General Relativity exercises

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October 15, 2019

We set $c = 1$.

1 Sheet 1

1.1 Lorentz transformations

1.1.1 Inverses

We can consider a Lorentz boost with velocity v in the x direction, and we look at its representation in the (t, x) plane (since the y and z directions are unchanged). Its matrix expression looks like:

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} , \tag{1}$$

where $\gamma = 1/\sqrt{1 - v^2}$. The inverse of this matrix can be computed using the general formula for a 2x2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} . \tag{2}$$

The determinant of Λ is equal to $\gamma^2(1 - v^2) = 1$, therefore the inverse matrix is:

$$\Lambda = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix} . \tag{3}$$

1.1.2 Invariance of the spacetime interval

Our Lorentz transformation is

$$\mathrm{d}t' = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{4a}$$
$$\mathrm{d}x' = \gamma(-v\,\mathrm{d}t + \mathrm{d}x) \tag{4b}$$
$$\mathrm{d}y' = \mathrm{d}y \tag{4c}$$
$$\mathrm{d}z' = \mathrm{d}z \tag{4d}$$

and we wish to prove that the spacetime interval, defined by $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is preserved: $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$. Let us write the claimed equality explicitly:

$$-\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{5a}$$

1.1.3 Tensor notation pseudo-orthogonality

The invariance of the spacetime interval $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$ can be also written as $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x'^{\,\mu}\,\mathrm{d}x'^{\,\nu}$. By making the primed differentials explicit we have:

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho\,\mathrm{d}x^\rho\,\Lambda^\nu{}_\sigma\,\mathrm{d}x^\sigma\ , \tag{6}$$

but the dummy indices on the LHS can be changed to ρ and σ, so that both sides are proportional to $\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$. Doing this we get:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = (\Lambda^\top)_\rho{}^\mu\eta_{\mu\nu}\Lambda^\nu{}_\sigma\,, \tag{7}$$

or, in matrix form, $\eta = \Lambda^\top\eta\Lambda$.

1.1.4 Explicit pseudo-orthogonality

For simplicity but WLOG we consider a boost in the x direction with velocity v and Lorentz factor γ. The matrix expression to verify is:

$$\begin{bmatrix}\gamma & -v\gamma \\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\gamma & -v\gamma \\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix} \tag{8a}$$
$$\begin{bmatrix}\gamma & -v\gamma \\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-\gamma & v\gamma \\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix} \tag{8b}$$
$$\begin{bmatrix}-\gamma^2 + \gamma^2 v^2 & v\gamma^2 - v\gamma^2 \\ v\gamma^2 - v\gamma^2 & -v\gamma^2 + \gamma^2\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}\,, \tag{8c}$$

which by $\gamma^2 = 1/(1-v^2)$ confirms the validity of the expression.

1.2 Muons

1.2.1 Nonrelativistic approximation

The survival probability is given by $\mathbb{P}(t) = \exp\left(-t/2.2 \times 10^{-6}\,\mathrm{s}\right)$. If the ground is $h = 15\,\mathrm{km}$ away, then the muon will reach it in $t = h/v = 15\,\mathrm{km}/(0.995c) \approx 5.03 \times 10^{-5}\,\mathrm{s}$, therefore $\mathbb{P}(t) \approx 1.2 \times 10^{-10}$.

1.2.2 Relativistic effects: ground perspective

The observer on the ground will see the muon having to traverse the whole $h = 15\,\mathrm{km}$, but the muon's time will be dilated for them by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$.

1.2.3 Relativistic effects: muons perspective

The muons in their system will observe lenght contraction, with respect to Lorentz boost, by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$. This result is the same of the one predicted by ground observer, with respect to relativity principle.

1.3 Radiation

1.3.1 New angle

In the source frame the radiation velocity components are $u'_x = \cos\theta', u'_y = \sin\theta'$. From the composition of velocities we obtain:

$$u_y = \sin\theta = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\sin\theta'}{\gamma_v(1 + v\cos\theta')} \tag{9a}$$

$$u_x = \cos\theta = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma_v(\mathrm{d}x' + v\,\mathrm{d}t')}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\cos\theta' + v}{1 + v\cos\theta'}\,, \tag{9b}$$

hence:

$$\frac{1}{\tan\theta} = \frac{\gamma_v}{\tan\theta'} + \frac{\gamma_v v}{\sin\theta'}\,. \tag{10}$$

1.3.2 Angle plot and relevant limits

See the jupyter notebook in the `python` folder for plots. For $v = 0$ we have $\theta = \theta'$ as we expected, while for $v = 1$, $\theta = 0$.

