Advanced astrophysics notes

Jacopo Tissino, Giorgio Mentasti

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1 Introduction

Teachers: Paola Marigo, Michele Trabucchi.
paola.marigo@unipd.it and michele.trabucchi@unipd.it

1.1 Topics

They are selected topics in stellar physics.

1. Stellar pulsations and Astroseismiology (dr. Michele Trabucchi);
2. stellar winds (dr. Paola Marigo);
3. final fates of massive & very massive stars (dr. Paola Marigo).

Basics in Stellar Physics: "astrophysics II" inside the bachelor's degree in astronomy (second semester). It can be taken as an optional course.

Material:

1. *Introduction to stellar winds* by Lamers, Cassinelli.
2. *Stellar Atmospheres: Theory and observations* (lecture notes from 1996).

and more on Paola Marigo's site.

1.1.1 Stellar Winds

Moving flows of materials ejected by stars. 20 to 2×10^3 km/s.

See *Bubble Nebula* in Cassiopea, there is a $45M_\odot$ star ejecting stellar wind at 1700 km/s.

Diagram: luminosity vs effective temperature. We see the *main sequence*. We can also plot the *mass loss rate*, $\dot{M} > 0$ in solar masses/year. An other important parameter is v_∞.

Diagram: mass loss (or gain) rate vs age of star.

Stellar winds affect stellar evolution, the dynamics of the interstellar medium, the chemical evolution of galaxies.

Momentum is approximately injected with $\dot{M}v$, kinetic energy with $^1/_2\dot{M}v^2$. Within 1×10^8 yr around half of the infalling matter is reemitted.

1.1.2 Contents

We will start with the basic theory of stellar winds, and then: *coronal*, *line-driven* and *dust-driven* winds.

1.2 Final fates of massive & very massive stars

Masses over $10M_\odot$.

1.3 Stellar oscillations

. . . see slides.

Material: slides on moodle or Marigo's page.

1. *Pulsating stars* by Catelan & Smith (introductory);
2. *Theory of stellar pulsation* by Cox (harder).

Written exam, partial exam on stellar pulsation.

2 Variability in Astronomy

First observations of variable stars: $\sim$ 1600, omicron-Ceti. It changes in magnitude by 6 orders of magnitude.

Others are found from the 1600 onwards, but since the XX century the reason is unknown. Is it *rotation*, *eclipses*?

For some it sure are eclipses, but the Cepheids are different. See δ-Cephei, asymmetric continuous curve. What if stars *pulsate*?

The *light curve* is the luminosity curve over time.

We can also look at the *phased* light curve. Of course we need the period: the phase is

$$\varphi = \frac{(t - t_0) \bmod \Pi}{\Pi} \tag{1}$$

where Π is the period. $E(t) = \lfloor (t - t_0)/\Pi \rfloor$ is the epoch.

We can then measure the period, but if the light curve is multiperiodic we can subtract the model from the curve to see if there are additional periods: this is *prewhitening*.

We can also look at the luminosity in Fourier space.

Of course there are issues with observational gaps (day-night, full moon): aliases; accuracy, duration of observations...

Also, the period can change in time.

Things have improved a great deal with large-scale surveys and space suveys.

2.1 Classification

By variability type: regular, semi-regular or irregular.

By intrinsic variability: extrinsic, external to the star: eclipses, transits, microlensing, rotation; intrinsic: rotation, eclipses (self-occultation), eruptive and explosive variables, oscillations, secular variations (?).

Whether rotation is to be considered intrinsic or extrinsic is a matter of taste.

Oscillations can be classified by several criteria.

The geometry can be radial (classical pulsators) or non-radial.

The restoring force can be the pressure gradient or the gravitational force (bouyancy, not gravitational waves).

The excitation mechanisms can be different.

The evolutionary phase and mass of the star can also be different.

3 Summary of stellar structure & evolution

Eulerian: properties of a gas are fields, the position is the position of an observer. To differentiate position with respect to time is meaningless: position is an independent variable. $f = f(r^i, t)$.

Lagrangian: we follow an element of fluid: $\mathrm{d}r^i \big/ \mathrm{d}t = v^i$. We can identify univocally these fluid elements.

When treating stellar structure & evolution, we look at mass layers $\mathrm{d}m$. $f = f(m, t)$. Do note that m is the mass of the whole full sphere under a certain layer, not the mass of the shell.

In the lagrangian case, we use the convective derivative $\mathrm{d}/\mathrm{d}t = \partial_t + v^i \partial_i$ where v^i is the velocity defined before.

3.1 Equations of stellar structure

We write these in the spherically symmetric case.

The continuity equation is:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} . \tag{2}$$

Momentum conservation is given by:

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} , \tag{3}$$

where

Energy conservation is given by:

$$\frac{\mathrm{d}L}{\mathrm{d}m} = \varepsilon - \varepsilon_\nu - \varepsilon_g , \tag{4}$$

where L is the luminosity, ε is the rate of nuclear energy generation per unit mass, while ε_ν is the rate of energy loss due to neutrino emission per unit mass, and ε_g is the work done by the gas per unit mass & time.

The energy transfer equation is:

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P} \nabla , \tag{5}$$

where $\nabla = \partial \log T / \partial \log P$ is the temperature gradient, which has contributions from radiation, conduction, convection...

With the diffusion approximation, we can write the gradient as

$$\nabla = \nabla_{\mathrm{rad}} = \frac{3}{16\pi a c G} \frac{\kappa_R L P}{m T^4} , \tag{6}$$

where a is a constant depending on the Stefan-Boltzmann constant and the speed of light.

where κ_R is the Rosseland mean opacity, given by

$$\frac{1}{\kappa_R} = \frac{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T} \frac{1}{\kappa_\nu} \, \mathrm{d}\nu}{\displaystyle\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T} \, \mathrm{d}\nu} \tag{7}$$

Substituting in the result in (6) we get:

$$L = -\frac{64\pi^2 a c}{3} r^4 \frac{T^3}{\kappa} \frac{\partial T}{\partial m} \tag{8}$$

where κ is a generalized opacity, the harmonic mean of the Rosseland opacity κ_R and the convective opacity $\kappa_c = 4acT^3/(3\rho\lambda_c)$.

Where λ_c is…?

If we need to deal with convection, this defies any simple modeling. There are instability criteria: where is it relevant? This is given by Ledoux's criterion,

$$\nabla_{\text{rad}} > \nabla_{\text{ad}} - \frac{\chi_\mu}{\chi_T} \nabla_\mu \,, \tag{9}$$

where:

$$\nabla_\mu = \frac{\mathrm{d} \log \mu}{\mathrm{d} \log P} \tag{10a}$$

$$\nabla_{\text{ad}} = \left(\frac{\partial \log T}{\partial \log P} \right)_{\text{ad}} \tag{10b}$$

$$\chi_\mu = \left(\frac{\partial \log P}{\partial \log \mu} \right)_{\rho, T} \tag{10c}$$

$$\chi_T = \left(\frac{\partial \log P}{\partial \log T} \right)_{\rho, \mu} \tag{10d}$$

which are thermodynamic parameters.

how are these called? What do they mean?

In the convective core, $\nabla \approx \nabla_{\text{ad}}$, but outside of it we need something else.

Mixed-length theory model convection with "bubbles":

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Figure 22.8 in some PDF: run of adiabatic, radiation gradients vs $\log T$.

We compute ∇_{ad} and ∇_{rad} and see whether the region is convective or radiative.

We can move from the Eulerian and Lagrangian formalisms using the continuity equation. In the Eulerian formalism:

$$m(r) = \int_0^\infty 4\pi r^2 \rho(x) \, \mathrm{d}x \ . \tag{11}$$

We need several *constitutive equations* for the parameters ρ, c_P (heat capacity of stellar matter), the opacity κ, the nuclear transformation rate r_{ij} and the rate of generation of nuclear energy ε. These can all be considered as functions of P, T, μ.

We define:

$$\mu^{-1} = \sum_i (1 + \nu_i) \frac{X_i}{A_i} \tag{12}$$

(CHECK)

The variables X, Y and Z represent the abundances of H, He and metals, and satisfy $X + Y + Z = 1$.

We may need to know the metal mixture inside Z, but often we can approximate it as the Sun's distribution.

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \sum_j \left(r_{ji} - r_{ij} \right) \tag{13}$$

3.2 Classification of stars

Low mass stars have between 0.8 and 2 solar masses. Intermediate mass stars have masses between 2 and 8 $M_\odot$. Massive stars have masses of over $8 M_\odot$. (Add characteristics of these).

3.2.1 Low-mass star evolution

See slides for figures. What are Hayashi lines?

3.2.2 Intermediate-mass star evolution

Idem

4 Stellar oscillations

Today we will look at typical time-scales, the period-mean density relation, the energy equation and perturbation theory w/ linearization.

No derivations of the equations in the exam.

4.1 Time-scales

The *free fall* time scale is:

$$\tau_{\text{dyn}} \sim \left(\frac{R}{g} \right)^{1/2} = \left(\frac{R^3}{GM} \right)^{1/2}, \tag{14}$$

this is associated with pulsation. It is calculated using the travel time of a mass in free fall across the stellar radius accelerated by constant acceleration equal to surface acceleration.

We note that $\tau_{\text{dyn}} \propto \overline{\rho}^{-1/2}$. For the Sun it is about 25 min.

The *thermal* time scale is the relaxation time of deviations from thermal equilibrium:

$$\tau_{\text{th}} \sim E_{\text{th}}/L\,. \tag{15}$$

It is calculated as the time required for a star to irradiate all its energy.

Proof. Roughly, $3\int_V P \,\mathrm{d}V = -\Omega$ with $\Omega = -\int_M \cdots$ □

Typically, $\tau_{\text{th}} \sim GM^2/(LR) \sim 10^7$ ... It is much larger than the dynamic time scale.

The *nuclear* time scale is even longer.

This allows us to say that oscillations will not be heavily affected by thermal conduction, and even less by nuclear processes: the pulsations will be almost *adiabatic*.

The best candidate for these oscillations are *sound waves*: is the adiabatic speed of sound roughly right?

The speed of sound is:

$$v_s^2 = \Gamma_1 \frac{P}{\rho} \tag{16}$$

If the gas follows the perfect equation of state

$$\frac{P}{\rho} = \frac{k_B}{m_H}\frac{T}{\mu}\,, \tag{17}$$

we get

$$v_s^2 = \frac{\Gamma_1 k_B T}{m_H \mu}\,. \tag{18}$$

Typical values are $\mu \sim (2X + 3Y/4 + Z/2)^{-2} \sim 0.6$, $\Gamma_1 = 5/3$, and $T_{\text{He}} \sim 4.5 \times 10^4\,\mathrm{K}$.

So we get $v_s \sim 32.2\,\mathrm{km/s}$.

The timescale is $\Pi \sim 2R/v_s \sim 22\,\mathrm{d}$, while $\Pi_{\text{obs}} = 5.336\,\mathrm{d}$, in terms of orders of magnitude it works.

We can use the equation for the sound speed in the virial theorem, and get:

$$\Omega = -3 \frac{\int_M v_s^2/\Gamma_1 \,\mathrm{d}m}{\int_M \mathrm{d}m} M = -3 \left\langle \frac{v_s^2}{\Gamma_1} \right\rangle M \tag{19}$$

If Γ_1 and v_s are independent, we can compute their averages separately.

This allows us to write Π wrt the moment of inertia, $I = \int_r r^2 \,\mathrm{d}m\,(r)$:

$$\Pi \sim \left(\frac{I_{\text{osc}}}{-\Omega}\right)^{1/2} \tag{20}$$

This is further evidence that we are dealing with a dynamical phenomenon.

Of course the speed of sound changes throughout the interior of the star. We compute the period as the travel time of sound waves throughout the diameter:

$$\Pi = 2 \int_0^R \mathrm{d}t(r) = 2 \int_0^R \frac{\mathrm{d}r}{\sqrt{\Gamma_1(r) P(r) / \rho(r)}}\,, \tag{21}$$

since $\mathrm{d}t = \mathrm{d}r\,/v_s$.

We also rewrite the differential equation for P substituting $m = \rho r$. Doing this we get

$$\Pi \overline{\rho}^{1/2} = \sqrt{\frac{3\pi}{2\Gamma_1 G}} \tag{22}$$

Which confirms Ritter's relation $\Pi \propto \overline{\rho}^{-1/2}$.

