Astroparticle physics notes

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Contents

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Introduction

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There are two courses, for Astrophysics and Cosmology and for Physics, which bear the same name. The other one is by Francesco D'Eramo: it assumes a knowledge of Quantum Field Theory.

This course, instead, only requires knowledge of Quantum Mechanics. The first part of this course is devoted to an introduction about the basics of Quantum Field Theory and gauge theories.

"By the end of the 20th century [...] we have a comprehensive, fundamental theory of all observed forces of nature which has been tested and might be valid from the Planck length scale of 10^{-33} cm to the edge of the universe 10^{28} cm". David Gross, 2007.

The task in APP is to be able to discuss such a fundamental theory.

First of all, we need to address the two standard models: the ΛCDM model for cosmology and the Standard Model of particle physics.

There are points of friction between the two Standard Models. There are also several questions: neutrinos' mass, what caused inflation...

These problems have a common denominator: the interplay between particle physics, cosmology and astrophysics. What we seek is new physics, beyond the two standard models.

Books: M.E. Peskin, "Concepts of Elementary Particle Physics". The book is addressed to students who are not experts in QFT and particle physics, rather, it provides the fundamental knowledge for these topics.

The exam is a colloquium, an oral exam, for which we can prepare a presentation on a specific topic. There is no issue if we do not precisely remember a specific formula, it is about going deep in the concepts.

An overview of the astroparticle physics landscape

Fundamental particles: the SM of particle physics Elementary particles make up ordinary matter. Fermions have spin $1/2$, and are composed of quarks:

$$\begin{bmatrix} u & c & t \\ d & s & b \end{bmatrix}, \tag{1a}$$

leptons:

$$\begin{bmatrix} \nu_e & \nu_\mu & \nu_\tau \\ e & \mu & \tau \end{bmatrix}. \tag{2a}$$

The muon and tau particles are similar to electrons, but with higher mass.

These particles' interactions are mediated by 12 vector bosons, which have spin 1: these are "radiation" (the term is outdated).

1. gluons (g) mediate the strong nuclear interaction, there are 8 of them;
2. the $W^\pm$ and Z^0 bosons mediate the weak interaction;
3. the photon (γ) mediates the electromagnetic interaction.

There was a need for a mechanism to provide mass to the weak bosons and the fermions: this is accounted for by the Higgs boson, which is a scalar (that is, it has spin 0). This realizes the electroweak symmetry breaking.

The issue is that gravity is missing. In order to describe it in this scheme we would need a way to quantize it: all of these particles are actually excitations of quantum fields.

There are two marvelous 20th century theories, but they are not compatible.

Unification of interactions In 1687 Newton unified two domains of interactions: the terrestrial phenomena and the celestial phenomena, establishing the universality of gravitational interactions.

In 1865 Sir Maxwell unified electricity and magnetism into electromagnetism.

In 1967 Glashow, Weinberg and Salam propose the Standard Model of Particle Physics, unifying the Electromagnetic and Weak interactions. This is not a true unification: it is more appropriate to say that they are "mixed together" into the Electroweak interaction.

In the Standard Model, there is a kind of "frontier" around 100 GeV: below this energy, we see two interactions: the electromagnetic and the weak interaction. They are very much different: photons are massless, so the interaction has an infinite range, while the weak bosons are massive.

How can these be unified? We will see; above 100 GeV this apparent profound difference disappears in favour of the electroweak interaction. This is a phase transition.

Is the electroweak interaction above 100 GeV massless or not? Above this energy there is still a difference between the coupling constants of the two interactions. Above this energy, the W and Z bosons are no longer massive.

Above 100 GeV the strong interaction is separated from the electroweak one. Maybe there is an energy at which the electroweak interaction is unified with the strong one? We shall explore this topic: there are theories (Grand Unified Theories, GUT) in which there is such a unification.

As the energy increases, the coupling of the strongest interactions becomes weaker.

The energy scale, however, is very large: around 10^{16} GeV: this is a "science fiction" energy scale, it is extremely large. This is close to the Planck mass: $M_P \sim 10^{19}$ GeV, so we might not be able to describe this energy range with vanilla SM.

The Standard Model of Cosmology Now we can work backwards in our energy scale: as time progresses forward from the Big Bang, the energy of particles decreases.

The symmetry group of the Grand Unified Theory is broken, so we get subgroups; at each transition some symmetry is broken.

The EM + weak into electroweak transition is not speculative: we have observed it at the LHC. On the other hand, the electroweak + strong into GUT transition is speculative.

When, in the expansion of the universe, we reach an energy per particle of ≈ 1 GeV we have a new transition: the quark-hadron transition, so free quarks become confined into hadrons such as protons and neutrons.

Around 1 MeV we have a new transition: nucleosynthesis.

Then, we reach recombination, which is when the radiation we see as the CMB is released.

There must be new physics somewhere: there is no room in the SM for dark matter, the matter-antimatter asymmetry, the mass of neutrinos.

Chapter 1

The particle physics Standard Model

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For this lecture, a good reference is Peskin, chapter 2 [Pes19].

We wish to describe what we described yesterday as "matter" and "radiation".

The problem is similar to the one we have in classical mechanics, an initial value problem: given the positions and velocities of the particles at a certain starting time t_0 we wish to compute their state at a later time t.

This classical description in which the particles are not wavelike fails at the microscopic level: we want to give a quantum description of such a system of particles. We will derive it from the classical description using the standard tools of quantization. We start with a refresher of the classical description.

1.0.1 The classical description of a system of particles

Our aim is to compute and solve the equations of motion. The usual approach is to use Hamilton's variational principle: it is the principle of least action, but it is not usually referred to as such: we are actually not *minimizing* the action but finding a *stationary point* for it. This could also be a maximum or a saddle point.

The action functional S depends on the coordinates $q_i(t)$ of the particles at at time t, on the derivatives of these positions $\dot{q}_i(t)$ which represent the velocities of the particles at a time t. We usually write $S\big[q_i(t), \dot{q}_i(t)\big]$.

If we fix $q(t_0)$ and $q(t_f)$, the positions at some initial and final time t_f, we can then trace out a path $q(t)$ and perturb it by $\delta q(t)$; we fix $\delta q(t_0) = \delta q(t_f) = 0$.

