Astroparticle physics notes

Jacopo Tissino

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Contents

1 The particle physics Standard Model **5**
1.1 The classical description of a system of particles 5
1.1.1 A relativistic reminder 7
1.2 Symmetries and conservation laws 8
1.2.1 Continuous transformations: space translations 11
1.2.2 Discrete symmetries 13
1.3 Relativistic wave equations 15
1.3.1 Spin 1 17
1.3.2 Spin 1/2 18
1.3.3 Photon-fermion coupling 23
1.3.4 Scattering 25
1.3.5 A decay example 28
1.3.6 Gauge symmetries in QED 32
1.4 QCD 33
1.4.1 Non-abelian gauge theory: Yang-Mills 35
1.4.2 Quantum Chromo Dynamics 37
1.4.3 Neutrinos 46

2 Early Universe **49**
2.1 Basics of cosmology 49
2.1.1 The ΛCDM model 50
2.2 Big Bang Nucleosynthesis 53
2.3 Dark Matter 55
2.3.1 WIMP candidates for Dark Matter 57
2.3.2 Axions 58
2.4 Matter-antimatter asymmetry 60

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Introduction

Professor Antonio Masiero, `antonio.masiero@unipd.it`

There are two courses, for Astrophysics and Cosmology and for Physics, which bear the same name. The other one is by Francesco D'Eramo: it assumes a knowledge of Quantum Field Theory.

This course, instead, only requires knowledge of Quantum Mechanics. The first part of this course is devoted to an introduction about the basics of Quantum Field Theory and gauge theories.

> "By the end of the 20th century [...] we have a comprehensive, fundamental theory of all observed forces of nature which has been tested and might be valid from the Planck length scale of 10^{-33} cm to the edge of the universe 10^{28} cm".
> David Gross, 2007.

The task in APP is to be able to discuss such a fundamental theory.

First of all, we need to address the two standard models: the ΛCDM model for cosmology and the Standard Model of particle physics.

There are points of friction between the two Standard Models. There are also several questions: neutrinos' mass, what caused inflation...

These problems have a common denominator: the interplay between particle physics, cosmology and astrophysics. What we seek is new physics, beyond the two standard models.

Books: Peskin [Pes19], "Concepts in Elementary Particle Physics". The book is addressed to students who are not experts in QFT and particle physics, rather, it provides the fundamental knowledge for these topics.

The exam is a colloquium, an oral exam, for which we can prepare a presentation on a specific topic. There is no issue if we do not precisely remember a specific formula, it is about going deep in the concepts.

An overview of the astroparticle physics landscape

Fundamental particles: the SM of particle physics Elementary particles make up ordinary matter. Fermions have spin 1/2, and are composed of quarks:

$$\begin{bmatrix} u & c & t \\ d & s & b \end{bmatrix}, \tag{1a}$$

leptons:

$$\begin{bmatrix} \nu_e & \nu_\mu & \nu_\tau \\ e & \mu & \tau \end{bmatrix}. \tag{2a}$$

The muon and tau particles are similar to electrons, but with higher mass.

These particles' interactions are mediated by 12 vector bosons, which have spin 1: these are "radiation" (the term is outdated).

1. gluons (g) mediate the strong nuclear interaction, there are 8 of them;
2. the $W^{\pm}$ and Z^0 bosons mediate the weak interaction;
3. the photon (γ) mediates the electromagnetic interaction.

There was a need for a mechanism to provide mass to the weak bosons and the fermions: this is accounted for by the Higgs boson, which is a scalar (that is, it has spin 0). This realizes the electroweak symmetry breaking.

The issue is that gravity is missing. In order to describe it in this scheme we would need a way to quantize it: all of these particles are actually excitations of quantum fields.

There are two marvelous 20th century theories, but they are not compatible.

Unification of interactions In 1687 Newton unified two domains of interactions: the terrestrial phenomena and the celestial phenomena, establishing the universality of gravitational interactions.

In 1865 Sir Maxwell unified electricity and magnetism into electromagnetism.

In 1967 Glashow, Weinberg and Salam propose the Standard Model of Particle Physics, unifying the Electromagnetic and Weak interactions. This is not a true unification: it is more appropriate to say that they are "mixed together" into the Electroweak interaction.

In the Standard Model, there is a kind of "frontier" around 100 GeV: below this energy, we see two interactions: the electromagnetic and the weak interaction. They are very much different: photons are massless, so the interaction has an infinite range, while the weak bosons are massive.

How can these be unified? We will see; above 100 GeV this apparent profound difference disappears in favour of the electroweak interaction. This is a phase transition.

Is the electroweak interaction above 100 GeV massless or not? Above this energy there is still a difference between the coupling constants of the two interactions. Above this energy, the W and Z bosons are no longer massive.

Above 100 GeV the strong interaction is separated from the electroweak one. Maybe there is an energy at which the electroweak interaction is unified with the strong one? We shall explore this topic: there are theories (Grand Unified Theories, GUT) in which there is such a unification. Also, from the 1980s there started to appear string theories, in which gravity is also unified to the other interactions, so that there is only one fundamental parameter, the "string tension".

As the energy increases, the coupling of the strongest interactions becomes weaker.

The energy scale, however, is very large: around 10^{16} GeV: this is a "science fiction" energy scale, it is extremely large. This is close to the Planck mass: $M_P \sim 10^{19}$ GeV, so we might not be able to describe this energy range with vanilla SM.

The Standard Model of Cosmology The standard model in cosmology describes a **Hot Big Bang**: the ancestral temperature was very high, and gradually decreased. Now we can work backwards in our energy scale: as time progresses forward from the Big Bang, the energy of particles decreases and our particles undergo various phase transitions.

The symmetry group of the Grand Unified Theory is broken, so we get subgroups; at each transition some symmetry is broken.

The EM + weak into electroweak transition is not speculative: we have observed it at the LHC. On the other hand, the electroweak + strong into GUT transition is speculative.

When, in the expansion of the universe, we reach an energy per particle of $\approx 1\,\mathrm{GeV}$ we have a new transition: the quark-hadron one, when free quarks become confined into hadrons such as protons and neutrons.

Around $1\,\mathrm{MeV}$ we have a new transition: nucleosynthesis, where protons and neutrons become confined into nuclei.

Then, at the eV scale, we reach recombination, which is when the radiation we see as the CMB is released. This is where nuclei and free electrons form hydrogen atoms.

There must be new physics somewhere: there is no room in the SM for dark matter, the matter-antimatter asymmetry, the mass of neutrinos.

Chapter 1

The particle physics Standard Model

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For this lecture, a good reference is Peskin, chapter 2 [Pes19].

We wish to describe what we described yesterday as "matter" and "radiation".

The problem is similar to the one we have in classical mechanics, an initial value problem: given the positions and velocities of the particles at a certain starting time t_0 we wish to compute their state at a later time t.

This classical description in which the particles are not wavelike fails at the microscopic level: we want to give a quantum description of such a system of particles. We will derive it from the classical description using the standard tools of quantization. We start with a refresher of the classical description.

1.1 The classical description of a system of particles

Our aim is to compute and solve the equations of motion. The usual approach is to use Hamilton's variational principle: it is the principle of least action, but it is not usually referred to as such: we are actually not *minimizing* the action but finding a *stationary point* for it. This could also be a maximum or a saddle point.

The action functional S depends on the coordinates $q_i(t)$ of the particles at at time t, on the derivatives of these positions $\dot{q}_i(t)$ which represent the velocities of the particles at a time t. We usually write $S\left[q_i(t), \dot{q}_i(t)\right]$.

If we fix $q(t_0)$ and $q(t_f)$, the positions at some initial and final time t_f, we can then trace out a path $q(t)$ and perturb it by $\delta q(t)$; we fix $\delta q(t_0) = \delta q(t_f) = 0$.

Under this perturbation of the path $q \to q + \delta q$, the action changes to $S \to S + \delta S$. We then ask that $\delta S = 0$.

S is an action: is dimensions are those of an energy times a time. In terms of the Lagrangian L, the action is defined as

$$S\left[q_i(t), \dot{q}_i(t)\right] = \int_{t_0}^{t_f} L(q_i(t), \dot{q}_i(t)) \, \mathrm{d}t \,, \tag{1.1}$$

which means that the Lagrangian must have the dimensions of an energy.

Moving on from classical mechanics to classical field theory, we will make use of a quantity called the Lagrangian density $\mathscr{L}$, such that we recover the Lagrangian as:

$$L = \int \mathscr{L}\Big(\phi(\vec{x}), \partial_\mu \phi(\vec{x})\Big)\, \mathrm{d}^3 x \,. \tag{1.2}$$

From a finite number of particles we move to considering a field $\phi(\vec{x})$: this means that, in a certain sense, we are considering an "infinite number of particles", the values of the field at each point in space.

The dependence of the Lagrangian on the q_i and $\dot{q}_i$ shifted to a dependence on the spacetime coordinates x and their 4-derivatives $\partial_\mu x$. It could depend on many fields simultaneously, we omit this dependence for simplicity. Now, this Lagrangian density has the dimensions of an energy per unit volume.

Then, the action, computed in a region Ω of 4-dimensional spacetime, is

$$S = \int_\Omega \mathrm{d}^4 x\, \mathscr{L}\Big(\phi(x), \partial_\mu(\phi(x))\Big) \,. \tag{1.3}$$

Now that we have established the notation, we can apply the action principle: we consider an infinitesimal variation of the field $\phi \to \phi + \delta\phi$. We require this variation to vanish not only at the initial and final time, but over all the boundary $\partial\Omega$:

$$\delta\phi\Big|_{\partial\Omega} = 0 \,. \tag{1.4}$$

Then, it can be shown with an integration by parts that imposing $\delta S = 0$ in the region Ω is equivalent to the Euler-Lagrange equations:

$$\frac{\partial \mathscr{L}}{\partial \phi(x)} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial\Big(\partial_\mu \phi(x)\Big)}\right) = 0 \,. \tag{1.5}$$

If we have many fields ϕ_r, then we have a set of E-L equations for each of them. This is still classical: for example, classical (relativistic) electrodynamics is formulated in this way.

In order to write the Hamiltonian formulation of the theory we need the momenta, which are

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}(x)} \,, \tag{1.6}$$

where a dot denotes a time derivative. Using these we define the Hamiltonian density by

$$\mathscr{H}(x) = \pi(x)\dot{\phi}(x) - \mathscr{L}(\phi, \partial_\mu \phi) \,, \tag{1.7}$$

and similarly to the Lagrangian we have the full Hamiltonian H

$$H = \int \mathrm{d}^3 x\, \mathscr{H} \,. \tag{1.8}$$

Now, as we quantize the theory the fields (ϕ and π) go from classical fields to Heisenberg-picture **operators**. As classical fields, ϕ and π satisfy Poisson bracket relations:

$$\{\phi(\vec{x},t),\pi(\vec{x}',t)\} = \delta^{(3)}(\vec{x},\vec{x}')\{\phi(\vec{x},t),\phi(\vec{x}',t)\} \qquad = 0 = \{\pi(\vec{x},t),\pi(\vec{x}',t)\}\,. \tag{1.9}$$

The operatorial version, instead, will need to satisfy the classical commutation relations, which are found by substituting the Poisson bracket with a commutator divided by $i\hbar$ [Tis20, section 2.4.2]:

$$\{\phi,\pi\} \to \frac{1}{i\hbar}[\hat{\phi},\hat{\pi}]\,. \tag{1.10}$$

So, we get the following:

$$[\phi(\vec{x},t),\pi(\vec{x}',t)] = i\hbar\delta(\vec{x}-\vec{x}') \tag{1.11a}$$

$$[\phi(\vec{x},t),\phi(\vec{x}',t)] = [\pi(\vec{x},t),\pi(\vec{x}',t)] = 0\,. \tag{1.11b}$$

Note that the commutation relations are computed *at equal time*.

1.1.1 A relativistic reminder

The energy-momentum four-vector is

$$p^\mu = \begin{bmatrix} E \\ \vec{p}c \end{bmatrix}, \tag{1.12a}$$

where the greek index μ can take values from 0 to 3. A four-vector is an element of the tangent bundle to the spacetime manifold; concretely speaking, under a coordinate transformation which is locally linear and represented by a Lorentz matrix $\Lambda^\mu{}_\nu$ a four-vector transforms as $p^\mu \to \Lambda^\mu{}_\nu p^\nu$.

The metric signature used here is the the mostly minus one. So, $p^\mu q_\mu = E_p E_q - \vec{p}\cdot\vec{q}$, since we raise and lower indices using the metric $\eta_{\mu\nu}$.

The square norm of the 4-momentum is $p \cdot p = p^2 = E^2 - |\vec{p}|^2 c^2$. It is Lorentz invariant.

In the rest frame of the observer, $p^\mu = [E_0, \vec{0}]$, and this E_0 is just (c^2 times) the mass of the particle: this is the *definition* of mass.

When the relation is satisfied we have

$$p^2 = E^2 - |p|^2 c^2 = \left(mc^2\right)^2 \tag{1.13a}$$

$$E = c\sqrt{|p|^2 + (mc)^2}\,. \tag{1.13b}$$

When this relation is satisfied we say we are *on shell*: for virtual particles, instead, this might not be satisfied. This is allowed because virtual particles are described in a *quantum* theory: because of the uncertainty principle, we have uncertainty in energy if we consider short times and uncertainty in momentum if we consider small position intervals. This uncertainty means that we cannot enforce the on-shell condition as an exact equality if we are considering processes on these small scales, such as those which we find in quantum field theory.

We will use natural units: $\hbar = c = 1$.

This means that we equate energies (eV) and angular velocities (Hz); also we equate times (s) and lengths (m).

The rest energy of the electron is $m_e \approx 511\,\text{keV}$. Let us consider an electron with a momentum p equal to its mass m_e: then, its uncertainty in position is of the order

$$\frac{\hbar}{pc} = \frac{1}{m_e} \approx 4 \times 10^{-11}\,\text{cm}\,. \tag{1.14}$$

The dimensions of the lagrangian density, in natural units, are those of an energy to the fourth power, or a length to the -4, or a mass to the fourth.

Another useful exercise is to calculate the coupling of the electromagnetic field:

$$V(r) = \frac{e^2}{4\pi\epsilon_0 r} = \frac{e^2}{4\pi}\frac{1}{r}\,, \tag{1.15}$$

since we set $\epsilon_0 = \mu_0 = 1$. We can introduce the electromagnetic α: this is

$$\alpha = \frac{e^2}{4\pi} \times \frac{1}{\hbar c}\,. \tag{1.16}$$

This then becomes adimensional: $\alpha \approx 1/137$. It represents the strength of the electromagnetic interaction: the strength of the coupling of the photon to the electron. The fact that it is $\sim 10^{-2}$ is important: it allows us to work in a perturbative way, in powers of α.

What is the coupling of the strong and weak interactions? This will be discussed.

Next time, we will discuss symmetries and symmetry breaking.

1.2 Symmetries and conservation laws

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Our aim is to describe the fundamental constituents of matter with a Quantum Field Theory. The method used to derive the equations of motion is a variational principle: we will find a Lagrangian density for various particles, and then apply the variational principle to find their equations of motion.

A guiding principle on the description of these fundamental particles is based on using their symmetries. We have Nöether's theorem in Quantum Field Theory: from these symmetries we are able to find conserved quantities.

These symmetries are described with groups, since we can compose their application; the theory describing groups is very rich. For this lecture we will base ourselves on Peskin's chapter 2 [Pes19].

A group G is a set of elements endowed with an operation. The set of elements can be either discrete or continuous. Examples of discrete transformations are the parity transformation P: $P\vec{x} = -\vec{x}$ and the time swap T: $Tx^\mu = (-x^0, \vec{x})$. Continuous symmetries, on the other hand, are parametrized by one or more continuous-valued parameters.

We distinguish:

1. **spacetime** symmetries: groups which transform our coordinate system for spacetime, such as Lorentz and Poincaré transformations;
2. **internal** symmetries: groups which transform a certain field, or a certain property of our quantum system.

For our set to be a group, we need to be able to define an operation — we will usually call it multiplication — between the elements of the group, such that if $a, b \in G$ then $ab \in G$. Also, we must have

1. associativity: $(ab)c = a(bc)$;
2. existence of the identity $\mathbb{1}$, such that $\mathbb{1}a = a\mathbb{1} = a$;
3. existence of inverses: there exists a^{-1} such that $aa^{-1} = a^{-1}a = \mathbb{1}$.

What is of interest to us is the association of the group with a transformation which is a symmetry: this is called a *representation*, which associates to each $g \in G$ a unitary operator U_g acting on the quantum states. We ask that this representation should preserve the group structure, that is to say, $U_{gh} = U_g U_h$ and $U_{g^{-1}} = U_g^{-1}$.

We call a transformation a symmetry if, after performing the transformation, the dynamics of the system do not change.

For a quantum mechanical system, we are interested in the observables: these are described by operators, whose eigenvalues are the observations we make, and which in the Heisenberg picture evolve like

$$-i\hbar \frac{\mathrm{d}}{\mathrm{d}t} O(t) = [H, O(t)]\,; \tag{1.17}$$

if an operator commutes with the Hamiltonian, $[H, O] = 0$, then the operator's expectation value on any state is constant — which is to say, the operator is constant.

