Advanced astrophysics notes

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Stellar oscillations

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1.1 Introduction

The course is held by Paola Marigo, Michele Trabucchi.
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1.1.1 Topics

They are selected topics in stellar physics.

1. Stellar pulsations and Astroseismiology (dr. Michele Trabucchi);
2. stellar winds (dr. Paola Marigo);
3. final fates of massive & very massive stars (dr. Paola Marigo).

For information about the basics in stellar physics refer to the course "astrophysics II" inside the bachelor's degree in astronomy (second semester). It can be taken as an optional course.

Material:

1. *Introduction to stellar winds* by Lamers, Cassinelli.
2. *Stellar Atmospheres: Theory and observations* (lecture notes from 1996).

and more on Paola Marigo's site.

Stellar oscillations

. . . see slides.
Material: slides on moodle or Marigo's page.

1. *Pulsating stars* by Catelan & Smith (introductory);

2. *Theory of stellar pulsation* by Cox (harder).

Written exam, partial exam on stellar pulsation.

Reference books can be found in the Moodle: they are ordered by difficulty, Catelan to Aerts to Salaris.

Stellar Winds

They are moving flows of materials ejected by stars, with speeds generally between 20 to 2×10^3 km/s.

See, for example, the *Bubble Nebula* in Cassiopea, there is a $45 M_\odot$ star ejecting stellar wind at 1700 km/s.

Diagram: luminosity vs effective temperature. We see the *main sequence*. We can also plot the *mass loss rate*, $\dot{M} > 0$ in solar masses/year. Another important parameter is v_∞, the asymptotic terminal velocity of the wind.

Diagram: mass loss (or gain) rate vs age of star.

Stellar winds affect stellar evolution, the dynamics of the interstellar medium, the chemical evolution of galaxies.

Momentum is approximately injected with $\dot{M}v$, kinetic energy with $1/2 \dot{M} v^2$. Within 1×10^8 yr around half of the infalling matter is reemitted.

We will start with the basic theory of stellar winds, and then discuss *coronal*, *line-driven* and *dust-driven* winds.

Final fates of massive & very massive stars

Masses over $10 M_\odot$.

1.2 Stellar oscillations

1.2.1 Variability in Astronomy

The first observations of variable stars happended around the year 1600: Fabritius observed the star omicron-Ceti, in the constellation of Cetus. It changes in magnitude by 6 orders of magnitude: several authors report it as a "new star" in the 16-hundreds, before finally in 1667 Bullialdus puts the pieces together and figures out that the star is periodic, with a period of 333 days.

The star o-Ceti was also called Mira, and it is considered a prototype for these long-period variables: they are called *Miras*.

Others are found from the 1600 onwards, but up to the XX century the reason is still unknown. Is it *rotation*, *eclipses*?

For some the cause was discovered to be indeed eclipses, but the Cepheids are different. See for example the δ-Cephei type: we have an asymmetric continuous curve, with no clearly recognizable *dip*, which we would expect to see if there was an eclipsing system. What if stars *pulsate*?

In order to investigate these phenomena, we need to define the *light curve*: it is the luminosity curve over time.

We can also look at the *phased* light curve: in order to plot it however we need the period. The phase is defined as

$$\varphi = \frac{(t - t_0) \bmod \Pi}{\Pi} \in [0, 1) \tag{1.1}$$

where Π is the period. $E(t) = \lfloor (t - t_0)/\Pi \rfloor$ is the epoch.

So, we can plot the magnitude against φ: we will get several curves in the same plot.

We can then measure the period, but if the light curve is multiperiodic we can subtract the model from the curve to see if there are additional periods: this is *prewhitening*.

We can also look at the luminosity in Fourier space, or more generally use other period measuring techniques, such as the Lomb-Scargle periodogram or *phase dispersion minimization*.

A curiosity: how is phase dispersion minimization actually implemented? Minimizing the area of the convex hull of the data seems error-prone, and it would be nice to have an algorithm which did not rely on the residuals from a *model*. Maybe: for each point compute the distance to the k nearest neighbours, add all of these together and minimize this?

Of course there are issues with observational gaps (day-night, full moon): aliases; accuracy, duration of observations...

Also, the period can change in time.

Things have improved a great deal with large-scale surveys and space suveys.

We also have to account for the Nyquist frequency: if we have n observations spaced with a constant interval Δt we will only be able to measure the frequency with a resolution of $\Delta f = (n\Delta t)^{-1}$.

A useful technique for the assessment of a true period is to plot the observed luminosity at a fixed phase with varying (integer) epoch: if the period was assessed exactly, we expect this to be constant. If we see a straight line, then we know we are under or overestimating the period. If we see some other curve, with this diagram we can start to figure out how the period is changing.

Classification of variable stars

By variability type: regular, semi-regular or irregular.

By variability class: *extrinsic*, external to the star: eclipses, transits, microlensing, rotation; *intrinsic*: rotation, eclipses (self-occultation), eruptive and explosive variables, oscillations, secular variations.

Whether rotation is to be considered intrinsic or extrinsic is a matter of taste.

Oscillations can be classified by several criteria.

The geometry can be *radial* (classical pulsators, such as cepheids, RR Lyrae, Miras) or *non-radial*.

The restoring force can be the pressure gradient (p-modes) or the gravitational force (so, bouyancy) (g-modes).

The excitation mechanisms can be different.

The evolutionary phase and mass of the oscillating star can also be different. We distinguish these populations by the sky region in which we see them.

1.2.2 Summary of stellar structure & evolution

In the *Eulerian* view, properties of a gas are fields, the position is the position of an observer. To differentiate position with respect to time is meaningless: position is an independent variable. Any function is a function of position and time: $f = f(r^i, t)$.

In this case, then, the mass underneath a certain layer is

In the *Lagrangian* view, we follow an element of fluid, which has a velocity $\mathrm{d}r^i \big/ \mathrm{d}t = v^i$. We can identify univocally these fluid elements (since the time-evolution is deterministic).

When treating stellar structure & evolution, we identify the fluid elements as mass layers $\mathrm{d}m$. Any function is then a function of mass and time: $f = f(m, t)$. Do note that m is the mass of the whole full sphere under a certain layer, not the mass of the shell.

In the lagrangian case, the expression for the total derivative with respect to time is given by the convective derivative $\mathrm{d}/\mathrm{d}t = \partial_t + v^i \partial_i$ where v^i is the velocity defined before.

Equations of stellar structure

We write these in the spherically symmetric case, using the Lagrangian formalism.

The *continuity equation* is:

$$\boxed{\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \,. \qquad (1.2)}$$

In order to switch between the Lagrangian view, in which the derivatives are done with respect to the mass m, and the Eulerian one, in which we differentiate with respect to the radius r, we use the continuity relation.

Momentum conservation is given by:

$$\boxed{\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \,, \qquad (1.3)}$$

which is the equation for hydrostatic equilibrium: P is the pressure. In the absence of hydrostatic equilibrium, the equation reads:

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2} \,. \tag{1.4}$$

Energy conservation is given by:

$$\boxed{\frac{\mathrm{d}L}{\mathrm{d}m} = \varepsilon - \varepsilon_\nu - \varepsilon_g \,, \qquad (1.5)}$$

where L is the luminosity, ε is the rate of nuclear energy generation per unit mass, while ε_ν is the rate of energy loss due to neutrino emission per unit mass, and ε_g is the work done by the gas per unit mass & time, which can be written as

$$\epsilon_g = \frac{\partial u}{\partial t} - \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\,, \tag{1.6}$$

The *energy transfer* equation is:

$$\boxed{\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla\,,} \tag{1.7}$$

where $\nabla = \partial \log T / \partial \log P$ is the temperature gradient, which has contributions from radiation, conduction, and convection.

With the diffusion approximation, we can write the gradient as

$$\nabla = \nabla_{\text{rad}} = \frac{3}{16\pi acG}\frac{\kappa_R LP}{mT^4}\,, \tag{1.8}$$

where a is a constant depending on the Stefan-Boltzmann constant and the speed of light.

where κ_R is the Rosseland mean opacity, given by

$$\frac{1}{\kappa_R} = \frac{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{1}{\kappa_\nu}\,\mathrm{d}\nu}{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\,\mathrm{d}\nu}\,, \tag{1.9}$$

where B_ν is the Planck function:

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left(\exp\left(\frac{h\nu}{k_B T}\right) - 1\right)^{-1}\,, \tag{1.10}$$

which can be written like this or multiplied by 4π, if we wish to integrate over all solid angles. It does not matter here, since any constant factor simplifies.

Substituting in the result in (1.7) we get:

$$L = -\frac{64\pi^2 ac}{3} r^4 \frac{T^3}{\kappa_R}\frac{\partial T}{\partial m}\,. \tag{1.11}$$

This can be improved by substituting κ_R with κ, a generalized opacity, which is given by the harmonic mean of the Rosseland opacity κ_R and the convective opacity $\kappa_c = 4acT^3/(3\rho\lambda_c)$.

The term λ_c here is the proportionality factor in the equation for the conductive energy flux in terms of the temperature gradient: $\vec{F}_c = -\lambda_c \vec{\nabla} T$.

If we need to deal with convection, this defies any simple modeling. There are instability criteria: where is conduction relevant? This is given by Ledoux's criterion,

$$\nabla_{\text{rad}} > \nabla_{\text{ad}} - \frac{\chi_\mu}{\chi_T}\nabla_\mu\,, \tag{1.12}$$

where:

$$\nabla_\mu = \frac{\mathrm{d}\log\mu}{\mathrm{d}\log P} \tag{1.13a}$$

$$\nabla_{\mathrm{ad}} = \left(\frac{\partial\log T}{\partial\log P}\right)_{\mathrm{ad}} \tag{1.13b}$$

$$\chi_\mu = \left(\frac{\mathrm{d}\log P}{\mathrm{d}\log\mu}\right)_{\rho,T} \tag{1.13c}$$

$$\chi_T = \left(\frac{\mathrm{d}\log P}{\mathrm{d}\log T}\right)_{\rho,\mu} \tag{1.13d}$$

which are thermodynamic parameters.

TODO: Add commentary about this stuff

In the convective core, $\nabla \approx \nabla_{\mathrm{ad}}$, but outside of it we need something else.

Mixed-length theory models convection with "bubbles".

Beyond these equations, we need the constitutive equations for:

1. the density ρ;
2. the heat capacity of stellar matter c_P;
3. the opacities (radiative and conductive) κ;
4. the transformation rates between nuclear species i and j: r_{ij};
5. the generation rate of nuclear energy ϵ

in terms of the pressure P, temperature T and chemical potential μ.

1 October 2019

Figure 22.8 in some PDF: run of adiabatic, radiation gradients vs $\log T$.

We compute ∇_{ad} and ∇_{rad} and see whether the region is convective or radiative.

We can move from the Eulerian and Lagrangian formalisms using the continuity equation. In the Eulerian formalism:

$$m(r) = \int_0^\infty 4\pi r^2 \rho(x)\,\mathrm{d}x\ . \tag{1.14}$$

We define the mean molecular weight μ with:

$$\mu^{-1} = \sum_i (1+\nu_e(i))\frac{X_i}{A_i}\,, \tag{1.15}$$

where $\nu_e(i)$ is the number of free electrons coming from element i, X_i is the abundance by mass fraction of the element i, and A_i is its mass number.

The variables X, Y and Z represent the abundances of H, He and metals, and satisfy $X + Y + Z = 1$.

We may need to know the metal mixture inside Z, but often we can approximate it as the Sun's distribution.

The time evolution of the various elements' fractions is given by

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \sum_j \left(r_{ji} - r_{ij} \right) \tag{1.16}$$

Classification of stars

1. Low mass stars have between 0.8 and 2 solar masses. They develop an electron-degenerate core after their time on the Main Sequence.

2. Intermediate mass stars have masses between 2 and 8 $M_\odot$. They start burning helium in a non-degenerate core, then they develop a degenerate C−O core.

3. Massive stars have masses of over $8 M_\odot$. They start burning carbon in a non-degenerate core.

What does *degenerate* mean in this context?

What are Hayashi lines?

1.2.3 Time-scales

The *free fall* time scale is:

$$\tau_{\text{dyn}} \sim \left(\frac{R}{g} \right)^{1/2} = \left(\frac{R^3}{GM} \right)^{1/2}. \tag{1.17}$$

It is associated with pulsation. It is calculated using the travel time of a mass in free fall across the stellar radius accelerated by constant acceleration equal to surface acceleration.

We note that $\tau_{\text{dyn}} \propto \overline{\rho}^{-1/2}$, since $\overline{\rho} \propto M/R^3$. For the Sun it is about 1.6×10^3 s.

The *thermal* time scale is the relaxation time of deviations from thermal equilibrium:

$$\tau_{\text{th}} \sim E_{\text{th}}/L\,. \tag{1.18}$$

It is calculated as the time required for a star to irradiate all its energy.

Proof. We use the virial theorem to estimate the thermal (so, local kinetic) energy T: we know that $T = -V/2$ where V is the total potential energy of the star. V can be computed as

$$V = -\int \frac{Gm}{r(m)} \, \mathrm{d}m\,, \tag{1.19}$$

and since $r(m) \approx \sqrt[3]{3m/4\pi\rho}$ if the density is constant, we get

$$V = -\int Gm\sqrt{\frac{4\pi\rho}{3}}m^{-1/3}\,\mathrm{d}m \tag{1.20a}$$

$$= -\frac{3}{5}G\sqrt[3]{\frac{4\pi\rho}{3}}M^{5/3} = -\frac{3}{5}\frac{GM^2}{R} \sim -\frac{GM^2}{R}\,. \tag{1.20b}$$

Therefore, $T \sim -GM^2/2R$. □

Typically, $\tau_{\text{th}} \sim GM^2/(LR) \sim 10^7 M^2/(LR)$ years. For the Sun we have $\tau_{\text{th}} \sim 5 \times 10^{14}$ s. It is much larger than the dynamic time scale.

The *nuclear* time scale is even longer: it is calculated using the efficiency of nuclear fusion of H to He, which is about $\epsilon \sim 0.7\%$, and the fraction of hydrogen in the star, which is about $M_H = 10\% \times M_\odot$. Using these numbers, we get for the Sun:

$$\tau_{\text{nuc}} = \frac{\epsilon c^2 M_H}{L_\odot} \approx 3 \times 10^{17}\,\text{s}\,, \tag{1.21}$$

This allows us to say that oscillations will not be heavily affected by thermal conduction, and even less by nuclear processes: the pulsations will be almost *adiabatic*.

The best candidate for these oscillations are *sound waves*: is the adiabatic speed of sound roughly right?

The speed of sound is given by:

$$v_s^2 = \frac{\partial P}{\partial \rho} = \frac{P}{\rho}\frac{\partial \log P}{\partial \log \rho} = \Gamma_1 \frac{P}{\rho}\,, \tag{1.22}$$

where Γ_1 is the adiabatic exponent, such that $P = \rho^{\Gamma_1}$.

If the gas follows the perfect equation of state

$$\frac{P}{\rho} = \frac{k_B}{m_H}\frac{T}{\mu}\,, \tag{1.23}$$

where μ is the average mass of a gas particle in atomic mass units (units of m_H), so we have $\rho = \mu N m_H / V$. Substituting this in, we get

$$v_s^2 = \frac{\Gamma_1 k_B T}{m_H \mu}\,. \tag{1.24}$$

The mean molecular weight is calculated taking account of the fact that the electrons appear in the count of particles, but we can neglect their mass: therefore, we get a $Z+1$ in the formula, since there are Z electrons and one nucleus. So the value becomes

$$\mu = \left(\sum_j \frac{X_j}{A_j}(1+Z_j)\right)^{-1} \approx (2X + 3Y/4 + Z/2)^{-1} \sim 0.6\,, \tag{1.25}$$

where we approximated that for each metal $(1+Z)/A \sim 1/2$ and we used the values of the mass fractions of the Sun: $X = 0.7381$, $Y = 0.2485$ and $Z = 0.0134$.[1]

[1]See https://arxiv.org/abs/0909.0948

Typical values for the other parameters are $\Gamma_1 = 5/3$, and $T_{\mathrm{He}} \sim 4.5 \times 10^4\,\mathrm{K}$.

So we get $v_s \sim 32.2\,\mathrm{km/s}$.

The timescale for a perturbation to go from one side of the star to the other is $\Pi \sim 2R/v_s \sim 22\,\mathrm{d}$, while the observed value is $\Pi_{\mathrm{obs}} = 5.336\,\mathrm{d}$, so in terms of orders of magnitude it works. We could say that from a more thorough analysis we would see that the vibration does not actually go all the way from an edge of the star to the other.

We can use the equation for the sound speed in the virial theorem, which for a star relates the gravitational potential energy Ω to the integral of the pressure:

$$\Omega = -3 \int P \, \mathrm{d}V = -3 \int \frac{P}{\rho} \, \mathrm{d}m \,, \tag{1.26}$$

and substitute in $P/\rho = v_s^2/\Gamma_1$:

$$\Omega = -3 \frac{\int_M v_s^2/\Gamma_1 \, \mathrm{d}m}{\int_M \mathrm{d}m} M = -3 \left\langle \frac{v_s^2}{\Gamma_1} \right\rangle M \,, \tag{1.27}$$

We multiply and divide by $M = \int \mathrm{d}m$

where the brackets denote an average weighted by the mass distribution.

If Γ_1 and v_s are independent, we can compute their averages separately: we approximate and do this. Then, we can substitute in our expression for the average $\left\langle v_s^2 \right\rangle \sim \langle v_s \rangle^2$ into $\Pi \sim 2R/v_s$: we get

$$\langle v_s \rangle^2 = -\frac{\Omega \Gamma_1}{3M} \,, \tag{1.28}$$

so

$$\Pi \sim \frac{2R}{\langle v_s \rangle} = \sqrt{\frac{-4R^2 \times 3M}{\Omega \Gamma_1}} \,, \tag{1.29}$$

This means we are writing the period Π with respect to something resembling the moment of inertia, $I = \int_r r^2 \, \mathrm{d}m\,(r) \sim R^2 M$:

$$\Pi \sim \left(\frac{I_{\mathrm{osc}}}{-\Omega} \right)^{1/2} \tag{1.30}$$

This is further evidence that we are dealing with a dynamical phenomenon.

We can refine our model: the speed of sound changes throughout the interior of the star. We compute the period as the travel time of sound waves throughout the diameter:

$$\Pi = 2 \int_0^R \mathrm{d}t(r) = 2 \int_0^R \frac{\mathrm{d}r}{\sqrt{\Gamma_1(r) P(r)/\rho(r)}} \,, \tag{1.31}$$

since $\mathrm{d}t = \mathrm{d}r\,/v_s$; the factor of 2 comes from the fact that the sound wave must go from one side of the star to the other.

We also integrate the hydrostatic balance equation, which reads

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G m \rho}{r^2} = -\frac{G \rho^2 4 \pi r}{3} \,, \tag{1.32}$$

with respect to r: then we get

$$P(r) - \underbrace{P(R)}_{=0} = \int_R^r \frac{\mathrm{d}P}{\mathrm{d}r}\,\mathrm{d}r = \frac{2\pi\rho^2 G}{3}\left(R^2 - r^2\right), \tag{1.33}$$

which we can plug into our formula for the period to find:

$$\Pi = 2\int \mathrm{d}r \left(\Gamma_1 \frac{P}{\rho}\right)^{-1/2} \tag{1.34a}$$

$$= 2\int \mathrm{d}r \left(\Gamma_1 \frac{2\pi\rho G}{3}\left(R^2 - r^2\right)\right)^{-1/2} \tag{1.34b}$$

$$= \sqrt{\frac{6}{\Gamma_1 \pi G \overline{\rho}}} \underbrace{\int \frac{\mathrm{d}r}{\sqrt{R^2 - r^2}}}_{\pi/2} \tag{1.34c}$$

$$= \sqrt{\frac{3\pi}{2\Gamma_1 G \overline{\rho}}}, \tag{1.34d}$$

which confirms Ritter's relation $\Pi \propto \overline{\rho}^{-1/2}$. Since the product $\Pi\overline{\rho}^{1/2}$ is approximately constant, we give it the name

$$\mathcal{Q} = \Pi\sqrt{\overline{\rho}} \approx \sqrt{\frac{3\pi}{2\Gamma_1 G}}. \tag{1.35}$$

Ritter's relation is consistent with the statement that the period of the oscillations is of the order of the dynamical characteristic time of the star, since

$$\tau_{\mathrm{dyn}} = \sqrt{\frac{R^3}{GM}} \qquad \text{while} \qquad \overline{\rho}^{-1/2} \approx \left(\frac{M}{4\pi R^3/3}\right)^{-1/2} = \sqrt{\frac{4\pi R^3}{3M}}. \tag{1.36}$$

This gives an estimate of $\Pi \sim 8.5\,\mathrm{d}$ for a δ-Cephei star, which we have to compare to the observed period of $\Pi_{\mathrm{obs}} = 5.466\,\mathrm{d}$. We have definitely improved our estimate. The prediction of this model is that *dense stars pulstate faster*.

This works for acoustic modes, such as those found in δ-Cephei, o-Ceti and SX-Phe stars, but if we consider non-radial g-modes such as those found in variables of type ZZ Ceti it stops working. Then, the estimate given by this model can be off by three orders of magnitude.

1.2.4 The energy equations

7 October 2019

We introduce the *mirror principle*: when the core contracts or expands, the envelope does the opposite.

The shell must remain at around the same temperature to maintain equilibrium: contracting the core would increase the temperature, therefore the envelope exapands. This heuristic argument is actually derived from simulations.

The relevant time scale for oscillations is the free-fall, dynamical time scale.

We come back to the energy equation

$$\frac{\partial L}{\partial m} = \varepsilon - \varepsilon_\nu - \varepsilon_g \tag{1.37}$$

we incorporate the nuclear energy generation rate and the energy lost as neutrino production into an effective energy generation rate per unit mass $\varepsilon - \varepsilon_\nu = \varepsilon_{\text{eff}}$ and express the energy absorbed by the stellar layer as

$$\varepsilon_g = \frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon_{\text{eff}} - \frac{\partial L}{\partial m}\,. \tag{1.38}$$

This makes the meaning of this transfer equation clearer. Using the first and second laws of thermodynamics, and recalling some thermodynamical values: the specific heat at constant volume $c_V = \left(\frac{\partial Q}{\partial T}\right)_V$, the equation of state exponents χ_T and χ_ρ which satisfy: $P = T^{\chi_T}$ and $P = \rho^{\chi_\rho}$ and the adiabatic exponents $\Gamma_{1,2,3}$, which are defined by

$$\Gamma_1 = \gamma_{\text{ad}} = \left(\frac{\partial \log P}{\partial \log \rho}\right)_s \tag{1.39a}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{1}{\nabla_{\text{ad}}} = \left(\frac{\partial \log P}{\partial \log T}\right)_s \tag{1.39b}$$

$$\Gamma_3 - 1 = \left(\frac{\partial \log T}{\partial \log \rho}\right)_s \tag{1.39c}$$

$$\tag{1.39d}$$

and satisfy

$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}\,. \tag{1.40}$$

These are all *exponents* in some power law. We use log values since our variables change by orders of magnitude.

We start with the definition of the entropy differential: $\mathrm{d}Q$ is not an exact differential but $\mathrm{d}Q\,/T = \mathrm{d}S$ is. So, we express it using the first law of thermodynamics: $\mathrm{d}Q = \mathrm{d}E + P\,\mathrm{d}V$. We assume the internal energy E to be a function of the volume V and of the temperature T, so we will have:

$$\mathrm{d}E = \frac{\partial E}{\partial V}\,\mathrm{d}V + \frac{\partial E}{\partial T}\,\mathrm{d}T\ , \tag{1.41}$$

which we can substitute into the expression for the entropy differential:

$$\mathrm{d}S = \frac{1}{T}\left(\frac{\partial E}{\partial V}\,\mathrm{d}V + \frac{\partial E}{\partial T}\,\mathrm{d}T + P\,\mathrm{d}V\right) \tag{1.42a}$$

$$
\begin{aligned}
&= \left(\frac{1}{T} \frac{\partial E}{\partial V} + \frac{P}{T} \right) \mathrm{d}V + \frac{1}{T} \frac{\partial E}{\partial T} \, \mathrm{d}T && (1.42\mathrm{b}) \\
&= \frac{\partial S}{\partial V} \, \mathrm{d}V + \frac{\partial S}{\partial T} \, \mathrm{d}T \,, && (1.42\mathrm{c})
\end{aligned}
$$

so we have identified the partial derivatives of the entropy. By Schwarz's lemma, we then have the equality:

$$
\begin{aligned}
\frac{\partial^2 S}{\partial T \partial V} &= \frac{\partial^2 S}{\partial V \partial T} && (1.43\mathrm{a}) \\
\frac{\partial}{\partial T} \left(\frac{1}{T} \frac{\partial E}{\partial V} + \frac{P}{T} \right) &= \frac{\partial}{\partial V} \left(\frac{1}{T} \frac{\partial E}{\partial T} \right) && (1.43\mathrm{b}) \\
\frac{1}{T} \left(\cancel{\frac{\partial^2 E}{\partial T \partial V}} + \frac{\partial P}{\partial T} \right) - \frac{1}{T^2} \left(\frac{\partial E}{\partial V} + P \right) &= \cancel{\frac{1}{T} \frac{\partial^2 E}{\partial V \partial T}} && (1.43\mathrm{c}) \\
\frac{\partial E}{\partial V} &= T \frac{\partial P}{\partial T} - P \,, && (1.43\mathrm{d})
\end{aligned}
$$

but we can turn the derivatives with respect to V with ones with respect to $\rho \propto V^{-1}$, by

$$
\frac{\partial}{\partial V} = \frac{\partial \rho}{\partial V} \frac{\partial}{\partial \rho} = -\rho^2 \frac{\partial}{\partial \rho} \,, \tag{1.44}
$$

so after dividing through by ρ we find:

$$
\begin{aligned}
\rho \frac{\partial E}{\partial \rho} &= -\frac{T}{\rho} \frac{\partial P}{\partial T} + \frac{P}{\rho} && (1.45\mathrm{a}) \\
&= -\frac{P}{\rho} \frac{\partial \log P}{\partial \log T} + \frac{P}{\rho} && (1.45\mathrm{b}) \\
\rho \frac{\partial E}{\partial \rho} &= -\frac{P}{\rho} \left(\frac{\partial \log P}{\partial \log T} - 1 \right) && (1.45\mathrm{c}) \\
\frac{\partial E}{\partial \log \rho} &= -\frac{P}{\rho} (\chi_T - 1) \,. && (1.45\mathrm{d})
\end{aligned}
$$

We used the fact that

$$
\frac{\partial}{\partial \log x} = \frac{\partial x}{\partial \log x} \frac{\partial}{\partial x} = x \frac{\partial}{\partial x} \,. \tag{1.46}
$$

Now, we can write the first law of thermodynamics for the specific energy density $E(\rho, T)$:

$$
\begin{aligned}
\mathrm{d}Q &= \mathrm{d}E - \frac{P}{\rho^2} \, \mathrm{d}\rho && (1.47\mathrm{a}) \\
&= \frac{\partial E}{\partial \log \rho} \, \mathrm{d}\log \rho + \frac{\partial E}{\partial \log T} \, \mathrm{d}\log T - \frac{P}{\rho} \, \mathrm{d}\log \rho && (1.47\mathrm{b}) \\
&= \left(-\frac{P}{\rho} (\chi_T - 1) - \frac{P}{\rho} \right) \mathrm{d}\log \rho + \frac{\partial E}{\partial \log T} \, \mathrm{d}\log T && (1.47\mathrm{c})
\end{aligned}
$$

$$= -\frac{P\chi_T}{\rho}\,\mathrm{d}\log\rho + \frac{\partial E}{\partial \log T}\,\mathrm{d}\log T\,, \tag{1.47d}$$

which in the adiabatic ($\mathrm{d}Q = 0$) case reduces to

$$\Gamma_3 - 1 \overset{\text{def}}{=} \frac{\partial \log T}{\partial \log \rho} = -\frac{P\chi_T}{\rho}\frac{\partial \log T}{\partial E} = -\frac{P\chi_T}{\rho T c_V} \tag{1.48a}$$

$$= -\frac{1}{\rho} \times \underbrace{\frac{P\chi_T}{T}}_{\frac{\partial P}{\partial T}} \times \underbrace{\frac{1}{c_V}}_{\frac{\partial T}{\partial E}} \tag{1.48b}$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial E}\,, \tag{1.48c}$$

where we used the fact that $c_V = \partial E/\partial T$. This can also be written as

$$\frac{\partial E}{\partial P} = \frac{1}{\rho(\Gamma_3 - 1)}\,. \tag{1.49}$$

If we drop the hypothesis of adiabaticity, we can study the variation with respect to time of Q, both when writing $E = E(\rho, T)$ and when writing $E = E(\rho, P)$. In the first case we can use equation (1.47a), "dividing through by $\mathrm{d}t$" (more formally, applying the differential covector equation to the vector ∂_t), after some manipulation we can bring out a factor $T\,\partial E/\partial T$ to get:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = T\frac{\partial E}{\partial T}\left(\frac{\mathrm{d}\log T}{\mathrm{d}t} + \frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{T\frac{\partial E}{\partial T}}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right), \tag{1.50}$$

and similarly if we express $E = E(\rho, P)$ we find

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = P\frac{\partial E}{\partial P}\left(\frac{\mathrm{d}\log P}{\mathrm{d}t} + \frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{P\frac{\partial E}{\partial P}}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right). \tag{1.51}$$

We can simplify these two expressions by recalling some results from before, plus two more expression we now derive for the coefficients Γ_1 and $\Gamma_3 - 1$: we assume adiabaticity, and get

$$0 = \mathrm{d}S = \frac{1}{T}\left(\frac{\partial E}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial E}{\partial P}\,\mathrm{d}P - \frac{P}{\rho^2}\,\mathrm{d}\rho\right) \tag{1.52a}$$

$$0 = \left(\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}\right)\mathrm{d}\log\rho + P\frac{\partial E}{\partial P}\,\mathrm{d}\log P \tag{1.52b}$$

$$\Gamma_1 = \left.\frac{\mathrm{d}\log P}{\mathrm{d}\log\rho}\right|_{\text{ad}} = \frac{\frac{P}{\rho} - \rho\frac{\partial E}{\partial \rho}}{P\frac{\partial E}{\partial P}} \tag{1.52c}$$

and

$$0 = \mathrm{d}S = \frac{1}{T}\left(\frac{\partial E}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial E}{\partial T}\,\mathrm{d}T - \frac{P}{\rho^2}\,\mathrm{d}\rho\right) \tag{1.53a}$$

$$0 = \left(\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}\right)\mathrm{d}\log\rho + T\frac{\partial E}{\partial T}\,\mathrm{d}\log T \tag{1.53b}$$

$$\Gamma_3 - 1 = \left.\frac{\mathrm{d}\log T}{\mathrm{d}\log\rho}\right|_{\mathrm{ad}} = \frac{\frac{P}{\rho} - \rho\frac{\partial E}{\partial \rho}}{T\frac{\partial E}{\partial T}}\,, \tag{1.53c}$$

The identifications are:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = T\underbrace{\frac{\partial E}{\partial T}}_{c_V}\left(\frac{\mathrm{d}\log T}{\mathrm{d}t} + \underbrace{\frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{T\frac{\partial E}{\partial T}}}_{-(\Gamma_3-1)}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right), \tag{1.54}$$

and

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = P\underbrace{\frac{\partial E}{\partial P}}_{1/\rho(\Gamma_3-1)}\left(\frac{\mathrm{d}\log P}{\mathrm{d}t} + \underbrace{\frac{\rho\frac{\partial E}{\partial \rho} - \frac{P}{\rho}}{P\frac{\partial E}{\partial P}}}_{-\Gamma_1}\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right). \tag{1.55}$$

Finally, we substitute in the equation of energy conservation:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m}\,, \tag{1.56}$$

to find the equations for the evolution of the energy and pressure:

$$\boxed{\begin{aligned} \frac{\partial\log P}{\partial t} &= \Gamma_1\frac{\partial\log\rho}{\partial t} + \frac{\rho}{P}(\Gamma_3 - 1)\left(\varepsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m}\right) && (1.57\mathrm{a}) \\ \frac{\partial\log T}{\partial t} &= (\Gamma_3 - 1)\frac{\partial\log\rho}{\partial t} + \frac{1}{c_V T}\left(\varepsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m}\right) && (1.57\mathrm{b}) \end{aligned}}$$

1.2.5 Linear perturbation theory

Say we have a solution for these equations, we look at linear perturbations of them. This makes sense: the main solution is basically static on the pulsation time-scales.

The perturbed model is $f = f(m)$, the unperturbed one is $f_0(m)$. The Lagrangian perturbation is $\delta f(m,t) = f(m,t) - f_0(m,t)$.

Let us consider specific cases for f: the radial displacement is $\delta r(m,t)$. The position of the layer at time t is $r = r_0 + \delta r$.

We can write:

$$r = r_0\left(1 + \frac{\delta r}{r_0}\right) = r_0(1+\zeta)\,, \tag{1.58}$$

where we define $\zeta = \delta r / r_0$.

In general the fractional perturbation $\delta f / f_0$ is assumed to be $\ll 1$. So, $\delta f / f_0 \sim \delta_f / f$. Formally, we only consider terms which are of first order in either perturbed function. We will insert expressions which are functions of perturbations of all our variables, and thus get linear differential equations.

Properties of Lagrangian perturbations

In general for a Lagrangian perturbation we have the following useful properties:

1. we can use the properties of derivatives: $\delta(f^n) = n f_0^{n-1} \delta f$;
2. we can use the properties of logarithmic derivatives:
$$\frac{\delta(\prod f_i)}{\prod f_i} = \sum \frac{\delta f_i}{f_i}\,; \tag{1.59}$$
3. δ commutes with partial derivation.

Continuity equation

Let us try the continuity equation, substituting in $r = r_0(1+\zeta)$ and $\rho = \rho_0(1 + \delta\rho/\rho_0)$.

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{1.60a}$$

$$\frac{\partial}{\partial m}\big(r_0(1+\zeta)\big) = \frac{1}{4\pi r_0^2}(1+\zeta)^{-2}\left(1 + \frac{\delta\rho}{\rho_0}\right)^{-1} \tag{1.60b}$$

and we use $(1+x)^n \approx 1 + nx$ plus the zeroth order equation: $\partial r_0/\partial m = 1/4\pi r_0^2 \rho_0$. With these, we find:

$$4\pi\rho_0^2\left(\frac{\partial r_0}{\partial m}(1+\zeta) + r_0\frac{\partial\zeta}{\partial m}\right) = (1-2\zeta) - \frac{\delta\rho}{\rho_0} \tag{1.61}$$

We can collapse the equation into:

$$\boxed{\frac{\delta\rho}{\rho_0} = -3\zeta - 4\pi r_0^3 \rho_0 \frac{\partial\zeta}{\partial m}} \tag{1.62}$$

or, the density perturbation is proportional with a negative constant to the radial perturbation, plus a term proportional to $\partial\zeta/\partial m$. If there is a positive gradient of radial perturbation, the corresponding layer expands.