This works for acoustic modes, but if we consider non-radial g-modes it stops working, such as variables of type ZZ Ceti.

4.2 The energy equations

Just an overview for now: we will consider the star as a thermodynamic engine.

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Reference books can be found in the Moodle: they are ordered by difficulty, Catelan to Aerts to Salaris.

The exam for this part of the course: it might be around the end of october.

The subscripts on last lecture *were* inverted after all.

The *mirror principle*: when the core contracts or expands, the envelope does the opposite.

The shell must remain at around the same temperature to maintain equilibrium: contracting the core would increase the temperature, therefore the envelope exapands. This heuristic argument is actually derived from simulations.

The relevant time scale for oscillations is the free-fall, dynamical time scale.

We come back to the energy equation

$$\frac{\partial L}{\partial m} = \varepsilon - \varepsilon_\nu - \varepsilon_g \tag{23}$$

we incorporate $\varepsilon - \varepsilon_\nu = \varepsilon_{\text{eff}}$ and call $\varepsilon_g = \frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon_{\text{eff}} - \frac{\partial L}{\partial m}$.

This makes the meaning of this transfer equation clearer. Using the first and second laws of thermodynamics, and recalling some thermodynamical values $c_V =$

$\left(\frac{\partial Q}{\partial t}\right)_V$, χ_T, χ_ρ, $\Gamma_{1,2,3}$. These are all *exponents* in some power law. We use log values since our variables change by orders of magnitude.

The final result we get from manipulations is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{P}{\rho(\Gamma_3 - 1)}\left(\frac{\partial \log P}{\partial t} - \Gamma_1 \frac{\partial \log \rho}{\partial t}\right) \tag{24a}$$

$$\frac{\partial \log P}{\partial t} = \Gamma_1 \frac{\partial \log \rho}{\partial t} + \frac{\rho}{P}(\Gamma_3 - 1)\left(\varepsilon_{\text{eff}} - \frac{\partial L}{\partial m}\right) \tag{24b}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = c_V T\left(\frac{\partial \log T}{\partial t} - (\Gamma_3 - 1)\frac{\partial \log \rho}{\partial t}\right) \tag{24c}$$

$$\frac{\partial \log T}{\partial t} = (\Gamma_3 - 1)\frac{\partial \log \rho}{\partial r} + \frac{1}{c_V T}\left(\varepsilon_{\text{eff}} - \frac{\partial L}{\partial m}\right) \tag{24d}$$

The second and fourth of these equations substitute our equation of energy conservation.

Say we have a solution for these equations, we look at linear perturbations of them. This makes sense: the main solution is basically static on the pulsation time-scales.

The perturbed model is $f = f(m)$, the unperturbed one is $f_0(m)$. The Lagrangian perturbation is $\delta f(m,t) = f(m,t) - f_0(m,t)$.

Let us consider specific cases for r: the radial displacement is $\delta r(m,t)$. The position of the layer at time t is $r = r_0 + \delta r$.

We can write:

$$r = r_0\left(1 + \frac{\delta r}{r_0}\right) = r_0(1+\zeta)\,. \tag{25}$$

In general the fractional perturbation $\delta f/f_0$ is assumed to be $\ll 1$. So, $\delta f/f_0 \sim \delta_f/f$. We will insert expressions which are functions of perturbations of all our variables, and thus get linear differential equations.

Let us try the continuity equation: $r = r_0(1+\zeta)$ and $\rho = \rho_0(1 + \delta\rho/\rho_0)$.

$$\frac{\partial}{\partial m}\big(r_0(1+\zeta)\big) = \frac{1}{4\pi r_0^2}(1+\zeta)^{-2}\left(1 + \frac{\delta\rho}{\rho_0}\right)^{-1} \tag{26}$$

and we use $(1+x)^n \approx 1 + nx$:

$$4\pi\rho_0^2\left(\frac{\partial r_0}{\partial m}(1+\zeta) + r_0\frac{\partial \zeta}{\partial m}\right) = (1 - 2\zeta) - \frac{\delta\rho}{\rho_0} \tag{27}$$

We can collapse the equation into:

$$\frac{\delta\rho}{\rho_0} = -3\zeta - 4\pi r_0^3\rho_0\frac{\partial\zeta}{\partial m} \tag{28}$$

or, the density perturbation is proportional with a negative constant to the radial perturbation, plus a term proportional to $\partial\zeta/\partial m$. If there is a positive gradient of radial perturbation, the corresponding layer expands.

Let us also perturb the momentum conservation equation, assuming hydrostatic equilibrium $\partial^2 r_0 \Big/ \partial t^2$.

We have a purely geometric term $-16\pi r_0^2 \zeta\, \partial P_0/\partial m = 4\zeta G m / r_0^2$: when a layer moves outwards it expands. There is a restoring force toward equilibrium.

Let us also consider the $\log P$ form of the energy conservation equation. We also perturb the adiabatic exponents and the $\mathrm{d}Q/\mathrm{d}t$ term.

After difficult manipulations we get back an equation which relates the changes in density and pressure to the change in energy: we manipulate until we get something which is similar to the original equation.

In general for a Lagrangian perturbation $\delta(f^n) = n f_0^{n-1} \delta f$. We have rules for the perturbation of product which work just like logarithmic derivatives. δ commutes with partial derivatives.

In the radiative case with the diffusion approximation we can perturb the luminosity: $L \propto r^4 T^4 \kappa_R^{-1} \frac{\partial \log T}{\partial m}$.

$$\frac{\delta L}{L_0} = 4\zeta + 4\frac{\delta T}{T_0} - \frac{\delta \kappa_R}{\kappa_{R,0}} + \frac{\frac{\partial}{\partial m}\frac{\delta T}{T}}{\frac{\partial \log T}{\partial m}} \tag{29}$$

In the end, we have a set of four linear PDE equations (written as a 5-equation system).

These describe implicitly how the properties of the star change over time.

Pulsation usually affects mostly the outer layers of a star.

Moving on to

5 Adiabatic oscillations

Explioting the adiabatic approximation we will get the LAWE: Linear Adiabatic Wave Equation: a single equation which summarizes the 4.

This can be solved explicitly.

We suppose that each layer does not lose nor gain heat: $\delta(\varepsilon_{\mathrm{eff}} - \partial L/\partial m) = 0$.

Is this approximation justified? The term multiplying $\delta(\varepsilon_{\mathrm{eff}} - \partial L/\partial m)$ in the perturbed energy equation is $\rho/P(\Gamma_3 - 1) = \chi_T/(c_V T)$. Usually $\chi_T \sim 1$, $\Gamma_1 \sim 1$.

This term, $\chi_T/(c_V T)\delta(\varepsilon_{\mathrm{eff}} - \partial L/\partial m)$, is of the order $1/\tau_{\mathrm{th}}$, the thermal time scale of this layer, while the term before, $\Gamma_1\, \partial/\partial t\ (\delta\rho/\rho)$, is of the order $1/\tau_{\mathrm{dyn}}$, the dynamical time scale.

Therefore, we neglect the second part. This only works for the star as a whole, not for single layers. There are stellar layers which are *strongly* non-adiabatic (driving layers). We will need some non-adiabatic theory to explain how pulsations *start*.

[Insert derivation of the LAWE, which can be found in the slides].

$$r\frac{\partial^2 \zeta}{\partial t^2} = 4\pi r^2 \zeta \frac{\partial}{\partial m}\left((3\Gamma_1 - 4)P\right) + \frac{1}{r}\frac{\partial}{\partial m}\left(16\pi^2 \Gamma_1 P \rho r^6 \frac{\partial \zeta}{\partial m}\right) \tag{30}$$

We decompose: $\zeta(m,t) = \eta(m)e^{i\sigma t}$ with a constant σ: putting this into the LAWE we simplify the exponentials and get the space dependent form of the LAWE. It is a Storm-Liouville equation.

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Last lecture we started talking about linearization & perturbation theory.

We will have analytic solutions for the adiabatic case, with the additional hypotheses of either $\Gamma_1 = 4/3$ or $\Gamma_1 > 4/3$ and homogeneity.

The justification of the adiabatic approx might be asked at the exam.

After plugging our ansatz in the LAWE we should take the real part.

What are the boundary conditions we should set?

1. $\delta r = 0$ at $r = 0$;
2. $\partial\eta/\partial r = 0$ at $r = 0$, which allows us to fix many divergences at the center;
3. $(4 + R^3\sigma^2/(GM))\eta + \delta P/P = 0$ at $r = R$;
4. $\eta = \delta r/r = 1$ at $r = R$.

What is the $\frac{\partial^2 \eta}{\partial r^2}$ stuff about?

We use the Eulerian form of the LAWE to figure out the surface boundary conditions. We assume that all perturbation are in phase, and write them all as proportional to $\exp(i\sigma t)$. The pressure scale height is defined as:

$$H_P = -\left(\frac{\partial \log P}{\partial r}\right)^{-1} \tag{31}$$

and represents "how far we should move in the star for the pressure to change e-fold". By inserting this in the equation we see that $H_P \to 0$ when $r \to R$: the pressure changes very quickly in the photosphere of the star. Therefore, the term multiplying it must also go to zero.

The last condition comes from the fact that we want our study to give us *periods*, not *amplitudes*: we cannot find those out, so we normalize. The LAWE is 1-homogeneous!

The LAWE can be written compactly with a linear operator $\mathcal{L}$:

$$\mathcal{L}(\eta) = \sigma^2 \eta\,, \tag{32}$$

therefore the eigenvalue is the square of the pulsation. There are infinitely many solutions to the LAWE, only (finitely many?) fulfill the boundary condition. The eigenvalues are real [$\mathcal{L}$ is Hermitian], have a wavefunction associated: $\eta_m(r)$ corresponding to σ_m^2.

If $\sigma^2 > 0$ we have an oscillating solution, if $\sigma^2 < 0$ we have an exponential collapse or explosion since the solution is proportional to $\exp(i\sigma t)$.

We label solutions by *radial order* $m \in \mathbb{N}$: $m = 0$ has the lowest frequency, and then we have overtones. We choose the labels so that $\sigma_{m_1} < \sigma_{m_2} \iff m_1 < m_2$.

The radial order m is also the number of nodes.

The eigenfunctions are orthogonal wrt the scalar product

$$\langle \eta_m | \eta_n \rangle = \int_0^R \eta_m \eta_n \rho r^4 \,\mathrm{d}r \tag{33}$$

Possibly there is a 4π missing in order for this to be consistent with the following?

The functions ζ_m are orthogonal wrt the same product. The system is linear: we can write a general solution as a superposition.

We can define the moment of inertia:

$$J_m = \int_0^M |\zeta_m|^2 r^2 \,\mathrm{d}m \tag{34}$$

and the following holds:

$$\sigma_m^2 = \frac{1}{J_m} \int_0^M \zeta_m^* \mathcal{L} \zeta_m r^2 \,\mathrm{d}m \tag{35}$$

5.1 Simplifications

5.1.1 Period-mean density relation

If $\eta = \text{const}$, and ρ and Γ_1 are also constanst, we immediately get:

$$\sigma^2 = (3\Gamma_1 - 4)\frac{GM}{r^3} \tag{36}$$

and by inserting the mean density formula we get the period-mean density relation: this is consistent with our previous assumptions.

5.1.2 Polytropic model

It is a gas sphere with the following constitutive equaition:

$$P = K_n \rho^{1+1/n} = K_n \rho^{\frac{n+1}{n}} \tag{37}$$

with varying *polytropic* index n. It models spheres with different mass distributions: $n = 0$ is constant density, $n = 5$ is infinite central density, $n = 3$ is the Eddington standard model, which is reasonable for the Sun and stars on the main sequence.

With these assumptions we can explicitly solve for the wavefunctions, and make predictions of the fractional modulus of the oscillations at a certain radius wrt the modulus at the surface (which can be found only experimentally).

The overtones die out toward the center even faster than the fundamental: these oscillations are very much a *surface phenomenon*.

Beyond the ηs we can also plot the pressure perturbations: these will not be normalized.

