$

Functional derivatives describes how the output of the functional changes as the argument goes from $q(x)$ to $q(x)+\eta(x)$, where $\eta(x)$ is small. This will be a linear functional of η to first order, so we define the functional derivative with the expression

$$F[q+\eta] - F[q]\Big|_{\text{linear order}} = \int \eta(y)\frac{\delta F}{\delta q(y)}\,\mathrm{d}y\ . \tag{0.2.3}$$

The analogy to finite-dimensional spaces is as follows: the functional derivative $\delta F/\delta q(y)$ corresponds to the *gradient* $\nabla^i F$, while the integral in the previous expression corresponds to the *directional derivative* $(\nabla^i F)\eta^j g_{ij}$. The metric is present since the gradient is conventionally defined with a vector-like upper index; in our infinite-dimensional space the scalar product is given by the integral.

Practically speaking, the most convenient way to calculate a functional derivative is by taking $\eta(x)$ to be such that it only differs from zero in a small region near y, and let us define

$$\delta\omega = \int \eta(x)\,\mathrm{d}x\ . \tag{0.2.4}$$

Then, we define

$$\frac{\delta F}{\delta q(y)} = \lim_{\delta\omega\to 0}\frac{F[q+\eta]-F[q]}{\delta\omega}\,. \tag{0.2.5}$$

In order for the limit to be computed easily, it is convenient for $\eta(x)$ to be in the form $\delta\omega \times$ fixed function, so that we are only changing the normalization as we shrink $\delta\omega$. A common choice is then

$$\eta(x) = \delta\omega\delta(x-y)\,. \tag{0.2.6}$$

If we apply this procedure to the identity functional $q \to q$, we find

$$\frac{\delta q(x)}{\delta q(y)} = \lim_{\delta\omega\to 0}\frac{q(x)+\delta\omega\delta(x-y)-q(x)}{\delta\omega} = \delta(x-y)\,. \tag{0.2.7}$$

The variable q is one-dimensional, if instead we wanted to consider a multi-dimensional coordinate system q_α by the same reasoning we would find

$$\frac{\delta q_\alpha(x)}{\delta q_\beta(y)} = \delta_{\alpha\beta}\delta(x-y)\,. \tag{0.2.8}$$

An example: the functional derivative of a functional F_n defined by

$$F_n[q] = \int f(x_1,\dots,x_n)q(x_1)\dots q(x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n\ , \tag{0.2.9}$$

where f is a symmetric function of its arguments, is given by

$$\frac{\delta F_n}{\delta q(y)} = n \int f(x_1, \dots, x_{n-1}, y) q(x_1) \dots q(x_{n-1}) \, \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \,, \tag{0.2.10}$$

a function of y.

A **linear transformation** is in the form

$$q(x) = \int K(x, y) q'(y) \, \mathrm{d}y \,. \tag{0.2.11}$$

If this transformation has an inverse, which is characterized by the kernel K^{-1}, then we must have the orthonormality relation

$$\int K(x, y) K^{-1}(y, z) \, \mathrm{d}y = \int K^{-1}(x, y) K(y, z) \, \mathrm{d}y = \delta(x - z) \,. \tag{0.2.12}$$

We can do **Legendre transforms**: if we have a functional F we can differentiate with respect to the coordinate q to find

$$\frac{\delta F[q]}{\delta q(x)} = p(x) \,, \tag{0.2.13}$$

in analogy to the momentum in Lagrangian mechanics. Then, we can map $F[q]$ to a new functional $G[p]$ which will only depend on the momentum:

$$G[p] = F[p] - \int q(x) p(x) \, \mathrm{d}x \,. \tag{0.2.14}$$

We can also define functional integration, by

$$\int F[q] \, [\mathrm{d}q] = \int \hat{F}(\{q_i\}) \prod_i \mathrm{d}q_i \,. \tag{0.2.15}$$

On the right-hand side we are using the expression of the functional as a function of infinitely many variables which we discussed above; we are then integrating over each of the coordinates in this infinite dimensional function space. The infinite-dimensional measure is also often denoted as $\mathcal{D}q$.

