General Relativity notes

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1 Special relativity

Definition 1.1. *An inertial frame is one in which Newton's laws hold: a free body moves with acceleration $a^i = 0$.*

Newton's first law establishes the *existence* of inertial frames.

Proposition 1.1. *The frames O and O' are both inertial frames iff O' moves with constant velocity wrt O.*

Proposition 1.2. *Coordinate transformations between inertial frames are Lorentz boosts, which in some coordinate frame can be written as*

$$t' = \gamma_v \left(t - \frac{vx}{c^2} \right) \tag{1a}$$

$$x' = \gamma_v (x - vt) \tag{1b}$$

$$y' = y \tag{1c}$$

$$z' = z \,, \tag{1d}$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$.

If $v \ll c$, so $v/c \sim 0$, they simplify to the identity for t, y, z and $x' = x - vt$: these are Galilean transformations.

If we have two events, x^μ and y^μ, they occur with some time and space separation $\Delta x^\mu = x^\mu - y^\mu$. We can compute $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$, where

$$\eta_{\mu\nu} = \operatorname{diag}\left(-c^2, 1, 1, 1 \right) . \tag{2}$$

Proposition 1.3. *Under Lorentz transformations Δs^2 is invariant.*

We can classify separations between events as

- time-like when $\Delta s^2 < 0$;
- null-like when $\Delta s^2 = 0$;
- space-like when $\Delta s^2 > 0$.

We can draw spacetime diagrams. A light cone is the set of points which are null-like separated from a select point. Things can be only causally related to events inside the light-cone (with $\Delta s^2 \geq 0$).

1.1 Time dilation

Take two events which occur at the same location for O'. In the primed frame they will have coordinates $x^\mu = (t_0, x_0)$ and $y^\mu = (t_1, x_0)$.

Definition 1.2. *The* proper time *between these two events is* $t_1 - t_0 \overset{def}{=} \Delta\tau$.

We now see that $\Delta s'^{\,2} = -c^2 \Delta\tau^2$. Then, any other observer will see the same $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 = \Delta s'^{\,2}$.

This directly implies that $\Delta\tau \leq \Delta t$ for any observer, since $\Delta\tau^2 = \Delta t^2 - \Delta x^2/c^2$. This effect is called *time dilation*.

By how much exacly is time dilated? Of course $\Delta x = v\Delta t$, therefore $\Delta t = \gamma_v \Delta\tau$.

-> Muon problem.

Inverse Lorentz transformation have the same expression, but with $v \to -v$. This can be proved both mathematically by solving the equations and phisically by reasoning about their meaning. There is no preferential inertial frame.

A Lorentz transformation can be written in matrix form in the (ct, x) plane as:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \tag{3}$$

since there is an angle θ such that $\gamma = \cosh\theta$ and $\gamma\beta = \sinh\theta$: the angle θ will be $\theta = \tanh^{-1}(v/c)$. This is true because $\gamma^2 - \beta^2\gamma^2 = 1$.

After a boost the ct' and x' axes are respectively the lines $ct = x/\beta$ and $ct = \beta x$.

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Last lecture we saw the fact that the ct' and x' axes are rotated by equal angles from the ct and x axes towards the $ct = x$ axis.

1.2 Relativity of simultaneity

Consider two events which are simultaneous in the O' frame. Their times in this frame are $t'_A = t'_B$.

In the O frame, instead, we have

$$ct_{A,B} = \frac{v}{c} x_{A,B} + \underbrace{\sqrt{1 - \frac{v^2}{c^2}} ct'_{A,B}}_{\text{a constant}}, \tag{4}$$

so the events are not simultaneous in the O frame.

1.3 Length contraction

If in the O frame, A occurs at $t, x = 0$ while B occurs at $t = 0$, $x = L$, then L is the measured length of their spatial interval by O. We assume that this is the frame in which the object is moving, and we transform into a frame in which it is stationary: O'.