If we perform a transformation in the form

$$|\psi\rangle \to |\psi'\rangle = U\,|\psi\rangle\,, \tag{1.18}$$

then the operators will change by

$$O \to O' = U^\dagger O U\,. \tag{1.19}$$

Note that whether we have $U^\dagger O U$ or $U O U^\dagger$ does not matter, since we ask observables O to be Hermitian, so $O = O^\dagger$.

We know that these transformations must always be unitary, because the conservation of probability implies that we must have $\langle\psi|\psi\rangle = \text{const}$: so,

$$U^\dagger U = \mathbb{1}\,. \tag{1.20}$$

This can be also stated as $U^\dagger = U^{-1}$.

So, the function associating a unitary operator U to an element g of the group is called its *unitary representation*.

A transformation G is a symmetry if $\forall a \in G$ we have

$$[U(a), H] = 0\,, \tag{1.21}$$

that is, the unitary representation of the group element always commutes with the Hamiltonian.

If we have a state $|\psi\rangle$ with energy $H|\psi\rangle = E|\psi\rangle$, then the transformation commuting with the Hamiltonian means that $|\psi'\rangle = U|\psi\rangle$ has the same energy:

$$H(U|\psi\rangle) = HU|\psi\rangle \overset{[H,U]=0}{=} UH|\psi\rangle = UE|\psi\rangle = E\Big(u\,|\psi\rangle\Big)\,, \tag{1.22}$$

so the eigenvalue of $U|\psi\rangle$ is the same as that of $|\psi\rangle$.

Now, we can move to an example, taken from Peskin [Pes19, eq. 2.38 onward]. Consider the discrete group $\mathbb{Z}_2$, which only has the elements 1 and -1, with the same multiplication rules as those we would have if these elements were integers. So, the group is closed with respect to multiplication. It can be easily checked that this is indeed a group based on our definition.

In order for this to be of interest to us, we can consider a quantum mechanical system and find a unitary representation acting on its Hilbert space.

Let us suppose we have a QM system with a basis made of two states $|\pi^+\rangle$ and $|\pi^-\rangle$. Let us define the *charge conjugation* operator C, by:

$$C\left|\pi^+\right\rangle = \left|\pi^-\right\rangle \qquad \text{and} \qquad C\left|\pi^-\right\rangle = \left|\pi^+\right\rangle\,. \tag{1.23}$$

So, we can find a unitary representation of $\mathbb{Z}_2$ in this system: we need to define $U(1)$ and $U(-1)$. We define

$$U(1) = \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad U(-1) = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,, \tag{1.24a}$$

where the matrices are to be interpreted as acting on vectors expressed to the basis $\big\{|\pi_+\rangle\,, |\pi_-\rangle\big\}$.

So, we can say that our unitary representation looks like

$$\mathbb{Z}_2 \to \{\mathbb{1}, C\}\,. \tag{1.25}$$

Now, if $[C, H] = 0$ (and $\mathbb{1}$ commutes with H, which is always the case) then we say that "H has the symmetry $\mathbb{Z}_2$": this implies that the energies of the two $\pi_\pm$ particles are equal.

The interesting question to determine will be whether this is actually the case for our given group.

Groups can be subdivided into abelian and non-abelian ones. A group is abelian if for every a, b in G we have $ab = ba$, or equivalently, $[a, b] = 0$. It is not if this is not the case, that is, there exist a, b such that $ab \neq ba$.

The condition on the elements directly translates to a condition on the matrices of the unitary representation. If we have commuting matrices, we can simultaneously diagonalize them: for example, in the case of $\mathbb{Z}_2$ we can go to a basis in which

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,, \tag{1.26a}$$

specifically the states on which this matrix will act will need to be

$$|\pi_1\rangle = \frac{|\pi^+\rangle + |\pi^-\rangle}{\sqrt{2}} \qquad \text{and} \qquad |\pi_2\rangle = \frac{|\pi^+\rangle - |\pi^-\rangle}{\sqrt{2}}\,, \tag{1.27}$$

since then $C\,|\pi_1\rangle = |\pi_1\rangle$ (we write $C = +1$) and $C\,|\pi_2\rangle = -\,|\pi_2\rangle$ (we write $C = -1$). We will often use this notation, confusing operator and eigenvalue.

In the case of nonabelian groups it is not in general possible to diagonalize all the matrices; we can however do a change of basis and write the matrices as a block matrix with the smallest possible blocks:

$$U_R = \begin{bmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & \dots \end{bmatrix}, \tag{1.28a}$$

where the matrices U_i are called the **irreducible unitary representations of** G. The dimension of the matrices U_i tells us the dimension of these irreducible unitary representations.

Add more details on irreps — maybe not here? They can be found in professor Rigolin's intro to groups.

Do note that some elements of a nonabelian group can commute: for example, in the rotation group we have

$$[J^i, J^j] = \epsilon^{ijk} J_k\,, \tag{1.29}$$

so if we take $i = j$, that is, we consider rotations along the same axis, they will commute since then the Kronecker symbol is equal to zero.

1.2.1 Continuous transformations: space translations

An element of the group can be written as

$$U(a) = e^{-iaP}\,, \tag{1.30}$$

where the operator P, whose eigenvalue is the momentum, is called the generator of the transformation.

If we consider a plane wave we can clearly see how this action works: if we start from

$$\langle x|p\rangle = e^{ipx}\,, \tag{1.31}$$

we can apply the operator $U(a)$ to $|p\rangle$, which will yield e^{-ipa} (since eigenvectors of an operator are also eigenvectors of its exponential): so we find

$$\langle x|\, U(a)\, |p\rangle = e^{ip(x-a)}\,, \tag{1.32}$$

which means that by acting with this operator we have effectively performed a translation with displacement a.

If our system is invariant under translations, then Nöether's theorem tells us that the momentum is conserved.

In order to be a physical observable P needs to be Hermitian: $P = P^\dagger$.

So, the adjoint of the transformation $U(a)$ is

$$U^\dagger(a) = \sum_n \left(\frac{(-iaP)^n}{n!} \right)^\dagger = \sum_n \frac{(iaP^\dagger)^n}{n!} = e^{iaP^\dagger} = e^{iaP} = U^{-1}(a)\,, \tag{1.33}$$

which confirms the fact that the transformation is unitary.

Let us suppose that the momentum operator P commutes with the Hamiltonian: $[P, H] = 0$. Then,

$$[U(a), H] = 0\,, \tag{1.34}$$

that is, the Hamiltonian is translation-invariant.

All this is to say that a constant of motion O corresponds to an operator O which commutes with the Hamiltonian. This is formalized by Nöther's theorem, which establishes the equivalence between symmetries and conservation laws:

$$[O, H] = 0 \iff [U_O, H] = 0\,. \tag{1.35}$$

As an example, take the group G of 3D rotations. They depend on a continuous parameter $\vec{\alpha}$, just like translations depended on the parameter a.

The rotation is written as

$$U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{J}}\,, \tag{1.36}$$

where the components of the angular momentum have the following commutation relations:

$$[J^i, J^j] = i\epsilon^{ijk} J^k\,. \tag{1.37}$$

We will be able to compose the representations of rotations:

$$U(\vec{\beta})U(\vec{\alpha}) = U(\vec{\gamma})\,. \tag{1.38}$$

This space of 3D rotations is called SO(3), since every rotation corresponds to a 3x3 matrix which is a rotation matrix — it is orthogonal and has determinant 1.

Now, we seek **representations** of these rotations: so, we choose the dimension d of a quantum-mechanical vector and describe how it changes upon the action of the unitary matrices found by exponentiating certain d-dimensional generators J^i, which must have the algebra discussed above.

If we look for 1D representations of the generators J^i the only option we find is $J^i = 0$, which means that we are not actually performing a rotation. This is because scalars commute with each other. Which states transform this way? These are scalar states, with spin 0.

For 2D representations, we have

$$J^i = \frac{1}{2}\sigma^i \,, \tag{1.39}$$

where the σ^i are the Pauli matrices.

We can also find 3D representations, which look like

$$J^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad J^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \qquad J^1 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.40a}$$

and represent a spin 1 particle. In general, spin s corresponds to a $2s+1$-dimensional representation.

A rotation in 2D, represented by an element of SO(2), corresponds to a phase shift, so we can say that it is equivalent to an element of U(1). This then allows us to see that SO(2) is abelian.

In general, we write for a unitary $n \times n$ representation

$$U(n) \to e^{-i\alpha^n t^a} \,, \tag{1.41}$$

where the generators t^a are Hermitian matrices corresponding to Hermitian operators. In particular, conventionally we say that one of these is the identity: $t^0 = \mathbb{1}$ (which must always be included in the group, lest we lose closure).

So, we omit it and say that we have $n^2 - 1$ generators for the SU(n) group. We shall see that each of these generators corresponds to a particle, and for the weak interaction we will have $2^2 - 1 = 3$ particles, while for the strong one we will have $3^2 - 1 = 8$.

In general, if t^a are the generators of an abstract Lie group, we can describe the algebra of the group by

$$\left[t^a, t^b\right] = i f^{abc} t^c \,, \tag{1.42}$$

so, the commutator is decomposed into a linear combination of the generators, whose coefficients f^{abc} are called the **structure constants** of the group.

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Yesterday we mentioned the fact that for group symmetries we can find corresponding conservation laws.

Invariance under spacetime translations gives us the conservation of 4-momentum p^μ. Invariance under the Lorentz group gives us conservation of angular momentum (for rotation).

1.2.2 Discrete symmetries

The symmetry in which we reverse the spatial coordinates is called parity P, if we reverse the time coordinate we have time reversal T, and later we will discuss the internal symmetry of charge conjugation C.

More precisely, if $x^\mu = \left(x^0, \vec{x}\right)$ we have

$$P x^\mu = \left(x^0, -\vec{x}\right) \tag{1.43a}$$

$$T x^\mu = \left(-x^0, \vec{x}\right) . \tag{1.43b}$$

These are Lorentz transformations represented by matrices with negative determinants: $\det \Lambda_P = \det \Lambda_T = -1$. This means that we cannot obtain these transformations with a composition of proper orthochronal Lorentz transformations — those defined to be continuously connected to the identity —, since the determinant of a Lorentz transformation is always ± 1, and it is a continuous function of the Lorentz transformation.

In quantum mechanics, these must be interpreted like operators. Their eigenvalues will be ± 1.

These transformations being symmetries is an experimental question. P and T are symmetries of the strong and electromagnetic interactions.

The weak interaction, instead, does not conserve P.

We defined the charge conjugation operator C by its action on the $\left|\pi^+\right\rangle$ and $\left|\pi^-\right\rangle$, on the basis defined by them its matrix representation was σ_x. It commuted with the Hamiltonian.

In general, this swaps not only the electric charge but all the quantum numbers, including the "hypercharges" such as baryon number. The charge conjugation operator C gives us the antiparticle of a certain particle, so it must flip all of the charges. There are, however, theories in which, say, lepton number conservation is broken but baryon number is conserved. The precise meaning of this operator depends on the theory.

If we constructed our experiments with antimatter would we get the same physics?

It was experimentally determined that weak interactions violated parity, and it was thought that baryon number was conserved: however in the early universe we would expect to have equal amounts of matter and antimatter, but this is not what we see — we are made of matter, and we do not see tha gamma ray background we would expect to see if there were matter and antimatter spacetime regions.

Another important concept is that of the *intrinsic parity* of a particle: consider a state of a particle A with momentum $\vec{k}$, denoted as $\left|A(\vec{k})\right\rangle$. Then, upon application of P we can have

$$P \left|A(\vec{k})\right\rangle = \pm \left|A(-\vec{k})\right\rangle , \tag{1.44}$$

where the sign depends on the properties of the particle.

We can apply several "mirrors" to our system: this amounts to the composition of the operators. There is a theorem in Quantum Field Theory: if a QFT is consistent, then it must obey CPT symmetry. After passing through all of the three mirrors, the system has the same physical properties. The order of the three symmetries does not matter: they commute.

The conservation of the electric charge Q, for example, is connected to $\mathrm{U}(1)_{\mathrm{em}}$ symmetry. This is an *exact* symmetry, which we expect not to break down at higher energy.

On the other hand, C, P and T are not exact symmetries since there exist some interactions which violate them (separately: P and CP are violated, so T is also violated by the CPT theorem).

1.3 Relativistic wave equations

In quantum mechanics we describe the dynamics of a quantum system using the Schrödinger equation:

$$E = \frac{\vec{p}^2}{2m} + V\,, \tag{1.45}$$

where $E = i\partial_t \psi$ and $\vec{p}^2 = -\vec{\nabla}^2\psi$.

This is non (special) relativistic: if we perform a Lorentz boost the equation does not remain in the same form. This is explicit, in that we have an addition of first time derivatives and second space derivatives, while in SR time and space are on the same footing.

Also, this equation describes the dynamics of one electron. In elementary particle physics it is no longer *consistent* to only consider one particle: when the energies are of the order of the mass of the particles we can create antiparticles or other particles. So, in general there is no reason to expect that the number of particles is conserved in particle physics.

Formally, it is hard to define from the Schrödinger equation a conserved quantity connected to the probability of finding the particle. It does not account for the possibility that the particle might appear or disappear. For example, we can have matter-antimatter collisions: this is called *annihilation*, and the two incoming particles disappear completely.

The equation $E = \vec{p}^2/2m$ is intrinsically nonrelativistic; the corresponding relativistic relation is $E^2 = \vec{p}^2 + m^2$.

We can make the same canonical quantization substitutions to get

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi(t, \vec{x}) = 0\,. \tag{1.46}$$

Why should we use $E^2 = \vec{p}^2 + m^2$ instead of, say, $E = \sqrt{\vec{p}^2 + m^2}$? The problem is that it is hard to see what could be meant by the square root of an operator. Anyways, later we shall use the approach of "taking the square root" in order to solve the KG equation. We can write this in a manifestly covariant way as

$$\left(\partial^\mu\partial_\mu + m^2\right)\phi(t, \vec{x}) = 0\,, \tag{1.47}$$

where $\partial_\mu = (\partial_t, \vec{\nabla})$, whose square $\partial^\mu\partial_\mu = \partial_t^2 - \vec{\nabla}^2 = \Box$ is the Dalambertian.

This is the Klein-Gordon equation. As long as ϕ is a scalar, this is a scalar Lorentz invariant equation.

It should be used to describe a spin-0 particle, since we are not accounting for spin: the only spin-0 elementary particle known is the Higgs Boson. All other spin-0 particles are composite.

A crucial fact in the KG equation is the fact that the energy can in principle be both positive and negative: $E = \pm\sqrt{\vec{p}^2 + m^2}$. A seemingly reasonable approach would be to not bother treating the "unphysical" $E < 0$ solution; however this is wrong, the negative energy solution is important.

This was a great open debate last century. We will be given the solution, but it would not have easy to figure it out.

This is the reason why people started discussing antiparticles.

An interesting question now is: can we define an action

$$S[\phi(x)] = \int \mathrm{d}^4x \, \mathscr{L}(\phi, \partial_\mu \phi) \tag{1.48}$$

whose actions of motion are the KG equation? The answer is yes, with

$$\mathscr{L} = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2\right). \tag{1.49}$$

To show that this is the case is left as an exercise. Notice that we talk of "Lagrangians" but we always mean Lagrangian densities. The corresponding Hamiltonian density $\mathscr{H}$ has a vacuum state we can call $|0\rangle$, which corresponds to the absence of particles.

We will then have states describing n particles of mass m.

Next time, we will move from the KG equation with ϕ being just a wavefunction to it being an operator, which can act on the vacuum creating particles.

This is sometimes called "second quantization", to distinguish it from the first quantization, in which operators act on wavefunctions.

Tuesday 2020-3-24, compiled 2020-06-25

Last week, we moved from the usual Schrödinger equation to the Klein Gordon equation by moving from $E = p^2/2m$ to $E^2 = p^2 + m^2$. The latter is covariant under Lorentz transformations.

There are, however, two issues with the Schrödinger equations: one is that it is not relativistic, while the second one is more subtle.

The Schrödinger equation describes the evolution of a single particle in time, while when we deal with elementary particles a one-particle description is unsuitable: we must have conservation of probability, so we are unable to describe situations in which a particle disappears by decaying into other particles, or we have inelastic collisions.

This is the reason why we need a multiparticle description. An example of the issues which arise in the single-particle relativistic description is the Klein paradox.

So, we will introduce the so-called second quantization formalism. We are going to interpret ϕ as a **quantum field operator**, instead of a scalar field. Let us make this explicit. We take the KG equation from the first quantization to the KG equation describing a Quantum Field Theory.

$\phi(x)$ will be an operator which can destroy or create particles. We start from the quantization of the Hamiltonian density, $\mathscr{H}$: we consider its ground state $|0\rangle$. This corresponds to a state in which there is no particle, and is called the **vacuum** state.

We will also have higher-energy states, in which we will have one or more particles: how do we describe these? A one particle state $\left|\varphi_1(p)\right\rangle$ can be acted upon by the operator

$\phi(x)$: it is destroyed, yielding a state which is proportional to the vacuum. Formally, we have

$$\langle 0| \phi(x) \left|\varphi_1(p)\right\rangle = e^{-ipx}\,, \tag{1.50}$$

where p^0 is the positive energy: $p^0 = +E_p = +\sqrt{\vec{p}^2 + m^2}$. For a more in-depth discussion of the second-quantization formalism, see the Theoretical Physics notes [Tis20, section 2.4].