Momentum conservation

Let us also perturb the momentum conservation equation; the unperturbed solution will be at hydrostatic equilibrium, so $\partial^2 r_0 \big/ \partial t^2 = \frac{\partial r_0}{\partial t} = 0$, which means

$$\frac{\partial P_0}{\partial m} = -\frac{Gm}{4\pi r_0^4}\,. \tag{1.63}$$

Substituting in we find that, to linear order:

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2} \tag{1.64a}$$

$$\frac{\partial^2}{\partial t^2}\big(r_0(1+\zeta)\big) = -4\pi\big(r_0(1+z)\big)^2 \frac{\partial}{\partial m}\left(P_0\left(1+\frac{\delta P}{P_0}\right)\right) - \frac{Gm}{r_0^2(1+\zeta^2)} \tag{1.64b}$$

$$r_0\frac{\partial^2 \zeta}{\partial t^2} = -4\pi r_0^2(1+2\zeta)\left(\frac{\partial P_0}{\partial m}\left(1+\frac{\delta P}{P_0}\right) + P_0\frac{\partial}{\partial m}\left(\frac{\delta P}{P_0}\right)\right) - \frac{GM}{r_0^2}(1-2z) \tag{1.64c}$$

$$\frac{r_0}{4\pi r_0^2}\frac{\partial^2 \zeta}{\partial t^2} = -(1+2\zeta)\left(\frac{\partial P_0}{\partial m}\left(1+\frac{\delta P}{P_0}\right) + P_0\frac{\partial}{\partial m}\left(\frac{\delta P}{P_0}\right)\right) + (1-2z)\frac{\partial P_0}{\partial m} \tag{1.64d}$$

Used equation (1.63).

$$\begin{aligned}\frac{1}{4\pi r_0}\frac{\partial^2 \zeta}{\partial t^2} &= -\left(\frac{\partial P_0}{\partial m}\left(1+\frac{\delta P}{P_0}\right) + P_0\frac{\partial}{\partial m}\left(\frac{\delta P}{P_0}\right)\right) + \frac{\partial P_0}{\partial m} + \\ &\quad - 2\zeta\left(\frac{\partial P_0}{\partial m}\left(1+\frac{\delta P}{P_0}\right) + P_0\frac{\partial}{\partial m}\left(\frac{\delta P}{P_0}\right)\right) - 2\zeta\frac{\partial P_0}{\partial m}\end{aligned} \tag{1.64e}$$

Split the terms into different orders of ζ

$$\begin{aligned}\frac{1}{4\pi r_0}\frac{\partial^2 \zeta}{\partial t^2} &= -\frac{\partial}{\partial m}(\delta P) + \\ &\quad - 4\zeta\frac{\partial P_0}{\partial m} - 2\zeta\frac{\partial}{\partial m}(\delta P)\end{aligned} \tag{1.64f}$$

$$r_0\frac{\mathrm{d}^2\zeta}{\mathrm{d}t^2} = -4\pi r_0^2\left(\frac{\partial}{\partial m}(\delta P) + 4\zeta\frac{\partial P_0}{\partial m}\right). \tag{1.64g}$$

Neglected a second order term

The final equation then looks like

$$\boxed{r_0\frac{\partial^2 \zeta}{\partial t^2} = -4\pi r_0^2\left(\underbrace{P_0\frac{\partial}{\partial m}\left(\frac{\delta P}{P_0}\right) + \frac{\delta P}{P_0}\frac{\partial P_0}{\partial m}}_{\frac{\partial \delta P}{\partial m}} + 4\zeta\frac{\partial P_0}{\partial m}\right),} \tag{1.65}$$

and we can now give a physical interpretation of the various terms.

We have a term $-16\pi r_0^2\zeta\,\partial P_0/\partial m = 4\zeta Gm/r_0^2$. This, by itself, is a force moving the system away from equilibrium: the equation with only that term on the RHS is precisely like a harmonic repulsor, $\ddot{\zeta} = \omega^2 z$ with

$$\omega^2 = \frac{4Gm}{r_0^3}\,. \tag{1.66}$$

This term is of geometric origin: as the layer moves outwards it expands, and the expansion is favoured by the decrease in the gravitational potential and the corresponding increase in pressure due to the increase of the area of the layer.

The equation with only the other term looks like

$$\ddot{\zeta} = -4\pi r_0 \frac{\partial \delta P}{\partial m} = -4\pi r_0 \left(P_0 \frac{\partial}{\partial m} \left(\frac{\delta P}{P_0} \right) + \frac{\delta P}{P_0} \frac{\partial P_0}{\partial m} \right). \tag{1.67}$$

According to the slides, the action of this term is to be split in two: for the second, a restoring force towards equilibrium, since if the pressure decreases as the layer expands then the force is inward, and a more vague interpretation for the first bit, stating that "a non uniform variation of δP from a layer to the next generates a change in the pressure gradient": but it *is* a change in the pressure field…

Energy conservation

Let us also consider the expression for the time derivative of $\log P$ coming from the energy conservation equation: equation (1.57a); the perturbed equation for the time derivative of $\log T$ (equation (1.57b)) is analogously derived. Besides ρ and P, we also perturb the adiabatic exponents and the $\mathrm{d}Q/\mathrm{d}t = \epsilon_{\text{eff}} - \partial L/\partial m$ term.

After difficult manipulations we get back an equation which relates the changes in density and pressure to the change in energy: we manipulate until we get something which is similar to the original equation:

$$\frac{\partial}{\partial t} \left(\frac{\delta P}{P_0} \right) = \Gamma_{1,0} \frac{\partial}{\partial t} \left(\frac{\delta \rho}{\rho_0} \right) + \frac{\rho_0}{P_0} (\Gamma_{3,0} - 1) \delta \left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m} \right) \tag{1.68a}$$

$$\frac{\partial}{\partial t} \left(\frac{\delta T}{T_0} \right) = (\Gamma_{3,0} - 1) \frac{\partial}{\partial t} \left(\frac{\delta \rho}{\rho_0} \right) + \frac{1}{c_V T} \delta \left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m} \right), \tag{1.68b}$$

where we note that the first index of the adiabatic exponents denotes *which exponent it is*, while the second indicates that it is the unperturbed value.

The last T in the temperature equation is not unperturbed nor perturbed...?

Luminosity equation

In the radiative case with the diffusion approximation we can perturb the luminosity equation, (1.11). It is much more convenient not to calculate δL but $\delta L/L_0$ instead: this allows us to use the logarithmic derivative properties of the perturbation; also, since

$$T^3 \frac{\partial T}{\partial m} = T^4 \frac{\partial \log T}{\partial m} \tag{1.69}$$

we use the latter expression, which is more convenient. So we have $L = \prod f_i$, with

$$f_i = \left\{ -\frac{64\pi^2 ac}{3}, r^4, \kappa_R^{-1}, T^4, \frac{\partial \log T}{\partial m} \right\} . \tag{1.70}$$

Now we can apply the rule given in equation (1.59): we find

$$\frac{\delta L}{L_0} = 4\zeta + 4\frac{\delta T}{T_0} - \frac{\delta \kappa_R}{\kappa_{R,0}} + \left(\frac{\partial \log T}{\partial m} \right)_0^{-1} \frac{\partial}{\partial m} \left(\frac{\delta T}{T_0} \right) , \tag{1.71}$$

where we used the simplification

$$\begin{aligned}
\delta \left(\frac{\partial \log T}{\partial m} \right) &= \delta \left(\frac{1}{T} \frac{\partial T}{\partial m} \right) && (1.72a) \\
&= \delta \left(\frac{1}{T} \right) \frac{\partial T_0}{\partial m} + \frac{1}{T_0} \delta \left(\frac{\partial T}{\partial m} \right) && (1.72b) \\
&= -\frac{\delta T}{T_0} \frac{\partial T_0}{\partial m} + \frac{1}{T_0} \frac{\partial \delta T}{\partial m} && (1.72c) \\
&= \frac{\partial}{\partial m} \left(\frac{\delta T}{T_0} \right) . && (1.72d)
\end{aligned}$$

Inverse application of the product rule

The last step is to assume a certain dependence of the Rosseland mean opacity on the temperature and density: specifically, it is a "Kramers-like" expression, given by

$$\kappa_R \propto \rho^n T^{-s} , \tag{1.73}$$

which can be substituted into our expression: the proportionality factor does not matter, and we get additional temperature and density terms:

$$\boxed{\frac{\delta L}{L_0} = 4\zeta + (4+s)\frac{\delta T}{T_0} - n\frac{\delta \rho}{\rho_0} + \left(\frac{\partial \log T}{\partial m} \right)_0^{-1} \frac{\partial}{\partial m} \left(\frac{\delta T}{T_0} \right) ,} \tag{1.74}$$

In the end, we have a set of four linear PDE equations (written as a 5-equation system). These describe implicitly how the properties of the star change over time.

Pulsation usually affects mostly the outer layers of a star.

1.3 Adiabatic oscillations

1.3.1 Derivation of the LAWE

Exploiting the adiabatic approximation we will get the Linear Adiabatic Wave Equation (LAWE): a single equation which summarizes the 4 and can be solved explicitly.

Recall the heat transfer equation (1.38): if we suppose that each layer does not lose nor gain heat, $\mathrm{d}Q/\mathrm{d}t = 0$, this implies that $\delta(\varepsilon_{\text{eff}} - \partial L/\partial m) = 0$.

Is this approximation justified? The term multiplying $\delta(\varepsilon_{\text{eff}} - \partial L/\partial m)$ in the perturbed energy equation (1.68a) is $\rho(\Gamma_3 - 1)/P = \chi_T/(c_V T)$. Usually $\chi_T \sim 1$, while the density perturbation term is multiplied by $\Gamma_1 \sim 1$.

This term, $\chi_T/(c_V T)\delta(\varepsilon_{\text{eff}} - \partial L/\partial m)$, is of the order $1/\tau_{\text{th}}$, the thermal time scale of this layer, while the term before, $\Gamma_1\, \partial/\partial t\ (\delta\rho/\rho)$, is of the order $1/\tau_{\text{dyn}}$, the dynamical time scale.

Is this just because the first term contains a time derivative while the second one does not?

Therefore, we neglect the second part. This only works for the star as a whole, not for single layers. There are stellar layers which are *strongly* non-adiabatic (driving layers). We will need some non-adiabatic theory to explain how pulsations *start*.

So, the energy conservation equations become

$$\frac{\delta P}{P} = \Gamma_1 \frac{\delta\rho}{\rho} \qquad \text{and} \qquad \frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta\rho}{\rho}, \tag{1.75}$$

which we can substitute into the momentum conservation equation (1.65) to find

$$r_0 \frac{\partial^2 \zeta}{\partial t^2} = -4\pi r_0^2 \left(P_0 \frac{\partial}{\partial m}\left(\Gamma_1 \frac{\delta\rho}{\rho}\right) + \Gamma_1 \frac{\delta\rho}{\rho}\frac{\partial P_0}{\partial m} + 4\zeta \frac{\partial P_0}{\partial m} \right), \tag{1.76}$$

and now we can use the continuity equation (1.62) which gives us an expression for $\delta\rho/\rho$: inserting it we get

$$\begin{aligned} r_0 \frac{\partial^2 \zeta}{\partial t^2} &= -4\pi r_0^2 \left[P_0 \frac{\partial}{\partial m}\left(\Gamma_1\left(-3\zeta - 4\pi r_0^3 \rho_0 \frac{\partial\zeta}{\partial m}\right)\right) + \right. \\ &\qquad \left. + \Gamma_1\left(-3\zeta - 4\pi r_0^3\rho_0\frac{\partial\zeta}{\partial m}\right)\frac{\partial P_0}{\partial m} + 4\zeta\frac{\partial P_0}{\partial m}\right] \end{aligned} \tag{1.77a}$$

$$\begin{aligned} r_0 \frac{\partial^2 \zeta}{\partial t^2} &= 4\pi r_0^2 \Bigg(\underbrace{(3\Gamma_1 - 4)\zeta\frac{\partial P}{\partial m}}_{①} + \underbrace{4\pi r_0^3 \Gamma_1 \rho \frac{\partial\zeta}{\partial m}\frac{\partial P}{\partial m}}_{②} + \\ &\qquad + \underbrace{3P\frac{\partial}{\partial m}(\Gamma_1\zeta)}_{③} + \underbrace{4\pi P \frac{\partial}{\partial m}\left(\Gamma_1 r^3 \rho \frac{\partial\zeta}{\partial m}\right)}_{④} \Bigg), \end{aligned} \tag{1.77b}$$

and now we can can manipulate the terms inside the parentheses; we start to drop the zero indices, any quantity not being δ'd is meant to be unperturbed. The first and third terms are respectively given by

$$① = \frac{\partial P}{\partial m}\zeta(3\Gamma_1 - 4) = \zeta\frac{\partial}{\partial m}\big((3\Gamma_1 - 4)P\big) - 3\zeta P \frac{\partial \Gamma_1}{\partial m} \tag{1.78}$$

Backwards derivative of a product

and

$$③ = 3P\frac{\partial}{\partial m}(\Gamma_1\zeta) = 3P\Gamma_1\frac{\partial\zeta}{\partial m} + 3P\zeta\frac{\partial\Gamma_1}{\partial m}, \tag{1.79}$$

so we can see that the highlighted terms cancel. Also, the other term in equation (1.79) can be rewritten using the continuity equation (1.2):

$$3P\Gamma_1 \frac{\partial \zeta}{\partial m} = 12\pi r^2 \rho \Gamma_1 P \frac{\partial \zeta}{\partial m} \frac{\partial r}{\partial m}, \tag{1.80}$$

so we can see that if we expand the fourth term in (1.77b) we find a thing that is equal to it:

$$④ = 4\pi P \frac{\partial}{\partial m}\left(r^3 \times \Gamma_1 \rho \frac{\partial \zeta}{\partial m}\right) = 4\pi P \times 3r^2 \frac{\partial r}{\partial m} \Gamma_1 \rho \frac{\partial \zeta}{\partial m} + 4\pi P r^3 \frac{\partial}{\partial m}\left(\Gamma_1 \rho \frac{\partial \zeta}{\partial m}\right), \tag{1.81}$$

so we will have twice that contribution in the final result.

For now then, we have shown that

$$① + ③ + ④ = \zeta \frac{\partial}{\partial m}((3\Gamma_1 - 4)P) + 2 \times 12\pi P r^2 \frac{\partial r}{\partial m} \Gamma_1 \rho \frac{\partial \zeta}{\partial m} + 4\pi P r^3 \frac{\partial}{\partial m}\left(\Gamma_1 \rho \frac{\partial \zeta}{\partial m}\right). \tag{1.82}$$

Now then, the full equation reads

$$\begin{aligned} r_0 \frac{\partial^2 \zeta}{\partial t^2} = 4\pi r^2 \zeta \frac{\partial}{\partial m}((3\Gamma_1 - 4)P) + 16\pi^2 r_0^5 \Gamma_1 \rho \frac{\partial \zeta}{\partial m} \frac{\partial P}{\partial m} + \\ + 96\pi^2 P r^4 \frac{\partial r}{\partial m} \Gamma_1 \rho \frac{\partial \zeta}{\partial m} + 16\pi^2 P r^5 \frac{\partial}{\partial m}\left(\Gamma_1 \rho \frac{\partial \zeta}{\partial m}\right), \end{aligned} \tag{1.83a}$$

and we can notice a certain similarity between the highlighted terms: consider the expression

$$\frac{\partial}{\partial m}\left(16\pi^2 \Gamma_1 P \rho r^6 \frac{\partial \zeta}{\partial m}\right), \tag{1.84}$$

to which we can apply the general expression, which holds for nonzero differentiable functions of a certain variable x (and if it is interpreted as a limit, even if the functions go to 0):

$$\frac{\partial}{\partial x}\left(\prod_i f_i\right) = \left(\prod_i f_i\right) \sum_i \frac{1}{f_i} \frac{\partial f_i}{\partial x}, \tag{1.85}$$

so

$$\frac{\frac{\partial}{\partial m}\left(16\pi^2 \Gamma_1 P \rho r^6 \frac{\partial \zeta}{\partial m}\right)}{16\pi^2 \Gamma_1 P \rho r^6 \frac{\partial \zeta}{\partial m}} = \frac{1}{P}\frac{\partial P}{\partial m} + \frac{1}{\Gamma_1 \rho\, \partial \zeta / \partial m} \frac{\partial}{\partial m}\left(\Gamma_1 \rho \frac{\partial \zeta}{\partial m}\right) + \frac{1}{r^6}\frac{\partial r^6}{\partial m} \tag{1.86a}$$

$$= \frac{1}{P}\frac{\partial P}{\partial m} + \frac{1}{\Gamma_1 \rho\, \partial \zeta / \partial m} \frac{\partial}{\partial m}\left(\Gamma_1 \rho \frac{\partial \zeta}{\partial m}\right) + \frac{6}{r}\frac{\partial r}{\partial m}, \tag{1.86b}$$

so

$$\frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6\frac{\partial \zeta}{\partial m}\right) = 16\pi^2 r^5\Gamma_1\rho\frac{\partial \zeta}{\partial m}\frac{\partial P}{\partial m} + \\ + 16\pi^2 r^5 P\frac{\partial}{\partial m}\left(\Gamma_1\rho\frac{\partial \zeta}{\partial m}\right) + 96\pi^2 r^4\Gamma_1\rho\frac{\partial \zeta}{\partial m}P\frac{\partial r}{\partial m}\,. \tag{1.87a}$$

Now, we can finally write the simplest form of the LAWE:

$$\boxed{r\frac{\partial^2\zeta}{\partial t^2} = 4\pi r^2\zeta\frac{\partial}{\partial m}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6\frac{\partial \zeta}{\partial m}\right),} \tag{1.88}$$

Next time, we will decompose: $\zeta(m,t) = \eta(m)e^{i\sigma t}$ with a constant σ: putting this into the LAWE we simplify the exponentials and get the space dependent form of the LAWE.

The LAWE is a Storm-Liouville equation.

8 October 2019

An ansatz for the LAWE

Last lecture we started talking about linearization & perturbation theory.

We use a complex spinning ansatz, which is by itself not physical, however its conjugate is also a solution so after we have found a solution we just need to take the real part.

Substituting $\zeta(m,t) = \eta(m)\exp(i\sigma t)$ into the LAWE we find that the exponentials simplify:

$$-r\sigma^2\eta = 4\pi r^2\eta\frac{\partial}{\partial m}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2\Gamma_1 P\rho r^6\frac{\partial \eta}{\partial m}\right). \tag{1.89}$$

Since we have an explicit r, it is convenient to rewrite this in the Eulerial formalism, using the continuity equation: we get

$$-r\sigma^2\eta = \frac{\eta}{\rho}\frac{\partial}{\partial r}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r^3\rho}\frac{\partial}{\partial r}\left(\Gamma_1 P r^4\frac{\partial \eta}{\partial r}\right) \tag{1.90a}$$

$$\sigma^2\eta = -\frac{\eta}{\rho r}\frac{\partial}{\partial r}(3\Gamma_1 P - 4P) - \frac{1}{r^4\rho}\frac{\partial}{\partial r}\left(\Gamma_1 P r^4\frac{\partial \eta}{\partial r}\right), \tag{1.90b}$$

Slide 4.37: there is a minus sign missing, right?

We will have analytic solutions for the adiabatic case, with the additional hypotheses of either $\Gamma_1 = 4/3$ or $\Gamma_1 > 4/3$ and homogeneity.

[The justification of the adiabatic approx might be asked at the exam.]

Boundary conditions for the LAWE

What are the boundary conditions we should set? The final result is:

1. $\delta r = 0$ at $r = 0$;
2. $\partial \eta / \partial r = 0$ at $r = 0$;
3. $(4 + R^3\sigma^2/(GM))\eta + \delta P/P = 0$ at $r = R$;
4. $\eta = \delta r/r = 1$ at $r = R$.

Let us show why. The first condition has an immediate physical basis: there cannot be radial displacement at $r = 0$ since there is nowhere to go at $r < 0$, and if there was positive radial displacement δr it would mean there is a vacuum in the region $r < \delta r$, which is unphysical because of the pressure there.

So, by the first condition, we need the term

$$\frac{1}{r^4 \rho} \frac{\partial}{\partial r}\left(\Gamma_1 P r^4 \frac{\partial \eta}{\partial r}\right) = \frac{\Gamma_1 P}{\rho} \frac{\partial^2 \eta}{\partial r^2} + 4 \frac{\Gamma_1 P}{\rho r} \frac{\partial \eta}{\partial r} + \frac{\Gamma_1}{\rho} \frac{\partial \eta}{\partial r} \frac{\partial P}{\partial r} + \frac{P}{\rho} \frac{\partial \eta}{\partial r} \frac{\partial \Gamma_1}{\partial r} \tag{1.91}$$

to be well behaved (nonsingular) at $r = 0$.

We know that $\partial P/\partial r$ and $\partial \Gamma_1/\partial r$ are zero at $r = 0$, so we can neglect the terms containing them. This is because any physical quantity must approach $r = 0$ with zero $\mathrm{d}/\mathrm{d}r$ slope, in order for it to be differentiable at the center.

In the second term we have a division by r: in order for it not to diverge we must ask that $\partial \eta/\partial r = 0$ at $r = 0$.

Since η is ζ times a phase, this also means that $\partial \zeta/\partial r = 0$ at $r = 0$. We can plug this into the linearized continuity equation, the Eulerian form of (1.62):

$$\frac{\delta \rho}{\rho} = -3\zeta - r\frac{\partial \zeta}{\partial r} \tag{1.92a}$$

$$0 = \frac{\partial \zeta}{\partial r} = -\frac{1}{r}\left(3\zeta + \frac{\delta \rho}{\rho}\right) \tag{1.92b}$$

$$\implies 0 = 3\zeta + \frac{\delta \rho}{\rho}, \tag{1.92c}$$

into which we can substitute the linearized adiabatic expressions for the temperature and pressure perturbations (1.75): so we find

$$3\zeta = -\frac{1}{\Gamma_1} \frac{\delta P}{P} = -\frac{1}{\Gamma_3 - 1} \frac{\delta T}{T}, \tag{1.93}$$

We use the Eulerian form of the momentum conservation (1.65) to figure out the surface boundary conditions: it reads

$$r\sigma^2 \eta = \frac{1}{\rho}\left(P \frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) + \frac{\delta P}{P} \frac{\partial P}{\partial r} + 4\eta \frac{\partial P}{\partial r}\right) \tag{1.94a}$$

$$\frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) = \frac{\rho}{P}\left(r\sigma^2\eta\right) - \frac{1}{P}\left(\frac{\delta P}{P}\frac{\partial P}{\partial r} + 4\eta\frac{\partial P}{\partial r}\right) \tag{1.94b}$$

$$= \frac{\rho r\sigma^2\eta}{P} - \frac{\delta P}{P}\frac{\partial \log P}{\partial r} - 4\eta\frac{\partial \log P}{\partial r} \tag{1.94c}$$

$$= -\frac{\partial \log P}{\partial r}\left(-\frac{\rho r\sigma^2\eta}{\partial P/\partial r} + 4\eta + \frac{\delta P}{P}\right), \tag{1.94d}$$

since we assumed that all perturbations are *in phase* to the radial one, and thus wrote them all as proportional to $\exp(i\sigma t)$, which then simplified. We define the *pressure scale height* as:

$$H_P = -\left(\frac{\partial \log P}{\partial r}\right)^{-1}, \tag{1.95}$$

it represents "how far we should move in the star for the pressure to change e-fold".

We insert this into the equation; also we use the Eulerian momentum conservation equation

$$\frac{\partial P}{\partial r} = -\rho\frac{Gm}{r^2}, \tag{1.96}$$

so that we find

$$\frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) = \frac{1}{H_P}\left(\left(4 + \frac{r^3\sigma^2}{Gm}\right)\eta + \frac{\delta P}{P}\right), \tag{1.97}$$

Since outside the star the pressure is zero, we must have that $H_P \to 0$ when $r \to R$: the pressure must change by an infinite number of e-folds to reach zero, so the displacement needed for a single e-fold change must go to zero. Therefore, the term multiplying $1/H_P$ must also go to zero to avoid divergences: so we ask

$$\left(4 + \frac{r^3\sigma^2}{Gm}\right)\eta + \frac{\delta P}{P} \to 0 \tag{1.98}$$

as $r \to R$. Since we are considering the mass variable at the edge of the star, we also must have $m = M$.

The last condition, $\eta = \delta r/r = 1$ at $r = R$ comes from the fact that we want our study to give us *periods*, not *amplitudes*: we cannot find those out, so we normalize in a way that is convenient. The LAWE is 1-homogeneous!

Solving the LAWE

The LAWE can be written compactly with a linear operator $\mathcal{L}$:

$$\mathcal{L}(\eta) = \sigma^2\eta, \tag{1.99}$$

where the expression for $\mathcal{L}$ can be read off equation (1.90b). The equation is in the form of a Sturm-Liouville differential equation; in general those have the form

$$\frac{\partial}{\partial r}\left(p(r)\frac{\partial \eta}{\partial r}\right) + \lambda t(r)\eta - s(r)\eta = 0. \tag{1.100}$$

Figure out signs! Why does nothing make sense?

So, the eigenvalue σ^2 is the square of the pulsation. There are infinitely many solutions to the LAWE, only some (finitely many?) fulfill the boundary condition. The eigenvalues are real since $\mathcal{L}$ is Hermitian, and they have a wavefunction associated: $\eta_m(r)$ corresponding to σ_m^2.

If $\sigma^2 > 0$ we have an oscillating solution, if $\sigma^2 < 0$ we have an exponential collapse or explosion since the solution is proportional to $\exp(i\sigma t)$.

We label solutions by *radial order* $m \in \mathbb{N}$: $m = 0$ has the lowest frequency, and then we have overtones. We choose the labels so that $\sigma_{m_1} < \sigma_{m_2} \iff m_1 < m_2$.

The radial order m is also the number of nodes.

The eigenfunctions are orthogonal wrt the scalar product

$$\langle \eta_m | \eta_n \rangle = \int_0^R \eta_m \eta_n \rho r^4 \,\mathrm{d}r = f(n)\delta_{nm} \tag{1.101}$$

Possibly there is a 4π missing in order for this to be consistent with the following?

The functions ζ_m are orthogonal wrt the same product. The system is linear: we can write a general solution as a superposition.

We can define the moment of inertia:

$$J_m = \int_0^M |\zeta_m|^2 r^2 \,\mathrm{d}m \tag{1.102}$$

and then we can recover the eigenvalue by:

$$\sigma_m^2 = \frac{1}{J_m} \int_0^M \zeta_m^* \mathcal{L} \zeta_m r^2 \,\mathrm{d}m = \frac{\langle \zeta_m | \mathcal{L} | \zeta_m \rangle}{\langle \zeta_m | \zeta_m \rangle} \tag{1.103}$$

1.3.2 Simplifications

Period-mean density relation

If $\eta = \text{const}$, and ρ and Γ_1 are also constanst, we can remove the $\partial \eta / \partial r$ terms: so we have

$$\sigma^2 \eta + \frac{\eta}{\rho r} \frac{\partial}{\partial r} \big(P(3\Gamma_1 - 4) \big) = 0 \tag{1.104a}$$

$$\sigma^2 = -\frac{3\Gamma_1 - 4}{\rho r} \frac{\partial P}{\partial r} \tag{1.104b}$$

$$\sigma^2 = (3\Gamma_1 - 4) \frac{Gm}{r^3}\,, \tag{1.104c}$$

and by inserting the mean density formula $\overline{\rho} = M / \left(4\pi R^3 / 3 \right)$ we get the period-mean density relation: $\sigma = 2\pi / T$, so we have $T^{-2} \propto \overline{\rho}$, the period-mean density relation.

Actually, this simplified model gives us $m/r^3 = \text{const}$: the density in this case must be constant and equal to $\overline{\rho}$ (even if we did not use the assumption before). The precise relation is given by

$$T^2 \overline{\rho} = \frac{3\pi}{(3\Gamma_1 - 4)G}\,, \tag{1.105}$$

Polytropic model

It is a gas sphere with the following constitutive equaition:

$$P = K_n \rho^{1+1/n} = K_n \rho^{\frac{n+1}{n}} \tag{1.106}$$

with varying *polytropic* index n.

This can be derived as a solution for the Lane-Emden equation, which comes from Poisson's equation for the gravitational potential: so we do the following manipulation

Laplacian in spherical coordinates under spherical symmetry

Used the hydrostatic equilibrium equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{1.107a}$$
$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r} \right) = 4\pi G\rho \tag{1.107b}$$
$$-\frac{1}{r^2}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} \right) = 4\pi G\rho \tag{1.107c}$$
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} \right) = -4\pi G\rho r^2\,, \tag{1.107d}$$

so if we assume $\rho = \rho_c \theta^n$ the polytropic equation of state will read

$$P = K_n \rho^{1+1/n} = K_n \rho_c^{1+1/n} \theta^{n(1+1/n)} = K_n \rho_c^{1+1/n} \theta^{n+1}\,, \tag{1.108}$$

where ρ_c is the central density, so we have $\theta(r=0) = 1$ and $\theta'(r=0) = 0$. Inserting this into Poisson's equation gives us

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho_c \theta^n}\frac{\mathrm{d}}{\mathrm{d}r}\left(K_n \rho_c^{1+1/n}\theta^{n+1} \right) \right) = -4\pi G\rho r^2 \tag{1.109a}$$
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 (n+1) K_n \rho_c^{1/n} \frac{\mathrm{d}\theta}{\mathrm{d}r} \right) = -4\pi G \rho r^2\,, \tag{1.109b}$$

so we can adimensionalize the radial coordinate by putting inside of it all the numbers: $r = \alpha\xi$, with α chosen so that the equation turns into

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) + \theta^n = 0\,. \tag{1.110}$$

This is always solvable, and it is solvable analytically for certain values of n.

It models spheres with different mass distributions: $n = 0$ is constant density, $n = 5$ is infinite central density, $n = 3$ is the Eddington standard model, which is reasonable for the Sun and stars on the main sequence.

With these assumptions we can explicitly solve for the wavefunctions, and make predictions of the fractional modulus of the oscillations at a certain radius wrt the modulus at the surface (which can be found only experimentally).

The overtones die out toward the center even faster than the fundamental: these oscillations are very much a *surface phenomenon*.

Beyond the ηs we can also plot the pressure perturbations: these will not be normalized.

Slide 4.108: What is ω? Should we not have $\sigma = \omega$ of the mode? It seems like $\mathcal{P}$ is the period, and the numbers are consistent with $\mathcal{P} = 2\pi/\sigma$

Answer, from the first equation in Schwarzschild 1941[a]: ω and σ are related by

$$\omega = \frac{\sigma}{\sqrt{\pi G \Gamma_1 \rho_{0c}}}, \tag{1.111}$$

so ω is adimensional: it is, in a certain sense, an adimensionalized frequency. From the data in slide 4.108 we can derive the value of $\Gamma_1 \rho_{0c} = (\sigma/\omega)^2/(\pi G) \approx 5.184 \times 10^{14}\,\mathrm{kg/m^3}$.

[a] https://ui.adsabs.harvard.edu/abs/1941ApJ....94..245S/abstract

Numerical RR Lyrae models

We integrate the stellar structure equations numerically, so instead of a simple polytropic equilibrium model our equilibrium model is numerical. This is done looking at an RR Lyrae variable stars.

What are their characteristics?

We find that the ratio between the period of the fundamental and of the first overtone is around $P_1/P_0 \approx 0.743 \div 0.745$ in observed RR Lyrae, while the LAWE model predicts 0.731. The observed period is around $P_0 \approx 0.5668\,\mathrm{d}$, while the model predicts $P_0 \sim 0.6454\,\mathrm{d}$.

We can see that $\sigma_m - \sigma_{m-1} \approx$ const when m gets large, see figure 1.1.

The wavefunctions die out faster than the polytropic model when $r \to 0$.

There are "bumps" in the pressure perturbation plot: these are the partial ionization regions of H and He.

These appear because we start from a solution of the stellar structure equations, where all the properties of stellar matter were used, to start off with the LAWE.

1.4 Non-adiabatic oscillations

1.4.1 Driving and damping layers

How can we tell, theoretically, how stable and how wide the various modes are? We expect to see the stable modes, and not to see the unstable ones.

Let us start from the Lagrangian momentum conservation:

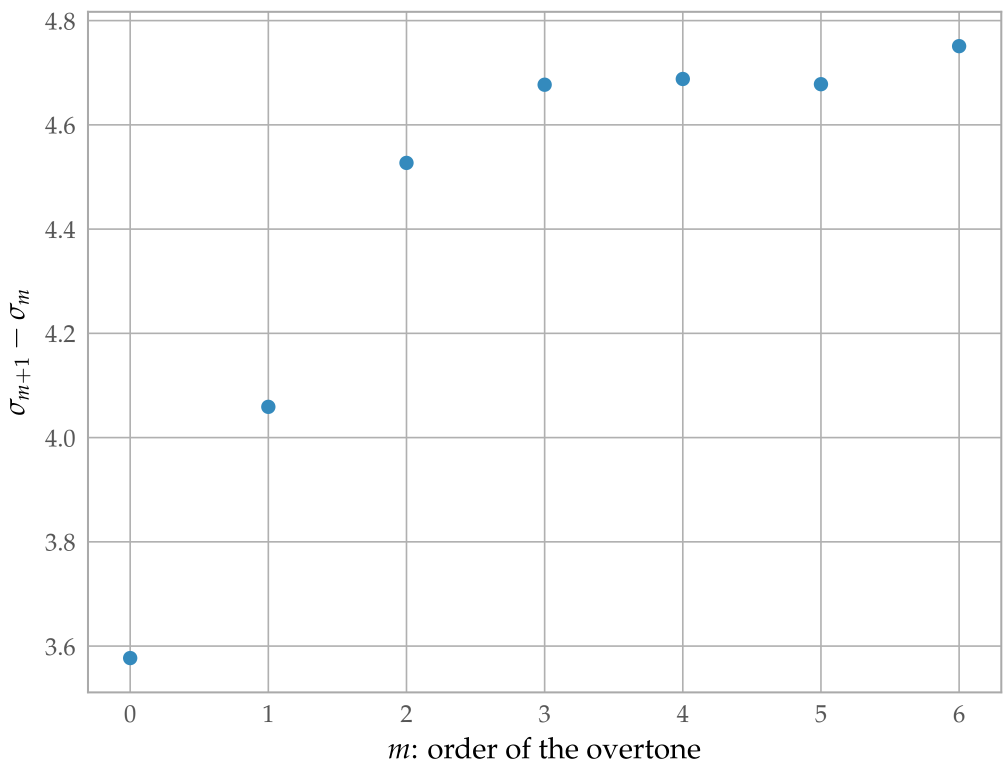

Figure 1.1: RR Lyrae frequency differences. The frequencies are measured in rad/d.