Under this perturbation of the path $q \to q + \delta q$, the action changes to $S \to S + \delta S$. We then ask that $\delta S = 0$.

S is an action: is dimensions are those of an energy times a time. In terms of the Lagrangian L, the action is defined as

$$S\big[q_i(t), \dot{q}_i(t)\big] = \int_{t_0}^{t_f} L(q_i(t), \dot{q}_i(t))\, \mathrm{d}t \ , \tag{1.1}$$

which means that the Lagrangian must have the dimensions of an energy. We will make use of a quantity called the Lagrangian density:

$$L = \int \mathscr{L}\Big(\phi(\vec{x}), \partial_\mu \phi(\vec{x})\Big)\, \mathrm{d}^3x \ . \tag{1.2}$$

From a finite number of particles we move to considering a field $\phi(\vec{x})$: this means that, in a certain sense, we are considering an infinite number of particles.

The dependence of the Lagrangian on the q_i and $\dot{q}_i$ shifted to a dependence on the spacetime coordinates x and their 4-derivatives $\partial_\mu x$. It could depend on many fields simultaneously, we omit this dependence for simplicity. Now, this Lagrangian density has the dimensions of an energy per unit volume.

Then, the action, computed in a region Ω of 4-dimensional spacetime, is

$$S = \int_\Omega \mathrm{d}^4x \, \mathscr{L}\Big(\phi(x), \partial_\mu(\phi(x))\Big) \,. \tag{1.3}$$

Now that we have established the notation, we can apply the action principle: we consider an infinitesimal variation of the field $\phi \to \phi + \delta\phi$. We require this variation to vanish not only at the initial and final time, but over all the boundary $\partial\Omega$:

$$\delta\phi\Big|_{\partial\Omega} = 0 \,. \tag{1.4}$$

Then, imposing $\delta S = 0$ is equivalent to the Euler-Lagrange equations:

$$\frac{\partial\mathscr{L}}{\partial\phi(x)} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}}{\partial\Big(\partial_\mu\phi(x)\Big)}\right) = 0 \,. \tag{1.5}$$

If we have many fields ϕ_r, then we have a set of E-L equations for each of them. This is still classical: for example, classical (relativistic) electrodynamics is formulated in this way.

The momenta are

$$\pi(x) = \frac{\partial\mathscr{L}}{\partial\dot{\phi}(x)} \,, \tag{1.6}$$

where a dot denotes a time derivative and using this we define the Hamiltonian density by

$$\mathscr{H}(x) = \pi(x)\dot{\phi}(x) - \mathscr{L}(\phi, \partial_\mu\phi) \,, \tag{1.7}$$

and similarly to the Lagrangian we have the full Hamiltonian H

$$H = \int \mathrm{d}^3x \, \mathscr{H} \,. \tag{1.8}$$

Now, these fields (ϕ and π) are classical fields: we will apply classical quantization to them. They will need to satisfy the classical commutation relations:

$$\big[\phi(\vec{x},t), \pi(\vec{x}',t)\big] = i\hbar\delta(\vec{x}-\vec{x}') \tag{1.9a}$$

$$\big[\phi(\vec{x},t), \phi(\vec{x}',t)\big] = \big[\pi(\vec{x},t), \pi(\vec{x}',t)\big] = 0 \,. \tag{1.9b}$$

1.0.2 A relativistic reminder

the energy-momentum four-vector is

$$p^\mu = \begin{bmatrix} E \\ \vec{p}c \end{bmatrix}, \tag{1.10a}$$

where the greek index μ can take values from 0 to 3.

The metric signature used here is the the mostly minus one. So, $p^\mu q_\mu = E_p E_q - \vec{p} \cdot \vec{q}$, since we raise and lower indices using the metric $\eta_{\mu\nu}$.

The square norm of the 4-momentum is $p \cdot p = p^2 = E^2 - \left|\vec{p}\right|^2 c^2$. It is Lorentz invariant.

In the rest frame of the observer, $p^\mu = \left[E_0, \vec{0}\right]$, and this E_0 is just (c^2 times) the mass of the particle: this is the *definition* of mass.

When the relation is satisfied we have

$$p^2 = E^2 - \left|p\right|^2 c^2 = \left(mc^2\right)^2 \tag{1.11a}$$

$$E = c\sqrt{\left|p\right|^2 + (mc)^2}\,. \tag{1.11b}$$

When this relation is satisfied we say we are *on shell*: for virtual particles, instead, this might not be satisfied.

We will use natural units: $\hbar = c = 1$.

This means that we equate energies (eV) and angular velocities (Hz); also we equate times (s) and lengths (m).

The rest energy of the electron is $m_e \approx 511\,\text{keV}$. Let us consider an electron with a momentum p equal to its mass m_e: then, its uncertainty in position is of the order

$$\frac{\hbar}{pc} = \frac{1}{m_e} \approx 4 \times 10^{-11}\,\text{cm}\,. \tag{1.12}$$

The dimensions of the lagrangian density, in natural units, are those of an energy to the fourth power, or a length to the -4, or a mass to the fourth.

Another useful exercise is to calculate the coupling of the electromagnetic field:

$$V(r) = \frac{e^2}{4\pi\epsilon_0 r} = \frac{e^2}{4\pi}\frac{1}{r}\,, \tag{1.13}$$

since we set $\epsilon_0 = \mu_0 = 1$. We can introduce the electromagnetic α: this is

$$\alpha = \frac{e^2}{4\pi} \times \frac{1}{\hbar c}\,. \tag{1.14}$$

This then becomes adimensional: $\alpha \approx 1/137$. It represents the strength of the electromagnetic interaction: the strength of the coupling of the photon to the electron. The fact that it is $\sim 10^{-2}$ is important: it allows us to work in a perturbative way, in powers of α.

What is the coupling of the strong and weak interactions? This will be discussed.

Next time, we will discuss symmetries and symmetry breaking.

1.1 Symmetries and conservation laws

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Our aim is to describe the fundamental constituents of matter with a Quantum Field Theory. The method used to derive the equations of motion is a variational principle: we will find a Lagrangian density for various particles, and then apply the variational principle to find their equations of motion.

