Astrophysics and cosmology notes

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Contents

Introduction and relevant material

These are the (yet to be) revised notes for the course “Fundamentals of Astrophysics and Cosmology” held by professor Sabino Matarrese in fall 2019 at the university of Padua.

They are based on the notes I took during lectures, complemented with notes from the previous years.

They will be revised by the professor in the future, as of yet they have not.

The exam is a traditional oral exam, there are fixed dates but they do not matter: on an individual basis we should write an email to the professor to set a date and time.

Material There is a dropbox folder with notes by a student from the previous years [Pac18] and handwritten notes by the professor.

There are many good textbooks, for example “*Cosmology*” by Lucchin and Coles [LC02].

Chapter 1

Cosmography

1.1 The cosmological principle

Thursday 2019-10-3, compiled 2020-09-14

The basis for the modern treatment of cosmology is the **Copernican principle**: roughly stated, it is "*we do not occupy a special, atypical position in the universe*". We will discuss the validity of this later in this section. It is extremely useful to make such an assumption since it endows our model of the universe with a great deal of symmetry, which makes its mathematical treatment manageable.

As we will discuss shortly, this principle can be combined with our observations of isotropy to yield:

Proposition 1.1.1 (Cosmological principle)**.** *Every* comoving observer *observes the Universe around them at a* fixed time *in their reference frame as being* homogeneous *and* isotropic*.*

Comoving means moving coherently with the absolute reference frame, which is defined as the rest frame of the cosmic fluid, which determines the geometry of the universe.

When we observe the Cosmic Microwave Background (CMB) we see that we are surrounded by radiation distributed like a blackbody of temperature $T_{\text{CMB}} \approx 2.725\,\text{K}$ [Fix09].

This radiation is not uniformly distributed in the sky: we see a *dipole modulation* of around $\Delta T \approx 3.4\,\text{mK}$ [Col18]. This is due to the Doppler effect: the Solar System is *not* comoving with respect to the CMB.

In fact, we can measure the *peculiar velocity* of the Solar System this way: it comes out to be around $c\Delta T / T_{\text{CMB}} \approx 370\,\text{km/s}$. This cannot be explained by the movement of the Sun through the galaxy, nor by the movement of the galaxy through the Local Group: the Local group is actually moving with respect to the absolute reference. In fact, the velocities of the Sun with respect to the Local Group and the velocity of the Local Group with respect to the CMB are almost directed in opposite directions, so the velocity of the LG with respect to the CMB can be measured to be $\approx 620\,\text{km/s}$ [Col18, Table 3].

So, the absolute reference frame can be experimentally defined as the frame of the observer who sees the CMB with zero dipole moment.

The CMB has anisotropies of the order of $20\,\mu\text{K}$ (root mean square) [Wri03] at higher order multipole moments: it is uniform to approximately 1 part in 10^5.

The word "time" in **fixed time** refers to the proper time of a comoving observer, which is called *cosmic time*.

Homogeneity means that the characteristics of the universe as observed from a point are the same as they would be as observed from any other point.

Isotropy means that the characteristics of the universe as observed in a certain direction are the same as they would be if they were observed in another direction.

On the validity of the cosmological principle The principle is expected to hold only on very large[1] scales: at small scales we see structures, such as galaxies or our Solar System, so we surely do not have homogeneity.

Change citation for homogeneity length scale?

How can we talk about homogeneity if we can only look at the universe from a single point? We must *assume* that any other observer would also see isotropy as we do: this is precisely what the Copernican principle tells us.

Isotropy around every point implies homogeneity. We observe isotropy, and with the assumption that we are typical observers we obtain homogeneity.

In the end, this assumption is the basis of modern cosmology: it has to be made before any cosmological study starts: it might not be completely correct, but it allows us to make falsifiable predictions, so we shall keep it until the models it allows us to create do not match observations anymore.

1.2 The geometry of spacetime

The best description for gravity we have so far is given by the general theory of relativity (GR). In it, spacetime is modelled as a 3+1-dimensional semi-Riemannian manifold with a line element which is generally given by the expression $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$ and which.

Latin indices can take values from 0 to 3, and we adopt the "mostly minus" metric signature.

Such a spacetime can have up to $4(4+1)/2 = 10$ global continuous symmetries, which can be classified into:

(a) 1 time translation;

(b) 3 Lorentz boosts;

(c) 3 spatial translations;

(d) 3 spatial rotations.