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2} \tag{1.112}$$

and apply to it the identity: $1/2 \frac{\partial}{\partial t} v^2 = \frac{\partial r}{\partial t} \frac{\partial^2 r}{\partial t^2}$, by multiplying everything by $\partial r / \partial t$, and then we integrate everything with respect to m: we find

$$\int_M \frac{1}{2} \frac{\partial}{\partial t} \left(v^2 \right) \mathrm{d}m = - \int_M 4\pi r^2 \frac{\partial P}{\partial m} \frac{\partial r}{\partial t} \mathrm{d}m - \int_M \frac{Gm}{r^2} \frac{\partial r}{\partial t} \mathrm{d}m \,. \tag{1.113}$$

Now we apply some manipulations: first of all, the following:

$$\int_M \left(- \frac{Gm}{r^2} \frac{\partial r}{\partial t} \right) \mathrm{d}m = \int_M \frac{\partial}{\partial t} \left(\frac{Gm}{r} \right) \mathrm{d}m \tag{1.114a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_M \frac{Gm}{r} \mathrm{d}m \tag{1.114b}$$

$$= - \frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{1.114c}$$

where Ω is the total gravitational potential energy, while

$$-\int_M 4\pi r^2 \frac{\partial P}{\partial m}\frac{\partial r}{\partial t}\,\mathrm{d}m = -\int_M \frac{\partial}{\partial m}\left(4\pi r^2 P\frac{\partial r}{\partial t}\right)\mathrm{d}m + \int_M P\frac{\partial}{\partial m}\left(4\pi r^2\frac{\partial r}{\partial t}\right)\mathrm{d}m \tag{1.115a}$$

$$= \underbrace{-\left.4\pi r^2 P\frac{\partial r}{\partial t}\right|_{m=0}^{m=M}}_{=0} + \int_M \frac{4\pi}{3}P\frac{\partial}{\partial m}\frac{\partial}{\partial t}\left(r^3\right)\mathrm{d}m \tag{1.115b}$$

$$= \int_M P\frac{\partial}{\partial t}\underbrace{\left(4\pi r^2\frac{\partial r}{\partial m}\right)}_{1/\rho}\mathrm{d}m \tag{1.115c}$$

Commuted m and r derivatives

$$= \int_M P\frac{\partial}{\partial t}\left(\rho^{-1}\right)\mathrm{d}m\,, \tag{1.115d}$$

so in the end we get

$$\frac{\partial}{\partial t}\int_M \frac{v^2}{2}\,\mathrm{d}m = -\frac{\mathrm{d}\Omega}{\mathrm{d}t} + \int_M P\frac{\partial}{\partial t}\frac{1}{\rho}\,\mathrm{d}m \tag{1.116}$$

We integrate in time over a pulsation period, which cancels out the gravitational potential term which is conservative:

$$-\int_\Pi \frac{\mathrm{d}\Omega}{\mathrm{d}t}\,\mathrm{d}t = 0\,. \tag{1.117}$$

If we also divide by the period, we are computing an average over a period Π. The term

$$\left\langle \frac{\partial}{\partial t}\int_M \frac{v^2}{2}\,\mathrm{d}m\right\rangle_\Pi = \frac{1}{\Pi}\int_\Pi \frac{\partial}{\partial t}\int_M \frac{v^2}{2}\,\mathrm{d}m\,\mathrm{d}t = \frac{\Delta E_{\mathrm{kin}}}{\Pi} \tag{1.118}$$

is the average power converted to kinetic energy; the variation in kinetic energy is also the work done on the system: $\Delta E_{\mathrm{kin}} = W$. So, taking the average on both sides of the equation we get

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t}\right\rangle_\Pi = \frac{1}{\Pi}\int_\Pi\int_M P\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)\mathrm{d}m\,\mathrm{d}t\,. \tag{1.119}$$

Some layers will provide energy to the oscillation motion (*drive* it), some others will *damp* it. These are characterized by the sign of their contribution the RHS of this equation.

If it is positive, we have instability since more and more work is being done one the star every period; if it is negative the pulsations will tend to die out, giving stability, since they lose kinetic energy every pulse.

The average time scale of change of the perturbations is

$$\kappa \stackrel{\text{def}}{=} \frac{1}{\tau} = -\frac{1}{2}\frac{\left\langle \mathrm{d}W/\mathrm{d}t\right\rangle_\Pi}{\langle \delta\psi\rangle_\Pi}\,, \tag{1.120}$$

where $\langle \delta\psi\rangle$ is the "average pulsational energy per pulsation cycle":

so, the integral of the absolute value of $\mathrm{d}W/\mathrm{d}t$? This ψ was not defined...
Also, this way τ can be negative: maybe the proper definition is $\tau = 1/|\kappa|$?

So, if $\kappa < 0$ we are looking at a driving layer, since then $\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle > 0$, while $\kappa > 0$ means we are in a damping layer.

The term $\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle_\Pi$ can also be interpreted as the net heat gain fed into mechanical work during a pulsation cycle: this can be derived from the first law of thermodynamics (written in its per-unit-mass form, and with the differentials being applied to time derivatives):

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + P\frac{\mathrm{d}(\rho^{-1})}{\mathrm{d}t}, \tag{1.121}$$

so if this is averaged over a period the internal energy change is approximately zero (thermal (i. e. internal) energy changes on the thermal timescale, which is much longer than the dynamical timescale), so we have

$$\left\langle \frac{\mathrm{d}Q}{\mathrm{d}t} \right\rangle_\Pi = \left\langle P\frac{\mathrm{d}}{\mathrm{d}t}(\rho^{-1}) \right\rangle, \tag{1.122}$$

and therefore we can rewrite equation (1.119) as

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle_\Pi = \frac{1}{\Pi}\int_\Pi \int_M \frac{\mathrm{d}Q}{\mathrm{d}t}\,\mathrm{d}m\,\mathrm{d}t\,, \tag{1.123}$$

where Q is a heat per unit mass, so its integral in $\mathrm{d}m$ is the total heat change across all stellar layers.

In the adiabatic case, we had

$$\frac{\partial}{\partial t}\left(\frac{\delta P}{P} - \Gamma_1\frac{\delta\rho}{\rho}\right) = 0\,, \tag{1.124}$$

(see equation (1.68a), in which we neglected the heat variation terms in the adiabatic case) so the perturbations were in phase.

Now we add a term to the time derivatives: the pressure and density perturbations will stop being in phase. The sign of the heat variation term gives us the difference between *driving* heat transfer and *damping* heat transfer. Equation (1.68a) will now look like

$$\frac{\partial}{\partial t}\frac{\delta P}{P} = \Gamma_1\frac{\partial}{\partial t}\frac{\delta\rho}{\rho} + \frac{P}{\rho}(\Gamma_3 - 1)\delta\frac{\mathrm{d}Q}{\mathrm{d}t}, \tag{1.125}$$

so the instants of minimum pressure and minimum density will not be synchronized anymore, since there is that extra $\mathrm{d}Q/\mathrm{d}t$ term.

In a PV diagram, we can see that these correspond to right and left oriented loops (as opposed to the loops with zero total signed area we had in the adiabatic case).

The total work performed on the star is given by the area of the loops: the area is positive for a loop going clockwise, and negative for a loop going counter-clockwise.

If the term $\delta\Big(\mathrm{d}Q/\mathrm{d}t\Big) > 0$ we are looking at a *driving layer*, since at the time of maximum compression ($\partial/\partial t\ (\delta\rho/\rho) = 0$) we will have $\partial/\partial t\ (\delta P/P) > 0$: the maximum pressure perturbation will happen as the layer is expanding, so the expansion will be amplified. The symmetric reasoning gives us the fact that if $\delta\Big(\mathrm{d}Q/\mathrm{d}t\Big) < 0$ we have a *damping layer*.

There tend to be more driving layers in the outer parts of the star, especially peaking in power generation in the partial Helium and Hydrogen ionization regions.

What is going on in these regions? Why do they act this way?

The star is effectively a themal engine converting heat into work; this will result in an increased overall entropy of the star, and a smoothing of its temperature gradient, however:

1. the time scales on which this process occurs are much larger than the time scales on which oscillating motions are created and destroyed (specifically, the entropy changes on the thermal time scale);

2. then energies of the oscillations are much smaller than the global thermal energy of the star.

therefore this process is typically not relevant.

Mon Oct 14 2019

1.4.2 The LNAWE

First of all, let us recall the linearized equations of stellar structure:

$$\frac{\delta\rho}{\rho} = -3\zeta - 4\pi r^3 \rho \frac{\partial \zeta}{\partial m} \tag{1.126a}$$

$$r\ddot{\zeta} = -4\pi r^2 \left[\left(4\zeta + \frac{\delta P}{P} \right) \frac{\partial P}{\partial m} + P \frac{\partial}{\partial m} \left(\frac{\delta P}{P} \right) \right] \tag{1.126b}$$

$$\frac{\partial}{\partial t} \left(\frac{\delta P}{P} \right) = \Gamma_1 \frac{\partial}{\partial t} \left(\frac{\delta \rho}{\rho} \right) + \frac{\rho}{P} (\Gamma_3 - 1) \delta \left(\epsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m} \right) \tag{1.126c}$$

$$\frac{\partial}{\partial t} \left(\frac{\delta T}{T} \right) = (\Gamma_3 - 1) \frac{\partial}{\partial t} \left(\frac{\delta \rho}{\rho} \right) + \frac{1}{c_V T} \delta \left(\epsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m} \right) \tag{1.126d}$$

$$\frac{\delta L}{L} = 4\zeta - n \frac{\delta \rho}{\rho} + (s+4) \frac{\delta T}{T} + \left(\frac{\partial \log T}{\partial m} \right)^{-1} \frac{\partial}{\partial m} \left(\frac{\delta T}{T} \right), \tag{1.126e}$$

using which we can derive after long manipulations the Linear Non-Adiabatic Wave

Equation, which in its Lagrangian formulation reads:

$$\ddot{\zeta} = 4\pi r \left(\dot{z} \frac{\partial}{\partial m} ((3\Gamma_1 - 4)P) - \frac{\partial}{\partial m} \left(\rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right) \right) \right) + \frac{1}{r^2} \frac{\partial}{\partial m} \left(16\pi^2 \Gamma_1 P \rho r^6 \frac{\partial \dot{\zeta}}{\partial m} \right), \tag{1.127}$$

where we used the fact that $\mathrm{d}Q/\mathrm{d}t = \epsilon_{\text{eff}} - \partial L/\partial m$. On the other hand, the Eulerian formulation is

$$\ddot{\zeta} = \frac{1}{r\rho} \left(\dot{\zeta} \frac{\partial}{\partial r} ((3\Gamma_1 - 4)P) - \frac{\partial}{\partial r} \left(\rho(\Gamma_3 - 1)\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right) \right) \right) + \frac{1}{r^4 \rho} \frac{\partial}{\partial r} \left(r^4 \Gamma_1 P \frac{\partial \dot{\zeta}}{\partial r} \right). \tag{1.128}$$

The procedure to derive the LNAWE is as follows:

1. substitute the continuity equation (1.126a) into the P, ρ form of the energy conservation equation (1.126c);
2. substitute the P, ρ form of the energy conservation equation (1.126c) into the time derivative of the Eulerian form of the momentum conservation equation (1.126b);
3. simplify.

1.4.3 Solving the LNAWE

Our ansatz for the LNAWE will still be of the form $\zeta(r, t) = \eta(r) e^{i\sigma t}$, but now we insert $\sigma = \omega + i\kappa$: this means we also consider *damped* exponential solutions and *diverging* exponential solutions. We want to simplify the exponentials, so we must assume that

$$\delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right) = \delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\text{sp}} e^{i\sigma t}. \tag{1.129}$$

Why do we assume that the heat derivative perturbation is *in phase* with the displacement? Maybe we do not, and the $_{\text{sp}}$ heat variation is complex?

With this substitution we get:

$$-i\sigma^3 \eta = \frac{i\sigma\eta}{r\rho} \frac{\partial}{\partial r} ((3\Gamma_1 - 4)P) - \frac{1}{r\rho} \frac{\partial}{\partial r} \left(\rho(\Gamma_3 - 1)\delta \left(\frac{\partial Q}{\partial t} \right)_{\text{sp}} \right) + \frac{i\sigma}{r^4 \rho} \frac{\partial}{\partial r} \left(r^4 \Gamma_1 P \frac{\partial \eta}{\partial r} \right). \tag{1.130}$$

The time scales for these parameters are $\omega \sim \omega_{\text{ad}} \sim \tau_{\text{dyn}}$, while $\kappa \sim 1/\tau_{\text{th}}$: therefore $\omega \gg |\kappa|$.

Using this result, we can make some useful *quasi-adiabatic* approximations: in the LNAWE we will identify the LAWE operator $\mathcal{L}$, and replace its application to the wavefunction with the corrisponding eigenvalue. Basically, we will consider the thermal contribution to be small and work "to first order" with it.

$$-i\sigma^3\eta = -i\sigma\left(-\frac{1}{r\rho}\frac{\partial}{\partial r}((3\Gamma_1-4)P) - \frac{1}{r^4\rho}\frac{\partial}{\partial r}r^4\Gamma_1 P\frac{\partial\eta}{\partial r}\right)+ \\ -\frac{1}{r\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3-1)\delta\left(\frac{\partial Q}{\partial t}\right)_{\text{sp}}\right) \tag{1.131a}$$

$$-i\sigma^3\eta = -i\sigma\mathcal{L}(\eta) - \frac{1}{r\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3-1)\delta\left(\frac{\partial Q}{\partial t}\right)_{\text{sp}}\right) \tag{1.131b}$$

$$\mathcal{L}(\eta) - \sigma^2\eta = \frac{i}{r\sigma\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3-1)\delta\left(\frac{\partial Q}{\partial t}\right)_{\text{sp}}\right), \tag{1.131c}$$

so we can see that the eigenvalue of the LAWE operator cannot be σ^2 now. We take this equation, multiply it by ηr^2 and integrate it over the whole star in dm: we get

$$i\sigma^3\int\eta^2r^2\,\mathrm{d}m - i\sigma\int\eta\mathcal{L}(\eta)r^2\,\mathrm{d}m = \int\frac{r}{\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3-1)\delta\left(\frac{\partial Q}{\partial r}\right)_{\text{sp}}\right)\eta\,\mathrm{d}m \stackrel{\text{def}}{=} C\,, \tag{1.132}$$

where we defined the *work integral* C. We only look at the first order terms in κ: so we make the approximation $i\sigma^3 \approx \omega^2(i\omega-3\kappa)$, while (to first order, but also exactly) $i\sigma = i\omega-\kappa$. We substitute these two, and then make the key manipulation: we substitute $\mathcal{L}(\eta)$ with $\omega^2\eta$. The only thing missing is the definition: $J \stackrel{\text{def}}{=} \int\eta^2r^2\,\mathrm{d}m$. So we get

$$\omega^2(i\omega-3\kappa)J - (i\omega-\kappa)\omega^2J = C \tag{1.133a}$$

$$-3\kappa+\kappa = \frac{C}{J\omega^2} \tag{1.133b}$$

$$\kappa = -\frac{C}{2\omega^2J}\,. \tag{1.133c}$$

Now we make some considerations on the expression of C and integrate by parts, getting:

$$C = \int_M\frac{1}{r\rho}\frac{\partial}{\partial r}\left(\rho(\Gamma_3-1)\delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\text{sp}}\right)\eta r^2\,\mathrm{d}m \tag{1.134a}$$

$$= \int_R\frac{\partial}{\partial r}\left(\rho(\Gamma_3-1)\delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\text{sp}}\right)4\pi r^3\eta\,\mathrm{d}r \tag{1.134b}$$

d$m = 4\pi r^2\rho\,$dr

$$= \rho(\Gamma_3-1)\delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\text{sp}}4\pi r^3\eta\Big|_{r=0}^{r=R} - \int_R\rho(\Gamma_3-1)\delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\text{sp}}\frac{\partial}{\partial r}\left(4\pi r^3\eta\right)\mathrm{d}r \tag{1.134c}$$

$$= -\int_R\rho(\Gamma_3-1)\delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\text{sp}}4\pi r^2\left(3\eta+r\frac{\partial\eta}{\partial r}\right)\mathrm{d}r \tag{1.134d}$$

$\rho = 0$ if $r = R$

$$= \int_R\rho(\Gamma_3-1)\delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\text{sp}}4\pi r^2\left(\frac{\delta\rho}{\rho}\right)_{\text{sp}}\mathrm{d}r \tag{1.134e}$$

From equation (1.126a): $\frac{\delta\rho}{\rho} = -3\zeta - r\frac{\partial\zeta}{\partial r}$

$$= \int_R (\Gamma_3 - 1) \left(\frac{\delta\rho}{\rho} \right)_{\text{sp}} \delta \left(\frac{\mathrm{d}Q}{\mathrm{d}t} \right)_{\text{sp}} \mathrm{d}m \tag{1.134f}$$

$$= \int_M \left(\frac{\delta T}{T} \right)_{\text{sp}} \delta \left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m} \right) \mathrm{d}m \,, \tag{1.134g}$$

From the time-integrated equation (1.126d), neglecting the heat transfer term

and now this expression allows us to study the mechanisms which create perturbations: the coefficient to calculate is κ, which we now know to be given by

$$\kappa = -\frac{1}{2\omega^2 J} \int_M \left(\frac{\delta T}{T} \right)_{\text{sp}} \delta \left(\epsilon_{\text{eff}} - \frac{\partial L}{\partial m} \right) \mathrm{d}m \,, \tag{1.135}$$

which we will study in the next section.

1.4.4 Driving mechanisms

The energy of the vibrations comes from the internal thermal energy of the star, which ultimately comes from thermonuclear reactions. We can rewrite equation (1.135) as

$$\kappa = -\frac{1}{2\omega^2 J} \underbrace{\int_M \left(\frac{\delta T}{T} \right)_{\text{sp}} \delta\epsilon_{\text{eff}} \, \mathrm{d}m}_{\text{energy generation}} + \frac{1}{2\omega^2 J} \underbrace{\int_M \left(\frac{\delta T}{T} \right)_{\text{sp}} \frac{\partial \delta L}{\partial m} \, \mathrm{d}m}_{\text{energy transfer}} \,. \tag{1.136}$$

The ϵ-mechanism is about energy generation, which is assumed to be due to nuclear reactions, without considering neutrino processes: this can happen if the magnitude of the temperature and density perturbations are large enough.

The κ-γ-mechanism is about considering the regions where the luminosity gradient and temperature gradient are discordant. This means that the considered stellar layer is absorbing or emitting; this is usually assumed to be happening through free-free interactions (bremsstrahlung and inverse bremsstrahlung).

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ϵ mechanism

We now concern ourselves with the energy generation term of equation (1.136).

In general the effective energy generation per unit mass is given by

$$\epsilon_{\text{eff}} = \epsilon_{\text{nuc}} - \epsilon_\nu \,, \tag{1.137}$$

but we assume that the neutrino loss of energy contribution in the perturbation is negligible, so when we perturb we will have $\delta\epsilon_{\text{eff}} = \delta\epsilon_{\text{nuc}}$. We also have the approximate relation

$$\frac{\delta T}{T} = (\Gamma_3 - 1) \frac{\delta\rho}{\rho} \,, \tag{1.138}$$

which implies, since $\Gamma_3 > 1$, that the temperature and density relative perturbations are concordant in sign.

Why can we use this relation? Does it have something to do with working "at first order in $\mathrm{d}Q/\mathrm{d}t$"?

Now, the line of reasoning goes like this: the nuclear energy generation perturbation is concordant in sign with the temperature perturbation. So, the term

$$\left(\frac{\delta T}{T}\right)_{\mathrm{sp}} \delta\epsilon_{\mathrm{eff}} \geq 0\,, \tag{1.139}$$

which means that the contribution to κ will be negative (because of the minus sign in equation (1.136)): so, since the dependence of the radial perturbation looks like $\zeta \propto \exp(-\kappa t)$, the perturbation is *amplified*: this is a driving layer.

The reasoning in slide 6.08 seems way too contorted: plus, it mentions the layers "absorbing energy", but we cannot actually talk about that without considering the luminosity gradient perturbation…

Are the fluctuations in ϵ_{nuc} actually enough to power pulsations? *No*. An illustrative example is given by RR-Lyrae stars. There, we can look at the relative magnitude of the realative temperature variation, $\delta T/T$, at various layers in the star: it is of the order 10^{-1} or even more at the surface, but quickly drops as we move towards the center; in the Hydrogen burning shell and in the Helium burning core it is of the order $10^{-9} \div 10^{-7}$, which implies that the actual temperature fluctuations are of the order $\delta T \sim 1\,\mathrm{K}$: way too little to amplify pulsations.

κ-γ mechanism

Today we look at the κ-γ-mechanism, which is about the term

$$\int_M \left(\frac{\delta T}{T}\right)_{\mathrm{sp}} \frac{\partial \delta L}{\partial m}\,\mathrm{d}m\,, \tag{1.140}$$

and we will have driving layers ($\kappa < 0$) in the regions in which the two terms multiplied in the integrand are discordant.

The mean Rosseland opacity is approximated by a law in the form:

$$\kappa_R \approx \overline{\kappa}_R \rho^n T^{-s}\,, \tag{1.141}$$

with $n \approx 1$, $s \approx 7/2$ in the case of free-free absorption (that is, inverse & direct bremsstrahlung) in a non-degenerate, totally ionized gas. This seems to be a good approximation for $4 < \log_{10} T < 8$.

This allows us to relate the perturbations in κ_R to those in T:

$$\frac{\delta\kappa_R}{\kappa_R} \approx n\frac{\delta\rho}{\rho} - s\frac{\delta T}{T} \tag{1.142a}$$

$$\approx \left(n - s(\Gamma_3 - 1)\right)\frac{\delta\rho}{\rho}\,, \tag{1.142b}$$

and if we substitute in $\Gamma_3 \approx 5/3$ with the other terms we get

$$\frac{\delta\kappa_R}{\kappa_R} \approx -\frac{4}{3}\frac{\delta\rho}{\rho}\,, \tag{1.143}$$

which means that the relative mean opacity perturbation is *discordant* in sign with the pressure one, and thus with the temperature one.

So, up to a positive constant, the integrand looks like

$$-\frac{\delta\kappa_R}{\kappa_R}\frac{\partial\delta L}{\partial m}\,, \tag{1.144}$$

and we want it to be < 0. However, if under the perturbation the layer is *more* opaque than usual ($\delta\kappa_R > 0$) then it will let through *less* light than usual, so we will have $\delta(\partial L/\partial m) < 0$. This can be rendered more formal by considering the luminosity equation (1.11): the dependence of L on κ_R is inverse.

So, the global contribution to κ will be positive, therefore our layer will be *damping*.

κ-mechanism However, layers for which the contribution of this term is negative are present in certain regions: where there is ionization, we can have regions for which the gradient $\partial\kappa_R/\partial T$ reverses: it is usually negative, but it can become locally positive. This provides an additional channel for energy stocking: it is like a *dam* for energy. The ionization energy is then released through mechanical work. This is called the κ-mechanism.

γ-mechanism The γ-mechanism, on the other hand, involves a decrease of Γ_3 which brings it near to 1: recall,

$$\Gamma_3 - 1 = \left.\frac{\partial \log T}{\partial \log \rho}\right|_{\text{adiabatic}}\,, \tag{1.145}$$

so if it is small that means that the change in temperature, i.e. internal energy, associated with a compression ($\delta\rho > 0$) is small: this can happen in the outer layers of the star, which have partial ionization regions. The work of the compression is partially absorbed in order to ionize the atoms in these layers: if Γ_3 is close enough to 1, we can have

$$n - s(\Gamma_3 - 1) > 0 \tag{1.146}$$

even with $s > 0$.

κ-γ-mechanism We look at the linearized equations of continuity and radiative transfer for an expression for the gradient of δL, neglecting the temperature perturbation gradient term for simplicity: then, starting from equation (1.126e) we have

$$\frac{\delta L}{L} = 4\zeta - n\frac{\delta\rho}{\rho} + (s+4)\frac{\delta T}{T}\,, \tag{1.147}$$

while from equation (1.126a) we get

$$\frac{\delta\rho}{\rho} = -3\zeta\,. \tag{1.148}$$

Combining these, and using $\delta T/T = (\Gamma_3 - 1)\delta\rho/\rho$ we find:

$$\frac{\delta L}{L} = \left(-\frac{4/3+n}{\Gamma_3 - 1} + s + 4\right)\frac{\delta T}{T}\,, \tag{1.149}$$

and now we make the following consideration: on average, in the outer layers of the stars we have no nuclear energy generation ($\epsilon_{\mathrm{nuc}} \approx 0$) and the heat variation is negligible ($\mathrm{d}Q/\mathrm{d}t \approx 0$). So, we have $\partial L/\partial m \approx 0$ by the energy conservation equation (1.5): using this, we can freely bring L outside of derivatives with respect to m, in order to write

Why?

$$\frac{\partial \delta L}{\partial m} = L\frac{\partial}{\partial m}\left[\left(s + 4 - \frac{4/3+n}{\Gamma_3 - 1}\right)\frac{\delta T}{T}\right]. \tag{1.150}$$

Outside the regions of partial ionization, typical values for the parameters are $\Gamma_3 \sim 1.6$, $s \sim 7/2$, $n \sim 1$. Therefore, we get

$$\frac{\partial \delta L}{\partial m} \approx 3.61 \times L\frac{\partial}{\partial m}\left(\frac{\delta T}{T}\right), \tag{1.151}$$

and a function and its derivative have the same sign, right? so we approximate the derivative of $\partial/\partial m\ (\delta T/T)$ with $\delta T/T$

which means that

$$\frac{\partial \delta L}{\partial m} \sim 3.6L\frac{\delta T}{T}\,, \tag{1.152}$$

so they are concordant in sign, so the layer is a damping layer.

In partial ionization regions, instead, $\kappa_R \propto \rho^n T^{-s}$ with negative s: so it is much easier for the terms to be discordant. Keeping $n = 1$ and $\Gamma_3 = 1.6$, we need $s < -0.11$ in order to have a driving layer. However, Γ_3 also decreases in these layers, so the threshold for s is higher: for example, with $n = 1$ and $\Gamma_3 = 1.4$ we only need $s < 1.83$.

So, the κ in the κ-γ-mechanism refers to the decrease in s while the γ refers to the decrease in Γ_3, right?

Opacity bump mechanism The κ-γ-mechanism cannot explain the pulsation of hotter stars than RR-Lyr, Cepheids, δ-Scuti or SX-Phe, such as β-Ceph, GW Vir and sdBV stars: in these, the stellar material is much more ionized on average: in the regions which, for cooler stars, are partial ionization regions the gas is fully ionized, while the partial ionization regions have moved further out towards the surface, if they have not disappeared entirely.

One could consider the partial second ionization layers: this has been thoroughly considered, and it has been found that they cannot provide a significant enough contribution.

There is an opacity bump at $5 < \log_{10}(T) < 6.5$, which was found in the eighties by Simon: the old graph for $\kappa(T)$ had spikes around $T \sim 10^4$ and $T \sim 10^{4.7}$, and a new spike was found at $T \sim 10^{5.4}$. Do note that these are temperatures reached when looking a bit inside the star, not at the surface (although close to it).

Convective blocking Convection is slow to adapt, so it cannot move away the heat brought by the pulsations. Then, the heat must be turned into work. This happens when the period of the pulsations is of the order of the thermal timescale?

Not sure about this

Convective driving Convective driving is called the δ-mechanism. It happens when the convective timescale is much shorter than the period of the pulsation, so the convection can mix the gas.

And this drives pulsations?

Stochastic driving There also are *stochastically driven* oscillations: they happen in stars which are intrinsically stable, but the pulsations may be fed by convective turbulence. This can happen in Sun-like Main Sequence stars, or in solar-like red giants, such as ξ-Hydrae stars.

Strange modes There are very luminous *strange modes*, very dim convectively driven modes.

The classical instability strip

The pulsation region has boundaries: for the κ-mechanism:

- if the star is too hot, the regions of partial ionization get too close to the surface;
- if the star is too cold, they are too far in: in their outer region the pulsation is damped by convective heat transfer.

So, there is an optimal region for the partial ionization layers to be.
What is the optimal region? We will only look at the fundamental mode. We define

$$\phi(m) \equiv \frac{1}{L(\Pi/2\pi)} \int_m^M c_V T \, \mathrm{d}m \;, \tag{1.153}$$

which represents the thermal balance of a single oscillation: $L(\Pi/2\pi)$ is the luminosity radiated in a radian of the pulsation cycle ($1/2\pi$ cycles), while $c_V T = T\left(\partial Q/\partial T\right) \approx Q$, the heat per unit mass, which when integrated from m to M, the whole mass of the star, gives us the global heat variation of the layers above m.

We are in the helium ionization region: there are no nuclear reactions, so we set energy generation to zero in equation (1.126d): we get

$$\frac{\partial}{\partial t}\left(\frac{\delta T}{T}\right) = (\Gamma_3 - 1)\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho}\right) - \frac{1}{c_V T}\frac{\partial \delta L}{\partial m}\,. \tag{1.154}$$

Now we assume that the temperature, density and luminosity perturbations are oscillating, so they have a certain starting value, and the temporal dependence is absorbed in a factor $e^{i\sigma t}$. Then, we get

$$i\sigma\frac{\delta T}{T} = (\Gamma_3 - 1)i\sigma\frac{\delta\rho}{\rho} - \frac{1}{c_V T}\frac{\partial \delta L}{\partial m} \tag{1.155a}$$

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta\rho}{\rho} + \frac{i}{\sigma c_V T}\frac{\partial \delta L}{\partial m}\,. \tag{1.155b}$$

Now, consider the following: with our definition of ϕ, we can write:

$$\frac{\partial}{\partial\phi} = \frac{\partial m}{\partial\phi}\frac{\partial}{\partial m} = -\frac{L(\Pi/2\pi)}{c_V T}\frac{\partial}{\partial m}\,; \tag{1.156}$$

do note that ϕ is adimensional, because of the way we define $L(\Pi/2\pi)$:

$$L(\Pi/2\pi) = L/\sigma\,, \tag{1.157}$$

which is an energy. With these facts in mind, plus the fact that approximately $\partial L/\partial m = 0$, we can do the following manipulation:

$$\frac{i}{\sigma c_V T}\frac{\partial \delta L}{\partial m} = -\frac{i}{\sigma L(\Pi/2\pi)}\frac{\partial \delta L}{\partial \phi} \tag{1.158a}$$

$$= -i\frac{\partial}{\partial\phi}\left(\frac{\delta L}{L}\right)\,. \tag{1.158b}$$

Now, we can use the chain rule on the derivative in $\partial\phi$ using any auxiliary variable we like: we choose $x = r/R$, the fractionary radius, so we get

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta\rho}{\rho} - i\left(\frac{\partial\phi}{\partial x}\right)^{-1}\frac{\partial}{\partial x}\left(\frac{\delta L}{L}\right)\,, \tag{1.159}$$

where δT is actually complex, since the perturbations are out of phase.

We can make some considerations on the variations of the terms in the equation:

$$\phi \approx \frac{1}{L(\Pi/2\pi)}\frac{4}{3}\pi R^3\left(1 - x^3\right)\rho c_V T \propto (1 - x^3)\rho c_V T\,, \tag{1.160}$$

so

$$\frac{\partial\phi}{\partial x} \propto -x^2 \rho c_V T\,. \tag{1.161}$$

The x^2 dependence, however, is not very important: if the internal energy of a layer does not change, we can write the first law of thermodynamics as

$$\mathrm{d}Q = P\,\mathrm{d}\left(\frac{1}{\rho}\right) \implies P \sim \frac{\mathrm{d}Q}{\mathrm{d}(\rho^{-1})} \sim Q \times \rho\,, \tag{1.162}$$

and $c_V T \sim Q$, so we have:

$$\frac{\partial \phi}{\partial x} \propto -x^2 P\,, \tag{1.163}$$

and while x goes to zero in the center, the pressure increases a lot.

From what I could gather in 5 minutes, the Sun's pressure profile is very roughly given by

$$P(x) \sim 2.5 \times 10^{11}\,\mathrm{Pa} \exp(-\lambda x)\,, \tag{1.164}$$

where $\lambda \sim 9$. Now, if we plot $\exp(-9x)x^2$, it becomes small both at the surface and at the center, and has a maximum around $x \sim 0.2$. So, the reasoning still makes sense, but you have to consider the very center of the star as a special case.

On the surface, the imaginary term is negligible. In the interior, it is relevant.

So, near the surface $\partial x / \partial \phi$ is very large: therefore the term multiplying it, $\partial / \partial x\ (\delta L / L)$, must become small in order for the temperature perturbation not to diverge. This means that the luminosity perturbation cannot change much: it is "frozen in". Anyway, the imaginary part of the RHS of equation (1.159) is much larger than the real part.

Right?

On the other hand, near the center the term $\partial x / \partial \phi$ becomes very small, while the spatial derivative of the perturbation does not become very large, so on balance the term remains small: the oscillations are then quasi-adiabatic.

Between these two regimes, we have a *transition region* in which the two terms in equation (1.159) are similar in terms of order of magnitude: depending on the temperature of the star, this can can occur before or after the ionization region, when going radially outward.

The classical instability strip is the region in the Hertzsprung-Russell diagram in which the partial ionization region roughly coincides with the transition region.

Why is the perturbation "frozen in" if its spatial derivative is small? we could have $\delta L / L$ constant with respect to x and it would still evolve in time!

Why are the boundaries of the classical instability strip tilted? It seems like the region is defined by some relation like $\log L = -(\text{large \#}) \log T$, why is this so?

1.4.5 Star botany

RR Lyræ

These are stars with periods between $0.2 \div 1\,\mathrm{d}$, absolute visual magnitude around 0.6, and mean effective temperature around $6000 \div 7250\,\mathrm{K}$.