1.3.3 Radiation speed invariance

Are the components of the velocity, which we called $\sin\theta$ and $\cos\theta$, actually normalized? Let us check:

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= \frac{(\frac{\sin\theta'}{\gamma_v})^2 + (\cos\theta' + v)^2}{(1 + v\cos\theta')^2} && \text{(11a)}\\
&= \frac{(1 - v^2)\sin^2\theta' + \cos^2\theta' + v^2 + 2v\cos\theta'}{(1 + v\cos\theta')^2} && \text{(11b)}\\
&= \frac{1 + v^2(1 - \sin\theta') + 2v\cos\theta'}{(1 + v\cos\theta')^2} = 1\,, && \text{(11c)}
\end{aligned}$$

therefore the square modulus of the speed of the radiation is still c, as we could have assumed earlier.

1.3.4 Isotropic emission

Since the angular distribution of emission varies when changing inertial reference, we might suppose that every system in relative motion respect to O with $v \neq 0$ observes nonisotropic emission.

This can be seen by noticing that for $v \simeq 1$ we have that in the observer system there is almost only emission at an angle $\theta = 0$. In general, since there is a Lorentz γ factor multiplying a function of the angle in the radiation emission frame O', the cotangent of the angle in the observation frame O must get larger and larger as the relative velocity v increases, therefore the radiation gets compressed towards angles with large cotangents: $\theta \sim 0$.

See the jupyter notebook in the `python` folder for interactive plots :)

2 Sheet 2

2.1 Constant acceleration

2.1.1 Coordinate velocity

We are given the position as a function of time,

$$x(t) = \frac{\sqrt{1 + \kappa^2 t^2} - 1}{\kappa}\,, \tag{12}$$

and we can directly compute its derivative

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\kappa t}{\sqrt{\kappa^2 t^2 + 1}}\,. \tag{13}$$

It is clear from the expression that $|v| < 1$ for all times, while v approaches 1 at positive temporal infinity and -1 at negative temporal infinity.

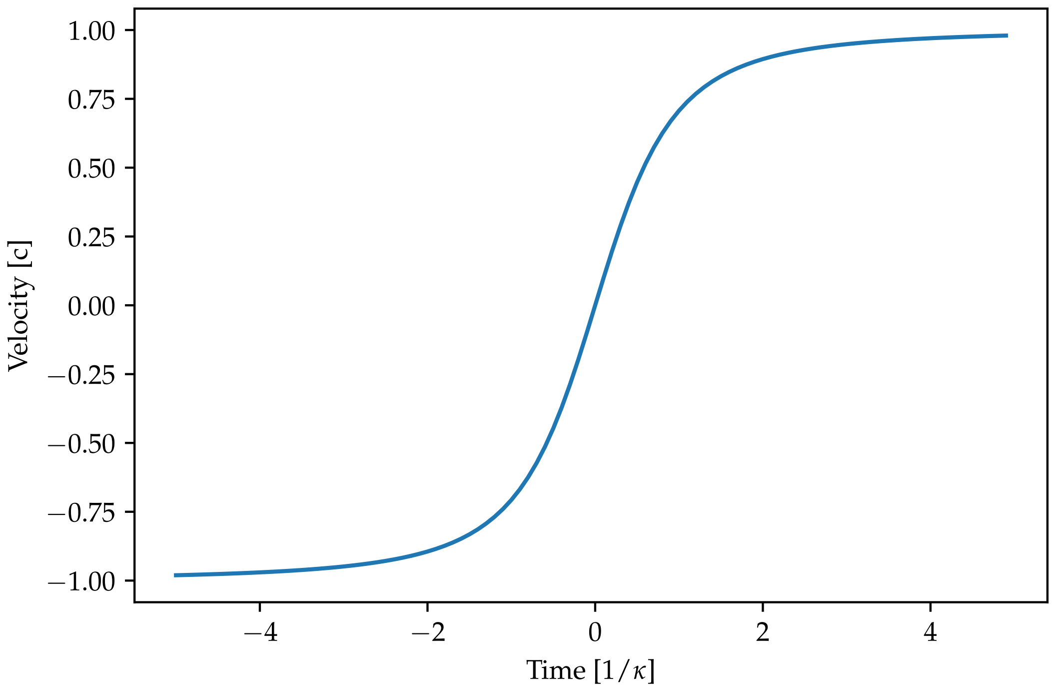

Figure 1: Velocity as a function of coordinate time t

2.1.2 Components of the 4-velocity

The Lorentz factor γ is given by

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{\kappa^2 t^2}{\kappa^2 t^2+1}}} = \sqrt{\kappa^2 t^2 + 1}\,, \tag{14}$$

therefore the four-velocity is given by:

$$u^\mu = \begin{bmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix}. \tag{15}$$

