Gravitational physics notes

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Contents

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0.1 Introduction

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Reading material: slides (to be used as an index of what is treated in the course), Hobson, Michele Maggiore.

This is a general class on gravitational physics and GW, it does not really follow any textbook: the field is young so there is not really a textbook.

The slides will be provided before lectures. There will be no home assignments.

The idea is that formulas are important, detailed calculations and derivations are not.

The target is to be able to read a research paper on GW and understand it. We will not go into very much detail on any topic: the program of the class is very large.

0.1.1 Topics

Understanding what GW are: how they are described, how they are generated, what is their physical effect.

Some astrophysical and cosmological GW courses. The professor's background is more in experimental physics than in astrophysics and cosmology.

Interactions of GW with light and matter: ideas, techniques, experiments to detect GWs, especially GW interferometers.

Analysis of GW signals.

What we can learn from GW, overview of the most significant recent papers.

What follows is a long, somewhat divulgative introduction.

Einstein thought their detection impossible. Now we can not only *detect* them, we can actually *observe* them.

They are a test of GR in *extreme* conditions, where the weak-field approximation does not apply.

We can derive properties of matter in these extreme conditions, such as the equation of state for a neutron star.

GWs are "ripples" in the metric of spacetime, described by a quadrupole formula: the quadrupole is

$$Q_{jk} = \int \rho x_j x_k \, \mathrm{d}^3 x \,, \tag{1}$$

and then the perturbation propagates like

$$h_{jk} = \frac{2}{r} \frac{\mathrm{d}^2 Q_{jk}}{\mathrm{d}t^2} \,. \tag{2}$$

What generates GWs are non-spherically symmetric perturbations: by Jebsen-Birkhoff, if we have spherical symmetry there is no perturbation in the vacuum metric.

They "stretch" space by squeezing one direction and stretching a perpendicular one.

The typical relative scale of these perturbations is

$$\frac{\Delta L}{L} \sim 10^{-21} \,, \tag{3}$$

which is *really small*: if we multiply it by the radius of the Earth's orbit we get a length on the order of the size of an atom.

An interesting thing which could emit in the $\sim 1\,\mathrm{Hz}$ range are extreme Mass Ratio inspirals: we have what is effectively a test particle in a strong gravitational field.

We have different kinds of interferometers: for now we have used ground interferometers, there are also space detectors like LISA, Pulsar Timing Arrays at higher frequencies, and inflation probes (?).

In binary systems, we have different stages in the pulsation: an almost stationary one, the inspiral, the coalescence, and finally the ringdown.

In the early years, it was thought that GWs might be a coordinate artifact which could be "gauged away".

In 1959, Joseph Weber proposed a "resonant bar" detector. These are based upon a sound principle: one of the last ones was AURIGA, the issue was that the sensitivity was insufficient and they are only sensitive in a specific frequency range.

GWs were detected indirectly using Hulse-Taylor pulsars: they measured the energy loss of a binary pulsar-NS system, which implied the loss of energy through gravitational wave emission.

The famous graph is not a fit line, it is the prediction based upon the measured orbital parameters.

Now we use laser interferometers: they are broad-band (a couple orders of magnitude, from 10 Hz to 1 kHz), they are inherently differential (as opposed to the single-mode excitation of a resonant bar).

We can use Fabry-Perrot cavities in order to amplify effective length. There is also a power recycling mirror in order for the light not to go back to the laser: modern lasers are on the order of 100 kW, so there is a huge amount of power circulating in the cavities.

We can plot the sensitivity of the interferometers. On the x axis we put the frequency of the incoming wave. On the y axis we put the amplitude spectral density $h(f)$, which is measured in $\text{Hz}^{-1/2}$.

Why?

The curve describes where the noise dominates. We can plot both the theoretical sensitivity and the measured one.

The signal comes out buried in noise, we must extract it in some way, like by correlating to a standard test signal.

Friday 2020-3-13, compiled 2020-03-13

What we will do in the first lessons:

1. A quick review of GR;
2. linearization and GW in free space;
3. the physical effect of GW: free falling reference frames, detector frame;
4. GW sources : binary systems, multipole expansion and quadrupole approximation, GW back reaction: energy & momentum loss, Hulse-Taylor pulsar;
5. GW sources: a rotating rigid body.