A guiding principle on the description of these fundamental particles is based on using their symmetries. We have Nöether's theorem in Quantum Field Theory: from these symmetries we are able to find conserved quantities.

These symmetries are described with groups, since we can compose their application; the theory describing groups is very rich. For this lecture we will base ourselves on Peskin's chapter 2 [Pes19].

A group G is a set of elements endowed with an operation. The set of elements can be either discrete or continuous. Most of the time, in theoretical physics, we use continuous groups.

We distinguish:

1. **spacetime** symmetries: groups which transform our coordinate system for spacetime, such as Lorentz and Poincaré transformations;
2. **internal** symmetries: groups which transform a certain field, or a certain property of our quantum system.

For our set to be a group, we need to be able to define an operation — we will usually call it multiplication — between the elements of the group, such that if $a, b \in G$ then $ab \in G$. Also, we must have

1. associativity: $(ab)c = a(bc)$;
2. existence of the identity $\mathbb{1}$, such that $\mathbb{1}a = a\mathbb{1} = a$;
3. existence of inverses: there exists a^{-1} such that $aa^{-1} = a^{-1}a = \mathbb{1}$.

What is of interest to us is the association of the group with a transformation which is a symmetry: this is called a *representation*.

We call a transformation a symmetry if, after performing the transformation, the dynamics of the system do not change.

For a quantum mechanical system, we are interested in the observables: these are described by operators, whose eigenvalues are the observations we make, and which in the Heisenberg picture evolve like

$$\frac{\mathrm{d}}{\mathrm{d}t}O(t) = [H, O(t)]\,; \tag{1.15}$$

if an operator commutes with the Hamiltonian, $[H, O] = 0$, then the operator is constant.

If we perform a transformation in the form

$$|\psi\rangle \to |\psi'\rangle = U\,|\psi\rangle\ , \tag{1.16}$$

then the operators will change by

$$O \to O' = U^\dagger O U \,. \tag{1.17}$$

Note that whether we have $U^\dagger O U$ or $U O U^\dagger$ does not matter, since we ask observables O to be Hermitian, so $O = O^\dagger$.

We know that these transformations must always be unitary, because the conservation of probability implies that we must have $\langle \psi | \psi \rangle = \text{const}$: so,

$$U^\dagger U = \mathbb{1} \,. \tag{1.18}$$

This can be also stated as $U^\dagger = U^{-1}$.

So, the function associating a unitary operator U to an element g of the group is called its *unitary representation*.

A transformation G is a symmetry if $\forall a \in G$ we have

$$[U(a), H] = 0 \,, \tag{1.19}$$

that is, the unitary representation of the group element always commutes with the Hamiltonian.

If we have a state $|\psi\rangle$ with energy $H|\psi\rangle = E|\psi\rangle$, then the transformation commuting with the Hamiltonian means that $|\psi'\rangle = U|\psi\rangle$ has the same energy.

Now, we can move to an example, taken from Peskin [Pes19]. Consider the discrete group $\mathbb{Z}_2$, which only has the elements 1 and -1, with the same multiplication rules as those we would have if these elements were integers. So, the group is closed with respect to multiplication. It can be easily checked that this is indeed a group based on our definition.

In order for this to be of interest to us, we can consider a quantum mechanical system and find a unitary representation acting on its Hilbert space.

Let us suppose we have a QM system with a basis made of two states $|\pi^+\rangle$ and $|\pi^-\rangle$. Let us define the *charge conjugation* operator C, by:

$$C\left|\pi^+\right\rangle = \left|\pi^-\right\rangle \qquad \text{and} \qquad C\left|\pi^-\right\rangle = \left|\pi^+\right\rangle \,. \tag{1.20}$$

So, we can find a unitary representation of $\mathbb{Z}_2$ in this system: we need to define $U(1)$ and $U(-1)$. We define

$$U(1) = \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad U(-1) = \sigma_x - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \,, \tag{1.21a}$$

where the matrices are to be interpreted as acting on vectors expressed to the basis $\{|\pi_+\rangle, |\pi_-\rangle\}$.

So, we can say that our unitary representation looks like

$$\mathbb{Z}_2 \to \{\mathbb{1}, C\} \,. \tag{1.22}$$

Now, if $[C, H] = 0$ (and $\mathbb{1}$ commutes with H, which is always the case) then we say that "H has the symmetry $\mathbb{Z}_2$".

The interesting question to determine will be whether this is actually the case for our given group.

Groups can be subdivided into abelian and non-abelian ones. A group is abelian if for every a, b in G we have $ab = ba$, or equivalently, $[a, b] = 0$. It is not if this is not the case, that is, there exist a, b such that $ab \neq ba$.

The condition on the elements directly translates to a condition on the matrices of the unitary representation. If we have commuting matrices, we can simultaneously diagonalize them: for example, in the case of $\mathbb{Z}_2$ we can go to a basis in which

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{1.23a}$$

specifically the states need to be

$$|\pi_1\rangle = \frac{|\pi^+\rangle + |\pi^-\rangle}{\sqrt{2}} \qquad \text{and} \qquad |\pi_2\rangle = \frac{|\pi^+\rangle - |\pi^-\rangle}{\sqrt{2}}, \tag{1.24}$$

since then $C|\pi_1\rangle = |\pi_1\rangle$ (we write $C = +1$) and $C|\pi_2\rangle = -|\pi_2\rangle$ (we write $C = -1$). We will often use this notation, confusing operator and eigenvalue.

In the case of nonabelian groups it is not in general possible to diagonalize all the matrices; we can however write the matrices as a block matrix:

$$U_R = \begin{bmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & \dots \end{bmatrix}. \tag{1.25a}$$

Do note that some elements of a nonabelian group can commute: for example, in the rotation group we have

$$[J^i, J^j] = \epsilon^{ijk} J_k, \tag{1.26}$$

so if we take $i = j$, that is, we consider rotations along the same axis, they will commute since then the Kronecker symbol is equal to zero.

1.1.1 Space translations

An element of the group can be written as

$$U(a) = e^{iaP}, \tag{1.27}$$

where the operator P, whose eigenvalue is the momentum, is called the generator of the transformation.