In the primed frame their coordinates will be:

$$x'_A = \gamma_v \left(x_A - \frac{v}{c} ct_A \right) \tag{5a}$$

$$x'_B = \gamma_v \left(x_B - \frac{v}{c} ct_B \right), \tag{5b}$$

therefore $x'_B - x'_A = \gamma_v (x_B - x_A)$: the length is contracted in the O frame, since $\gamma \geq 1$.

1.4 Addition of velocities

Two observers see an object moving with $v' = \mathrm{d}x'/\mathrm{d}t'$ and $v = \mathrm{d}x/\mathrm{d}t$ respectively. Their relative velocity is u. Differentiating we get:

$$v' = \frac{\gamma(\mathrm{d}x - v\,\mathrm{d}t)}{\gamma\left(\mathrm{d}t - \frac{u\mathrm{d}x}{c^2}\right)} = \frac{v - u}{1 - \frac{uv}{c^2}}. \tag{6}$$

Two interesting limits of this formula are: $v' = v - u$ if $u \ll c$ or $v \ll c$; and $v' = c$ if $v = c$ for whatever u.

1.5 Tensor notation

The position four-vector is $x^\mu = (ct, x, y, z)$. The Euclidean scalar product is given by $x \cdot y = \delta_{\mu\nu} x^\mu x^\nu$. If we substitute the identity $\delta_{\mu\nu}$ with another metric we can find a more general metric space.

The Minkowski metric is $\eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, 1)$. The separation 4-vector is $\mathrm{d}x^\mu = (c\,\mathrm{d}t\,, \mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z)$.

Using Einstein summation notation, we can write the spacetime interval as $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$.

Specifically for the Minkowski metric we have the relation $\eta_{\mu\nu} = \eta^{\mu\nu}$: it is its own inverse. For a general metric $g_{\mu\nu}$ this will not hold.

How do we express the Lorentz boosts? They preserve $\mathrm{d}s^2$, therefore they look like $x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu$, with the $(1, 1)$ tensors $\Lambda_\mu{}^\nu$ satisfying $\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$. This is called the *pseudo-orthogonality* relation.

The metric allows us to raise and lower indices. Raising an index in the pseudo-orthogonality relation gives us: $\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta \eta^{\beta\sigma} = \delta_\alpha{}^\sigma$, therefore $\eta_{\mu\nu}\Lambda^\nu{}_\beta \eta^{\beta\sigma}$ is the inverse of a Lorentz transformation.

Four-vectors can also have their indices down, and they will transform according to the inverse of Lorentz transformations:

$$
\begin{aligned}
(\eta_{\alpha\mu} x^\mu)' &= \eta_{\alpha\mu}\Lambda^\mu{}_\nu x^\nu && \text{(7a)}\\
&= \Lambda_{\alpha\sigma}\delta^\sigma{}_\nu x^\nu && \text{(7b)}\\
&= \Lambda_{\alpha\sigma}\eta^{\sigma\beta}\eta_{\beta\nu} x^\nu && \text{(7c)}\\
&= \Lambda_\alpha{}^\beta x_\beta\,. && \text{(7d)}
\end{aligned}
$$

We will write our laws as tensorial equations, which are covariant.

By pseudo-orthogonality, the scalar product $A_\mu B^\mu$ is a covariant (that is, invariant) scalar. Of course it is equal to $A^\mu B_\mu$.

Definition 1.3 (Tensor). *A* (p, q) tensor *is an object* $M^{\nu_1\dots\nu_q}_{\mu_1\dots\mu_p}$ *with many components indexed by* $p + q$ *indices, which transforms as:*

$$
M^{\nu_1\dots\nu_q}_{\mu_1\dots\mu_p} \to \Lambda_{\mu_1}{}^{\mu_1'} \dots \Lambda_{\mu_p}{}^{\mu_p'} \Lambda^{\nu_1}{}_{\nu_1'} \dots \Lambda^{\nu_q}{}_{\nu_q'} M^{\nu_1'\dots\nu_q'}_{\mu_1'\dots\mu_p'} \tag{8}
$$

under Lorentz transformations $\Lambda_\mu{}^\nu$.

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Last lecture we introduced tensors.

