Compact Object Astrophysics

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Contents

Tuesday 2020-9-29, compiled 2020-10-06

Introduction

Tuesdays and Wednesday at 14.30 PM in room P1A, Paolotti building. 22 people.

This course overlaps with "Computational Astrophysics" by professor Mapelli.

The examination is an oral one, done either online or live.

We start with a brief overview of the final fates of massive stars. We have white dwarfs, neutron stars and black holes under the category of "compact objects", but white dwarfs are not really that compact.

We then discuss accretion onto compact objects, and neutron stars. An open question: what is the EOS of ultradense neutron matter?

"Accretion power in astrophysics", "The physics of Compact Objects", "Astrofisica Relativistica I & II", "Astrofisica delle Alte Energie".

0.1 A journey into the life of a massive star

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Stars whose mass is $M \gtrsim 8 M_\odot$ go supernova at the end of their life. During their lifetime, hydrogen fuses through two channels: the p-p chain and the CNO cycle.

In the CNO cycle, four protons turn into a ^{4}He nuclide, two positrons, two electron neutrinos using heavier nuclides as catalysts.

The critical temperature above which the CNO cycle dominates is around $T_c \sim 2 \times 10^7\,\mathrm{K}$. For the Sun, less than $8\,\%$ of energy production is through the CNO cycle.

When the temperature of the core reaches a value around $1 \div 2 \times 10^8\,\mathrm{K}$ and the density is around $10^8 \div 10^9\,\mathrm{g/cm^3}$, helium starts to burn in the 3α process, becoming ^{12}C. The Q-value here is around $7.27\,\mathrm{MeV}$.

As soon as we have carbon, this can fuse with an α particle giving rise to a nucleus of oxygen with $Q \approx 7.16\,\mathrm{MeV}$. This oxygen can further catch an α particle, making a ^{20}Ne nuclide.

Then, we have a temperature around $5 \times 10^8\,\mathrm{K}$ and a density around $3 \times 10^6\,\mathrm{g/cm^3}$. Carbon starts to fuse with itself, making sodium, magnesium, and more neon (plus an α particle).

If you want carbon burning to proceed in a steady way, it must occur in a nondegenerate electron gas. This occurs only if the star is quite massive, more than $8 M_\odot$. Otherwise, it is an explosive process.

Now the core temperature reaches 10^9 K. The energy of a typical photon is quite high, $h\nu \sim k_B T \sim 100\,\mathrm{keV}$. Suppose there are neon nuclei in the core (this will be the case since they are a product of fusion).

It is not hard for a Neon to lose an α particle through photodissociation, this produces an Oxygen. The energy required for this is of the order 4.7 MeV, at the high energy tail we have a few photons at this energy.

This is the "neon burning phase", after which we have oxygen and magnesium. Oxygen is the next candidate for nuclear burning, and after a further contraction the star starts burning it. It fuses with itself to produce ^{28}Si plus an α, or ^{32}S.

Sulfur cannot fuse with itself, the potential barrier is too high. Through successive α captures, the star synthesizes elements in the "iron peak": iron, nickel, cobalt.

The core tries to contract in the attempt to get them to burn, but they have the maximum possible binding energy per nucleon. So, the contraction continues.

If the mass of the contracting core exceeds the Chandrasekhar limit, it cannot become an electron-degenerate object. The mass of the core is always in excess of this limit mass for the stars which are massive enough to reach this stage of stellar burning.

Iron is photodissociated to make helium nuclei first, then bare protons, electrons and neutrons. Protons and electrons can combine into neutrons. The core becomes more and more neutrons rich, but the reaction also produces neutrinos, which can fly away.

The *neutron* degeneracy pressure can stop the collapse in certain cases: this is how a neutron star is formed. The threshold between neutron stars and black holes is hard to determine, but generally speaking with $8M_\odot < M < 25M_\odot$ a neutron star is formed, while for larger masses the core collapses further to form a black hole.

The freefall velocity is a significant fraction of the speed of light. What are the statistics? how many NS and BH are there in our galaxy?

We can model the distribution of star masses in our galaxy with the distribution, the IMF, as a Salpeter IMF,[1] which is given by

$$N(m) \propto m^{-\alpha}\,, \tag{0.1.1}$$

where $\alpha \approx 2.35$. Then, we can calculate the number of stars which have more than $8M_\odot$ by integrating: we find something proportional to $8^{-1.35}$, while the number of stars which have more than $25M_\odot$ we get something proportional $25^{-1.35}$. These will give us the amount of compact objects. The proportionality constant depend on the minimum and maximum mass of stars, but we can calculate the ratio of the two without concern for it. We find

$$\frac{N_{BH}}{N_{NS} + N_{BH}} = \left(\frac{8}{25}\right)^{-1.35} \approx 0.2\,. \tag{0.1.2}$$

The present rate of supernova explosions in the galaxy is around 1 per century, or $10^{-2}\,\mathrm{yr}^{-1}$. In the age of the galaxy (around 10^{10} yr), we will then have had around 10^8 compact objects.

[1] See the evil organization in Mission Impossible.