If we take a plane wave, for example,

$$\langle x|p\rangle = e^{ipx}, \tag{1.28}$$

then we have

$$\langle x| U(a) |p\rangle = e^{ip(x-a)}. \tag{1.29}$$

If our system is invariant under translations, then Nöether's theorem tells us that the momentum is conserved.

In order to be a physical observable P needs to be Hermitian: $P = P^\dagger$.

So, the adjoint of the unitary transformation is

$$U^\dagger(a) = \sum_n \left(\frac{(-iaP)^n}{n!} \right)^\dagger = \sum_n \frac{(iaP^\dagger)^n}{n!} = e^{iaP^\dagger} = e^{iaP} = U^{-1}(a)\,, \tag{1.30}$$

which confirms the fact that the transformation is unitary.

Let us say that the momentum operator P commutes with the Hamiltonian: $[P, H] = 0$. Then,

$$[U(a), H] = 0\,. \tag{1.31}$$

All this is to say that a constant of motion O corresponds to an operator O which commutes with the Hamiltonian.

An example: the group G of 3D rotations. They depend on a continuous parameter $\vec{\alpha}$, just like translations depended on the parameter a.

The rotation is written as

$$U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{J}}\,, \tag{1.32}$$

where the components of the angular momentum have the following commutation relations:

$$[J^i, J^j] = \epsilon^{ijk} J^k\,. \tag{1.33}$$

We can multiply rotation matrices:

$$U(\vec{\beta})U(\vec{\alpha}) = U(\vec{\gamma})\,. \tag{1.34}$$

This space of 3D rotations is called SO(3), since every rotation corresponds to a 3x3 matrix which is a rotation matrix — so it is orthogonal and has determinant 1.

We can look for 1D representations of the generators J^i: we get $J^i = 0$, which means that we are not actually performing a rotation. Which states transform this way? These are scalar states, spin 0.

For 2D representations, we have

$$J^i = \frac{1}{2}\sigma^i\,, \tag{1.35}$$

where the σ^i are the Pauli matrices.

We can also find 3D spin-1/2 representations, which look like

$$J^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad J^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \qquad J^1 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,, \tag{1.36a}$$

A rotation in 2D, represented by an element of SO(2), corresponds to a phase shift, so we can say that it is equivalent to an element of U(1). This then allows us to see that SO(2) is abelian.

In general, we write for a unitary $n \times n$ unitary representation

$$U(n) \to e^{-i\alpha^n t^a}\,, \tag{1.37}$$

where the generators t^a are Hermitian matrices corresponding to Hermitian operators. In particular, one of these is the identity: $t^0 = \mathbb{1}$.

So, we omit it and say that we have $n^2 - 1$ generators for the SU(n) group. We shall see that each of these generators corresponds to a particle, and for the weak interaction we will have $2^2 - 1 = 3$ particles, while for the strong one we will have $3^2 - 1 = 8$.

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Yesterday we mentioned the fact that for group symmetries we can find corresponding conservation laws.

Invariance under spacetime translations gives us the conservation of 4-momentum p^μ. Invariance under the Lorentz group gives us conservation of angular momentum (for rotation).

1.1.2 Discrete symmetries

The symmetry in which we reverse the spatial coordinates is called parity P, if we reverse the time coordinate we have time reversal T, and later we will discuss the internal symmetry of charge conjugation C.

More precisely, if $x^\mu = \left(x^0, \vec{x}\right)$ we have

$$Px^\mu = \left(x^0, -\vec{x}\right) \tag{1.38a}$$

$$Tx^\mu = \left(-x^0, \vec{x}\right). \tag{1.38b}$$

These are Lorentz transformations represented by matrices with negative determinants: $\det \Lambda_P = \det \Lambda_T = -1$. This means that we cannot obtain these transformations with a composition of orthochronal Lorentz transformations — those defined to be continuously connected to the identity —, since the determinant of a Lorentz transformation is always ± 1, and it is a continuous function of the Lorentz transformation.

In quantum mechanics, these must be interpreted like operators. Their eigenvalues will be ± 1.

These transformations being symmetries is an experimental question. P and T are symmetries of the strong and electromagnetic interactions.

The weak interaction, instead, does not conserve P.

We defined the charge conjugation operator C by its action on the $\left|\pi^+\right\rangle$ and $\left|\pi^-\right\rangle$, on the basis defined by them its matrix representation was σ_x. It commuted with the Hamiltonian.

In general, this swaps not only the electric charge but all the quantum numbers, including the "hypercharges" such as baryon number. The charge conjugation operator C gives us the antiparticle of a certain particle, so it must flip all of the charges. There are, however, theories

in which, say, lepton number conservation is broken but baryon number is conserved. The precise meaning of this operator depends on the theory.

If we constructed our experiments with antimatter would we get the same physics?

It was experimentally determined that weak interactions violated parity, and it was thought that baryon number was conserved: however in the early universe we would expect to have equal amounts of matter and antimatter, but this is not what we see — we are made of matter, and we do not see tha gamma ray background we would expect to see if there were matter and antimatter spacetime regions.

We can apply several "mirrors" to our system: this amounts to the composition of the operators. There is a theorem in Quantum Field Theory: if a QFT is consistent, then it must obey CPT symmetry. After passing through all of the three mirrors, the system has the same physical properties. The order of the three symmetries does not matter: they commute.

The conservation of the electric charge Q, for example, is connected to $\mathrm{U}(1)_{\mathrm{em}}$ symmetry. This is an *exact* symmetry, which we expect not to break down at higher energy.

On the other hand, C, P and T are not exact symmetries since there exist some interactions which violate them (separately: P and CP are violated, so T is also violated by the CPT theorem).

1.2 Relativistic wave equations

In quantum mechanics we describe the dynamics of a quantum system using the Schrödinger equation:

$$E = \frac{\vec{p}^2}{2m} + V\,, \tag{1.39}$$

where $E = i\partial_t\psi$ and $\vec{p}^2 = -(2m)^{-1}\vec{\nabla}^2\psi$.

This is non (special) relativistic: if we perform a Lorentz boost the equation does not remain in the same form.