An example of those is the EM tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_x & B_y \\ -E_y/c & B_x & 0 & -B_z \\ -E_z/c & -B_y & B_z & 0 \end{bmatrix}, \tag{9}$$

which, it can be checked, transforms as a $(0,2)$ tensor. Also, we can define the current vector $j^\mu = (c\rho, j^i)$. Then, the Maxwell equations read:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \qquad \text{and} \qquad \partial_{[\mu} F_{\nu\rho]} \,. \tag{10}$$

They are covariant!

1.6 Particles in motion

In Newtonian mechanics, the motion of a particle is described by a function of time $x^i = x^i(t)$.

In special relativity, we introduce the concept of *worldline*. It must be parametrized with respect to some parameter λ, such that $x^\mu = x^\mu(\lambda)$. A preferred choice for λ is the proper time of the particle, $\lambda = \tau$.

We then define the 4-velocity:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \,. \tag{11}$$

It is a tensor since it is the product of a scalar and a tensor.

Multiplying $u^\mu u_\mu$ we always get $-c^2$, since:

$$u^\mu u_\mu = \frac{\mathrm{d}x^\mu \, \mathrm{d}x_\mu}{\mathrm{d}\tau^2} = -c^2 \frac{\mathrm{d}s^2}{\mathrm{d}s^2} \tag{12}$$

We can make the expression explicit using $\mathrm{d}\tau = \gamma \, \mathrm{d}t$, which gives us $u^\mu = (\gamma c, \gamma v^i)$. In the frame of the particle, $u^\mu = (c, 0)$.

The *four-momentum* of a particle is defined as:

$$p^\mu = m u^\mu = (m\gamma c, m\gamma v^i) \,. \tag{13}$$

The component p^0 is mc at $v = 0$. What does it mean? we can expand it for small v:

$$\frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}} \sim mc\left(1 + \frac{v^2}{2c^2}\right) = mc + \frac{1}{c}\frac{mv^2}{2} \,. \tag{14}$$

We get the mass, plus a kinetic energy term: more explicitly, $cp^0 = mc^2 + {}^1\!/\!{}_2 mv^2$.

We can rewrite Newton's first law in SR:

Proposition 1.4 (Newton I)**.** *A free particle moves with constant u^μ, or*

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \tag{15}$$

To express this in an easier way we introduce the 4-acceleration:

$$a^\mu \stackrel{\text{def}}{=} \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} \tag{16}$$

We now with to introduce the concept of a path minimizing proper time. Recall Snell's law, which allows us to relate the angles of incidence of light when it passes between one medium to another, if they have different indices of refraction:

$$\frac{\sin(\theta_2)}{\sin(\theta_1)} = \frac{n_1}{n_2} = \frac{v_2}{v_1}\,. \tag{17}$$

This can be shown to be equivalent to light minimizing the time it takes to move from a point in one medium to a point in the other.

Analogously, saying that a massive particle travels along the worldline which minimizes τ is equivalent to Newton's first principle.

We want to perturb a generic worldline x^μ with some $\mathrm{d}x^\mu$, and consider the proper time functional τ which gives the proper time of a generic trajectory: we impose

$$\frac{\tau[x^\mu + \varepsilon^\mu] - \tau[x^\mu]}{|\varepsilon^\mu|} = \frac{\delta\tau}{\delta x^\mu} \stackrel{!}{=} 0\,, \tag{18}$$

where a limit $|\varepsilon^\mu| \to 0$ is implied, and only the linear terms are considered.

The proper time functional for paths between A and B is given $\tau = \int_A^B \mathrm{d}\tau$. We can rewrite it as:

$$\tau = \int_A^B \mathrm{d}\tau\, \frac{\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = \int_A^B \mathrm{d}\tau\, \frac{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}{\mathrm{d}\tau^2}\,. \tag{19}$$

We now consider a perturbation $\varepsilon^\mu = \delta^\mu_1 \delta x$:

$$\tau_{AB}[x+\varepsilon] = \int_A^B \mathrm{d}\tau \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\mathrm{d}\delta x}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \right]. \tag{20}$$

We can discard a second order term $\left(\mathrm{d}\delta x/\mathrm{d}\tau\right)^2$, and subtract off $\tau_{AB}[x]$: we are left with

$$\delta\tau = -\frac{2}{c^2}\int_A^B \mathrm{d}\tau\, \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\delta x}{\mathrm{d}\tau} \tag{21}$$

Now, we integrate by parts, disregard the boundary terms since the endpoints of the path cannot be deformed, and get:

$$\frac{\delta \tau_{AB}}{\delta x} = +\frac{2}{c^2} \int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} \,, \tag{22}$$

which proves the equivalence for this type of perturbation, the others are analogous.

