Astrophysics and cosmology notes

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3 October 2019

Sabino Matarrese ♡. There is a dropbox folder with notes by a student from the previous years. Also, there are handwritten notes by Sabino.

Textbooks: there are many. A good one is "Cosmology" by Lucchin and Coles.

On the astroparticle side, there is another book: I did not catch its name.

Exam: traditional oral exam, there are fixed dates but they do not matter: on an individual basis we should write an email to set a date and time.

In october the lessons of GR and this course on fridays are swapped.

1 Cosmology

The cosmological principle (or Copernican principle): *we do not occupy a special, atypical position in the universe*. We will discuss the validity of this. A more formal statement is:

Proposition 1.1 (Cosmological principle)**.** *Every* comoving observer *observes the Universe around them at a* fixed time *as being* homogeneous *and* isotropic*.*

We are allowed to use this principle only on very large scales. Comoving is a proxy for the absolute reference frame. When we observe the CMB we see that we are surrounded by radiation of temperature $\sim 3\,\mathrm{K}$. There is a *dipole modulation* though: around a milliKelvin of difference. This is due to the Doppler effect: we are *not* comoving with respect to the CMB.

We, however, ignore the anisotropies on the order of a μK.

The velocity needed to expain the Doppler effect is of the order of $630\,\mathrm{km\,s^{-1}}$. This is *after* correcting for the motion of everything up to a galactic scale.

So, we must assume that to get a *comoving* observer we should launch it in a certain direction at an extremely high velocity. So, we *do* have a preferred frame.

Fixed time refers to the proper time of the comoving observer.

This refers to scales on the order of 100 Mpc.

How can we talk about homogeneity if we can only look at the universe from a single point? We assume that any other observer would also see isotropy as we do.

Isotropy around every point is equivalent to homogeneity. We observe isotropy, we assume homogeneity.

What if we are not typical? a different assumption is that our observer status is *random* according to some distribution... This discussion then starts to involve the anthropic principle.

We must however always keep in mind that these assumptions have to be made before any cosmological study starts.

In GR, our line element will generally be $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$. The *preferred* form of the metric is the one written in the comoving frame: the Robertson-Walker line element,

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right)\right) \tag{1}$$

where $a(t)$ is called the *scale factor*. Sometimes it is convenient to write $\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2 = \mathrm{d}\Omega^2$. The angular part is the same (just rescaled) as the Minkowski one.

The parameter k is a constant, which can be ± 1 or 0.

The parameter r is not a length: we choose our variables so that $a(t)$ is a length, while the stuff in the brackets in (1) is adimensional (and so is r).

- $k = 1$ are called closed universes;
- $k = -1$ are open universes;
- $k = 0$ are called flat universes.

The set of flat universes with $k = 0$ has zero measure.

In a 4D flat spacetime we have the 10 degrees of freedom of Lorentz transformations: Minkowski spacetime is *maximally symmetric*.

Other geometries are De Sitter and Anti de Sitter spacetimes.

For the actual universe, however, not all of these symmetries hold: the universe seems not to be symmetric under time translation, and we have a preferential velocity. 4 symmetries are broken (time translation and Lorentz boosts), 6 hold (rotations and spatial translations): our assumption is that there is a 3D maximally symmetric space.

Under these assumptions, we can derive the RW line element.

Say we have a cartesian flat 2D plane: the flat metric for this is $\mathrm{d}l^2 = a^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$. The constant a is included since we want r to be adimensional.

The surface of a sphere has the following line element: $\mathrm{d}l^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$, where $a^2 = R^2$, the square radius of the sphere.

For a hyperboloid, we will have: $\mathrm{d}l^2 = a^2\left(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\varphi^2\right)$, therefore the only difference is that trigonometric functions become hyperbolic ones.

For both of these, let us define the variable: $r = \sin\theta$ in the spherical case, and $r = \sinh\theta$ in the hyperbolic case. Then, these line elements become respectively:

$$\mathrm{d}l^2{}_{\text{sphere}} = a^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\,\mathrm{d}\varphi^2\right) \tag{2a}$$

$$\mathrm{d}l^2{}_{\text{hyperboloid}} = a^2\left(\frac{\mathrm{d}r^2}{1+r^2} + r^2\,\mathrm{d}\varphi^2\right). \tag{2b}$$

We can rewrite the RW element as:

$$\mathrm{d}l^2 = c^2\,\mathrm{d}t^2 - a^2\begin{cases}\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega^2\\ \mathrm{d}\chi^2 + \chi^2\,\mathrm{d}\Omega^2\\ \mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\Omega^2\end{cases} \tag{3}$$

where if $k = +1$ then $r = \sin\chi$, if $k = 0$ then $r = \chi$, and if $k = -1$ then $r = \sinh\chi$. If we wish to use cartesian coordinates we will have:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(1 + \frac{k|x|^2}{4}\right)^{-2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{4}$$

Universes in which a is a constant are called *Einstein spaces*. We can change time variable, defining $\mathrm{d}t = a(\eta)\,\mathrm{d}\eta$, where $a(\eta) \overset{\text{def}}{=} a(t(\eta))$: so, we will have

$$\mathrm{d}s^2 = a^2(\eta)\left(c^2\,\mathrm{d}\eta^2 - \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega\right)\right). \tag{5}$$

The parameter η is called *conformal time*: RW is said to be *conformal* to Minkowski. Conformal geometry is particularly useful for systems which have no characteristic length.

Photons do not have a characteristic length: they do not perceive the expansion of spacetime.

The photons of the CMB look like they are thermal: they were thermal originally, and remained such despite the expansion of the universe.

For now we did not use any dynamics, but we will insert them later.

The Friedmann equations are:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{6a}$$

$$\ddot{a} = -\frac{4\pi G}{3}a\left(\rho + \frac{3P}{c^2}\right) \tag{6b}$$

$$\dot{\rho} = \frac{-3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \tag{6c}$$

where dots denote differentiation with respect to the proper time of a consmological observer, t, which is called *cosmic time*. These imply that the energy density $\rho = \rho(t)$ and the isotropic pressure $P = P(t)$ only depend on t.

An important parameter is $H(t) \stackrel{\text{def}}{=} \dot{a}/a$, the *Hubble parameter*. We can write an equation for it from the first Friedmann one:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{7}$$

If $k = 0$, then there we have a critical energy density $\rho_C(t) = 3H^2(t)/(8\pi G)$: we call $\Omega(t) = \rho(t)/\rho_C(t)$. Is Ω larger or smaller than 1? This tells us about the sign of k; while measuring k directly is *very* hard. The former is a "Newtonian" measurement, while the latter is a "GR" measurement.

We define:

$$H_0 = H(t_0) = 100h \times \mathrm{km\,s^{-1}\,Mpc^{-1}} \tag{8}$$

where h is a number: around 0.7, while t_0 just means *now*.

1.1 Energy density

How do we measure it? We want the energy density *today* of *galaxies*: ρ_{0g}. This is $\mathscr{L}_g \langle M/L\rangle$, where $\mathscr{L}_g$ is the mean (intrinsic, bolometric) luminosity of galaxies per unit volume, while M/L is the mass to light ratio of galaxies. It is measured in units of $M_\odot/L_\odot$. Reference values for these are $M_\odot \sim 1.99 \times 10^{33}\,\mathrm{g}$, while $L_\odot \sim 3.9 \times 10^{33}\,\mathrm{erg\,s^{-1}}$.

How do we measure this? We have a trick:

$$\mathscr{L}_g = \int_0^\infty \mathrm{d}L\, L\Phi(L) \tag{9}$$

where $\Phi(L)$ is the number of galaxies per unit volume and unit luminosity: the *luminosity function*. With of our observations we estimate the shape of $\Phi(L)$. We know that the integral must converge, so we can bound the shape of Φ (?).

4 October 2019

From yesterday we recall that $k = 0$ iff $\rho(t) = \rho_C(t)$.

We can write $H(t_0) = H_0 = 100h \times \mathrm{km\,s^{-1}\,Mpc^{-1}}$. Do note that $1\,\mathrm{Mpc} = 3.086 \times 10^{22}\,\mathrm{m}$.

In the American school, the pupils of Hubble thought $h \sim 0.5$, while the French school thought $h \sim 1$. Now, we know that $h \sim 0.7$. Some people find $h \approx 0.67 \div 68$, others find $h \approx 0.62$.

If we have H we can find $\rho_{0C} = h^2 \times 1.88 \times 10^{-28}\,\mathrm{g\,m^{-3}}$. We have defined $\Omega(t) \stackrel{\text{def}}{=} \rho/\rho_C$: recall that $\mathrm{sign}(\Omega - 1) = \mathrm{sign}(k)$.

So we want to measure the energy density in galaxies to figure out what Ω is.

There is a professor called Schechter who introduced a *universal* luminosity function of the universe.

$$\Phi(L) = \frac{\Phi^*}{L^*}\left(\frac{L}{L^*}\right)^{-\alpha} \exp\left(-\frac{L}{L^*}\right) \tag{10}$$

These can be fit by observation: we find $\Phi^* \approx 10^{-2}h^3\mathrm{Mpc}^{-3}$, $L^* \approx 10^{10}h^{-2}L_\odot$ and $\alpha \approx 1$.

[Plot of this function: sharp drop-off at $L = L^*$]

The integral for $\mathscr{L}_g$ converges despite the divergence of $\Phi(L)$ as $L \to 0$: so we do not need to really worry about the low-luminosity cutoff.