It is a stage, which lasts no more than 10^8 yr, and it is observed only in clusters older than 10^{10} yr.

They are classified by a, b, or c according to the shape of the light curve, its amplitude, its period:

1. RRa: sharp rise, large amplitude: the fundamental;
2. RRb: similar to RRa with smaller amplitude, longer period: the fundamental;
3. RRc: more symmetric light curve, short periods, low amplitudes: they pulsate in the first overtone.

We have also RRd: bimodal, RRe: second overtone.

These characteristics are seen well in a Bailey diagram, in which we scatter plot the stars with magnitude on the y axis and period on the x axis.

[Argument for the different amplitudes at different wavelengths: to understand]

We can do spectral analysis on a set of several different stars: we find that, when applying a spectral filter to the star and doing Fourier analysis on that light curve, we can identify a relation in the form

$$M_\nu = k_\nu \log \Pi \,, \tag{1.165}$$

where M_ν is the magnitude measured in that particular spectral band, while k_ν is a constant depending on the band. k_ν can be either positive or negative, and it is increasing with ν.

[Qualitative part of the lecture.]

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We were talking about RR Lyrae varibles, classifying them. In a Bailey diagram we plot variables as points with the coordinates amplitude and period.

We can tell whether a star is a RR Lyr variable, and then we can use it as a standard candle.

We have nonlinear models which accurately predict the light curves. There are HUMPs, BUMPs, JUMPs and LUMPs in the light curves: they tell us about the propagation of shock waves in the star.

We also have the Blazhko Effect: when it is not present, the light curve is the same at every cycle; when it is present the light curve changes, it is modulated every cycle. This implies that the phased light curve scatter plot is very spread, no matter how well we fix the period. This is a poorly understood effect, it might have something to do with magnetic fields.

About half of RRab have this effect, very few RRc have it.

In a given Globular Cluster, we look at the distribution of RR Lyr with respect to period: we can see two distinct clusters, corresponding to the fundamental and first harmonic (RRab and RRc respectively). We can characterize them with respect to the Oosterhoff group (the one of the fundamental): if it is big with respect to the first harmonic it is a type I, if it is comparable it is a type II.

We can distinguish these by making a histogram of the average period of GCs: OoII are clustered around 0.65 d, while OoI around 0.55 d. There is a distinct gap between the two.

Metal-rich Globular Clusters tend to few RR Lyrae, metal poor-ones tend to have more, unless their horizontal branch is very extended to the blue, in which case they may have few RR Lyr.

The Oosterhoff gap can be seen in metallicity as well.

The period, theoretically, should only change on evolutionary (i. e. very long) time scales, however we observe much faster period changes, especially in stars which exhibit the Blazhko effect.

So far, RR Lyrae have been found in local dwarf galaxies, in the Magellanic Clouds, in M31 (Andromeda Galaxy), in M33 (Triangulum Galaxy).

RR Lyr beyond the Local Group have not yet been observed.

1.4.6 Classical Cepheids

They massive stars ($4 \div 9 M_\odot$) are younger than RR Lyr, typically around $10^7 \div 10^8$ yr. Their absolute visual magnitude M_V can change in a range of typically -2 to -6. Their periods are typically beteen $0.5 \div 135$ d.

They are very important for the *cosmic distance ladder* determination, because of the Period-Luminosity relation, which can be used to measure distances: period is not affected by reddening.

The light curves of Cepheids are characterized by a sharp rize and a shallow fall in magnitude if the star is vibrating on the fundamental, a more symmetric light curve if it is vibrating on the first overtone. A second overtone pulsation is rarely found, and it generally has a much lower amplitude. In classical Cepheids we also find multimodal pulsators.

As with RR Lyrae, the amplitude of the pulsation is larger at higher light frequencies such as the UV, and decreases through visible and IR.

Cepheids with periods between 6 and 16 d we often have a *bump*, which could be caused by resonances between the fundamental and the second overtone or by echoes from the core.

Some Cepheids, like some RR Lyrae, also show the Blazhko effect (very long-lived modulations in the light curve).

The Period-Luminosity relation

The fundamental relation we start from is the Stefan-Boltzmann law:

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 , \tag{1.166}$$

which can be converted into a relation involving the bolometric magnitude: we first of all divide through by the solar values for the parameters, and then take the log:

$$\log\left(\frac{L}{L_\odot}\right) = 2\log\left(\frac{R}{R_\odot}\right) + 4\log\left(\frac{T_{\text{eff}}}{T_{\text{eff},\odot}}\right) , \tag{1.167}$$

and we use the definition of bolometric magnitude:

$$M_{\text{bol}} - M_{\text{bol},\odot} = -2.5 \log \frac{L}{L_\odot}\,, \tag{1.168}$$

so we substitute it in:

$$M_{\text{bol},\odot} - M_{\text{bol}} = 2.5\left(2\log\left(\frac{R}{R_\odot}\right) + 4\log\left(\frac{T_{\text{eff}}}{T_{\text{eff},\odot}}\right)\right) \tag{1.169a}$$

$$M_{\text{bol}} = -5\log\left(\frac{R}{R_\odot}\right) - 10\log\left(\frac{T_{\text{eff}}}{T_{\text{eff},\odot}}\right) + \text{const}\,, \tag{1.169b}$$

and now we can use the period-mean density relation:

$$\Pi^2\overline{\rho} = \mathcal{Q} = \text{const} \implies 2\log\Pi + \log\overline{\rho} = \log\mathcal{Q}\,, \tag{1.170}$$

but $\overline{\rho} \sim M/R^3$, so this becomes

$$2\log\Pi + \log M - 3\log R - \log\mathcal{Q} = \text{const} \tag{1.171a}$$

$$\log R = \frac{2}{3}\log\Pi + \frac{1}{3}\log M - \frac{2}{3}\log\mathcal{Q} + \text{const}\,. \tag{1.171b}$$

We can now substitute this into the other equation; we get

$$M_{\text{bol}} + 5\left(\frac{2}{3}\log\Pi + \frac{1}{3}\log M - \frac{2}{3}\log\mathcal{Q} + \text{const}\right) + 10\log T_{\text{eff}} = 0 \tag{1.172a}$$

Multiply by 3/10

$$\frac{3}{10}M_{\text{bol}} + \log\Pi + \frac{1}{2}\log M - \log\mathcal{Q} + 3\log T_{\text{eff}} = \text{const}\,, \tag{1.172b}$$

where we started dropping the adimensionalizing divisions by the solar values of the parameters, since they are "constants".

Now, we make the assumption of a mass-luminosity relation like that of Main Sequence stars: $\log M_{\text{bol}} = -8\log M + \text{const}$, so we substitute $\log M$ with $-M_{\text{bol}}/8$. The number multiplying M_{bol} is then $3/10 - 1/16 \approx 0.24$. So, in the end, we get

$$\log(\Pi) = -0.24 M_{\text{bol}} - 3\log(T_{\text{eff}}) + \log(Q) + \text{const}\,, \tag{1.173}$$

which is really useful because the period can be measured precisely, while the *absolute* bolometric magnitude is really difficult.

Since the stars which satisfy the P-L relations have different temperatures we have a certain spread of the P-L relation scatter plot.

Is T_{eff} not very much correlated with M_{bol}?

The derivation of this might be asked at the exam.

The PL relation is quite well respected experimentally: if we plot classical Cepheids' $\log\Pi$ vs $\log\langle L\rangle$ we see two distincts lines, which correspond to fundamental pulsators and first overtone pulsators.

1.5 Non-radial oscillations and astroseismiology

This is just an introduction, a full course might be given at the PhD level.

We use similar assumptions as before, except we lose the spherical symmetry. Specifically:

1. we consider perturbations to a spherically symmetric equilibrium model;
2. we make the Cowling approximation: we consider the unperturbed gravitational potential.

The equations describing the oscillations are

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\zeta_r\right) - \frac{g}{v_s^2}\zeta_r + \left(1 - \frac{L_\ell^2}{\sigma^2}\right)\frac{P'}{v_s^2\rho} = 0 \tag{1.174a}$$

$$\frac{1}{\rho}\frac{\partial P'}{\partial r} + \frac{g}{v_s^2\rho}P' + \left(N^2 - \sigma^2\right)\zeta_r = 0\,, \tag{1.174b}$$

where:

1. the displacement vector is separated into radial and horizontal components: $\vec{\zeta} = \zeta_r\hat{e}_r + \vec{\zeta}_h$;
2. any perturbed quantity changes in two ways: we write

$$\delta f(\vec{r}_0 + \delta\vec{r}) = f'(\vec{r}_0) + \vec{\delta r}\cdot\vec{\nabla} f_0\,, \tag{1.175}$$

Does the prime denote a derivative? how are these things defined?

3. L_ℓ is the Lamb frequency, or acoustic frequency: $L_\ell^2 = \ell(\ell+1)(v_s/r)^2$
4. N is the Brunt-Väisälä frequency, or bouyancy frequency:

$$N^2 = -g\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - \frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right), \tag{1.176}$$

where the term inside brackets is called the *Schwarzschild discriminant* $A(r)$, and it measures the non-adiabaticity of the system, since it is proportional (with a positive constant) to $\nabla - \nabla_{\mathrm{ad}}$. Recall Schwarzschild's criterion: we have convectively unstable regions if $A(r)$ is positive. So, N^2 is *negative* in regions which are convectively unstable according to Schwarzschild's criterion.

Our ansatz for the solution is similar to what is done in quantum mechanics: we have a radial part with a quantum number n, and spherical harmonics indexed by the angular order ℓ and the azimuthal order m, with $|m| \leq \ell$. The temporal periodicity will still look like an oscillating complex exponential, but its frequency will depend on the quantum numbers: so, in the end we will have

$$\zeta(\vec{r}) = R_n(r) Y_{\ell m}(\Omega) \exp(i\sigma_{n\ell m} t) , \tag{1.177}$$

where R and Y are the radial and angular components of the perturbation, $\Omega = (\theta, \varphi)$ and $Y_{\ell m}$ are the spherical harmonics, the solutions $Y \colon S^2 \to \mathbb{C}$ to $\nabla^2 Y = -\ell(\ell+1)Y$, satisfying $\partial_\varphi Y = im$: these are restrictions on the unit sphere of $\nabla^2 Y = 0$.

We can approximate the differential equations locally as a single differential equation (by assuming that all the parameters are constant: we do the following change of variables:

$$\frac{\partial \widetilde{\zeta}}{\partial r} = h \frac{r^2}{v_s^2} \left(\frac{L_\ell^2}{\sigma^2} - 1 \right) \widetilde{\eta} \tag{1.178a}$$

$$\frac{\partial \widetilde{\eta}}{\partial r} = \frac{1}{r^2 h} \left(\sigma^2 - N^2 \right) \widetilde{\zeta} , \tag{1.178b}$$

which can be combined into

$$\frac{\partial^2 \widetilde{\zeta}}{\partial r^2} = -k_r^2 \widetilde{\zeta} , \tag{1.179}$$

with a wavenumber given by

$$k_r^2 = -\frac{1}{v_s^2 \sigma^2} \left(L_\ell^2 - \sigma^2 \right) \left(\sigma^2 - N^2 \right) . \tag{1.180}$$

We have several families of solutions:

1. p-modes, mostly radial, pressure waves, high frequency: $\sigma^2 > L_\ell^2, N^2$;
2. g-modes, mostly horizontal, buoyancy, low frequency, $\sigma^2 < L_\ell^2, N^2$;
3. f-modes; without nodes.

We can derive equations for this case similarly to the LAWE.

This is because k_r^2 is negative iff σ^2 is either smaller or larger than both N^2 and L_ℓ^2; these are respectively g and p-modes. If σ^2 is between them, we get a real exponential as a solution, which decays quickly. This is called the *evanescent regime*.

p-modes are characterized by a *turning point*: below a certain radius, the mode is in the evanescent regime. For g-modes we have the opposite behaviour: they only exist in the core, and are evanescent for large radii.

We can plot N and L_ℓ as a function of radius.

Mixed modes arise when the evanescent region is thin, therefore the mode can tunnel through.

The frequencies are degenerate in m, but the degeneracy is split when the star is rotating.

In the slide for "asymptotic behaviour", the y axis is frequency.

We analyze the power spectra of stars.

Tue Oct 22 2019

For the physics undergraduate students: "Fundamental Astronomy" by Karttunen, Kröger, Öja, Poutanen, Donner.

1.5.1 Red variable stars

They include the ξ Hydrae, SR and Mira stars. These are low temperature stars.

They are evolved stars, and most all of the evolved stars are at least somewhat variable.

Miras, SRVs and OSARGs have very long periods, on the order of a year.

We need good estimates for the radius: because of the period-mean density relation, the fundamental at a certain radius can correspond to the first overtone at a larger radius.

We have different period-luminosity relations corresponding to which overtone we see. Using Weisenheit indices, which compensate for self-reddening, these relations are even more evident.

We can simulate galaxies, and observe the same patterns.

The linear models are not appropriate for the fundamental. Today, we do 3D models.

1.6 Summary

- $\tau_{\text{dyn}} \propto 1/\sqrt{\overline{\rho}}$;
- the inequality between the timescales;
- variability is due to mechanical, acoustic phenomena!
- derivation of the period-mean density relation;
- the balance of heat absorption;
- meaning of perturbation theory;
- the final result: the perturbed structure equations;
- the adiabatic approximation: justification, cases in which it does not hold: driving layers, stability;
- ideas of how the LAWE is solved (not boundary conditions): the Sturm-Liouville problem;
- the meaning of the solutions of the LAWE: nodes, the shape of the eigenfunctions, orthogonality;
- stability conditions: inequalities which tell us whether a star pulsates or not, driving and damping layers, phase lag;
- NOT the derivation of the LNAWE, but qualitative characterization of its solutions; the expression of the coefficient κ;

- the ϵ and κ-γ mechanisms: orders of magnitude;
- the opacity bump mechanism;
- the difference between self-excited and stochastically driven oscillation;
- the classical instability strip: not important to know the derivation;
- RR Lyrae, typical parameters;
- Cepheids: derivation of the mass-luminosity relation;
- NOT nonradial oscillations and astroseismiology;
- red variables.

Chapter 2

Stellar winds

2.1 Introduction

One should be able to follow this part even without a strong background in stellar astrophysics.

Bubble Nebula in Cassiopeia: a 45 $M_\odot$ star is ejecting mass at $1.7 \times 10^6\,\mathrm{m/s}$.

Some important quantities: we introduce

1. $\dot{M}$ is the mass loss rate:
$$\dot{M} = -\frac{\mathrm{d}M}{\mathrm{d}t} > 0\,; \tag{2.1}$$
2. v_∞ is the terminal wind velocity, in the limit of radial infinity.

The gas initially escapes from the star at low (subsonic, $\sim 1\,\mathrm{km/s}$) velocity; then it is accelerated. It is accelerated, and in the far field when no more forces are acting on it it approaches v_∞. We describe it with a *velocity law*: $v(r)$, and physically since the force is always radially outward we have
$$\frac{\mathrm{d}v}{\mathrm{d}r} > 0 \tag{2.2}$$
for any r. A typical law is something like:
$$v(r) = v_0 + (v_\infty - v_0)\left(1 - \frac{R_*}{r}\right)^\beta, \tag{2.3}$$
where $\beta \approx 0.8$, and R_* is the radius of the photosphere (where, from infinity, we have an opacity of $\tau = 1$).

In an H-R diagram we can plot the mass loss rate using color: we see that it increases when going up the main sequence, and is also high in the RGB.

Probably the thing is that the mass loss rate increases through the later stages of stellar evolution…

The relation $\dot{M}(M)$ seems to be something like a power law: what is it?

The momentum input can come either from a force, like radiation pressure (line driven winds and dust driven winds) or from heating.

Stellar winds can be characterized by their temperature, measured with respect to the effective temperature of the star; velocity, measured with respect to the escape velocity, and density:

1. late type supergiant stars have *cold*, *slow* and *dense* winds;
2. luminous hot stars have *cold*, *fast* and *dense* winds;
3. cool dwarfs and giants have *hot*, *fast* and *tenuous* winds.

We can also characterized them by their driving mechanisms:

1. coronal winds: driven by gas pressure;
2. line driven winds: driven by radiation pressure on highly ionized atoms, O and B stars;
3. dust driven winds: due to radiation pressure on dust grains, solid particles, which are very opaque.

For line-driven winds we have a relations between luminosity and wind momentum: it is a powerlaw. Specifically, what is measured is $\dot{M}_\infty\sqrt{R/R_\odot}$ versus the visual magnitude.

We will not consider winds driven by pulsation, sound waves and Alfvèn waves (magnetic winds).

2.2 Wind structure equations

We will assume:

1. spherical symmetry;
2. stationarity;
3. no magnetic fields.

The equation of continuity with the hypothesis of stationarity is given by $\dot{M} = 4\pi r^2 \rho(r) v(r) \equiv$ const.

If we differentiate $\dot{M}$ with respect to r we get 0 on the LHS (since, by continuity, the mass loss rate across any layer is constant), and on the RHS:

$$0 = 2\rho v + rv\frac{\mathrm{d}\rho}{\mathrm{d}r} + r\rho\frac{\mathrm{d}v}{\mathrm{d}r}\,, \tag{2.4}$$

where we simplified the 4π. If we divide by $\rho v r$ we get:

$$\frac{2}{r} + \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = 0\,, \tag{2.5}$$

and then this gives us

$$2\log(r)' + \log(v)' + \log(\rho)' = 0\,, \tag{2.6}$$

where the prime denotes derivatives with respect to r. So, the gradients of the velocity and density are related.

The force per unit volume is given by

$$F = \rho \frac{\mathrm{d}v}{\mathrm{d}t}\,, \tag{2.7}$$

and if we divide by ρ we get the force per unit mass:

$$f = \frac{F}{\rho} = \frac{\mathrm{d}v}{\mathrm{d}t}\,. \tag{2.8}$$

Under stationarity ($\partial_t = 0$), we have:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v(r)\frac{\mathrm{d}v}{\mathrm{d}r}\,. \tag{2.9}$$

The conservation of momentum gives us the Euler equation:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = \underbrace{-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}}_{f_P} \underbrace{-\frac{GM}{r^2}}_{f_g} + f(r)\,, \tag{2.10}$$

where $f(r)$ is a generic unspecified external force, which we assume to be *outward* (no dissipative effects!), while f_P and f_g are respectively the force due to the pressure gradient and to gravitation.

Do note that while f_P has a minus sign, $\mathrm{d}P/\mathrm{d}r < 0$ so the force due to the pressure gradient is outward. On the other hand, $f_g < 0$. The first principle of thermodynamics gives us:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} + P\frac{\mathrm{d}\rho^{-1}}{\mathrm{d}t}\,, \tag{2.11}$$

where Q is the specific heat, u is the specific internal energy.

The internal energy, for an ideal gas, scales linearly with the temperature:

$$u = \frac{3}{2}\frac{k_B T}{\mu m_u} = \frac{3}{2}\frac{\mathcal{R}T}{\mu}\,, \tag{2.12}$$

where μ is the mean molecular weight, while $m_u \approx m_H$ is the atomic unit of mass. Using these, we define the specific gas constant $\mathcal{R} = k_B/m_u$.

The mean molecular weight is defined as the average weight of a molecule in atomic mass units: $\mu = \overline{m}/m_H$, and for a neutral gas it can be calculated as

$$\frac{1}{\mu m_H} = \frac{\sum_j N_j}{M_{\text{tot}}} = \sum_j \frac{N_j}{N_j m_j}\frac{N_j m_j}{M_{\text{tot}}} = \sum_j \frac{N_j}{N_j A_j m_H} X_j\,, \tag{2.13}$$

so

$$\frac{1}{\mu} = \sum_j \frac{N_j}{A_j}\,, \tag{2.14}$$

where we have used the facts that $m_j = A_j m_H$, we defined the mass fraction $X_j = N_j m_j / M_{\text{tot}}$: the fraction of the total mass which is of the species j. For an ionized gas the calculation is the same except we need to include a factor $(1 + z_j)$, at the numerator in the sum, where z_j is the atomic number of the element, since for every ionized atom we have an additional particle (the electron).

Using the mass fractions for the Sun we get approximately $\mu \approx 1.3$ for neutral gas and $\mu \approx 0.62$ for fully ionized gas.

We will also assume that the gas pressure follows the ideal gas law:

$$P = \frac{k_B T \rho}{\mu m_u} = \frac{\mathcal{R} T \rho}{\mu}\,, \tag{2.15}$$

where we used the fact that $N = M/\overline{m}$, and $\overline{m} = \mu m_H$.

By stationarity, all time derivatives can be written as $\frac{\mathrm{d}}{\mathrm{d}t} = v \frac{\mathrm{d}}{\mathrm{d}r}$.

We define

$$q(r) = \frac{\mathrm{d}Q}{\mathrm{d}r}\,, \tag{2.16}$$

the heat input or loss per unit mass per unit distance in the wind (since Q is already a specific heat). Inserting this in the previous equation, we get the following expression for the energy equation:

$$\boxed{q = \frac{3}{2} \frac{\mathcal{R}}{\mu} \frac{\mathrm{d}T}{\mathrm{d}r} + P \frac{\mathrm{d}\rho^{-1}}{\mathrm{d}r}\,,} \tag{2.17}$$

This can be incorporated, using the momentum conservation as well, into the global energy equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\underbrace{\frac{v^2}{2}}_{A} + \underbrace{\frac{5}{2}\frac{\mathcal{R}T}{\mu}}_{B} \underbrace{- \frac{GM}{r}}_{C} \right) = f(r) + q(r)\,,} \tag{2.18}$$

where the expression inside the brackets is the total internal energy.

Here is the derivation of the formula: the momentum conservation equation (2.10) can be written as:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{P}{\rho}\right) + P \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\rho}\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{GM}{r}\right) + f\,, \tag{2.19}$$

Inverse Leibniz rule on the P, ρ term.

so we can substitute in our expression from the energy equation for $P \,\mathrm{d}\rho^{-1} \Big/ \mathrm{d}r$: we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v^2}{2} + \frac{P}{\rho} - \frac{GM}{r}\right) = f + q - \frac{3}{2}\frac{\mathcal{R}}{\mu}\frac{\mathrm{d}T}{\mathrm{d}r}\,, \tag{2.20}$$

but P/ρ is precisely equal to $\mathcal{R}T/\mu$, and the molecular weight and the gas constant are constants, so we can bring that term inside the derivative on the LHS in order to get the 5/2 factor in the final result.

The terms Ⓐ, Ⓑ and Ⓒ describe the different types of energy we can have: Ⓐ is kinetic energy, Ⓑ is chemical potential energy (enthalpy) while Ⓒ is gravitational potential energy.

We can change the total energy by either adding heat (increasing q) or by adding momentum (increasing f).

Equation (2.18) can be written in integral form, by integrating from a generic radius r_0: a useful choice is often the photospheric radius. We introduce the notation $e(r) =$ Ⓐ+Ⓑ+Ⓒ for the total energy.

Near the photosphere $\left|\text{Ⓒ}\right| \gg$ Ⓐ + Ⓑ, so $e(r_0) \approx GM/r_0 < 0$ (this is not obvious theoretically: it is an experimental fact that the gas is slow-moving near the surface).

At large radii Ⓑ,Ⓒ $\to 0$: $e(r_\infty) \approx v_\infty^2/2 > 0$.

If we integrate from r_0, the radius of the photosphere, to infinity then we get:

$$\frac{v_\infty^2}{2} = -\frac{GM}{r_0} + \int_{r_0}^{\infty} f(r) + q(r)\,\mathrm{d}r\ , \tag{2.21}$$

so the sum of the two integrals must be large enough for the gas to escape the gravitational well.

2.3 Coronal winds

The corona is very hot: it goes to 10^6 K, but with very low density. We know this experimentally by seeing the emission lines of highly ionized elements like $\mathrm{Fe}^{+5\div 13}$.

Why is this the case? We *do not know*. Magnetism?

If the gas particles in the corona have enough kinetic energy to reach terminal velocity and escape the gravitational well, they form what is known as the coronal wind. This is a good model for the Sun: its wind's asymptotic velocity is around 500 km/s, so it reaches the Earth in around 3 d. For the sun $\dot{M} \sim 1 \times 10^9$ kg/s: this means that it loses 0.016 % of its total mass per 10 Gyr, which is appriximately the lifespan of the Sun.

Not 0.1 %, as the slides say!

2.3.1 Isothermal winds

Now we will treat isothermal winds. These calculations were first done by Parker.

The assumption of constant temperature already gives us $T(r) \equiv T$, which acts as our energy equation: there must be an energy input exactly equal to

$$q = P \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\rho} : \tag{2.22}$$

this is a type of wind which is driven by gas pressure only.

Is this actually the case? it is not mentioned in the slides, but it seems to be a natural consequence of the hypothesis.

This is a good model for low $\dot{M}$, which does not affect the total mass of the star.

Tomorrow we will discuss full solutions in this case.

Recall the equations from last time: continuity, momentum and energy conservation in the isothermal case:

$$\dot{M} = 4\pi r^2 \rho v \tag{2.23a}$$

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} \tag{2.23b}$$

$$T(r) \equiv T\,. \tag{2.23c}$$

Also, by differentiating the ideal gas law $P\mu = \mathcal{R}T\rho$ we get

$$\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathcal{R}T}{\mu} \frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r}\,, \tag{2.24}$$

Divided through by ρ.

since the temperature is constant.

Using this, we manipulate the momentum equation into

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho} \frac{\mathcal{R}T}{\mu} \frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2}\,, \tag{2.25}$$

but we know by the continuity equation that the density gradient must correspond to the velocity gradient:

$$-\frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\,, \tag{2.26}$$

which we can substitute into the equation: we get

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\mathcal{R}T}{\mu} \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) - \frac{GM}{r^2}\,. \tag{2.27}$$

It is a known fact that the isothermal speed of sound is given by

$$a^2 = \frac{\partial P}{\partial \rho} = \frac{\partial}{\partial \rho} \left(\frac{\mathcal{R}\rho T}{\mu} \right) = \frac{\mathcal{R}T}{\mu} \implies a = \sqrt{\frac{\mathcal{R}T}{\mu}}\,, \tag{2.28}$$

so

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = a^2 \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) - \frac{GM}{r^2}\,. \tag{2.29}$$

Now we move all the terms which are proportional to the velocity gradient on the LHS:

$$\frac{\mathrm{d}v}{\mathrm{d}r}\left(v - \frac{a^2}{v}\right) = \frac{2a^2}{r} - \frac{GM}{r^2}, \tag{2.30}$$

which can be written as

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r}\left(v^2 - a^2\right) = \frac{2a^2}{r} - \frac{GM}{r^2}, \tag{2.31}$$

so the Jacobian of the differential equation is zero is singular in $v = a$: if $v = a$ we must have $2a^2 r = GM$, which fixes the radius to the so-called Parker radius: $r_P = GM/2a^2$, which is obtained by setting the LHS of the equation to zero. Close to the star, the speed is subsonic ($v < a$), so the denominator D is negative; also the numerator N is negative in

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2a^2/r - GM/r^2}{v^2 - a^2} \stackrel{\text{def}}{=} \frac{N}{D}, \tag{2.32}$$

so on the whole N/D is positive, which is consistent with our assumption $\mathrm{d}v/\mathrm{d}r > 0$, which we make since we are considering winds, as opposed to accretion.

Why is the numerator negative? This is equivalent to saying

$$2\frac{\mathcal{R}T}{\mu} < \frac{GM}{r}, \tag{2.33}$$

which means that Ⓑ $\times 4/5 <$ Ⓒ, (using the notation from equation (2.18)) which holds since, as we wrote there, Ⓑ $\ll$ Ⓒ near the stellar radius.

Far from the star the numerator is positive, so the speed must be supersonic, so that $N > 0$ and we can still have $N/D > 0$ everywhere, guaranteeing that the velocity gradient is always positive.

So, the only physical solution is transsonic.

The critical velocity *must* be attained at the Parker radius in order to have a physically meaningful transsonic solution; always-subsonic and always supersonic solutions are mathematically possible but not usually observed.

The velocity gradient at the critical point can be found by de l'Hôpital's rule to be

$$\left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r_P} = \pm\frac{a^3}{GM}. \tag{2.34}$$

This is calculated by expanding in r; however note that we must vary both r and v when we differentiate with respect to r. We find:

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}r} &= v\frac{2a^2/r - GM/r^2}{v^2 - a^2} && \text{(2.35a)}\\
&= \left(a + \underbrace{\frac{\mathrm{d}v}{\mathrm{d}r}}_{\text{Second order}}\right)\frac{-2a^2/r_P^2 + 2GM/r_P^3}{2a\,\mathrm{d}v/\mathrm{d}r} && \text{(2.35b)}
\end{aligned}$$

Differentiated above and below at $r = r_P$ and $v = a$

$$\left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2 = -\frac{a^2}{r_P^2} + \frac{GM}{r_P^3} \tag{2.35c}$$

Simplified the $2a$

$$= -a^2\left(\frac{2a^2}{GM}\right)^2 + GM\left(\frac{2a^2}{GM}\right)^3 \tag{2.35d}$$

Substituted $r_P = 2a^2/GM$

$$= -4\frac{a^6}{(GM)^2} + 8\frac{a^6}{(GM)^2} \tag{2.35e}$$

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \pm 2\frac{a^3}{GM}\,. \tag{2.35f}$$

Claim 2.3.1 (Exercise)**.** *The speed of sound at the critical point equals half of the escape velocity at that radius.*

The escape velocity is given by

$$v_{\text{esc}}^2 = \frac{2GM}{r}\,, \tag{2.36}$$

but we know that $r_P = GM/2a^2$: so

$$a^2 = \frac{GM}{2r_P}\,, \tag{2.37}$$

which means that, if we calculate the escape velocity at the Parker radius, we have $v_{\text{esc}}^2/a^2 = 4$, so $v_{\text{esc}}/a = 2$.

There are exactly two transsonic solutions: if we trace a cross in the r, v plane centered on the critical point and speed of sound we see that all solutions meet it perpendicular to it. The solution with the decreasing velocity gradient is an accretion solution: what is plotted is the *absolute value* of the velocity, the accretion solution has always-negative absolute velocity gradient.

Always-supersonic solutions and always-subsonic ones are also found, but they do not obey the monotonicity of the velocity, so we discard them.

The boundary condition is the velocity at some r_0: the problem is second-order, but if we select the specific transsonic solution we eliminate the necessity of one boundary condition. The choice we make for $v_0 = v(r_0)$ is key, and nontrivial.

If we have the density, velocity and radius of the lower boundary we have *fixed* the accretion rate: $\dot{M} = 4\pi r_0^2 \rho_0 v_0$.

This is actually fixed by only specifying the gravity, temperature and density at the star's edge. It would seem like we also should have v_0, but we do not actually need it, as we will show in a moment.

With these hypotheses, we can solve the momentum equation analytically: we get

$$\frac{v}{v_0}\exp\left(-\frac{v^2}{2a^2}\right) = \left(\frac{r_0}{r}\right)^2 \exp\left(\frac{GM}{a^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right) \tag{2.38a}$$

$$= \left(\frac{r_0}{r}\right)^2 \exp\left(\frac{2r_c}{r_0} - \frac{2r_c}{r}\right). \tag{2.38b}$$

So, to calculate the Mach number we need to solve

$$Me^{-M^2/2} = \frac{v_0}{a}\left(\frac{r_0}{r}\right)^2 \exp\left(\frac{2r_c}{r_0} - \frac{2r_c}{r}\right), \tag{2.39}$$

but there seems to be an issue: the left hand side is bounded (it attains its maximum value of $e^{-1/2} \approx 0.6$ at $M = 1$) while the RHS is maximized at $r = r_c$, where it is equal to

$$\frac{v_0}{a}\left(\frac{r_0}{r_c}\right)^2 \exp\left(\frac{2r_c}{r_0} - 2\right), \tag{2.40}$$

which is dependent on v_0! If we plug in solar values for the parameters, $T = 1 \times 10^6\,\mathrm{K}$, $\mu = 0.62$ and select an initial velocity like $v_0 = 1\,\mathrm{km/s}$ we get that the RHS's maximum is around 35!

This means that, in order to always have solutions, we need to fix v_0 to set the maxima to be equal. This gives us

$$v_0 = e^{-1/2}\left(\frac{1}{a}\left(\frac{r_0}{r_c}\right)^2 \exp\left(\frac{2r_c}{r_0}\right)e^{-2}\right)^{-1} \tag{2.41a}$$

$$= a\left(\frac{r_c}{r_0}\right)^2 \exp\left(-\frac{2r_c}{r_0} + \frac{3}{2}\right). \tag{2.41b}$$

Specifically, we can see that for the solar values mentioned before $1\,\mathrm{km/s}$ was an overestimate, and with the values we fixed the initial velocity is actually more like $17\,\mathrm{m/s}$.

At large distances, we get $v \to 2a\sqrt{\ln(r/r_0)}$.

To derive this, take the log of the equation and take the limit $r \to \infty$.

$$-\frac{M^2}{2} + \log M = \log\frac{v_0}{a} + 2\log\frac{r_0}{r} + \frac{2r_c}{r_0} - \frac{2r_c}{r}, \tag{2.42}$$

so the RHS diverges as $-\log r$ does for $r \to \infty$, which means that the LHS must also diverge and become very large and negative. The polynomial term $-M^2/2$ dominates the logarithmic one. So, asymptotically we have $-M^2/2 \approx 2\log r_0/r$, which means $M \approx 2\sqrt{\log(r/r_0)}$.

This is reported incorrectly in the slides as $v \to 2a\log(r/r_0)$.

This is unphysical (the velocity diverges!?) and due to the fact that we assume a constant temperature (and, thus, energy input) even for diverging r.