The generalization of Newton's second law, which at low speeds is $F^i = ma^i$, can be similarly restated as $\delta S = 0$, for the action $S = \int \mathrm{d}\tau$.

1.7 Motion of light rays

For light we cannot compute u^μ with the previous definition, since its proper time is always zero.

Instead, we *define* u^μ to be a normalized null-like vector, such that $x^\mu = \lambda u^\mu$ for some λ.

We know from quantum mechanics that $E = \hbar\omega$, where $\hbar = h/(2\pi)$ and $\omega = 2\pi/T = 2\pi f$.

The momentum is proportional to the wavevector k^i: $p^i = \hbar k^i / c$. The relativistic generalization of this fact is

$$p^\mu = \left(\frac{\hbar\omega}{c}, \frac{\hbar k^i}{c} \right) = \frac{\hbar k^\mu}{c} \,. \tag{23}$$

Since the momentum of light must be null we have that necessarily $\omega = |k|$.

1.8 Doppler effect

We take a special case: radiation goes in the same direction as the observer. In the O frame we have $k^\mu = (\omega, \omega, 0, 0)$.

The observer, moving with velocity v, measures $k^{\prime,\mu}$. This can be easily computed with a Lorentz transformation: $k^{\prime,\mu} = \Lambda^\mu{}_\nu k^\nu$.

We are mostly interested in $k^{\prime,0} = \omega'$: it comes out to be $\omega' = \gamma\omega + (-\gamma\beta)\omega = (1 - v/c)\gamma\omega$.

Some notes: at slow speeds $\omega' \approx (1 - v/c)\omega$; we have $f' < f$ when source and observer are moving away from each other.

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1.9 Bases

In Euclidean 2D geometry we can choose, for example, the basis $e_1 = (1, 0)^\top$ and $e_2 = (0, 1)^\top$. This basis is orthonormal with respect to the scalar product $g_{\mu\nu} = \delta_{\mu\nu}$: $e_{(\alpha)} \cdot e_{(\beta)} = e^\mu_{(\alpha)} e^\nu_{(\beta)} g_{\mu\nu} = g_{(\alpha)(\beta)}$.

I use parentheses around indices to denote the fact that they are not tensorial indices, but instead denote which basis vector we are considering. We express our vectors in components with respect to this basis.

In SR, we can do the same: our coordinate basis can be given by $e^\mu_{(\alpha)} = \delta^\mu_{(\alpha)}$. Now, the orthonormality $e_{(\alpha)} \cdot e_{(\beta)} = g_{(\alpha)(\beta)}$ holds with respect to $g_{\mu\nu} = \eta_{\mu\nu}$.

1.10 Observers & observations

Every observer will be characterized by their trajectory $x^\mu(\tau)$. We can associate a coordinate system with the observer: the one in which the observer's own 4-velocity u^μ is the time-like unit vector (rescaled by a factor of c: $u^\mu = ce_{(0)}$).

When the observer sees a particle with $p^\mu = (E_p/c, p^i)$ they measure the energy of the particle to be $p^0 c$: in this frame this is $E_{\text{measured}} = -e^\mu_{(0)} p_\mu c = -u^\mu p_\mu c$. Do note that this is a covariant expression, while p^0 is not: the energy of a particle with 4-momentum p^μ measured by an observer with 4-velocity u^μ is an invariant.

In the rest frame of the observer, their own 4-velocity is $(c, \vec{0}) = c(1, \vec{0})$. In the rest frame of the particle, its own energy is measured to be mc^2. The measured energy by an observer such that the product of the 4-velocities of the particle and of the observer is $-\gamma c^2$ is $m\gamma c^2$.