The result of the integral is $\mathscr{L}_g = \Phi^* L^* \Gamma(2 - \alpha)$ where Γ is the Euler gamma function, and $\Gamma(2 - 1) = 1$. Numerically, we get $(2.0 \pm 0.7) \times 10^{18} h L_\odot \mathrm{Mpc}^{-3}$.

Spiral galaxies are characterized by rotation.

We plot the velocity of rotation of galaxies v against the radius R. This is measured using the Doppler effect.

We'd expect a roughly linear region, and then a region with $v \sim R^{-1/2}$: we apply $GM(R) = v^2(R)R$ (this comes from Kepler's laws or from the virial theorem). This implies

$$v(R) \propto \sqrt{\frac{M(R)}{R}} \tag{11}$$

So in the inside of the galaxy, where $M(R) \propto R^3$, $v \propto R$, while outside of it $M(R)$ is roughly constant, so $v \propto R^{-1/2}$.

Instead of this, we see the linear region and then $v(R)$ is approximately constant. Is Newtonian gravity wrong? (GR effects are trivial at these scales).

An option is MOND: they propose that there is somthing like a Yukawa term at Megaparsec distances. They are wrong for some other reasons.

Another option is that what we thought was the galaxy, from our EM observations, is actually smaller than the real galaxy. We'd need mass obeying $M(R) \sim R$:

since $M(R) = 4\pi \int_0^{R_{\max}} \mathrm{d}R \, R^2 \rho(R)$, we need $\rho(R) \propto R^{-2}$. This is a *thermal* distribution (?): we call it the *dark matter halo*.

People tend to believe that this matter is made up of beyond-the-standard-model particles, like a *neutralino*. An alternative is the *axion*.

The total density of DM is ~ 5 times more than that of regular matter.

If galaxies are not spiral, we look at other things: the Doppler broadening of spectral lines gives us a measure of the RMS velocity.

1.2 Virial theorem

Later in the course we will obtain the (nonrelativistic) virial theorem:

$$2T + U = 0 \tag{12}$$

This holds when the inertia tensor stabilizes.

The kinetic energy is $T = {}^3\!/\!{}_2 M \left\langle v_r^2 \right\rangle$: we expect the radial velocity to account for one third of total energy by equipartition. M is the total mass of the energy.

The potential energy is $U = -GM^2/R$. Substituting this in we get $3 \left\langle v_r^2 \right\rangle = GM/R$.

If we account for the extra DM mass, we get $\langle M/L \rangle \approx 300 h M_\odot / L_\odot$. In order to have $\Omega = 1$, we'd need 1390.

So measuring the number density of galaxies and their velocities we get a way to measure Ω_0.

So, only 5% of the energy budget is given by baryionic matter (not all of which is visible), while around 27% is dark matter.

In order to comply with observation, it must be:

$$0.013 \leq \Omega_\mathrm{b} h^2 \leq 0.025 \,, \tag{13}$$

where Ω_b corresponds to the baryionic density: so the universe *cannot* be made only of baryons.

Dark matter likes "clumping": we characterize it by this property.

In the end we have $\Omega = 1 = \Omega_b + \Omega_{DM} + \Omega_{DE}$ (do note that the value of 1 is measured, not theoretical!)

The Friedmann equation would imply deceleration if $\rho, P \geq 0$: dark energy seems to have *negative pressure*.

What about radiation? The CMB appears to be Planckian:

$$\rho_{0\gamma} = \frac{\sigma_r T_{0\gamma}^4}{c^2} = 4.8 \times 10^{-34} \, \mathrm{g} \, \mathrm{cm}^{-3} \tag{14}$$

where $\sigma_r = \pi^2 k_B^4 / (15 \hbar^3 c^3)$, while $\sigma_{SB} = \sigma_r c/4$.

We are going to show that if neutrinos were massless, their temperature would be $T_\nu = (4/11)^{1/3} T_\gamma$.

However, we know that for sure $\sum m_\nu \leq 0.12\,\mathrm{eV}$.

We have:

$$\rho_\nu = 3 N_\nu \frac{\langle m_\nu \rangle}{10\,\mathrm{eV}} 10^{-30}\,\mathrm{g\,cm^{-3}} \tag{15}$$

(??? to check)

1.3 The Hubble law

It is very simply

$$v = H_0 d\,, \tag{16}$$

where v is the velocity of objects far from us, and d is their distance from us. Can we derive this from the Robertson-Walker line element? It was actually derived first by Lemaitre.

We drop the angular part in the FLRW line element (for $k = 0$):

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2 \tag{17}$$

So the distance $d = a(t) r$: therefore $\dot{d} = \dot{a} r = \frac{\dot{a}}{a} d$. This is Newtonian and rough, but it seems to work.

Definition 1.1 (Redshift)**.** *The redshift z is defined by*

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}\,, \tag{18}$$

where λ_0 and λ_e are the observed and emission wavelengths.

We can show that $1 + z = a_0 / a_e$. Therefore, $\nu_o / \nu_e = a_e / a_0$.

So, we can define (?) the luminosity distance:

$$d_L = \sqrt{\frac{L}{4\pi\ell}}\,, \tag{19}$$

where L is the observed luminosity, while ℓ is the apparent luminosity.

How do we relate the luminosity distance and the scale factor? Geometrically we derive:

$$\ell = \frac{L}{4\pi r^2 a^2} \left(\frac{a_e}{a_0} \right)^2 \tag{20}$$

since we can just integrate over angles the FLRW element: the value of k does not enter into the equation. The corrective factor comes from the frequency dependence of energy and the fact that power is energy over time, which also changes.

Therefore:

$$d_L = \frac{a_o^2}{a_e} r = a(1+z) r\,. \tag{21}$$

Thu Oct 10 2019

Today at 17 there is a colloquium at Aula Rostagni.

Next week, also at 17, there is a meeting at San Gaetano (for the general public).

We just had a Nobel prize in cosmology: he predicted the existence of the CMB, and two in astrophysics, they found the first exoplanet.

Send the professor an e-mail to get access to his Dropbox folder with many relevant texts.

We come back to the RW metric:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right), \tag{22}$$

where $k = 0, \pm 1$ and r is the dimensionless comoving radius.

The distance can be calculated as:

$$d_P(t) = a(t)\frac{r}{\sqrt{1-kr^2}}. \tag{23}$$

We are allowed to consider radial trajectories, however this does not account for the fact that we cannot actually measure with a space-like measuring stick.

A better definition is

$$d_P = a(t)\int_0^r \frac{\mathrm{d}\widetilde{r}}{\sqrt{1-k\widetilde{r}^2}}. \tag{24}$$

This is called the *proper distance* (the subscript P is for "proper"): the integration, at zero $\mathrm{d}\Omega$ and zero $\mathrm{d}t$, of the radial part of the metric.

On the other hand (OTOH), the flux of photons can give us the *luminosity distance*, which is actually measurable.

We want to derive the Hubble law ($v = H_0 d$) mathematically. It can also be restated as $cz = H_0 d$.

why?

For a fixed source at $d = ar$, naïvely we would have:

$$\dot{d} = \dot{a}r = \frac{\dot{a}}{a}ar = H_0 d \tag{25}$$

We can "move away from the current epoch by Taylor expanding".

The scale factor $a(t)$ can be written as

$$a(t) \simeq a_0 + \dot{a}_0(t-t_0) + 1/2\ddot{a}(t-t_0)^2 \tag{26a}$$

$$= a_0\left(1 + \frac{\dot{a}}{a_0}|_{t_0}(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)\right), \tag{26b}$$

where $q_0 \stackrel{\text{def}}{=} -\ddot{a_0} a_0/(\dot{a_0})^2$ is called the *deceleration parameter* by historical reasons: people thought they would see deceleration when first writing this.

(the Hubble *constant* is not constant wrt *time* , but wrt *direction*!)

The deceleration parameter is actually measured to be negative.

Now, $1+z = a_0/a$ can be expressed as:

$$1 + z \simeq \left(1 + \frac{\dot{a}}{a_0}|_{t_0}(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)\right)^{-1} . \tag{27}$$

Do note that this is derived by using at most second order in the distance interval: when expanding we cannot trust terms of order higher than second. Expanding with this in mind we get:

$$1 + z \simeq 1 - H_0 \Delta t + \frac{q_0}{2} H_0^2 \Delta t^2 + H_0^2 \Delta t^2 , \tag{28}$$

therefore

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t)^2 . \tag{29}$$

Rearranging the equation we get

$$t_0 - t = z\left(H_0 + \left(1 + \frac{q_0}{2}\right) H_0 z\right)^{-1} , \tag{30}$$

but as before we must expand to get only the relevant orders:

$$t_0 - t = H_0(t_0 - t) + \left(1 + \frac{q_0}{2} H_0^2 (t_0 - t)^2\right) \tag{31}$$

We would like the time interval to disappear: for photons $\mathrm{d}s^2 = 0$, therefore in that case $c^2 \,\mathrm{d}t^2 = a^2(t) \,\mathrm{d}r^2 /(1 - kr^2)$. Taking a square root and integrating:

$$\int_t^{t_0} \frac{c \,\mathrm{d}t}{a(t)} = \pm \int_r^0 \frac{\mathrm{d}\widetilde{r}}{\sqrt{1 - kr^2}} , \tag{32}$$

where the plus or minus sign comes from. . .

The integral on the RHS can be solved analytically: it is

$$\begin{cases} \arcsin r \qquad k = 1 \\ r \qquad k = 0 \\ \operatorname{arcsinh} r \qquad k = -1 \end{cases} \tag{33}$$

in all cases, it is just r up to *second* order (since the next term in the expansion of an arcsine or hyperbolic arcsine etc is of third order).