2.1.3 Proper time

The relation between coordinate and proper time is given by the definition of the first component of the four-velocity: $u^0 = \mathrm{d}t/\mathrm{d}\tau = \gamma$, therefore $\mathrm{d}\tau = \mathrm{d}t\,/\gamma$. Integrating this relation we get:

$$\tau = \int \mathrm{d}\tau = \int \frac{\mathrm{d}t}{\gamma} = \frac{\operatorname{arcsinh}(\kappa t)}{\kappa}\,, \tag{16}$$

where the constant of integration is selected by imposing $t = 0 \iff \tau = 0$. Notice that, as we would expect, when expanding up to second order near $t = \tau = 0$ we have $t \sim \tau$, since in that region the velocity is much less than unity.

The inverse relation is given by $t = \sinh(\kappa\tau)/\kappa$. Using this, we can write:

$$x(t(\tau)) = \frac{\cosh(\kappa\tau) - 1}{\kappa}\,. \tag{17}$$

2.1.4 Four-acceleration

Now, we wish to compute the four-acceleration. There are many ways to approach this: an easy one is to simply find the explicit expression $u^\mu(\tau)$ and to differentiate it. The expression we get is:

$$a^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}u^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \sqrt{\sinh^2(\kappa\tau) + 1} \\ \frac{\sqrt{\kappa^2 t^2+1}\,\sinh(\kappa\tau)}{\sqrt{\sinh^2(\kappa\tau)+1}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\kappa\sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa\cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

which is a bit unwieldy but it can be used to check two important facts: $a^\mu a_\mu = \text{const}$ and $a^\mu u_\mu = 0$. The first of the two is:

$$a^\mu a_\mu = -(a_0)^2 + (a_1)^2 = \kappa^2\cosh^2(\kappa\tau) - \frac{\kappa^2\sinh^2(2\kappa\tau)}{2\left(\cosh(2\kappa\tau) + 1\right)} = \kappa^2\,, \tag{19}$$

which tells us that the constant acceleration $\sqrt{a^\mu a_\mu} = \kappa$.

Also, we verify the orthogonality to the four-velocity:

$$a^\mu u_\mu = -\frac{\sqrt{2}\kappa\sqrt{\sinh^2(\kappa\tau) + 1}\,\sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau) + 1}} + \kappa\sinh(\kappa\tau)\cosh(\kappa\tau) = 0\,. \tag{20}$$

2.1.5 Local velocity & acceleration

We can apply a Lorentz boost corresponding to this velocity: it will be given by the matrix:

$$\begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

where v and γ are those found before. Without doing any calculations we could already say that the transformed velocity will be equal to the time-like unit vector, while the acceleration will be equal to κ times the unit x-directed vector.

The velocity becomes:

$$(u^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{22}$$

as we expected.

The acceleration instead becomes:

$$(a^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}\kappa \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa \cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \kappa \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

2.2 Fixed target collision

2.2.1 Center of mass momenta

In the CoM frame, the momenta of the two protons are respectively $(E_p, \pm p, 0, 0)^\top = m_p(\gamma, \pm v, 0, 0)$, where $E_p^2 = m_p^2 + p^2$. The total CoM energy is $-(p_A^\mu + p_B^\mu)^2 = 2m_p^2$.