Now, let us consider the complex conjugate of this matrix element: we swap the bra and ket and take the adjoint of the operator, to get

$$\left\langle \varphi_1(p)\right| \phi^\dagger(x) |0\rangle = e^{ipx}\,. \tag{1.51}$$

If we interpret $\phi^\dagger$ as acting on the right on the vacuum, we must say that it creates a particle at the location x, with indeterminate momentum.

This means that, if the first equation describes a particle propagating with momentum p^μ, this new equation will now describe a particle propagating with momentum $-p^\mu$.

This will now have a negative energy. This was a problem historically, now we give the solution directly.

The field ϕ can be either real or complex: that is, ϕ can be either equal to $\phi^\dagger$ or not. Let us consider the complex case: we introduce a particle $\left|\varphi_2(p)\right\rangle$, such that it is destroyed by the operator $\phi^\dagger$:

$$\langle 0| \phi^\dagger(x) \left|\varphi_2(p)\right\rangle = e^{-ipx}\,, \tag{1.52}$$

so now this particle has the same mass m, but — as it is shown by Peskin [Pes19, sec. 3.5], this new particle $\left|\varphi_2\right\rangle$ differs from $\left|\varphi_1\right\rangle$ for the charge, which is now opposite.

This means that the complex field describes a **charged** particle. The particle described by $\phi^\dagger$ is called the **antiparticle** of that described by ϕ.

Do note that charge does not exclusively mean electric charge! We can also have other kinds of charges. For example, neutrinos have the charge of lepton number.

Particles can be their own antiparticles, if they have zero charge. So, we can interpret the negative energy solution as the presence of an antiparticle: a negative energy particle would correspond to a positive energy antiparticle.

A generic field theory is defined by its action, which is written from the density Lagrangian: a free Lagrangian is

$$\mathscr{L} = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \varphi - m^2 \varphi^2\right), \tag{1.53}$$

and if we impose $\operatorname{var}(S) = 0$ we get precisely the KG equation as our equation of motion.

1.3.1 Spin 1

We discuss 3D vectors V^i, which transform under rotations $R^i{}_j \in SO(3)$ as

$$V^i(x) \to V'^i(x') = R^i{}_j V^j(R^{-1}x)\,. \tag{1.54}$$

What? x is controvariant...

Now, the matrix element from before reads

$$\langle 0| V^i(x) \left|v(p,\epsilon)\right\rangle = \epsilon^i e^{-ipx} , \tag{1.55}$$

where we need to account for the momentum p and the polarization ϵ^i. If we want to move to a relativistic description, we get the same thing, with the spatial index i being replaced by a 4-dimensional index μ:

$$\langle 0| V^\mu(x) \left|v(p,\epsilon)\right\rangle = \epsilon^\mu e^{-ipx} . \tag{1.56}$$

The problem is now the normalization of these states: what is the value of $\langle v|v\rangle$? This will be proportional to $\epsilon^\mu \epsilon_\mu$; but in general this is neither positive definite nor negative definite, since ϵ^μ could be timelike or spacelike *a priori*. We know that the photon has two physical helicities, which are transverse degrees of freedom. Since we know that those are physical and spacelike (with negative norm in our convention), we are tempted to say that the probability must be a positive multiple of $-\epsilon^\mu \epsilon_\mu > 0$: but then the timelike polarization has negative probability, and the longitudinal one does not belong (since it is known from classical electromagnetism that the photon only has the two transverse ones). The longitudinal polarization and the timelike one must be somehow forbidden.

This is in general a problem. It can be solved by a hack which is called the Gupta-Bleuer condition [Tis20, sec. 2.8].

The electromagnetic four-potential is defined as

$$A^\mu = \left(\varphi(x), \vec{A}\right) , \tag{1.57}$$

and the EM field strength is $F^{\mu\nu} = 2\partial^{[\mu} A^{\nu]}$. So, we have

$$F^{i0} = -\nabla^i \varphi - \partial_t A^i = E^i \tag{1.58a}$$

$$F^{ij} = -2\nabla^{[i} A^{j]} = \epsilon^{ijk} B^k . \tag{1.58b}$$

The Maxwell equations follow from the density Lagrangian

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu , \tag{1.59}$$

where j^μ is an external current. This yields $\partial_\mu F^{\mu\nu} = j^\nu$.

The photon is a massless vector boson. This description applies in general to a massless vector field. Are there massive vector fields? Yes, the weak interaction is described by a massive vector boson $W^\pm_\mu$.

1.3.2 Spin 1/2

Now we discuss spin 1/2 particles: we will need to have first derivatives on either side, as opposed to the second equations in the KG equation.

Our ansatz for what will be called the Dirac equation is:

$$i\partial_t = -i\vec{\alpha} \cdot \vec{\nabla} + \beta m \,. \tag{1.60}$$

Is it possible to find four numbers (3 encoded in the vector $\vec{\alpha}$, one more in β) so that the square of this relation is $E^2 = p^2 + m^2$ and that we still retain Lorentz invariance?

If we try to impose these conditions, we find that there are no solutions: there are no such four numbers.

We can, however, find a solution if we allow $\vec{\alpha}$ and β to be matrices: specifically, 4×4 matrices, which must obey the **anticommutation relations**:

$$\left\{\alpha_i, \alpha_j\right\} = 2\delta_{ij} \qquad \text{and} \qquad \left\{\alpha_i, \beta\right\} = 0 \,, \tag{1.61}$$

where we used the commutator: $\{a, b\} = ab + ba$, and now we define

$$\gamma^0 = \beta \qquad \text{and} \qquad \gamma^i = \beta\alpha^i \,, \tag{1.62}$$

which must be 4D, as we said, and the simplest representation is called the Dirac representation:

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} \qquad \text{and} \qquad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \,. \tag{1.63a}$$

It can be verified that then

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \,. \tag{1.64}$$

We can move between representations of these matrices using unitary transformations. Inserting these, we find:

$$\left[i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla} - m\mathbb{1}\right] \varphi(x) = 0 \,, \tag{1.65}$$

which means that the wavefunction must be 4-dimensional as well, since we are acting on it with a 4D operator. We are going to call such an object a **spinor**. If we denote $\gamma^\mu = \left(\gamma^0, \vec{\gamma}\right)$ we can write the Dirac equation as

$$\left(i\gamma^\mu \partial_\mu - m\right) \varphi(x) = 0 \tag{1.66a}$$

$$\left(i\partial\!\!\!/ - m\right) \varphi(x) = 0 \,, \tag{1.66b}$$

where we have defined the notation $x\!\!\!/ = \gamma^\mu x_\mu$. As we shall see tomorrow morning, this equation is extremely rich in structure.

In the Westminster abbey, this equation is inscribed as a homage to Paul Dirac.

Today we are going to examine the third case of wave equations: we discussed the KG equation for a scalar particle, then we moved to a vector particle, and now we are going to consider a spin-1/2 particle.

Wednesday 2020-3-25, compiled 2020-06-25

One has to prove that the equation $\left(i\partial\!\!\!/ - m\right)\varphi(x) = 0$ is Lorentz-covariant. Also, solutions of the Dirac equation are also solutions of the Klein-Gordon equation, therefore they satisfy $E^2 = p^2 + m^2$. The latter is relatively easy to prove: we apply the operator $i\partial\!\!\!/ + m$ to the Dirac equation and find

$$\left(i\partial\!\!\!/ + m\right)\left(i\partial\!\!\!/ - m\right)\psi(x) = 0 \tag{1.67}$$

$$\left(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2\right)\psi(x) = 0 \tag{1.68}$$

$$(\Box + m^2)\psi(x) = 0\,, \tag{1.69}$$

where we used the fact that, since the derivatives $\partial_\mu\partial_\nu$ commute, we can substitute $\gamma^\mu\gamma^\nu$ with half of their anticommutator, which equals $\eta^{\mu\nu}$.

Since we have proven the equivalence, we expect the Dirac equation to also have negative energy solutions.

Proving the covariance is harder, even though the equation looks covariant, since γ^μ does not transform as a vector *a priori*.

The object $\psi(x)$ is a spinor: it is neither a scalar, nor a vector. A priori we can say that its transformation law will look like

$$\psi(x) \to \psi'(x) = S(\Lambda)\psi(x)\,, \tag{1.70}$$

where $S(\Lambda)$ is a unitary transformation associated with the Lorentz transformation Λ, and by imposing the covariance of the Dirac equation we find that we must have

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \left(\Lambda^{-1}\right)^\mu_{\ \nu}\gamma^\nu\,. \tag{1.71}$$

So, in order to find the explicit form of $S(\Lambda)$ we write an infinitesimal Lorentz transformation: it can be shown that we can write it as

$$\Lambda^\mu_{\ \nu} = \eta^\mu_{\ \nu} + \delta\omega^\mu_{\ \nu}\,, \tag{1.72}$$

where $\delta\omega^\mu_{\ \nu}$ is an antisymmetric matrix, if it has nonzero $0i$ components it gives boosts, if it has nonzero ij components it gives rotations. The result for the form of S is:

$$S = \mathbb{1} + \frac{1}{8}\left[\gamma_\mu, \gamma_\nu\right]\delta\omega^{\mu\nu}\,. \tag{1.73}$$

If we make a rotation, for example, we get

$$\psi(x) \to \exp\left(i\frac{i}{2}\left[\gamma_i, \gamma_j\right]\delta\omega^{ij}\right)\psi(x)\,, \tag{1.74}$$

and typically one uses the shorthand rotation

$$\frac{i}{2}\left[\gamma_i, \gamma_j\right] \stackrel{\text{def}}{=} \sigma_{ij}\,. \tag{1.75}$$

If, for example, we want to perform a rotation by an angle φ around the z axis we get

$$\psi'(x') = S(\Lambda)\psi(x) = \exp\left(\frac{i}{2}\varphi\sigma_{12}\right)\psi(x)\,, \tag{1.76}$$

where

$$\sigma_{12} = \frac{i}{2}\left[\gamma^1, \gamma^2\right] = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}, \tag{1.77}$$

where we used the fact that $\left[\sigma_i, \sigma_j\right] = i\epsilon_{ijk}\sigma_k$.

This means that a spinor $\psi(x)$ reacts in a peculiar way to rotations: it rotates by an angle $\varphi/2$ if we perform a Lorentz rotation of an angle φ; its periodicity is 4π.

We introduce the *chiral representation* of the gamma matrices:

$$\gamma^0 = \begin{bmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix} \qquad \text{and} \qquad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}. \tag{1.78a}$$

Then, for Lorentz boosts we have

$$\sigma_{0i} = \frac{1}{2}[\gamma_0, \gamma_i] = -i\begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \tag{1.79a}$$

while for rotations we have

$$\sigma_{ij} = \frac{i}{2}\left[\gamma_i, \gamma_j\right] = \epsilon_{ijk}\begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, \tag{1.80a}$$

which is useful since it gives us block-diagonal matrices. So, we can interpret the spinor as being made up of two components:

$$\psi(x) = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \eta \\ \omega \end{bmatrix}, \tag{1.81a}$$

on which Lorentz transformations act independently. This is relevant since, for example, if we deal with an electron, we will describe it with a 4-component spinor, however we will be able to divide it into two components e_L and e_R, which are two component spinors on which we can act independently. We have effectively divided our representation of the Lorentz group into the sum of two irreps, of dimension $(1/2, 0)$ and $(0, 1/2)$ respectively.

This will become very concrete when we will discuss how many degrees of freedom were present in the original plasma.

Since a spinor ψ also solves the KG equation, it will be able to be written as

$$\psi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} e^{-ipx}, \tag{1.82a}$$

but what are the relations between the coefficients? We consider a simple case, that of $\vec{p} = 0$, so that we are in the rest frame of the particle.

Then, the Dirac equation reads:

$$\left(\gamma^0 E - m\right) \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0\,, \tag{1.83a}$$

so we need to choose a representation for the γ^0 in order to write this explicitly. We choose the Dirac representation, in which

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}. \tag{1.84a}$$

Then, the equation reads

$$\begin{bmatrix} E-m & 0 & 0 & 0 \\ 0 & E-m & 0 & 0 \\ 0 & 0 & -E-m & 0 \\ 0 & 0 & 0 & -E-m \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} e^{-ipx} = 0\,, \tag{1.85a}$$

so we get a solution in which $E = m$, and a solution in which $E = -m$. So, in general we write the two linearly independent solutions, respectively with positive and negative energy, as

$$\psi = \begin{bmatrix} \xi \\ 0 \end{bmatrix} e^{-imt} \qquad \text{and} \qquad \Psi = \begin{bmatrix} 0 \\ \eta \end{bmatrix} e^{+imt}. \tag{1.86a}$$

The negative energy solution, as we will see, represents the antiparticle of the Dirac fermion.

The assumption we made, $\vec{p} = 0$, does not actually mean we lose generality: we can simply boost into the rest frame of the particle. If we do this, we get the general

$$\begin{bmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m \end{bmatrix} \begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} e^{-imt+i\vec{p}\cdot\vec{x}} = 0\,, \tag{1.87a}$$

so we can decompose our solution into

$$\psi = u^s e^{-ipx} \qquad\qquad E > 0 \tag{1.88}$$

$$\psi = v^s e^{ipx} \qquad\qquad E < 0\,, \tag{1.89}$$

where s is an index denoting which 2D unit vector we are considering, that is, $s = 1, 2$. So, now we come to the interpretation: we introduce the existence of an antifermion, which corresponds to the solution to the Dirac equation with negative energy, but it has the opposed momentum: so, it has positive energy. Then, both of our solutions have positive energy and evolve forward in time, and both have the same mass.

Now, we make the jump to second quantization: we start interpreting $\psi(x)$ as an operator, which can destroy a one-state particle or create a particle starting from the vacuum.

We will have

$$\langle 0| \psi(x) \left| e^-(p,s) \right\rangle = u^s(p) e^{-ipx} \,. \tag{1.90}$$

Now, the one-particle state $\left|e^-(p,s)\right\rangle$ is promoted to a spinor. On the other hand, we have the creation operator $\psi^\dagger$:

$$\left\langle e^-(p,s) \right| \psi^\dagger(x) |0\rangle = u^{s\dagger} e^{-ipx} \,. \tag{1.91}$$

Now, the tricky question is to introduce the negative-energy solution. The $\psi^\dagger$ operator will destroy this state, while ψ will create it. So we will write an equation like

$$\langle 0| \psi^\dagger(x) \left| e^+(p,s) \right\rangle = v^{s\dagger}(p) e^{-ipx} \,, \tag{1.92}$$

where we would write $v^s(p)$ if we were considering the negative energy particle, instead we are looking at the antiparticle.

Now, we will be able to operate with $\psi(x)$ on the vacuum, to find

$$\left\langle e^+(p,s) \right| \psi(x) |0\rangle = v^s(p) e^{ipx} \,. \tag{1.93}$$

1.3.3 Photon-fermion coupling

Yesterday we discussed the Lagrangian of the free photon field,

$$\mathscr{L} \propto F^{\mu\nu} F_{\mu\nu} \,, \tag{1.94}$$

but as we said there can also be coupling to external currents, which we did not quantize. However, now we quantized the electron: so, can we construct the external current j_μ in the coupling term $j_\mu A^\mu$?

The first attempt would be to write something like

$$j^\mu \sim \psi^\dagger \gamma^\mu \psi \sim e^+ \gamma^\mu e^- \,, \tag{1.95}$$

but we would need to check whether it is a vector: in fact, it does not transform correctly.

Is this at least a Hermitian operator? Well, its adjoint is

$$\left(\psi^\dagger \gamma^\mu \psi \right)^\dagger = \psi^\dagger (\gamma^\mu)^\dagger \psi \,, \tag{1.96}$$

but

$$(\gamma^\mu)^\dagger = \left(\gamma^0, -\gamma^i \right) \neq \gamma^\mu \,. \tag{1.97}$$

One finds that the correct definition is to have

$$\overline{\psi} \stackrel{\text{def}}{=} \psi^\dagger \gamma^0 , \tag{1.98}$$

and then

$$j^\mu = \overline{\psi} \gamma^\mu \psi \tag{1.99}$$

is the correct definition. This, then, transforms as a 4-vector; it can also be shown starting from Dirac's equation (and its conjugate) that this is a conserved current: $\partial_\mu j^\mu = 0$.

Now, in order to couple the EM field to the electron we will use minimal coupling:

$$\partial_\mu \to \mathrm{D}_\mu = \partial_\mu + ieA_\mu , \tag{1.100}$$

so the Dirac equation will read

$$(i\not{\mathrm{D}} - m)\psi = 0 \tag{1.101a}$$

$$\Big(i\gamma^\mu \Big(\partial_\mu + ieA_\mu\Big) - m\Big)\psi = 0 . \tag{1.101b}$$

From which density Lagrangian can we derive the Dirac equation? It turns out to be

$$\mathscr{L} = \overline{\psi}\Big(i\gamma^\mu \mathrm{D}_\mu - m\Big)\psi , \tag{1.102}$$

so if we want to describe both the EM field, the electron and their interaction, we have

$$\mathscr{L}(e, A_\mu) = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{free EM field}} + \underbrace{\overline{\psi}(i\not{\partial} - m)\psi}_{\text{electron}} - \underbrace{e\overline{\psi}\not{A}\psi}_{\text{electron-EM coupling}} \tag{1.103}$$

$$= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\not{\mathrm{D}} - m)\psi . \tag{1.104}$$

This is the density Lagrangian of Quantum ElectroDynamics. This is the first interacting QFT which was constructed, and it was extraordinarily successful.