On the other side, we have:

$$\frac{1}{a_0}\int_{t_0}^{t} c\,\mathrm{d}\widetilde{t}\left(1+H_0(\widetilde{t}-t_0)-\frac{q_0}{2}H_0^2(\widetilde{t}-t_0)^2\right)^{-1} \tag{34}$$

therefore, neglecting *third* and higher order terms, we have:

$$\frac{c}{a_0}\left((t_0-t)+\frac{1}{2}H_0(t_0-t)^2+o(|t_0-t|^2)\right)=r\,, \tag{35}$$

since the term proportional to q_0 only gives a third order contribution.

The explicit form of the luminosity distance was

$$d_L=a_0^2\frac{r}{a}=a_0(1+z)r\,. \tag{36}$$

The term a_0 should disappear at the end of every calculation: it is a bookkeeping parameter. Moving on:

$$d_L=a_0(1+z)\frac{c}{a_0H_0}\left(z-\frac{1}{2}(1+q_0)z^2\right), \tag{37}$$

but this also contains cubic terms:

$$d_L\simeq\frac{c}{H_0}\left(z+\frac{1}{2}(1-q_0)z^2+o(z^2)\right). \tag{38}$$

Therefore:

$$cz=H_0\left(d_L+\frac{1}{2}(q_0-1)\frac{H_0}{c}d_L^2\right), \tag{39}$$

and we can notice that the relation is approximately linear and independent of acceleration for low redshift, but we can detect the acceleration at higher redshift. Typically we need to go at around 10 Mpc.

The parameter q_0 appears to be negative now.

We can do better than that if we go from *cosmography* to *cosmology*, by understanding what causes the acceleration of the expansion of the universe.

Let us expand on the concept of reshift: photons are emitted with a certain wavelength λ_e, at a comoving radius r from us, and detected at λ_o. The line element for the photon is $\mathrm{d}s^2=0$, therefore $c\,\mathrm{d}t\,/a(t)=\pm\,\mathrm{d}r\,\sqrt{1-kr^2}$.

As before, we can integrate this relation from the emission to the absorption: we call it $f(r)$ (it can be any of the functions shown before).

$$\int_t^{t_0}=\frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}=f(r) \tag{40}$$

If we map $t\to t+\delta t$ and $t_0\to\delta t_0$ in the integration limits, the integral must be constant since it only depends on r.

When equating these two we can simplify the original integral, rearrange the integration limits and get:

$$\int_{t}^{t+\delta t} \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}, \tag{41}$$

which can we cast into

$$\frac{c\delta t}{a(t)} = \frac{c\delta t_0}{a(t_0)}. \tag{42}$$

Since the frequency must be proportional to the inverse of the time intervals δt or δt_0, we have

$$\nu_e a(t_e) = \nu_o a(t_o), \tag{43}$$

therefore

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a_0}{a}. \tag{44}$$

1.4 A Newtonian derivation of the Friedmann equations

It is useful to do it first this way, pedagogically.

Let us take a uniform spacetime with density ρ. We consider a sphere, and take all the mass inside the sphere away.

Consider Birkoff's theorem: the gravitational field of a spherically symmetric body is always described by the Schwarzschild metric. This can also be applied to a hole: in this case, it tells us that the metric inside the cavity is the Minkowski metric.

The mass taken away will be $M(\ell) = {}^{4\pi}/_{3}\rho\ell^3$, where $\vec{l} = a(t)\vec{r}$ is the radius of the sphere.

We suppose that the gravitational field is *weak*:

$$\frac{GM(\ell)}{\ell c^2} \ll 1. \tag{45}$$

We put a test mass on the surface of the sphere. What is the motion of the mass due to the gravitational field from the center? It will surely be radial, therefore

$$\ddot{\ell} = -\frac{GM(\ell)}{l^2} = -\frac{4\pi G}{3}\rho\ell. \tag{46}$$

This seems to give us a net force even though by our hypotheses there should be none, this is actually not an issue since the unit vectors in our equations will go away in the end.

Then, we have

$$\ddot{a}r = -\frac{4\pi G}{3}\rho a r \to \ddot{a} = -\frac{4\pi G}{3}\rho a. \tag{47}$$

This is part of a Friedmann equation: the isotropic pressure term cannot be recovered, it's like the speed of light is infinite now.

Now consider

$$\dot{\ell}\ddot{\ell} = -\frac{GM}{\ell^2}\dot{\ell}\,, \tag{48}$$

therefore

$$\frac{1}{2}\dot{\ell}^2 = \frac{4\pi}{3}G\rho\ell^3 + \text{const}\,, \tag{49}$$

and integrating we get

$$\dot{a}^2 r^2 = \frac{8\pi G}{3}\rho a^2 r^2 + \text{const} \tag{50}$$

or, removing the r^2 term, which is a constant,

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \text{const}\,. \tag{51}$$

We know that this new constant is related to the energy per particle.

A universe with negative k expands forever and so on.

The number k was badly defined to be only ± 1 or 0: its newtonian version is much better represented by $k_N = kc^2$.

Fri Oct 11 2019

On Oct 31 Marco Peloso will not give his lecture, so Sabino will do both lectures.

Recalling last lecture: we consider a universe with constant density ρ...

We can also recover the third Friedmann equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right), \tag{52}$$

without the last term, which yet again will come from the relativistic consideration.

We will consider *ideal fluids*. From thermodynamics we have:

$$\mathrm{d}E + p\,\mathrm{d}V = 0\,. \tag{53}$$

We can write $E = \rho c^2 a^3$. Then, this becomes $\mathrm{d}(\rho c^2 a^3) + p\,\mathrm{d}a^3 = 0$; expanding:

$$c^2 \mathrm{d}\rho a^3 + c^2\rho\,\mathrm{d}a^3 + p\,\mathrm{d}a^3 = 0\,. \tag{54}$$

This amounts to the third Friedmann equation, exactly.

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{55a}$$

$$\ddot{a} = -\frac{4\pi G}{3}a\left(\rho + \frac{3P}{c^2}\right) \tag{55b}$$

$$\dot{\rho} = -\frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \tag{55c}$$

The equations are in terms of the three parameters a, ρ and p, which are all functions of time.

The third equation comes from the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$.

Rewriting the first equation:

$$\ddot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{56}$$

and

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\dot{\rho}a^2 + \frac{16\pi G}{3}\rho\dot{a}a \tag{57}$$

We then substitute in the expression we have for $\dot{\rho}$ from the third FE. Everything is multiplied by $\dot{a}$: if it is not zero we have

$$\frac{\ddot{a}}{a} = -4\pi G\rho - \frac{4\pi G p}{c^2} + \frac{8\pi G}{3}\rho = -4\pi G\left(\frac{\rho}{3} + \frac{P}{c^2}\right). \tag{58}$$

The equation system is underdetermined. We have to make an assumption: we will assume our fluid is a *barotropic* perfect fluid: $p \overset{!}{=} p(\rho)$.

Very often this equation of state will look like $p = w\rho c^2$, for a constant w (not ω!). This is related to the adiabatic constant γ by $\gamma = w + 1$.

We are going to assume homogeneity and isotropicity.

Some possible equations of state are $p \equiv 0$, or $w = 0$: this means $\gamma = 1$. This is a *dust pressureless fluid*.

What is the speed of sound of our fluid? We only have the adiabatic speed of sound $c_s^2 = \partial p / \partial \rho$, where the derivative is to be taken at constant entropy and is just a total derivative in the barotropic case.

Also in the baryionic case $p = 0$ is a good approximation.

If $p = 0$ we can simplify:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \implies \rho \propto a^{-3}. \tag{59}$$

More generally, not assuming $w = 0$ we get:

$$\dot{\rho} = -3(1+w)\rho\frac{\dot{a}}{a} \implies \rho \propto a^{-3(1+w)} = a^{-3-3\gamma} \tag{60}$$

Another case is a *gas of photons*: in that case $p = \rho c^2/3$, so $w = 1/3$, $\gamma = 4/3$, $c_s^2 = c^2/3$: the speed of sound is $c/\sqrt{3}$. These photons are thermal: perturbations can propagate (even without interactions with matter…).

In this case we get $\rho \propto p \propto a^{-4}$.

Stiff matter is $p = \rho c^2$, $w = 1$, $\gamma = 2$ and $c_s = c$. This is an incompressible fluids: it is so difficult to set this matter in motion that once one does it travels at the speed of light. Now, $\rho \propto p \propto a^{-6}$.

A possible case is $p = -\rho c^2$: $w = -1$ and $\gamma = 0$: we cannot compute a speed of sound. Now ρ and p are constants. This is the case of dark energy (?).

This can be interpreted as an interpretation of the cosmological constant Λ.

Now we relace the last FE with $w = \text{const}$, $\rho(t) = \rho_* \big(a(t)/a_*\big)^{-3(1+w)}$.

Now, if we substitute into the second FE we get that gravity is attractive ($\ddot{a} < 0$) iff $w > -1/3$.

[Plot: ρ vs a: the cosmological constant is constant, matter is decreasing, radiation is decreasing faster].

In this plot, a can be interpreted as the time. We can insert the spatial curvature in the plot: it decreases, but slower than matter. Now, the dark energy in the universe is more important than the curvature.

Let us solve the first FE: inserting the third one we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_* \left(\frac{a}{a_*}\right)^{-3(1+w)} - \frac{kc^2}{a^2}\,. \tag{61}$$

We defined the parameter $\Omega = \frac{8\pi G \rho}{3H^2} = \rho/\rho_C$. Experimentally this is very close to 1. The Einstein-de Sitter model is one where we take $\Omega \equiv 1$: negligible spatial curvature. This amounts to making $k = 0$.