5.1.3 A concrete example

This is done looking at an RR Lyrae variable. We integrate the stellar structure equations numerically. We can see that $\sigma_m - \sigma_{m-1} \approx$ const when m gets large.

The wavefunctions die out faster than the polytropic model when $r \to 0$.

There are "bumps" in the pressure plot: these are the partial ionization regions of H and He.

These appear because we start from a solution of the stellar structure equations, where all the properties of stellar matter were used, to start off with the LAWE.

6 Non-adiabatic oscillations

How can we tell, theoretically, how stable and how wide the various modes are? We expect to see the stable modes, and not to see the unstable ones.

Let us start from the Lagrangian momentum conservation:

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r} \tag{38}$$

and apply to it the identity: $1/2 \frac{\partial}{\partial t} v^2 = \frac{\partial r}{\partial t} \frac{\partial^2 r}{\partial t^2}$, by multiplying everything by $\partial r / \partial t$.

We then integrate everything with respect to m and apply some manipulations [see slides].

$$\frac{\partial}{\partial t} \int_M \frac{v^2}{2} \, \mathrm{d}m = -\frac{\mathrm{d}\Omega}{\mathrm{d}t} + \int_M P \frac{\partial}{\partial t} \frac{1}{\rho} \, \mathrm{d}m \tag{39}$$

We integrate a pulsation period, which cancels out the gravitational potential term which is conservative.

$$\left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle_\Pi = \frac{1}{\Pi} \int_\Pi \int_M P \frac{\partial}{\partial t} \left(\frac{1}{\rho} \right) \mathrm{d}m \, \mathrm{d}t \tag{40}$$

Some layers will provide energy to the oscillation motion (*drive* it), some others will *damp* it. These are characterized by the sign of the RHS of this equation.

If it is positive, we have instability; if it is negative the pulsations will tend to die out, giving stability.

The average time scale of change of the perturbations is

$$\kappa \stackrel{\text{def}}{=} \frac{1}{\tau} = -\frac{1}{2} \frac{\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle_\Pi}{\left\langle \delta\psi \right\rangle_\Pi} \tag{41}$$

The term $\left\langle \mathrm{d}W/\mathrm{d}t \right\rangle_\Pi$ can also be interpreted as the net heat gain fed into mechanical work during a pulsation cycle.

In the adiabatic case, we had $\frac{\partial}{\partial t}\left(\frac{\delta P}{P} - \Gamma_1 \frac{\delta \rho}{\rho} \right) = 0$: the perturbations were in phase.

Now we add a term to the time derivatives: the pressure and density perturbations will stop being in phase. The sign of the heat variation term gives us the difference between *driving* heat transfer and *damping* heat transfer.

In a PV diagram, we can see that these correspond to right and left oriented loops (as opposed to the loops with zero total signed area we had in the adiabatic case).

The star is effectively a themal engine converting heat into work; this will result in an increased overall entropy of the star, and a smoothing of its temperature gradient, however:

1. the timescales on which this process occurs are much larger than the timescales on which oscillating motions are created and destroyed;
2. then energies of the oscillations are much smaller than the global thermal energy of the star.

therefore this process is typically not relevant.

Mon Oct 14 2019

Summary of past lectures: we discussed:

1. the different time scales of stellar evolution, and their impact on the adiabatic approximation;
2. easy approximations for stellar pulsation;
3. acoustic approximations;
4. Ritter's relation;
5. linearized structure equations: the adiabatic approximation (arguments for it!), deriving the LAWE;
6. examples of solutions to the LAWE and stability conditions;
7. the LNAWE: today;

We consider the LNAWE: inside of $\zeta(r,t) = \eta(r)e^{i\sigma t}$ we insert $\sigma = \omega + i\kappa$: this means we also consider *damped* exponential solutions and *diverging* exponential solutions.

The time scales for these parameters are $\omega \sim \omega_{\mathrm{ad}}$, while $\kappa \sim 1/\theta_{\mathrm{th}}$: therefore $\omega \gg |\kappa|$.

Using this result, we can make some useful *quasi-adiabatic* approximations: in the LNAWE we identify the LAWE operator $\mathcal{L}$, and replace its application to the wavefunction with the corrisponding eigenvalue.

We only look at the first order terms in κ. We define the *work integral* C, and then we derive

$$\kappa = -\frac{C}{2\omega^2 J} \,. \tag{42}$$

We make some considerations on the expression of C, integrate by parts, getting:

$$C = \int_M \left(\frac{\delta T}{T}\right) \delta\left(\epsilon_{\mathrm{eff}} - \frac{\partial L}{\partial m}\right) \mathrm{d}m \;, \tag{43}$$

which allows us to study the mechanisms which create perturbations.

The energy of the vibrations comes from the internal thermal energy of the star, which ultimately comes from thermonuclear reactions.

The two terms in C come respectively from energy generation and transfer.

The ϵ-mechanism is about energy generation, which is assumed to be due to nuclear reactions, without considering neutrino processes: this can happen if the magnitude of the temperature and density perturbations are large enough.

The κ-γ-mechanism: we look at regions where the luminosity gradient and temperature gradient are discordant. This means that the considered stellar layer is absorbing or emitting; this is usually assumed to be happening through free-free interactions (bremsstrahlung and inverse bremsstrahlung).

Tue Oct 15 2019

Today we look at the κ-γ-mechanism.

This deals with the term

$$\int_M \frac{\delta T}{T} \frac{\partial \delta L}{\partial m} \, \mathrm{d}m \,, \tag{44}$$

which keeps account of the regions in which the two terms multiplied in the integrand are discordant.

The mean Rosseland opacity is approximated by a law in the form:

$$\kappa_R \simeq \overline{\kappa}_R \rho^n T^{-s} \,, \tag{45}$$

with $n \approx 1$, $s \approx 7/2$ in the case of free-free absorption (inverse & direct bremsstrahlung) in a non-degenerate, totally ionized gas.

This allows us to relate the perturbations in ρ to those in T. However, in this law δT is positive iff $\delta\rho$ is positive: *damping layers*.

Driving layers, however, are present in certain regions: where there is ionization, it provides an additional channel for energy stocking: it is like a *dam* for energy.

The ionization energy is released through mechanical work.

We look at the linearized equations of continuity and radiative transfer for an expression for the gradient of δL: in the outer regions of the star we have $\partial L / \partial m \approx 0$. In these regions $\kappa_R \propto \rho^n T^{-s}$ with negative s.

There is an opacity bump at $5 < \log(T) < 6.5$, which was found in the eighties by Simon. Do note that these are temperatures reached when looking a bit inside the star, not at the surface (although close to it).

There are very luminous *strange modes*, very dim convectively driven modes.

Convective driving is called the δ-mechanism.

There also are *stochastically driven* oscillations: they are intrinsically stable.

The pulsation region has boundaries: for the κ-mechanism:

- if the star is too hot, the regions of partial ionization get too close to the surface;
- if the star is too cold, they are too far in: in the outer region the pulsation is damped.

What is the optimal region? We will only look at the fundamental mode. We define

$$\phi(m) \equiv \frac{1}{L(\Pi/2\pi)} \int_m^M c_V T \, \mathrm{d}m \,, \tag{46}$$

which represents the thermal balance of a single oscillation.

We are in the helium ionization region: there are no nuclear reactions, so we set energy generation to zero.

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta\rho}{\rho} - i\left(\frac{\partial \phi}{\partial x}\right)^{-1} \frac{\partial}{\partial x}\left(\frac{\delta L}{L}\right) , \tag{47}$$

where δT is actually complex, since the perturbations are out of phase.

On the surface, the imaginary term is negligible. In the interior, it is relevant.

The pulsation region is the one in which the ionization region can build up an energy excess.

6.1 RR Lyræ

Now for some star zoology: RR Lyræ. It is a stage, which lasts no more than 10^8 yr.

They are classified by a, b, or c according to the shape of the light curve, its amplitude, its period:

1. RRa: sharp rise, large amplitude: the fundamental;
2. RRb: similar to RRa with smaller amplitude, longer period: the fundamental;
3. RRc: more symmetric light curve, short periods, low amplitudes: they pulsate in the first overtone.

We have also RRd: bimodal, RRe: second overtone.

[Argument for the different amplitudes at different wavelengths: to understand]

[Qualitative part of the lecture.]

Mon Oct 21 2019

We were talking about RR Lyrae varibles, classifying them. In a Bailey diagram we plot variables as points with the coordinates amplitude and period.

We can tell whether a star is a RR Lyr variable, and then we can use it as a standard candle.

We have nonlinear models which accurately predict the light curves. There are HUMPs, BUMPs, JUMPs and LUMPs in the light curves: they tell us about the propagation of shock waves in the star.

We also have the Blazhko Effect: when it is not present, the light curve is the same at every cycle; when it is present the light curve changes, it is modulated every cycle. This is a poorly understood effect, it might have something to do with magnetic fields.

In a given Globular Cluster, we look at the distribution of RR Lyr with respect to period: we can see two distinct clusters, corresponding to the fundamental and first harmonic. We can characterize them with respect to the Oosterhoff group (the one of the fundamental): if its is big with respect to the first harmonic it is a type I, if it is comparable it is a type II.

6.2 Classical Cepheids

They are younger than RR Lyr. They are very important if the *cosmic distance ladder*.

We have a very important Period-Luminosity relation: this can be used to measure distances.

Period is not affected by reddening.

The fundamental relation is

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 \,, \tag{48}$$

which can be converted into a relation involving the bolometric magnitude. In the end, using the mass luminosity relation of main sequence stars, we have:

$$\log(\Pi) = -0.24 M_{\text{bol}} - 3\log(T_{\text{eff}}) + \log(Q) + \text{const} \,. \tag{49}$$

The derivation of this might be asked at the exam.

7 Non-radial oscillations and astroseismiology

This is just an introduction, a full course might be given at the PhD level.

We use the same assumptions as before, except we lose the spherical symmetry.

We have several families of solutions:

1. p-modes, mostly radial, pressure;
2. g-modes, mostly horizontal, buoyancy;
3. f-modes; without nodes.

We can derive equations for this case similarly to the LAWE.

[Equations]

We get two different velocities of propagation: $L_\ell = \ell(\ell+1)(v_s/r)^2$ is the Lamb frequency, while $N^2 = -g\Big(\mathrm{d}\ln\rho/\mathrm{d}r - 1/\Gamma_1 \,\mathrm{d}\ln P/\mathrm{d}r\Big)$ is the Brunt-Väisälä frequency.

We can approximate the differential equations locally as a single differential equation, which is a harmonic oscillator for the perturbation, with wavenumber

$$k_r^2 = -\frac{1}{v_s^2\sigma^2}\Big(L_\ell^2 - \sigma^2\Big)\Big(\sigma^2 - N^2\Big) \,. \tag{50}$$

This is negative iff σ^2 is either smaller or larger than both N^2 and L_ℓ^2; these are respectively g and p-modes. If σ^2 is between them, we get a real exponential as a solution. This is called the *evanescent regime*.

p-modes are characterized by a *turning point*: below a certain radius, the mode is in the evanescent regime. For g-modes we have the opposite behaviour: they only exist in the core, and are evanescent for large radii.

We can plot N and L_ℓ as a function of radius.

Mixed modes arise when the evanescent region is thin, therefore the mode can tunnel through.

The frequencies are degenerate in m, but the degeneracy is split when the star is rotating.

In the slide for "asymptotic behaviour", the y axis is frequency.

We analyze the power spectra of stars.

Tue Oct 22 2019

For the physics undergraduate students: "Fundamental Astronomy" by Karttunen, Kröger, Öja, Poutanen, Donner.

7.1 Red variable stars

They include the ξ Hydrae, SR and Mira stars. These are low temperature stars.

They are evolved stars, and most all of the evolved stars are at least somewhat variable.

Miras, SRVs and OSARGs have very long periods, on the order of a year.

We need good estimates for the radius: because of the period-mean density relation, the fundamental at a certain radius can correspond to the first overtone at a larger radius.

We have different period-luminosity relations corresponding to which overtone we see. Using Weisenheit indices, which compensate for self-reddening, these relations are even more evident.

We can simulate galaxies, and observe the same patterns.

The linear models are not appropriate for the fundamental. Today, we do 3D models.