2.2.2 Center of mass velocity

The momentum of particle B will be given by $p^\mu = m_p u^\mu = (m_p\gamma, m_p\gamma v, 0, 0)^\top$. Therefore, $\gamma v = p/m_p$. Solving this we get:

$$v = \frac{p}{m_p}\sqrt{\frac{1}{(p/m_p)^2 + 1}} = \frac{p}{E_p}, \tag{24}$$

2.2.3 Lab frame momenta

The momentum of particle B in its own rest frame will just be $(m_p, 0, 0, 0)^\top$. The momentum of particle A instead will be given by a boost in the x direction with velocity $-v$:

$$(p_A^\mu)_{\text{lab}} = \begin{bmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_p \\ p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma E_p + v\gamma p \\ v\gamma E_p + \gamma p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_p \frac{1+v^2}{\sqrt{1-v^2}} \\ 2\gamma p \\ 0 \\ 0 \end{bmatrix}, \tag{25}$$

2.3 Weak field gravitational time dilation

2.3.1 Time dilation expression

It is more intuitive geometrically to deal with a pulse sent from A to B, for which we expect the time dilation to work in the opposite sense:

$$\Delta t_B = \Delta t_A (1 + gh) , \tag{26}$$

up to first order in gh and $g\Delta t_A$, since $(1+gh)(1-gh) = 1 - (gh)^2 = 1$ to first order in gh.

We know that the paths of the observers are two curves of constant acceleration: we know their explicit expression from equation (12), and additionally we assume that they are separated by a space interval h:

$$x_A(t) = \frac{\sqrt{1+(gt)^2} - 1}{g} \tag{27a}$$

$$x_B(t) = \frac{\sqrt{1+(gt)^2} - 1}{g} + h \,. \tag{27b}$$

At $t = 0$ Alice sends a pulse, which then reaches Bob at a time t_1. After a time Δt_A, she sends another, which then reaches Bob at a time t_2. Right now, we are referring to all times as measured in the rest frame of Alice at $t = 0$. These times can be found by imposing that the space and time separation between the events of the pulse being sent and received are equal, since it travels at light speed: the equations which represent this are $x_B(t_1) = t_1$ and $x_B(t_2) - x_A(\Delta t_A) = t_2 - \Delta t_A$. Substituting the expressions for the positions:

$$t_1 = \frac{\sqrt{1+(gt_1)^2} - 1}{g} + h \tag{28a}$$

$$t_2 - \Delta t_A = \frac{\sqrt{1+(gt_2)^2} - 1}{g} + h - \left(\frac{\sqrt{1+(g\Delta t_A)^2} - 1}{g} \right) . \tag{28b}$$

Now, it is just a matter of calculation to solve these equations, expand up to first order in the adimensional parameters gh and $g\Delta t_A$ and one recovers the desidered expression for $\Delta t_B = t_2 - t_1$.

There is one more consideration to make though: what about the Lorentz time dilation for Bob? This it actually a *second order effect*.

Claim 2.1. *The time interval measured by Bob in his frame at $t \sim t_1$ is the same as the one measured in the rest frame of Alice at $t = 0$ up to first order in gh and $g\Delta t_B$.*

Proof. We perform a Lorentz boost to the velocity of Bob at $t = t_1$: this is given by equation (13), and is equal to:

$$v = \frac{gt}{\sqrt{(gt)^2 + 1}}, \tag{29}$$

with a Lorentz factor of $\gamma = \sqrt{(gt)^2 + 1}$ (see equation (14)).

The temporal separation between the two events is Δt_B, while the spatial separation is $\Delta x_B \approx v\Delta t_B$ to first order. The boost, in the (t, x) plane, looks like:

$$\begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix}' = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix} = \begin{bmatrix} \Delta t_B \left(\sqrt{(gt)^2 + 1} - (gt)^2 / \sqrt{(gt)^2 + 1} \right) \\ -gt\Delta t_B + \sqrt{(gt)^2 + 1} gt\Delta t / \sqrt{(gt)^2 + 1} \end{bmatrix}, \tag{30}$$

therefore as we would expect the spatial separation is eliminated, while expanding the factor multiplying the temporal one near $gt = 0$ we get:

$$\sqrt{(gt)^2 + 1} - (gt)^2 / \sqrt{(gt)^2 + 1} = 1 + O((gt)^2), \tag{31}$$

which proves our result. □

2.3.2 Gravitational time dilation

By the equivalence principle, the effects measured in a uniformly accelerating frame at g are the same as those measured in a gravitational field with constant acceleration g. The gravitational field in such a frame is given by $\Phi = gh$, where h is the height (with arbitrary zero point): the result follows.

2.3.3 Twins and gravitation

The gravitational time dilation is given by:

$$\Delta t = t_{\text{elapsed}} \frac{g\Delta h}{c^2} \approx 1\,\text{yr} \frac{10\,\text{m/s} \times 100m}{(3 \times 10^8\,\text{m/s})^2} \approx 3.5 \times 10^{-7}\,\text{s}. \tag{32}$$