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_* a_*^{3(1+w)} a^{-(1+3w)} \tag{62}$$

therefore $\dot{a} = \pm A a^{\frac{1+3w}{2}}$, or $a^{\frac{1+3w}{2}}\,\mathrm{d}a = A\,\mathrm{d}t$. A solution is:

$$a(t) = a_* \left(1 + \frac{3}{2}(1+w)H_*(t - t_*)\right)^{\frac{2}{3(1+w)}} \tag{63}$$

where $H_*^2 = \frac{8\pi G}{3}\rho_*$, coupled to:

$$\rho(t) = \rho_* \left(1 + \frac{3}{2}(1 + wH_*(t - t_*))\right)^{-2} \tag{64}$$

There is a time where the bracket in $a(t)$ is zero: we call it as t_{BB} define it by

$$1 + \frac{3}{2}(1+w)H_*(t_{\mathrm{BB}} - t_*) = 0\,. \tag{65}$$

Since the curvature scalar is $R \propto H^2$, at t_{BB} the curvature is diverges.

Hakwing & Ellis proved that if $w > -1/3$ we unavoidably must have a Big Bang.

We can define a new time variable by $t_{\text{new}} \equiv (t - t_*) + 2H_*^{-1}/(3(1+w))$. Then, we can just write:

$$a \propto t_{\text{new}}{}^{\frac{2}{3(1+w)}} \tag{66}$$

and this allows us to get rid of t_*.

Inserting this new time variable, we get

$$\rho(t) = \frac{1}{6(1+w)^2 \pi 4 t^2} \tag{67}$$

and the Hubble parameter is:

$$H(t) = \frac{2}{3(1+w)t} \tag{68}$$

Some cases are:

$$\begin{cases} w = 0 \implies a \propto t^{2/3} \\ w = 1/3 \implies a \propto t^{1/2} \\ w = 1 \implies a \propto t^{1/3} \end{cases} \tag{69}$$

The *De Sitter* universe is one where $w \to 1$: $a(t) \propto \exp(Ht)$ and $H = \text{const.}$ (CHECK)

stuff

Thu Oct 17 2019

If we neglect spatial curvature, which is small, we can write the luminosity distance as an integral which we can compute:

$$d_L \equiv \left(\frac{L}{4\pi\ell} \right)^{1/2} . \tag{70}$$

Our metric is the FLRW line element. Then, we can write d_L as:

$$d_L = a_0 (1+z) r(z) . \tag{71}$$

We now define the *conformal time* τ: we want to impose $a^2(\tau)\, \mathrm{d}\tau^2 = \mathrm{d}t^2$. Then, the RW line element becomes:

$$\mathrm{d}s^2 = a^2(\tau) \left(c^2\, \mathrm{d}\tau^2 - \frac{\mathrm{d}r^2}{1 - kr^2} - r^2\, \mathrm{d}\Omega^2 \right) . \tag{72}$$

This is very important when we talk about zero-mass particles, with no intrinsic length scale. Using the variable χ, we have:

$$ds^2 = a^2(\tau)\left(c^2\,d\tau^2 - d\chi^2 - f_k^2(\chi)\,d\Omega^2\right), \tag{73}$$

where $f_k(\chi) = r$ is equal to $\sin(\chi)$, χ or $\sinh(\chi)$ if k is equal to 1, 0 or -1.

If we look at photons moving radially, we get

$$ds^2 = 0 = a^2(\tau)\left(c^2\,d\tau^2 - d\chi^2\right), \tag{74}$$

therefore $c^2\,d\tau^2 = d\chi^2$: setting $c = 1$, we get $\tau(t_0) - \tau(t_e) = \chi(r_e) - \chi(0)$, where a subscript e means "emission".

$$d\tau = \frac{dt}{a} = \frac{da}{a\dot{a}}, \tag{75}$$

and now recall $a = a_0/(1+z)$: differentiating this we get

$$da = -\frac{a_0}{(1+z)^2}\,dz\,, \tag{76}$$

$$\frac{da}{a^2} = \frac{da\,(1+z)^2}{a^2} = -\frac{dz}{a_0}, \tag{77}$$

which means

$$de = -\frac{dz}{a_0 H(z)}\,. \tag{78}$$

??? probably there is wrong stuff here

The Hubble parameter is given by

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{79}$$

with density $\rho(t) = \rho_r(t) + \rho_m(t) + \rho_\Lambda$, where the first term scales like a^{-4}, the second a^{-3}, the third is constant. Thus they scale like $(1+z)^4$, $(1+z)^3$ and so on.

Then we can write a law for the evolution of $H(z) = H_0 E(z)$. Recall the definition of $\Omega(t)$: we can look at the $\Omega_i(t)$ for i corresponding to matter, radiation and so on:

$$\Omega_i(z) = \frac{8\pi G \rho_i(t)}{3H^2(z)} = \frac{8\pi G}{3H^2}\frac{\rho_i(z)}{E^2(z)} \stackrel{\text{def}}{=} \Omega_{i,0}\frac{(1+z)^\alpha}{E^2(z)}\,. \tag{80}$$

In the case of radiation, $p = \rho c^2/3$, and then $\alpha = 4$.

For matter $p = 0$: $\alpha = 3$.

For the cosmological constant Λ: $\alpha = 0$.

For spatial curvature we have $\alpha = 2$.

For the Ω) corresponding to the curvature we define: $\Omega_k = -tc^2/(a^2H^2)$. We must have

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k \,. \tag{81}$$

Recall the definition of $E^2(z)$:

$$E^2(z) = \frac{H^2}{H_0^2} = \Omega_{k,0} + \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(q+z)^3 \,. \tag{82}$$

and to get E we just take the square root. We have $\tau(t_0) - \tau(t_e) = \chi(r_e)$. Integrating:

$$\chi(r) = c\int_{a_t}^{a} \frac{\mathrm{d}a}{a\dot{a}} = \int \frac{\mathrm{d}z'}{E(z')} \,, \tag{83}$$

therefore

$$r = f_k\left(\frac{c}{a_0H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right) . \tag{84}$$

Two weeks ago we defined the luminosity distance: now we can compute it.

How do we decide which k to use?

Now, suppose we are looking at a certain far-away object with angular size $\Delta\theta$: we fix r in the RW line element, and look at a constant time: then we get a linear size corresponding to the angular one of

$$\mathrm{d}s = a(t) r \Delta\theta \,, \tag{85}$$

which, when divided by $\Delta\theta$, is called angular diameter $D_A = ar = a_0 r(z)/(1+z)$. This changes with distance... (?)

$$d_L = a_0(1+z)r \,, \tag{86}$$

then

$$\frac{d_L}{d_A} = (1+z)^2 \,. \tag{87}$$

Einstein thought that the universe had to be static.

Recall the Friedmann equations (6). Now, if we look at static solutions for matter ($p = 0$): the third equation becomes the identity, the derivatives of a are zero: therefore the second equation gives us $\rho \equiv 0$: there cannot be matter.

Now we know that the universe is neither static nor stationary.

Einstein modified his equations in order to get a static non-empty solution.

The Einstein equations read

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{88}$$

when $c = 1$, where the Einstein tensor $G_{\mu\nu}$ can be defined with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,. \tag{89}$$

Einstein added a term $-\Lambda g_{\mu\nu}$ to the LHS of the Einstein equations. This is allowed since

1. it is symmetric;
2. it has zero covariant divergence, since Λ is constant while $\nabla_\mu g^{\mu\nu} = 0$.

Then, we can rewrite the EE with a modified stress-energy tensor, to which we add $\Lambda g_{\mu\nu}/8\pi G$. Comparing this to an ideal fluid tensor

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} , \tag{90}$$

we get $\rho \to \rho + \Lambda/8\pi G$ and $p \to p + \Lambda/8\pi G$.

Inserting this into the Friedmann equations we get:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \,, \tag{91}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \Lambda \,, \tag{92}$$

while in the third equation, in the contribution $\widetilde{\rho} + \widetilde{p}$ the two Λ terms cancel. Then for the first equation we get

$$\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} = \frac{k}{a^2} \,, \tag{93}$$

and for the second:

$$4\pi G \rho = \Lambda \,. \tag{94}$$

Then,

$$\Lambda \left(\frac{1}{3} + \frac{2}{3}\right) = \Lambda = \frac{k}{a^2} \,, \tag{95}$$

and we want a solution with $k = 1$, $\Lambda > 0$.

A candidate for the cosmological constant term is the vacuum energy in QFT: however the estimate given there is around 10^{120} times off.

The next topic is a solution of the Friedmann equations. We will try to do it with $p = 0$: $\rho \propto a^{-3}$.

Fri Oct 18 2019

From yesterday: other consequences of inserting Λ. Now we have a different approach from Einstein's: we insert the constant not in order to get a static universe but just as a measurable parameter of our theory.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} + \frac{k}{a^2}\,, \tag{96}$$

with $\rho = \rho_0 (a_0/a)^3$. The more the universe expands, the more the cosmological constant term dominates, since the other terms are inversely dependent on a.

The asymptotic state, neglecting spatial curvature, gives us a steady-state solution:

$$a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}} t\right)\,. \tag{97}$$

This is called a *de Sitter* solution. An alternative is found if we *do* consider the spatial curvature, with $k = \pm 1$. The two other solutions are also called *de Sitter* and look like cosines (?): they can be mapped into each other.

How?

These belong to the maximally symmetric solutions: Minkowski, dS and Anti de Sitter: the latter has $\Lambda < 0$.

