Theoretical physics notes

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Contents

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General information

Written & oral exam.

The oral is optional: but is there grade truncation?

The suggested book is D'Auria & Trigiante [DT11]. For the second part it is also useful to have a look at Mandl & Shaw [MS10].

Live question time in Zoom at half past 11 on Mondays.

Things which will be taken for granted: four-vectors, Lorenz and Poincaré groups, basics of QM, basics of linear operators.

Contents

This course will deal with the basics of Relativistic Quantum Field Theory.

We will discuss the Lagrangian formalism for a Classical Field Theory. We will quantize these theories using canonical quantization, specifically for a scalar, a Dirac fermion, and a vector boson.

Then, we will introduce interactions in our Lagrangian: we will use the S-matrix expansion, and Feynman diagrams.

Chapter 1

Relativistic Quantum Field Theory

1.1 The nonrelativistic wave equation

We will review the derivation of the nonrelativistic Schrödinger equation. We find it starting from the correspondence principle: we start from the expression of the energy

$$E = \frac{p^2}{2m} + V(x)\,, \tag{1.1}$$

and substitute the energy with $E \to i\partial_t$, the momentum with $\vec{p} \to -i\vec{\nabla}_x$ and the position with the position operator $\vec{x}$, all acting on the wavefunction. With this we get

$$i\frac{\partial\psi}{\partial t}(\vec{x},t) = \left(\frac{-\nabla^2}{2m} + V(\vec{x})\right)\psi(\vec{x},t)\,. \tag{1.2}$$

We still need to assign a meaning to the wavefunction: this is given by the Bohr condition, which tells us that the probability density of finding the particle in a specific region is

$$\rho(\vec{x},t) = \left|\psi(\vec{x},t)\right|^2 \geq 0\,. \tag{1.3}$$

This probability density must be normalized as an initial condition:

$$\mathbb{P}(t_0) = \int_{\mathbb{R}^3} \mathrm{d}^3x\, \rho(\vec{x},t_0) = 1\,, \tag{1.4}$$

and we wish to show that it will also be normalized at later times:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3x\, \frac{\partial}{\partial t}\left|\psi(\vec{x},t)\right|^2 \tag{1.5}$$

$$= \int_{\mathbb{R}^3} \mathrm{d}^3x \left(\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right). \tag{1.6}$$

Using the Schrödinger equation we can substitute in the expression for the derivative of the wavefunction:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \frac{1}{i} \left(-\frac{\nabla^2}{2m} + V \right) \psi - \frac{1}{i} \psi \left(-\frac{\nabla^2}{2m} + V \right) \psi^* \right\} \tag{1.7}$$

$$= \frac{i}{2m} \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \nabla^2 \psi - 2m\psi^* V \psi - \psi \nabla^2 \psi^* + 2m\psi V \psi^* \right\}, \tag{1.8}$$

and we use the fact that

$$\psi^* V \psi = \psi V \psi^* = \left(\psi^* V \psi \right)^*, \tag{1.9}$$

which is true since V is a symmetric operator: it has real eigenvalues. This allows us to simplify the terms which include V, and we find:

There seem to be some $2m$ factors missing in the formula in the notes.

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \frac{i}{2m} \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right\} \tag{1.10}$$

$$= \frac{i}{2m} \int_{\mathbb{R}^3} \left\{ \nabla_{\vec{x}} \cdot \left[\psi^* \vec{\nabla} \psi - \psi \left(\vec{\nabla} \psi^* \right) \right] \right\}, \tag{1.11}$$

so we can define

$$\vec{j}(\vec{x}, t) = -\frac{i}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right), \tag{1.12}$$

so that our equation now reads

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = -\int_{\mathbb{R}^3} \mathrm{d}^3x \, \vec{\nabla}_x \cdot \vec{j} = \int_{\partial\mathbb{R}^3} \vec{j} \cdot \hat{n} \, \mathrm{d}^2x = 0, \tag{1.13}$$

since the wavefunction is integrable: that is, it goes to zero *quickly* as $|\vec{x}| \to \infty$. Therefore, $\left|\vec{j}\right| \to 0$ as $|\vec{x}| \to \infty$. For a more detailed explanation, see the Quantum Mechanics notes by Manzali [Man19, page 147].