The density profile can be found from the continuity equation to be

$$\frac{\rho}{\rho_0} \exp\left(\frac{1}{2}\left(\frac{v_0 \rho_0 r_0^2}{a \rho r^2}\right)^2\right) = \exp\left(2 r_c \left(\frac{1}{r} - \frac{1}{r_0}\right)\right), \tag{2.43}$$

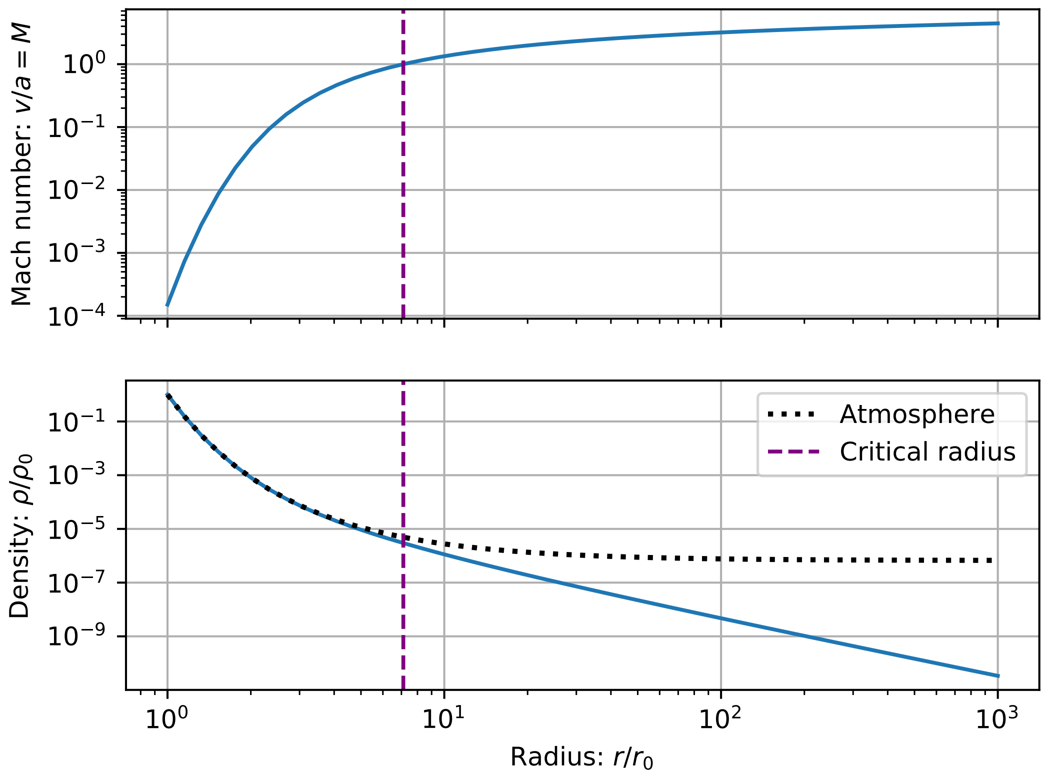

Figure 2.1: Velocity profile (solution to (2.38a)) and density wind profile (solution to (2.43)), plus atmosphere (hydrostatic) profile (shown in (2.46)).

Now we look at the structure of the wind in the subcritical region.

In the corresponding slide: the dashed line in the density profile is the density we'd expect in a hydrostatic atmosphere, while the solid one is the solution.

The hydrostatic density structure is given by

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0, \tag{2.44a}$$

since we set the (log) velocity gradient to zero. Manipulating this we get

$$\frac{r^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{GM}{a^2}, \tag{2.45}$$

So, integrating this we see that the density profile follows a decreasing exponential law:

$$\frac{\rho(r)}{\rho_0} = \exp\left(\frac{-(r-r_0)}{H_0}\frac{r_0}{r}\right), \tag{2.46}$$

where H_0 is the scale height, $H_0 = \mathcal{R}T/(\mu g_0)$ with $g_0 = GM/r_0^2$. The length scale at which this decreases is defined by H_0.

The density profile in the subsonic region is very well approximated by this hydrostatic profile: as we can see in figure 2.1 in the subsonic region the log-velocity gradient is small, so the pressure contribution dominates.

The mass loss rate is our main prediction: we have $\dot{M} = 4\pi r_0^2 \rho_0 v_0 = 4\pi r_c^2 \rho_c a$.

In order to compute it at the critical radius, we can use the density profile equation:

$$\dot{M} = 4\pi r_c^2 a \rho_0 \exp\left(\frac{-(r_c-r_0)}{H_0}\frac{r_0}{r_c}\right). \tag{2.47}$$

This is an approximation, justified by the fact that in the subsonic region the density profile of the wind is almost the same as in hydrostatic equilibrium. There is a correction factor of $\exp(-1/2)$.

If we consider this numerically, we find that the exponential is the dominant part: the mass loss rate becomes lower when the critical point moves outward. We can specify it by fixing

1. the temperature at the corona, T_C;
2. the radius at the bottom of the corona, r_0;
3. the stellar mass M;
4. the density at the bottom of the corona ρ_0.

In the slides there are numerical estimates. As H_0 increases, the density profile is less steep: the density remains high at larger radii.

Why is the dependence on $(r_c - r_0)/H_0$ so strong? The idea seems to be that in order for the wind to expel a significative quantity of material it needs to give it a large velocity in a small space, otherwise the wind becomes supersonic when its density is very small.

Mon Nov 04 2019

2.3.2 Isothermal winds with an external force

Last time we found that the only physical solution to the stellar wind is the transsonic solution.

Now, let us add an external force:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} + f(r)\,, \tag{2.48}$$

so we get

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2\frac{a^2}{r} - \frac{GM}{r^2} + f(r)}{v^2 - a^2}\,, \tag{2.49}$$

with the speed of sound $a = \sqrt{\mathcal{R}T/\mu}$. How does the velocity gradient change? With an outward force, we expect the velocity gradient to be less steep in the subsonic region (the velocity decreases slower: the numerator is *less negative*). Adding this force is formally equivalent to modifying the gravitational field by making it weaker. This increases the pressure scale height, so the density gradient becomes less steep, so the velocity gradient becomes less steep as well.

In the supersonic region, instead the gradient will be larger: the numerator is *more positive* and higher velocities are reached.

The critical radius changes: it is the solution to

$$r_C = \frac{GM}{2a^2} - \frac{f(r_C)r_C^2}{2a^2}\,, \tag{2.50}$$

which will shift inward as f goes from 0 to positive. This is shown in figure 2.2.

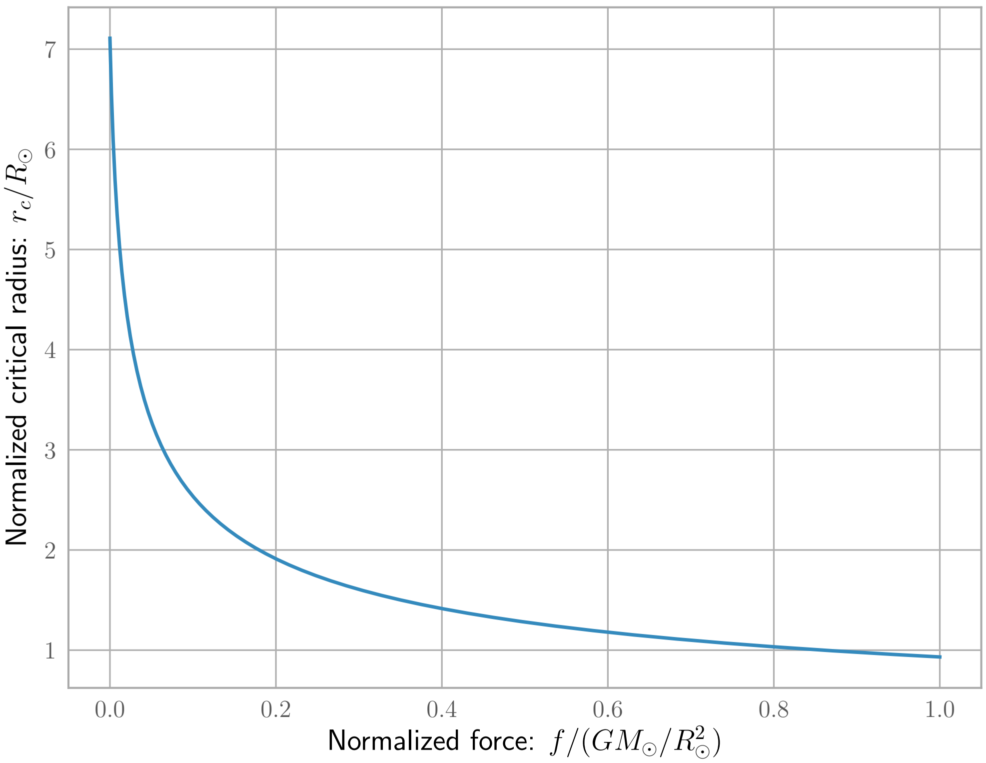

Figure 2.2: Critical radius position in function of a constant force, expressed in units of the gravitational acceleration at the surface. Here, we assume solar parameters and $T = 10^6\,$K.

The adiabatic speed of sound is the same, it does not depend on f. Also, the velocity gradient is less steep as we said before. These two facts combined mean that the velocity at the corona, v_0, must be larger.

Since $\rho'/\rho + v'/v + 2/r = 0$, and the critical radius velocity is $v = a$ regardless of the value of f, and the critical radius decreases if we have $f > 0$, we must have that ρ_c, the density at the critical radius, must be smaller than that with $f = 0$.

How do we expect the mass loss rate to change? from the continuity equation at the bottom of the corona, we get $\dot{M} = 4\pi r_0^2 \rho_0 v_0$, and everything on the RHS is fixed but v_0, so when we increase f the LHS must incrase as well.

Let us consider some explicit law scaling as $f \propto r^{-2}$, like a radiative force:

$$g_{\text{rad}} \propto \kappa_F \times \left(\frac{r}{R}\right)^{-2} . \tag{2.51}$$

This is the same as changing the mass of the star, since it scales like the gravitational force. We then take our force to be $f(r) = A/r^2$. Our momentum equation, with the pressure gradient substituted in from the differentiated ideal gas law and the continuity equation, becomes

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \left(\frac{2a^2}{r} - \frac{GM}{r^2} + \frac{A}{r^2}\right) / \left(v^2 - a^2\right), \tag{2.52}$$

which is the same as the equation without force, with a smaller effective mass: $M_{\mathrm{eff}} = M(1 - A/GM)$.

The correction is usually called the Eddington ratio:

$$\Gamma = \frac{A}{GM} = \frac{A/r^2}{GM/r^2}, \tag{2.53}$$

the acceleration of the force divided by the gravitational one.

If A is a consant, the critical radius becomes

$$r_C = \frac{GM}{2a^2}(1 - \Gamma), \tag{2.54}$$

however in general A is not taken as a constant, instead it is modelled with a Heaviside theta function, so that it only activates after a certain radius.

As Γ increases, the critical velocity is reached faster, and the velocity at the corona is greater. The density profile gets less and less steep: the density scale height decreases.

We approximate $A(r) = A[r \geq r_d]$ for some r_d.[1] The justification for this model is that below the dust condensation region there is only gas, above it there is dust which is more opaque to radiation. We are still assuming the wind is isothermal.

The critical point depends on Γ:

$$\frac{r_C}{1 - \Gamma(r_C)} = \frac{GM}{2a^2}. \tag{2.55}$$

If the extra force switches on *outside* the critical region, the mass loss rate is *unchanged*, since it only depends on the subsonic region.

Is there the possibility to have more than one critical radius with only outward force?

There is a maximum value of Γ such that the critical radius r_C is equal to the radius of the star.

As Γ increases towards this value, the mass loss rate increases greatly. If Γ is larger than this value, and v_0 is subsonic, then the RHS of the momentum equation is *negative*: the velocity gradient is negative, the velocity decreases.

2.3.3 Non-isothermal winds

Now, we consider the possibility that our winds are *not isothermal*. This will change the structure of the wind, by the introduction of an additional pressure gradient.

It will change the speed of sound and thus the Mach number.

[1] Using the Iverson bracket here!

It is useful to define the energy per unit mass e:

$$e(r) = \frac{v^2(r)}{2} - \frac{GM}{r} + \frac{\gamma}{\gamma - 1}\frac{\mathcal{R}T}{\mu}\,, \tag{2.56}$$

where $\gamma/(\gamma-1) = 5/2$ for a monoatomic gas, which has $\gamma = 3/2$.

In the lower boundary of the wind the velocity is much less than the escape velocity ($v \ll v_{\text{esc}}$), and also the thermal velocity of the particles is not enough for them to escape the gravitational well: $\mathcal{R}T/\mu \ll v_{\text{esc}}$ at the surface of the star, while far from the star we have $v \gg v_{\text{esc}}$.

If we have an isothemal wind, some energy must be added in order to prevent the adiabatic cooling of the gas, lift it from the potential well, and to increase its kinetic energy.

If a force is applied it increases the momentum, but the heat transmission q also appears in the momentum equation, which is $\Delta e = \int f + q\,\mathrm{d}r$. Heat transmission changes the pressure profile, which affects the momentum, even though q does not appear explicitly in the momentum equation.

We define the total heat deposition Q and the total work done by the force W:

$$Q(r) = \int_{r_0}^{r} q(\widetilde{r})\,\mathrm{d}\widetilde{r} \qquad \text{and} \qquad W(r) = \int_{r_0}^{r} f(\widetilde{r})\,\mathrm{d}\widetilde{r}\,, \tag{2.57}$$

and we will have $e(\infty) - e(r_0) = Q(\infty) + W(\infty)$.

The most general momentum equation is given by:

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \left(2\frac{c_s^2}{r} - \frac{GM}{r} + f - (\gamma-1)q\right) \Big/ \left(v^2 - c_s^2\right), \tag{2.58}$$

where we introduce the adiabatic speed of sound $c_s = \sqrt{\gamma a^2}$, a being the *isothermal* speed of sound. In general $-(\gamma - 1) < 0$, therefore if we add heat this is equivalent to pushing *inward*.

If either f or q depend on the velocity gradient $\mathrm{d}v/\mathrm{d}r$ then the sonic point can *decouple* from the critical point.

There are cases in which we have multiple critical points (specifically, multiple zeros of the denominator).

The momentum equation plus the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v^2}{2} + \frac{5}{2}\frac{RT}{\mu} - \frac{GM}{r}\right) = f(r) + q(r)\,, \tag{2.59}$$

can be solved numerically, and if we impose smooth passage through the critical point this yields the mass loss rate.

Qualitatively, the results are the same as in the isothermal case. Adding either momentum or energy to the subsonic region of the wind increases the bottom-of-the-corona velocity and the mass loss rate. Doing it in the supersonic region has no effect.

This ends our general introduction to stellar winds.

2.3.4 Exercises

Now we will do a couple of exercises to get familiar with the theory.

Exercise

The wind is isothermal. The solar wind has a mean coronal temperature of 1.5×10^6 K and a mass loss rate of 2×10^{-14} solar masses per year. The bottom of the corona is at $r_0 \approx 1.003 R_\odot$, where the density is $\rho(r_0) = 10^{-14}\,\mathrm{g/cm^3}$.

Calculate the potential energy, the kinetic energy and the enthalpy of the gas at r_0.

Calculate the same quantities at the critical point. Which of these energies has absorbed the largest fraction of the energy input?

We can use the continuity equation $\dot{M} = 4\pi\rho_0 r_0^2 v_0$ to get

$$v_0 = \frac{\dot{M}}{4\pi r_0^2 \rho_0} \approx 21\,\mathrm{m/s}\,, \tag{2.60}$$

and with this we can calculate

$$e(r_0) = -\frac{GM}{r_0} + \frac{1}{2}v_0^2 + \frac{5}{2}\frac{RT}{\mu}\,. \tag{2.61}$$

We get:

$$E_{\text{kin, 0}} = \frac{v_0^2}{2} \approx 212\,\mathrm{J/kg}\,, \tag{2.62}$$

while for the gravitational energy we'd need the mass of the star. Assuming it is equal to the solar mass, we find

$$E_{\text{grav, 0}} = -\frac{GM}{r_0} \approx -1.9 \times 10^{11}\,\mathrm{J/kg}\,. \tag{2.63}$$

The mean molecular weight of the gas for the Sun is something like $\mu = 0.62$ (we count electrons in it). Then we get

$$E_{\text{chem, 0}} = E_{\text{chem, crit}} = \frac{5}{2}\frac{RT}{\mu} \approx 5.0 \times 10^7\,\mathrm{J/kg}\,. \tag{2.64}$$

The enthalpy is the same everywhere in the flow, since the flow is isothermal.

The values at the critical radius are calculated with the same formula. The velocity will be the speed of sound $a = \sqrt{RT/\mu} \approx 4.5 \times 10^3\,\mathrm{m/s}$.

Then we find:

$$E_{\text{kin, crit}} = \frac{a^2}{2} \approx 1.0 \times 10^7\,\mathrm{J/kg} \approx 5 \times 10^4 \times E_{\text{kin, 0}}\,. \tag{2.65}$$

The critical radius is given by $r_c = GM/(2a^2)$: so, we get

$$E_{\text{grav, crit}} = -\frac{GM}{r_c} = -2a^2 \approx -4.0 \times 10^7\,\text{J/kg} \approx 2.1 \times 10^{-4} \times E_{\text{grav, 0}}\,. \tag{2.66}$$

Qualitatively, at the corona we have

$$\left|E_{\text{grav, 0}}\right| \gg E_{\text{chem, 0}} \gg E_{\text{kin, 0}}\,, \tag{2.67}$$

while at the critical point they are similar, and specifically

$$E_{\text{chem, crit}} \gtrsim \left|E_{\text{grav, crit}}\right| \gtrsim E_{\text{kin, crit}}\,. \tag{2.68}$$

Exercise

A star with $T_{\text{eff}} = 3200\,\text{K}$, $R_* = 30R_\odot$, $L_* = 85L_\odot$ and $M_* = 6M_\odot$ has an isothermal corona of $T = 10^6\,\text{K}$ with a density at the lower boundary of $10^{-13}\,\text{g/cm}^3$.

Calculate the energy per unit mass at the bottom of the corona at $r_0 = R_*$.

Calculate the location of the critical point, r_c, and the mass loss rate.

Calculate the energy per gram gained by the wind between r_0 and r. What fraction of the stellar luminosity is used to drive the wind up to the critical point?

The energy per unit mass at the bottom of the corona is given by

$$e(r) = -\frac{GM}{r_0} + \frac{1}{2}v_0^2 + \frac{5}{2}\frac{RT}{\mu}\,, \tag{2.69}$$

but we cannot use this formula since we do not have v_0 nor $\dot{M}$. However, we can approximate the density profile as an exponential, applying the formula

$$\dot{M} = 4\pi r_c^2 a\rho_c \tag{2.70a}$$

$$= 4\pi r_c^2 a\rho_0 \exp\left(-\frac{r_c - r_0}{H_0}\frac{r_0}{r_c}\right), \tag{2.70b}$$

where we have: the length scale $H_0 = RTr_0^2/(GM\mu)$, the critical velocity $a = \sqrt{RT/\mu}$, and the critical radius $r_c = GM/(2a^2)$.

Plugging these in, we find:

$$\dot{M} = 4\pi\left(\frac{GM\mu}{2RT}\right)^2\sqrt{\frac{RT}{\mu}}\rho_0 \exp\left(\frac{\mu GM}{RT}\left(\frac{2RT}{GM\mu} - \frac{1}{r_0}\right)\right) \tag{2.71a}$$

$$= \pi(GM)^2\left(\frac{\mu}{RT}\right)^{3/2}\rho_0 \exp\left(2 - \frac{\mu GM}{RTr_0}\right). \tag{2.71b}$$

One temperature should probably be the effective temperature, but I do not really know what that means.

Then, the mass loss rate can be calculated, since we have all of these quantities. It comes out to be barely anything, since $\mu GM/RTr_0$ is very large and we have a negative exponential of it…

2.4 Wind types

Coronal winds These are driven by gas pressure due to high temperature. Stars with a convection zone right under the photosphere can have coronas of a few million Kelvin degrees. These are rather well described as isothermal, however the temperature decreases slowly because of conduction.

Dust driven winds For these winds, the driving mechanism is the radiation pressure on the dust grains, which are continuum absorbers. The gas component is dragged along by momentum transfer. This happens for cool stars, whose envelopes have temperatures of less than a thousand degrees Kelvin, in which the dust grains can actually be formed. These can be well modelled as winds with a force like $f = [r \geq r_d] A/r^2$, where the radius r_d is the one after which the temperature is low enough for the grains to form.

Line-driven winds These are winds of hot stars, driven by radiation pressure on spectral lines of abundant ions which have many in the UV and far UV. The pressure depends on the Doppler effect: the ions can absorb different wavelengths of light at different radii because of it. The force depends on the velocity gradient: it cannot be modelled with what we discussed so far.

Pulsation driven winds Stars such as Miras and those in the Asymptotic Giant Branch may pulsate: the atmosphere is tossed up and then falls back. Because of the low gravity, they fall slowly and are hit by an outmoving layer before they have completed their fall. This means that for each pulsation cycle they get a “kick”.

This can be a very efficient driving method if we account for dust formation. The shockfront from the pulsation cycle basically travels outwards to infinity.

Sound wave driven winds They are modelled in a way that is similar to coronal winds. The resulting amplitudes are usually small.

Alfvén Wave Driven winds Due to magnetic fields: if the points at which the field lines make contact with the surface move, than a magnetic wave travels forward with a speed $v_A = B/\sqrt{4\pi\rho} \gg a$. This can be very efficient, and result in high wind speeds. It is relevant for stars which do not have strong radiation pressure (ie not more than a thousand times more luminous than the Sun).

Magnetic rotating winds Material is "flung out" moving along magnetic field lines. The terminal velocity is heavily dependent on the strength of the field.

2.5 Line driven winds

Now, we introduce the next topic. We deal with hot, luminous stars: at the top left of the HR diagram.

[Picture from the slides]

We talk of line-driven winds for hot stars. These are winds driven by *spectral lines*. The blackbody emission for these stars is mainly at high frequencies, like the UV.

One may ask: hydrogen is much more abundant than heavier elements like carbon, nitrogen. . . why do we see the spectral lines for these heavier elements?

This is because hydrogen is completely ionized at these temperatures, and helium is also.

The strongest lines are far in the UV, where the stellar flux is low: only few atoms are hit, the rest of the gas is dragged along.

A peculiar characteristic of the spectrum is the so-called P-Cygni profile.

Now, an overview of the formation of spectral lines: there are 5 processes

1. Line scattering: if it comes from the ground state of the atom, it is called *resonance scattering*, which is the main phenomenon.

2. Emission by recombination: the ion recombines to an excited state.

3. Emissional from collisional or photo-excitation. A photon is absorbed, it excites the atom which then descends to a lower energy level.

4. Pure absorption and then de-excitation of an already excited atom.

5. Masering by stimulated emission: it can happen in a very narrow set of circumstances: an excited atom is hit by a photon which has exactly the same energy as the one between the atom's state and the ground state, so the atom is deexcited and now there are two photons. This happens when there are many excited atoms, and when the velocity gradient is very small — otherwise, the photons are Doppler-shifted out of the right frequency.

2.5.1 Spectral lines and P-Cygni profiles

The spectrum of a star can be roughly approximated as a blackbody spectrum: Wien's law, then, tells us that the maximum of the spectral intensity occurs for a frequency which is proportional to the temperature of the star.

Specifically, the blackbody spectrum has the shape:

$$I_\nu \propto \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}, \tag{2.72}$$

so its maximum can be found by differentiating: setting $x = h\nu/k_B T$, this yields the equation $x/3 = 1 - e^{-x}$, which can be solved numerically to see that the solutions are $x = 0$ and $x \approx 2.821\,439\,37$. We keep the positive solution: then we can see that

$$\nu_{\max} \approx 2.82 \frac{k_B T}{h}. \tag{2.73}$$

This does *not* correspond to the maximum of the intensity expressed in terms of the wavelength: that formula looks like

$$I_\lambda \propto \frac{\lambda^{-5}}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}, \tag{2.74}$$

whose maximum can be found similarly to before: if we set $x = \lambda k_B T/hc$, the equation to solve becomes

$$-5x\left(e^{1/x} - 1\right) + e^{1/x} = 0, \tag{2.75}$$

which can be solved numerically to yield $x \approx 1/4.965\,114\,231\,74$, so

$$\frac{1}{\lambda_{\max}} \approx 4.97 \frac{k_B T}{hc}. \tag{2.76}$$

These are both artefacts of the fact that we are plotting probability distributions: the variable matters, since to go from one to the other we must multiply by $\left|\mathrm{d}\nu/\mathrm{d}\lambda\right|$. A figure which is more representative since it is variable-independent is the *average* energy of a photon:

$$E_{\text{mean}} \approx 3.832 k_B T. \tag{2.77}$$

Many of the strongest absorption lines of abundant elements such as Carbon, Nitrogen, Oxygen, Silicon, Sulfur and Iron are in the UV, where the flux of stars is usually low. However, if the star is very hot, then the peak of the Planckian is near or in the UV band: specifically, the peak of the distribution in wavelength moves into the UV when the temperature is somewhere between 7000 K and 8000 K.

The spectral lines in winds are characterized by a large width: the wavelength shift is due to the Doppler effect from the outflowing motion.

Spectral line formation

Line scattering A process like:

$$|0\rangle + \gamma_{\text{in}} \to |1\rangle \to |0\rangle + \gamma_{\text{out}}\,, \tag{2.78}$$

the energies of the in and out photons are almost the same, they may be slightly different due to the Doppler shift due to thermal motion. The direction of the photons may be different. If $|0\rangle$ is indeed the ground state of the system, this is called *resonance scattering*.

Emission by recombination An electron collides with an ion (which is in a high-energy continuum state), and causes it to recombine into a less energetic state:

$$e^- + |\text{cont}\rangle_{\text{ion}} \to |0\rangle + \gamma\,, \tag{2.79}$$

it may pass through one or several excited state(s) on its way to the ground state. For each deexcitation we have a spectral line photon. We can see this as emission lines in a spectrum.

What is HEX emission?

Emission from collisional or photo-excitation Collisional excitation is a process like:

$$|0\rangle \stackrel{\text{collision}}{\to} |2\rangle \to |1\rangle + \gamma \to |0\rangle + 2\gamma \tag{2.80}$$

with $m < n$. The collisional excitation is efficient at high temperatures and densities, in plasmas: because of this process we see emission lines from hot coronas.

Photo excitation is a process like:

$$|0\rangle + \gamma \to |2\rangle \to |1\rangle + \gamma \to |0\rangle + 2\gamma\ : \tag{2.81}$$

it occurs if a photon excites the ground state into a higher state, but the descent to lower energy states occurs in more then one step: then a higher-energy photon is absorbed and two or more lower-energy ones are emitted. In principle we could see emission lines from this process in stellar winds, but in practice the most likely thing to happen is simply resonance scattering, which does not significantly change the energy of the photon.

Pure absorption A process like

$$|1\rangle + \gamma \to |2\rangle \to |0\rangle + \gamma\,, \tag{2.82}$$

which is not relevant for stellar winds since it needs an excited atom to start and there are few, the vast majority are in their ground state.

Masering by stimulated emission A process like

$$|1\rangle + \gamma \to |0\rangle + 2\gamma\,, \tag{2.83}$$

where the two emitted photons are equal both in energy and direction to the incoming one.

If there is no velocity gradient, this process can generate a strong and narrow emission line. A velocity gradient inhibits it because of the Doppler effect. This process is relevant for the winds of cool stars.

P-Cygni profiles

We have components due both to absorption and emission: these make up the so-called P-Cygni profile: a spectral line characterized by a blue-shifted absorption and a red-shifted emission with respect to the base spectrum profile.

They are mainly found for the resonance lines of C IV, N V, Si II and Mg II.

We consider a source which emits a spherically symmetric wind, continuum radiation and the gas has an absorption line at some wavelength λ_0.

This wavelength is shifted to $\lambda = \lambda_0(1+v)$, where v is the velocity of the gas relative to the emission: it is positive if the gas is receding away from the emitter, and negative otherwise.

The region of the wind which is directed at us corresponds to a blue-shifted absorption, which is caused by atoms moving towards us at velocities $0 \leq v \leq +v_\infty$: so the absorption looks something like $-k[0 \leq v \leq v_\infty]$ (this is very rough, but can serve as a first model).

For line emission, its profile will be symmetric centered at $v = 0$ (since most of the radiation comes from the photosphere), and the spectral profile looks something like a Gaussian centered on the original wavelength.

We can gather data from these profiles: for example, the maximum Doppler shift of the absorption corresponds to the maximum velocity v_∞.

We can also infer the number densities of the chemical species in the wind:

$$n_i = \frac{X_i \rho}{A_i m_u}, \tag{2.84}$$

by comparing the observed profile with simulations: then we will have the density profile $\rho(r) = \sum_i n_i m_i$, and since we also have the velocity profile $v(r)$ we can derive the mass loss rate using the continuity equation.

Absorption lines The opacity of the spectral lines is several orders of magnitude larger than the continuum opacity: for example, the opacity of the IV line of Carbon is $\sim 10^6 \kappa_{es}$.

When a photon is absorbed, the momentum increases by $\Delta p = h\nu/c = h/\lambda$. The increase in velocity when a typical metal ($m \sim 20 m_H$) absorbs a typical UV photon (wavelength 10^{-7} m) is of the order $\Delta v \approx 2 \times 10^{-1}$ m/s.

Typically, due to the redistribution of momentum among atoms, the *effective* change in velocity is something like four orders of magnitude less: 2×10^{-5} m/s, since the ions involved in this process typically contribute around 10^{-4} to the total mass of the gas (and 10^{-5} to the number: they are mostly heavy, while most of the gas is light). This is because the light elements - Hydrogen and Helium - are either fully ionized (H^+ and He^{++}) so they have no electrons, or have absorption lines in the far UV where the flux is low (He^+).

All the gas is then accelerated by the radiation.

In order to accelerate the gas to 2×10^6 m/s we'd need 10^{11} photons. If the distance to accelerate to terminal velocity is about three sun radii, the time to accelerate is of the order 10^4 s. So we need around 10^7 photons per second: the typical lifetime of the transition must be at most of the order 10^{-7} s.

So only transitions with oscillator strengths $f \gtrsim 0.01$ will contribute significantly.

Oscillation strength is defined as the ratio of the quantum to the classical transition probability. It is not really clear how the number 0.01 is found, but the idea is that only permitted transitions contribute, which makes sense.

The photons contribute energy and momentum to the gas: all of what they transfer, however, does not significantly impact the luminosity of the star, which is typically decreased by a single percent.

2.5.2 Acceleration and mass loss due to one line

Let us calculate the radiation pressure due to one line. Suppose that we have a strong absorption line at a wavelength λ_0 from an abundant ion, like C IV, and further suppose that the peak of the Planckian of the star is at λ_0. For now, let us assume that the line is so strong that any photon at its wavelength is readily absorbed. We want to compute the mass loss.

The momentum equation is

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} + f(r)\,, \tag{2.85}$$

and we need to compute the line-driven $f(r)$.

The emitted wavelengths which are absorbed are all the ones between λ_0 and $\lambda_0(1 - v_\infty/c)$, which the gas sees as being at λ_0 at some point in its acceleration. Equivalently, we can say that all the photons at frequencies ν_0 to $\nu_0(1 + v_\infty/c)$ are absorbed

If our velocity range goes from $v = 0$ ar $r = R$ to $v = v_\infty$ at $r = \infty$, then the total energy absorbed per unit time by the gas is:

$$L_{\text{line}} = \int_{\nu_0}^{\nu_0(1+v_\infty/c)} \underbrace{F_\nu 4\pi R^2}_{L_\nu} \mathrm{d}\nu \approx L_{\nu_0}\Delta\nu = L_{\nu_0}\nu_0 v_\infty/c\,, \tag{2.86}$$

where F_ν is the monochromatic flux (specific spectral intensity in $\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}$) at the photosphere, whose radius is R. We approximate the flux to be almost constant with respect to ν: this is valid as long as $\beta_\infty = v_\infty/c$ is small.

L_{ν_0} is the specific luminosity per unit frequency at ν_0.

The variation of momentum due to this process is

$$\frac{\mathrm{d}p_{\text{rad}}}{\mathrm{d}t} = \frac{L_{\text{line}}}{c} = \frac{L_{\nu_0}\nu_0 v_\infty}{c^2}\,, \tag{2.87}$$

and the total momentum loss in the wind per second is

$$\frac{\mathrm{d}p_{\text{wind}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(Mv_\infty) = \dot{M}v_\infty\,, \tag{2.88}$$

so if we approximate the wind's momentum to be all due to the line, we can equate these two terms: then we get

$$\frac{L_{\nu_0}\nu_0}{c^2} = \dot{M}\,. \tag{2.89}$$

Now, the Planck function has the property that $L_{\nu_0}\nu_0 \approx 0.62 L_{\text{tot}}$ if ν_0 is the peak (in frequency) of the Planckian. Then we see that the mass loss rate due to one strong line is linear in the luminosity of the star. The formula then reads:

$$0.62 L_{\text{tot}} \approx \dot{M} c^2 \,. \tag{2.90}$$

These are additive: if we can approximate the mass loss rate and luminosity with other means, we can find the total number of spectral lines.

The figure 0.62 is a red herring: the product νL_ν can increase further, its maximum is slightly higher than $0.7L$. It stays above $0.5L$ in the region from $2.36 k_B T$ to $5.98 k_B T$, which is rather wide.

Last time we estimated the radiation pressure due to one line.

If we know $\dot{M}$ and the luminosity L, then we can calculate the effective number of lines: $N_{\text{eff}} = \dot{M} c^2 / L$.

There will need to be more than these: they cannot *all* be at the peak of the Planckian.

For a hot luminous star with $L \sim 10^6 L_\odot$ the mass loss rate for a single line is $L/c^2 \approx 7 \times 10^{-8} M_\odot/\text{yr}$. The observed mass loss rate is a hundred times higher, so we can say that $N_{\text{eff}} \sim 10^2$.

If *all* of the star's light were to be absorbed by lines, then the momentum variation of the wind, $\dot{M} v_\infty$, will equal the total force applied by the star's emission: L/c.

Lamers calls this a balance of momenta but dimensionally they are forces!

So, we define the *efficiency parameter*:

$$\eta = \frac{\dot{M} v_\infty}{L/c} = \frac{\dot{M}}{\dot{M}_{\text{max}}} \,, \tag{2.91}$$

which represents the ratio between the real wind's mass loss rate and the theoretical maximum.

We can also look at the efficiency in the transmission of energy: we look at the variation of a certain type of energy, and compare it to the star's luminosity.

$$\eta_{\text{pot}} = \frac{\dot{E}_{\text{pot}}}{L} = \frac{\dot{M} G M_*}{R_* L} \tag{2.92a}$$

$$\eta_{\text{kin}} = \frac{\dot{E}_k}{L} = \frac{\dot{M} v_\infty^2}{2L} \tag{2.92b}$$

$$\eta_{\text{th}} = \frac{5 \dot{M} R T_w}{2 \mu L} \,, \tag{2.92c}$$

while the $\eta = \eta_{\text{mom}} = \dot{M} v_\infty c / L$ we defined before was the efficiency in the transmission of *momentum*. The main point is: *momentum transfer is efficient, energy transfer is not*. To see this, let us consider a star which, like the example from before, operated with $\eta_{\text{mom}} = 1$. This is realistic, and matches observations. Then, let us compute η_{kin}: it is

$$\eta_{\text{kin}} = \frac{\dot{M}v_\infty^2}{2L} = \frac{v_\infty^2}{2L}\frac{L}{v_\infty c} = \frac{v_\infty}{2c} , \tag{2.93}$$

which, as we know, is very small: the wind's terminal speed is very much subluminal.