In AdS, do we have $a \propto \exp(ikt)$?

The *no-hair cosmic theorem* is actually a conjecture: it states that asymptotically only the dark energy contribution is relevant: all the matter and everything else is forgotten.

$$a \propto \left(\sinh(At)\right)^{2/3}\,, \tag{98}$$

where we define $2A/3 = \sqrt{\Lambda/3}$ is a solution which *interpolates* between the current — matter dominated — universe and the asymptotic one, since the hyperbolic sine is locally a simple exponential.

We can rewrite the Friedmann equation as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k\,, \tag{99}$$

and now we will solve it with $k = \pm 1$. In general, for an ODE like $y = f(y')$ with f' continuous we introduce $y' \equiv p$ with $p \neq 0$: then $y = f(p)$, which implies

$$y' = \frac{\mathrm{d}f}{\mathrm{d}p} p'\,, \tag{100}$$

and then

$$p = \frac{\mathrm{d}f}{\mathrm{d}p} p' \implies \frac{\mathrm{d}x}{\mathrm{d}p} = \frac{1}{p}\frac{\mathrm{d}f}{\mathrm{d}p}\,, \tag{101}$$

which gives by integration the solution:

$$x = \int \mathrm{d}p \, \frac{1}{p} \frac{\mathrm{d}f}{\mathrm{d}p} \qquad \text{with} \qquad y = f(p) \, . \tag{102}$$

Using this for our problem, we get $\dot{a}^2 = Aa^{-1} - k$, where $A \equiv 8\pi G a_0^3 \rho_0 / 3$. We can rewrite this as

$$a = \frac{A}{p^2 + k} \qquad \text{where} \qquad p = \dot{a} \, . \tag{103}$$

Then we get:

$$\dot{a} = p = -2Ap\dot{p} \frac{1}{(p^2 + k)^2} \, , \tag{104}$$

and using our formula

$$t = -2A \int \frac{\mathrm{d}p}{(p^2 + k)^2} \, . \tag{105}$$

In order to go forward, we distinguish the cases: it $k = +1$, then $p = \tan(\theta)$, therefore $1 + p^2 = \sec^2 \theta$ which implies $\mathrm{d}p = \mathrm{d}\theta \sec^2 \theta$.

For the time:

$$t = \int -2A \, \mathrm{d}\theta \cos^2(\theta) = -A\big(\theta + \sin(\theta)\cos(\theta)\big) + \text{const} \, , \tag{106}$$

and we apply the trigonometric identity $\sin(\theta)\cos(\theta) = \sin(2\theta)$:

$$t = -\frac{A}{2}\big(2\theta + \sin(2\theta)\big) + \text{const} \, , \tag{107}$$

now we can define $2\theta = \pi - \alpha$, therefore $\sin(2\theta) = \sin(\alpha)$: this gives us $t = A/2(\alpha - \sin(\alpha))$ and $p = 1/\tan(\alpha/2) = \tan(\pi/2 - \alpha/2)$.

So we almost have our solution:

$$a = \frac{A}{1 + \tan^2(\pi/2 - \alpha/2)} = A\cos^2(\frac{\pi}{2} - \frac{\alpha}{2}) = \frac{A}{2}\big(1 + \cos(\pi - \alpha)\big) = \frac{A}{2}\big(1 - \cos(\alpha)\big) \, , \tag{108}$$

which should be complemented with the equation we found for t.

Reinserting the constants we have:

$$a = a_0 \frac{\Omega_0}{2(\Omega_0 - 1)}\big(1 - \cos(\alpha)\big) \, , \tag{109}$$

and

$$t = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}\big(\alpha - \sin(\alpha)\big) \, , \tag{110}$$

but now we switch from α to θ for historical reasons.

[Plot: $a(\theta)$ vs θ].

We have $\dot{a} > 0$ when $0 \leq \theta \leq \theta_m = \pi$, while $\dot{a} < 0$ when $\theta_m \leq \theta \leq 2\pi$. This is the *turn-around* angle. The angles 0 and 2π correspond to the Big Bang and the Big Crunch.

At θ_m we have:

$$a = a_0 \frac{\Omega_0}{\Omega_0 - 1}, \tag{111}$$

and

$$t = \frac{\pi}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}, \tag{112}$$

therefore if we set $a = 1$ at the current time we get

$$t_0 = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \left(\arccos\left(\frac{2}{\Omega_0} - 1 \right) - \frac{2}{\Omega_0} (\Omega_0 - 1)^{1/2} \right), \tag{113}$$

In a pure matter model we would have $a \propto t^{2/3}$, which would imply $H_0 = 2/(3t_0)$ or $t_0 = 2/(3H_0)$: this does not make sense! It would give us an age of the universe of the order 10^9 yr, while the measured age of the universe is 1.4×10^{10} yr. This is for a flat universe.

For a given H_0, we would have a smaller t_0 with a closed universe.

For $k = -1$ we do exactly the same steps with hyperbolic functions instead of trigonometric ones: we get

$$a(\psi) = a \frac{\Omega_0}{2(1 - \Omega_0)} (\cosh \psi - 1), \tag{114}$$

with $\cosh \psi = 2/\Omega_0 - 1$ and

$$t(\psi) = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} (\sinh \psi - \psi). \tag{115}$$

The same reasoning as before gives us a $t_0 > 2/(3H_0)$! This is then more attractive.

Radiation's energy density $\rho_r(a)$ can be seen as a function of the redshift: $\rho_r(z) = \rho_{0r}(1+z)^4$, since $(1+z) = a_0/a$.

For matter $\rho_m(z) = \rho_{0m}(1+z)^3$, for Λ instead $\rho_\Lambda(z) = \rho_{0\Lambda}$.

Let us define a moment called the *equality redshift* z_{eq}. This is when $\rho_r(z_{\text{eq}}) = \rho_m(z_{\text{eq}})$. This means that

$$(1 + z_{\text{eq}}) = \frac{\rho_{0,m}}{\rho_{0,r}} = \frac{\Omega_{0,m}}{\Omega_{0,r}}, \tag{116}$$

where we divided and multiplied by the critical density.

We know that $\Omega_{0,m}$ is around 0.3, while for the radiation it would be easier to measure the density. Neutrinos are matter now but they were radiation at the time of equality (???).

Accounting for everything, we think that

$$1 + z_{\text{eq}} \simeq 2.3 \times 10^4 \Omega_{0,m} h^2 \,. \tag{117}$$

This means that the recombination of electrons and protons into Hydrogen happened when the universe was already *matter dominated*.

Another time is z_Λ, defined by: $\rho_m(z_\Lambda) = \rho_\Lambda(z_\Lambda)$.

$$1 + z_\Lambda = \left(\frac{\rho_{0,\Lambda}}{\rho_{0,m}}\right)^{1/3} \simeq \left(\frac{0.7}{0.3}\right)^{1/3} , \tag{118}$$

which implies that $z_\Lambda \approx 0.33$.

2 The thermal history of the universe

The model which won out is the *hot Big Bang* model.

Consider radiation: we know from Stefan's law that $\rho_r \propto T^4$, while $\rho_r \propto a^{-4}$. Therefore we would expect *Tolman's law* to hold: $T \propto 1/a$.

By *total number of galaxies* we mean the galaxies in our past light cone.

When (in natural units) we get temperature of the order of a certain elementary particle, then statistically that type of particle will usually be ultra-relativistic.

The number density of particles is:

$$n = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, f(\vec{q}) \,, \tag{119}$$

in units where $c = \hbar = k_B = 1$. The parameter g is the number of helicity states, q is the three-momentum. The function $f(\vec{q})$ is a pdf in phase space. We will have to integrate over position and momentum: the metric we will get in the end must not depend on anything but time, by the cosmological principle.

There is no general rule for g: for photons, we only have two spin states; the "rule" $g = 2s + 1$ is not actually applied, for photons $\vec{s} = 0$ is unphysical, for gravitons $|\vec{s}| \leq 1$ is unphysical. g accounts for all internal degrees of freedom: for atoms we also have vibration, rotation…

The energy density is

$$\rho = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 \vec{q} \, E(q) f(\vec{q}) \,, \tag{120}$$

where $E^2 = q^2 + m^2$. For photons $E = q$, for nonrelativistic particles $E \approx m + q^2/2m$.

The adiabatic pressure is

$$P = \frac{g}{(2\pi)^3} \int \mathrm{d}^3\vec{q}\, \frac{q^2 f(\vec{q})}{3E}\,, \tag{121}$$

which comes from a consideration of the diagonal components T_{ii} of the stress energy tensor of particles. Alternatively, we can say that this comes from imposing $\mathrm{d}E = P\,\mathrm{d}V + \mathrm{d}Q$.

This definition gives us $P = \rho/3$ for photons directly.

In full generality the distribution is

$$f(\vec{q}) = \left(\exp\left(\frac{E-\mu}{T} \right) \pm 1 \right)^{-1}\,, \tag{122}$$

where we have a plus for fermions, and a minus for bosons. Here, μ is the chemical potential: is becomes relevant when the gas becomes hot and dense. It can be introduced as a Lagrange multiplier for changes in number of particles.

The Planck distribution is given by:

$$f_k(\vec{q}) = \left(\exp\left(\frac{q}{T} \right) - 1 \right)^{-1}\,, \tag{123}$$

since they are bosons with no chemical potential:

In general we can say that if for some species we have the reaction $i + j \leftrightarrow k + l$, then $\mu_i + \mu_j = \mu_k + \mu_l$. We can deduce them by the known relations: for example, from the annihilation of electron and positron we can derive $\mu_{e^+} = -\mu_{e^-}$. This rule is not trivial; it is an ansatz of thermodynamical equilibrium to get a solution of the Boltzmann equation which allows us to write the Saha equations.