The Earth moves with speed $10^{-4}c$ around the Sun.

Now we can start using $c = 1$. We can put the c back whenever we want with dimensional analysis.

2 Gravity

2.1 The Equivalence Principle

Just like Newton supposedly thought about universal gravity when, while looking at the sky, an apple fell on his head; Einstein supposedly thought up the equivalence principle when he saw a man falling from a rooftop.

Proposition 2.1 (Equivalence principle)**.** *Experiments in a small free falling system over a short amount of time give the same result as experiments in an inertial frame in empty space.*

Why "small"? The gravitational field is not really homogeneous. The idea is that gravity can only be removed *locally*.

If we were to see that objects fall differently even in the same neighbouhood then we would lose the EP.

Definition 2.1. *The* inertial *mass is an object's resistance to motion:* $m_{inertial} = F^i/a^i$.

Definition 2.2. *The* gravitational *mass is the one which defines the gravitational force on an object:* $m_{gravitational} = |F|r^2/(GM)$.

These are *a priori* different, but experimentally equal: in general the gravitational acceleration is given by

$$a^i = \frac{GMr^i}{r^3}\frac{m_{\text{gravitational}}}{m_{\text{inertial}}} \tag{24}$$

If the ratio of masses depended on the material, this could vary.

We can do a torsion pendulum experiment: the torsion applied by the Coriolis effect on a pendulum depends on the inertial mass, while its restoring force depends on the gravitational mass. Experimentally we have measured them to be equal with an accuracy of 10^{-12}.

A person on a rocket accelarating at g experiences the same acceleration as a person standing on Earth.

2.2 Gravitational redshift

We treat it now in a weak field approximation.

Alice sends radiation to Bob from a higher altitude on Earth. Alice sends it with frequency f, Bob receives it with f'. They are at rest with respect to one another: there is no kinematic Doppler effect here.

We do this by applying the equivalence principle! We imagine A and B to be standing in a rocket which is accelerating at g: there is no more gravity.

Bob will receive a greater frequency: $f' > f$. This can be seen by imagining two consecutive wavefronts as two particles. Alice sends them $\Delta t_A = 1/f$ apart, Bob receives them as $\Delta t_B = 1/f'$ apart.

If the rocket is at rest, the time for the radiation to reach B is h/c; if the rocket is moving then the time is $< h/c$.

When the second wavefront starts moving the rocket is already going: the second wave starts later but it has less distance to travel. Therefore $\Delta t_A > \Delta t_B$, which implies $f' > f$.

Claim 2.1. *The first terms in the expansion are:*

$$f' = f\left(1 + \frac{gh}{c^2} + O\left(\left(\frac{gh}{c^2}\right)^2\right)\right). \tag{25}$$

2.3 Potentials

In electromagnetism, the potential energy between a charge Q and a test charge q is $U = kQq/r$: then we define the electromagnetic potential $V = U/q$ which has the advantage of being test-charge independent.

Similarly, we define the gravitational potential $\Phi = U/m = gh$. Then, the second order term in the formula for the redshift becomes $\Delta\Phi c^{-2}$: now we can properly say that this *weak field* means $\Delta\Phi c^{-2} \ll 1$.

This is surely the case for the cases we can treat concretely. If two people are separated by 1 km of difference in altitude, they have $\Delta\Phi c^{-2} \approx 10^{-13}$: the difference they will experience is one second in a million years.

Our expression from Φ in the newtonian approximation is $\Phi = GM/r$.

We can say even now by dimensional analysis that $GM/(rc^2)$ is the parameter which tells us how relevant the gravitational effects are: if it is similar to 1 we must consider GR.

This is very close to the expression for the Schwarzschild radius: it is $r = M$ in natural units $c = G = 1$, while the correct expression is $r = 2M$: that one can actually be recovered exactly if we calculate the radius at which the escape velocity is equal to c.

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3 The mathematical description of curved spacetime

The metric describes the spacetime. It depends on the coordinates.