We'd expect $\eta_{\text{mom}} \leq 1$, but actually sometimes the momentum efficiency can be larger: for WR stars we have $\eta_{\text{mom}} \approx 59$.

If across all the spectrum of the star we have completely optically thick absorption, then the momentum of the wind is equal to the momentum of radiation:

$$\dot{M}_{\text{max}} v_\infty = \frac{L}{c} \implies \dot{M}_{\text{max}} = \frac{L}{c v_\infty} , \tag{2.94}$$

and typically v_∞ is of the order of 2, 3 times the escape velocity at the photosphere: $v_\infty \approx 3\sqrt{2GM/R}$. For the luminosity we consider $L = 10^5 L_\odot$.

Then, the maximum mass loss rate estimated by (2.94) is similar to the observed one. The ratio $\eta_{\text{momentum}} = \dot{M}/\dot{M}_{\text{max}}$ is typically $0.5 \div 1$, but for some stars it can be of the order 10^1 to 10^2.

This is called the *single scattering upper limit*: we assume that scattering is isotropic, therefore we'd expect that after the first scattering the photon does not contribute anymore. This is not actually the case: *multiple scattering* can contribute, enhancing the maximum mass loss rate by the optical depth of the wind, τ_w, which is equal to

$$\tau_w = \int_{r_c}^{\infty} \kappa \rho \, \mathrm{d}r , \tag{2.95}$$

so $\dot{M}_{\text{max, multiple scattering}} = \tau_w \dot{M}_{\text{max, single scattering}}$.

This typically gives us enhancement factors of the order 2 to 6. The quantity τ_w is adimensional, since the units of κ are cm^2/g.

Even though we have this corrective factor, the efficiency is always expressed with respect to the single scattering cross section. Multiple scattering theory accounts for all of the increased mass loss rate, even up to $\tau \sim 10^2$.

How does it account for the approximate isotropy of the radiation after the first scattering?

Radiation pressure due to one line

Now, we derive the expression for the radiative acceleration privided by one line in a moving atmosphere.

We have a unit volume, of $1\,\text{cm}^3$, it has a velocity gradient inside it from v to $v + \Delta v$, and it will be heated by a monochromatic flux given by $F_\nu = I_\nu/(4\pi r^2)$.

What is the acceleration g_{line}? First we need to know the absorption properties of the medium. The absorption coefficient per cubic centimeter of gas is

$$\kappa_\nu = \frac{\pi e^2}{m_e c 4\pi\epsilon_0} f n_i \phi(\nu) , \tag{2.96}$$

where $\pi e^2/(4\pi\epsilon_0 m_e c) \approx 2.654 \times 10^{-2}\,\mathrm{cm^2/s}$ is the frequency integrated cross section[2] of the classical oscillator (for the electron):

$$\frac{\pi e^2}{m_e c} = \sigma = \int_0^\infty \sigma(\nu)\,\mathrm{d}\nu \;; \tag{2.97}$$

f is the oscillator strength, which depends on the line (it is a correction needed to account for quantum effects); n_i is the number density of atoms which can absorb the line, and $\phi(\nu)$ is the Gaussian profile of the absorption coefficient, centered at the rest frequency ν_0, and normalized so that $\int \phi(\nu)\,\mathrm{d}\nu = 1$: therefore the units of ϕ are 1/Hz.

This κ_ν is not the same as the one in the expression $\tau = \int \kappa\rho\,\mathrm{d}r$: this one actually is $\kappa\rho$ already, since it has the units 1/m.

The typical profile function is a *Doppler profile*: its width is of an order based on the thermal velocity of the atoms (divided by c), which can be approximated as much less than the wind velocity (as long as there is no turbulence): so, we apply the *Sobolev approximation*, and estimate $\phi(\nu) \sim \delta(\nu - \nu_0)$.

So we need to consider lines which do not overlap.

Newton's second law, expressed as a function of force, momentum and energy *per unit volume*, reads:

$$F_{\mathrm{rad}} = \frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}t} = \frac{1}{c}\frac{\mathrm{d}E_{\mathrm{rad}}}{\mathrm{d}t}\,, \tag{2.98}$$

where $\mathrm{d}E_{\mathrm{rad}}/\mathrm{d}t$ is the radiative energy absorbed per unit time and volume by the line.

Then, the acceleration due to the line is:

$$g_{\mathrm{line}} = \frac{F_{\mathrm{rad}}}{\rho} = \frac{1}{c\rho}\frac{\mathrm{d}E_{\mathrm{rad}}}{\mathrm{d}t}\,. \tag{2.99}$$

We still need to calculate the radiative energy absorbed: for a single optically thick line it is given by $\mathrm{d}E_{\mathrm{rad}}/\mathrm{d}t = F_\nu \kappa_{\mathrm{line}}$, so in the end the line acceleration is

$$g_{\mathrm{line}} = \frac{\kappa_{\mathrm{line}} F_\nu}{c\rho}\,. \tag{2.100}$$

This κ_{line} has the dimensions of $\mathrm{m^2/s}$: it is not the same as a regular opacity, which is a cross-section per unit mass.

2.5.3 Optically thick and optically thin case

A more general expression is:

$$g_{\mathrm{rad}} = \frac{F_{\nu_0}\nu_0}{c}\Big(1 - \exp\big(-\tau_{\nu_0}\big)\Big)\frac{\mathrm{d}v}{\mathrm{d}r}\frac{1}{c\rho}\,; \tag{2.101}$$

[2] We introduce the factor of $1/4\pi\epsilon_0$ because like civilized people we prefer the International System of units. However, from here on out we will use Gaussian units (set $4\pi\epsilon_0 = 1$) since I do not want to change all the formulas.

here F_{ν_0} is the monochromatic flux from the star emitted at the line rest frequency ν_0, the quantity

$$\frac{\nu_0}{c}\frac{\mathrm{d}v}{\mathrm{d}r} \tag{2.102}$$

is the width of the frequency band that can be absorbed, while $1 - \exp(-\tau)$ is the probability that the absorption occurs in our cubic centimeter: there τ is the optical depth (integrated up to infinity).

The product of these three terms gives an energy absorption rate $\mathrm{d}E_{\mathrm{rad}}/\mathrm{d}t$, if we multiply by $1/c$ we get absorbed momentum, if we multiply by $1/\rho$ we get acceleration.

Optically thin line Let us start from optically thin lines: they absorb only part of the radiative flux.

Then τ_{ν_0} is small, so we get $\exp(-\tau_{\nu_0}) \sim 1 - \tau_{\nu_0}$, therefore $1 - \exp(-\tau_{\nu_0}) \approx \tau_{\nu_0}$.

So what? there seems to be a factor ρ missing...

So, we can substitute (2.96) into (2.99), and we get:

$$g_{\mathrm{line}} = \frac{F_\nu}{c}\frac{n_i}{\rho}\left(f\frac{\pi e^2}{m_e c}\right) \sim \frac{L_\nu}{r^2}\frac{n_i}{\rho}, \tag{2.103}$$

where L_ν is the monochromatic luminosity of the star: we have $F_\nu \sim L_\nu/r^2$, the factor of 4π does not make much of a difference. We can approximate n_i/ρ as a constant with respect to the radius: it depends on the ionization ratio of the gas, which will not change much. Therefore, we have $g_{\mathrm{line}} \propto L_\nu/r^2$.

Optically thick line In this case, the assumption is that $\tau_\nu \gg 1$: all the radiation which is at the appropriate frequency will get absorbed.

The absorption in a unit volume will be proportional to the difference in velocities at its edges, which is then proportional to the gradient of the velocity.

The formula we then get is

$$g_{\mathrm{line}} = \frac{F_\nu}{c}\frac{\nu_0}{c}\frac{1}{\rho}\frac{\mathrm{d}v}{\mathrm{d}r} \propto \frac{L_\nu}{r^2}\frac{1}{\rho}\frac{\mathrm{d}v}{\mathrm{d}r}, \tag{2.104}$$

so we have the same dependence as before with an additional factor.

2.5.4 Castor-Abbott-Klein formalism

We can reduce drastically the number of spectral lines we have to account for if we assume that, since the density of the wind is very low, collisional excitation is negligible: only lines from the ground state, low energy states and metastable states have to be accounted for. Even with this consideration, the number of lines to consider is still in the hundreds of thousands.

We need to sum over all the optically thick and thin lines and everything in between.

In order to do this, Castor, Abbott and Klein (CAK) showed in 1975 that a reasonable estimate is:

$$g_L \sim \left(\rho^{-1}\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{\alpha} \sim \left(\frac{vr^2}{\dot{M}}\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{\alpha}, \tag{2.105}$$

From the continuity equation $\dot{M} \propto r^2\rho v$.

where α is a parameter quantifying the optical thickness of the line: it goes from 0 for an optically thin line to 1 for an optically thick line.

The expression proposed by CAK was $g_L = g_e M(t)$, where g_e is the radiative acceleration from continuum phenomena (mostly electron scattering), while the *force multiplier* M is in the form:

$$M(t) = kt^{-\alpha}s^{\delta}, \tag{2.106}$$

where k, α, δ are called *force multiplier parameters*. The variables t and s will be defined shortly.

The radiative acceleration due to electron scattering is given by

Maybe we could keep a consistent notation between things? Why write first g_e and then $g_L(e)$? Frustrating.

$$g_e = \frac{\kappa_e}{c}\frac{L_*}{4\pi r^2} = \Gamma_e\frac{GM_*}{r^2}, \tag{2.107}$$

where Γ_e is the so-called *Eddington factor*:

$$\Gamma_e = \frac{\kappa_e}{4\pi cG}\frac{L_*}{M_*}, \tag{2.108}$$

which is the luminosity divided by the Eddington luminosity.

The scattering opacity κ_e is given by

$$\kappa_e \approx \frac{\sigma_e}{m_H}, \tag{2.109}$$

which is measured in $\mathrm{cm^2g^{-1}}$. The value $\sigma_e = 6.65 \times 10^{-25}\,\mathrm{cm^2}$ is the Thomson scattering cross section for electrons. This ratio gives $0.39\,\mathrm{cm^2/g}$, the value used by CAK is $0.325\,\mathrm{cm^2/g}$, which is of the same order of magnitude and probably accounts for some other effects.

The t in $M(t)$ is defined by

$$t \equiv \kappa_e v_{\text{thermal}}\rho\frac{\mathrm{d}r}{\mathrm{d}v} = \kappa_e\sqrt{\frac{2k_BT}{m_H}}\rho\frac{\mathrm{d}r}{\mathrm{d}v} = \kappa_e\sqrt{\frac{2k_BT}{m_H}}\left(\frac{1}{\rho}\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{-1}, \tag{2.110}$$

and is inversely proportional to the velocity gradient divided by the density: so, since it is elevated to the $-\alpha$, it gives the same contribution as the term in (2.105).

The s in the definition of $M(t)$ contains information about the degree of ionization: it is

$$s \equiv \frac{10^{-11}\rho}{m_H W}, \tag{2.111}$$

where W is the *geometrical dilution factor*:

$$W(r) = \frac{1}{2}\left(1 - \sqrt{1 - \left(\frac{R_*}{r}\right)^2}\right) \sim \left(\frac{R_*}{2r}\right)^2, \tag{2.112}$$

and now we present a proof for the fact that this encodes the radial dependence of the observed intensity. It is a ratio of solid angles:

$$W(r) = \frac{\int_0^\Omega \mathrm{d}\Omega}{4\pi}, \tag{2.113}$$

where the integration limit Ω encodes the solid angle subtended by the star. Assuming simmetry with respect to the azimuthal angle, we can rewrite it as:

$$W(r) = \frac{2\pi}{4\pi}\int_0^{\theta_1} \sin\theta \,\mathrm{d}\theta = \frac{1}{2}\int_1^{\cos(\theta_1)} (-\,\mathrm{d}x) = \frac{1}{2}\left(1 - \sqrt{1 - (R/r)^2}\right), \tag{2.114}$$

so $W(r)$ is the solid angle fraction subtended by a star with radius R at a distance r.

The complete expression for an ensemble of lines is given by

$$g_L = \frac{\kappa_e}{c}\frac{L_*}{4\pi r^2} k t^{-\alpha} s^\delta, \tag{2.115}$$

Simulations show that $M(t)$ decreases with t with some kind of power law: the approximation $\log M \sim -\alpha \log t$ is justified.

Numerical simulations show that $\alpha \sim 0.5 \div 0.6$, independent of temperature. The δ parameter is almost always of the order $\delta \sim 0.1$.

The force multiplier is also linearly dependent on the metallicity of the star: $M_n(t_n) = M_n(t_n)_\odot (Z/Z_\odot)$.

A typical velocity gradient is something like

$$v(r) = v_\infty \left(1 - \frac{r_0}{r}\right)^\beta, \tag{2.116}$$

with $\beta \sim 0.7$. Then, the profile of g_L can be computed (using the continuity equation): it is in the form

$$g_L \sim r^{-2}\left(\rho \frac{\mathrm{d}r}{\mathrm{d}v}\right)^{-\alpha} \sim r^{2(\alpha-1)}\left(v \frac{\mathrm{d}v}{\mathrm{d}r}\right)^\alpha, \tag{2.117}$$

$r^2 v \rho = \text{const}$ by continuity

where $\alpha \sim 0.6$. The r dependence is something like

$$g_L \sim r^{-2}\left(1 - \frac{r_0}{r}\right)^{\alpha(2\beta-1)}, \tag{2.118}$$

which we can plot.

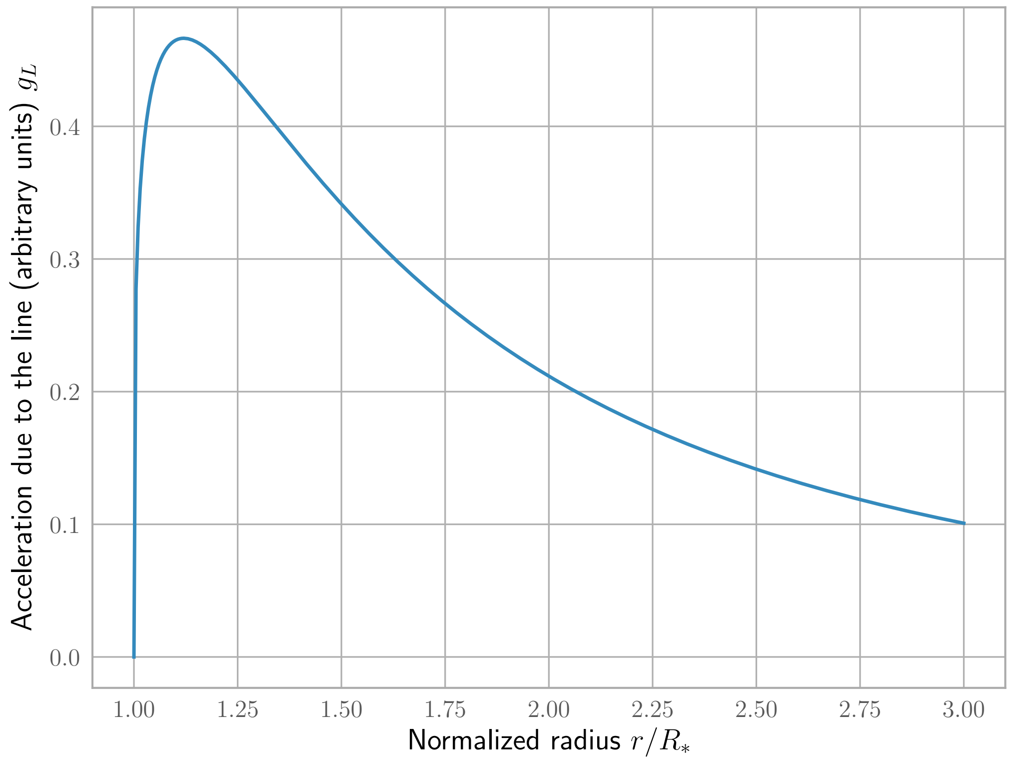

Figure 2.3: Line acceleration profile for $\alpha = 0.6$ and $\beta = 0.7$.

Which lines drive the winds?

We classify the driving lines according to which series of the spectral lines of hydrogen they are closest to, even though hydrogen is not the element which is ionized and responsible for line driven winds. Recall: the wavenumbers of the lines are given by

$$\lambda^{-1} = R_H \left(\frac{1}{n^2} - \frac{1}{m^2} \right) , \tag{2.119}$$

so, the series which are relevant for us are:

1. $n = 1$ gives the Lyman lines, which are around $\lambda \sim 1000\,\text{Å}$ — these drive the winds of the very hottest stars;

2. $n = 2$ gives the Balmer lines, which are around $\lambda \sim 4000\,\text{Å} \div 6000\,\text{Å}$ — these drive the winds of stars between 20 kK and 30 kK;

3. $n = 3$ gives the Paschen lines, but actually even the winds of stars below 10 kK are driven by lines in the Balmer continuum.

As the effective temperature decreases, the Planckian shifts to longer wavelengths, and since the gas is on average less ionized, the absorption lines move to longer wavelengths as well.

Which elements contribute most? The dependence on temperature is significant. A synthesis is as follows:

1. below 25 kK the iron-group elements are most important;
2. above 25 kK Carbon, Nitrogen, Oxygen, Neon and Calcium are more important.

Hydrogen and helium hardly matter at all.

2.5.5 Applications to hot luminous stars

Yesterday we introduced the CAK formalism.

We consider a spherically symmetric, stationary problem, and we assume that the star looks pointlike to us. Also, we assume that the process is isothermal: $\forall r : T = \text{const}$.

The momentum conservation equation is

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{GM}{r^2} - \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + g_c + g_L \,, \tag{2.120}$$

where we distinguished the continuum acceleration g_c and the line acceleration g_L, which we assume is expressible in the form described by CAK: $g_L = g_e k t^{-\alpha} s^{\delta}$.

We assume that the gas follows the ideal gas law:

$$P = \frac{R\rho T}{\mu} \,, \tag{2.121}$$

which implies

$$-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{RT\rho}{\mu}\right) = -\frac{1}{\rho}\frac{RT}{\mu}\frac{\mathrm{d}\rho}{\mathrm{d}r} \,, \tag{2.122}$$

where we can recognize the square speed of sound $a^2 = RT/\mu$. Using the continuity equation then we get

$$-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = a^2 \left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\right) . \tag{2.123}$$

For the continuum and line acceleration we use the CAK formalism:

$$g_c = \frac{GM}{r^2}\Gamma_e \,, \tag{2.124}$$

and

$$g_L = \frac{\kappa_e}{c}\frac{L}{4\pi r^2} k t^{-\alpha} s^{\delta} \,, \tag{2.125}$$

so in the end we get:

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{GM}{r^2} + a^2 \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) + \frac{GM}{r^2} \Gamma_c + \frac{\kappa_e}{c} \frac{L}{4\pi r^2} k t^{-\alpha} s^{\delta} . \tag{2.126}$$

Now recall

$$t = C_T \rho \frac{\mathrm{d}r}{\mathrm{d}v} = \frac{C_T \dot{M}}{4\pi} \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{-1} , \tag{2.127}$$

where we defined the constant $C_T = \kappa_e \sqrt{2 k_B T_{\text{eff}} / m_H}$.

We plug this into the momentum conservation equation and multiply by r^2:

$$v r^2 \frac{\mathrm{d}v}{\mathrm{d}r} = -GM(1 - \Gamma_e) + a^2 r^2 \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) + C \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha} , \tag{2.128}$$

where

$$C = \frac{\kappa_e}{c} \frac{L}{4\pi} k \left(C_T \frac{\dot{M}}{4\pi} \right)^{-\alpha} \left(\frac{10^{11} \rho}{m_H W} \right)^{\delta} . \tag{2.129}$$

Also, we can bring a term to the RHS: we get

$$\left(1 - \frac{a^2}{v^2} \right) v r^2 \frac{\mathrm{d}v}{\mathrm{d}r} = -GM(1 - \Gamma_e) + 2a^2 r + C \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha} . \tag{2.130}$$

This equation has a critical point corresponding to the speed of sound, similarly to the ones we encountered before, but this time it is nonlinear.

Now, we make a simplifying assumption: we neglect the gas pressure. This is reasonable at large distances from the star. It also can be shown that the main physical characteristics we will find are the same we would find if we did consider the gas pressure.

Since the speed of sound is $a^2 = \partial P / \partial \rho$ neglecting the pressure gradient means $a = 0$: removing the terms with a we get

$$r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} - C \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha} = -GM(1 - \Gamma_e) = \text{const} , \tag{2.131}$$

and if we denote $r^2 v \, \mathrm{d}v / \mathrm{d}r = D$, the equation can be written as

$$D - C D^{\alpha} = -GM(1 - \Gamma_e) . \tag{2.132}$$

Our parameters here are D and $C \propto \dot{M}^{-\alpha}$.

The system is equivalent to $C D^{\alpha} = D + GM(1 - \Gamma_e)$: so the solutions, when plotted with respect to D, are the intersections between a slope-1 straight line and a powerlaw (with index $0 < \alpha < 1$).

There are values of C such that there is no solution.

We select the value of C such that the system has a unique solution — we do this because, inspecting the equation more closely, we find that it is also a critical-point equation, and if we want a monotonic velocity gradient we must have the solution passing through the critical point.

This is not really justified: the critical point condition should be set by setting the numerator of the full equation, with a, to zero when $v = a$... I guess it works this way.

So we differentiate $D - CD^\alpha + GM(1-\Gamma_e)$ with respect to D, set it equal to 0 and find $C = D^{1-\alpha}/\alpha$.

Plugging this expression for C in we find a differential equation:

$$r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} = D = \frac{\alpha}{1-\alpha} GM_*(1-\Gamma_e)\,, \tag{2.133}$$

which can be solved by direct integration if we express it like:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(v^2\right) = \frac{\alpha}{1-\alpha} GM_*(1-\Gamma_e) \tag{2.134a}$$

$$v^2 = \int \frac{\alpha}{1-\alpha} 2GM_*(1-\Gamma_e)\,\mathrm{d}r\,, \tag{2.134b}$$

so the solution is

$$v(r) = \left(\frac{\alpha}{1-\alpha} 2GM_*(1-\Gamma_e)\left(\frac{1}{R_*} - \frac{1}{r}\right)\right)^{1/2} = v_\infty \sqrt{1 - \frac{R_*}{r}}\,. \tag{2.135}$$

this is a β-type law, with $\beta = 0.5$ and

$$v_\infty = \sqrt{\frac{\alpha}{1-\alpha} 2GM_* \left(\frac{1-\Gamma_e}{R_*}\right)} = v_{\text{esc}} \sqrt{\frac{\alpha}{1-\alpha}}\,, \tag{2.136}$$

since the escape velocity for electrons in the photosphere is

$$v_{\text{esc}} = \sqrt{\frac{2GM_*(1-\Gamma_e)}{R_*}}\,, \tag{2.137}$$

and we include Γ_e in the escape velocity since there is electron scattering there.

This gives us a rather complicated but well-defined expression for the mass loss rate, which appears in C, for which we have an expression in terms of D: we find

$$\dot{M} = \frac{4\pi}{C}\left(\frac{\kappa_e}{4\pi}\right)^{1/\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1-\alpha}{\alpha}} (k\alpha)^{1/\alpha} s^{\delta/\alpha} \left(\frac{L_*}{c}\right)^{1/\alpha} \big(GM_*(1-\Gamma_e)\big)^{(\alpha-1)/\alpha}\,, \tag{2.138}$$

in which the main dependences are $\dot{M} \propto L_*^{1/\alpha} M_*^{(\alpha-1)/\alpha}$.

For Main Sequence stars $L_* \sim M_*^{2.5}$, so we can turn the luminosity dependence into a mass dependence: $\dot{M} \sim M^{(3.5\alpha-1)/\alpha}$, or similarly into a luminosity dependence:

$$\dot{M} \sim L^{\frac{2}{5} + \frac{3}{5\alpha}}\,. \tag{2.139}$$

For $\alpha = 0.52$ we get approximately $\dot{M} \sim M^{1.58}$ and $\dot{M} \sim L^{1.55}$. This is the same result we would see if we did consider the gas pressure.

The terminal velocity does not depend on the mass loss rate An important result we found is the fact that D is a constant, regardless of the value of α (so, for any value of the optical thickness of the line).

The asymptotic velocity does not depend on the mass loss rate, even though the radiative acceleration does ($g_{\text{line}} \sim \dot{M}^{\alpha}$). This is because we are fixing the passage of the wind velocity law through the critical point.

Corrections due to the finite size of the star Now, we talk about the corrections due to the finite size of a star. Near the star, we cannot approximate it as a point source: the direction of the momentum provided by the radiation is not all radial anymore; a significant part of the momentum is in a non-radial direction.

Accounting for this is in general hard to do, the result is: we get a lower mass loss rate, and a higher terminal velocity.

The line acceleration formula picks up a correction factor D_f, which is derived geometrically:

$$D_f = \frac{(1+\sigma)^{\alpha+1} - (1+\sigma\mu_*^2)^{\alpha+1}}{(1+\mu_*^2)(\alpha+1)\sigma(1+\sigma)^{\alpha}}\,, \tag{2.140}$$

where $\mu_* = \cos(\theta) = \sqrt{1-(R_*/r)^2}$, with θ being half of the angle subtended by the star from the point of view we are considering. α is the same as before, while σ is defined as

$$\sigma = \frac{r}{v}\frac{\mathrm{d}v}{\mathrm{d}r} - 1\,. \tag{2.141}$$

The definition of μ_* is inconsistent in the slides, I suspect that the square needed to make it into a cosine of the angle was forgotten in Lamers' book.

So now our radiative acceleration will look like

$$g_{\text{line}} = \frac{\kappa_e}{c}\frac{L_*}{4\pi r^2} k t^{-\alpha} s^{\delta} D_f\,, \tag{2.142}$$

The correction factor can be both below and above unity, depending on the distance from the star and on β. As we'd expect in the $r \to \infty$ limit we have $D_f \to 1$. For most realistic values of β it starts off below 1 near the star, then rises above it (by not much, it becomes of the order 1.1) and then tends towards 1.

At this point, Lamer states that a decrease in the effective optical thickness of the line causes an increase of the line acceleration. This is counterintuitive: after all, if the line absorbs more it should also accelerate more, right? Actually, the effect is the opposite, as I will now show: we prove that if $\beta > 0.5$ then

$$\frac{\mathrm{d}g_{\text{line}}}{\mathrm{d}\alpha} < 0\,, \tag{2.143}$$

which means that as the optical thickness increases (α increases) the radiative acceleration decreases, and vice versa.

We know that $g_{\text{line}} \sim D^\alpha$, and let us use a β-type velocity law. Then, we have (setting $R_* = 1$ here):

$$\frac{dv}{dr} = \frac{1}{r^2}\left(1 - \frac{1}{r}\right)^{\beta-1}, \tag{2.144}$$

therefore

$$D^\alpha = \left(vr^2 \frac{dv}{dr}\right)^\alpha = \left(1 - \frac{1}{r}\right)^{\alpha(2\beta-1)}, \tag{2.145}$$

so

$$\frac{d}{d\alpha} D^\alpha = \frac{d}{d\alpha} \exp\bigl(\alpha \log D\bigr) = D^\alpha \log D \tag{2.146a}$$

$$= \left(1 - \frac{1}{r}\right)^{\alpha(2\beta-1)} (2\beta - 1) \log\left(1 - \frac{1}{r}\right), \tag{2.146b}$$

where, as long as $2\beta - 1 > 0$, all the terms are positive except for $\log(1 - 1/r)$, which is negative outside the star. This proves that an increase in optical depth decreases the radiative acceleration.

Typically, the mass loss rate variation due to this correction is similar to multiplying it by 1/2, while the escape velocity is approximately multiplied by 2 (although the multiplier varies with effective temperature).

Line driven winds' instability Phenomena which are not accounted for by this model are:

1. X-ray emission;

2. superionization (beyond the sixth line);

3. discrete absorption components & variability.

In general these winds are unstable to small-amplitude stellar oscillations: they will feel shocks.

These can be simulated: the time-averaged v_∞ and $\dot{M}$ are similar to those found in stationary models. The fact that the line acceleration has a positive dependence on the velocity gradient means that the velocity profile is unstable to perturbations. However, they are generally more or less sinusoidal so their effects cancel.

The takeaways are:

1. v_∞ and $\dot{M}$ in perturbed models are similar to those found in unperturbed ones;

2. the amplitude of the velocity perturbation is comparable to the velocity itself and the perturbations affect a large part of the wind: so, the P-Cygni profiles are broadened;

3. if the density is high the temperatures in the perturbed regions rise significantly: we then see X-ray emission and ionization of very energetic lines.

The bistability jump Now, let us treat the terminal vs escape velocities for O-B stars: we seem to have a linear relation between these, for each group of stars, where we group them by effective temperature.

If we plot $v_\infty / v_{\mathrm{esc}}$ in terms of T_{eff}, we get three horizontal regions:

1. under 10 kK the ratio is around 0.7: this corresponds to an effective $\alpha \approx 0.33$;
2. between 10 kK and 21 kK the ratio is around 1.3: this corresponds to an effective $\alpha \approx 0.63$;
3. over 21 kK the ratio is around 2.6: this corresponds to an effective $\alpha \approx 0.87$.

This is called the *bistability jump*, and it seems to be due to the fact that the strength of different lines, such as those in carbon, is heavily dependent on the ionization level of the gas.

We cannot reproduce all the data with a single α value, instead we need at least three. Simple CAK theory cannot fully explain the data.

We have the maximum mass loss rate when the equation

$$\dot{M} v_\infty = \tau_w \frac{L}{c} \tag{2.147}$$

with $\tau_w = 1$ is satisfied.

We can plot both sides of this equation for O-stars[3] and Wolf-Rayet stars of type WNL. O-stars' observations agree with the model (in that their mass loss rate is slightly lower than the luminosity), WNL observations seem not to (in that their $\dot{M} v_\infty$ is higher than L/c).

We can make a plot of initial mass vs metallicity and distinghish regions fo the final fates of stars. Metallicity characterizes the amount of heavier-than-Helium elements in the star.

The definition is

$$\text{Metallicity} = \frac{m_{\text{metals}}}{m_{\text{tot}}} = Z\,, \tag{2.148}$$

where m denotes the mass of element of a certain species. $1 = X + Y + Z$ are the fractional masses of hydrogen, helium and metals respectively, so $X = m_{\mathrm{H}} / m_{\mathrm{tot}}$ and so on.

Achievements of line driven wind theory

It correctly predicts $v_\infty \propto v_{\mathrm{esc}}$ and $\dot{M} \sim L^{1.5}$.

Unexplained issues are the bistability jump, the momentum problem and the metallicity dependence.

2.6 Dust driven winds

2.6.1 General considerations

Now, we will speak of dust driven winds: we move to the cool and luminous part of the H-R diagram, which is populated by red giants and so on.

[3] Hey now, you're an O star, get your game on, go play.

The main characteristics of these stars are:

1. mass between 0.7 and $2M_\odot$;
2. luminosity between 10^3 and $10^5 L_\odot$;
3. temperature between 2 and 3 kK;
4. mass loss rate between 10^{-7} and $10^{-4} M_\odot/\mathrm{yr}$;
5. asymptotic velocity between 20 and 30 km/s;
6. mass of the shells between 0.1 and $0.6M_\odot$;
7. pulsation period between 10^2 and 10^3 d.

Qualitatively, the driving mechanism is the radiation pressure on dust grains; dust can exist in low temperature regions of the atmosphere, with $T \sim 10^3\,\mathrm{K}$. Dust absorbs momentum and is accelerated outwards: it is very opaque.

The bulk of the outward travelling matter is gas: the dust is a small part, but it is dynamically coupled to the gas (it "drags it along", they share momentum).

A good approximation is a wind with a force $f \sim r^{-2}$. This mechanism critically depends on the dust formation radius: the distance at which the temperature becomes low enough for dust to form.

Dust grains can absorb photons in the continuum, they are then said to be "continuum driven" as opposed to "line driven". Then, this very much diminishes the role of Doppler shift in making it possible for the radiation to pass through much of the gas uninterrupted. Besides, the actual shift is low since the wind is not accelerated to very high velocities.

The wind speeds of cool stars become comparable to the escape speed near the transsonic points, as opposed to what happens for hot stars, for which this happens right after the wind has left the stellar surface.

The typical photons in this process are infrared, on the order of $\lambda \sim 1\,\mu\mathrm{m}$.

The dust causes significant reddening. We have low v_∞ and high $\dot{M}$.

Most of the momentum must be transferred to the wind in the subsonic region to account for the high $\dot{M}$, while little energy must be transferred in the supersonic region in order for the exit velocity to be low.

The formation of dust grains must be described with molecular chemistry, and it heavily depends on the ratios of chemical species.

An example: the Helix Nebula.

We define the Eddington factor: the ratio of radiative acceleration to gravity. In the case of radiation forces on dust, it is

$$\Gamma_d = \frac{\kappa_{\mathrm{rp}} L_*}{4\pi c G M_*}\,, \tag{2.149}$$

where the κ_{rp} is called the "radiation pressure mean opacity": it measures the capacity of the dust to absorb photons.

In order to have a wind, we must have Γ_d greater than unity after some radius, otherwise the net force on a single particle is inward.

To a good approximation, the transsonic transition corresponds to the point at which Γ_d becomes greater than one.

The plot of our model of Γ_d looks something like $\Gamma_d(r) = \Theta(r - r_{\text{sonic}}) \times 1.4$, where $r_{\text{sonic}} \sim (3 \div 4) r_*$, where Θ is the Heaviside theta.

2.6.2 Necessary conditions

What are the conditions in order to have a dust driven wind? We first need to form the dust grains: they need to survive against sublimation. Then, the dust must be coupled enough to the gas in order to drive it.

Condensation radius

There are two interesting temperatures. The first is the radiative equilibrium temperature T_{rad}, which is the temperature of the grains: it is the result of the balance between the radiative heating and cooling of the grains.

The second one is the condensations temperature T_{cond}, below it the grain becomes a solid. If $T_{\text{rad}} > T_{\text{cond}}$ we have immediate sublimation.