So, if the probability is equal to one at a certain time than it keeps being equal to one.

We can express this as a differential equation for the integrand: the *continuity equation*,

$$\frac{\partial}{\partial t} \left| \psi(\vec{x}, t) \right|^2 + \vec{\nabla} \cdot \vec{j} = 0. \tag{1.14}$$

Let us now consider the way to solve the free Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} = -\frac{\nabla^2 \psi}{2m}\,. \tag{1.15}$$

We start from an ansatz of the equation being factorizable: $\psi(\vec{x}, t) = \chi(t)\varphi(\vec{x})$. So, we get

$$i\frac{\partial \psi_0}{\partial t} = \varphi(\vec{x}) i \frac{\partial \chi}{\partial t} \tag{1.16}$$

on the LHS, and

$$H_0(\psi) = -\chi(t)\frac{\vec{\nabla}^2}{2m}\varphi(\vec{x}) \tag{1.17}$$

on the RHS. Dividing both by $\psi = \chi\varphi$ we get

$$i\frac{1}{\chi}\frac{\partial \chi}{\partial t} = -\frac{1}{\varphi}\frac{\vec{\nabla}^2}{2m}\varphi\,, \tag{1.18}$$

and since these are dependent only on time (for the LHS) and only on position (for the RHS) they must be separately constant: let us call their value E. Therefore, we can integrate them to get

$$\frac{\partial \chi}{\partial t} = -iE\chi \implies \chi(t) = \chi(0)\exp(-iEt) \tag{1.19}$$

and

$$\nabla^2 \varphi = -2mE\varphi \implies \varphi(\vec{x}) = \varphi(0)\exp\left(i\vec{k}\cdot\vec{x}\right). \tag{1.20}$$

Here , $\vec{k}$ is a 3D vector such that $\left|\vec{k}\right|^2 = 2mE$.

This is called the *dispersion relation*. So, the full solution, which is called a *monochromatic* solution, is

$$\psi(\vec{x}, t) = \exp\left(-i\left(Et - \vec{k}\cdot\vec{x}\right)\right), \tag{1.21}$$

where $\left|\vec{k}\right|^2 = 2mE$.

The general solution will be a continuous superposition of solutions of this form:

$$\psi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\, \widetilde{\varphi}(\vec{k})\, \exp\left(-i\left(\omega_k - \vec{k}\cdot\vec{x}\right)\right)\Bigg|_{\omega_k = \frac{\left|\vec{k}\right|^2}{2m}}\,. \tag{1.22}$$

Our conventions for the Fourier transform are:

$$\varphi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, \widetilde{\varphi}(\vec{k}) \exp\Big(-i\vec{k}\cdot\vec{x}\Big) \widetilde{\varphi}(\vec{x}) \quad = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3x \, \varphi(\vec{k}) \exp\Big(i\vec{k}\cdot\vec{x}\Big) , \tag{1.23}$$

so we use the symmetric definition. Other conventions have factors $(2\pi)^{-3}$ on one side and nothing on the other; it is the same but the we must be consistent.

It is a theorem that $|\varphi|^2 = |\widetilde{\varphi}|^2$, where the square norm of φ, $|\varphi|^2$, is just the integral of $\varphi^*\varphi$ over all 3D space.

The 3D dirac delta function is defined as

$$\delta^3(\vec{x}-\vec{y}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k \exp\Big(-i\vec{k}\cdot(\vec{x}-\vec{y})\Big) , \tag{1.24}$$

and the 3D delta in the momentum space is perfectly analogous.

The Schrödinger equation is manifestly *non relativistic*: we started from the nonrelativistic expression $E = p^2/2m + V$, so we should expect so. In the differential equation we have a second spatial derivative and a first temporal derivative: there is no way to write such an equation covariantly.

This kind of law of physics is only invariant under *galilean transformations*, which do not change time.

Bibliography

[DT11] R. D'Auria and M. Trigiante. *From Special Relativity to Feynman Diagrams*. Springer, 2011.

[MS10] F. Mandl and G. Shaw. *Quantum Field Theory*. 2nd ed. John Wiley and Sons, 2010.

[Man19] F. Manzali. *Appunti Di Fisica Teorica*. 2019. URL: https://einlar.github.io/note.html#fisica-teorica.