Also, this equation describes the dynamics of one electron. In elementary particle physics it is no longer *consistent* to only consider one particle: when the energies are of the order of the mass of the particles we can create antiparticles or other particles. So, in general there is no reason to expect that the number of particles is conserved in particle physics.

Formally, it is hard to define from the Schrödinger equation a conserved quantity connected to the probability of finding the particle. It does not account for the possibility that the particle might appear or disappear. For example, we can have matter-antimatter collisions: this is called *annihilation*, and the two incoming particles disappear completely.

The equation $E = \vec{p}^2/2m$ is intrinsically nonrelativistic; the corresponding relativistic relation is $E^2 = \vec{p}^2 + m^2$.

We can make the same canonical quantization substitutions to get

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi(t,\vec{x}) = 0\,. \tag{1.40}$$

Why should we use $E^2 = \vec{p}^2 + m^2$ instead of, say, $E = \sqrt{\vec{p}^2 + m^2}$? The problem is that it is hard to see what could be meant by the square root of an operator. Anyways, we shall use the approach of "taking the square root" in order to solve the KG equation. We can write this in a manifestly covariant way as

$$\left(\partial^\mu \partial_\mu + m^2\right)\phi(t, \vec{x}) = 0\,, \tag{1.41}$$

where $\partial_\mu = (\partial_t, \vec{\nabla})$, whose square $\partial^\mu \partial_\mu = \partial_t^2 - \vec{\nabla}^2 = \Box$ is the Dalambertian.

This is the Klein-Gordon equation. As long as ϕ is a scalar, this is a scalar Lorentz invariant equation.

It should be used to describe a spin-0 particle, since we are not accounting for spin: the only spin-0 elementary particle known is the Higgs Boson. All other spin-0 particles are composite.

A crucial fact in the KG equation is the fact that the energy can in principle be both positive and negative: $E = \pm\sqrt{\vec{p}^2 + m^2}$. A reasonable approach would be to not bother treating the "unphysical" $E < 0$ solution; however this is wrong, the negative energy solution is important.

This was a great open debate last century. We will be given the solution, but it would not have easy to figure it out.

This is the reason why people started discussing antiparticles.

An interesting question now is: can we define an action

$$S[\phi(x)] = \int \mathrm{d}^4 x\, \mathscr{L}(\phi, \partial_\mu \phi) \tag{1.42}$$

whose actions of motion are the KG equation? The answer is yes, with

$$\mathscr{L} = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2\right). \tag{1.43}$$

To show that this is the case is left as an exercise. Notice that we talk of "Lagrangians" but we always mean Lagrangian densities. The corresponding Hamiltonian density $\mathscr{H}$ has a vacuum state we can call $|0\rangle$, which corresponds to the absence of particles.

We will then have states describing n particles of mass m.

Next time, we will move from the KG equation with ϕ being just a wavefunction to it being an operator, which can act on the vacuum creating particles.

This is sometimes called "second quantization", to distinguish it from the first quantization, in which operators act on wavefunctions.

1.3 Relativistic wave equations

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Last week, we moved from the usual Schrödinger equation to the Klein Gordon equation by moving from $E = p^2/2m$ to $E^2 = p^2 + m^2$. The latter is covariant under Lorentz transformations.

There are, however, two issues with the Schrödinger equations: one is that it is not relativistic, while the second one is more subtle.

The Schrödinger equation describes the evolution of a single particle in time, while when we deal with elementary particles a one-particle description is unsuitable: we must have conservation of probability, so we are unable to describe situations in which a particle disappears by decaying into other particles, or we have inelastic collisions.

This is the reason why we need a multiparticle description. An example of the issues which arise in the single-particle relativistic description is the Klein paradox.

So, we will introduce the so-called second quantization formalism. We are going to interpret ϕ as a **quantum field operator**, instead of a scalar field. Let us make this explicit. We take the KG equation from the first quantization to the KG equation describing a Quantum Field Theory.

$\phi(x)$ will be an operator which can destroy or create particles. We start from the quantization of the Hamiltonian density, $\mathscr{H}$: we consider its ground state $|0\rangle$. This corresponds to a state in which there is no particle, and is called the **vacuum** state.

We will also have higher-energy states, in which we will have one or more particles: how do we describe these? A one particle state $\left|\varphi_1(p)\right\rangle$ can be acted upon by the operator $\phi(x)$: it is destroyed, yielding a state which is aligned with the vacuum. Formally, we have

$$\langle 0|\,\phi(x)\left|\varphi_1(p)\right\rangle = e^{-ipx}\,, \tag{1.44}$$

where p^0 is the positive energy: $p^0 = +E_p = +\sqrt{\vec{p}^2 + m^2}$.

Now, let us consider the complex conjugate of this matrix element: we swap the bra and ket and take the adjoint of the operator, to get

$$\left\langle\varphi_1(p)\right|\phi^\dagger\,|0\rangle = e^{ipx}\,. \tag{1.45}$$

This means that, if the first equation describes a particle propagating with momentum p^μ, this new equation will now describe a particle propagating with momentum $-p^\mu$.

This will now have a negative energy. This was a problem historically, now we give the solution directly.

Let us introduce a particle $\left|\varphi_2(p)\right\rangle$, such that

$$\langle 0|\,\phi^\dagger(x)\left|\varphi_2(p)\right\rangle = e^{-ipx}\,, \tag{1.46}$$

so now this particle has the same mass m, but this new particle $\left|\varphi_2\right\rangle$ differs from $\left|\varphi_1\right\rangle$ only for the charge, which is now opposite.

Add clarification: in general we change the sign of the coupling, which is in front of the V in the Lagrangian.

Do note that charge does not exclusively mean electric charge! We can also have other kinds of charges. For example, neutrinos have the charge of lepton number.

Particles can be their own antiparticles, if they have zero charge. So, we can interpret the negative energy solution as the presence of an antiparticle: a negative energy particle would correspond to a positive energy antiparticle.

A generic field theory is defined by its action, which is written from the density Lagrangian: a free Lagrangian is

$$\mathscr{L} = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\varphi - m^2\varphi^2\right), \tag{1.47}$$

and if we impose $\text{var}(S) = 0$ we get precisely the KG equation as our equation of motion.