$T_{\text{cond}} \sim \text{const}$ is not a bad approximation; the radiative equilibrium temperature instead decreases moving outwards from the star. There is a radius at which they are equal, and for larger radii we have $T_{\text{rad}} < T_{\text{cond}}$. This critical radius is called the dust condensation radius r_d.

Let us suppose that we are at $r > r_d$. The grain then can form, and it can then gain momentum $h\nu/c$ and energy $h\nu$ from the Sun's photons, and transfer it to the gas molecules. The energy is not really relevant: it is absorbed and then reemitted isotropically. The momentum, on the other hand, is radial.

We can get a lower limit to the mass loss rate $\dot{M} \gtrsim 10^{-7} M_\odot/\text{yr}$ because of the gas-dust coupling condition. Deriving this is complicated, but qualitatively it is since if there is less mass loss rate there is also less gas density, so there is less transfer.

Drift speed and gas-dust coupling

Also, we have an upper limit on the drift speed of a grain through the gas, which is defined as the difference in velocity between dust and gas: $w_{\text{drift}} = u_{\text{grain}} - v_{\text{gas}}$. If it is too high, collisions with the gas particles will destroy the gas grains.

The drift speed decreases if there is more dynamical coupling between gas and dust, and the dynamical coupling depends on the density of the gas: the precise relation is $w_{\text{drift}} \propto \rho_{\text{gas}}^{-1/2}$. The density decreases moving outward, so there is a radius beyond which the collisions exceed the limiting energy at which they are able to destroy the grains, so there is no further increase in the wind speed, since the dust and gas get progressively more decoupled as the density decreases.

2.6.3 Dust grain composition

The properties of the grains are very important, but studying the processes in grain formation and growth is hard.

There are both processes of accretion of the gas onto the dust particle and erosion by sputtering: collisions of the gas onto the grain.

There will be a distribution of grain sizes, typically from $0.05 \div 0.1\,\mu\mathrm{m}$: we can sometimes make the small grain approximation.

The composition of the grains depends on the materials in the gas, particularly on the C/O (Carbon to Oxygen) ratio (defined as a ratio of number of atoms).

1. If $\mathrm{C/O} < 1$ typically we form silicate grains. The stars with this ratio are called M-type stars, they are oxygen-rich.
2. If $\mathrm{C/O} > 1$ typically we form carbonaceous grains. The stars with this ratio are called C-type stars.
3. If $\mathrm{C/O} \sim 1$ the star is called an S-type star.

Stars in the AGB evolve through these: they start off as M-type, move through S type into C-type.

If we make an infrared color-magnitude diagram, we can see that oxygen-rich stars are redder, while carbon-rich ones are bluer.

The phase we are interested in is called TP-AGB: Thermally Pulsating Asymptotic Giant Branch.

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2019-11-26

Pulsation assisted winds The density ρ_c at the condensation radius r_C is critical: it determines the mass loss rate $\dot{M}$. If it is too low no wind can be generated. Therefore, there is a need for a mechanism to increase the density scale height: these winds are *pulsation assisted*, the driving mechanism are shocks occurring in pulsating stars. So, $\dot{M}$ is very sensitive to these pulsations, which "launch" the gas to a height sufficient for it to form dust grains.

The wind starts to be accelerated at the critical radius r_C, and its acceleration is so fast that it is reasonable to assume that this critical radius is the same as the sonic radius at which the gas reaches $v = a$. This is the dust formation radius and also the condensation radius.

The momentum equation of the dust driven wind reads:

$$v \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{GM}{r^2} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = g_{\mathrm{rad}} = \frac{GM}{r^2}\Gamma_d\,, \tag{2.150}$$

with the Γ_d defined in (2.149), proportional to the opacity κ_{rp}.

2.6.4 Maximum mass loss rate: the single scattering limit

In this case we also have a maximum mass loss rate given by

$$\dot{M}_{\max} v_\infty = \frac{L}{c}\,, \tag{2.151}$$

which is rather low since the asymptotic velocity is low.

As was the case with line driven winds, multiple scattering can enhance this maximum mass loss rate by a factor depending on the optical depth $\tau_w = \int_{r_s}^\infty \kappa\rho\,\mathrm{d}r$, however now we also have a dependence on Γ_d:

$$\dot{M}_{\max} = \frac{L}{v_\infty c}\left(\frac{\Gamma_d - 1}{\Gamma_d}\right)\tau_w\,. \tag{2.152}$$

Here is the derivation of this expression, from Lamers: we integrate the momentum equation multiplied by $4\pi r^2\rho$ from the star's radius R_* to infinity, but we split certain integrals into the contributions from R_* to r_s and from r_s to infinity. We get:

$$0 = \int_{R_*}^\infty \left[4\pi r^2\rho v\frac{\mathrm{d}v}{\mathrm{d}r} + 4\pi GM\rho + 4\pi r^2\frac{\mathrm{d}P}{\mathrm{d}r} - 4\pi GM\Gamma_d\right]\mathrm{d}r \tag{2.153a}$$

$$\begin{aligned}0 = \int_{R_*}^\infty 4\pi r^2\rho v\frac{\mathrm{d}v}{\mathrm{d}r}\,\mathrm{d}r + \int_{R_*}^{r_s}\left[\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2}\right]4\pi r^2\rho\,\mathrm{d}r\, + \\ + \int_{r_s}^\infty \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}4\pi r^2\rho\,\mathrm{d}r + \int_{r_s}^\infty \frac{GM}{r^2}(1-\Gamma_d)\rho 4\pi r^2\,\mathrm{d}r\ ,\end{aligned} \tag{2.153b}$$

but $4\pi r^2\rho\,\mathrm{d}r = \mathrm{d}m$, so the second integral can be written as

$$\int_{R_*}^{r_s} \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2}\,\mathrm{d}m\ , \tag{2.154}$$

which is precisely 0 if we have hydrostatic balance, and we do under r_s. So this term can be neglected. Similarly, the third integral is

$$\int_{r_s}^\infty \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\,\mathrm{d}m\ , \tag{2.155}$$

and it is also negligible since the pressure gradient is very low beyond the sonic point.

The first integral can be written as

$$\int_{R_*}^\infty v\frac{\mathrm{d}v}{\mathrm{d}r}\,\mathrm{d}m\ , \tag{2.156}$$

and we can recognize the expression of the convective derivative: $\dot{v} = v\,\mathrm{d}v/\mathrm{d}r$ if the wind is stationary. The contribution to this integral at R_* is negligible: the term which remains is $\dot{M}v_\infty$. So, we are left with

$$\dot{M}v_\infty \approx -4\pi GM(1-\Gamma_d)\int_{r_s}^\infty \rho\,\mathrm{d}r\ , \tag{2.157}$$

and we define $\tau_w = \int_{r_s}^{\infty} \rho\kappa_{rp}\,\mathrm{d}r$ we can express this as

$$\dot{M}v_\infty \approx 4\pi GM \frac{\Gamma_d - 1}{\kappa_{rf}} \tau_w\,, \tag{2.158}$$

and now we can insert the expression for κ_{rf} which is found by inverting the definition of Γ_d: we get

$$\kappa_{rp} = \frac{4\pi cGM}{L}\Gamma_d\,, \tag{2.159}$$

so

$$\dot{M}v_\infty \approx 4\pi GM \frac{L}{4\pi cGM} \frac{\Gamma_d - 1}{\Gamma_d} \tau_w \tag{2.160a}$$

$$\approx \frac{L}{c} \frac{\Gamma_d - 1}{\Gamma_d} \tau_w\,, \tag{2.160b}$$

and far from the star $\Gamma_d \gg 1$, so the expression simplifies to

$$\dot{M} \approx \frac{L}{cv_\infty} \tau_w\,. \tag{2.161}$$

It would seem that τ can be arbitrarily large, but this is not the case: by conservation of energy, we can see that

$$\frac{1}{2}\dot{M}v_\infty^2 < L \tag{2.162a}$$

$$\dot{M} < \frac{2L}{v_\infty^2}\,, \tag{2.162b}$$

so, comparing the formulas, we see that $\tau_w < 2c/v_\infty$. More precise considerations allow us to drop the factor 2.

2.6.5 Dust grain opacities

In this lecture the attention was focused on the importance of C/O ratio in the formation of dust grains in dust driven winds.

In AGB stars the core, mainly made of heavier elements such as carbon and oxygen, is sourrounded by a helium envelope and, more externally, a hydrogen envelope. Between these areas there exist some convective flows that can bring oxygen and carbon from the core to the more external parts of the star. This process is called the *third dredge-up*.

Reaching thanks to convection these external layers, which lay at a lower temperature, the heavy elements can form molecular bounds. In particular the strongest bounded molecule that can be formed is carbon monoxyde (CO). This implies that, if the star has a C/O ratio > 1 (i.e. it has more abundance of carbon over oxygen) almost all the oxygen is used to form carbon monoxyde, while the rest of the carbon can form other molecules: for this reason, in these stars we observe carbon grains.

Viceversa, in oxygen-rich stars we have with the specular mechanism, which ends in the formation of silicate grains. At this point we started a discussion that will be continued in the following lecture about dust grains' opacity, focusing first on cross section.

We must distinguish two different kinds of opacities:

1. the absorptive grain opacity κ_d: this accounts for pure absorption *without* scattering, it determines the pressure, so it defines the condensation radius;

2. the radiation pressure mean opacity k_{rp}:[4] this accounts for both absorption and scattering. Scattering does not affect the temperature of the gas, but it *does* affect the radiative acceleration.

2.6.6 Dust grain cross section

Let a be the radius of the grain, and Q the efficency of the cross section, which depends both on a and on the wavelength λ of the light. The distribution of the grain radii is as such:

$$n(a)\,\mathrm{d}a \approx Ka^{-3.5}[a_{\text{min}} < a < a_{\text{max}}]\,\mathrm{d}a \ , \tag{2.163}$$

where $a_{\text{min}} \approx 5\,\text{nm}$ while $a_{\text{max}} \approx 250\,\text{nm}$, while k is a normalization factor.

In this case we have a cross section $c = \pi a^2$ and, considering both absorption and scattering cross sections c_a, c_s, we have

$$c_{\text{tot}} = c_a + c_s = \pi a^2 (Q_a(a,\lambda) + Q_c(a,\lambda)) \tag{2.164}$$

where $Q_a \ll Q_s$ for IR wavelengths: typically the absorptive efficiency Q_a can be modelled by $Q_a \propto a\lambda^{-p}$, where the exponent p is around 1 in the infrared — it then approaches 2 for longer wavelengths;[5] on the other hand the scattering efficiency is in the Rayleigh regime, $Q_s \propto \lambda^{-4}$.

Now we can define the global absorption opacity

$$k_\lambda = \frac{1}{\rho}\int_{a_{\text{min}}}^{a_{\text{max}}} Q_a \pi a^2 n\,\mathrm{d}a \ , \tag{2.165}$$

the scattering opacity

$$\sigma_\lambda = \frac{1}{\rho}\int_{a_{\text{min}}}^{a_{\text{max}}} Q_s \pi a^2 n\,\mathrm{d}a \ , \tag{2.166}$$

and the mean opacity

$$k = \int \mathrm{d}\lambda\, k_\lambda \tag{2.167}$$

which is the relevant quantity for the momentum equation.

Coming back to concepts from last time: the cross section is given by the product of the geometrical cross section times an adimensional efficiency; we define it both for absorption and scattering.

[4] Do note that this is a latin letter k, while the other opacity is a greek κ.

[5] Actually the exponent is very much dependent on the chemical composition of the dust, this is an average.

$$C^{\mathrm{A,\,S}}(a,\lambda) = \pi a^2 Q^{\mathrm{A,\,S}}(a,\lambda)\,, \tag{2.168}$$

while for the total cross section we have

$$C^{\mathrm{TOT}}(a,\lambda) = \pi a^2 \Big(Q^{\mathrm{A}}(a,\lambda) + Q^{\mathrm{S}}(a,\lambda) \Big)\,. \tag{2.169}$$

We are speaking about dust grains. If $\lambda \gg a$, then the following holds:

$$Q^A \propto a\lambda^{-p}\,, \tag{2.170}$$

where p depends on the grain composition, and approaches 2 for increasing λ for any composition.

At infrared wavelengths, we have $Q^S \sim \lambda^{-4}$, so $Q^S \ll Q^A$ there.

These are the wavelengths we should consider: stars with dust driven winds are cool, with $T_{\mathrm{eff}} \sim 3\,\mathrm{kK}$, so their Planckian peaks are around $1\,\mu\mathrm{m}$.

The values of p are not the result of any theoretical model, they are observed.

2.6.7 Planck mean opacity

We define

$$Q_p^A(a,T) = \frac{\int_0^\infty Q^A(a,\lambda) B_\lambda(T)\,\mathrm{d}\lambda}{\int_0^\infty B_\lambda(T)\,\mathrm{d}\lambda}\,, \tag{2.171}$$

where we use the Planck function $B_\lambda(T)$; with it we define

$$\kappa_p(T) = \frac{1}{\rho}\int_{a_{\mathrm{min}}}^{a_{\mathrm{max}}} Q_p^A(a,T)\pi a^2 n(a)\,\mathrm{d}a\,, \tag{2.172}$$

which is called the Planck mean absorption coefficient. This quantity will enter into the balance between cooling and heating of the single grain. We are effectively doing a mean over energy, since

$$B_\lambda\,\mathrm{d}\lambda = \mathrm{d}E \tag{2.173}$$

for a blackbody.

For hydrodynamical calculations we need to account both for scattering and absorption: then we need some geometric considerations, since there is a strong dependence on the scattering angle. This is the definition of the radiation pressure mean efficiency:

$$Q_{\mathrm{rp}}(a) = \frac{\int_0^\infty \Big(Q^A(a,\lambda) + (1-g_\lambda) Q^S(a,\lambda) \Big) F_\lambda\,\mathrm{d}\lambda}{\int_0^\infty F_\lambda\,\mathrm{d}\lambda}\,. \tag{2.174}$$

Here we are accounting for both the absorption efficiency and the scattering efficiency. Similarly, we can define the radiation pressure mean opacity κ_{rp}:

$$\kappa_{\mathrm{rp}} = \frac{\int_0^\infty \big(\kappa_\lambda + (1-g_\lambda)\sigma_\lambda\big) F_\lambda\,\mathrm{d}\lambda}{\int_0^\infty F_\lambda\,\mathrm{d}\lambda}\,, \tag{2.175}$$

where F_λ is the monochromatic flux: if we have spherical symmetry it is given by

$$F_\lambda = \frac{L_\lambda}{4\pi r^2} . \tag{2.176}$$

Do note that the F_λ is not generally known *a priori*: it is a solution to the radiative transfer equation in the presence of dust.

The parameter g_λ is the *mean cosine* of the scattering angle: if it is equal to 1 we have forward scattering, if it is equal to -1 we have backward scattering, while if it is equal to 0 we have isotropic scattering.

We are shown a plot of g_λ in terms of the wavelength: it decreases from 0.8 to 0 as λ goes from $0.1\,\mu\text{m}$ to $10\,\mu\text{m}$: less energetic photons have little momentum, so they are less likely to keep moving in the same direction they were before, the isotropic thermal motion of the gas can easily sway them.

The temperaure of the grain is determined by the balance of the heating and cooling rates: heating occurs because of collisions with fast-moving gas particles or because of the absorption of radiation.

Cooling, on the other hand, occurs because of collisional energy transfer or by emission of thermal radiation.

We make an approximation: we only consider radiative processes, and estimate the temperature of the dust grain with the radiative equilibrium temperature.

The radiative equilibrium equation is

$$\int_0^\infty \kappa_\lambda B_\lambda(T_d)\,\mathrm{d}\lambda = \int_0^\infty \kappa_\lambda J_\lambda\,\mathrm{d}\lambda \ , \tag{2.177}$$

where $\kappa_\lambda B_\lambda(T_d)$ is the radiative cooling per unit wavelength of the grains, while $\kappa_\lambda J_\lambda$ models the radiative heating:

$$J_\lambda = \frac{1}{4\pi}\int_0^{4\pi} I_\lambda\,\mathrm{d}\Omega = W(r)B(T_{\text{eff}})\,, \tag{2.178}$$

where $W(r)$ is the geometrical dilution factor:

$$W(r) = \frac{1}{2}\left(1 - \sqrt{1 - \left(\frac{R}{r}\right)^2}\right) \sim \left(\frac{R}{2r}\right)^2 \qquad \text{as } r \gg R\,, \tag{2.179}$$

where R is the radius of the star, r is our considered radial position.

This allows us to fix r and compute $T_d(r)$.

In both terms we make the dependence on Q_p^A explicit:

$$\int_0^\infty \pi a^2 Q_p^A(a, T_d) B_\lambda(T_d)\,\mathrm{d}\lambda = \int_0^\infty \pi a^2 Q_p^A(a, T_{\text{eff}}) B_\lambda(T_{\text{eff}}) W(r)\,\mathrm{d}\lambda \ . \tag{2.180}$$

We are assuming that diffusion of heat between gas grains is negligible: we simplify some terms.

$$Q_p^A(a, T_d)\int_0^\infty B_\lambda(T_d)\,\mathrm{d}\lambda = W(r) Q_p^A(a, T_{\text{eff}})\int_0^\infty B_\lambda(T_{\text{eff}})\,\mathrm{d}\lambda \ , \tag{2.181}$$

and we know that the integral of the Planck function is given by $\sigma_{\mathrm{SB}} T^4$, where σ is the Stefan-Boltzmann constant. In the end then our expression is

$$Q_p^A(a, T_d) T_d^4 = T_{\mathrm{eff}}^4 Q^A(a, T_{\mathrm{eff}}) W(r)\,, \tag{2.182}$$

so, far from the star,

$$T_d(r) \sim T_{\mathrm{eff}} \left(\frac{R}{2r}\right)^{1/2} \left(\frac{Q_p^A(a, T_{\mathrm{eff}})}{Q_p^A(a, T_d)}\right)^{1/4} \tag{2.183}$$

or

$$T_d^4 Q_p^A = T_{\mathrm{eff}}^4 Q^A(a, T_{\mathrm{eff}}) W(r)\,. \tag{2.184}$$

2.6.8 Condensation radius and dust temperature

An immediate application is to find the condensation radius: what is the radius at which the temperature becomes low enough so that the grains are not broken up by thermal motion?

We calculate this by substituting the condensation temperature, which is $T_c \sim 1 \div 1.5 \times 10^3\,\mathrm{K}$, in place of the dust temperature T_d.

Then, the only dependence on the radius is in $W(r)$: we are still in the far-from-the-star approximation, so we find:

$$r_c \sim \frac{R}{2} \left(\frac{T_{\mathrm{eff}}}{T_c}\right)^2 \sqrt{\frac{Q_p^A(a, T_{\mathrm{eff}})}{Q_p^A(a, T_c)}}\,, \tag{2.185}$$

and this allows us to see that typically the condensation radius is around $2 \div 4$ star radii, for stars at a few thousand Kelvin.

In order to do this estimate we also need to assume the functional dependence of $Q^A \propto \lambda^{-p}$: then we get

$$T_d = T_{\mathrm{eff}} W(r)^{\frac{1}{4+p}}\,, \tag{2.186}$$

so the condensation radius becomes

$$r_c \approx \frac{R_*}{2} \left(\frac{T_d}{T_{\mathrm{eff}}}\right)^{-\frac{4+p}{2}}\,. \tag{2.187}$$

Tomorrow we will speak of the combined dust & wind equation.

2.6.9 The momentum equations for combined flow

In the treatment of dust driven winds' momentum equations, we must consider grains of different sizes as different fluids, which are dynamically coupled to the gas. To do this, we have to implement some simplifying assumptions:

1. a fixed radius a of dust grains
2. a radial profile of flow velocity for the grains $u(r)$
3. the existence of a drag force $f_{\text{drag}}(r)$: this models the effective drag on the dust due to its coupling to the gas.

In this case the equation for the momentum can be written as

$$u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{GM}{r^2} + Q_{rp}\pi a^2 \frac{L}{cm_d 4\pi r^2} - \frac{f_{\text{drag}}}{m_d}, \tag{2.188}$$

where m_d is the mass of a single grain.

The acceleration of a dust grain is given by

$$g_{\text{rad}} = \frac{\kappa \mathscr{F}}{c}, \tag{2.189}$$

where $\mathscr{F} = L/4\pi r^2$ is the radiative flux, $\sigma = \pi a^2 Q_{rp}$ is the cross section for the absorption of radiation and $\kappa = \sigma/m_d$ is the opacity per unit mass.

The thermal speed of sound is given by

$$\frac{m a_{th}^2}{2} = \frac{1}{2}\mu m_H a_{th}^2 = k_B T, \tag{2.190}$$

we want to study flows for which the drift velocity $w_{\text{drift}} = u - v$ is much higher than this a_{th}; in this definition u is the velocity of the dust, v the one of the gas, while μ is the mean molecular weight: it would be around 2.4 for a solar mixture with molecular hydrogen and stuff, but instead it is $\mu \approx 1.3$ since shocks dissociate hydrogen.

What is the expression of the drag force? we have two limiting cases:

1. if the drift velocity w_{dr} is $\gg$ than the thermal speed a_{th} then the expression is

$$f_{\text{drag}} = \pi a^2 \rho w_{\text{dr}}^2; \tag{2.191}$$

2. if instead $w_{\text{dr}} \ll a_{\text{th}}$, then

$$f_{\text{drag}} = \pi a^2 \rho w_{\text{dr}} a_{\text{th}}. \tag{2.192}$$

These expressions follow from the form of the relative dynamic pressure: ρw_{dr}^2 or $\rho a_{\text{th}} w_{\text{dr}}$ respectively. So, we take the drag force to be an average of these two cases:

$$f_{\text{drag}} = \pi a^2 \rho w_{\text{dr}} \sqrt{w_{\text{dr}}^2 + a_{th}^2}. \tag{2.193}$$

Imposing now that at infinity the velocity gradient is null, we have

$$0 = Q_{rp}\pi a^2 \frac{L}{cm_d 4\pi r^2} - \frac{f_{\text{drag}}}{m_d} \tag{2.194a}$$

$$= Q_{rp}\pi a^2 \frac{L}{cm_d 4\pi r^2} - \pi a^2 \frac{\rho}{m_d} w \sqrt{w_{\text{dr}}^2 + a_{th}^2}, \tag{2.194b}$$

which leads to an expression for the drift velocity: we write it in the limit $w_{\rm dr} \gg a_{\rm th}$, so we can ignore the $a_{\rm th}^2$ in the square root: we get

$$w_{\rm drift} = \sqrt{Q_{rp} \frac{L}{4\pi r^2 \rho c}} \tag{2.195}$$

and we can substitute in $4\pi r^2 \rho$ from the continuity equation:

$$w_{\rm drift} = \sqrt{Q_{rp} \frac{Lv}{\dot{M}c}} \,. \tag{2.196}$$

This makes sense: recall that the meaning of $w_{\rm dr}$ is the *difference* in velocity between dust and gas. So, if the density is high the dust is more tightly bound to the gas. This dependence of the drift velocity on the mass loss rate is confirmed by simulations.

Let us now write the momentum equation for the gas (whose velocity is v):

$$v\frac{{\rm d}v}{{\rm d}r} = -\frac{1}{P}\frac{{\rm d}P}{{\rm d}r} - \frac{GM}{r^2} + n_d \frac{f_{\rm drag}}{\rho} \tag{2.197a}$$

$$= -\frac{1}{P}\frac{{\rm d}P}{{\rm d}r} - \frac{GM}{r^2} + \frac{n_d}{\rho}\frac{Q_{RP}\pi a^2 L}{4\pi r^2 c} \tag{2.197b}$$

$$= -\frac{1}{P}\frac{{\rm d}P}{{\rm d}r} - \frac{GM}{r^2}(1-\Gamma_d) \tag{2.197c}$$

where n_d is the number density of the dust; also, we defined the corrective factor Γ_d:

$$\Gamma_d = \frac{n_d}{\rho}\frac{LQ_{\rm rp}}{4\pi cGM} = \frac{\kappa_{\rm rp}L}{4\pi cGM} \,, \tag{2.198}$$

which encodes the pull against gravity driven by the drag force. The physical interpretation of this solution is that radiation transfers momentum to dust, that can transfer it to the gas, reaching a terminal velocity.

The expression for $f_{\rm drag}$ we used was

$$f_{\rm drag} = \pi a^2 \rho w_{\rm dr}^2 \tag{2.199a}$$

$$= \pi a^2 \rho \frac{Q_{\rm rp}L}{4\pi r^2 \rho c} \tag{2.199b}$$

$$= \pi a^2 \frac{Q_{\rm rp}L}{4\pi r^2 c} \,. \tag{2.199c}$$

We know from previous lectures that mass loss rate is lower for static atmosphere, because the scale factor H_r is very small: the density is small at the dust formation radius. Again, pulsation is an important mechanism which enhances the mass loss rate of dust driven winds.

2.6.10 The supersonic structure of dust driven winds

The continuity equations for the gas and dust respectively read:

$$\dot{M} = 4\pi r^2 \rho v \tag{2.200a}$$

$$\dot{M}_d = 4\pi r^2 n_d m_d u\,, \tag{2.200b}$$

where u is the velocity of the dust, and v is that of the gas.

So, we define the ratio of the dust to gas densities:

$$\delta_{dg} = \frac{n_d m_d}{\rho} = \frac{\dot{M}_d}{\dot{M}} \frac{v}{u}\,, \tag{2.201}$$

where the mass loss rates are roughly constant after the dust has been formed; on the other hand the velocities are variable. At the condensation radius r_s the velocities are equal, so we have $\delta^0_{dg} = \dot{M}_d / \dot{M}$.

We can get a lower limit for the luminosity by the condition that $\Gamma_d > 1$: this is written as

$$\Gamma_d = \frac{n_d \pi a^2 Q_{\text{rp}} L}{\rho 4\pi c G M} > 1 \tag{2.202a}$$

$$L > \frac{4\pi \rho c G M}{n_d \pi a^2 Q_{\text{rp}}} = \frac{4\pi c G M}{k_F}\,, \tag{2.202b}$$

where

$$k_F = \frac{n_d}{\rho} \pi a^2 Q_{\text{rp}}\,. \tag{2.203}$$

Chapter 3

Massive and very massive stars

Mon Dec 09 2019

3.1 Massive stars generalities

Here is a classification of stars' final fates by initial mass M:

1. $0.8M_\odot < M < 2M_\odot$ are low-mass stars: Carbon-Oxygen white dwarves;
2. $2M_\odot < M < 6 \div 8M_\odot$ are intermediate-mass stars: Carbon-Oxygen white dwarves;
3. $8M_\odot < M < 10 \div 11M_\odot$ are quasi-massive stars: Electron Capture SuperNovae or Oxygen-Neon white dwarves;
4. $10M_\odot < M < 30M_\odot$ are massive stars: Core Collapse Supernovae (BH or NS);
5. $30M_\odot < M < 100M_\odot$ are massive stars: Black Holes;
6. $100M_\odot < M < 5 \times 10^4 M_\odot$ are very massive stars: their fate depends on the Helium content in their core:
   (a) $M_{\text{He}} < 65M_\odot$: Pair Instability SuperNovae;
   (b) $65M_\odot < M_{\text{He}} < 133M_\odot$: Pair Creation SuperNovae with no remnant;
   (c) $M_{\text{He}} > 133M_\odot$: Black Holes;
7. $5 \times 10^4 M_\odot < M < 10^5 M_\odot$ are super massive stars: supernova explosion, with no remnant.

For very massive stars, a rule of thumb is $M \approx 2.2M_{\text{He}}$.

Massive stars are pivotal in the description of:

The **chemical evolution of galaxies**: they light up (increase the temperature of) the regions of stellar formation, aiding it; they produce most of the heavy elements; they mix up the interstellar medium; they produce neutron stars and black holes.

The **cosmology** of population III stars: reionization of the early ($z < 5$) universe; massive remnants such as black holes possibly being the progenitors of Active Galactic Nuclei.

In **High energy astrophysics**, the production of heavy and long-lived unstable isotopes which can be observed in Gamma Ray Bursts.

The most important equations for stellar evolution are the usual ones:

- continuity equation
$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi r^2 \rho}\,, \tag{3.1}$$
- momentum equation
$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\,, \tag{3.2}$$
- energy transport equation
$$\frac{\mathrm{d}L}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}\frac{T}{P}\nabla\,, \tag{3.3}$$
where ∇ has contributions from convection, encoded in ∇_{ad}, and from radiative transport:
$$\nabla_{\mathrm{rad}} = \frac{3\kappa}{16\pi acG}\frac{L_r P}{mT^4}\,, \tag{3.4}$$
- energy conservation equation
$$\frac{\mathrm{d}L}{\mathrm{d}m} = \epsilon_{\mathrm{nuc}} - \epsilon_\nu - T\frac{\mathrm{d}s}{\mathrm{d}t}\,, \tag{3.5}$$
- chemical species' evolution equation:
$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = A_i \frac{m_u}{\rho}\sum_j \left(r_{ji} - r_{ij}\right)\,. \tag{3.6}$$

and we need constitutive relations for the following parameters:

- pressure $P(\rho, T, X_i)$;
- opacity $k(\rho, T, X_i)$;
- energy production $\epsilon(\rho, T, X_i)$,

where ρ is the density, T is the temperature and X_i are the chemical species' mass fractions.

Hereafter we consider very massive stars ($M > 80M_\odot$), laying in the main sequence of H-R diagram. In the first time of their life, these stars burn hydrogen in the CNO cycle. When H ends up in the core, it starts the burning of helium in the triple alpha reaction, producing carbon. Note that in this reaction thermal (i.e. weak-interactions origined) neutrinos are produced. Then it starts the synthesis of oxygen from carbon, neon from oxygen, magnesium from neon, silicium and iron from magnesium.[1] These elements lay in the star having the heaviest in the core and the others in order of mass.

[1]Che al mercato mio padre comprò.

They are elements with mass numbers 1, 4, 12, 16, 20, 24, 28: if helium is present, they are created by adding an α particle each time. Beryllium ($A = 8$) is missing: this is because ^{8}Be is unstable, so the thing which actually happens is that while it is formed an other α particle comes by and it makes an excited state of ^{12}C which is close in energy.

We said before that pressure is one among the most important variables we shall consider. Its dependence on some other thermodynamical variables is strongly affected by the electron characterization:

$$P_R = \frac{aT^4}{3} \qquad \text{in the relativistic case} \tag{3.7a}$$

$$P_{\text{class}} = \frac{R\rho T}{\mu} \qquad \text{in the classical perfect-gas case} \tag{3.7b}$$

$$P_{\text{deg,NR}} = \rho^{\frac{5}{3}} \qquad \text{for a degenerate gas of non relativistic electrons} \tag{3.7c}$$

$$P_{\text{deg,R}} = \rho^{\frac{4}{3}} \qquad \text{for a degenerate gas of relativistic electrons} \tag{3.7d}$$

The main nuclear stages of stellar evolution are

1. core nuclear burning, moving to;
2. nuclear fuel exhaustion, moving to;
3. core contraction, moving to;
4. core heating, back up.

When we get to iron, this process stops. The contraction raises the temperature: the two are related by a law like $T_{\text{core}} \propto \rho_{\text{core}}^{1/3}$.

3.1.1 Neutrino production

Now we consider neutrino producion in this phase of the star, since it has been shown that $\epsilon_{\text{nuc}} \approx \epsilon_\nu$, i.e. almost all the power due to nuclear reactions is brought away by neutrinos. For strong-interaction born neutrinos we have that the free path is

$$l_\nu = \frac{1}{\sigma_\nu n} = \frac{\mu m_u}{\rho \sigma_\nu} \approx 3000 R_\odot \,, \tag{3.8}$$

since $\sigma_\nu \approx 10^{-48}\,\text{m}^{-2}$. We are assuming $\mu \sim 1.3$, and $\rho \sim 10^6\,\text{g/cm}^3$.

This means that neutrinos in average do not interact with the star and can not give back energy to it. This explains the strong neutrinos luminosity (higher than electromagnetic) of almost all the stars, and the consequent mass loss rates.

The main mechanism that can produce weak-interaction or thermal neutrinos are the following:

- photo-neutrinos production ($\gamma + e^- \to \nu + \bar{\nu} + e^-$) happens at high temperatures ($T > 2 \times 10^8\,\text{K}$) — it is basically a Compton process;

- pair-neutrinos production ($e^+ + e^- \to \nu + \bar{\nu}$) happens at very high temperatures ($T \gtrsim 10^9\,\mathrm{K}$);
- plasma-neutrinos production ($\gamma \to \nu + \bar{\nu}$) happens for degenerate matter with very high density;
- bremsstrahlung-neutrinos production (similar, $\gamma \to \nu + \bar{\nu}$) happens if we have low temperatures but very high density.

In general these processes are very rare in nature, since the probability of emitting a neutrino/antineutrino pair instead of a photon in any process which generally emits a photon is:

$$\frac{\mathbb{P}(\nu\bar{\nu})}{\mathbb{P}(\gamma)} = 3 \times 10^{-18} \left(\frac{E_\nu}{m_e c^2} \right)^4 \tag{3.9}$$

but in high-temperature situations ($E_\nu > 511\,\mathrm{keV}$) the quantity in parentheses can become larger than 1: the process becomes relevant. Neutrinos produced anywhere in the core essentially amount to lost energy: if the temperature of the core exceeds $T_c = 5 \times 10^8\,\mathrm{K}$, the luminosity of neutrino emission exceeds the electromagnetic one.

For this reason neutrinos are important for the core-cooling process and the speed-up of core reactions.

As we said, during nuclear burning all the (specific) energy produced is brought away by neutrinos, $\epsilon_{\mathrm{nuc}} \approx \epsilon_\nu$: each elements's burning has a characteristic curve $\epsilon(T)$: these are quite steep, and at a given ϵ we need higher and higher temperatures as we reach heavier elements. The neutrino curve $\epsilon_\nu(T)$ instead is shallower, and intersects them all: the intersections define the burning temperatures.

The nuclear timescale is much shorter this way: it is given by

$$\tau_{\mathrm{nuc}} = \frac{E_{\mathrm{nuc}}}{\dot{E}_{\mathrm{nuc}}} \approx \frac{E_{\mathrm{nuc}}}{L_\nu} \ll \frac{E_{\mathrm{nuc}}}{L}\,, \tag{3.10}$$

since $L_\nu \gg L$. If the cooling happened only electromagnetically the star would need a much longer time to burn.