Its predictions for the anomalous magnetic moment of the electrons were exceptional: we can solve it perturbatively to different orders in

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137} . \tag{1.105}$$

Tuesday 2020-3-31, compiled 2020-06-25

We have associated spin $1/2$ fermions to matter fields. Spin 1 particles, instead, are vector bosons, such as the photon γ, the gluon g and the weak $W^\pm$ and Z^0 bosons. These are the radiation fields. We will explore this in more detail.

This connection comes from the **spin-statistics** theorem: it states that particles with integer spin obey Bose-Einstein statistics, while particles with half-integer spin obey Fermi-Dirac statistics. Particles which are "matter" (electrons, quarks and such) are fermions, while particles which are "force carriers" (photons, weak-interaction W and Z particles, gluons) are bosons.

This is due to the fact that in order to consistently quantize Dirac's theory we need to use **anticommutators** instead of field commutators to replace the Poisson brackets of the classical theory.

Up until now, we have observed only particles with spins 0, 1/2 and 1.

We could have a symmetry called supersymmetry, which connects fermions and bosons.

The graviton has spin 2; in supergravity the graviton has a fermion partner called the "gravitino" with spin 3/2.

1.3.4 Scattering

How do we normalize the states in relativistic theory? Classically we did

$$\langle p_1 | p_2 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\,, \tag{1.106}$$

in the relativistic case instead we will do

$$\langle p_1 | p_2 \rangle = 2E_{p_1} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\,. \tag{1.107}$$

This is a Lorentz invariant normalization (although not manifestly so — see Peskin [Pes19, sec. 3.5] for a proof).

The relativistic volume element is given by

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \to \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p}\,, \tag{1.108}$$

where we integrated over the variable $p^0 = E$, which removed the $\delta(p^2 - m^2) = \delta(E - \sqrt{p^2 + m^2})/2E$.

This way, we have the completeness relation

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle\langle p| = \mathbb{1}\,. \tag{1.109}$$

States have the dimension of an energy to the -1, field operators have the dimension of an energy.

There are two main kinds of processes which are considered in particle physics. The first is a **decay** process: we have a particle A decaying into a possibly multi-particle state f. We are interested in the decay *rate* of this process.

The probability of survival of particle A at time t is in the form $\mathbb{P}(t) = \exp(-t/\tau_A)$, so we define the decay rate

$$\Gamma_A = \frac{1}{\tau_A}\,, \tag{1.110}$$

which has the dimensions of a frequency, or equivalently an energy.

Generally the decay of a certain particle species can happen through different **channels**, that is, into different kinds of particles. We can define the branching ratios as

$$BR(A \to f) = \frac{\Gamma(A \to f)}{\Gamma_A}\,. \tag{1.111}$$

Another process of interest is a **scattering** process of $n \to m$ particles. There is no particle number conservation: we can create and destroy as many particles as we like. These types of processes are described by their *cross section*, which is an effective area corresponding to how aligned the trajectories of the incoming particles must be in order for them to interact.

This cross section allows us to compute the average time for an interaction to occur: if two particles' interaction becomes so rare that they cannot interact within a Hubble time then they are said to have *decoupled*.

We start by considering fixed-target experiments: we want to know how many events per second we will have, which will be given by

$$\frac{\text{\# events}}{\text{second}} = n_A \times v_A \times \sigma \,, \tag{1.112}$$

where σ is the cross section, v_A is the velocity of the incoming particles, while n_A is the number density of particles in the beam. The number density of the particles in the target is accounted for inside of σ. This tells us that σ has the dimensions of an area.

If we have two beams of particles coming towards each other, the term will look like $n_A n_B (v_A - v_B) \ell_B A_b \sigma$.

In general, we will be interested in the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^3 p_1 \, \mathrm{d}^3 p_2 \dots \mathrm{d}^3 p_n} \,. \tag{1.113}$$

This is just a definition, and it is not covariant, if we want to integrate we still need to use the covariant momentum element. We can integrate it in $\mathrm{d}^3 p_1 \dots \mathrm{d}^3 p_n$ in order to recover the total cross section, but inside it we have more information about the angular properties of the process.

For a scattering process like $A + B \to 1 + \dots + n$ we will need to compute things like

$$\langle 12 \dots n | \, T \, | AB \rangle = \mathcal{M}(A + B \to 1 + \dots + n)(2\pi)^4 \delta^{(4)}(P_A + P_B - \sum p_i) \,, \tag{1.114}$$

where T is the time evolution, and we defined the invariant scattering amplitude $\mathcal{M}$. If we want to compute the width Γ_A for a decay process, we need to define the phase space integral:

$$\int \mathrm{d}\Pi_n = \underbrace{\prod_i \int \frac{\mathrm{d}^3 p_i}{2E_i (2\pi)^3} (2\pi)^4 \delta^{(4)}(P_A - \sum_i p_i)}_{\text{phase space integral}} \,, \tag{1.115}$$

so we will have what is called Fermi's golden rule:

$$\Gamma_A = \frac{1}{2M_A} \int \mathrm{d}\Pi_n \left| \mathcal{M}(a \to f) \right|^2 , \tag{1.116}$$

since in order to get the probability we need to take the square of the amplitude.

Now, the Feynman amplitude $\mathcal{M}$'s dimensionality can be inferred from equation (1.114): the dimension of a state is $1/[M]$, the dimension of a four dimensional delta function is $1/[M]^4$, while the time evolution operator is dimensionless, so we have

$$[M]^{-n-1} = [\mathcal{M}][M]^{-4} \implies [\mathcal{M}] = [M]^{3-n}\,, \tag{1.117}$$

where n is the number of outgoing particles. Here by $[M]$ we mean the dimensions of a mass, energy, inverse length or inverse time. The dimension of the phase space element can similarly be found to be $[M]^{2n-4}$; so we can check that Fermi's golden rule is dimensionally consistent: its dimensions read

$$[\Gamma_A] = [M]^{-1}[M]^{2n-4}\Big([M]^{3-n}\Big)^2 = [M]\,, \tag{1.118}$$

which makes sense, since the decay rate is an inverse time.

The expression we gave is **polarized**, that is, by deciding on the initial and final states we are fixing the spins of the particles. This may be useful in some cases, but often we cannot control the spins of the incoming particles, and/or we cannot select only outgoing ones with a certain spin configuration. So, what is done usually is to **average** over the initial polarizations' probabilities, and to **sum** over the final ones. Note that in doing this we must add probabilities, not amplitudes: in this context we are dealing with classical mixtures.

For cross sections, Fermi's golden rule reads:

$$\sigma(A+B\to f) = \frac{1}{2E_A E_B |v_A - v_B|}\int \mathrm{d}\Pi_2 \left|\mathcal{M}(A+B\to f)\right|^2\,, \tag{1.119}$$

where (if we are in the COM frame) the phase space integral for the two final particles case is (nontrivially! see the TP notes [Tis20, sec. 4.2.3]) given by:

$$\int \mathrm{d}\Pi_2 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)\delta(E_{CM} - E_1 - E_2) \tag{1.120a}$$

$$= \frac{1}{8\pi}\frac{2p}{E_{CM}}\int\frac{\mathrm{d}\Omega}{4\pi}\,. \tag{1.120b}$$

The dimensionality check now gives $[\mathcal{M}] = [M]^{2-n}$, so

$$[\sigma] = [M]^{-2}[M]^{2n-4}\Big([M]^{2-n}\Big)^2 = [M]^{-2}\,, \tag{1.121}$$

which makes sense: it is a length squared, and lengths are inverse masses.

A kind of process which appears often is a **resonance**: something like $A+B\to X\to A+B$, where we have a certain short-lived intermediate state X, whose decay rate is Γ. The nonrelativistic Breit-Wigner formula describes this: it is given by

$$\mathcal{M} \sim \frac{1}{E - E_X + i\Gamma/2}\,, \tag{1.122}$$

where E_X is the energy of X, while E is the center-of-mass energy of the process.

If we perform a Fourier transform of this expression we get

$$\psi(x) = \int \frac{dE}{2\pi} \frac{e^{-iEt}}{E - E_X + i\Gamma/2} \tag{1.123}$$

$$= 2\pi i \operatorname{Res}_{E=E_X+i\Gamma/2} \left(\frac{e^{-iEt}}{E - E_X - i\Gamma/2} \right) \tag{1.124}$$

$$= ie^{-i(E_X - i\Gamma/2)t} = ie^{-iE_X t} e^{-\Gamma t/2} , \tag{1.125}$$

so the probability of finding the resonant state, $|\psi|^2$, decays as $e^{-\Gamma t}$.

The paradigmatic process for these kinds of interactions is the process $e^+e^- \to \mu^+\mu^-$.

1.3.5 A decay example

Wednesday 2020-4-1, compiled 2020-06-25

This and next week we will finish the introduction to particle physics, then we will start discussing the open problems in cosmology and astroparticle physics.

We consider the following process:

$$e^+e^- \to \mu^-\mu^+ , \tag{1.126}$$

where the mass of the electron is around $m_e \sim 0.5\,\text{MeV}$, the mass of the muon is around $m_\mu \sim 100\,\text{MeV}$.

Digression: there are different families of fermions (leptons and quarks), the first encompasses e, ν_e, u, d; the second encompasses μ, ν_μ, c, s and the third encompasses τ, ν_τ, t, b. The characteristics of the four members of the family are well-known, and between families the characteristics are the same: the only thing which varies between the families is the mass.

So, Rabi famously asked "who ordered the fermions"?

Coming back to our problem: the state $|e^+e^-\rangle$ must be annihilated by the EM current $j^\mu_{EM} = \overline{\psi}_e \gamma^\mu \psi_e$; it is then converted to a photon, which however is not on mass shell — it cannot be, since its momentum must be that of the electron-positron pair, so a timelike vector. It is then called a *virtual photon*: it can exist, as long as it does so for a short time.

Then, this photon decays to a muon-antimuon pair: then we will have a term $\langle \mu^-\mu^+ | j^\mu_(\mu\rangle$. The index between parenthesis is not a Lorentz one, it just means that this is a muonic current, different from the electronic one.

Let us call p_- and p_+ the momenta of the electron and positron, and p'_- and p'_+ those of the muon and antimuon.

The momentum q of the photon cannot have $q^2 = 0$, but this is fine: it is just an excitation.

The physics of the process is all contained in the matrix element $\mathcal{M}(e^+e^- \to \mu^+\mu^-)$. How do we calculate it? We will not go into details here, but it can be directly derived from the Feynman diagram of the interaction [Tis20, sec. 4.1.2]: we have

$$\mathcal{M}(e^+e^- \to \mu^+\mu^-) = (-e) \left\langle \mu^-\mu^+ \right| j^\mu \left|0\right\rangle \frac{1}{q^2} (-e) \left\langle 0\right| j_\mu \left| e^+e^- \right\rangle , \tag{1.127}$$

where the $-e$ factor is because of the EM coupling to the photon. The Breit-Wigner factor looks like

$$\frac{1}{P^2 - M_R^2}, \tag{1.128}$$

but for the photon we have no mass, therefore we only get a factor $1/q^2$.

The Feynman diagrams are just a way to collect the Feynman rules needed to compute the process, they are not meant to represent how the process "looks like".

Let us take the ultrarelativistic limit, in which the energy of the process is much larger than the muon's mass. If this is the case, then we can set $m_e = m_\mu = 0$.

Let us consider the Dirac equation, in the case in which the mass m is equal to zero: then we get

$$i\partial\!\!\!/\psi = 0\,. \tag{1.129}$$

Let us use the chiral representation for the γ matrices:

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{bmatrix}, \tag{1.130a}$$

where we mean by $\sigma^0 = \mathbb{1}$, $\sigma^\mu = (\sigma^0, \sigma^i)$ and $\overline{\sigma}^\mu = (\sigma^0, -\sigma^i)$.

Let us then split the spinor ψ into

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}, \tag{1.131a}$$

where $\psi_{L,R}$ are two-component spinors. This allows us to write two two-dimensional equations:

$$i\overline{\sigma}^\mu \partial_\mu \psi_L = 0 \tag{1.132a}$$

$$i\sigma^\mu \partial_\mu \psi_R = 0\,, \tag{1.132b}$$

where we get no interaction terms between the two: if we have no mass the equations decouple.

We can do the same thing if the Dirac equation is coupled to the EM field, since the issue is with the structure of the γ^μ, it does not matter if we have $\gamma^\mu \mathrm{D}_\mu$ or $\gamma^\mu \partial_\mu$; however we will write the decoupled solution for now.

So, for the right-handed spinor we have:

$$\left(i\partial_t + i\vec{\sigma} \cdot \vec{\partial}\right)\psi_R\,, \tag{1.133}$$

which is solved by a plane wave:

$$\psi_R = u_R(p) e^{-iEt + i\vec{p}\cdot\vec{x}}\,. \tag{1.134}$$

Let us suppose the equation reads

$$\left(E - p\sigma^3\right) u_R = \begin{bmatrix} E-p & 0 \\ 0 & E+p \end{bmatrix} u_R = 0\,, \tag{1.135a}$$

so we must have two solutions: they look like

$$\psi_R = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-iEt+iEx_3} \qquad \text{and} \qquad \psi_R = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{+iEt+iEx_3}\,. \tag{1.136a}$$

The solution $\psi_R \sim \exp(-iEt + iEx_3)$ describes a right-handed electron with spin eigenvalue $+1/2$ along the direction of motion (this is called the *helicity* [Tis20, sec. 1.4.9])

On the other hand, the solution $\psi_R \sim \exp(iEt + iEx_3)$ describes a right-handed electron with spin eigenvalue $s = -1/2$.

Our quantum field operator ψ_R acts as on the right by destroying a right-handed electron:

$$\langle 0| \psi_R \left| e_R^-(p) \right\rangle = u_R(p) e^{-ipx}\,. \tag{1.137}$$

Then, we can have it acting on the left by destroying a left-handed positron:

$$\left\langle e_L^+(p) \right| \psi_R \left|0\right\rangle = v_L(p) e^{+ipx}\,. \tag{1.138}$$

If we were to repeat the analysis for the other spinor, we would get the specular result.

The Lagrangian can be written as

$$\mathscr{L} = \psi_R^\dagger (i\sigma \cdot \partial)\psi_R + \psi_L^\dagger (i\overline{\sigma} \cdot \partial)\psi_L - m\left(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R\right)\,, \tag{1.139}$$

so the coupling between the left and right handed fermions depends on the mass, if we are in a situation in which $T \gg m$ they effectively decouple.

In order to describe this spin, we introduce the helicity quantum number, which is defined as

$$h = \hat{p} \cdot \vec{s}\,, \tag{1.140}$$

the projection of the spin along the direction of motion. For ψ_R, we have the $(1,0)$ state with helicity $h = 1/2$, while the state $(0,1)$ is a positron with helicity $h = -1/2$.

If $m = 0$, then helicity is exactly conserved. At high energies, it is suppressed by a factor m/E.

Let us compute the cross section: we must calculate

$$\langle 0| j^\mu \left| e_R^-(p_-) e_L^+(p_+) \right\rangle\,, \tag{1.141}$$

where we have a term $j^\mu = \overline{\psi}\gamma^\mu \psi = \psi^\dagger \gamma^0 \gamma^\mu \psi$; the term $\gamma^0\gamma^\mu$ reads

$$\gamma^0\gamma^\mu = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{bmatrix} = \begin{bmatrix} \overline{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{bmatrix}\,, \tag{1.142}$$

so we get

$$j^{\mu} = \psi_L^{\dagger}\overline{\sigma}^{\mu}\psi_L + \psi_R^{\dagger}\sigma^{\mu}\psi_R \,. \tag{1.143}$$

We describe the process in the center-of-mass frame, and we choose to align the axes so that the electron and positron have momenta $p^{\mu} = (E, 0, 0, \pm E)$ respectively (minus for the positron).

The wavefunction $\psi_R^{\dagger}$ annihilates the positron e_L^+ yielding a term $v_L^{\dagger}(p_+)$, the wavefunction ψ_R annihilates the electron e_R^- yielding a term $u_R(p_-)$.

Then, we are left with

$$\begin{aligned} v_L^{\dagger}(p_+)\sigma^{\mu}u_R(p_-) &= \sqrt{2E}\begin{bmatrix} 0 & 1 \end{bmatrix}(\mathbb{1}, \vec{\sigma})\sqrt{2E}\begin{bmatrix} 1 \\ 0 \end{bmatrix} && (1.144) \\ &= 2E(0, 1, i, 0)^{\mu}\,, && (1.145) \end{aligned}$$

so if we define the vector $\vec{\epsilon}_+ = (\hat{1} + i\hat{2})/\sqrt{2}$ we can write

$$\langle 0|\, j^{\mu} \left| e_R^-(p_-)e_L^+(p_+) \right\rangle = 2E\sqrt{2}(0, \vec{\epsilon}_+)^{\mu}\,, \tag{1.146}$$

while we would have

$$\langle 0|\, j^{\mu} \left| e_L^-(p_-)e_R^+(p_+) \right\rangle = -2E\sqrt{2}(0, \vec{\epsilon}_-)^{\mu}\,, \tag{1.147}$$

with $\vec{\epsilon}_- = (\hat{1} + i\hat{2})/\sqrt{2}$.