How much can we trust this figure? Kind of, the true number is closer to 10^9, about 1 % of the number of stars in the galaxy.

The galaxy roughly looks like a cylinder with radius $R \sim 60\,\mathrm{kpc}$ and height $H \sim 1\,\mathrm{kpc}$. Its volume will then be $V = 2\pi R^2 H \approx 10^{13}\,\mathrm{pc}^3$. Then, the number density of compact objects is around $0.1\,\mathrm{pc}^{-3}$.

The typical separation between them will be something like 20 pc. Compact objects are close, common! Beware!

The closest compact object we know of is a neutron star 60 pc away: this is on the same order of magnitude.

Compactness The gravitational radius characterizing an object is

$$R_g = \frac{GM}{c^2}\,, \tag{0.1.3}$$

while the Schwarzschild radius is $2R_g$. For the Sun, this is approximately 1.5 km. It's small.

The value R_g/R is 0.5 for black holes, 0.15 for neutron stars, 10^{-4} for white dwarfs.

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As we said earlier, massive stars go type-2 supernova: this corresponds to Core-Collapse.

Compact objects are quite common in the galaxy.

A compact object is one for which the ratio of the gravitational radius $R_g = GM/c^2$ is comparable to the radius of the true object. For a white dwarf, the ratio is of the order of 10^3.

We then need GR in order to deal with them. Let us quickly go over exact solutions of the Einstein Field Equations.

This lecture, we consider the vacuum Schwarzschild solution. The most general line element which is spherically symmetric (invariant under spatial rotations) must be made up of elements which are themselves invariant under spatial rotations. We will use spherical coordinates: r, θ, φ, t.

In flat spacetime, the line element reads

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,, \tag{0.1.4}$$

and our Schwarzschild solution will need to reduce to this in some limit.

The spatial line element is given by

$$\mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r} = \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{0.1.5}$$

Then, the most general spherically symmetric line element will read

$$\mathrm{d}s^2 = F(r,t)\,\mathrm{d}t^2 + M(r,t)\,\mathrm{d}r^2 + G(r,t)\,\mathrm{d}r\,\mathrm{d}t + C(r,t)r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,, \tag{0.1.6}$$

however, we can redefine the radial coordinate in order to remove the function multiplying the angular term, so we get

$$\mathrm{d}s^2 = F\,\mathrm{d}t^2 + M\,\mathrm{d}r^2 + G\,\mathrm{d}r\,\mathrm{d}t + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,. \tag{0.1.7}$$

We can also introduce a new time variable:

$$\mathrm{d}t' = \mathrm{d}t + \psi(r,t)\,\mathrm{d}r\ . \tag{0.1.8}$$

If $\psi = rG/F$, then we remove the mixed term, and then we are left with the expression

$$\mathrm{d}s^2 = -B(r,t)\,\mathrm{d}t^2 + A\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\ . \tag{0.1.9}$$

However, we have not yet determined the two functions, and we have not said anything about the Einstein Field Equations, which are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,. \tag{0.1.10}$$

In vacuo, the stress-energy tensor vanishes. The curvature scalar must vanish (we can show this by contracting the EFE with the inverse metric), so the equations reduce to $R_{\mu\nu} = 0$. We restrict ourselves to the static case.

The linearly independent components of the Ricci tensor read

$$R^0_0 = \frac{B''}{2AB} - \frac{A'B'}{4A''B} - \frac{B'^2}{4AB^2} + \frac{B'}{rAB} = 0 \tag{0.1.11}$$

$$R^1_1 = \frac{B''}{2AB} - \frac{A'B'}{4A''B} - \frac{B'^2}{4AB^2} + \frac{A'}{rA^2} = 0 \tag{0.1.12}$$

$$R^2_2 = \frac{1}{4rA}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{r^2}\left(\frac{1}{A} - 1\right). \tag{0.1.13}$$

Computing $R^0_0 - R^1_1 = 0$ we find

$$\frac{1}{rA}\left(\frac{B'}{B} + \frac{A'}{A}\right) = 0 \tag{0.1.14}$$

$$\frac{\mathrm{d}\log(AB)}{\mathrm{d}r} = 0\,, \tag{0.1.15}$$

so AB is constant. Without losing generality we can take $A = 1/B$, since if this is not the case we can just rescale the radial or temporal coordinate until it is.

Then, we can compute

$$R^2_2 = \frac{B}{2r}\left(\frac{B'}{B} + \frac{B'}{B}\right) + \frac{1}{r^2}(B-1) = 0 \tag{0.1.16}$$

$$B' + \frac{B}{r} - \frac{1}{r} = 0 \tag{0.1.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(rB) = 1\,, \tag{0.1.18}$$

so $rB(r) = r + C$, or equivalently

$$B(r) = \frac{C}{r} + 1\,. \tag{0.1.19}$$

After this, we can already substitute into the metric:

$$ds^2 = -\left(1 + \frac{C}{r}\right) dt^2 + \frac{1}{1 + C/r} dr^2 + r^2 d\Omega^2 . \tag{0.1.20}$$

For any value of C, $B \to 1$ as $r \to \infty$: the metric reduces to the flat one asymptotically. Right now C is an arbitrary constant, however in the weak field limit it is known that

$$g_{00} = -\left(1 + 2\frac{\phi}{c^2}\right) , \tag{0.1.21}$$

where $\phi = -GM/r$ is the Newtonian gravitational field. Equating this expression to the one for g_{00}, we find

$$g_{00} = -\left(1 - \frac{2GM}{rc^2}\right) \implies C = -\frac{2GM}{c^2} . \tag{0.1.22}$$

The constant M in the classical case is the mass of the source, however we are computing a vacuum solution. This is the mass we would compute if we were to measure the orbits of objects around the compact object.