0.2 A quick review of GR

We start from special relativity. The "old" way to do transformations are galilean transformations: in 2D they are

$$t' = t \tag{4}$$

$$x' = x - vt\,. \tag{5}$$

There are issues with these, such as the invariance of the speed of light. So, we use Lorentz transformations:

$$ct' = \gamma(ct - \beta x) \tag{6}$$

$$x' = \gamma\big(x - \beta c t\big)\,, \tag{7}$$

where $\beta = v/c \leq 1$, c being the speed of light, and $\gamma = 1/\sqrt{1-\beta^2} \geq 1$.

These preserve the spacetime interval

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2\,. \tag{8}$$

The intervals between two events can be spacelike ($\Delta s^2 > 0$), null ($\Delta s^2 = 0$) or timelike ($\Delta s^2 < 0$).

We can express this using an infinitesimal time interval

$$\mathrm{d}s^2 = \mathrm{d}x^\mu\, \eta_{\mu\nu}\, \mathrm{d}x^\nu\ , \tag{9}$$

where we use Einstein summation convention. We are going to use the mostly plus metric convention.

We can define the differential *proper time* along a curve, by

$$c^2\,\mathrm{d}\tau^2 = -\,\mathrm{d}s^2 = c^2\,\mathrm{d}t^2\left(1-\beta^2\right) = \frac{c^2}{\gamma^2}\,\mathrm{d}t^2\ , \tag{10}$$

which means that $\mathrm{d}\tau = \mathrm{d}t\,/\gamma$. We can use this as a *covariant* parametrization of a spacetime curve.

For curved spacetime, we model it as a 4D semi-Riemannian manifold with signature $(1,3)$. Since it is a manifold, the parametrization of points in spacetime must be a homeomorphism, and we ask for the *transition maps* between two regions of spacetime to be infinitely differentiable. The set of local charts is called an atlas. The charts are maps from $\mathbb{R}^4$ to the manifold.

The metric is a function of the point at which we are, and (the way it changes) describes the local geometry of the manifold. Only the symmetric part of the metric appears in the spacetime interval, therefore we say that the metric is always symmetric without losing any generality.

The metric is a bilinear form at each point of the manifold, and it transforms as a $(0,2)$ tensor. The components of this tensor in our chosen reference frame are $g_{\mu\nu}$. The choice of coordinates is arbitrary and tricky.

In a neighborhood of a point we can always choose a reference frame (Riemann normal coordinates) such that $g_{\mu\nu} = \eta_{\mu\nu}$, and $g_{\mu\nu,\alpha} = 0$ (partial derivatives calculated *at that point*), but the second derivatives $g_{\mu\nu,\alpha\beta}$ cannot all be set to zero.

Vectors in a manifold are defined in the tangent space *at a point*. Formally, we define curves parametrically as $X^\mu(\lambda)$.

Then, we define the tangent vector to the curve as the *directional derivative* operator along the curve:

$$\vec{v}(f) = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_C = \frac{\partial f}{\partial x^\mu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\,, \tag{11}$$

for any scalar field f. The motivation for this definition, as opposed to just taking the tangent vector to the curve, is the fact that there is no *intrinsic* way to to that.

If we define a curve using a coordinate as a parameter, with the other coordinates staying constant along the curve, this is called a *coordinate curve*.

Vectors defined at different points are in different spaces, we cannot compare them directly.

Tangent vectors to coordinate lines are called coordinate basis vectors $e_{(\mu)}$, where μ is not a vector index but instead it spans the basis vectors. So, any vector can be written as a linear combination as $\vec{v} = v^\mu e_\mu$. We also have $e_\mu \cdot e_\nu = g_{\mu\nu}$, so, in order to find the components of the scalar product $v \cdot w$ we need to do $v^\mu w^\nu g_{\mu\nu}$.

This is because $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}s \cdot \mathrm{d}s = \left(\mathrm{d}x^\mu\, e_\mu\right) \cdot (\mathrm{d}x^\nu\, e_\nu)$.

An orthonormal basis in one for which $e_\mu \cdot e_\nu = \eta_{\mu\nu}$.

Dual basis vectors e^μ are defined by $e^\mu e_\nu = \delta^\mu_\nu$.

We write a co-vector (or dual vector) as a linear combination of these: $v = v_\mu e^\mu$.

Then, we can raise and lower indices like

$$g_{\mu\nu} v^\mu w^\nu = v \cdot w = v_\mu e^\mu \cdot w^\nu e_\nu = v_\mu w^\nu \delta^\mu_\nu = v_\mu w^\nu \,. \tag{12}$$

The inverse metric is defined as $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$.