Fri Oct 25 2019

Do we wish to have a part on stellar astrophysics right now, before going on with cosmology? Let him know.

Are we in thermal equilibrium? In general, no. This must be considered when dealing with CMB anisotropies.

QFT must be dealt with not only at zero temperature, but also at finite temperatures. People started doing this in the seventies.

The time variable is then periodic, with period $2\pi\beta$, where $\beta = 1/(k_B T)$.

Is this connected to imaginary time?

This is "in-in" instead of "in-out": there are no equilibrium states before or after the interaction.

We will simplify: we assume thermal equilibrium at any time in the evolution. A key point in cosmologi is the *absence* of time translation invariance, therefore energy is not conserved.

This part will come from Weinberg's book.

We use units where $c = k_B = \hbar = 1$. The number density of a certain species is

$$n(\dots) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q\, f(\vec{q})\,, \tag{124}$$

where $f(\vec{q})$ is the phase space distribution of the particles, while g is the number of helicity states of that species. The dots will be explained later: they are about the parametrization of the phase space distribution.

For any species, $E = \sqrt{m^2 + |q|^2}$: this applies *on shell*, for the classical equations of motion: when there are quantum fluctuations it does not hold. The energy density is

$$\rho(\dots) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q\, E(\vec{q}) f(\vec{q})\,, \tag{125}$$

while the pressure is

$$P(\dots) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q\, f(\vec{q}) \frac{q^2}{3E}\,. \tag{126}$$

Because of isotropy, in the cases we need to consider, the dependence on $\vec{q}$ is actually a dependence on $|\vec{q}| \equiv q$.

What is our ansatz for the phase space distribution?

$$f(q) = \frac{1}{\exp\left(\frac{E-\mu}{T}\right) \mp 1}\,, \tag{127}$$

where the sign is $-$ for bosons, $+$ for fermions.

The chemical potential deals with the flux of particles. We recover the Planck distribution when $E = q$ and $\mu = 0$: this tells us that photons do not have any chemical potential. If we measured $\mu \neq 0$ that would be called a *spectral distortion*, but is seems like that is not the case.

The rule for the sum of the chemical potentials only holds at equilibrium.

We can relate some chemical potentials by reactions, and this tells us about the conserved quantities which follow from the symmetry group of our theory.

If there was a global charge, phenomenologically we'd expect global magnetic fields, but we see them only of the order of the $10\,\mu\mathrm{G}$.

So we have an upper bound on the global charge density (?)

We can estimate the orders of magnitude for the various species in the universe.

In the end: very often, $\mu/T \ll 1$. Therefore, the dependence of the number density, energy density and pressure would be also on μ but we forget it, and keep only the temperature dependence: we write $n(T)$, $\rho(T)$ and $P(T)$.

From the second principle of thermodynamics we know that the entropy in a certain volume V at temperature T, $S(V,T)$ is given by:

$$\mathrm{d}S = \frac{1}{T}\big(\mathrm{d}\rho(T,V) + P(T)\,\mathrm{d}V\big)\,. \tag{128}$$

Then we have

$$\frac{\partial S}{\partial V} = \frac{1}{T}\big(\rho(T) + P(T)\big)\,, \tag{129}$$

and

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{\mathrm{d}\rho(T)}{\mathrm{d}T}\,. \tag{130}$$

Recall the Pfaff relations: in order for the differential to be exact it needs to be closed, which means that the second partial derivatives need to commute:

$$\frac{\partial}{\partial T}\left(\frac{1}{T}\big(\rho(T) + P(T)\big)\right) = \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{\mathrm{d}\rho(T)}{\mathrm{d}T}\right)\,. \tag{131}$$

We can proceed with the calculation, to get:

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{1}{T}\big(\rho(T) + P(T)\big)\,. \tag{132}$$

Cosmology has not entered into the picture yet, but it can by the third Friedmann equation, which can be rewritten as

$$a^3\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(a^3(P+\rho)\Big)\,, \tag{133}$$

and these two, when put together, are equivalent (check!) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{T}(\rho + P)\right) = 0\,, \tag{134}$$

therefore this quantity is a constant of motion.

(For the RW line element, the square of the determinant $\sqrt{-g} = a^3$, so we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{-g}\frac{\rho + P}{T}\right) = 0\,, \tag{135}$$

)

So the quantity which is differentiated is constant. If we plug this back into the differential expression for the entropy, we get:

$$\mathrm{d}S = \mathrm{d}\left(\frac{(\rho + P)V}{T}\right)\,, \tag{136}$$

therefore the differentiated quantities are equal up to a constant, but since $V \propto a^3$ we get that the *entropy is constant*.

$$S \equiv S(a^3, T) = \frac{a^3}{T}(\rho + P) = \text{const}\,. \tag{137}$$

Since $\rho \propto T^4$ and $\rho \propto a^{-4}$, we expect Tolman's law: $Ta \sim 1$.

Also, for photons $P = \rho/3$: so we get $S \propto {}^4\!/_3(a^3/T)T^4 \propto T^3 a^3 = \text{const}$.

We only consider photons since they have a much larger number density.

Is this because the suppression is exponential while the ratio of energies is somehow polynomial?

Now we wish to do the integrals for the thermodynamical quantities:

$$\int \mathrm{d}^3\vec{q} = 4\pi \int_0^\infty \mathrm{d}q\, q^2\,, \tag{138}$$

so we get

$$n(T) = \frac{g}{(2\pi^2} \int \mathrm{d}q\, q^2 f(q) \tag{139a}$$

$$\rho(T) = \frac{g}{2\pi^2} \int \mathrm{d}q\, q^2 f(q) E(q) \tag{139b}$$

$$P(T) = \frac{g}{6\pi^2} \int \mathrm{d}q\, q^2 f(q) \frac{q^2}{E(q)}\,, \tag{139c}$$

which can be solved analytically in the ultrarelativistic and nonrelativistic approximations.

Ultrarelativistic means $q \gg m$: actually we compare these to the temperature: we define $x = q/T$, and then the energy is $\sqrt{x^2 + m^2/T^2}$: we suppose $m/T \ll 1$.

The momentum will not always be large, but the cases in which it is large give a much greater contribution if the temperature is large.

Ultrarelativistic We approximate $E(q) \sim q$, so

$$n(T) = \frac{g}{2\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^2 \Big(\exp\big(q/T\big) \mp 1\Big)^{-1}\,, \tag{140}$$

$$\rho(T) = \frac{g}{2\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^3 \Big(\exp\big(q/T\big) \mp 1\Big)^{-1}\,, \tag{141}$$

$$P(T) = \frac{g}{6\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^3 \Big(\exp\big(q/T\big) \mp 1\Big)^{-1}\,, \tag{142}$$

so we can see that in this approximation, which is equivalent to $m \approx 0$, we get matter behaving like radiation: $P = \rho/3$. The result of the integrals depends on the type of the particles, and it is

$$n(T) = \begin{cases} \frac{\xi(3)}{\pi^2} g T^3 & BE \\ \frac{3}{4}\frac{\xi(3)}{\pi^2} g T^3 & FD \end{cases}, \tag{143}$$

and we get the proportionality to T^3. For the energy density:

$$\rho(T) = \begin{cases} \frac{\pi^2}{30} g T^4 & BE \\ \frac{7}{8}\frac{\pi^2}{30} g T^4 & FD \end{cases}, \tag{144}$$

and for the pressure we just divide by 3. For photons, $g = 2$ and they are bosons, so we get

$$\rho_\gamma = \frac{\pi^2}{15} T^4 \,. \tag{145}$$

Nonrelativistic Now $m \gg T$, so we get $E \approx m + \frac{q^2}{2m} + O((q/\sqrt{m})^4)$ by expanding.

So, can we just substitute this? It would seem like this makes the exponential very large, so the difference between bosons and fermions becomes negligible. So, to zeroth order in (q^2/m) we get

$$f \approx \exp\left(-\frac{m-\mu}{T} \right), \tag{146}$$

and then

$$n = \frac{g}{2\pi^2} \exp\left(-\frac{m-\mu}{T} \right) \int_{\mathbb{R}^+} \mathrm{d}q \, q^2 \,, \tag{147}$$

which diverges. This is the ultraviolet catastrophe: we need the *Planckian*, by considering the first order in q^2/m: then

$$n = \frac{g}{2\pi^2} \exp\left(-\frac{m-\mu}{T} \right) \int_{\mathbb{R}^+} \mathrm{d}q \, q^2 \exp\left(-\frac{q^2}{2mT} \right) = g \left(\frac{mT}{2\pi} \right)^{3/2} \exp\left(\frac{\mu - m}{T} \right), \tag{148}$$

where we applied the identity

$$\int_{\mathbb{R}} \mathrm{d}x \, x^2 \exp\left(-\alpha x^2 \right) = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \,. \tag{149}$$

To recover the energy density and pressure we could do $\rho = mn$ and $P = nT$.

Do these come out of doing the other two integrals?

Therefore, $P = T\rho/m$, so $P \ll \rho$: for nonrelativistic particles, we can deal with them as *noninteracting dust*.

$P = nT$ is just the ideal gas law.

If we compare relativistic particles to nonrelativistic ones, the former dominate the latter in terms of all of these three quantities.

Thu Oct 31 2019

Neutrinos have very low mass: therefore they become relativistic very quickly.

We saw last time the case of either bosons' or fermions' number density, energy density and pressure.