This is then surely a *gravitational* mass, is it also an *inertial* mass? Can we show this in GR?

Then, we can write the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right) dr^2 + r^2 d\Omega^2 . \tag{0.1.23}$$

This is derived by assuming time-independence, however the result is the same even in the time-dependent case by the Jebsen-Birkhoff theorem (which we will prove in a moment). The element g_{rr} diverges as $r \to R_g = 2GM/c^2$, however this does not represent any physical divergence: the scalar $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \propto r^{-6}$ does not diverge there.

There are coordinates which do not diverge near the horizon: one classical choice employs the "tortoise" coordinates, which are the same for r, θ, φ as the Schwarzschild ones, while the time becomes (setting $G = c = 1$)

$$t = t' - 2M \log\left(1 - \frac{r'}{2M}\right) . \tag{0.1.24}$$

Substituting this into the metric yields (dropping the primes for clarity):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{4M}{r} dr\, dt + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 . \tag{0.1.25}$$

There is no pathology at $r = 2M$ anymore, so it was not a physical divergence. The temporal coefficient g_{00} is the same: it can be shown that it is an invariant under coordinate transformations.

If we take two points which are very close along a particle trajectory, they must be separated by an interval $ds^2 < 0$.

If we consider a (nongeodesic!) radial path described by $r(t)$, we can compute the corresponding line element by neglecting the angular part:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{4M}{r}\,\mathrm{d}r\,\mathrm{d}t + \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 \tag{0.1.26}$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = -\left(1 - \frac{2M}{r}\right) + \frac{4M}{r}\frac{\mathrm{d}r}{\mathrm{d}t} + \left(1 + \frac{2M}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 . \tag{0.1.27}$$

Now, the question we ask is: is it possible for the particle trajectory to be timelike or lightlike ($\mathrm{d}s^2 \leq 0$) and outgoing ($\mathrm{d}r/\mathrm{d}t > 0$) under these conditions? If this is the case, the signs of the three terms read

$$\underbrace{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}_{<0?} = -\left(1 - \frac{2M}{r}\right) \underbrace{+\frac{4M}{r}\frac{\mathrm{d}r}{\mathrm{d}t}}_{>0} \underbrace{+\left(1 + \frac{2M}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2}_{>0} , \tag{0.1.28}$$

so we can see that the equality can be satisfied (a positive number cannot equal a negative one!) as long as the first term on the right-hand side is negative, which means $r > 2M$. If $r \leq 2M$, on the other hand, this cannot be the case: a radial trajectory below the horizon *cannot* be outward.

This is what "horizon" means: it is a *semi-permeable* membrane, particles can surpass it only in one direction.

Jebsen-Birkhoff This theorem states that the Schwarzschild solution also describes the spacetime around an object in the spherically-symmetric but time-*dependent* case. Let us give a sketch of its proof, omitting some tedious calculations. If we write out the components of the Ricci tensor, we find something in the form

$$R^0_0 = R^0_0\Big|_{\text{static}} + \dot{A}(\dots) \tag{0.1.29}$$

$$R^1_1 = R^1_1\Big|_{\text{static}} + \dot{A}(\dots) , \tag{0.1.30}$$

while R^2_2 and R^3_3 are the same. Also, the term R^1_0 does not vanish unlike the static case, and is equal to

$$R^1_0 = -\frac{\dot{A}}{rA^2} = 0 , \tag{0.1.31}$$

so $\dot{A} = 0$: the equations then are the same as the static case ones! This, however, is not the end, since now the equation

$$\frac{A'}{A} + \frac{B'}{B} = 0 \tag{0.1.32}$$

is not necessarily solved by $\log A = -\log B$, since a prime denotes a *partial* derivative with respect to r, so in general we will have $\log A + \log B = f(t)$, some generic function of time. The metric will then read

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) f(t)\,\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 , \tag{0.1.33}$$

but we can simply rescale the time coordinate to $t \to \sqrt{f}t$ in order for this to reduce to the usual expression. This theorem was originally discovered by the Norwegian physicist Jebsen, and only later popularized in a textbook by Birkoff [JR05].

Bibliography

[JR05] Nils Voje Johansen and Finn Ravndal. *On the Discovery of Birkhoff's Theorem*. Version 2. Sept. 6, 2005. arXiv: physics/0508163. URL: http://arxiv.org/abs/physics/0508163 (visited on 2020-03-08).