8 Summary

- $\tau_{\text{dyn}} \propto 1/\sqrt{\overline{\rho}}$;
- the inequality between the timescales;
- variability is due to mechanical, acoustic phenomena!
- derivation of the period-mean density relation;

- the balance of heat absorption;
- meaning of perturbation theory;
- the final result: the perturbed structure equations;
- the adiabatic approximation: justification, cases in which it does not hold: driving layers, stability;
- ideas of how the LAWE is solved (not boundary conditions): the Sturm-Liouville problem;
- the meaning of the solutions of the LAWE: nodes, the shape of the eigenfunctions, orthogonality;
- stability conditions: inequalities which tell us whether a star pulsates or not, driving and damping layers, phase lag;
- NOT the derivation of the LNAWE, but qualitative characterization of its solutions; the expression of the coefficient κ;
- the ϵ and κ-γ mechanisms: orders of magnitude;
- the opacity bump mechanism;
- the difference between self-excited and stochastically driven oscillation;
- the classical instability strip: not important to know the derivation;
- RR Lyrae, typical parameters;
- Cepheids: derivation of the mass-luminosity relation;
- NOT nonradial oscillations and astroseismiology;
- NOT red variables.

Mon Oct 28 2019

9 Stellar Winds

With Paola Marigo, now.

One should be able to follow this part even without a strong background in stellar astrophysics.

Bubble Nebula in Cassiopeia: a 45 $M_\odot$ star is ejecting mass at $1.7 \times 10^6\,\mathrm{m/s}$.

Some important quantities: we introduce

1. $\dot{M}$ is the mass loss rate:

$$\dot{M} = -\frac{\mathrm{d}M}{\mathrm{d}t} > 0\,; \tag{51}$$

2. v_∞ is the terminal wind velocity, in the limit of radial infinity.

The gas initially escapes from the star at low (subsonic, $1\,\mathrm{km/s}$) velocity; then it is accelerated. It is accelerated, and in the far field when no more forces are acting on it it approaches v_∞. We describe it with a *velocity law*: $v(r)$, and physically since the force is always radially outward we have

$$\frac{\mathrm{d}v}{\mathrm{d}r} > 0 \tag{52}$$

for any r. A typical law is something like:

$$v(r) = v_0 + (v_\infty - v_0)\left(1 - \frac{R_*}{r}\right)^\beta, \tag{53}$$

where $\beta \approx 0.8$, and R_* is the radius of the photosphere (where, from infinity, we have an opacity of $\tau = 1$).

In an H-R diagram we can plot the mass loss rate.

The relation $\dot{M}(M)$ seems to be something like a power law: what is it?

We have Massive, Cool Luminous and Solar-type stars.

The momentum input can come either from a force, like radiation pressure; or from heating.

We have:

1. coronal winds: driven by gas pressure;
2. line driven winds: driven by radiation pressure on highly ionized atoms, O and B stars;
3. dust driven winds: due to radiation pressure on dust grains, solid particles, which are very opaque.

Are ionized atoms more opaque to radiation?

We will not consider winds driven by pulsation, sound waves and Alfvèn waves (magnetic winds).

We will assume:

1. spherical symmetry;
2. stationarity;

3. no magnetic fields.

The equation of continuity with the hypothesis of stationarity is given by $\dot{M} = 4\pi r^2 \rho(r) v(r) \equiv \text{const}$.

If we differentiate $\dot{M}$ with respect to t we get 0 on the LHS, and on the RHS:

$$0 = 2\rho v + rv\frac{\mathrm{d}\rho}{\mathrm{d}r} + r\rho\frac{\mathrm{d}v}{\mathrm{d}r}\,, \tag{54}$$

where we simplified the 4π. If we divide by $\rho v r$ we get:

$$\frac{2}{r} + \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = 0\,, \tag{55}$$

and then this gives us

$$2\log(r)' + \log(v)' + \log(\rho)' = 0\,, \tag{56}$$

where the prime denotes derivatives with respect to r. The gradients of the velocity and density are related.

The force per unit volume is given by

$$F = \rho\frac{\mathrm{d}v}{\mathrm{d}t}\,, \tag{57}$$

and if we divide by ρ we get the force per unit mass:

$$f = \frac{F}{\rho} = \frac{\mathrm{d}v}{\mathrm{d}t}\,. \tag{58}$$

Under stationarity ($\partial_t = 0$), we have:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v(r)\frac{\mathrm{d}v}{\mathrm{d}r}\,. \tag{59}$$

The conservation of momentum gives us the Euler equation:

$$v\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} + f(r)\,, \tag{60}$$

where $f(r)$ is a generic unspecified external force, which we assume to be *outward* (no dissipative effects!).

The projection of $\nabla_\mu T^{\mu\nu} = 0$ along the flow of the fluid u^ν, which is a timelike Killing vector field, gives us the conservation of the energy: the 1st law of thermodynamics.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} + P\frac{\mathrm{d}\rho^{-1}}{\mathrm{d}t}, \tag{61}$$

where Q is the specific heat, u is the specific internal energy.

The internal energy, for an ideal gas, scales linearly with the temperature:

$$u = \frac{3}{2}\frac{k_B T}{\mu m_u} = \frac{3}{2}\frac{RT}{\mu}, \tag{62}$$

where μ is the mean molecular weight, while m_u is the atomic unit of mass: $R = k_B/m_u$.

We will also assume that the gas pressure follows the ideal gas law:

$$P = \frac{k_B T\rho}{\mu m_u} = \frac{RT\rho}{\mu}. \tag{63}$$

All time derivatives can be written as $\frac{\mathrm{d}}{\mathrm{d}t} = v\frac{\mathrm{d}}{\mathrm{d}r}$.

We define

$$q(r) = \frac{\mathrm{d}Q}{\mathrm{d}r}, \tag{64}$$

the heat input or loss per unit mass per unit distance in the wind. Inserting in the previous equation, we get:

$$q = \frac{3}{2}\frac{R}{\mu}\frac{\mathrm{d}T}{\mathrm{d}r} + P\frac{\mathrm{d}\rho^{-1}}{\mathrm{d}r}. \tag{65}$$

These can be incorporated in the global energy equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v^2}{2} + \frac{5}{2}\frac{RT}{\mu} - \frac{GM}{r}\right) = f(r) + q(r), \tag{66}$$

where the expression inside the brackets is the total internal energy. The terms are called A, B, C. This clarifies the claim from before: we can change the internal energy by applying a force.

Was the claim from before not the reverse?

Equation (66) can be written in integral form, by integrating from a generic radius r_0, we could choose the photospheric radius.

Near the photosphere $|C| \gg A + B$, so $e(r_0) \approx GM/r_0 < 0$.

At large radii $B, C \to 0$: $e(r_\infty) \approx v_\infty^2/2$.

If we integrate from r_0 at the photosphere to infinity then we get:

$$\frac{v_\infty^2}{2} = -\frac{GM}{r_0} + \int_{r_0}^{\infty} f(r) + q(r)\,\mathrm{d}r, \tag{67}$$

so the sum of the two integrals must be large enough for the gas to escape the gravitational well.

Can't this be made stronger by simply swapping the equality with a less than sign, and integrating not only from r_0 but from any radius?

Now we will treat isothermal winds. These calculations were first done by Parker.

The assumption of constant temperature already gives us $T(r) \equiv T$, and then we do not need the energy equation: there must be an energy input.

So we do not consider the energy equation because energy *is not conserved?*

This is a good model for low $\dot{M}$, which does not affect the total mass of the star.

Tomorrow we will discuss full solutions in this case.

Now, let us talk about the corona. It is very hot: it goes to 10^6 degrees, but with very low density. We see this experimentally by seeing highly ionized elements like $\mathrm{Fe}^{+5\div 13}$.

Why is this the case? We *do not know*. Magnetism?

Tue Oct 29 2019

Recall the equations from last time: continuity, momentum and energy conservation in the isothermal case:

$$\dot{M} = 4\pi r^2 \rho v \tag{68a}$$

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} \tag{68b}$$

$$T(r) \equiv T\,. \tag{68c}$$

The term $\rho^{-1}\,\mathrm{d}P/\mathrm{d}r$ is equal to

$$\frac{RT}{\mu}\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\,, \tag{69}$$

so we manipulate the equation into

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{RT}{\mu}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2}\,, \tag{70}$$

but we know by the continuity equation that

$$-\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\,, \tag{71}$$

which we can substitute into the equation: we get

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{RT}{\mu} \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) - \frac{GM}{r^2} \,. \tag{72}$$

The isothermal speed of sound is given by

$$a^2 = \frac{\partial P}{\partial \rho} = \frac{\partial}{\partial \rho} \left(\frac{R \rho T}{\mu} \right) = \frac{RT}{\mu} \,, \tag{73}$$

so

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -a^2 \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) - \frac{GM}{r^2} \,. \tag{74}$$

Now we put all the terms which are proportional to the velocity gradient on the LHS:

$$\frac{\mathrm{d}v}{\mathrm{d}r} \left(v - \frac{a^2}{v} \right) = \frac{2a^2}{r} - \frac{GM}{r^2} \,, \tag{75}$$

which is just

$$\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} \left(v^2 - a^2 \right) = \frac{2a^2}{r} - \frac{GM}{r^2} \,, \tag{76}$$

so the Jacobian of the differential equation is zero is singular in $v = a$: if $v = a$ we must have $2a^2 r = GM$, which fixes the radius to the Parker radius: $r_P = GM/2a^2$. Close to the star, the speed is subsonic and the numerator is negative in

$$\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{N}{D} \,, \tag{77}$$

which is consistent with our assumption $\mathrm{d}v/\mathrm{d}r > 0$, which we make since we are considering winds, as opposed to accretion.

Far from the star the numerator is positive, so the speed must still be supersonic.

The critical velocity *must* be attained at the Parker radius in order to have a phyisically meaningful solution.

The velocity gradient at the critical point can be found by de l'Hôpital's rule to be

$$\left. \frac{\mathrm{d}v}{\mathrm{d}r} \right|_{r_P} = \pm \frac{a^3}{GM} \,. \tag{78}$$

So, the only physical solution is transsonic.

Claim 9.1 (Exercise). *The speed of sound at the critical point equals half of the escape velocity at that radius.*

The boundary condition is the velocity at some r_0.

There are other solutions, but if we trace a cross in the r, v plane centered on the critical point and speed of sound we see that all solutions meet it perpendicular to it.

Accretion solutions are also found in this diagram: what is plotted is the *absolute value* of the velocity. They have always-negative absolute velocity gradient.

Always-supersonic solutions and always-subsonic ones are also found, but they do not obey the monotonicity of the velocity.

The choice we make for $v_0 = v(r_0)$ is key, and nontrivial.

If we have the density, velocity and radius of the lower boundary we have fixed the accretion rate: $\dot{M} = 4\pi r_0^2 \rho_0 v_0$. We can fix these by fixing the constant temperature T and the velocity v_0 at the critical radius r_0 and assuming our solution is transsonic.

What is stated in the slides seems different. What are we fixing?

We can solve the momentum equation analytically: we get

$$\frac{v}{v_0} \exp\left(\frac{-v^2}{2a^2}\right) = \left(\frac{r_0}{r}\right)^2 \exp\left(\frac{GM}{a^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right). \tag{79}$$

At large distances, we get $v \to 2a \ln(r/r_0)$.

Now we look at the structure of the wind in the subcritical region.

In the corresponding slide: the dashed line in the density profile is the density we'd expect in a hydrostatic atmosphere, while the solid one is the solution.

The hydrostatic density structure is given by

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0\,, \tag{80a}$$

manipulating this we get

$$\frac{r^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{GM}{a^2}\,, \tag{81}$$

So the density profile follows a decreasing exponential law:

$$\frac{\rho(r)}{\rho_0} = \exp\left(\frac{-(r - r_0)}{H_0}\frac{r_0}{r}\right), \tag{82}$$

where H_0 is the scale height, $H_0 = RT/(\mu g_0)$ with $g_0 = GM/r_0^2$. The length scale at which this decreases is defined by H_0.

The density profile in the subsonic region is very well approximated by this hydrostatic profile.

The mass loss rate is our main prediction: we have $\dot{M} = 4\pi r_0^2 \rho_0 v_0 = 4\pi r_c^2 \rho_c a$.