On the other hand, the terms $e_R^- e_R^+$ and $e_L^- e_L^+$ do not contribute (in our $m = 0$ approximation).

The muons are massless fermions as well in our treatment, so we get analogous terms:

$$\begin{aligned} \left\langle \mu_R^-(p'_-)\mu_L^+(p'_+) \right| j^{\mu}\,|0\rangle &= 2E\sqrt{2}\big(0, \vec{\epsilon}\,'^{*}_{+}\big)^{\mu} && (1.148) \\ \left\langle \mu_L^-(p'_-)\mu_R^+(p'_+) \right| j^{\mu}\,|0\rangle &= -2E\sqrt{2}\big(0, \vec{\epsilon}\,'^{*}_{-}\big)^{\mu}\,, && (1.149) \end{aligned}$$

so in the end we find

$$\begin{aligned} \mathcal{M}(e_L^- e_R^+ \to \mu_R^- \mu_L^+) &= -\frac{e^2}{q^2} 2(2E)^2 \vec{\epsilon}\,'^{*}_{+} \cdot \vec{\epsilon}_+ && (1.150) \\ &= -2e^2 \vec{\epsilon}\,'^{*}_{+} \cdot \vec{\epsilon}_+\,, && (1.151) \end{aligned}$$

since $q = 2E$. The scalar product here depends on the direction of emission of the muons in the center of mass frame, θ; we find the absolute value $|\mathcal{M}|^2 = e^4(1 \pm \cos\theta)^2$, depending on whether we are looking at an $LR \to LR$ process or $LR \to RL$ process. The unpolarized (spin-averaged) differential cross section comes out to be:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{1}{2}\frac{\pi\alpha^2}{2E_{CM}^2}\Big(1 + \cos^2\theta\Big)\,, \tag{1.152}$$

which can be integrated across the sphere to get the total cross section for the process $e^- e^+ \to \mu^- \mu^+$:

$$\sigma = \frac{4\pi}{3} \frac{\alpha^2}{E_{CM}^2} \,. \tag{1.153}$$

We could have guessed the dependence on α^2 / E_{CM}^2, but for the numerical factor we needed to do the full computation. This is because the cross section is a length square, so it must depend on the inverse square of our only energy parameter, E_{CM}.

Also, the coupling was fixed: we are working in QED, so we only have a coupling constant: e, so we will have terms $e^2/4\pi = \alpha$ inside of $\mathcal{M}$, so we will get α^2 inside of $|\mathcal{M}|^2$. The 4πs will cancel because of the phase-space angular integrals.

Once we have done this, can we generalize it? suppose we want to compute the cross section $\sigma(e^+ e^- \to \text{hadrons})$. How could we do it?

We can make a similar kind of reasoning: the process will look like $e^- e^+ \to q\overline{q}$, and while the coupling for the first vertex in the Feynman diagram will be $-e$ the one for the second vertex will look like Q_q, the charge of these quarks. We calculate the unpolarized cross section, counting all the quarks which can be produced at a fixed COM energy: we assume $E \sim 100\,\text{GeV}$, so all the quarks except for the top are candidates; so the computation goes:

$$\frac{\sigma(e^+ e^- \to q\overline{q})}{\sigma(e^+ e^- \to \mu^+ \mu^-)} \approx \sum_q Q_q^2 = \underbrace{2\frac{4}{9}}_{u,c} + \underbrace{3\frac{1}{9}}_{d,s,b} = \frac{11}{9} \,. \tag{1.154}$$

In experiments, however, we get a cross section ratio which is $11/3$, 3 times larger than expected: this indicates that we have a different type of charge, color charge, which means we have a multiplicity of 3 for each quark.

We shall see that this is related to certain kinds of internal symmetries of our field theory.

1.3.6 Gauge symmetries in QED

Tuesday 2020-4-7, compiled 2020-06-25

Last time we wrote the Lagrangian of QED:

$$\mathscr{L}_{QED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \overline{\psi} \Big(i\gamma^\mu \mathrm{D}_\mu - m \Big) \psi \,, \tag{1.155}$$

where $\mathrm{D}_\mu = \partial_\mu + ieA_\mu$.

This possesses several symmetries: Lorentz (actually, Poincaré) invariance, P (parity), C (charge conjugation), T (time inversion).

We are also interested in the *internal* symmetries of this Lagrangian: all the aforementioned symmetries were spacetime ones.

We know that $\overline{\psi} = \psi^\dagger \gamma^0$, so if we transform $\psi \to e^{i\theta} \psi$ the Lagrangian is unchanged, since we also have $\overline{\psi} \to e^{-i\theta} \overline{\psi}$. Here, we are taking a *constant* phase angle θ: it comes out of the derivative unchanged. This is called a $U(1)$ **global** symmetry, since it is the same everywhere in space and since a phase is the same as a 1×1 unitary matrix.

The following section differs from the notes. Let us ignore the interactions of electrons with the EM fields in QED. Our world is made of electrons, and we want to describe these free propagating electrons. We take the Lagrangian

$$\mathscr{L}_{\text{Dirac}} = \overline{\psi}\Big(i\gamma^\mu \partial_\mu - m\Big)\psi\,, \tag{1.156}$$

which still has the symmetry $\psi \to e^{i\alpha}\psi$.

Is this invariant also with respect to a *local* $U(1)$ symmetry? This looks like $\psi(x) \to e^{i\alpha(x)}\psi(x)$, where $\alpha(x)$ is a continuous spacetime scalar function.

This is a "promotion" of the symmetry: why? It seems like a global symmetry is a more general thing... However, the global symmetry is a special case of the local one.

This local symmetry is called a $U(1)$ *gauge* symmetry. Properly speaking, the global symmetry is also a gauge one but it is commonly called just a global symmetry.

Substituting in, we get

$$e^{-i\alpha(x)}\overline{\psi}\Big[i\partial_\mu\gamma^\mu - m\Big]e^{i\alpha(x)}\psi = \overline{\psi}\Big[i\partial_\mu\gamma^\mu - m\Big]\psi + \overline{\psi}\psi i\gamma^\mu\partial_\mu\alpha\,, \tag{1.157}$$

so we see that the Lagrangian is *not* invariant under this gauge symmetry in general.

If we want the symmetry to hold, we need to introduce a *compensating* field to cancel out the term.

The answer is that the thing to add is a vector field A_μ. Then, if we transform

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha \tag{1.158}$$

this compensates the change, as long as the vector is coupled to the fermion with a term $\overline{\psi}e\gamma^\mu A_\mu\psi$ in the Lagrangian. This is a profound result: if we wanted to describe a world with only electrons, and we want to have this electron be symmetric with respect to the $U(1)$ gauge symmetry $e \to e^{i\alpha(x)}e$ then we *must* have a vector coupled to it.

Then, we should also insert a term describing the propagation of the free vector A_μ, the kinetic term $\propto F^{\mu\nu}F_{\mu\nu}$.

A quote: "And she said 'let there be symmetry', and there was light".

We could have just proven that the QED Lagrangian is invariant with respect to $U(1)$ symmetry: however, this reasoning illustrates the point that the symmetry requires the insertion of photons.

One might ask: how do you know that this is the correct symmetry? The method is trial and error.

Could we have something which is more complicated than a spin 1 mediator? It is basically a guessing game, we see what works.

1.4 QCD

This section can also be followed from Peskin [Pes19, sec. II.11].

This $U(1)$ gauge symmetry is an abelian symmetry, but we also have non-abelian ones: we denote by T^a the generators of the group, their Lie algebra is defined by

$$\left[T^a, T^b\right] = if^{abc}T^c \,. \tag{1.159}$$

The *structure coefficients* f^{abc} are manifestly antisymmetric in their first two indices. It can also be shown that they are fully antisymmetric. If the group is abelian, we have f^{abc} identically, but this need not be the case.

To say that these are generators means that any infinitesimal transformation can be written as

$$\Phi \to (1 + i\alpha^a t^a_R)\Phi \,, \tag{1.160}$$

where α^a are the parameters of the infinitesimal transformation, while t^a_R are Hermitian matrices of dimension d_R which make up the representation of the group. There are d_G of them, where d_G is the dimension of the group.

The finite unitary transformation mapping $\Phi \to U(\alpha)\Phi$ can be recovered from here by

$$U(\alpha) = e^{i\alpha^a t^a_R} \,. \tag{1.161}$$

We will be interested in Lie groups $SU(n)$ with $N \geq 2$, which are $N \times N$ unitary matrices with determinant 1.

For example, recall that $SU(2)$ has a 2-to-1 correspondence with $SO(3)$. $SU(2)$ describes the rotation of spinors, the generators of their rotation are $\sigma^i/2$.

Last time we discussed the annihilation of e^+e^- into hadrons: we can get protons and neutrons, pions, kaons…

However we can simplify by discussing only the creation of quarks.

When we compute the cross sections, our calculation seems to be wrong by a factor 3. If we multiply it by 3 we get the correct result. So there are three types of quarks: we categorize them by "color", even though it has nothing to do with colors.

We associated QED with $U(1)$: is there a group corresponding to Quantum Chromo Dynamics? Can we do this with the weak interaction as well?

Since there are three quarks, we are drawn to represent them as triplets. Since we know that unitary matrices are nice, we try $SU(3)$. We call this symmetry $SU(3)_{\text{color}}$.

We start by giving some general results for N-dimensional groups $SU(N)$: we normalize their representation by imposing

$$\operatorname{Tr}\left[t^a_N t^b_N\right] = \frac{1}{2}\delta^{ab} \,, \tag{1.162}$$

where t^a_N are the Hermitian generators of an N-dimensional unitary representation. If we generalize to an R-dimensional representation, we will have

$$\operatorname{Tr}\left[t^a_G t^b_G\right] = C(R)\delta^{ab} \,, \tag{1.163}$$

where $C(R)$ is some constant depending only on the dimension of the representation.

A special representation we can choose is the **adjoint** representation, which is the one under which the generators of the Lie algebra transform; it is defined by:

$$\left(t_G^a\right)^{bc} \stackrel{\text{def}}{=} i f^{abc} . \tag{1.164}$$

By making use of the Jacobi identities we can show that this is indeed a valid representation of the group, its dimension is that of the group and we have:

$$\operatorname{Tr}\left[t_G^a t_G^b\right] = f^{acd} f^{bcd} = C(G)\delta^{ab} , \tag{1.165}$$

where the constant $C(G)$ is just the dimension N for the adjoint representation. For $SU(N)$ the dimension is N^2-1: the representation consists of N^2-1 matrices, each $N \times N$.

1.4.1 Non-abelian gauge theory: Yang-Mills

Consider the Lagrangian

$$\mathscr{L} = \overline{\psi} i \gamma^\mu \partial_\mu \psi . \tag{1.166}$$

This can describe an electron. Now, let us add an index j in the group G, according to which the particle transforms in the R-dimensional (in our case $R=3$) representation:

$$\mathscr{L} = \overline{\psi}_j i \gamma^\mu \partial_\mu \psi_j . \tag{1.167}$$

Under the local action of a group G the particle transforms like

$$\psi_j(x) \xrightarrow{G} \psi_j^{'}(x) = \left(1 + i\alpha^a(x) t_R^a\right)_{jk} \psi_k , \tag{1.168}$$

where a is an index going from 1 to N^2-1.

We have the same problem we had with $U(1)_{\text{em}}$: we need to cancel the term coming from the derivative of $\alpha(x)$, with a one-index object. The variation in the Lagrangian is

$$\delta\mathscr{L} = \overline{\psi}_j i \gamma^\mu \left(i \partial_\mu \alpha^a(x) t^a_{R,jk}\right) \psi_k . \tag{1.169}$$

The solution is the same: we introduce a coupling to the derivative, which takes the form

$$\mathrm{D}_\mu \to \partial_\mu - i g A_\mu^a t_R^a , \tag{1.170}$$

where g is the strength of the interaction, which is also written as g_s in the case of the strong force. The index a goes from 1 to $N^2-1=8$: we must introduce a vector field A for each *generator* of the required symmetry.

The beautiful thing is the fact that starting only from the symmetry requirement we can get a full theory.

We now attribute actual *existence* to these quantum fields: they are interaction bosons. They must transform like

$$A_\mu^a(x) \to A_\mu^a(x) + \frac{1}{g}\partial_\mu \alpha^a(x) + \overbrace{A_\mu^b f^{abc} \alpha^c(x)}^{\text{new nonabelian term}} \tag{1.171}$$

$$= A_\mu^a(x) + \frac{1}{g} \mathrm{D}_\mu \alpha^a(x) \,. \tag{1.172}$$

This is derived making use of the adjoint representation definition (1.164): we are equating

$$A_\mu^b f^{abc} \alpha^c = -i A_\mu^b t_R^b \alpha^a \tag{1.173}$$

$$A_\mu^b f^{abc} \alpha^c = -i A_\mu^b (i f^{bca}) \alpha^c \tag{1.174}$$

$$A_\mu^b f^{abc} \alpha^c = + A_\mu^b f^{abc} \alpha^c \,. \tag{1.175}$$

We can do cyclic permutations of the indices of f^{abc}.

If we assign physical reality to these fields we must give them kinetic terms in the Lagrangian, which will then be

$$\boxed{\mathscr{L} = -\frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a + \overline{\psi}\Big(i\gamma^\mu \mathrm{D}_\mu - m \Big) \psi \,,} \tag{1.176}$$

which looks the same as the QED Lagrangian, but we must be careful: what is $F_{\mu\nu}^a$? We might think it is

$$F_{\mu\nu}^a = 2\partial_{[\mu} A_{\nu]}^a \,, \tag{1.177}$$

but this is not enough: inside the covariant derivative in $\mathrm{D}_\mu \alpha$ in the transformation law we also have A_μ, so we get an additional term, and the final formula looks like

$$F_{\mu\nu}^a = 2\partial_{[\mu} A_{\nu]}^a + g f^{abc} A_\mu^b A_\nu^c \,. \tag{1.178}$$

This corresponds to the fact that, while there is no photon-photon interaction since $U(1)$ is abelian, $SU(3)$ is not: so, we do have gluon-gluon interaction. Notice that this expression has abelian symmetries as a special case: the term fAA is antisymmetric in the two bosons, so if they commute it vanishes. The square of this field strength is then both Lorentz and gauge invariant.

This field strength can be interpreted as the Riemann tensor of the Lie group manifold:

$$\Big[\mathrm{D}_\mu, \mathrm{D}_\nu \Big] = -ig F_{\mu\nu}^a t_R^a \,. \tag{1.179}$$

This is crucial in very high energy situations, such as in the early universe. The dynamics of the field are now more complicated than the ones we find in electrodynamics due to the nonlinear terms.

The wavefunction ψ which appears in the Lagrangian will transform in some d-dimensional representation of $G = SU(3)_c$. It will have indices like $\psi_{\alpha,i}$: α is a four-dimensional spinorial index, while i is a three-dimensional color index.

The strong-interaction coupling constant g_s is dimensionless, we also define the parameter

$$\alpha_s = \frac{g_s^2}{4\pi} \,. \tag{1.180}$$

The coupling term of the gluons with the fields can be made explicit as

$$\overline{\psi}_{\alpha,i} \gamma_{\alpha\beta}^\mu A_\mu^a (t^a)_{ij} \psi_{\alpha\beta} \,, \tag{1.181}$$

where μ is a Lorentz index, i and j are color indices, while α and β are spinorial indices.

1.4.2 Quantum Chromo Dynamics

Wednesday 2020-4-8, compiled 2020-06-25

Asymptotic freedom is not a political slogan from the sixties!

With this lecture and the next we should finish discussing the standard model of particle physics.

Today we are going to explore a property which is profoundly different between EM and the strong interaction.

We introduced the quantity $\alpha_{EM} = e^2/4\pi$: how do we measure this parameter? We know that the interaction term looks like

$$\mathscr{L} \sim e j^\mu A_\mu \,, \tag{1.182}$$

so we can measure e using the cross sections of processes, such as $e^+e^- \to \mu^+\mu^-$; the free parameters are just e and the COM energy, so we are done. But how do we do it without an accelerator? We could do the Millikan drop experiment, for example.

But we are doing Quantum Field Theory: beyond physical particles we can also have virtual particles, which despite the name can influence physical processes: inside the "black box" within which the interactions occur we can have off-shell processes.

In fact, in the aforementioned decay the photon is off mass shell; but we can go beyond. The photon can create and then annihilate an e^+e^- pair.

The vacuum is a *quantum* vacuum, which does correspond to a minimal energy but it is filled by the continuous creation and annihilation of these particle-antiparticle pairs, since there is an indetermination between time and energy. It is important that after being created these pairs are indeed destroyed. This is called *vacuum polarization*

The pairs are virtual but they have an effect. The coupling constant α has a numerical value of around $1/137$, and the loop diagrams of increasing number of loops depend on increasing powers of α. The fact that α is small allows us to work perturbatively.

This works, the predictions of QED are extremely precise and correspond to experiment. For instance, the anomalous magnetic moment of the electron: in a QFT, when an electron interacts with a photon, there can be other particles.

The polarization of the vacuum creates a screening effect. So, we expect the effective α_{EM} to be larger. We compare the Millikan experiment ($E \sim 0$), where we find $\alpha \sim 1/137$, with LEP: now $E \sim 100\,\text{GeV}$, and we find $\alpha \sim 1/128$!