This integral will not always exist, however in the cases in which it does we can change variables. Let us consider a linear change of variable, whose kernel is $K(x, y)$, such that (compactly written) $q = Kq'$.

Then, we want to compute the integral

$$\int F[Kq'] \, [\mathrm{d}Kq'] \tag{0.2.16}$$

as an integral in $[\mathrm{d}q]$: in order to do so, we need to relate the two functional measures. We start by expressing both q and q' in terms of an orthonormal basis ϕ_i: inserting this into the linear transformation law we get

$$q(x) = \int K(x, y) q'(y) \, \mathrm{d}y \tag{0.2.17}$$

$$\sum_i q_i \phi_i(x) = \int K(x,y) \sum_j q_j' \phi_j(y)\, \mathrm{d}y \tag{0.2.18}$$

$$\sum_i q_i \underbrace{\int \phi_i(x)\phi_k(x)\, \mathrm{d}x}_{\delta_{ik}} = \sum_j q_j' \underbrace{\int K(x,y)\phi_j(y)\phi_k(x)\, \mathrm{d}y\, \mathrm{d}x}_{k_{jk}} \tag{0.2.19}$$

$$q_k = \sum_j q_j' k_{jk}\,. \tag{0.2.20}$$

Then, the measure will transform with the determinant $\det K = \det k$, which we can now express as an infinite product of the eigenvalues of k:

$$[\mathrm{d}q] = \left|\frac{\partial q}{\partial q'}\right| [\mathrm{d}q'] = \det K [\mathrm{d}q']\,. \tag{0.2.21}$$

Usually functional integrals cannot be computed analytically; the exception is given by Gaussian integrals, which generalize the finite-dimensional result

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} A_{ij} x_i x_j + i b_j x_j\right) \mathrm{d}x_1 \dots \mathrm{d}x_j = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right). \tag{0.2.22}$$

Here A_{ij} is an n-dimensional real matrix (which WLOG can be taken to be symmetric) while b_i is an n-dimensional vector. The result comes from a transformation of the coordinates according to the finite-dimensional

This can be interpreted as a "functional" (still finite-dimensional, so just a function, but we will generalize soon) of b_i; we write it with an additional normalization N for convenience:

$$Z[b] = N \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} A_{ij} x_i x_j + b_j x_j\right) \mathrm{d}x_1 \dots \mathrm{d}x_j = N \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right), \tag{0.2.23}$$

and if we rescale the normalization N so that $Z[\vec{0}] = 1$ we get

$$Z[b] = \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right). \tag{0.2.24}$$

The infinite-dimensional generalization of this result amounts to replacing all the sums (expressed implicitly with Einstein notation here) with integrals; also conventionally we change the names of the variables to $x \to q$, $A \to K$, $b \to J$:

$$Z[J] = N \int \mathcal{D}q \exp\left(-\frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, K(x,y) q(x) q(y) + i \int \mathrm{d}x\, q(x) J(x)\right) \tag{0.2.25}$$

$$= \exp\left(-\frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, J(x) J(y) K^{-1}(x,y)\right). \tag{0.2.26}$$

Let us now give some examples of applications of this result: $K(x,y) = \sigma^{-2}\delta(x-y)$ means $K^{-1}(x,y) = \sigma^2 \delta(x-y)$, so

$$Z[J] = \exp\left(-\frac{\sigma^2}{2} \int \mathrm{d}x\, J^2(x)\right). \tag{0.2.27}$$

This, as we shall see, can be used to give us a description of white noise, which is uncorrelated in momentum space.

Let us consider another example, whose physical application is to describe the motion of a massive scalar boson with Lagrangian

$$\mathscr{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2}_{\mathscr{L}_0} + \mathscr{L}_I(\phi)\,, \tag{0.2.28}$$

where the self-interaction term is some non-quadratic function of ϕ, often taken to be proportional to ϕ^3 or ϕ^4.