1.3.1 Spin 1

We discuss 3D vectors V^i, which transform under rotations $R_{ij} \in SO(3)$ as

$$V^i \to V'^i = R_{ij} V^j \,. \tag{1.48}$$

Now, the matrix element from before reads

$$\langle 0 | \, V^i(x) \, | v(p,\epsilon) \rangle = \epsilon^i e^{-ipx} \,, \tag{1.49}$$

where we need to account for the momentum p and the polarization ϵ^i. If we want to move to a relativistic description, we get the same thing, with the spatial index i being replaced by a 4-dimensional index μ:

$$\langle 0 | \, V^\mu(x) \, | v(p,\epsilon) \rangle = \epsilon^\mu e^{-ipx} \,. \tag{1.50}$$

The problem is now the normalization of these states: what is the value of $\langle v|v\rangle$? With the definition we gave, this is equal to $\epsilon^\mu \epsilon_\mu$, not 1! In general this will not be positive. Take the photon: then, we have the two helicities, which are transverse degrees of freedom. For example, we can have $\epsilon^\mu = (0,1,1,0)$, which in our choice of metric, which is the mostly negative one.

So, we could impose $\langle v|v\rangle = -\epsilon^\mu \epsilon_\mu$: but what if we make a boost?

The square modulus is covariant though!

This is in general a problem.

The electromagnetic field strength is defined as

$$A^\mu = \left(\varphi(x), \vec{A} \right) , \tag{1.51}$$

and the EM field strength is $F^{\mu\nu} = 2\partial^{[\mu} A^{\nu]}$. So, we have

$$F^{i0} = -\nabla^i \varphi - \partial_t A^i = E^i \tag{1.52a}$$

$$F^{ij} = -2\nabla^{[i} A^{j]} = \epsilon^{ijk} B^k \,. \tag{1.52b}$$

The Maxwell equations follow from the density Lagrangian

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu \,, \tag{1.53}$$

where j^μ is an external current. This yields $\partial_\mu F^{\mu\nu} = j^\nu$.

The photon is a massless vector boson. This description applies in general to a massless vector field. Are there massive vector fields? Yes, the weak interaction is described by a massive vector boson $W^\pm_\mu$.

1.3.2 Spin 1/2

Now we discuss spin $1/2$ particles: we will need to have first derivatives on either side, as opposed to the second equations in the KG equation.

Our ansatz for what will be called the Dirac equation is:

$$i\partial_t = -i\vec{\alpha}\cdot\vec{\nabla} + \beta m\,. \tag{1.54}$$

Is it possible to find four numbers (3 encoded in the vector $\vec{\alpha}$, one more in β) so that the square of this relation is $E^2 = p^2 + m^2$ and that we still retain Lorentz invariance?

If we try to impose these conditions, we find that there are no solutions: there are no such four numbers.

We can, however, find a solution if we allow $\vec{\alpha}$ and β to be matrices: specifically, 4×4 matrices, which must obey the **anticommutation relations**:

$$\left\{\alpha_i, \alpha_j\right\} = 2\delta_{ij} \qquad \text{and} \qquad \left\{\alpha_i, \beta\right\} = 0\,, \tag{1.55}$$

and now we define

$$\gamma^0 = \beta \qquad \text{and} \qquad \gamma^i = \beta\alpha^i\,, \tag{1.56}$$

which must be 4D, as we said, and the simplest representation is called the Dirac representation:

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} \qquad \text{and} \qquad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix}. \tag{1.57a}$$

It can be verified that then

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,. \tag{1.58}$$

We can move between representations of these matrices using unitary transformations. Inserting these, we find:

$$\left[i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\mathbb{1}\right]\varphi(x) = 0\,, \tag{1.59}$$

which means that the wavefunction must be 4-dimensional as well, since we are acting on it with a 4D operator. We are going to call such an object a **spinor**. If we denote $\gamma^\mu = \left(\gamma^0, \vec{\gamma}\right)$ we can write the Dirac equation as

$$\left(i\gamma^\mu\partial_\mu - m\right)\varphi(x) = 0 \tag{1.60a}$$

$$\left(i\not{\partial} - m\right)\varphi(x) = 0\,, \tag{1.60b}$$

where we have defined the notation $\not{x} = \gamma^\mu x_\mu$. As we shall see tomorrow morning, this equation is extremely rich in structure.

In the Westminster abbey, this equation is inscribed as a homage to Paul Dirac.

Wednesday 2020-3-25, compiled 2020-03-31

Today we are going to examine the third case of wave equations: we discussed the KG equation for a scalar particle, then we moved to a vector particle, and now we are going to consider a spin-1/2 particle.

One has to prove that the equation $\left(i\partial\!\!\!/ - m\right)\varphi(x) = 0$ is Lorentz-covariant. Also, solutions of the Dirac equation are also solutions of the Klein-Gordon equation, therefore they satisfy $E^2 = p^2 + m^2$.

The first is hard, the second is comparatively easy to prove. The first is hard, even though it looks covariant, since γ^μ does not transform as a vector *a priori*. Nevertheless, these are both true statements.

The result is that under a Lorentz transformation we have

$$\psi(x) \to \psi'(x) = S(\Lambda)\psi(x)\,, \tag{1.61}$$

where $S(\Lambda)$ is a unitary transformation associated with the Lorentz transformation Λ, defined by

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \left(\Lambda^{-1}\right)^\mu_{\ \nu}\gamma^\nu\,. \tag{1.62}$$

So, in order to do this we need to write an infinitesimal Lorentz transformation

$$\Lambda^\mu_{\ \nu} = \eta^\mu_{\ \nu} + \delta\omega^\mu_{\ \nu}\,, \tag{1.63}$$

where $\delta\omega^\mu_{\ \nu}$ is an antisymmetric matrix, if it has nonzero $0i$ components it gives boosts, if it has nonzero ij components it gives rotations. The result for the form of S is:

$$S = \mathbb{1} + \frac{1}{8}\Big[\gamma_\mu, \gamma_\nu\Big]\delta\omega^{\mu\nu}\,. \tag{1.64}$$

If we make a rotation, for example, we get

$$\psi(x) \to \exp\left(i\frac{i}{2}\Big[\gamma_i, \gamma_j\Big]\delta\omega^{ij}\right)\psi(x)\,, \tag{1.65}$$

and typically one uses the shorthand rotation

$$\frac{i}{2}\Big[\gamma_i, \gamma_j\Big] \stackrel{\text{def}}{=} \sigma_{ij}\,. \tag{1.66}$$

[see notes for an explicit example]

A spinor $\psi(x)$ reacts in a peculiar way to rotations: it rotates by an angle $\varphi/2$ if we perform a Lorentz rotation of an angle φ. This means that its periodicity is 4π.