In the nonrelativistic case instead we must consider the second order in $q/\sqrt{m}$, and we get the Boltzmann suppression factor out of the integral: $\exp(-m/T)$.

In the nonrelativistic case there is no distinction between bosons and fermions.

What happens in the part of the history of the universe where all the particles were in thermal equilibrium?

If the timescales of the interactions are much larger than the cosmological events timescales (such as the energy of the universe), then those interactions do not happen. These are called *decoupled*.

Let us consider ultrarelativistic particles which are not *decoupled*, in the early universe which is radiation dominated (here "radiation" refers to all kinds of ultra-relativistic particles).

In this case, from the "conservation of the stress-energy tensor" we know that $\rho \propto a^{-4}$.

Our equation is

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{150}$$

and we want to neglect the last term. The equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_* \left(\frac{a}{a_*}\right)^{-4} - \frac{k}{a^2}\,, \tag{151a}$$

but it is not enough to look at the slopes: we can, however, have information about the normalization as well from present-day observations. If the second term is much smaller than the first today, then it was even more so in the far past. We can rewrite the equation as

$$1 = \Omega_{\text{tot}} - \frac{k}{a^2H^2} \equiv \Omega_{\text{tot}} + \Omega_{\text{curvature}}\,, \tag{152}$$

where $\Omega_{\text{curvature}} \equiv -k/(a^2H^2)$.

So we neglect the second term: approximately we then have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\text{rad}}\,. \tag{153}$$

We know that $a(t) \propto t^{1/2}$, therefore $\dot{a}/a = H = 1/2t$.

Wait, how does this work?

So, we have

$$\frac{1}{4t^2} = \frac{8\pi G}{3} g_* \frac{\pi^2}{30} T^4 , \tag{154}$$

where

$$g_* = g_*(T) = \sum_{i \in BE} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i \in FD} g_i \left(\frac{T_i}{T}\right)^4 , \tag{155}$$

where we insert a correction factor in order to not consider particles which are nonrelativistic (?). BE and FD denote the bosonic and fermionic degrees of freedom respectively.

The i spans the different types of particles in the SM. The T_i are the thermalization temperatures of the various species.

The plot of g_* in terms of the temperature goes "down in steps", as more and more species become nonrelativistic.

Then we have a formula for temperature in terms of time:

$$\frac{1}{2t} = \left(\frac{8\pi G}{3}\right)^{1/2} g_*^{1/2} \left(\frac{\pi^2}{30}\right)^{1/2} T^2 , \tag{156}$$

therefore we get:

$$t \approx 0.301 g_*^{-1/2} \frac{m_P}{T^2} \approx \left(\frac{T}{\mathrm{MeV}}\right)^{-2} s , \tag{157}$$

where $m_P = G^{-1/2}$ is the Planck mass. Beware: there are different conventions for this! The factor $g_*^{-1/2}$ is of order 1 around $T \approx 1\,\mathrm{MeV}$, so we ignore it.

The number is found by:

$$0.301 \approx \frac{1}{2\sqrt{\frac{8\pi\pi^2}{3\times 30}}} . \tag{158}$$

The Planck mass is approximately $1.2 \times 10^{19}\,\mathrm{GeV}$. It is the scale at which we need to account for quantum gravitational effects.

When the universe is 1 second old, weak interactions decouple.

Why are we allowing different thermalization temperatures?

A hypothesis is that the particles exit the Planck epoch, $t \approx m_P$, thermalized. When decoupling occurs, they can stop being thermalized.

Another hypothesis is that they stop being thermalized at a certain temperature during inflation.

Some time ago we discussed entropy conservation.

The entropy density is given by

$$s \equiv S/V = (P + \rho)/T = \frac{4}{3}\frac{\rho}{T} = (2\pi^2/45) g_{*s} T^3 ; \tag{159}$$

since $V \propto a^3$ we have $sa^3 = \text{const}$.

We defined a new g_* to account for the different species:

$$g_{*s} \equiv \sum_{i \in BE} g_i \left(\frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{i \in FD} g_i \left(\frac{T_i}{T} \right)^3 , \tag{160}$$

so our new Tolman's law is $T a g_{*s}^{1/3} = \text{const}$.

When the neutrinos decouple, their temperature keeps scaling like $1/a$: they keep being thermalized with the photons even though they do not react.

When we reach $T \approx 0.5\,\text{MeV}$ electrons and positrons become thermalized: they decay into photons: the photon temperature T increases with respect to what would happen without this process (it actually just decreases slower).

The neutrinos are not updated when this process happens: their temperature then becomes consistently less than the one of the photons, but they keep scaling the same: one of their temperatures is a constant multiple of the other after this event.

Now we will calculate this multiple.

First, at $1\,\text{MeV}$, neutrinos decouple. Then, at $0.5\,\text{MeV}$, electrons and positrons decouple.

We require continuity at this transition: $T_>$ and $T_<$ must both be T_e.

$$sa^3 = \frac{2\pi^2}{45} g_{*s>} T_>^3 a_>^3 = \frac{2\pi^2}{45} g_{*s<} T_<^3 a_<^3 , \tag{161}$$

but we can drop all the factors we know to be continuous on the boundary:

isn't a different on either side of the boundary, if it is not instant?

we are left with

$$T_< = \left(\frac{g_{*s>}}{g_{*s<}} \right)^{1/3} T_< , \tag{162}$$

where we can compute

$$g_{*s>} = 2 + \frac{7}{8}(4) = \frac{11}{2} , \tag{163}$$

where we considered photons, electrons but not neutrinos (since they are separated). On the other hand,

$$g_{*s<} = 2 , \tag{164}$$

since electrons are not thermalized anymore. Therefore

$$T_< = \left(\frac{11}{4} \right)^{1/3} T_> , \tag{165}$$

which allows us to compute the neutrino temperature at any time:

$$T_\nu = T_\gamma \left(\frac{4}{11} \right)^{1/3} , \tag{166}$$

since they scale the same.

Now let us compute $\rho_r(T = 0.1\,\mathrm{MeV})$. Let us assume that the global temperature is the one of the photons: we get

$$\begin{aligned} g_* &= \sum_{i \in BE} g_i \left(\frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i \in FD} g_i \left(\frac{T_i}{T} \right)^4 && \text{(167a)} \\ &= 2 + \frac{7}{8} \left(3 \times 2 \left(\frac{T_\nu}{T_\gamma} \right)^4 \right) && \text{(167b)} \\ &= 2 + \frac{21}{4} \left(\frac{4}{11} \right)^{4/3} \approx 3.36 \,, && \text{(167c)} \end{aligned}$$

since we need to consider neutrinos, which contribute to the total energy density, but not electrons which are not relativistic.

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Last hour we discussed the Planck mass $m_P \approx 1.2 \times 10^{19}\,\mathrm{GeV}$: it also defines a wavelength, $\lambda_P \approx 10^{-33}\,\mathrm{cm}$ and a timescale $t_P \approx 10^{-43}\,\mathrm{s}$ (both of these are $1/m_P$ in natural units).

When the age of the universe is of this order, our theories are not guaranteed to work.

2.1 The cosmological horizon

Let us consider null geodesics in a De Sitter universe with a RW metric: we can take them as radial, and find

$$c^2 \, \mathrm{d}t^2 = a^2(t) \frac{1}{1 - kr^2} \, \mathrm{d}r^2 \,, \tag{168}$$

therefore we get:

$$\int^t \frac{c \, \mathrm{d}t}{a(t)} = \int^r \frac{\mathrm{d}\widetilde{r}}{\sqrt{1 - k\widetilde{r}^2}} = f(r) \,, \tag{169}$$

but in order to get something which has the dimensions of a length we need to multiply by a:

$$d_{\mathrm{Hor}}(t) = a(t) \int_0^t \frac{c \, \mathrm{d}\widetilde{t}}{a(\widetilde{t})} \,. \tag{170}$$

"If this integral is convergent, we should be worried".

If we integrate from the beginning of time to now, we get the spatial (current) distance elapsed by a photon which started at the start of time. This is the radius of the largest region we could in principle observe. It is of the order of 3 Gpc.

This is not the case, for example, in Minkowski spacetime.

If there is an end of time, there is also a future horizon.

The quantity $d_{\mathrm{Hor}}(t)$ is increasing in time: this causes the Cosmological Horizon problem.

What is the problem?

An important moment is the *recombination of hydrogen*: the formation of the first Hydrogen atoms, so the first moment at which Compton scattering can occur. This causes the decoupling of radiation and baryionic matter. This is the moment at which the radiation in the CMB was emitted.

The CMB is very close to being uniform. It was emitted something like $t = 3.8 \times 10^5\,\mathrm{yr}$ after the BB.

Let us consider the past light cones from two points diametrically opposite with respect to us, at this point in time: they do not overlap, so the CMB cannot be causally correlated. However, it seems like it is!

What is the angular scale at which the light cones overlap?

We have a classification by Bianchi of non-isotropic universes (in 9 classes).

There is the Mixed Master Universe.

Let us consider

$$R_{\mathrm{Hor}} = \int_0^t \frac{c \, \mathrm{d}\widetilde{t}}{a(\widetilde{t})} \,. \tag{171}$$

We can hypothesize that there was a period where the comoving radius was decreasing with time. This would solve the problem. We can approximate

$$d_{\mathrm{Hor}}(t) = a(t) \int_0^t \frac{c \, \mathrm{d}\widetilde{t}}{a(\widetilde{t})} \approx ct \sim \frac{c}{H} \equiv d_{\mathrm{H}} \,, \tag{172}$$

where we defined the new *Hubble distance*. We also define the comoving Hubble radius: $r_H = c/(Ha) = c/\dot{a}$.