The Feynman path integral corresponding to this Lagrangian is given by the functional

$$Z[J] = N\int \mathcal{D}\phi \exp\left(i\int \mathscr{L}(\phi) + J\phi\,\mathrm{d}x\right). \tag{0.2.29}$$

Let us start with the non-interacting case, that is, we compute Z_0 with only the quadratic term in the Lagrangian. This can be expressed, in the formalism from before, using the kernel

$$K(x,y) = (-\Box_x - \mu^2)\delta(x-y)\,. \tag{0.2.30}$$

Now, the expression the functional is given in terms of K^{-1}: what is the inverse of this kernel? The definition reduces to

$$\int K(x,y)K^{-1}(y,z)\,\mathrm{d}y = \delta(x-z) \tag{0.2.31}$$

$$-\left(\Box_x + \mu^2\right)K^{-1}(x,z) = \delta(x-z)\,, \tag{0.2.32}$$

which is readily solved in momentum space, with a $+i\epsilon$ prescription in order to avoid the pole in the integration: what we find is called the *Green's function*,

$$K^{-1}(x,z) = G(x-z) = \frac{1}{(2\pi)^4}\int \frac{e^{-ik\cdot(x-z)}}{k^2+\mu^2-i\epsilon}\,\mathrm{d}k\,, \tag{0.2.33}$$

so the unperturbed functional reads

$$Z_0[J] = \exp\left(-\frac{i}{2}\int \mathrm{d}x\,\mathrm{d}y\,G(x-y)J(x)J(y)\right). \tag{0.2.34}$$

This by itself might not seem very useful, the motion of a free massive boson can be calculated with easier methods. However, the real power of this path integral is the possibility to write the interacting term perturbatively: the interaction Lagrangian is a function of ϕ, which is what we find if we perform a functional integration of the argument of the exponential in $Z_0[J]$ with respect to J; so we can express the full functional as

$$Z[J] = \exp\left(i\int \mathrm{d}x\,\mathscr{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right)\underbrace{\int \mathcal{D}\phi\exp\left(i\int \mathrm{d}x\,\left(\mathscr{L}_0 + J\phi\right)\right)}_{=Z_0[J]} \tag{0.2.35}$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\int \mathrm{d}x \, \mathscr{L}_I \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right]^n Z_0[J] \,. \tag{0.2.36}$$

We can use this to compute the Green's functions:

$$G(x_1, \dots, x_n) = \frac{1}{i^n} \frac{\delta Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \bigg|_{J=0} \,. \tag{0.2.37}$$

Review from the PI notes why this makes sense.

0.2.1 The probability density functional

We can interpret the quantity

$$\exp\left(-\frac{1}{2} (q, K, q) \right) \mathcal{D}q \tag{0.2.38}$$

as a *probability density functional* $\mathrm{d}P\,[q]$, since

1. it is positive definite;
2. it is normalized, as long as we set its integral, $Z[0]$, to 1;
3. it goes to zero as $q \to \pm\infty$.

If this is the case, then we ought to be able to compute the average value of a functional $F[q]$ as

$$\langle F[q] \rangle = \int F[q] \, \mathrm{d}P\,[q] = \int \mathcal{D}q \exp\left(-\frac{1}{2} (q, K, q) \right) F[q] \,, \tag{0.2.39}$$

which we can generalize to any non-gaussian probability density functional by replacing the exponential $\exp\left(-\frac{1}{2}(q,K,q)\right)$ with a generic $\mathcal{P}[q]$.

A useful kind of average we can compute is given by the N-point correlation function,

$$C^{(N)}(x_1, \dots, x_n) = \langle q(x_1) \dots q(x_n) \rangle \,. \tag{0.2.40}$$

With the formula we gave earlier, this can be computed as

$$C^{(N)}(x_1, \dots, x_n) = \int \mathcal{D}q \mathcal{P}[q] \prod_i q(x_i) \,. \tag{0.2.41}$$

Here we can make use of a trick: going back to the Gaussian probability case, consider the functional derivative

$$\frac{1}{i} \frac{\delta Z[J]}{\delta J(x_1)} \bigg|_{J=0} = \frac{1}{i} \left. \frac{\delta}{\delta J(x)} \right|_{J=0} \int \mathcal{D}q \exp\left(-\frac{1}{2} (q, K, q) + i (J, q) \right) \tag{0.2.42}$$

$$= \int \mathcal{D}q \exp\left(-\frac{1}{2}(q, K, q)\right) q(x_1) = \langle q(x_1) \rangle = C^{(1)}(x_1)\,, \tag{0.2.43}$$

which actually holds for any probability density functional, we did not make use of the gaussianity. So, in general we will be able to write

$$C^{(N)}(x_1 \dots x_n) = \frac{1}{i^N} \left.\frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)}\right|_{J=0}\,. \tag{0.2.44}$$