We introduce the *chiral representation* of the gamma matrices:

$$\gamma^0 = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}. \tag{1.67a}$$

Then, for Lorentz boosts we have

$$\sigma_{0i} = \frac{1}{2}[\gamma_0, \gamma_i] = -i\begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \tag{1.68a}$$

while for rotations we have

$$\sigma_{ij} = \frac{i}{2}\left[\gamma_i, \gamma_j\right] = \epsilon_{ijk}\begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, \tag{1.69a}$$

which is useful since it gives us block-diagonal matrices. So, we can interpret the spinor as being made up of two components:

$$\psi(x) = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \eta \\ \omega \end{bmatrix}, \tag{1.70a}$$

on which Lorentz transformations act independently. This is relevant since, for example, if we deal with an electron, we will describe it with a 4-component spinor, however we will be able to divide it into two components e_L and e_R, which are two component spinors on which we can act independently.

This will become very concrete when we will discuss how many degrees of freedom were present in the original plasma.

Since a spinor ψ also solves the KG equation, it will be able to be written as

$$\psi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} e^{-ipx}, \tag{1.71a}$$

but what are the relations between the coefficients? We consider a simple case, that of $\vec{p} = 0$, so that we are in the rest frame of the particle.

Then, the Dirac equation reads:

$$\left(\gamma^0 E - m\right)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0, \tag{1.72a}$$

so we need to choose a representation for the γ^0 in order to write this explicitly. We choose the Dirac representation, in which

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}. \tag{1.73a}$$

Then, the equation reads

$$\begin{bmatrix} E-m & 0 & 0 & 0 \\ 0 & E-m & 0 & 0 \\ 0 & 0 & -E-m & 0 \\ 0 & 0 & 0 & -E-m \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} e^{-ipx} = 0, \tag{1.74a}$$

so we get a solution in which $E = m$, and a solution in which $E = -m$. So, in general we write the two linearly independent solutions, respectively with positive and negative energy, as

$$\psi = \begin{bmatrix} \xi \\ 0 \end{bmatrix} e^{-imt} \qquad \text{and} \qquad \Psi = \begin{bmatrix} 0 \\ \eta \end{bmatrix} e^{+imt} . \tag{1.75a}$$

The assumption we made, $\vec{p} = 0$, does not actually mean we lose generality: we can simply boost into the rest frame of the particle. If we do this, we get the general

$$\begin{bmatrix} E - m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -E - m \end{bmatrix} \begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} e^{-imt + i\vec{p}\cdot\vec{x}} , \tag{1.76a}$$

so we can decompose our solution into

$$\psi = u^s e^{-i\vec{p}\cdot\vec{x}} \qquad E > 0 \psi \qquad = v^s e^{i\vec{p}\cdot\vec{x}} E \qquad < 0 , \tag{1.77}$$

so now we come to the interpretation: we introduce the existence of an antifermion, which corresponds to the solution to the Dirac equation with negative energy, but it has the opposed momentum: so, it has positive energy. Then, both of our solutions have positive energy and evolve forward in time, and both have the same mass.

Now, we make the jump to second quantization: we start interpreting $\psi(x)$ as an operator, which can destroy a one-state particle or create a particle starting from the vacuum.

We will have

$$\langle 0 | \psi(x) \left| e^-(p,s) \right\rangle = u^s(p) e^{-ipx} . \tag{1.78}$$

Now, the one-particle state $\left| e^-(p,s) \right\rangle$ is promoted to a spinor. On the other hand, we have the creation operator $\psi^\dagger$:

$$\left\langle e^-(p,s) \right| \psi^\dagger(x) \left| 0 \right\rangle = u^{s^\dagger} e^{-ipx} . \tag{1.79}$$

Now, the tricky question is to introduce the negative-energy solution. Then, the $\psi^\dagger$ operator will destroy this state, while ψ will create it. So we will write an equation like

$$\langle 0 | \psi^\dagger(x) \left| e^+(p,s) \right\rangle = v^{s^\dagger}(p) e^{-ipx} , \tag{1.80}$$

where we would write $v^s(p)$ if we were considering the negative energy particle, instead we are looking at the antiparticle.

Now, we will be able to operate with $\psi(x)$ on the vacuum, to find

$$\left\langle e^+(p,s) \right| \psi(x) \left| 0 \right\rangle = v^s(p) e^{ipx} . \tag{1.81}$$

Yesterday we discussed the Lagrangian of the free photon field,

$$\mathscr{L} \propto F^{\mu\nu}F_{\mu\nu}\,, \tag{1.82}$$

but as we said there can also be coupling to external currents, which we did not quantize. However, now we quantized the electron: so, can we construct the external current j_μ in the coupling term $j_\mu A^\mu$?

The first attempt would be to write something like

$$j^\mu \sim \psi^\dagger \gamma^\mu \psi \sim e^+ \gamma^\mu e^-\,, \tag{1.83}$$

but we would need to check whether it is a vector: in fact, it does not transform correctly.

Is this at least a Hermitian operator? Well, its adjoint is

$$\left(\psi^\dagger \gamma^\mu \psi\right)^\dagger = \psi^\dagger (\gamma^\mu)^\dagger \psi\,, \tag{1.84}$$

but

$$(\gamma^\mu)^\dagger = \left(\gamma^0, -\gamma^i\right) \neq \gamma^\mu\,. \tag{1.85}$$

One finds that the correct definition is to have

$$\overline{\psi} \stackrel{\text{def}}{=} \psi^\dagger \gamma^0\,, \tag{1.86}$$

and then

$$j^\mu = \overline{\psi}\gamma^\mu\psi \tag{1.87}$$

is the correct definition. This, then, transforms as a 4-vector.