Tensors are geometrical objects which belong to the dual space to the cartesian product of n copies of the tangent space and m copies of the dual tangent space. The type of such a tensor is then said to be (n, m), and its rank is $n + m$. This definition means that the tensor is a *multilinear* transformation.

Once we have a coordinate system, we can move to another via a coordinate transformation

$$x'^\mu = x'^\mu\left(x^\mu\right) \implies \mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu \,. \tag{13}$$

A scalar does not transform: $\phi(x) = \phi'(x')$. A vector's component do transform: we find the transformation law by imposing $v = v'$ in components. This works both for covariant and contravariant vectors, and we find that these transform using either the Jacobian of the transformation or its inverse.

In order to compute derivatives we need to compare vectors in different tangent spaces: we need to "connect" infinitesimally close tangent spaces, and the tool to do so is indeed called a connection, or covariant derivative.

For a scalar field S the covariant derivative is $\nabla_\alpha S = \partial_\alpha S$.

A torsionless manifold is one in which

$$[\nabla_\mu, \nabla_\nu] S = 0 \,, \tag{14}$$

for a scalar field S. This means that

$$\nabla_{[\mu} \nabla_{\nu]} S = \nabla_{[\mu} \partial_{\nu]} S = \partial_{[\mu} \partial_{\nu]} S - \Gamma^\alpha_{[\mu\nu]} \partial_\alpha = 0 \implies \Gamma^\alpha_{[\mu\nu]} = 0 \,. \tag{15}$$

Parallel transport: intuitively, we move along a curve and keep the angle with respect to the tangent vector constant. Formally, if u^μ is the tangent vector to the curve and V^μ is the vector we want to transport, we set $u^\mu \nabla_\mu V^\nu = 0$.

The Riemann tensor is defined as the commutator of the covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta \,, \tag{16}$$

and it can be expressed in terms of the Christoffel symbols as

$$R^\mu_{\nu\rho\sigma} = -2\Big(\Gamma^\mu_{\nu[\rho,\sigma]} + \Gamma^\beta_{\nu[\rho}\Gamma^\mu_{\sigma]\beta} \Big) \,. \tag{17}$$

Geodesics: they are "the straightest possible path between two points". They stationarize the proper length. Formally, they are curves whose tangent vector is parallel-transported along the curve.

We actually do not need to say that the derivative of the tangent vector with respect to the parameter is zero: it can be nonzero, as long as it is parallel to the tangent vector.

So, we could say that

$$h_{\nu\rho}\Big(u^\mu \nabla_\mu u^\nu \Big) = 0 \,, \tag{18}$$

?

The path that a massive particle follows in the absence of external forces is a geodesic. We can describe the separation between two particles which follow geodesics: this is described by the equation of geodesic deviation. We take a geodesic x^μ and another $y^\mu = x^\mu + \xi^\mu$, with ξ^μ being (at least initially) small.

We can choose a coordinate system in which $\Gamma^\mu_{\nu\rho} = 0$. So,

$$\left.\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}u^2}\right|_P = 0 \,, \tag{19}$$

$$\left.\left(\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}u^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}y^\nu}{\mathrm{d}u} \frac{\mathrm{d}y^\rho}{\mathrm{d}u} \right)\right|_P = 0 \,, \tag{20}$$

where u is the tangent vector to the geodesics. We approximate the Christoffel symbols to first order as

$$\left.\Gamma^\mu_{\nu\rho}\right|_Q = \xi^\alpha \partial_\alpha \Gamma^\mu_{\nu\rho} \,. \tag{21}$$

If we subtract the two, we get

$$\ddot{\xi}^\mu + \Big(\partial_\alpha \Gamma^\mu_{\nu\rho} \Big) \dot{x}^\nu \dot{x}^\rho \xi^\alpha = 0 \,, \tag{22}$$

but the first term is not an intrinsic derivative: that would be given by

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{\xi}^\mu + \Gamma^\mu_{\nu\rho}\xi^\nu\dot{x}^\rho\right) = \ddot{\xi}^\mu + \left(\partial_\alpha\Gamma^\mu_{\nu\rho}\right)\xi^\alpha\dot{x}^\nu\dot{x}^\rho\,, \tag{23}$$

which means that

$$0 = \frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} + \left(\partial_\alpha\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\alpha}\right)\xi^\alpha\dot{x}^\mu\dot{x}^\nu = \frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} + R^\mu_{\nu\rho\sigma}u^\nu u^\rho\xi^\sigma\,. \tag{24}$$

The gravitational field is described by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}\,. \tag{25}$$