What is the theoretical relation between α and Q? It comes out to be

$$\alpha(Q) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi} \log\left(\frac{Q}{Q_0}\right)} \,, \tag{1.183}$$

where Q is the energy at which we measure.

What happens if we compute this for the strong interaction? We will have

$$\alpha_3 = \alpha_{\text{strong}} = \frac{g_s^2}{4\pi} \,. \tag{1.184}$$

The problem is that this is a large number: but our tools are perturbative! How do we compute the cross sections then? The thing is, with increasing energy this α becomes *smaller*! Why is this? The gluons are self interacting, because of the $A \wedge A$ term in the field strength of the strong interaction, as opposed to photons.

In the EM case, we had $\gamma \to e^+e^- \to \gamma$ as a virtual particle process; in the chromodynamics case instead we have $g \to g + g \to g$: it is the case (even though it is not easy to prove) that this loop contribution has the opposite sign! We have

$$\frac{\mathrm{d}g_s}{\mathrm{d}\log Q} = \beta(g_s) \qquad \text{where} \qquad \beta = -\frac{11}{3}C(G) + \frac{4}{3}n_f\frac{1}{2}\,, \tag{1.185}$$

where $C(G)$ is the Casimir of the adjoint representation, so for QCD it is 3, while n_f, the number of fermions, is 6. Then, the final result is negative!

This is called asymptotic freedom: α goes to zero with Q. If the energy is high enough, the quarks and gluons are effectively decoupled.

So, at the low energies we look at in atomic nuclei the quarks are confined: no one has ever observed a free quark, not even at LHC. This is called hadronization, since the quarks are always bound into hadrons. ALICE, at LHC, is looking for the phase transition between this "infrared slavery" of the quarks and their free state.

Since α_S decreases, while α_{EM} increases, there should be a point at which they cross and the electromagnetic interaction becomes stronger than the strong one.

Actually, it might be possible that all of the interactions are manifestations of the same kind of interaction.

The crucial point is that the strength of an interaction depends on the energy.

We know that the time needed for hadronization is of the order $\tau_{\text{had}} \sim 10^{-23}\,\text{s}$, while the time for a pion's decay is $\tau(\pi^+) \sim 3 \times 10^{-8}\,\text{s}$. The time for beta decay is $\tau(n \to p + e^- + \overline{\nu}_e) \sim 880\,\text{s}$.

This can give us a first signal for the fact that the weak interaction is *weak*.

We know that the neutron is *udd* in terms of quarks, while the proton is *uud*: so, in the beta decay one down quark must become an up quark. There is no term in QED or QCD in which this can happen: QED and QCD conserve flavour.

But we do see beta decay: so, we must introduce a new *weak interaction*.

Beta decay is experimentally observed to be a three-body process, which is why observations of it were the first indication of the existence of neutrinos. Beta-decay is not P-symmetric; on the other hand, QED and QCD are.

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Last lecture we started the discussion of a new kind of interaction: we had treated QED and QCD, now we introduced the nuclear weak interaction.

We would like to introduce a sort of "QWD", and we will see that this is possible in the standard model, by unifying the electromagnetic and weak interactions.

We have seen that fermions are described by a four-component spinor:

$$\Psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}\,, \tag{1.186a}$$

where $\psi_{L,R}$ are both two-component vector. The right handed part has helicity $h = +1/2$, the left handed part has helicity $h = -1/2$.

The puzzling part of the weak interaction is that it seems like the only part of the fermion entering β-decay is the left-handed part. In the late seventies, the professor's master thesis stated that this was a mystery. Still, we do not know why this interaction breaks parity.

We need to describe the experimental results: so, we need a projector onto the left-handed components. We already introduced the Dirac γ^μ matrices. Now we introduce

$$\gamma^5 = \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \tag{1.187a}$$

which has the important property that it anticommutes with all the Dirac matrices:

$$\left\{\gamma^5, \gamma^\mu\right\} = 0\,, \tag{1.188}$$

and actually we can write it as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{1.189}$$

By its form we can readily see that ψ_L is an eigenstate of γ^5 with eigenvalue -1, while ψ_R has eigenvalue $+1$. So, the matrix

$$\frac{\mathbb{1} - \gamma^5}{2} \tag{1.190}$$

projects a state onto the left-handed subspace. The Dirac matrices can be written as

$$\gamma^\mu \begin{bmatrix} 0 & \sigma^\mu \\ -\overline{\sigma}^\mu & 0 \end{bmatrix}, \tag{1.191a}$$

where

$$\sigma^\mu = \begin{bmatrix} \mathbb{1} & \sigma^\mu \end{bmatrix} \qquad \text{and} \qquad \overline{\sigma}^\mu = \begin{bmatrix} \mathbb{1} & -\sigma^\mu \end{bmatrix}. \tag{1.192}$$

Now, then, we discard the left handed components: for $m = 0$ we write

$$i\overline{\sigma}\partial\psi_L = 0 \tag{1.193a}$$

$$i\sigma\partial\psi_R = 0\,, \tag{1.193b}$$

so we can discard the second equation. We can introduce a current involving only the left-handed component:

$$j_L^{\mu+} = \nu_L^+\overline{\sigma}^\mu e_L + u_L^+\overline{\sigma}^\mu d_L + \dots\,. \tag{1.194}$$

Notice that we did not introduce here an index a as we had done for the color charge.

This is a charged current.

Why are there no terms involving σ^μ? We projected along the first two components, so in the terms

$$\overline{u}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)d = \overline{u}\gamma^\mu d_L \tag{1.195}$$

the last two terms vanish, we do not need to write them. This theory of weak interaction is called a $V - A$ theory: in the current we have the terms

$$\frac{1}{2}\left[\underbrace{\overline{u}\gamma^{\mu}d}_{V} - \underbrace{\overline{u}\gamma^{\mu}\gamma^{5}d}_{A}\right], \tag{1.196}$$

where the term V is a vector current, while the term A is an axial current.

We know that a density Lagrangian has a dimension of 4, so from the expression $\mathscr{L} = \overline{\psi}m\psi$ we get that the fermion must have dimension $3/2$. By dimension, we mean power of the mass.

When we have a term like $j^{\mu}j_{\mu} \times \text{const}$, the constant must have a dimension of -2. The constant is called G_F, and we have that the cross section goes like $\sigma \sim G_F^2 \sim [m]^{-4}$? this does not work, we need something with dimension 2 multiplying it, since the cross section has dimension -2.

Our only free parameter is s, the square of the center of mass energy. So, we will have

$$\sigma \sim G_F^2 s\,, \tag{1.197}$$

which means that if we increase the beam energy the cross section increases. s can diverge in principle, but σ is connected to a probability: this means that we have a unitarity violation.

This can be fixed with the introduction of a charged mediator W.

How massive must it be? We can measure $G_F \sim 10^{-5}\,\text{GeV}^{-2}$.

In the Breit-Wigner resonance formula we will need to insert

$$\frac{1}{q^2 - M_W^2}\,, \tag{1.198}$$

which avoids the divergence. In order for this to work, we need this boson to be of the order of $10\,\text{GeV}$ to $100\,\text{GeV}$.

Now we need to describe this as a gauge theory. The symmetry group will be $SU(2)_L$. We will have $2^2 - 1 =$ three generators.

This process is going to fail: the kinetic term of these vector bosons will look like

$$F^{\mu\nu,i}F^{i}_{\mu\nu}\,, \tag{1.199}$$

which will never have a quadratic term $M^2 W^{\mu} W_{\mu}$. So, the bosons will never be massive like we need. This is called the Intermediate Vector Boson theory.

Since our plain Yang-Mills theory does not have vector bosons, we can insert them manually. However, if we do we are explicitly breaking the gauge symmetry.

We lose renormalizability: renormalization means that we can absorb all the infinities into a finite number of parameters.

We have a dilemma: Yang-Mills theories are based on the power of symmetry, but they predict massless mediators.

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Last time we stopped at the dilemma of beauty versus pragmatism.

How do we provide a mass to the W vector boson?

As we were discussing, we could add a mass term and write

$$\mathscr{L} = \mathscr{L}_{\text{Yang-Mills}} + M^2 W^+_\mu W^{\mu,-} , \tag{1.200}$$

but this term *brutally* breaks the symmetry. This is explicit breaking of the symmetry.

So, we introduce **Spontaneous Symmetry Breaking**. This is not exclusive to HEP: it happens in ferromagnets for example. As we cool them, at a certain stage all the spins align.

We cannot predict the direction along which the spins will align.

Let us introduce the Lagrangian of a scalar field ϕ:

$$\mathscr{L} = \left(\partial^\mu \phi^*(x)\right)\left(\partial_\mu \phi(x)\right) + \mu^2 \left|\phi(x)\right|^2 + \lambda^4 \left|\phi(x)\right|^4 , \tag{1.201}$$

and to find the ground state we can move to the Hamiltonian:

$$\mathscr{H} = \pi(x)\phi(x) - \mathscr{L} . \tag{1.202}$$

Recover a few minutes.

The ground state is not symmetric under the whole symmetry of the Lagrangian.

There is a critical parameter: if μ^2 is positive we are in the symmetric phase of the system, if instead μ^2 is negative we transition to a new phase which is not symmetric. The system still has the symmetry, but the ground state does not.

Why do we only have terms in ϕ^2 and ϕ^4? we basically constructed the simplest potential which has the properties we want.

If $\mu^2 > 0$ we can perturb around the state without the quartic term, which is described by the KG equation.

Recover a few minutes

Goldstone theorem and VEV different from zero. We have two problems which solve each other.

If we explicitly break a $U(1)$ global symmetry, we have a Goldstone boson. If we break a $U(1)$ local symmetry, instead, we get a massless vector boson and a massless scalar.

The wonderful thing is that the two dof of the massless vector ad the single degree of freedom of the massless scalar couple to give a massive vector with three degrees of freedom. This spontaneous breaking of local gauge symmetry is the **Higgs mechanism**.

It is a "transmutation" of degrees of freedom.

Wednesday 2020-4-22, compiled 2020-06-25

Last time we considered the spontaneous breaking of a *global* symmetry: we saw the appearance, corresponding to the breaking of the $U(1)$ symmetry, of massless Goldstone bosons. Each broken generator has a corresponding Goldstone boson.

We had a quadratic term in the Lagrangian, and a quartic term.

What would $\mu^2 < 0$ physically mean? It would be a tachyonic particle. The Vacuum Expectation Value goes from 0 to v: in this case we have broken the $U(1)$ symmetry.

The imaginary part of the field ϕ corresponds to a massless scalar field.

Every time we have a certain global symmetry described by a group G with generators t^a, which is broken to a subgroup G' (which can also be just the identity) with generators t^i, we can identify the broken generators with Goldstone bosons.

We have the Dirac equation $\left(i\partial\!\!\!/ - m\right)\psi = 0$, where ψ has a global $U(1)$ symmetry. This can*not* be generalized to a local $U(1)$ symmetry, unless we introduce a compensating gauge field.

Let us consider the Lagrangian from yesterday for the scalar field with a kinetic, quadratic and quartic term. Can we do $\partial_\mu \to \mathrm{D}_\mu$ as we did with QED, to generalize the $U(1)$ symmetry of ϕ?

We can do it and encounter no issues.

What if $\mu^2 < 0$? The result is surprising: we do as before, perturbing around the vacuum $\phi = v$, writing $\phi = v + \sigma(x) + i\eta(x)$. This is around pages 92–93 of the notes.

We start from ϕ, which has 2 dof since it is a complex scalar, and A_μ, which also has 2 dof since it is a massless vector.

We would expect to get the fields σ and η: however, the field η does not appear, it is "eaten up" by the vector field A_μ. We only find a massive real scalar $\sigma(x)$, which has 1 dof, and a massive vector boson A_μ, which has 3 dof.

This is the *transmutation*, the scalar degree of freedom is absorbed to a degree of freedom in the vector field.

We can repeat this in the general case, with the group G being broken to G'. Suppose, for clarity, that this is $SU(2)$ being broken to $U(1)$.

We are curing two different problems: we have found massive gauge bosons, and removed the unphysical Goldstone bosons.

As $SU(2)$ is broken to $U(1)$, the 3 generators A^a_μ, with $a = 1,2,3$ are broken to give two massive vector bosons ($\mathrm{Z}^\pm$) and one massless vector boson (the photon), while the real field σ is the Higgs field.

We are missing the Z boson: this is just an example to clarify what this mechanism looks like, it is not what we actually will use: we will have to choose another symmetry group.

This will be a way to unify electromagnetic and weak interactions, and we will get to use a single coupling constant for our new electroweak theory.

The correct symmetry group for the electroweak theory is

$$SU(2)_L \otimes U(1)_{\text{hypercharge}}\,, \tag{1.203}$$

where the hypercharge is usually denoted as Y.

We have a doublet under $SU(2)$, we will see that one component of this doublet has charge $+$ while the other has charge 0.

We have three vector fields for $SU(2)$, which we call A^i_μ, and also a vector field for $U(1)$: B_μ.

The VEV of our field will be

$$\left\langle \phi \right\rangle = \begin{bmatrix} 0 \\ v \end{bmatrix}. \tag{1.204a}$$

The residual gauge symmetry can be identified with $U(1)_{\text{em}}$. Do note that $U(1)$-hypercharge *is* broken. However, a certain combination of the generators gives us the $U(1)$ electromagnetic. The combination is

$$T^3 + Y\,, \tag{1.205}$$

where T^3 is the third generator of $SU(2)$, while Y is the generator of hypercharge. A combination which is broken is

$$Q_z = T^3 - \sin^2(\theta_w) Q \,, \tag{1.206}$$

where θ_w is the Weak, or Wonder angle, defined by

$$\tan(\theta_w) = \frac{g'}{g} \,, \tag{1.207}$$

where g' and g are the coupling constants relative to $U(1)$ and $SU(2)$ respectively.

The model is called the Glashow-Weinberg-Salam model. This is the Standard Model of the electroweak interaction.

How is this unification since we have two coupling constants? The constants are not the coupling constants of weak and electromagnetic interaction. The photon is given by

$$A_\mu = \sin(\theta_w) A^3_\mu + \cos(\theta_w) B_\mu \,, \tag{1.208}$$

while the orthogonal combination is

$$Z^0_\mu = \cos(\theta_w) A^3_\mu - \sin(\theta_w) B_\mu \,, \tag{1.209}$$

so we cannot decouple the weak and electromagnetic bosons. This is why the symmetry is called the electroweak symmetry.

When the symmetry is broken, we are left with a long-range interaction and a short range one.

If μ^2 is a function of T, then we can get a phase transition. This is called the electroweak phase transition.

Standard Model Lagrangian:

$$\boxed{\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \overline{\psi} \not{D} \psi + \psi_i y_{ij} \psi_j \phi + \text{h. c.} + \left| \mathrm{D}_\mu \phi \right|^2 - V(\phi) \,.} \tag{1.210}$$

We have not seen the y_{ij} bit yet, it is responsible for the fermion masses.

Let us discuss the Standard Model of Particle Physics.

Our symmetry is

Tuesday 2020-4-28, compiled 2020-06-25

$$\text{Lorentz} \otimes SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \,. \tag{1.211}$$

Starting from the symmetry, the only other prescription needed are the quantum numbers of the various particles: we can make a table.

	Name	T_3 (SU(2))	Y	Q
Leptons (no $SU(3)$ charge)	ν_L	$+1/2$	$-1/2$	0
	e^-	$-1/2$	$-1/2$	-1
	$\cancel{\nu_R}$	0	0	0
	e_R^-	0	-1	-1
Higgs scalar	ϕ^+	$+1/2$	$+1/2$	$+1$
	ϕ^0	$-1/2$	$+1/2$	0
Quarks ($SU(3)_c$ triplets)	u_L	$+1/2$	$1/6$	$2/3$
	d_L	$-1/2$	$1/6$	$-1/3$
	u_R	0	$2/3$	$2/3$
	d_R	0	$-1/3$	$-1/3$

Figure 1.1: Particles

Here we write only one of the generations for both leptons and quarks. The quantum numbers for the other generations are the same. An unanswered question is *why* exactly there are three families.

We are always computing the electric charge as $Q = T_3 + Y$. Sometimes we write "1" for the right-handed quarks under T_3, this is not the eigenvalue but it is used to mean "singlet".

For the $SU(3)$ we have 8 generators, for $SU(2)$ we have three generators, for $U(1)$ we only have 1 generator.

The assignment of the charges cannot come from the symmetry, it must be determined experimentally.

The right-handed neutrino does not appear here: it should be invariant under any gauge transformation. This would then be the *sterile* neutrino: it would not interact with matter by any of the forces of the Standard Model.

It makes sense not to include it *a priori* then, however we can keep it in mind: it is a DM candidate.

Can we say anything about its possible mass? There are schemes in which the RH neutrino enters, and in which it can be assigned a mass. However, in general this is a free parameter: in some contexts we can give it a specific value, but we cannot say anything about its mass from basic Standard Model theory.

There are experiments at Fermilab which could point indirectly at their existence.

The rule is always:

> *Write all the terms in your Lagrangian which are invariant under your symmetry and which have dimension* 4.

Should we not include also some prescription for the couplings? No, this is included in the symmetry: if we have the coupling constants g_S, g (weak) and g' (hypercharge) we can determine everything else.