Now, in order to introduce the EM field we will use minimal coupling:

$$\partial_\mu \to \mathrm{D}_\mu = \partial_\mu + ieA_\mu\,, \tag{1.88}$$

so the Dirac equation will read

$$(i\not{\mathrm{D}} - m)\psi = 0 \tag{1.89a}$$

$$\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\psi = 0\,. \tag{1.89b}$$

From which density Lagrangian can we derive the Dirac equation? It turns out to be

$$\mathscr{L} = \overline{\psi}\left(i\gamma^\mu \mathrm{D}_\mu - m\right)\psi\,, \tag{1.90}$$

so if we want to describe both the EM field, the electron and their interaction, we have

$$\mathscr{L}(e, A_\mu) = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{free EM field}} + \underbrace{\overline{\psi}(i\not{\mathrm{D}} - m)\psi}_{\text{electron}} - \underbrace{e\overline{\psi}\not{A}\psi}_{\text{electron-EM coupling}}\,. \tag{1.91}$$

This is the density Lagrangian of Quantum ElectroDynamics. This is the first interacting QFT which was constructed, and it was extraordinarily successful.

Its predictions for the anomalous magnetic moment of the electrons were exceptional: we can solve it perturbatively to different orders in

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}\,. \tag{1.92}$$

Tuesday 2020-3-31, compiled 2020-03-31

Last time we saw our first field theory: QED.

We have associated spin $1/2$ fermions to matter fields. Spin 1 particles, instead, are vector bosons, such as the photon γ, the gluon g and the weak $W^{\pm}$ and Z^0 bosons. These are the radiation fields. We will explore this in more detail.

This connection comes from the **spin-statistics** theorem: it states that particles with integer spin obey Bose-Einstein statistics, while particles with half-integer spin obey Fermi-Dirac statistics.

Up until now, we have observed only particles with spins 0, $1/2$ and 1.

We could have a symmetry called supersymmetry, in which fermions and bosons are associated.

The graviton has spin 2; in supergravity the graviton has a fermion partner called the "gravitino" with spin $3/2$.

How do we normalize the states in relativistic theory? Classically we did

$$\langle p_1 | p_2 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\,, \tag{1.93}$$

in the relativistic case instead we will do

$$\langle p_1 | p_2 \rangle = 2E_{p_1}(2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\,. \tag{1.94}$$

The relativistic volume element is given by

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \to \int \frac{\mathrm{d}^4 p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_p}\,, \tag{1.95}$$

where we integrated over the variable $p^0 = E$, which removed the $\delta(p^2 - m^2) = \delta(E - \sqrt{p^2 + m^2})/2E$.

Let us consider a **decay** process: we have a particle A decaying into a possibly multi-particle state f. We are interested in the decay *rate* of this process.

Another process of interest is a **scattering** process of $n \to m$ particles. There is no particle number conservation: we can create and destroy as many particles as we like. These types of processes are described by their *cross section*, which is an effective area corresponding to how aligned the trajectories of the incoming particles must be in order to interact.

This cross section allows us to compute the average time for an interaction to occur: if two particles' interaction becomes so rare that they cannot interact within a Hubble time then they are said to have *decoupled*.

The probability of survival of particle A at time t is in the form $\mathbb{P}(t) = \exp(-t/\tau_A)$, so we define

$$\Gamma_A = \frac{1}{\tau_A}\,, \tag{1.96}$$

which has the dimensions of a frequency, or equivalently an energy.

We can define the branching ratios as

$$BR(A \to f) = \frac{\Gamma(A \to f)}{\Gamma_A}\,. \tag{1.97}$$

We can consider fixed-target experiments: we want to know how many events per second we will have, which will be given by

$$\frac{\#\text{ events}}{s} = n_A \times v_A \times \sigma\,, \tag{1.98}$$

where σ is the cross section, v_A is the velocity of the incoming particles, while n_A is the number density of particles in the target.

If we have two beams of particles coming towards each other, the term will look like $n_A n_B (v_A - v_B)\ell_B A_b \sigma$.

In general, we will be interested in the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^3 p_1\, \mathrm{d}^3 p_2 \ldots \mathrm{d}^3 p_n}\,. \tag{1.99}$$

This is just a definition, and it is not covariant, if we want to integrate we still need to use the covariant momentum element.

For a scattering process like $A + B \to 1 + \cdots + n$ we will need to compute things like

$$\langle 12\ldots n|\, T\, |AB\rangle = \mathcal{M}(A + B \to 1 + \cdots + n)(2\pi)^4 \delta^{(4)}(P_A + P_B - \sum p_i)\,, \tag{1.100}$$

where T is the time evolution, and we defined the invariant scattering amplitude $\mathcal{M}$. If we want to compute the width Γ_A for a decay process, we need to define the phase space integral:

$$\int \mathrm{d}\Pi_n = \underbrace{\prod_i \int \frac{\mathrm{d}^3 p_i}{2E_i (2\pi)^3} (2\pi)^4 \delta^{(4)}(P_A - \sum_i p_i)}_{\text{phase space integral}}\,, \tag{1.101}$$

so we will have what is called Fermi's golden rule:

$$\Gamma_A = \frac{1}{2M_A} \int \mathrm{d}\Pi_n \left|\mathcal{M}(a \to f)\right|^2\,, \tag{1.102}$$

since in order to get the probability we need to take the square of the amplitude.

[Discussion on the dimensionality of $\mathcal{M}$]

If we are in the COM frame, the phase space integral for the initial particles reads:

$$\int \mathrm{d}\Pi_2 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_1} \frac{1}{2E_2} (2\pi)\delta(E_{CM} - E_1 - E_2) \tag{1.103}$$

$$= \frac{1}{8\pi} \frac{2p}{E_{CM}} \int \frac{\mathrm{d}\Omega}{4\pi}\,. \tag{1.104}$$

The nonrelativistic Breit-Wigner formula is given by

$$\mathcal{M} \sim \frac{1}{E - E_R + i\Gamma/2}\,. \tag{1.105}$$

The paradigmatic process for these kinds of interactions is the process $e^+e^- \to \mu^+\mu^-$. In the noted it is sketched how one would go about discussing such a process.

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