Euclidean metric In 2D, it is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. We can express it as

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \, \mathrm{d}x_\mu = \mathrm{d}x^\mu \, \mathrm{d}x^\mu \, g_{\mu\nu} \,, \tag{26}$$

computed with $g_{\mu\nu} = \delta_{\mu\nu}$.

Polar coordinates Let us see how this changes when we change coordinates: for example, we can go to polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\big(y/x\big) \end{cases} \,. \tag{27}$$

Let us compute the differentials $\mathrm{d}x$ and $\mathrm{d}y$:

$$\mathrm{d}x = \mathrm{d}r\cos(\theta) - r\sin(\theta)\,\mathrm{d}\theta \,, \tag{28}$$

and

$$dy = dr\sin(\theta) + r\cos(\theta)\,d\theta\ . \tag{29}$$

Plugging these into the metric and simplifying we get

$$ds^2 = dr^2 + r^2\,d\theta^2\ . \tag{30}$$

It is not clear to see that these are equivalent. We then want to compute scalar quantities to characterize them.

Another issue is the fact that the polar metric is singular at the origin: we'd have to invert

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{31}$$

at $r = 0$: we cannot do it! This is a *coordinate singularity*. There is nothing wrong with the space; $\mathbb{R}^2$ is perfectly regular at $(0, 0)$, but our coordinate description fails: what value of θ should we assign to that point?

Spherical coordinates In $\mathbb{R}^3$ we have the same issue. We use:

$$\begin{cases} x = \cos(\theta)\cos(\varphi) \\ y = \cos(\theta)\sin(\varphi) \\ z = \sin(\theta) \end{cases}, \tag{32}$$

and it is a simple computation as before to see that

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin(\theta)^2 \end{bmatrix}, \tag{33}$$

therefore these coordinates are not defined on the *whole z axis*!

General coordinate transformations Now we consider a general transformation of space, which we denote by $x'^{\mu}(x^{\mu})$: we see how the metric should change in order for the spacetime distance to be invariant: we define $g'_{\mu\nu}$ by

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} \stackrel{!}{=} g'_{\mu\nu}\,dx'^{\mu}\,dx'^{\nu}\ . \tag{34}$$

This means that the metric changes as a tensor:

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}\,, \tag{35}$$

and we can see that it transforms as a $(0,2)$ tensor since the primes are in the denominator: it transforms with the *inverse* of the Jacobian matrix.

The inverse metric transforms as:

$$g^{\mu\nu} \to g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta} , \tag{36}$$

which we can check by proving that $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$ is conserved when we transform both the metric and the inverse metric. We get:

$$g'^{\mu\nu} g'_{\sigma\tau} \delta^{\sigma}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta} \delta^{\sigma}_{\nu} \frac{\partial x^{\pi}}{\partial x'^{\sigma}} \frac{\partial x^{\lambda}}{\partial x'^{\tau}} g_{\pi\lambda} , \tag{37}$$

which can be simplified using the relations between the partial derivatives:

$$\frac{\partial x'^{\nu}}{\partial x^{\beta}} \delta^{\sigma}_{\nu} \frac{\partial x^{\pi}}{\partial x'^{\sigma}} = \frac{\partial x^{\pi}}{\partial x^{\beta}} = \delta^{\pi}_{\beta} , \tag{38}$$

since we are multiplying the Jacobian with its inverse, thus we get the identity. One can make this reasoning more explicit by seeing it as an application of the chain rule: x^{μ} is a function of x'^{μ} which is a function of x^{μ}. Plugging this in we get

$$g'^{\mu\nu} g'_{\sigma\tau} \delta^{\sigma}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} g^{\alpha\beta} \delta^{\pi}_{\beta} \frac{\partial x^{\lambda}}{\partial x'^{\tau}} g_{\pi\lambda} , \tag{39}$$

and now we can apply our hypothesis that $g^{\alpha\beta} g_{\beta\lambda} = \delta^{\alpha}_{\lambda}$ to contract some more indices, get one more multiplication of a Jacobian with its inverse, which we can simplify in the same way as before to finally get the identity. This proves that the inverse metric is a $(2,0)$ tensor.