Then, we can use the density profile equation:

$$\dot{M} = 4\pi r_c^2 a \rho_0 \exp\left(\frac{-(r_c - r_0)}{H_0}\frac{r_0}{r_c}\right). \tag{83}$$

If we consider this numerically, we find that the exponential is the dominant part. The mass loss rate is lower when the critical point moves outward. We can specify it by fixing

1. the temperature at the corona, T_C;
2. the radius at the bottom of the corona, r_0;
3. the stellar mass M;
4. the density at the bottom of the corona ρ_0.

In the slides there are numerical estimates. As H_0 increases, the density profile is less steep: the density remains high at larger radii.

Mon Nov 04 2019

No lectures tomorrow, instead on Wednesday at 14.30.

Last time we found that the only physical solution to the stellar wind is the transsonic solution.

Now, let us add an external force:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} + f(r)\,, \tag{84}$$

so we get

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2\frac{a^2}{r} - \frac{GM}{r^2} + f(r)}{v^2 - a^2}\,, \tag{85}$$

with the speed of sound $a = \sqrt{RT/\mu}$. How does the velocity gradient change? We expect the velocity gradient to be less steep in the subsonic region (the velocity decreases slower: the numerator is *less negative*).

In the supersonic region, the gradient will be larger: the numerator is *more positive* and higher velocities are reached.

The critical radius changes: it is the solution to

$$r_C = \frac{GM}{2a^2} - \frac{f(r_C) r_C^2}{2a^2}\,, \tag{86}$$

which will shift inward as f goes from 0 to positive.

Then, we can show that the velocity at the corona must be larger: the critical radius is further inward, and the gradient is less steep.

How do we expect the mass loss rate to change? from the continuity equation at the bottom of the corona, we get $\dot{M} = 4\pi r_0^2 \rho_0 v_0$, and everything on the RHS is fixed but v_0, so when we increase it the LHS must incrase as well.

Let us consider some explicit law scaling as $f \propto r^{-2}$, like a radiative force:

$$g_{\text{rad}} = \kappa_F \times \text{stuff} \times \left(\frac{r}{R}\right)^{-2} . \tag{87}$$

This is the same as changing the mass of the star, since it scales like the gravitational force. We take our force to be $f(r) = A/r^2$. Our equation becomes

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \left(\frac{2a^2}{r} - \frac{GM}{r^2} + \frac{A}{r^2}\right) / \left(v^2 - a^2\right) , \tag{88}$$

so the effective mass is $M_{\text{eff}} = M(1 - A/GM)$.

This is usually called the Eddington ratio:

$$\Gamma = \frac{A}{GM} = \frac{A/r^2}{GM/r^2} , \tag{89}$$

the acceleration of the force divided by the gravitational one.

Are the units of A and GM not m^3s^{-2}?

The critical radius becomes

$$r_C = \frac{GM}{2a^2}(1 - \Gamma) , \tag{90}$$

however in general A is not taken as a constant, but activates only after a certain radius.

As Γ increases, the critical velocity is reached faster, and the velocity at the corona is greater. The density profile gets less and less steep: the density scale height decreases.

We approximate $A(r) = A[r \geq r_d]$ for some r_d.[1] Below the dust condensation region there is only gas, above it there is dust which is more opaque to radiation.

The critical point depends on Γ:

$$\frac{r_C}{1 - \Gamma(r_C)} = \frac{GM}{2a^2} . \tag{91}$$

If the extra force switches on *outside* the critical region, the mass loss rate is *unchanged*, since it only depends on the subsonic region.

Is there the possibility to have more than one critical radius with only outward force?

[1]Using the Iverson bracket here!

Wed Nov 06 2019

Now, we consider the possibility that our winds are *not isothermal*. This will change the structure of the wind, by the introduction of an additional pressure gradient.

It will change the speed of sound and thus the Mach number.

We use the energy per unit mass e:

$$e(r) = \frac{v^2(r)}{2} - \frac{GM}{r} + \frac{\gamma}{\gamma - 1}\frac{RT}{\mu}\,, \tag{92}$$

where $\gamma/(\gamma - 1) = 5/2$ for a monoatomic gas, which has $\gamma = 3/2$.

In the lower boundary of the wind the velocity is much less than the escape velocity ($v \ll v_{\text{esc}}$), while far from the star we have $v \gg v_{\text{esc}}$.

If we have an isothemal wind, some energy must be added in order to prevent the adiabatic cooling of the gas, lift it from the potential well, and to increase its potential energy.

If a force is applied it increases the momentum, but the heat transmission q does not appear in the momentum equation, which is $\Delta e = \int f + q \, \mathrm{d}r$. This is how it would seem, but heat transmission changes the pressure profile, which affects the momentum.

The most general momentum equation is given by:

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \left(2\frac{c_s^2}{r} - \frac{GM}{r} + f - (\gamma - 1)q \right) \Big/ \left(v^2 - c_s^2 \right)\,, \tag{93}$$

where we introduce the adiabatic speed of sound $c_s = \sqrt{\gamma a^2}$. In general $-(\gamma - 1) < 0$, therefore if we add heat this is equivalent to pushing *inward*.

If either f or q depend on the velocity gradient $\mathrm{d}v/\mathrm{d}r$ then the sonic point can *decouple* from the critical point.

There are cases in which we have multiple critical points (specifically, multiple zeros of the denominator).

The momentum equation plus the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{v^2}{2} + \frac{5}{2}\frac{RT}{\mu} - \frac{GM}{r} \right) = f(r) + q(r)\,, \tag{94}$$

can be solved numerically, and if we impose smooth passage through the critical point this yields the mass loss rate.

Qualitatively, the results are the same as in the isothermal case.

This ends our general introduction to stellar winds.

Now we will do a couple of exercises to get familiar with the theory.

Exercise

The wind is isothermal. The solar wind has a mean coronal temperature of 1.5×10^6 K and a mass loss rate of 2×10^{-14} solar masses per year. The bottom of the corona is at $r_0 \approx 1.003 R_\odot$, where the density is $\rho(r_0) = 10^{-14}\,\mathrm{g/cm^3}$.

Calculate the potential energy, the kinetic energy and the enthalpy of the gas at r_0.

Calculate the same quantities at the critical point. Which of these energies has absorbed the largest fraction of the energy input?

We can use the continuity equation $\dot{M} = 4\pi\rho_0 r_0^2 v_0$ to get

$$v_0 = \frac{\dot{M}}{4\pi r_0^2 \rho_0} \approx 21\,\mathrm{m/s}\,, \tag{95}$$

and with this we can calculate

$$e(r_0) = -\frac{GM}{r_0} + \frac{1}{2}v_0^2 + \frac{5}{2}\frac{RT}{\mu}\,. \tag{96}$$

We get:

$$E_{\text{kin, 0}} = \frac{v_0^2}{2} \approx 212\,\mathrm{J/kg}\,, \tag{97}$$

while for the gravitational energy we'd need the mass of the star. Assuming it is equal to the solar mass, we find

$$E_{\text{grav, 0}} = -\frac{GM}{r_0} \approx -1.9 \times 10^{11}\,\mathrm{J/kg}\,. \tag{98}$$

The mean molecular weight of the gas for the Sun is something like $\mu = 0.62$ (we count electrons in it). Then we get

$$E_{\text{chem, 0}} = E_{\text{chem, crit}} = \frac{5}{2}\frac{RT}{\mu} \approx 5.0 \times 10^7\,\mathrm{J/kg}\,. \tag{99}$$

The enthalpy is the same everywhere in the flow, since the flow is isothermal.

The values at the critical radius are calculated with the same formula. The velocity will be the speed of sound $a = \sqrt{RT/\mu} \approx 4.5 \times 10^3\,\mathrm{m/s}$.

Then we find:

$$E_{\text{kin, crit}} = \frac{a^2}{2} \approx 1.0 \times 10^7\,\mathrm{J/kg} \approx 5 \times 10^4 \times E_{\text{kin, 0}}\,. \tag{100}$$

The critical radius is given by $r_c = GM/(2a^2)$: so, we get

$$E_{\text{grav, crit}} = -\frac{GM}{r_c} = -2a^2 \approx -4.0 \times 10^7\,\mathrm{J/kg} \approx 2.1 \times 10^{-4} \times E_{\text{grav, 0}}\,. \tag{101}$$

Qualitatively, at the corona we have

$$\left|E_{\text{grav, 0}}\right| \gg E_{\text{chem, 0}} \gg E_{\text{kin, 0}}\,, \tag{102}$$

while at the critical point they are similar, and specifically

$$E_{\text{chem, crit}} \gtrsim \left|E_{\text{grav, crit}}\right| \gtrsim E_{\text{kin, crit}}\,. \tag{103}$$

Exercise

A star with $T_{\text{eff}} = 3200\,\mathrm{K}$, $R_* = 30R_\odot$, $L_* = 85L_\odot$ and $M_* = 6M_\odot$ has an isothermal corona of $T = 10^6\,\mathrm{K}$ with a density at the lower boundary of $10^{-13}\,\mathrm{g/cm^3}$.

Calculate the energy per unit mass at the bottom of the corona at $r_0 = R_*$.

Calculate the location of the critical point, r_c, and the mass loss rate.

Calculate the energy per gram gained by the wind between r_0 and r. What fraction of the stellar luminosity is used to drive the wind up to the critical point?

The energy per unit mass at the bottom of the corona is given by

$$e(r) = -\frac{GM}{r_0} + \frac{1}{2}v_0^2 + \frac{5}{2}\frac{RT}{\mu}\,, \tag{104}$$

but we cannot use this formula since we do not have v_0 nor $\dot{M}$. However, we can approximate the density profile as an exponential, applying the formula

$$\dot{M} = 4\pi r_c^2 a \rho_c \tag{105a}$$

$$= 4\pi r_c^2 a \rho_0 \exp\left(-\frac{r_c - r_0}{H_0}\frac{r_0}{r_c}\right)\,, \tag{105b}$$

where we have: the length scale $H_0 = RTr_0^2/(GM\mu)$, the critical velocity $a = \sqrt{RT/\mu}$, and the critical radius $r_c = GM/(2a^2)$.

Plugging these in, we find:

$$\dot{M} = 4\pi \left(\frac{GM\mu}{2RT} \right)^2 \sqrt{\frac{RT}{\mu}} \rho_0 \exp\left(\frac{\mu GM}{RT} \left(\frac{2RT}{GM\mu} - \frac{1}{r_0} \right) \right) \tag{106}$$

$$= \pi (GM)^2 \left(\frac{\mu}{RT} \right)^{3/2} \rho_0 \exp\left(2 - \frac{\mu GM}{RTr_0} \right). \tag{107}$$

One temperature should probably be the effective temperature, but I do not really know what that means.

Then, the mass loss rate can be calculated, since we have all of these quantities. It comes out to be barely anything, since $\mu GM / RTr_0$ is very large and we have a negative exponential of it...

Lamers-Cassinelli: chap 4, section 3: multiple critical points.

Next Tuesday, the 12th, after the lecture, we will likely move to the DFA in Via Marzolo until 14.30.

Now, we introduce the next topic. We deal with hot, luminous stars: at the top left of the HR diagram.

[Picture from the slides]

We talk of line-driven winds for hot stars. These are winds driven by *spectral lines*. The blackbody emission for these stars is mainly at high frequencies, like the UV.

One may ask: hydrogen is much more abundant than heavier elements like carbon, nitrogen...why do we see the spectral lines for these heavier elements?

This is because hydrogen is completely ionized at these temperatures, and helium is also.

The strongest lines are far in the UV, where the stellar flux is low: only few atoms are hit, the rest of the gas is dragged along.

A peculiar characteristic of the spectrum is the so-called P-Cygni profile.

Now, an overview of the formation of spectral lines: there are 5 processes

1. Line scattering: if it comes from the ground state of the atom, it is called *resonance scattering*, which is the main phenomenon.
2. Emission by recombination: the ion recombines to an excited state.
3. Emissional from collisional or photo-excitation. A photon is absorbed, it excites the atom which then descends to a lower energy level.
4. Pure absorption and then de-excitation of an already excited atom.

5. Masering by stimulated emission: it can happen in a very narrow set of circumstances: an excited atom is hit by a photon which has exactly the same energy as the one between the atom's state and the ground state, so the atom is deexcited and now there are two photons. This happens when there are many excited atoms, and when the velocity gradient is very small — otherwise, the photons are Doppler-shifted out of the right frequency.