The correlation functions, which as we discussed in an earlier section are crucial when discussing structure formation, can be "simply" calculated by functional differentiation as long as we have the generating functional $Z[J]$. This generating functional is very similar mathematically to a partition function in statistical mechanics, and it serves an analogous role: its derivatives allow us to characterize the dynamics of the system.

Now, any functional $\mathscr{F}[q]$ can be expressed through a functional Taylor series:

$$\mathscr{F}[q] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left.\frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)}\right|_{q=0} q(x_1) \dots q(x_n)\,, \tag{0.2.45}$$

so if we compute the average value $\langle \mathscr{F}[q] \rangle$ we find

$$\langle \mathscr{F}[q] \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left.\frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)}\right|_{q=0} \underbrace{\langle q(x_1) \dots q(x_n) \rangle}_{=C^{(N)}(x_1 \dots x_n)} \tag{0.2.46}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left.\frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)}\right|_{q=0} \left.\frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)}\right|_{J=0} \tag{0.2.47}$$

$$= \mathscr{F}\left[-i\frac{\delta}{\delta J}\right] Z[J]\Big|_{J=0}\,. \tag{0.2.48}$$

Formally this makes sense, but what does it mean to calculate the field at a derivation operator? Should this just be interpreted as a shorthand for the Taylor expansion or is there more to it?

An example: consider a Gaussian field whose partition function $Z[J]$ is given by

$$Z[J] = \exp\left(-\frac{1}{2}(J, K^{-1}, J)\right)\,. \tag{0.2.49}$$

Then, as before we can calculate the correlation functions through functional derivatives: the first ones are

$$\langle q(x) \rangle = \frac{1}{i} \left.\frac{\delta Z[J]}{\delta J(x)}\right|_{J=0} \tag{0.2.50}$$

$$= -i \int \mathrm{d}y\, K^{-1}(x, y) J(y) \exp\left(-\frac{1}{2}(J, K^{-1}, J)\right)\Big|_{J=0} = 0 \tag{0.2.51}$$

$$\langle q(y)q(x)\rangle = -\left.\frac{\delta^2 Z[J]}{\delta J(y)\delta J(x)}\right|_{J=0} \tag{0.2.52}$$

$$= -\left(-K^{-1}(x,y) + \int \mathrm{d}z\,\mathrm{d}w\, K^{-1}(x,z)K^{-1}(y,w)J(w)\right)\exp\left(-\frac{1}{2}(J,K^{-1},J)\right)\Bigg|_{J=0} \tag{0.2.53}$$

$$= K^{-1}(x,y)\,. \tag{0.2.54}$$

So, we have our result: *for a Gaussian variable, the two-point correlation function is the inverse of the kernel*. A similar, albeit quite long, calculation allows us to compute the N-point correlation function for the same Gaussian variable: we expand the exponential in $Z[J]$ in a power series, and when we differentiate it an even number of times we find

$$C^{2N}(x_1\dots x_{2N}) = \left[K^{-1}(x_1,x_2)K^{-1}(x_3,x_4)\dots K^{-1}(x_{2N-1},x_{2N})\right]_{\text{symmetrized}}, \tag{0.2.55}$$

where "symmetrized" means that we must sum over all the permutations of the variables x_i in the argument of the inverse kernels; on the other hand, the odd correlation functions C^{2N+1} all vanish since they correspond to the integrals of odd functions over all space.

We can also apply this process in reverse: starting from the two-point correlation function we can reconstruct the kernel, and with it the probability density functional $\mathrm{d}P\,[q]$.

In the Gaussian case, as long as we know the two-point function, which corresponds to the inverse kernel, we can reconstruct the N-point function. This can also be stated by saying that the "irreducible" N-point functions are all zero except for $N=2$, since all the higher ones can be reduced to that one.