The issue is that, before including the Higgs boson, everything is massless.

Gluons and quarks have the same "destiny": there is confinement when g_S becomes large.

The $SU(2) \times U(1)_Y$ symmetry is spontaneously broken to $U(1)_{\text{em}}$. If we had 4 massless DoF before, they must still be there but they have become massive.

These three particles would have become Goldstone bosons, but instead they become the longitudinal component of the three vector bosons.

The charged Higgs component ϕ^+ gives mass to the $W^\pm$ vector bosons of the electroweak interaction, the ϕ^0 gives mass to the Z^0 vector boson.

So, we know how to write the kinetic terms $F^{\mu\nu} F_{\mu\nu}$, and the mass terms $\overline{\psi} \not{D} \psi$.

Now we move to the terms coming from the spontaneous breaking of the symmetry.

We have the massless G^a and A^μ, and the massive $W^\pm$ and Z^0. These are the terms we've written so far: where do the $\overline{\psi} m \psi$ terms come from?

The kinetic term has dimension 2^2, which is fine, the wavefunction has dimension $3/2$ while the derivative has dimension 1. So, also a term $\overline{\psi} m \psi$ would be fine, dimension-wise: but is it symmetric? we can write it as $\overline{\psi}_L \psi_R$. Let us consider electrons for example.

But e_L is in an $SU(2)$ doublet, while the e_R is an $SU(2)$ singlet. So, the object is not a singlet. This is not invariant under $SU(2)_L$.

So, does this mean that fermions are massless? Before introducing ϕ, in the pure Yang-Mills theory, they indeed are.

There is only one kind of field which is invariant under spacetime transformations: a scalar. So, we could introduce a term like

$$\overline{\psi}^i_L \phi_i \psi_R \,, \tag{1.212}$$

where i is an index going from 1 to 2, an $SU(2)$ index. This is not a mass term, but an interaction term between the Higgs field and the electron-left and electron-right.

Then, the way the mechanism works is by the fact that the VEV of ϕ is $v \neq 0$, so at low energies we will see an effective mass term. The constant we put before this will look like

$$y^{ij} \overline{\psi}^i_L \phi \psi^j_R \,, \tag{1.213}$$

where now the indices i and j run over the possible fermions. We will then have

$$M_\ell = y_\ell v \,, \tag{1.214}$$

and the matrix will be $y^{ij} = \delta^{ij} y^j$.

Similarly we will have mass terms for the leptons, up and down quarks. These are free parameters of the theory, and they must be different. The VEV of the Higgs field is fixed by the masses of the W and Z bosons: so it is of the order $100\,\text{GeV}$.

But the mass of the electron is $m_e \sim 511\,\text{keV}$: so, we must have $y_e \sim 10^{-5}$.

The price we pay for this is the fact that the number of free parameters increases. This is a "dirty part" of the theory.

Why are these parameters so different? This is the *flavour problem*.

We could have a term which looks like

$$\overline{u}_L \gamma^\mu d_L W_\mu \,. \tag{1.215}$$

Thi is not the most general thing: we could write

$$\overline{u}_L U_L^{\dagger,(u)} \gamma^\mu U_L^{(d)} d_L W_\mu \,. \tag{1.216}$$

The product $U^\dagger U$ is called V_{CKM}, the Cabibbo matrix, which gives us CP violation in the Standard model.

It does seem, though, that the amount of CP violation in the Standard Model is too small to satisfy the Sacharov conditions.

There are many parameters: the couplings, the mass terms inside y^{ij}, the terms μ and λ inside of $V(\phi)$.

1.4.3 Neutrinos

Wednesday 2020-4-29, compiled 2020-06-25

The Yukawa Lagrangian is written as

$$\mathscr{L}_{\text{Yukawa}} = -y_e L_a^\dagger \phi_a e_R - y_d O_a^\dagger \phi_a d_r - y_a O_a^\dagger \epsilon_{ab} \phi_b^\dagger u_R + \text{h.c.}\,. \tag{1.217}$$

Commonly this is written as $\overline{\psi}_L \phi \psi_R$, this however does not mean $\overline{\psi} = \psi^\dagger \gamma^0$, since it is applied to a 2-component spinor.

The index a is an SU(2) index, and the Higgs field looks like

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} ; \tag{1.218a}$$

we have two terms like

$$\begin{bmatrix} \nu_L & e_L^- \end{bmatrix} \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} e_R \,, \tag{1.219a}$$

and also

$$\begin{bmatrix} \nu_R^c & e_R^+ \end{bmatrix} \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} e_R^- \,. \tag{1.220a}$$

This means that the neutrino is massless: the VEV of ϕ_a is 0 for ϕ^+, and v for ϕ^0. So, we get Dirac mass terms for the electron but not for the neutrino.

Ettore Majorana proposed a way to give a mass for the neutrino (or any neutral fermion, really).

The symmetries of the SM are, generally, Lorentz $\otimes$ Gauge. We then write all the terms we can which are invariant under these.

From these we can derive conservation laws, such as that for the Baryon number. We define

$$B(\text{Quark}) = \frac{1}{3} \qquad \text{and} \qquad B(\text{antiQuark}) = -\frac{1}{3}\,, \tag{1.221}$$

while for any other particle we assign 0. This is conserved, but we need not impose it: the most general Lagrangian we write with our symmetries has it. This is a global symmetry: $U(1)_B$.

The way to find it is to construct a baryonic current j_B^μ, in such a way that it is conserved in our Lagrangian, then the integral of j_B^0 will give the conserved charge.

This has heavy consequences: the lightest baryon is the proton. If it were to decay, this would not conserve baryon number.

If we are allowed to violate baryon number conservation we can have a decay like $p \to e^+ + \gamma$, which would occur and destabilize atoms.

Recover a couple minutes

We get a bound on the lifetime of the proton of

$$\tau_{\text{proton}} > 10^{32}\,\text{yr}\,. \tag{1.222}$$

We also have lepton number conservation. This is more interesting since we have neutrinos, which are hard to detect.

We will see a method to give mass to neutrinos, which will however violate lepton number conservation.

Parity and charge are massively violated.

CP symmetry moves us from e_L^- to e_L^+ to $(e^+)_R$.

If we have CP symmetry, then if in our model we have a ν_L we will also need to have a ν_R^c. That is, there are only two degrees of freedom between them.

In the SM we introduce only left-handed neutrinos. However, we can also introduce

$$\nu_R \iff (\nu^c)_L\,. \tag{1.223}$$

The beta decay is always

$$n \to p + e^- + \overline{\nu}_e\,, \tag{1.224}$$

and we look at the spectrum of the energies of the electron.

If the neutrinos are massless the curve gets to a certain point, if they are massive the curve stops a bit earlier. People have done this to a very high degree of precision.

What we have found up to now is $m_{\nu_e} < 2\,\text{eV}$ in these beta decay experiments. The result that the neutrinos are massive did not come from here.

From cosmic ray interactions we have charged pions, which decay into

$$\pi^+ \to \mu^+ + \nu_\mu\,. \tag{1.225}$$

Then, the muon decays into

$$\mu^+ \to e^+ + \nu_e + \overline{\nu}_\mu\,. \tag{1.226}$$

So, we expect twice as many ν_μ as ν_e.

We can build detectors in order to see these. We expect a muonic neutrino and antineutrino.

The state ν_μ is a charge eigenstate, not a mass eigenstate: in general it will be written as $a\nu_1 + bn_2 + c\nu_3$.

Its time-evolution will then exhibit an oscillation in the probability to still find a ν_μ, since the mass components oscillate at different frequencies.

The experiment OPERA found that some of the neutrinos from CERN going to Gran Sasso were converted to ν_τ and such.

Also, Kamiokande found a ratio different from 2 of ν_μ to ν_e.

So, if there is an oscillation it means that the neutrinos must indeed be massive, or at least some of them must be.

We put a Dirac mass term for the ν_R: the mass will look like

$$m_\nu = y_\nu v \,. \tag{1.227}$$

But we know that the masses of the neutrinos are very small, from beta decay but also from structure formation in the early universe.

So, since $v \sim 100\,\text{GeV}$ and $m_\nu \lesssim 1\,\text{eV}$ we must have $y_\nu \sim 10^{-11}$.

The Dirac mass term is 2+2 degrees of freedom.

If L is a lepton doublet, we can write a term $L\phi$.

However, L has an index: but we can consider a term

$$L^i \phi L^j \phi \epsilon_{ij} \,. \tag{1.228}$$

We can do this for the neutrinos but not for the other particles since they are uncharged.

So the term looks like

$$\nu_L \phi^0 \nu_L \phi^0 \,. \tag{1.229}$$

When we compute the VEV we get a mass for the neutrino.

This is for the left-handed neutrino. The dimension of this term is 5, since the fermions have dimension 3/2, while the Higgs has dimension 1: we should put a mass term in the denominator. We will then have a term which looks like

$$y_\nu^2 \frac{LL\phi^2}{M} \,, \tag{1.230}$$

so that in the end $m_\nu = y_\nu^2 v^2 / M$.

This M must be very large in order to account for the small neutrino mass.

This is a sort of seesaw model: the mass matrix looks something like

$$\begin{bmatrix} 0 & m_{\text{Dirac}} \\ m_{\text{Dirac}} & M \end{bmatrix} . \tag{1.231a}$$

The thing is: ν_R has no gauge numbers. The mass of the right-handed neutrino is not "protected" by the gauge symmetry, it can be as large as it likes.

It is important to find out whether the mass of the neutrino is due to a Majorana or Dirac mass term.

The experiments which can determine this are called *neutrinoless double beta decay*.

If this were found, it would mean that neutrinos have a Majorana mass.

This concludes the discussion of the particle physics standard model. Next week we are going to start looking at the standard model of cosmology.

Chapter 2

Early Universe

2.1 Basics of cosmology

We establish some basic notions in cosmology. We use natural units $\hbar = c = k_B = 1$, so that we have the equivalence of mass, energy, temperature and angular frequency. Tuesday 2020-5-5, compiled 2020-06-25

The Planck units are defined by also setting $G = 1$. The classic textbook is Kolb-Turner [KT94]. For something more recent, we have Gorbunov and Rubakov [GR11].

The FLRW metric is given by

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}\, dx^i\, dx^j \,, \tag{2.1}$$

where γ_{ij} is the metric of a unit 3-sphere, 3-hyperboloid or 3-plane.

The Hubble parameter is given by $H(t) = \dot{a}(t)/a(t)$. The Einstein equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \,. \tag{2.2}$$

The pressure is assumed to follow the law $P = w\rho$, and fluids are characterized by their w. The stress-energy tensor is $T_{\mu\nu} = \operatorname{diag}\big(()\rho, -P, -P, -P\big)$.

The Friedmann equations read

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.3a}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{2.3b}$$

$$P = P(\rho)\,. \tag{2.3c}$$

The dynamical $\ddot{a}$ Friedmann equation is a consequence of the first two.

If we have nonrelativistic matter, which is commonly called "dust", with $w = 0$, we get $\rho \propto a^{-3}$ and $a(t) \sim t^{2/3}$. So, the energy density scales like $\rho(t) \sim t^{-2}$.

The Hubble parameter, on the other hand, is given by $H(t) = 2/(3t)$. This is not actually verified, since the assumption we are making of the universe being filled with matter does not hold.

For relativistic matter, we have $P = \rho/3$, so the density scales like $\rho \sim a^{-4}$, $a(t) \sim t^{1/2}$, $H = 1/(2t)$ and $\rho = \frac{3}{32\pi G t^2}$.

The density scales like

$$\rho = \frac{\pi^2}{30} g_* T^4 , \tag{2.4}$$

and the Hubble parameter scales like

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8}{90}\pi^3 g_* . \tag{2.5}$$

Let us consider the vacuum: suppose we had a relation like

$$T_{\mu\nu} = \rho_{\text{vac}}\eta_{\mu\nu} , \tag{2.6}$$

so that $P = -\rho_{\text{vac}}$.

This is equivalent to the introduction of a $-8\pi G\Lambda g_{\mu\nu}$ term to the left-hand side of the field equations.

2.1.1 The ΛCDM model

The ingredients are:

1. Nonrelativistic matter: baryons, dark matter, and also neutrinos as long as $m_\nu \gtrsim -3\,\text{eV}$.
2. Relativity.
3. Dark energy: is it vacuum energy?

The Hubble parameter is given by

$$H^2 = \frac{8\pi G}{3}\left(\rho_M + \rho_{\text{rad}} + \rho_\Lambda + \rho_{\text{curv}}\right) . \tag{2.7}$$

The critical density is given by

$$\rho_c = \frac{3}{8\pi G} H_0^2 \approx 5 \times 10^{-6}\,\text{GeV}/\text{cm}^3 . \tag{2.8}$$

This is obtained by taking $h = 0.7$.

We can take the ratios of $\rho_i/\rho_c = \Omega_i$), this quantifies how much of the density of the universe is in the form of a certain kind of fluid.

What is the vacuum energy predicting by the Standard Model? We know that there is symmetry breaking.

What remains when we take the VEV are the scalar terms, since the vector fields are zero in the vacuum. Specifically, we are left with

$$\mu^2\phi^2 + \lambda\phi^4 . \tag{2.9}$$

The order of magnitude of the VEV of the Higgs is around 100 GeV. The dimensions of $\rho = E/L^3$ are those of an energy to the fourth power. So, our estimate will be $\rho \sim (100\,\text{GeV})^4$.

The estimate from cosmology is $10^{-10}\,\text{eV}^4$; the one from particle physics is 120 orders of magnitude larger.

Wednesday 2020-5-6, compiled 2020-06-25

Today we will discuss the **thermodynamics of the early universe**.

We start off by discussing particles in thermal equilibrium, but the interesting thing is the transition between this equilibrium and non-equilibrium.

We use the usual approximation of a dilute gas, we take a Boltzmann distribution function $f(p)$, from which we can compute n, ρ and P.

We can make some useful approximations in the relativistic limit $T \gg m$ and in the nonrelativistic limit $T \ll m$.

Add reference to Fundamentals notes, chapter 3

A very important parameter is g_*: it is the *effective* number of degrees of freedom: it is computed as

$$g_* = \sum_{i \in \text{bosons}} g_i \left(\frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i \in \text{fermions}} g_i \left(\frac{T_i}{T} \right)^4 , \tag{2.10}$$

since we need to weigh the contributions to the degrees of freedom according to whether the particles are in equilibrium or not, and when the particle decoupled.

Let us compute g_* in a couple of examples. First, let us consider $T \ll \text{MeV}$.

So, in the standard model we only have photons and the three neutrino species. So, we get

$$g_*(T \ll \text{MeV}) = 2 + \frac{7}{8} \times 3 \times 2 , \tag{2.11}$$

and we must ask: are these particles still coupled at the time we are considering? The temperature of the neutrinos is given by

$$T_\nu = \sqrt[3]{\frac{4}{11}} T_\gamma , \tag{2.12}$$

so in the end we get $g_* \approx 3.37$.

If we consider a temperature around $1\,\text{MeV} < T < 100\,\text{MeV}$, we also will have to account for (?).

In the era of radiation domination, we get

$$a(t) \propto \sqrt{t} , \tag{2.13}$$

and the Hubble parameter scales like

$$H \approx 166 \sqrt{g_*} \frac{T^2}{M_{\text{Pl}}} = \frac{T^2}{M_{\text{Pl}}} . \tag{2.14}$$

So, we get a time of the order of

$$t \approx \left(\frac{T}{\text{MeV}}\right)^2 \text{s}\,. \tag{2.15}$$

How do we quantitatively describe coupling? We take the rate Γ_i^ν from our particle to another, considering all possible processes.

Let us suppose that $T > M_W$, so roughly $T > 100\,\text{GeV}$.

The rate is given by

$$\Gamma_\nu = \sigma n v\,, \tag{2.16}$$

and the physics is really given by the cross section σ: the other factors are always roughly the same, $n \sim T^3$ and $v \sim 1$.

We have the coupling constant

$$\frac{e^2}{4\pi} = \alpha_{\text{em}}\,, \tag{2.17}$$

and the charge is $e = g\sin(\theta_w)$, where we know experimentally that $\sin^2\theta_w \approx 0.24$.

So, we know that the scaling is

$$\sigma \sim \left(\frac{g_w^2}{4\pi}\right)^2 \sim \left(\alpha_w^2\right), \tag{2.18}$$

and on the denominator we need a mass square: but the only energy scale we know of here is T, so we finally get $\sigma \sim \alpha_w^2/T^2$. So, in the end we find

$$\sigma \sim \frac{\alpha_w^2}{T^2} T^3 \times 1 \sim \alpha_w^2 T\,. \tag{2.19}$$

The result we get is: $T > H$ when

$$T < \frac{\alpha_w^2 M_{\text{Pl}}}{\sqrt{g_*}}\,. \tag{2.20}$$

In another case (which?) we have $\sigma \sim T^2$, so $\Gamma \sim T^5$.

We must compare to the expansion of the universe.

The end result is $\Gamma_\nu > H$ as long as $T > 1\,\text{MeV}$, roughly. So, for larger temperatures the neutrinos are coupled, for smaller temperatures they are decoupled. Below 1 MeV they completely decouple.

The "relic neutrinos" are colder than the relic photons. Moreover, their number density is smaller. The entropy density is given by

$$s = \frac{\rho + P}{T} = \frac{4}{3}\frac{\rho}{T}\,. \tag{2.21}$$

The temperature of neutrinos today is around $T_\nu^0 \approx 1.96\,\text{K}$.