Mon Nov 11 2019

Tomorrow we have the meeting at 13.30 in room C.

On the 19th the lecture is in room A (still @ Specola).

Now we deal with line driven winds. The main mechanism is line scattering.

9.1 P-Cygni profiles

They are spectra characterized by a blue-shifted absorption and a red-shifted emission.

We consider a source which emits a spherically symmetric wind. The region of the wind which is directed at us corresponds to a blue-shifted absorption. We basically have emission which is symmetric centered at $v = 0$ (since most of the radiation comes from the photosphere), and absorption which is caused by atoms moving towards us at velocities $0 \leq v \leq +v_\infty$: so the absorption is something like $-k[0 \leq v \leq v_\infty]$.

We can gather data from these profiles: for example, the maximum Doppler shift of the absorption corresponds to the maximum velocity.

We can also infer the number densities of the chemical species in the wind:

$$n_i = \frac{X_i \rho}{A_i m_u}, \tag{108}$$

assuming some parametric velocity profile $v(r)$ and density profile $\rho(r)$, by this we derive the mass loss rate.

The opacity of the spectral lines is several orders of magnitude larger than the continuum opacity: for example, the opacity of the IV line of Carbon is $\sim 10^6 \kappa_{es}$.

When a photon is absorbed, the momentum increases by $\Delta p = h\nu / c$. The increase in velocity when a typical metal absorbs an UV photon with wavelength 10^{-7} m is of the order $\Delta v \approx 20\,\mathrm{cm/s}$. Typically, due to the redistribution of momentum among atoms, this is something like four orders of magnitude less. All the gas is then accelerated by the radiation.

In order to accelerate the gas to $2000\,\mathrm{km/s}$ we'd need 10^{11} photons. If the distance to accelerate to terminal velocity is about three sun radii, the time to accelerate is of the order 10^4 s. So we need around 10^7 photons per second: the typical lifetime of the transition is of the order 10^{-7} s,

So only transitions with oscillator strengths $f \gtrsim 0.01$ will contribute significantly.

What is oscillator strength? for $\Delta t \approx 10^{-7}$ s we have $\Delta E = \hbar/\Delta t \approx 6$ neV...

The largest contribution is not the energy of the photons, but their momentum input.

Let us calculate the radiation pressure due to one line. The momentum equation is

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} + f(r)\,, \tag{109}$$

and we want to compute the line-driven $f(r)$.

Say we have a line @ wavelength λ_0, which we assume coincides with the peak of the Planckian function for the star. We also assume that the line is so strong that it absorbs or scatters *all* the photons at its wavelength.

So, photons at that wavelength have 0 mean free path?
It seems like that is not necessary actually.

How much mass loss can one optically thick line produce?

The emitted wavelengths which are absorbed are all the ones between λ_0 and $\lambda_0(1 - v_\infty/c)$.

If our velocity range goes from $v = 0$ ar $r = R$ to $v = v_\infty$ at $r = \infty$, then the total energy absorbed is:

$$L_{\text{line}} = \int_{\nu_0}^{\nu_0(1+v_\infty/c)} \underbrace{F_\nu 4\pi R^2}_{L_\nu} \,\mathrm{d}\nu \approx L_{\nu_0}\Delta\nu = L_{\nu_0}\nu_0 v_\infty/c\,, \tag{110}$$

where F_ν is the monochromatic flux (specific spectral intensity in $\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}$) at the photosphere, whose radius is R. We approximate the flux to be almost constant with respect to ν.

L_{ν_0} is the specific luminosity per unit frequency at ν_0.

The variation of radiative momentum is

$$\frac{\mathrm{d}p_{\text{rad}}}{\mathrm{d}t} = \frac{L_{\text{line}}}{c} = \frac{L_{\nu_0}\nu_0 v_\infty}{c^2}\,, \tag{111}$$

and the total momentum loss in the wind per second is

$$\frac{\mathrm{d}p_{\text{wind}}}{\mathrm{d}t} = \dot{M}v_\infty\,, \tag{112}$$

so if we approximate the wind to be momentum wind, we can equate these two terms: then we get

$$\frac{L_{\nu_0}\nu_0}{c^2} = \dot{M}\,. \tag{113}$$

Now, the Planck function has the property that $L_{\nu_0}\nu_0 = 0.62L$. Then we see that the mass loss rate of one strong line is linear in the luminosity of the star.

These are additive: if we can approximate the mass loss rate and luminosity with other means, we can find the total number of spectral lines.

Tue Nov 12 2019

Last time we estimated the radiation pressure due to one line.

If we know $\dot{M}$ and the luminosity L, then we can calculate the effective number of lines: $N_{\text{eff}} = \dot{M}c^2/L$.

We define the *efficiency parameter*:

$$\eta = \frac{\dot{M}v_\infty}{L/c}\,. \tag{114}$$

This can be split into various components due to several factors: we always divide by the total luminosity, and have

$$\eta_{\text{pot}} = \frac{\dot{E}_{\text{pot}}}{L} = \frac{\dot{M}GM_*}{R_*L} \tag{115a}$$

$$\eta_{\text{kin}} = \frac{\dot{E}_k}{L} = \frac{\dot{M}v_\infty^2}{2L} \tag{115b}$$

$$\eta_{\text{th}} = \frac{5\dot{M}RT_w}{2\mu L}\,, \tag{115c}$$

and we will need

$$\eta_{\text{moment}} = \frac{\dot{M}v_\infty c}{L}\,. \tag{116}$$

This is basically the energy emission budget, L, divided into its contributions.

The efficiency for non-momentum components is very low as compared to momentum efficency: what matters, again, is the transfer of momentum, not the transfer of energy.

We'd expect $\eta \leq 1$, but actually sometimes the momentum efficiency can be larger: in one case we have $\eta_{\text{momentum}} \approx 59$.

If across all the spectrum of the star we have completely optically thick absorption, then the momentum of the wind is equal to the momentum of radiation:

$$\dot{M}_{\max} v_\infty = \frac{L}{c} \implies \dot{M}_{\max} = \frac{L}{c v_\infty}, \tag{117}$$

and typically v_∞ is of the order of 2, 3 times the escape velocity at the photosphere: $v_\infty \approx 3\sqrt{2GM/R}$. For the luminosity we consider $L = 10^5 L_\odot$.

Then, the maximum mass loss rate estimated by (117) is similar to the observed one. The ratio $\eta_{\text{momentum}} = \dot{M}/\dot{M}_{\max}$ is typically $0.5 \div 1$, but for some stars it can be of the order 10^1 to 10^2.

This is called the *single scattering upper limit*: we assume that scattering is isotropic, therefore we'd expect that after the first scattering the photon does not contribute anymore. This is not actually the case: *multiple scattering* can contribute, enhancing the maximum mass loss rate by a factor τ_w, equal to

$$\tau_w = \int_{r_c}^{\infty} \kappa \rho \, \mathrm{d}r \,, \tag{118}$$

so $\dot{M}_{\text{max, multiple scattering}} = \tau_w \dot{M}_{\text{max, single scattering}}$.

This typically gives us enhancement factors of the order 2 to 6. The quantity τ_w is adimensional, since the units of κ are cm^2/g.

Even though we have this corrective factor, the efficiency is always expressed with respect to the single scattering cross section. Multiple scattering theory accounts for all of the increased mass loss rate, even up to $\tau \sim 10^2$.

Now, we derive the expression for the radiative acceleration privided by one line in a moving atmosphere.

We have a unit volume, of $1\,\text{cm}^3$, it has a velocity gradient inside it from v to $v + \Delta v$, and it will be heated by a monochromatic flux given by $F_\nu = I_\nu/(4\pi r^2)$.

What is the acceleration g_{line}? First we need to know the absorption properties of the medium. The absorption coefficient per cubic centimeter of gas is

$$\kappa_\nu = \frac{\pi e^2}{m_e c} f n_i \phi(\nu) \,, \tag{119}$$

where $\pi e^2/(m_e c) \approx 2.654 \times 10^{-2}\,\text{cm}^2/\text{s}$ is the frequency integrated cross section of the classical oscillator (for the electron):

$$\frac{\pi e^2}{m_e c} = \sigma = \int_0^\infty \sigma(\nu) \, \mathrm{d}\nu \,; \tag{120}$$

f is the oscillator strength, which depends on the line (it is a probability); n_i is the number density of atoms which can absorb the line, and $\phi(\nu)$ is the Gaussian profile of the absorption coefficient, centered at the rest frequency ν_0, and normalized so that $\int \phi(\nu) \, \mathrm{d}\nu = 1$: therefore the units of ϕ are $1/\text{Hz}$.

The typical profile function is a *Doppler profile*: its width is of an order based on the thermal velocity of the atoms, which can sometimes be approximated as much less than the wind velocity: so, we apply the *Sobolev approximation*, and estimate $\phi(\nu) \sim \delta(\nu - \nu_0)$.

So we need to consider lines which do not overlap.

We get

$$F_{\text{rad}} = \frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}t} = \frac{1}{c}\frac{\mathrm{d}E_{\text{rad}}}{\mathrm{d}t}, \tag{121}$$

where $\mathrm{d}E_{\text{rad}}/\mathrm{d}t$ is the radiative energy absorbed per unit time and volume by the line.

Then we get

$$g_{\text{line}} = \frac{F_{\text{rad}}}{\rho} = \frac{1}{c\rho}\frac{\mathrm{d}E_{\text{rad}}}{\mathrm{d}t}. \tag{122}$$

We still need to calculate the radiative energy absorbed: for a single optically thick line is $\mathrm{d}E_{\text{rad}}/\mathrm{d}t = F_\nu \kappa_{\text{line}}$.

A more general expression, which however still applies the Sobolev approximation, is

$$g_{\text{rad}} = \frac{F_{\nu_0}\nu_0}{c}\Big(1 - \exp\big(-\tau_{\nu_0}(\mu = 1)\big)\Big)\frac{\mathrm{d}v}{\mathrm{d}r}\frac{1}{c\rho}; \tag{123}$$

What is $\mu = 1$ about?

here F_{ν_0} is the monochromatic flux from the star emitted at the line rest frequency ν_0, the quantity

$$\frac{\nu_0}{c}\frac{\mathrm{d}v}{\mathrm{d}r} \tag{124}$$

is the width of the frequency band that can be absorbed, while $1 - \exp(\dots)$ is the probability that the absorption occurs in our cubic centimeter: there τ is the optical depth (integrated up to infinity).

The product of these three terms gives an energy absorption rate $\mathrm{d}E_{\text{rad}}/\mathrm{d}t$, if we multiply by $1/c$ we get absorbed momentum, if we multiply by $1/\rho$ we get acceleration.

Let us start from optically thin line: they absorb only part of the radiative flux.

Then we get $\exp(-\tau_{\nu_0}) \sim 1 - \tau_{\nu_0}$, therefore $1 - \exp(-\tau_{\nu_0}) = \tau_{\nu_0}$. This is proportional to $\rho\kappa \propto n_i$.

In the end, we get

$$g_{\text{line}} = \frac{F_\nu}{c}\frac{n_i}{\rho}\left(f\frac{\pi e^2}{m_e c}\right) \sim \frac{L_\nu}{r^2}\frac{n_i}{\rho}, \tag{125}$$

where L_ν is the monochromatic luminosity of the star: we have $F_\nu \sim L_\nu / r^2$. We can approximate n_i/ρ as a constant with respect to the radius: therefore, we have $g_{\text{line}} \propto L_\nu / r^2$.

Mon Nov 18 2019

Let us come back to

$$g_l = \frac{F_\nu}{c} \frac{n_i}{\rho} \frac{\pi e^2}{m_e c} f \sim \frac{L_\nu}{r^2} \frac{n_i}{\rho} \,. \tag{126}$$

We can reduce drastically the number of spectral lines we have to account for if we assume that the density of the wind is very low: only lines from the ground state have to be accounted for.