For it to be decreasing, the condition is

$$\dot{r}_H = -\frac{\ddot{a}}{\dot{a}^2} < 0\,, \tag{173}$$

therefore we need $\ddot{a} > 0$ for at least some time. We know that

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right), \tag{174}$$

therefore we need to have $\rho + 3P/c^2 < 0$. So, since the energy density is positive, the condition is $P < -\rho/3$.

Let us consider the paramter $\ddot{a}$: it is

$$\ddot{a} = \dot{a}H + a\dot{H} = a(H^2 + \dot{H}) > 0\,, \tag{175}$$

so the condition is $H^2 + \dot{H} > 0$. In terms of $P/\rho = w$ we have:

1. $\dot{H} < 0$ while $\dot{H} + H^2 > 0$: this corresponds to $-1 < w < -1/3$;
2. $\dot{H} = 0$: this is De Sitter: $a(t) = \exp(Ht)$, corresponding to $w = -1$;
3. $\dot{H} > 0$: here the solution is

$$a(t) = a_*\left(1 + \frac{3}{2}\left((1+w)H_*(t-t_*)\right)\right)^{2/(3(1+w))}, \tag{176}$$

   which corresponds to $= w < -1$ and means $a(t) \propto t^p$ for some $p > 1$. This is called power-law inflation.

Big Rip singularity: with $a(t) = |t - t_{as}|^{-\alpha}$ with $\alpha > 0$. This is called poli-inflation.

What is this about?

The third condition is very hard to achieve. The boundary at $w = -1$ is called the *phantom divide*.

Now, let us consider the *flatness problem*: this was first proposed bt Dick and Peebles in 1986.

The parameter $\Omega = \frac{8\pi G\rho(z)}{3H^2(z)}$ diverges from 1 as time increases. Measurements of Ω_{tot} gave approximately 0.1.

How is it possible that the universe is still so flat, even when the universe is so old? Oldness and flatness seem incompatible.

This is a type of *fine-tuning* problem. Typically, if there is a fine-tuning problem then it signals that we should improve our theory.

Let us assume that w is constant. Then, $\rho(z) = \rho_0(1+z)^{3(1+w)}$. Recall that

$$H^2(z) = \frac{8\pi G}{3}\rho(z) - \frac{k}{a^2} \tag{177}$$

$$H_0^2 = \frac{8\pi G}{3}\rho_0 - \frac{k}{a_0^2}\,, \tag{178}$$

the latter of which implies $1 = \Omega - k/(a_0^2 H_0^2)$. So,

$$H^2(z) = H_0^2\left(\frac{8\pi G}{3H_0^2}\rho(z) - \frac{k}{a^2 H_0^2}\right) \tag{179}$$

$$= H_0^2\left(\frac{\rho(t)}{\rho_0}\Omega_0\frac{a_0^2}{a}(1-\Omega_0)\right) \tag{180}$$

$$= H^2(z) = H_0^2(1+z)^2\Big(\Omega_0(1+z)^{1+3w} + (1-\Omega_0)\Big)\,, \tag{181}$$

so in the end

$$\Omega(z) = \frac{8\pi G\rho_0}{3H_0^2}\frac{\rho(z)}{\rho_0}\frac{H_0^2}{H^2(z)} \tag{182}$$

$$= \Omega_0(1+z)^{1+3w}\Big(1-\Omega_0+\Omega_0(1+z)^{1+3w}\Big)\,, \tag{183}$$

and since $\rho(z) = \rho_0(1+z)^{3(1+w)}$ we get

$$\Omega^{-1}(z) - 1 = \frac{\Omega_0\Big((1-\Omega_0)+\Omega_0(1+z)^{1+3w}\Big) - (1+z)^{1+3w}}{(1+z)^{1+3w}} \tag{184}$$

$$= (\Omega_0^{-1} - 1)(1+z)^{-(1+3w)}\,. \tag{185}$$

If we assume $w = 1/3$ for all times (which is false, but we do it to get a result that is close enough) we get

$$\Omega^{-1}(z) - 1 = (\Omega_0^{-1} - 1)\left(\frac{T_0}{T(z)}\right)^2. \tag{186}$$

If we compute T_{Planck}/T_0 we get approximately 4.5×10^{32}. Then when squaring we get 10^{-64}, without the approximation $w = 1/3$ we get 10^{-60}.

Then, we see that there is something deeply unnatural in the Friedmann model.

Thu Nov 07 2019

We talk about inflation again. The comoving horizon increases with time:

$$d_H(t) = a(t)\int_0^t \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})} \sim ct \sim \frac{c}{H} \equiv \text{Hubble horizon}\,, \tag{187}$$

it is also called the *past event horizon*. The reason why we can plot the history of the universe in a single spacetime diagram is because there is a well-defined transformation which brings an infinite interval to a finite one, using a hyperbolic arctangent: this gives us a *Penrose diagram*.

The comoving Hubble radius is $r_H = c/aH = c/\dot{a}$. This can grow.

If we have positive pressure $p > 0$, than the scale factor goes like $a \propto t^{2/(3(1+w))}$ with $w > 0$, then the comoving radius increases with time.

We can actually still get increasing comoving radii with a weaker condition: $p > -\frac{1}{3}\rho c^2$. This is the actual boundary (it can be checked looking at the derivative of a).

This is directly connected to the sign of the acceleration in the Friedmann equation.

If these conditions are always met, Hawking and Ellis proved that a Big Bang is inevitable.

The inflation hypothesis is that, even though now we have $\ddot{a} < 0$ now (or, it was so for some time: now the expansion seems to be accelerating), there was a period in which we had $\ddot{a} > 0$.

These are drawn as straight lines, but it is only qualitative: we are looking at the sign of the slope.

Then, there was a time in the past at which the comoving horizon was as large as it is now: now we will see how much inflation there must have been in order to solve the horizon problem up to now, ignoring the fact that now the universe's expansion is accelerating.

The inequality we want to impose is

$$r_H(t_i) \geq r_H(t_0) \,, \tag{188}$$

where t_0 is now while t_i is the beginning of inflation.

A sphere with comoving radius $d_H(t_i)$ will expand after inflation up to

$$d_H(t_i) \frac{a(t_f)}{a(t_i)} \,, \tag{189}$$

where t_f is the time of the end of inflation.

$$d_H(t_i) \frac{a(t_f)}{a(t_i)} \geq d_H(t_0) \frac{a(t_f)}{a(t_0)} \,. \tag{190}$$

We want to see what the limiting condition is.

$$Z_{\min} = \frac{d_H(t_0)}{d_H(t_i)} \frac{a_f}{a_0} = \frac{H_i}{H_0} \frac{a_f}{a_0} \,, \tag{191}$$

is Z a redshift?

$$z_{\text{min}} \frac{H_i}{H_0} \frac{a_f}{a_0} = \frac{H_i}{H_f} \frac{H_f}{H_0} \frac{a_f}{a_0}, \tag{192}$$

or

$$\frac{H_f}{H_i} z_{\text{min}} = \frac{H_f}{H_0} \frac{a_f}{a_0}, \tag{193}$$

in which we can insert our solution to the Friedmann equations, for the scale factor and Hubble parameter in function of time:

$$H(t) = H_* \left(\frac{a(t)}{a_*} \right)^{-\frac{3(1+w)}{2}}. \tag{194}$$

This can be found using the results we found some time ago: the expressions for a and H were equal up to a different thing multiplying the parenthesis, and a different exponent.

This is of course an approximation, but it works. A better number can be found by integrating numerically over different more realistic equations of state. We find:

$$z = \frac{a_f}{a_i} = , \tag{195}$$

Put earlier

$$\frac{H_f}{H_i} = z_{\text{min}}^{-\frac{3(1+w)}{2}}, \tag{196}$$

therefore we get

$$z_{\text{min}} = \left(\frac{H_f}{H_0} \frac{a_f}{a_0} \right)^{-\frac{2}{(1+3w_{\text{inf}})}}, \tag{197}$$

where w_{inf} is calculated at the time of matter-radiation equality.

So we get:

$$\frac{H_f}{H_0} = \frac{H_f}{H_{\text{eq}}} \frac{H_{\text{eq}}}{H_0} = \left(\frac{a_f}{a_{\text{eq}}} \right)^{-2} \left(\frac{a_{\text{eq}}}{a_0} \right)^{-3/2} = \left(\frac{a_f}{a_0} \right)^{-2} \left(\frac{a_0}{a_{\text{eq}}} \right)^{-1/2}, \tag{198}$$

which means that the minimum inflation redshift must be

$$z_{\min} = \left(\left(\frac{a_f}{a_0} \right)^{-1} \left(\frac{a_0}{a_{\mathrm{eq}}} \right)^{1/2} \right)^{\frac{-2}{1+3w_{\mathrm{inf}}}}, \tag{199}$$

so the result can be expressed in terms of temperatures:

$$\frac{a_0}{a_f} = \frac{T_f}{T_0} = \frac{T_f}{T_{\mathrm{pl}}} \frac{T_{\mathrm{pl}}}{T_0}, \tag{200}$$

where the T_{pl} is the Planck temperature, and $a_0/a_{\mathrm{eq}} = 1 + z_{\mathrm{eq}}$: in the end our result is

$$z_{\min} = \left(\frac{T_{\mathrm{pl}}}{T_0} (1 + z_{\mathrm{eq}}) \frac{T_f}{T_{\mathrm{pl}}} \right)^{-\frac{2}{1+3w_{\mathrm{inf}}}}. \tag{201}$$