3.1 Lengths, areas, volumes and so on

In 4D space, if we fix x^0 and x^3, we can consider an area element like $\mathrm{d}A = \sqrt{g_{11}}\,\mathrm{d}x\,\sqrt{g_{22}}\,\mathrm{d}x^2$.

Similarly, the 4-volume is just

$$\mathrm{d}v = \sqrt{-g_{00} g_{11} g_{22} g_{33}}\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = \sqrt{-g}\,\mathrm{d}^4 x , \tag{40}$$

where $g = \det g_{\mu\nu}$. We have shown it for a diagonal metric, but it can be shown that it holds for any general metric. The minus sign comes from the fact that our metric has signature -1 (and, as we will see, this holds in general): the determinant is then negative, and to take the square root we need a positive number.

Claim 3.1 (Unproven)**.** *A metric can be always diagonalized, at least locally.*

When it is diagonal, then $\mathrm{d}v = \sqrt{-g}\,\mathrm{d}^4x$. Then the quantity $\sqrt{-g}\,\mathrm{d}^4x$ is a scalar.

Intuitively, the claim follows from the equivalence principle: we put a free-falling observer in our space. They will perceive spacetime as being flat.

Under a diffeomorphism with the determinant of the jacobian equal to J we have the following transformation law for the determinant:

$$g' = J^{-2}g \qquad \Longrightarrow \qquad \sqrt{-g'} = J^{-1}\sqrt{-g}\,. \tag{41}$$

This can be expressed by saying that $\sqrt{-g}$ is a *tensor density* of weight -1.

Definition 3.1. *A tensor density of weight w transforms just like a tensor, except we need to multply by a factor J^w in the transformation, where J is the determinant of the Jacobian of the transformation.*

A (p,q) tensor density *of weight w is an object $M^{\nu_1\ldots\nu_q}_{\mu_1\ldots\mu_p}$ with many components indexed by $p+q$ indices, which transforms as:*

$$M^{\nu_1\ldots\nu_q}_{\mu_1\ldots\mu_p} \to J^w \Lambda_{\mu_1}{}^{\mu_1'} \ldots \Lambda_{\mu_p}{}^{\mu_p'} \Lambda^{\nu_1}{}_{\nu_1'} \ldots \Lambda^{\nu_q}{}_{\nu_q'} M^{\nu_1'\ldots\nu_q'}_{\mu_1'\ldots\mu_p'} \tag{42}$$

under diffeomorphisms with Jacobian matrix $\Lambda_\mu{}^\nu$, with determinant J.

When we transform the 4-volume element $\mathrm{d}^4x' = J\,\mathrm{d}^4x$, we get a Jacobian: the element d^4x is a *tensor density of weight 1*.

Then we see that the volume element $\sqrt{-g}\,\mathrm{d}^4x$ is a *scalar*, since when multiplying tensor densities their weights are added.

3.2 Vectors in curved spacetime

They are not objects *in spacetime*: instead they belong to the *tangent space*.

For each point x in the manifold we define a basis at that point: $\left\{e^\mu_{(\alpha)}\right\}(x)$, where μ is a vector index while (α) denotes which vector we are considering.

Any vector $a^\mu(x)$ can be then decomposed as

$$a^\mu(x) = a^{(\alpha)}(x)e^\mu_{(\alpha)}(x)\,. \tag{43}$$

Since spacetime is not flat, the dependence on x is not trivial.

The scalar product between the vectors can be expressed with respect to the basis:

$$a(x)\cdot b(x) = a^{(\alpha)}b^{(\beta)}e_{(\alpha)}\cdot e_{(\beta)}\,, \tag{44}$$

and we can select our basis so that at every point it is orthonormal:

$$e_{(\alpha)}\cdot e_{(\beta)} = \eta_{(\alpha)(\beta)}\,. \tag{45}$$

We *cannot sum vectors in different tangent spaces*. But we want to: we need to define derivatives! To solve this issue, we will introduce the notion of *parallel transport*.

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