We shall see that all the reduced N-point functions can be recovered starting from the *generating functional* of connected correlation functions:

$$\mathscr{W}[J] = \log Z[J]\,, \tag{0.2.56}$$

through the expansion

$$\mathscr{W}[J] = \sum_{n=1}^{\infty}\frac{i^n}{n!}\int \mathrm{d}x_1\dots\mathrm{d}x_n\, C_C^N(x_1\dots x_N)J(x_1)\dots J(x_n)\,. \tag{0.2.57}$$

These connected correlation functions C_C^N are not (in principle) related to the C^N from before.

What is the relation between them, though?

We could also have defined $\mathscr{W}[J] = i\log Z[J]$, this is a matter of convention.

Now, we define the *classical field*

$$q_{\mathrm{cl}}(x) = \frac{\delta\mathscr{W}[J]}{\delta J(x)}\,, \tag{0.2.58}$$

and the effective action $\Gamma[q_{\mathrm{cl}}]$ as the Legendre transform of $\mathscr{W}[J]$:

$$\Gamma[q_{\mathrm{cl}}] = \mathscr{W}[J] - \int \mathrm{d}x\, q_{\mathrm{cl}}(x)J(x)\,, \tag{0.2.59}$$

from which we can then recover $J(x)$ as

$$J(x) = -\frac{\delta\Gamma[q_{\text{cl}}]}{\delta q_{\text{cl}}(x)}\,. \tag{0.2.60}$$

In the Gaussian case we have

$$\mathscr{W}[J] = \log Z[J] = -\frac{1}{2}(J, K^{-1}, J)\,, \tag{0.2.61}$$

which, by direct comparison with the Taylor expansion, means that

$$C_C^2(x_1, x_2) = K^{-1}(x_1, x_2)\,, \tag{0.2.62}$$

while $C^N \equiv 0$ for any $N \neq 2$. Also, our expression for q_{cl} yields

$$q_{\text{cl}}(x) = -\int \mathrm{d}y\, K^{-1}(x, y) J(y)\,, \tag{0.2.63}$$

from which we can express $J(x)$ by using the direct kernel $K(x, y)$:

$$\int \mathrm{d}x\, q_{\text{cl}}(x) K(x, w) = -\int \mathrm{d}y\, \mathrm{d}w\, K^{-1}(x, y) K(x, w) J(y) = -\int \mathrm{d}w\, \delta(y - w) J(y) = -J(w)\,. \tag{0.2.64}$$

With an analogous procedure we can show that

$$(J, K^{-1}, J) = (q_{\text{cl}}, K, q_{\text{cl}})\,. \tag{0.2.65}$$

The effective action then reads

$$\Gamma[q_{\text{cl}}] = -\frac{1}{2}(J, K^{-1}, J) + (J, K^{-1}, J)\Big|_{J = J(q_{\text{cl}})} \tag{0.2.66}$$

$$= \frac{1}{2}(q_{\text{cl}}, K, q_{\text{cl}})\,. \tag{0.2.67}$$

Now we have the tools to consider actual probabilities: starting from our classical field q, we want to compute the probability that it takes on a certain value $q \in (\alpha, \alpha + \mathrm{d}\alpha)$ at a point x: this is expressed with a probability density function in the form

$$\frac{\mathrm{d}P_q}{\mathrm{d}\alpha} = P_{q(x)}(\alpha; x)\,. \tag{0.2.68}$$

We want to write this "$P(\alpha)\,\mathrm{d}\alpha$" in terms of the functional integral; in order to do so, we start from the Fourier transform

$$\int \mathrm{d}\beta \exp\big(i\beta\varphi\big) P_q(\beta; x) = \left\langle e^{i\beta\varphi} \right\rangle_\beta\,. \tag{0.2.69}$$

0.3 Applications

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[Zai83] M. H. Zaidi. "Functional Methods". In: *Fortschritte der Physik/Progress of Physics* 31.7 (1983), pp. 403–418. ISSN: 1521-3978. DOI: 10.1002/prop.2190310703. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/prop.2190310703 (visited on 2020-10-04).