Next week we will consider nucleosynthesis.

We should be able to finish by Wednesday June 3rd.

Tuesday 2020-5-12, compiled 2020-06-25

We keep exploring Standard Model Cosmology.

2.2 Big Bang Nucleosynthesis

Around a few hundred MeV to the GeV we have a phase transition.

Now we move to the infrared regime, where we have the so-called slavery: we go to the range from 1 MeV to few tens of keV. Neutrons and protons are the new protagonists. The first process is

$$\mathrm{n} + \mathrm{p} \to \mathrm{D} + \gamma\,, \tag{2.22}$$

the formation of deuterium. We are interested in the ratio of abundances of deuterium as opposed to hydrogen.

At a sufficiently high temperature, higher than the binding energy of the deuterium, the inverse process can occur.

So, we need to reach the moment at which the photodissociation does not happen anymore.

Then, more processes can occur, like

$$\mathrm{D} + \mathrm{p} \to {}^3\mathrm{He} + \gamma\,. \tag{2.23}$$

Similarly we can form tritium (^{3}H) from deuterium and a proton, and regular Helium (^{4}He) from two deuterii — this is the most stable of these nuclei.

The nucleosynthesis process heavily depends on the characteristics of the two standard models.

An interesting part of nuclear astrophysics is the computation of these numbers and cross-sections.

A tricky thing is also to actually measure the abundances of these elements, or more precisely to distinguish what is produced by primary nucleosynthesis and what instead is produces by secondary nucleosynthesis.

We will not go into the details, but instead we will comment on what parts of our models enter in the computations.

To start, let us say we have a temperature above the MeV. Weak interactions decouple around this scale: the timescale for a weak interaction to occur is longer than the lifetime of the universe at that point. We saw last time that $H \sim T^2/M_P$.

The distribution functions of the particles go like $e^{-m/T}$, so the ratio of

$$\frac{n_n}{n_p} = \frac{n}{p} \approx e^{-\Delta m/T}\,. \tag{2.24}$$

Here, it becomes relevant that $m_n - m_p \approx 13\,\mathrm{MeV} \neq 0$. This ratio of number densities applies as long as the neutrons and protons are in equilibrium and nonrelativistic.

At a certain point, the temperature gets as low as $T = T_D$, the decoupling temperature.

The proton, as far as we can tell, is stable; on the on the other hand the neutron is unstable: it can beta-decay into a proton.

At 1 MeV, which is around the moment of the freeze-out, the ratio n/p is around $1/6$.

What is the temperature at which the deuterium can stay bound? it will depend on the ratio n_B/n_γ, and also on the ratio E_γ/E_D. The binding energy of deuterium is around $E_D \approx 2.2\,\mathrm{MeV}$. To get stable deuterium we must wait until

$$\frac{n_\gamma}{n_B} e^{-\frac{2.2\,\mathrm{MeV}}{T}} < 1\,. \tag{2.25}$$

We actually have to wait quite a long time, since $n_\gamma \gg n_B$. This is since baryons are quite massive. The ratio comes out to be around one billionth: $n_\gamma/n_B \sim 10^9$. So, we need a strong drop in temperature: we find that the necessary temperature is around $0.1\,\mathrm{MeV}$.

Let us try to focus on what is of interest for us.

We must compare a weak interaction rate Γ_W with the expansion rate of the universe H. The first is computed according to the SM of particle physics, and crucially depends on the coupling strength.

The lifetime of a neutron is

$$\tau_n = (10.5 \pm 0.2)\,\mathrm{min}\,. \tag{2.26}$$

The coupling at the vertex:

$$G_F = \frac{\alpha_W^2}{M_W^4}\,, \tag{2.27}$$

where $M_W \approx 100\,\mathrm{GeV}$, while the Hubble parameter scales like

$$H \sim \frac{T^2}{M_P^4}\,. \tag{2.28}$$

We also need the coupling g_*, which can also be derived experimentally, and n_γ/n_B: the latter can be used as an output parameter. This will then tell us about Ω_B, the matter fraction, which is hard to measure in the modern universe.

Why do we directly measure the neutron lifetime? This is a hard computation since we both have the strong dynamics in the nucleons and the weak dynamics moving quarks between different flavours.

The number of degrees of freedom g_* is interesting: it could be calculated from the Standard Model, but we know that some new degrees of freedom must exist since no SNddM particle could be Dark Matter.

Finally we got $n/p \sim 1/7$ for the $0.1\,\mathrm{MeV}$ temperature; with this information we can compute

$$\frac{N_{\mathrm{He}}}{N_{\mathrm{H}}} = \frac{\frac{1}{2}n}{p-n}\,, \tag{2.29}$$

since the number of hydrogen nuclei is the same as the number of unpaired protons. We get

$$Y_4 = \frac{M_{\mathrm{He}}}{M_{\mathrm{H}} + M_{^4\mathrm{He}}} = \frac{4N_{\mathrm{He}}}{N_{\mathrm{H}} + 4N_{^4\mathrm{He}}}\,, \tag{2.30}$$

which can be expressed in terms of n/p.

We can plot, in terms of $\eta_B = n_\gamma / n_B$, the fraction of mass of ^{4}He to H, as well as that of deuterium and Lithium.

We finally get

$$\frac{n_B}{n_\gamma} \sim 6 \times 10^{-10}\,, \tag{2.31}$$

so we must have $\Omega_B \sim 4 \div 5\,\%$.

Based on the Helium-4 abundance, we can put a bound like $\left|\Delta N_{\text{eff}}^{\nu}\right| < 1/2$: our candidate must contribute as half a neutrino species.

So, when we propose a DM candidate, we must always ask whether it spoils nucleosynthesis.

This must be weighted by their temperature and whether these species are at equilibrium.

There are models in which we introduce a new particle which must decay into some other particle plus a photon. This is dangerous: how energetic are the photons which are produces? We must ensure that they do not destroy deuterium.

Nucleosynthesis is the furthest event we can describe. See "The first three minutes" by Weinberg.

Tomorrow we will discuss Dark Matter.

2.3 Dark Matter

Wednesday 2020-5-13, compiled 2020-06-25

Yesterday we described the formation of the first nuclei. The amount of baryons in terms of energy density is of the order of 4 to 5 %.

We know that the total value of $\Omega_{\text{tot}} \sim 1$. The density fraction due to dark energy is around 70 %, while $\Omega_{\text{matter}} \sim 30\,\%$.

So, most (something like 80 %) of the matter is not made of baryons.

We are then looking for non-baryonic matter with $\Omega \sim 25\,\%$.

There were interesting studies about how the lensing of light is affected by Dark Matter. All the evidence is based on our knowledge of gravitational interactions. Are we sure that Newton's law of gravitation is right on such large scales? There are MOND theories.

However, the hints about the existence of non-baryonic DM come from very different directions. We then take the standard view: Dark Matter exists.

Could the Higgs boson be the source of Dark Matter? No. It decays too fast.

We have similar problems for the Z boson. So, in the end we must have some other neutral particle.

So, neutrinos? Maybe. They are light, of course, but there are many.

We get

$$\Omega_\nu = \frac{\rho_\nu^0}{\rho_{\text{crit}}}\,, \tag{2.32}$$

where $n_\nu^0 \approx 339.5\,\mathrm{cm}^{-3}$. So, this comes out to be

$$\Omega_\nu = \frac{\sum_i m_{\nu_i}}{h^2 93.14\,\mathrm{eV}}\,. \tag{2.33}$$

Recall that "electron neutrino" is a current eigenstate, not a mass eigenstate. The bound we have is of the order $m_{\nu_e} \lesssim 1\,\mathrm{eV}$.

We also have bounds for the square of the mass differences. In the end, this means that the term $\sum m_\nu$ can be at most a few eV.

In spite of the fact that they are very numerous, neutrinos have too small a mass.

Dark Matter is also crucial in the formation of Large Scale Structure. It must "clump" in order to do so. This means that it must be nonrelativistic. But neutrinos are relativistic today!

We must ask that the mass of Dark Matter, M_X, be larger than the temperature of decoupling. This is then called "Cold Dark Matter".

Relativistic DM kills all of the density fluctuations.

If neutrinos were the source of DM, the first structures would form at a very large scale.

There are simulations used in order to compare these models.

What does the Dark Matter density profile look like in a galaxy? CDM would have a "cusp" in density at the center, a really heavy region.

Could we also have "Warm" Dark Matter? An intermediate situation between HDM and CDM.

Dark Matter around 1 keV in mass would be a candidate for this.

Let us consider **Thermal Dark Matter**: it has been in equilibrium with the plasma for some time in the early universe.

This means that at that time we can know the number density of those particles. Consider a heavy particle X, with mass larger than, say, 10 GeV, and which is stable or quasi-stable (that is, such that its lifetime is long compared to the age of the universe).

We have different kinds of equilibrium: kinematic and chemical equilibrium. If we have a scattering like

$$\mathrm{X} + \mathrm{A} \to \mathrm{X} + \mathrm{B}\,, \tag{2.34}$$

this does not change the number of particles X. It only distributes the momenta.

If the annihilation rate of X is lower than the expansion of the universe, it stays as it is. Then, we have the Boltzmann suppression factor in the distribution, $\exp(-m_X/T)$.

The final number of X is reduced by a factor $\exp(-m_X/T_{\text{freezeout}})$.

What is the window of the parameters of this particle which could give us the Ω_{DM} we observe? If the particle is a Weakly Interacting Massive Particle, we get the "WIMP miracle": with very simple assumptions, we get Ω between 1 and 10^{-1}.

The fact that the LHC has not seen any new particles at the electroweak scale poses a problem, even though the WIMP possibility looks good.

We can expect that there are more than one kind particle constituting DM. A possibility is the presence of a "Dark Sector" in the SM. It is generally assumed that there is a sort of "portal", which makes the ordinary matter sector allowing the sectors to communicate.

A very simple kind of dark sector would be constituted by right-handed neutrinos: they only interact with the Higgs boson, with terms like $\overline{\nu}_L H^0 \nu_R$.

2.3.1 WIMP candidates for Dark Matter

Tuesday 2020-5-19, compiled 2020-06-25

WIMP means Weakly Interacting Massive Particle, this is a so-called heavy stable relic of the early universe.

So, we require stability; this could also be a very long-lived particle, the only requirement is that it be still abundant now.

The cross section must also be small.

Let us consider a typical WIMP, with mass $M_x \sim 100\,\text{GeV}$.

Let us consider a scenario in which the particle remains in equilibrium even when the temperature drops below the mass of the particle.

We want to request that $T_f < M_X$, where T_f is the temperature of the freezeout.

The value of $\rho_X/\rho_C = \Omega_X$ is of the order $1/4$ today. If we had $n_X \sim n_\gamma$ before the freezeout, because then DM was radiation-like.

Today we have around a baryon every billion photons, and $\Omega_B \sim 5\,\%$. The statistics of the DM distribution are given by

$$(M_X T)^{3/2} \exp\left(-\frac{M_X}{T}\right) , \tag{2.35}$$

so the exponential suppression plays a very important role as the temperature decreases. We cannot over-close the universe, if we get $\Omega_X > 1$ the universe would become strongly closed.

Now, let us compute the freezeout temperature, assuming that the particle's mass is $100\,\text{GeV}$. We want to show that $T_f < M_X$.

What do we mean by "equilibrium" precisely? There are two possible meanings; one is the chemical equilibrium, and the other is the kinetic equilibrium.

The annihilation of X with its antiparticle changes the total number of particles. In discussing the freezeout temperature we are interested in the *chemical* equilibrium. So, we must ask: "what is the temperature at which the annihilation Γ_X is equal to the expansion rate of the universe H?"

The first is given by

$$\Gamma_X = \langle \sigma v \rangle \, n_X \, , \tag{2.36}$$

where

$$n_X = (M_X T)^{3/2} e^{-M_X/T} \, , \tag{2.37}$$

while $\langle \sigma v \rangle$ is model-dependent. Therein lies the physics of the problem. Without a specific model, we take something like

$$\sigma \sim \frac{\alpha_W^2}{M_w^4} \, . \tag{2.38}$$

We plug everything into the expression, using a standard cosmology for H. We can get an order-of-magnitude estimate. If we did the calculation more properly, we would be able to get the Boltzmann equation:

$$\frac{\mathrm{d}n_x}{\mathrm{d}t} + 3Hn_x = - \langle \sigma v \rangle \left(n_x^2 - \right) , \tag{2.39}$$

complete formula

which can be found in the notes. We find

$$\Omega_X h^2 \approx 10^{-10} \left(\frac{\mathrm{GeV}^{-2}}{\langle \sigma v \rangle} \right) \frac{1}{\sqrt{g(T_f)}} L \,, \tag{2.40}$$

where

$$L = \log \left(\frac{g_X M_X}{} \right) \dots \,, \tag{2.41}$$

complete formula

which is generally of the order of 20 to 30.

So in the end the temperature is given by

$$T_f \approx M_X L^{-1} \,. \tag{2.42}$$

We are, however, assuming that the annihilation goes like the weak interaction — is this accurate? If the value α_W were different, we could get the correct number for Ω_X. There are good reasons to think that there might be new physics in the electroweak scale.

How is the estimate for Ω_X derived?

Suppose we asked for $\Omega_X < 0.3$. Then, we need to reduce the number of X, n_X.

2.3.2 Axions

Wednesday 2020-5-20, compiled 2020-06-25

The search for WIMPs can be indirect or direct: direct searches look for DM interacting with a target with a large cross section, indirect searches look for the recoil of particles.

There might be a DM "wind" passing through the Earth. We would then expect a seasonal modulation of this effect, because of the Earth's orbit around the Sun.

Is there a modulation of this kind in the neutrino flux? Yes, at Gran Sasso they have shown this with very high confidence.

We have our SM particles and the "Dark Sector": there might be a "portal" between them, for instance the Higgs boson.

Suppose the Dark Sector had a $U(1)$ symmetry: then, we would have a sort of "dark photon". If this symmetry were spontaneously broken, we could then have a mixing between the two sectors.

A particular light pseudoscalar DM candidate is called the **axion**. Its interest is not only as a DM candidate, but also in other contexts: it is linked to BTSM physics below the electroweak scale.

It was introduced by particle physicists first, and then it was understood that it might be useful as DM.

Weak interactions violate CP symmetry. This is due to an effect related to the electroweak interaction.

In the QCD Lagrangian we have a kinetic term $G^a_{\mu\nu}G^{\mu\nu}_a$; then we can also add a term $G_{\mu\nu}\widetilde{G}^{\mu\nu}$, where $\widetilde{G}$ is the Hodge dual of the field strength. This would produce a CP-violating term. We can write it but people were not worried: this term can be written as a 4-divergence, so it could be removed.

However, it could be shown that due to instantons — nonperturbative quantum effects — there was an anomalous current, the term could not be neglected.

The term can still be written as a divergence, but because of the instanton the asymptotic values of the field are not zero anymore: therefore, the surface integral in the action is not zero anymore.

Then, a term $\theta_0 G\widetilde{G}$ was added. This also affected the phases of quark transitions. Then, we introduce a term

$$\overline{\theta} = \theta_0 + \arg\left(\det M\right), \tag{2.43}$$

where M is the quark mass matrix. This $\overline{\theta}$ will then multiply the $G\widetilde{G}$. This is a free parameter of the theory: the quark mass matrix depends on the Yukawa couplings, and θ_0 is completely free. This should be $\overline{\theta} < 10^{-9}$ to comply with experiment: it looks like fine-tuning!

There is no physical reason why the parameter should be small. This is weird: there must be some hidden symmetry.

We can introduce a $U(1)$ global symmetry, called the Pacci-Quinn symmetry. This is discussed by Rubakov.

Since the field H feels the symmetry, in terms like $\overline{Q}_L H d_R$. The spontaneous breaking of this symmetry yields a goldstone boson called the axion.

The mass of this pseudo-Goldstone boson depends on the scale at which this symmetry is broken, Q.

We can introduce additional scalars, which are singlets or doublets under SU(2), whose VEV is much larger than the electroweak scale.

If the photons inside a star could convert into axions, they could speed up the cooling down of a star: our stellar evolution observations then allow us to give a bound

$$f_{PQ} > 10^9\,\text{GeV}\,. \tag{2.44}$$

What happens to the axions after they are produced? The axion is light, since $m_a \sim 1/f_Q$. The mass is small, like $< 10^{-5}\,\text{eV}$. They oscillate, moving around the minimum in their potential. Most off the energy is contained in this oscillation, the mass contributes very little.

This is then a type of DM which is Cold, but also very light! In order to not over-close the universe we must also have $f_{PQ} < 10^{12}\,\text{GeV}$.

How would we detect axions? They interact with photons, so we can have an experiment in which we have photon conversion into an axion.

We have a spectacular effect which is called "light shining through a wall": we send a photon towards a wall, it will not pass through usually. Suppose that we put a source of a strong magnetic field: if there are axions, we could have photon-photon conversion into an axion, which can pass through the wall and then emerges as an axion. We then put another strong source of magnetic field which can deconvert the axion into a photon.

So far, this has not provided any evidence for the existence of the axion.

2.4 Matter-antimatter asymmetry

Wednesday 2020-6-3, compiled 2020-06-25

Sacharov conditions.

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