How do we deal with the radiation pressure for an ensemble of lines? We have the CAK formalism: it gives the estimate

$$g_L \sim \left(\rho^{-1} \frac{\mathrm{d}v}{\mathrm{d}r} \right)^\alpha \sim \left(\frac{v r^2}{\dot{M}} \frac{\mathrm{d}v}{\mathrm{d}r} \right)^\alpha \,, \tag{127}$$

where α is a parameter quantifying the optical thickness of the line: it goes from 0 for an optically thin line to 1 for an optically thick line. In general we have from the continuity equation $\dot{M} \propto r^2 \rho v$, which justifies the expression here.

The expression proposed by CAK was $g_L = g_e M(t)$, where g_e is the radiative acceleration from continuum phenomena (mostly electron scattering), while the *force multiplier* M is in the form:

$$M(t) = k t^{-\alpha} s^\delta \,, \tag{128}$$

where κ, α, δ are called *force multiplier parameters*.

The radiative acceleration due to electron scattering is given by

$$g_L(e) = \frac{\kappa_e}{c} \frac{L_*}{4\pi r^2} = \Gamma_e \frac{G M_*}{r^2} \,, \tag{129}$$

where Γ_e is the so-called *Eddington factor*:

$$\Gamma_e = \frac{\kappa_e}{4\pi c G} \frac{L_*}{M_*} \,, \tag{130}$$

which is the luminosity divided by the Eddington luminosity. The Compton opacity κ_e is given by

$$\kappa_e = \sigma_e \frac{m_e}{\rho} \,, \tag{131}$$

which is measured in $\mathrm{cm^2 g^{-1}}$. The value $\sigma_e = 6.65 \times 10^{-25}\,\mathrm{cm^2}$ is the Thomson scattering cross section for electrons.

The t in $M(t)$ is defined by

$$t \equiv \kappa_e v_{\text{thermal}} \rho \frac{\mathrm{d}r}{\mathrm{d}v} = \kappa_e \sqrt{\frac{2k_B T}{m_H}} \rho \frac{\mathrm{d}r}{\mathrm{d}v}, \tag{132}$$

and is inversely proportional to the velocity gradient.

The s in the definition of $M(t)$ contains information about the degree of ionization: it is

$$s \equiv \frac{10^{-11}\rho}{m_H W}, \tag{133}$$

where W is the *geometrical dilution factor*:

$$W(r) = 0.5\left(1 - \sqrt{1 - \left(\frac{R_*}{r}\right)^2}\right) \sim \left(\frac{R_*}{2r}\right)^2, \tag{134}$$

and now we present a proof for the fact that this encodes the radial dependence of the observed intensity. It is a ratio of solid angles:

$$W(r) = \frac{\int_0^\Omega \mathrm{d}\Omega}{4\pi}, \tag{135}$$

where the integration limit Ω encodes the solid angle subtended by the star. Assuming simmetry with respect to the azimuthal angle, we can rewrite it as:

$$W(r) = \frac{2\pi}{4\pi} \int_0^{\theta_1} \sin\theta \,\mathrm{d}\theta = \frac{1}{2} \int_1^{\cos(\theta_1)} (-\,\mathrm{d}x) = \frac{1}{2}\left(1 - \sqrt{1 - (R/r)^2}\right), \tag{136}$$

so $W(r)$ is the solid angle fraction subtended by a star with radius R at a distance r.

The complete expression for an ensemble of lines is given by

$$g_L = \frac{\kappa_e}{c} \frac{L_*}{4\pi r^2} k t^{-\alpha} s^\delta, \tag{137}$$

Simulations show that $M(t)$ decreases with t with some kind of power law: the approximation $\log M \sim -\alpha \log t$ is justified.

Numerical simulations show that $\alpha \sim 0.5$, independent of temperature. The δ parameter is almost always of the order $\delta \sim 0.1$.

We also expect a dependence on the metallicity: $M_n(t_n) = M_n(t_n)_\odot (Z/Z_\odot)$.

A typical velocity gradient is something like

$$v(r) = v_\infty \left(1 - \frac{r_0}{r}\right)^\beta, \tag{138}$$

with $\beta \sim 0.7$. Then, the profile of g_L can be computed (using the continuity equation): it is in the form

$$g_L \sim r^{-2} \left(\rho \frac{\mathrm{d}r}{\mathrm{d}v} \right)^{-\alpha} \sim r^{2(\alpha-1)} \left(v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha}, \tag{139}$$

where $\alpha \sim 0.6$. The r dependence is something like

$$g_L \sim r^{-0.8} \left(1 - \frac{r_0}{r} \right)^{0.21}, \tag{140}$$

which we can plot.

Tue Nov 19 2019

Yesterday we introduced the CAK formalism.

We consider a spherically symmetric, stationary problem, and we assume that the star looks pointlike to us. Also, we assume that the process is isothermal: $\forall r : T = \text{const}$.

The momentum conservation equation is

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{GM}{r^2} - \frac{1}{r} \frac{\mathrm{d}P}{\mathrm{d}r} + g_c + g_L, \tag{141}$$

where we distinguished the continuum acceleration g_c and the line acceleration g_L.

We assume that the gas follows the ideal gas law:

$$P = \frac{R \rho T}{\mu}, \tag{142}$$

which implies

$$-\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{RT\rho}{\mu} \right) = -\frac{1}{\rho} \frac{RT}{\mu} \frac{\mathrm{d}\rho}{\mathrm{d}r}, \tag{143}$$

where we can recognize the square speed of sound $a^2 = RT/\mu$. Using the continuity equation then we get

$$-\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}r} = a^2 \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right). \tag{144}$$

For the continuum and line acceleration we use the CAK formalism:

$$g_c = \frac{GM}{r^2} \Gamma_e, \tag{145}$$

and

$$g_L = \frac{\kappa_e}{c} \frac{L}{4\pi r^2} k t^{-\alpha} s^{\delta}\,, \tag{146}$$

so in the end we get:

$$v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{GM}{r^2} + a^2 \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) + \frac{GM}{r^2} \Gamma_c + \frac{\kappa_e}{c} \frac{L}{4\pi r^2} k t^{-\alpha} s^{\delta}\,. \tag{147}$$

Now recall

$$t = C' \rho \frac{\mathrm{d}r}{\mathrm{d}v} = C' \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{-1}, \tag{148}$$

where the constant $C' = \kappa_e \sqrt{2 k_B T_{\text{eff}} / m_H}$ encodes all the physical constants.

We plug this into the momentum conservation equation and multiply by r^2:

$$v r^2 \frac{\mathrm{d}v}{\mathrm{d}r} = -GM(1 - \Gamma_e) + a^2 r^2 \left(\frac{1}{v} \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r} \right) + C \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha}, \tag{149}$$

where

$$C = \frac{\kappa_e}{c} \frac{L}{4\pi} k \left(\kappa_e \sqrt{\frac{2 k_B T_{\text{eff}}}{m_H}} \frac{\dot{M}}{4\pi} \right)^{-\alpha} \left(\frac{10^{11} \rho}{m_H W} \right)^{\delta}. \tag{150}$$

Also, we can bring a term to the RHS: we get

$$\left(1 - \frac{a^2}{v^2} \right) v r^2 \frac{\mathrm{d}v}{\mathrm{d}r} = -GM(1 - \Gamma_e) + 2a^2 r + C \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha}. \tag{151}$$

Now, we make a simplifying assumption: we neglect the gas pressure. This is reasonable at large distances from the star. It also can be shown that the physical main points we will find are the same as we would find if we did consider the gas pressure. Since the speed of sound is $a^2 = \partial P / \partial \rho$ neglecting the pressure gradient means $a = 0$: removing the terms with a we get

$$r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} - C \left(r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} \right)^{\alpha} = -GM(1 - \Gamma_e) = \text{const}\,, \tag{152}$$

and if we denote $r^2 v \,\mathrm{d}v / \mathrm{d}r = D$, the equation can be written as

$$D - C D^{\alpha} = -GM(1 - \Gamma_e)\,. \tag{153}$$

Our parameters here are D and $C \propto \dot{M}^{-\alpha}$.

The system is equivalent to $CD^{\alpha} = D + GM(1 - \Gamma_e)$: so the solutions, when plotted with respect to D, are the intersections between a slope-1 straight line and a powerlaw.

There are values of C such that there is no solution.

We select the value of C such that the system has a unique solution — we do this because, inspecting the equation more closely, we find that it is also a critical-point equation, and if we want a monotonic velocity gradient we must have the solution passing through the critical point.

So we differentiate $D - CD^{\alpha} + GM(1 - \Gamma_e)$ with respect to D, set it equal to 0 and find $C = D^{1-\alpha}/\alpha$.

Plugging the solution in we find a differential equation:

$$r^2 v \frac{\mathrm{d}v}{\mathrm{d}r} = D = \frac{\alpha}{1-\alpha} GM_*(1 - \Gamma_e)\,, \tag{154}$$

which can be solved by separation of variables.

The solution is

$$v(r) = \left(\frac{\alpha}{1-\alpha} 2GM_*(1 - \Gamma_e) \left(\frac{1}{R_*} - \frac{1}{r} \right) \right)^{1/2} = v_\infty \sqrt{1 - \frac{R_*}{r}}\,. \tag{155}$$

this is a β-type law, with $\beta = 0.5$ and

$$v_\infty = \sqrt{\frac{\alpha}{1-\alpha} 2GM_* \left(\frac{1 - \Gamma_e}{R_*} \right)} = v_{\mathrm{esc}} \sqrt{\frac{\alpha}{1-\alpha}}\,, \tag{156}$$

since the escape velocity (at the photosphere) is

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM_*(1 - \Gamma_e)}{R_*}}\,, \tag{157}$$

and we include Γ_e in the escape velocity since there is electron scattering there.

This gives us a rather complicated but well-defined expression for the mass loss rate.

The main dependences are $\dot{M} \propto L_*^{1/\alpha} M_*^{(\alpha-1)/\alpha}$.

Copy formula for $\dot{M}$.

For Main Sequence stars $L_* \sim M_*^{2.5}$, so we can turn the luminosity dependence into a mass dependence: $\dot{M} \sim M^{(3.5\alpha - 1)/\alpha}$. For $\alpha = 0.52$ we get approximately $\dot{M} \sim M^{1.58}$.

Now, we talk about the corrections due to the finite size of a star. Near the star, we cannot approximate it as a point source. This is in general hard to do, the result is: we get a lower mass loss rate, and a higher terminal velocity. The line acceleration formula picks up a correction factor D_f, which is derived geometrically.

Copy formula for D_f.

The correction factor can be both below and above unity, depending on the distance from the star and on β. As we'd expect in the $r \to \infty$ limit we have $\beta \to 1$.

Typically, the mass loss rate variation due to this correction is similar to multiplying it by $1/2$, while the escape velocity is approximately multiplied by 2.

Phenomena which are not accounted for by this model are:

1. X-ray emission;
2. superionization (beyond the sixth line);
3. discrete absorption components & variability.

In general these winds are unstable: they will feel shocks. These can be simulated: the time-averaged v_∞ and $\dot{M}$ are similar to those found in stationary models.

Mon Nov 25 2019

Now, let us treat the terminal vs escape velocities for O-B stars: we seem to have a linear relation between these, for each group of stars, where we group them by effective temperature.

If we plot $v_\infty / v_{\text{esc}}$ in terms of T_{eff}, we get three horizontal regions. We cannot reproduce all the data with a single α value, instead we need at least three.

We have the maximum mass loss rate when the equation

$$\dot{M} v_\infty = \tau_w \frac{L}{c} \tag{158}$$

with $\tau_w = 1$ is satisfied.

We can plot both sides of this equation for O-stars[2] and Wolf-Rayet stars of type WNL. O-stars' observations agree with the model, WNL observations seem not to.

We can make a plot of initial mass vs metallicity and distinghish regions fo the final fates of stars. Metallicity characterizes the amount of heavier-than-Helium elements in the star.

The definition is

$$\text{Metallicity} = \frac{m_{\text{metals}}}{m_{\text{tot}}} = Z\,, \tag{159}$$

where m denotes the mass of element of a certain species. $1 = X + Y + Z$ are the fractional masses of hydrogen, helium and metals respectively, so $X = m_{\text{H}} / m_{\text{tot}}$ and so on.

[2] Hey now, you're an O star, get your game on, go play.

10 Dust driven winds

Now, we will speak of dust driven winds: we move to the cool and luminous part of the H-R diagram, which is populated by red giants and so on.