Burning stage	Dominant Process	T_c [keV]	ρ_c [g/cm^3]	L_γ [$10^4\ L_\odot$]	L_ν / L_γ	Duration [yr]
Hydrogen	H $\to$ He	3	5.9	2.1	0.0	1.2×10^7
Helium	He $\to$ C, O	14	1.3×10^3	6.0	1.7×10^{-5}	1.3×10^6
Carbon	C $\to$ Ne, Mg	53	1.7×10^5	8.6	1.0	6.3×10^3
Neon	Ne $\to$ O, Mg	110	1.6×10^7	9.6	1.8×10^3	7.0
Oxygen	O $\to$ Si	160	9.7×10^7	9.6	2.1×10^4	1.7
Silicon	Si $\to$ Fe, Ni	270	2.3×10^8	9.6	9.2×10^5	1.6×10^{-2}

Figure 3.1: Burning stages of a $15 M_\odot$ star, with neutrino and electromagnetic luminosities. Iron is the last to appear since it is at the peak of the binding energy per nucleon curve.

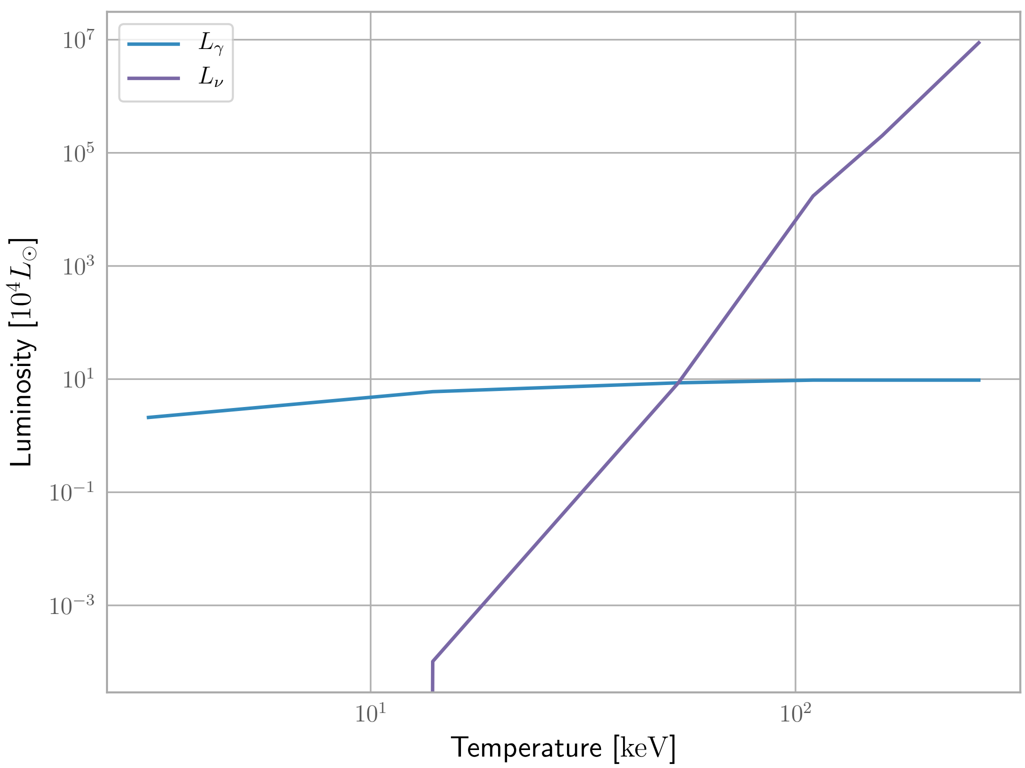

Figure 3.2: Electromagnetic and neutrino luminosities as the temperature increases.

So, as the later stages of stellar evolution come about, the temperature of the core increases, then neutrino production becomes much more relevant, more energy is lost to it, and therefore the last stages of stellar evolution are much more short-lived than they would be otherwise.

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We continue with the evolution of a $15M_\odot$ star. Due to the mechanism of neutrino production described last lecture, after a few tens of thousands of years after the start of carbon burning we have an iron core, since the burning was sped up. The evolution of the envelope is decoupled from that of the core: the process is too fast to affect it. Therefore, the position of the star in the HR diagram is unchanged.

We use Kippenhahn diagrams: in these, we have age on the x-axis and the mass coordinate on the y-axis. We can then show how the placement of the different regions of the star's interior evolves. We can, for example, have the negative logarithm of the time until the core's collapse of the star on the x-axis. There can be different burning stages simultaneously, in different shells.

3.1.2 Stellar winds in massive stars

Stellar winds are of general application in the study of stellar evolution.

We have **radiation driven winds** for OB-type stars and Blue Supergiants: these are very hot massive stars. So, we have lots of UV absorption from resonance transitions in the spectral lines of Fe, O, Si, C.

This leads to high asymptotic velocities $v_\infty \sim 4 \times 10^3\,\mathrm{km/s}$ and high mass loss rates $\dot{M} \sim 5 \times 10^{-5} M_\odot/\mathrm{yr}$. There is a dependence on metallicity like $\dot{M} \sim Z^{0.6}$. CAK theory deals well with these.

For red supergiants, instead, we have **dust driven winds**: the mass loss rates can be very high, as much as $10^{-4} M_\odot/\mathrm{yr}$. Stars with masses lower than $40 M_\odot$ spend a long time in the core-Helium burning phase as red supergiants: they have time to shed their outer layers; they are then observed as Wolf-Rayet stars, which are the very hottest.

3.1.3 Wolf-Rayet stars

The Humphrey-Davidson limit

As discussed in the last section, very luminous stars shed their outer layers: this is a plausible explanation of the empirical *Humphrey-Davidson Limit*, an upper limit on the luminosity of red supergiant stars: it is found at around $M_{\mathrm{bol}} \approx -9.5 \div -10$ for stars cooler than 15 kK.[2]

The relation between bolometric magnitude and visual magnitude is:

$$M_{\mathrm{bol}} = -2.5 \log L + 4.73\,, \tag{3.11}$$

where the luminosity is measured in solar luminosities. Then, the limit can be stated as saying that there are no red supergiants with *visual* luminosities larger than $10^{5.8} L_\odot$.

This limit depends on the effective temperature of the star, and it is a sharper bound than the Eddington luminosity.

The Eddington luminosity is calculated by setting

$$\Gamma_{\mathrm{Edd}} = \frac{a_{\mathrm{rad}}}{g} = \frac{\kappa L_{\mathrm{Edd}}}{4\pi R^2 c} \frac{R^2}{GM} = 1\,, \tag{3.12}$$

which implies

$$L_{\mathrm{Edd}} = \frac{cGM4\pi}{\kappa_e}\,. \tag{3.13}$$

It can be interpreted as a generalized Eddington limit: if the luminosity is higher the outer layers of the star are shed. Stars near the limit are unstable: they periodically undergo intense mass loss events, and are known as Luminous Blue Variables.

[2] http://articles.adsabs.harvard.edu/pdf/1979ApJ...232..409H

Wolf-Rayet stars

Wolf-Rayet stars have high effective temperatures ($T > 10^{4.8}$ K) and high luminosities, they expel a great quantity of mass through stellar winds.

The spectra of WR stars exhibit emission lines: we see little or no H, and an abundance of either He + N or C + O.

We distinguigh them into

1. WNL stars have some hydrogen on the surface (but its mass fraction is lower than 0.4) and lots of H and Ne;

2. WNE are similar but with no hydrogen;

3. WC stars have no hydrogen, little N and more He, C and O;

4. WO stars are similar but with more oxygen.

These are actually evolutionary stages: the outer layers are successively stripped by winds, and the inner ones are exposed.

We observe lots of ^{14}N in WNE stars, since the Nitrogen burning phase is the slowest process.

The ratio of ^{12}C to ^{16}O is heavily dependent on the mass loss rate: it increases with the mass loss rate.

3.1.4 Final fates by mass

Depending on the mass of the progenitor red supergiant, we have different behaviours: below $30M_\odot$ the star explodes as a red supergiant; more massive stars will experience more mass loss and explode as blue supergiants, further left on the HR diagram.

Here is the general progress of the evolution of stars with $M > 30M_\odot$:

1. $30M_\odot \lesssim M \lesssim 35M_\odot$ stars go from Red SuperGiants to WNL stars;

2. $35M_\odot \lesssim M \lesssim 40M_\odot$ stars go from Red SuperGiants to WNL stars to WNE stars;

3. $40M_\odot \lesssim M \lesssim 60M_\odot$ stars go from Red SuperGiants to WNL stars to WNE stars to WCO stars;

4. $60M_\odot \lesssim M$ stars go from WNL stars to WNE stars to WCO stars;

We see a simulation: a $15M_\odot$ star starts off on the high part of the Main Sequence, then quickly moves right becoming a red supergiant when its core collapses.

A $3M_\odot$ star instead moves away from the MS more slowly, and then after having moved right it quickly moves far left and cools, as a white dwarf.

3.2 The collapse of the iron core

Now, let us discuss the engine of the explosion of these massive stars. The explosion is triggered by an implosion: we are at the end of the Silicon-burning phase. The pressure of the iron core is mantained by the degenerate electrons.

The degenerate iron core starts off with density $\rho \approx 10^9\,\mathrm{g cm^{-3}}$, temperature $T \approx 10^{10}\,\mathrm{K}$, mass $M_{\mathrm{Fe}} \approx 1.5 M_\odot$ and radius $R \approx 8000\,\mathrm{km}$.

What triggers the collapse of the iron core?

In order to study its dynamical instability, we nee to look at the adiabatic exponent:

$$\gamma_{\mathrm{ad}} = \left(\frac{\partial \log P}{\partial \log \rho} \right)_{\mathrm{ad}} , \tag{3.14}$$

we have a dynamic instability when γ_{ad} falls below the critical value of $4/3$. It is a local quantity, it can be defined for each layer; we can use a global parameter to measure the instability:

$$\int \left(\gamma_{\mathrm{ad}} - \frac{4}{3} \right) \frac{P}{\rho} \, \mathrm{d}m \; , \tag{3.15}$$

which will tell us whether the system as a whole is stable. This is effecively an integral over the work of the pulsation: $\rho^{-1}\,\mathrm{d}m = \mathrm{d}V$, so $P/\rho\,\mathrm{d}m = \mathrm{d}W$.

We have two processes which are opposed: the first is **photodisintegration**, a process like

$$^{56}_{26}\mathrm{Fe} + \gamma \to 13\,^{4}_{2}\mathrm{He} + 4n \, , \tag{3.16}$$

in which a heavy nucleus is shattered into α particles. This is an *endothermic* process.

The other process is **electron capture**, or *neutronization*: $\mathrm{p}^+ + \mathrm{e}^- \to \mathrm{n} + \nu_e$. This is a weak-interaction process. This increases the mean molecular weight per electron μ_e, so the Chandrasekar mass decreases since it is inversely dependent on μ_e: it becomes

$$M_{\mathrm{ch}} = \frac{5.83}{\mu_e^2} \sim 1.26 M_\odot \tag{3.17}$$

for the iron core. A huge amount of neutrinos are produced.

Are these actually opposed? it seems like they both act to speed up the collapse; this is what is written in Janka et al: "Electron capture, β decay, and partial photodisintegration of iron-group nuclei to alpha particles cost the core energy and reduce its electron density. As a consequence, the collapse is accelerated."

3.2.1 The steps of the collapse

Now we follow a work by Janka et al (2007) at the Max Planck, which outlines the steps of the collapse.

The first step in the collapse happens after the iron core, which is only supported by electron degeneracy pressure since it cannot fuse iron esothermally, grows past its Chandrasekar mass, which as we have discussed can decrease by several mechanisms. Then, gravity overcomes this pressure and forces *electron capture* to happen. This

As the density of the core increases, it becomes opaque to neutrinos: their scattering cross section is

$$\sigma_\nu \approx 10^{-49} A^2 \left(\frac{\rho}{\mu_e} \right)^{2/3} \mathrm{cm}^2 \,, \tag{3.18}$$

so when their diffusion time becomes longer than the time of the collapse they do not have time to escape, on average. Here A is the mass number of the relevant nucleus, ρ is the density while μ_e is the mean molecular weight per electron. We have the formation of a shell below which neutrinos are trapped and above which they escape.

The formula as given is dimensionally inconsistent, I posit that it is to be interpreted ignoring the dimensionality of ρ.

In the meantime the collapse is proceding *homologously* (the different layers are falling simultaneously), until it reaches a density of around $\rho \sim 10^{14}\,\mathrm{gcm}^{-3}$: this is comparable to the density of the nucleus, which is around $2 \times 10^{14}\,\mathrm{gcm}^{-3}$: at this point the material becomes incompressible, since nuclear matter is much less compressible than the gas of electrons, protons and neutrons we had before.

A proto-neutron star is formed, and a shock front travels back towards the outer layers of the infalling iron core.

It is unlikely that the energy of the shock is already enough to blow off the envelope. This would be called the **prompt mechanism**: all modern simulations show it is not actually effective. This is because there are several endothermic or effectively endothermic processes taking place:

1. the dissociation of heavy nuclei into nucleons: the binding energy of iron is around $B \sim 492\,\mathrm{MeV}$, if we had a $1.4 M_\odot$ iron core then the energy needed to break up all of the iron nuclei would be

$$E = B \frac{1.4 M_\odot}{m_{\mathrm{Fe}}} \approx 24\,\mathrm{foe}\,, \tag{3.19}$$

   where m_{Fe} is the mass of an iron nuclide, and $1\,\mathrm{foe} = 10^{51}\,\mathrm{erg}$ (ten to the Fifty-One Erg);

2. neutrino prodution: the energy emitted in the form of neutrino radiation is essentially lost to the star, although the energy of the delayed neutrinos which are still trapped in the nucleus will contribute.

An alternative, more plausible mechanism driving the explosion is the **delayed neutrino-heating mechanism**: the shockwave travelling outward after the infalling matter hits nuclear density would stall under the inward pressure of the infalling material, but it is *revived* by the neutrino flux: these neutrinos carry most of the energy which was converted from gravitational potential energy.

These neutrinos keep streaming; when they are still inside they are degenerate and thus energetic, as they stream outward they are down-scattered in energy space.

As they are heated by neutrinos, the layers between the forming remnant and the shock front expand, creating a region of high temperature and low density. This process then keeps driving the full explosion.

Do note that the neutrinos need not convert all their energy to heat in this process: in simulations they convert to thermal energy around 10 % to 20% of the radiated neutrino energy. The supernova does not need a large fraction of the gravitational binding energy of the iron nucleus in order to explode.

The time needed to reach this stage from the start of the collapse is on the order of 10^{-1} s.

3.2.2 Advanced stellar burning phases and mass of the iron core

Now we start a new part: the advanced burning stages, after the He-burning phases.

There are four major burning phases:

1. Carbon;
2. Neon;
3. Oxygen;
4. Silicon.

The central burning region forms a convective core. Burning in external shells forms convective shells. After a certain type of fuel is locally exhausted, the burning shifts outward. This gives rise to a so-called "onion-skin" model.

As the initial mass of the star increases, the fraction of mass in the CO core decreases from around 0.35 to 0.15 as M goes from 10 to 120 solar masses. So, the mass of the CO core scales with the initial mass, but sublinearly so.

The mass of the iron core right before the SN explosion also increases with the stellar mass: as M goes from 10 to $120 M_{\odot}$ it goes from 1.2 to 1.8 solar masses. So, the higher the mass of the CO core was, the more compact is the structure of the presupernova stage.

More specifically the mass of the iron core M_{Fe} is between 1.2 and $1.45 M_{\odot}$ for a total initial mass below $40 M_{\odot}$, while for $M > 40 M_{\odot}$ we have M_{Fe} between 1.45 and $1.8 M_{\odot}$.

Hold on: is the mass of the iron core right before the explosion just the Chandrasekar mass? I'd think that, as long as there is silicon left to burn, iron would keep on being formed, until it reaches the Chandrasekar mass…

As the total mass increases, at a fixed radius we have more internal mass: the density *increases* with mass.

Explosion and fallback

Some of the material near the core, when the shock occcurs, falls back on it if it is inside a certain critical radius, if it is outside that radius it is thrown out. How much of it falls back depends on the physical properties of the core. For an average supernova, the energy of the matter thrown into the interstellar medium is of the order of 1 foe.

So, the questions we ask are these:

1. Do all massive stars actually explode?
2. What are their remnants?
3. What is the efficiency of the fall back?

The remnant can either be a black hole or a neutron star. If the fallback is very efficient we have an *implosion*, if it is inefficient we have an *explosion*. The possibilities are

1. Explosion and Neutron Star;
2. Implosion and Black Hole;
3. (rarely) Explosion and Black Hole.

There is no clear-cut law or mass thereshold. One thing to look at is the **bounce-compactness parameter**:

$$\xi_{M^*} = \left. \frac{M^*/M_\odot}{R(M^*)/10^7\,\mathrm{m}} \right|_{t=t_{\mathrm{bounce}}} : \tag{3.20}$$

it seems like this parameter's value being high is correlated to a black hole being formed, while it being low is correlated with succesful explosions. We usually look at $\xi_{2.5}$, which means we are basically considering the inverse of the radius of the region encompassing $2.5M_\odot$.

$\xi_{2.5}$ is a measure of compactness: the time-scale of black hole formation scales like $t_{\mathrm{BH}} \propto \xi_{2.5}^{-3/2}$, so if $\xi_{2.5}$ is high the time for black hole formation is short: if the shock revitalization time is held constant then we can have implosion into a BH if $t_{\mathrm{BH}} < t_{\mathrm{shock}}$, and explosion into a SN plus a neutron star remnant otherwise.

In the slides it is stated: "Successful SN: the shock is revived on a time longer than the time-scale for a BH formation.", but it seems to me like it is the other way around.

We can plot $\xi_{2.5}$ as a function of the Zero-Age Main Sequence (so, initial) or ZAMS mass of the star: it has peaks around $23M_\odot$ and $40M_\odot$.

This variable is effectivey equal to

$$\xi_{2.5} = 6772.2 \times \frac{G(2.5M_\odot)}{c^2 R}\,, \tag{3.21}$$

so it measures how relativistic the core is. Generally, $\xi \sim 0.5$, so it is not very relativistic.

A slightly more sophisticated approach is a two-parameter one: here we use the **normalized mass** M_4 and the **mass derivative** μ_4:

$$M_4 = \frac{m(s = 4)}{M_\odot} \tag{3.22a}$$

$$\mu_4 = \left. \frac{\mathrm{d}m \,/ M_\odot}{\mathrm{d}r \,/ 10^7\,\mathrm{m}} \right|_{s=4}, \tag{3.22b}$$

the normalized mass inside a dimensionless entropy per nucleon of $s = 4$.

These variables are meaningful because they describe the accretion rate of the proto-neutron star: $\dot{M} \propto \mu_4$ and the neutrino luminosity: $L_\nu \propto \mu_4 M_4$.

So, on the M_4 versus $M_4\mu_4$ plane there is a linear separation line, since at a given accretion rate $\dot{M}$ there is a critical neutrino luminosity L_ν above which we have a SN explosion.

3.2.3 The final fates of stellar layers

We can make a plot of the various final fates of the parts of the initial mass: some of it is expelled by winds, some of it is expelled through the supernova explosion, some of it ends up in the final remnant.

In relatively low-mass stars (20 to $30 M_\odot$) most of the ZAMS mass is expelled in the SN explosion.

NL00 models This is a model by Nugis and Lamers,[3] which considers a star with solar metallicity and a 1 foe SN explosion.

As the ZAMS mass increases, both the mass fraction expelled by the wind and the mass fraction left in the remnant increase.

The mass of the iron core is never very much larger than a couple of solar masses, but BHs can become more massive through fallback.

Then it seems that WR stars of more than $30 M_\odot$ will not explode: they will not eject material as supernovas.

3.2.4 Supernova taxonomy

Supernovae are classified based on whether they exhibit Hydrogen, Helium or Silicon spectral lines.

The classification is:

1. No Hydrogen:
    (a) Yes Silicon: Type Ia;
    (b) No Silicon:

[3] http://adsabs.harvard.edu/full/2000A%26A...360..227N

i. Yes Helium: Type Ib;
ii. No Helium: Type Ic;

2. Yes Hydrogen: type II.

Types Ib, Ic and II are Core Collapse Supernovae, while type Ia are thermonuclear explosions: a white dwarf in a binary attains the necessary conditions to reignite Carbon fusion.

The threshold between types Ib/Ic and type II is around $30 \div 35 M_\odot$. Below the threshold we have type II Supernovae.

So, type Ib/c supernovae were Wolf-Rayet stars before their explosion, while type II supernovae still had hydrogen burning outer shells.

These hotter supernovae of types Ib/c have large fallback, so they do not largely contribute to the metallicity of the interstellar medium. Typically, more massive stars are rarer: if we use the Salpeter Initial Mass Function to approximate the mass distribution of these stars we can estimate to see around 4.5 times as many SNII as SNIb/c.

The threshold between neutron star and black Hole formation is around 25 to $30 M_\odot$.

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The main uncertainties in SN theory

There are still many uncertain areas, both in the presupernova phase and about the explosion:

Presupernova phase The mass loss of stars in their Red and Blue Supergiant phases is unclear.

Convection is not well understood, especially when it is heavily coupled to nuclear burning.

The cross section of the reaction $^{12}C(\alpha, \gamma)^{16}O$[4] is unclear.

The effects of rotation are not well understood: it can cause convective overshooting, which means that bits of a convective layer have enough momentum to be "shot into" another layer.

Explosion In a binary system things are more complicated.

We do not have a self-consistent hydrodinamical model which includes neutrino transport.

Impact of the chemistry of a WR star on the fallback

The models from Langer in 1989 differ from those of Nugis and Lamers (NL00): Langer predicts a higher mass loss rate, which as was discussed in the treatment of WR stars entails a higher ^{12}C fraction and a lower overall mass of the CO core.

[4]This is a shorthand notation used in nuclear physics: $a(b,c)d$ means $a + b \to c + d$.

So, the core is less compact and has a lower binding energy in the models by Langer: the binding energy of the core does *not* increase monotonically with the ZAMS mass of the star, it has a peak of around $B \sim 6\,\text{foe}$ around $M_{\text{ZAMS}} \sim 30 M_\odot$ and then slightly decreases, while in the NL00 models B increases steadily up to 20 foe and more as M_{ZAMS} increases.

So, the nuclei in the models by Langer are less compact, therefore we predict less efficient fallback and formation of Neutron Stars as opposed to Black Holes.

These high-ZAMS-mass supernovae match well the observed type Ib/c supernovae.

As it turns out, both the models by Langer (LA89) and by Nugis-Lamers (NL00) are useful: LA89 describes stars with heavy mass loss in the pre-SN stages, while NL00 describe stars with less mass loss in the pre-SN stages, which are more likely to collapse into BHs for high initial mass.

LA89-type massive $M > 40 M_\odot$ stars generally have higher yields of heavy elements than NL00-type stars.

As the metallicity increases, the mass range resulting in failed supernovae (so, the resulting BHs) decreases.

3.3 Very massive stars

3.3.1 Generalities

Now, we discuss the evolution and final fate of very massive stars. They have masses in the range $100 M_\odot < M_{\text{in}} < 5 \times 10^4 M_\odot$. For these, we do not have a core collapse, but a thermonuclear explosion instead.

There is some evidence for the existence of these stars, for example in the cluster R136: the star R136a1 has an estimated mass of $315 M_\odot$, the highest known of any star.

The cluster R136 seems to be very young, with an age ($\lesssim 2\,\text{Myr}$).

In general, the fate of a (not necessarily massive) star is determined by its initial mass, chemical composition, and rotation rate. For very massive stars, the most relevant quantity is the mass of the core of Helium and other heavy elements, commonly denoted as "the Helium core", whose mass is determined by the mass loss rate throughout the star's evolution.

Astrophysical relevance of PCSNe In 2007 a Super Luminous supernova (SLSN) was observed: SLSNe can have luminosities brighter by a factor of 10 than the regular supernovae, reaching an absolute magnitude of more than $-21\,\text{mag}$ (the Sun's is $4\,\text{mag}$!).

Spectral analyses show that in the explosions of these stars we have a hefty production of Nickel. Beyond the increased luminosity, the light curve of these SLSNe also decreases slower than that of regular Core Collapse Supernovae — CCSNe.

Metallicity dependence We will need to understand under which metallicity regime we can have these SLSNe: there seems to be a metallicity threshold, since PCSNe evolve from very massive stars, as long as they avoid heavy mass loss. We have $\dot{M} \propto Z$, so if the metallicity is too high the star loses too much mass to produce a PCSN. The threshold is denoted as Z_{PCSN}. It seems like the threshold is around $Z_{\text{PCSN}} \sim 0.006 = Z_\odot / 2$.

Most of the observed PCSN are in regions with positive metallicity, for example SN 2007bi exploded in a galaxy with $Z = Z_\odot/3$.

These might be the first contributors to the presence of metal in the universe,[5], but measured abundances of elements in the early universe do not seem to match the simulated yields of PCSNe.

3.3.2 The engine of PCSNe

Pair Creation SNe occur when the temperature of the core is such that a large fraction of the energy goes into producin electron-positron pairs. This is called the Pair Instability.

This significantly decreases the pressure of the core, causing a contraction, followed by a thermonuclear explosion: formally, the pair production makes $\gamma_{\text{ad}} < 4/3$, which causes a gravitational collapse.

To see why this is the case, one can look at the LAWE: (1.88), if $\Gamma_1 = \gamma_{\text{ad}}$ is $< 4/3$ the differential equation for the radial perturbation ζ becomes a harmonic repulsor.

An important parameter in this process is the mass of the helium core.

We'd expect more and more massive stars to lose more and more mass loss through winds.

In order to have pair instability, we need a Helium core with mass such that $40M_\odot < M_{\text{He}} < 133M_\odot$.

If $M_{\text{He}} < 65M_\odot$ then we have Pulsation pair instability supernovae, above this we have Pair creation supernovae.

The star is able to eject most of the material in a series of pulses even if the nuclear energy is smaller than the binding energy: these are *pulsation* PISNe, while if we have $E_{\text{nuc}} > E_{\text{bind}}$ right away then it is a regular PCSN.

The pulses are usually separated by time intervals on the order of 10^5 s to 10^{11} s.

The pair creation triggers the collapse. Then, because of the heating induced by the released gravitational energy, we have a runaway thermonuclear reaction. If this exceeds the binding energy of the star, then we have complete disintegration.

For $M_{\text{He}} > 133M_\odot$ the infall produces a BH.

Radiation pressure Very massive stars are dominated by radiation pressure: we have the relation

$$\beta^{1/2}(1+\beta)^{3/2} \propto M\,, \tag{3.23}$$

where $\beta = P_{\text{rad}}/P_{\text{tot}}$. This relation for β looks complicated but if we plot it it is roughly linear.

Instead of P_{rad} in the slides we have P_{gas}, but from the context this is probably what was meant.

Radiation pressure scales like $P_{\text{rad}} = aT^4/3$, where $a = 4\sigma/c$ is the radiation constant.

[5]The Black Sabbath of the cosmos

The core's temperature T_c scales like

$$T_c \propto M^k \rho_c^{1/3}\,, \tag{3.24}$$

where ρ_c is the core density, M is the total star mass and k is a parameter depending on the equation of state: for an ideal gas it is $2/3$, if we only have radiation pressure it is $1/6$.

Pair instability occurs in the Oxygen burning phase, when the temperature increases beyond $10^9\,\mathrm{K}$.

The energy for a photon to decay into an electron-positron pair is $1\,\mathrm{MeV} \sim 10^{10}\,\mathrm{K}$, so when the temperature reaches values of the order $10^9\,\mathrm{K}$ the $> 1\,\mathrm{MeV}$ tail of the Planckian becomes significant.

At $T = 10^9\,\mathrm{K}$ the tail accounts for just $0.2\,\%$ of the flux, but already at $T = 2 \times 10^9\,\mathrm{K}$ it is already almost $15\,\%$ of the flux, and at $T = 3 \times 10^9\,\mathrm{K}$ it is $41\,\%$ of the flux.

The free photons and the $e^+ - e^-$ pairs reach equilibrium, decreasing the pressure (and so, the adiabatic index).

If this happens in a large enough region it can trigger an instability.

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The critical parameter governing if and how this instability gives rise to an explosion is the mass of the heavy (Helium) core.

The number of pulsation instabilties decreases as the He core mass increases. After $60 M_\odot$ only one pulsation is needed in order to blow off the whole core.

Typically the binding energies of the outer layers of very massive stars is rather low, on the order of $10^{48}\,\mathrm{erg}$ to $10^{49}\,\mathrm{erg}$, while a pulse generally has energies of the order $10^{49}\,\mathrm{erg}$ to $10^{51}\,\mathrm{erg}$.

After each of these pulses, the star contracts again and the process can repeat, if the temperatures reached are high enough. If it cannot start the pulsation again, it keeps evolving until it forms an iron core and undergoes a regular core collapse SN explosion.

The time between pulses increases dramatically as M_{He} increases; also the number of pulses decreases.

3.3.3 The population of PCSNe

People generally focus on low metallicity, very massive stars in order to study PCSNe, since we need low metallicity in order for the star not to lose too much mass and still have a massive enough He core.

So, people usually studied Population III stars, but recent developments show that even stars with a metallicity as high as $Z_\odot/3$ could be candidates for PCSN explosions; so based on stellar population models we should expect 1 supernova in 10^5 to be a PCSN in the local universe.

The super-luminous SNe observed in the local universe could possibly be PCSNe.

Many factors affect the growth of the oxygen core:

1. the rate of the nuclear reaction ${}^{12}\mathrm{C}(\alpha,\gamma)^{16}\mathrm{O}$, which affects the C/O ratio at the end of helium burning;
2. convective overshooting reduces the minimum mass for a PCSN to around $100 \div 120 M_\odot$;
3. rotation-induced chemical mixing also reduces the minimum stellar mass needed in order to have a PCSN;
4. rotation decreases the binding energy: it increases the maximum mass needed in order to have a PCSN.

3.3.4 Pair Creation Supernovae: the dynamics of the explosion

If $60 M_\odot \lesssim M_{\mathrm{He}} \lesssim 130 M_\odot$ a single pulse can completely disrupt the star.

Plots of the nuclear binding, gravitational binding and kinetic energies are useful to understand the process.

As soon as carbon burning ends, the nuclear binding energy sharply decreases because of explosive O and Si burning, making the star globally unbound.

The kinetic energy released by the explosion of a star with a global mass of $250 M_\odot$ is of the order of 44 foe; the ejected material reaches asymptotic velocities of the order $0.017c$.

3.3.5 Chemical Yields of the explosion

If we increase the initial mass of the star, the amount of Nickel produced increases: let us fix the metallicity as $Z = 0.001$; then, if the starting mass is $M = 150 M_\odot$ then the Nickel mass produced is $0.04 M_\odot$; if the starting mass is $M = 250 M_\odot$ then the Nickel mass produced is $19.3 M_\odot$.

We can plot the production factor of various nuclides relative to the solar rate of production: as the isotope charge number varies: we plot $\log(\mathrm{prod}/\mathrm{solar\ prod})$. If the starting metallicity of the star increases, the production rate of these elements decreases. Also, if the starting mass of the star increases these yields increase.

We see a zig-zag pattern: a large difference between even and odd nuclides, even nuclides are produced more. This is because of the lack of neutrons, which is quantified by the neutron excess parameter:

$$\eta = \sum_i \frac{(N_i - Z_i) X_i}{A_i}\,; \tag{3.25}$$

in PCSNe η is lower than in CCSNe, since the cores of the stars undergoing PCSN explosions are much less dense than those undergoing CCSN explosions, so neutronization is much less relevant.

Lots of Oxygen is left unburnt and thus is ejected in the explosion.

The amount of ejected material beyond iron is low.

The production of Nickel, mentioned above, is important because of the decay chain Ni $\to$ Co $\to$ Fe, which could explain the shape of the decaying light curve of the supernova.

The reactions, more specifically, are

$$^{56}\mathrm{Ni} + e^- \to {}^{56}\mathrm{Co} + \nu + \gamma \qquad t_{1/2} \approx 6.1\,\mathrm{d} \tag{3.26a}$$

$$^{56}\mathrm{Co} + e^- \to {}^{56}\mathrm{Fe} + \nu + \gamma \qquad t_{1/2} \approx 77\,\mathrm{d}\,, \tag{3.26b}$$

which matches well the exponentially decaying light curve after $50 \div 100\,\mathrm{d}$.

This constrains the amount of Nickel produced: for the lightcurves of CCSNe it is around $0.07 M_\odot$.

3.3.6 PISN and PCSN contribution to the chemical evolution of galaxies

We want to invesigate how relevant these SN explosions are for the chemical composition of low-metallicity galaxies, with metallicities around $Z_\odot/3 \div Z_\odot/10$, below the PISN critical metallicity.

I think this is what was meant, but I'm not sure about it. The slide seemed to imply that the critical metallicity for PISN *is* $Z_\odot/3$ to $Z_\odot/10$, which does not seem right.

In order to study this, we need to know how many PISN progenitors (i.e. very massive stars) form per unit time. They can be created as a result of very rapid mass accretion, or after mergers in binary star systems, or from stellar collisions.

The mass distribution of stars is not well known, an educated guess is a Salpeter-like mass function, $\varphi(M)\,\mathrm{d}m \sim M^{-2.5}\,\mathrm{d}m$. If we assume this, we predict that around 2 % of supernova explosions should be massive enough to form a PISN.

We can plot the metal yield multiplied by the probability to have a star of that mass as the mass of the star varies: this models the total contribution to the metallicity in the ISM of that specific mass range.

We include Core Collapse SNe (with $M < 50 M_\odot$), then there is a mass region in which mostly BHs are formed, and then we have PISNe from $120 M_\odot < M < 260 M_\odot$, then BHs again.

The total mass yield per unit initial mass is lower for PISNe than for CCSNe, but the mass range is larger so the total yields of the two types of supernova explosion are comparable.

Per-element yields We compare the per-element yields of four different models:

1. CCSN alone at $Z = 0$;
2. CCSN + PCSN at $Z = 0$;
3. CCSN alone at $Z = 0.002$;
4. CCSN at $Z = 0.002$ + PCSN at $Z = 0.001$.

We notice the following things when considering the heavy (after Oxygen) elements:

1. at higher metallicity the yields are in general higher;
2. we always see an even-odd effect, with even elements being produced more, and

3. this effect is less pronounced if we only look at CCSNe;
4. this effect is less pronounced at higher metallicity.
5. The production of elements after Iron is not dependent on the metallicity, and their production is negligible in PCSNe.

3.3.7 PCSNe light curves

We can model the light curves by simply solving the hydrodinamical and radiative transport equations.

The bolometric magnitude light curve follows the visual curve for a certain time period, before which it needs UV corrections, and after which it needs IR corrections.