Qualitatively, the driving mechanism is the radiation pressure on dust grains; dust can exist in low temperature regions of the atmosphere, with $T \sim 10^3$ K. Dust absorbs momentum and is accelerated outwards: it is very opaque.

The bulk of the outward travelling matter is gas: the dust is a small part, but it is dynamically coupled to the gas (it "drags it along", they share momentum).

A good approximation is a wind with a force $f \sim r^{-2}$. This mechanism critically depends on the dust formation radius: the distance at which the temperature becomes low enough for dust to form.

Dust grains can absorb photons in the continuum, they are then said to be "continuum driven" as opposed to "line driven". Then, this very much diminishes the role of Doppler shift in making it possible for the radiation to pass through much of the gas uninterrupted.

The typical photons in this process are infrared, on the order of $\lambda \sim 1\,\mu\text{m}$.

The dust causes significant reddening. We have low v_∞ and high $\dot{M}$.

Most of the momentum is then transferred in the subsonic region for the high $\dot{M}$, while little energy must be transferred in the supersonic region in order for the exit velocity to be low.

The formation of dust grains must be described with molecular chemistry, and it heavily depends on the ratios of chemical species.

An example: the Helix Nebula.

We define the Eddington factor: the ratio of radiative acceleration to gravity. In the case of radiation forces on dust, it is

$$\Gamma_d = \frac{\kappa_{\text{rp}} L_*}{4\pi c G M_*}, \tag{160}$$

where the κ_{rp} is called the "radiation pressure mean opacity": it measures the capacity of the dust to absorb photons.

Sometimes, we can have Γ_d greater than unity.

To a good approximation, the transsonic transition corresponds to the point at which Γ_d becomes greater than one.

The plot of Γ_d looks something like $\Gamma_d(r) = \Theta(r - r_{\text{sonic}}) \times 1.4$, where $r_{\text{sonic}} \sim (3 \div 4) r_*$, where Θ is the Heaviside theta.

What are the conditions in order to have a dust driven wind? We first need to form the dust grains: they need to survive against sublimation. Then, the dust must be coupled enough to the gas in order to drive it.

There are two interesting temperatures. The first is the radiative equilibrium temperature T_{rad}, which is the temperature of the grains: it depends on the radiative heating and cooling of the grains.

The second one is the condensations temperature T_{cond}, below it the grain becomes a solid. Above T_{cond} we have sublimation.

$T_{\text{cond}} \sim$ const is not a bad approximation; the radiative equilibrium temperature instead decreases. There is a radius at which they are equal, and for larger radii we have $T_{\text{rad}} < T_{\text{cond}}$. This critical radius is called the dust condensation radius r_d.

Let us suppose that we are at $r > r_d$. The grain then can form, and it can then gain momentum and energy from the Sun's photons, and transfer it to the gas molecules.

We can get a lower limit to the mass loss rate because of the gas-dust coupling condition. Deriving this is complicated, but qualitatively it is since if there is less mass loss rate there is also less gas density, so there is less transfer.

Also, we can have an upper limit on the speed of the dust driven wind. We have the maximal drift speed of a grain through the gas: the drift speed $u_{\text{grain}} - v_{\text{gas}} \propto \rho_{\text{gas}}^{-1/2}$, so there is a radius beyond which the collisions exceed the limiting energy at which they are able to destroy the grains, so there is no further increase in the wind speed, since the dust and gas get progressively more decoupled as the density decreases.

The properties of the grains are very important, but studying the processes in grain formation and growth is hard.

There are both processes of accretion of the gas onto the dust particle and erosion by sputtering, collisions of the gas onto the grain.

There will be a distribution of grain sizes, typically from $0.05 \div 0.1\,\mu\text{m}$: we can sometimes make the small grain approximation.

The composition of the grains depends on the materials in the gas, particularly on the C/O ratio (defined as a ratio of number of atoms).

If $C/O < 1$ typically we form silicate grains, while if it is larger than one we typically form carbonaceous grains.

Wed Nov 27 2019

(Based on few Giorgio notes) In this lecture the attention was focused on the importance of C/O ratio in the formation of dust grains in dust driven winds. In AGB stars the core, mainly made of heavier elements such as carbon and oxygen, is ourrounded by an helium envelope and, more externally, an hydrogen envelope. Between these areas there exist some convective flows that can bring oxygen and carbon from the core to the more external parts of the star. Reaching thanks to convection these external layers, which lay at a lower temperature, the heavy elements can form moleculary bounds. In particular the strongest bounded molecule that can be formed is carbon monoxyde (CO). This implies that, if the star has a C/O ratio >1 (i.e. it has more abbundance of carbon over oxygen) almost all the oxygen is used to form carbon monoxyde, while the rest of the carbon can form other molecules: for this reason, in these stars we observe carbon grains. Viceversa, in oxygen-rich stars we have with the specular mechanism silicate grains. At this point we started a discussion that will be continued in the following lecture about dust grains opacity, focusing first on cross section. Let a be the radius of the grain, and Q the efficency of the cross section. In this case we have a cross section $c = \pi a^2$ and, considering both absorption and scattering cross sections c_a, c_s, we have

$$c_t ot = c_a + c_s = \pi a^2 (Q_a(a, \lambda) + Q_c(a, \lambda)) \tag{161}$$

where $Q_a << Q_c$ for IR wavelengths. Now we can define the opacity

$$k_\lambda = \frac{\int_{a_{min}}^{a_{max}} Q_a \pi a^2 n da}{\rho} \tag{162}$$

and the mean opacity

$$k = \int d\lambda k_\lambda \tag{163}$$

which is the relevant quantity for the momentum equation

Mon Dec 02 2019

Coming back to concepts from last time: the cross section is given by the product of the geometrical cross section times an adimensional efficiency; we define it both for absorption and scattering.

$$C^{\mathrm{A,\,S}}(a, \lambda) = \pi a^2 Q^{\mathrm{A,\,S}}(a, \lambda)\,, \tag{164}$$

while for the total cross section we have

$$C^{\mathrm{TOT}}(a, \lambda) = \pi a^2 \Big(Q^{\mathrm{A}}(a, \lambda) + Q^{\mathrm{S}}(a, \lambda) \Big)\,. \tag{165}$$

We are speaking about dust grains. If $\lambda \gg a$, then the following holds:

$$Q^A \propto a \lambda^{-p}\,, \tag{166}$$

where p depends on the grain composition, and approaches 2 for increasing λ for any composition.

At infrared wavelengths, we have $Q^S \sim \lambda^{-4}$, so $Q^S \ll Q^A$ there.

The values of p are not the result of any theoretical model, they are observed.

Today we will speak more about mean opacities.

We define

$$Q_p^A(a,T) = \frac{\int_0^\infty Q^A(a,\lambda) B_\lambda(T)\,\mathrm{d}\lambda}{\int_0^\infty B_\lambda(T)\,\mathrm{d}\lambda}\,, \tag{167}$$

where we use the Planck function $B_\lambda(T)$; with it we define

$$\kappa_p(T) = \int_{a_{\min}}^{a_{\max}} Q_p^A(a,T)\pi a^2 n(a)\,\mathrm{d}a\,, \tag{168}$$

which is called the Planck mean absorption coefficient. This quantity will enter into the balance between cooling and heating of the single grain. We are effectively doing a mean over energy, since

$$B_\lambda\,\mathrm{d}\lambda = \mathrm{d}E \tag{169}$$

for a blackbody.

For hydrodynamical calculations instead we use the radiation pressure mean efficiency:

$$Q_{\mathrm{rp}}(a) = \frac{\int_0^\infty \Big(Q^A(a,\lambda) + (1-g_\lambda)Q^S(a,\lambda)\Big) F_\lambda\,\mathrm{d}\lambda}{\int_0^\infty F_\lambda\,\mathrm{d}\lambda}\,. \tag{170}$$

Here we are accounting for both the absorption efficiency and the scattering efficiency. Similarly, we can define the radiation pressure mean opacity κ_{rp}:

$$\kappa_{\mathrm{rp}} = \frac{\int_0^\infty \big(\kappa_\lambda + (1-g_\lambda)\sigma_\lambda\big) F_\lambda\,\mathrm{d}\lambda}{\int_0^\infty F_\lambda\,\mathrm{d}\lambda}\,, \tag{171}$$

where

$$F_\lambda = \frac{L_\lambda}{4\pi r^2} \tag{172}$$

is the monochromatic flux.

Also, g_λ is the *mean cosine* of the scattering angle: if it is equal to 1 we have forward scattering, if it is equal to -1 we have backward scattering, while if it is equal to 0 we have isotropic scattering.

We are shown a plot of g_λ in terms of the wavelength: it decreases from 0.8 to 0 as λ goes from 0.1 μm to 10 μm.

Do note that the F_λ is not generally known *a priori*: it is a solution to the radiative transfer equation.

The temperaure of the grain is determined by the balance of the heating and cooling rates: heating occurs because of collisions with fast-moving gas particles or because of the absorption of radiation.

Cooling, on the other hand, occurs because of collisional energy transfer or by emission of thermal radiation.

Are we modelling the dust grain as a black body?

We make an approximation: we only consider radiative processes, and estimate the temperature of the dust grain with the radiative equilibrium temperature.

The equilibrium equation is

$$\int_0^\infty \kappa_\lambda B_\lambda(T_d)\,\mathrm{d}\lambda = \int_0^\infty \kappa_\lambda y_\lambda\,\mathrm{d}\lambda\,, \tag{173}$$

where

$$y_\lambda = \frac{1}{4\pi}\int_0^{4\pi} I_\lambda\,\mathrm{d}\Omega = W(r)B(T_{\text{eff}})\,, \tag{174}$$

where $W(r)$ is the geometrical dilution factor:

$$W(r) = \frac{1}{2}\left(1 - \sqrt{1-\left(\frac{R}{r}\right)^2}\right) \sim \left(\frac{R}{2r}\right)^2 \qquad \text{as } r \gg R\,, \tag{175}$$

where R is the radius of the star, r is our considered radial position.

This allows us to fix r and compute $T_d(r)$.

The LHS of the balance equation corresponds to the radiative cooling, while the RHS corresponds to the radiative heating.

In both terms we make the dependence on Q_p^A explicit:

$$\int_0^\infty \pi a^2 Q_p^A(a,T_d)B_\lambda(T_d)\,\mathrm{d}\lambda = \int_0^\infty \pi a^2 Q_p^A(a,T_{\text{eff}})B_\lambda(T_{\text{eff}})W(r)\,\mathrm{d}\lambda\,. \tag{176}$$

We are assuming that diffusion of heat between gas grains is negligible: we simplify some terms.

$$Q_p^A(a,T_d)\int_0^\infty B_\lambda(T_d)\,\mathrm{d}\lambda = W(r)Q_p^A(a,T_{\text{eff}})\int_0^\infty B_\lambda(T_{\text{eff}})\,\mathrm{d}\lambda\,, \tag{177}$$

and we know that the integral of the Planck function is given by $\sigma_{\text{SB}}T^4$, where σ is the Stefan-Boltzmann constant. In the end then our expression is

$$Q_p^A(a,T_d)T_d^4 = T_{\text{eff}}^4 Q^A(a,T_{\text{eff}})W(r)\,, \tag{178}$$

so

$$T_d(r) \sim T_{\text{eff}}\left(\frac{R}{2r}\right)^{1/2}\left(\frac{Q_p^A(a,T_{\text{eff}})}{Q_p^A(a,T_d)}\right)^{1/4}\,. \tag{179}$$

An immediate application is to find the condensation radius: what is the radius at which the temperature becomes low enough so that the grains are not broken up by thermal motion?

Typically the condensation temperature is $T_c \sim 1 \div 1.5 \times 10^3\,\text{K}$.

The equation is

$$r_c \sim \frac{R}{2}\left(\frac{T_{\text{eff}}}{T_c}\right)^2 \sqrt{\frac{Q_p^A(a, T_{\text{eff}})}{Q_p^A(a, T_c)}}\,, \tag{180}$$

and this allows us to see that typically the condensation radius is around $2 \div 4$ star radii, for stars at a few thousand Kelvin.

Tomorrow we will speak of the combined dust & wind equation.

Tue Dec 03 2019

Missing lecture, to recover.