The metric of Minkowski spacetime, which has all of these 10 symmetries, reads:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\vec{x}^2 \qquad \text{(Cartesian coordinates)} \tag{1.1a}$$

[1] Larger than $260h^{-1}$Mpc ≈ 380 Mpc $\approx 1.2 \times 10^{25}$ m [YBK10], where $h \approx 0.68$ [Col16]: this will be made clearer in later sections, but is meant to give an idea of the length scales involved . The portion of Universe we can see is of order 10 Gpc.

$$= c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \qquad \text{(spherical coordinates)}\,, \tag{1.1b}$$

where $r = |x|$, while θ and φ are the spherical angles and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$.
Minkowski space is *maximally symmetric*, which means we could not have more symmetries than these. It is not the only possibility, there are three maximally symmetric spacetimes: Minkowski, de Sitter, anti-de Sitter.

In cosmology we lose time translation symmetry, since the universe is expanding, and Lorentz boost symmetry, since as we saw in the previous section we can tell at which speed we are moving with respect to the CMB.

We keep the purely spatial symmetries: so, our description of spacetime will be as a 3+1-dimensional manifold which, if we fix the temporal coordinate for a comoving observer, reduces to a *maximally symmetric* 3-dimensional space — for 3 dimensions the maximum number of symmetries is $3(3+1)/2 = 6$, which correspond to (c) and (d).

It can be shown that the most general form of a metric satisfying these conditions is the **Friedmann-Lemaître-Robertson-Walker line element**,[2] which, in the comoving frame, reads

$$\boxed{\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right).} \tag{1.2}$$

The coordinate r does not have the dimensions of a length: we choose our variables so that $a(t)$ is a length, while r is dimensionless.[3]

The parameter $a(t)$ is called the *scale factor*: varying it amounts to rescaling all of space. It has the dimensions of a length, and it depends on the cosmic time t.

The parameter k is a constant describing the spatial curvature, which can always be normalized to ± 1 or 0. Universes with:

- $k = 1$ are called closed universes;
- $k = -1$ are called open universes;
- $k = 0$ are called flat universes.

We can choose the normalization of k, but its sign is a constant. Positive values correspond to $k = 1$, negative values to $k = -1$. When computing probability distributions for k, we must not consider it to be a discrete variable but a continuous variable instead: so, the set of flat universes with $k = 0$ has zero measure, and thus zero probability with any probability density function.

[2] This is sometimes called just "Robertson-Walker", or RW, or FLRW.

[3] This is a matter of convention: if we normalize $|k| = 1$ then we will have $0 \leq r < 1$; we could also let k be arbitrarily large; then r would also be.

1.2.1 A bidimensional example

We consider surfaces: these are the simplest manifolds which can have intrinsic curvature.

Intrinsic curvature is described by the Riemann tensor, which has 20 independent components in 4D and 6 in 3D, so it is difficult to visualize.

On the other hand, in 2D has only 1 independent component: $R_{1212} = R \det g_{ab}$, where R is the scalar curvature.

The scalar curvature R has an immediate geometric interpretation: it is equal to $2/(r_1 r_2)$, where r_i are the radii of the osculating circles at the point; it is positive if the circles are in the same direction, as they are for a sphere, and negative if they are in different directions, as they are for a hyperboloid. For a flat surface we cannot define an osculating circle (at least not in both directions): its radius diverges, so the curvature vanishes.

The metric for a Cartesian flat 2D plane is $\mathrm{d}l^2 = a^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$. The constant a is included since we want r to be dimensionless.

The metric for the surface of a sphere is: $\mathrm{d}l^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$, where $a^2 = R^2$, the square radius of the sphere.

The metric for the surface of a hyperboloid is: $\mathrm{d}l^2 = a^2\left(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\varphi^2\right)$, therefore the only difference is that trigonometric functions become hyperbolic ones.

Do note that for the sphere $\theta \in [-\pi, \pi]$ while for the hyperboloid we can in principle have $\theta \in \mathbb{R}$: this is indicative of the fact that the sphere is bounded, while the hyperboloid is not.

For both of these, let us define the variable: $r = \sin\theta$ in the spherical case, and $r = \sinh\theta$ in the hyperbolic case.

As we change variable we do the following manipulation for the sphere:

$$\mathrm{d}r^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \mathrm{d}\theta^2 = \cos^2\theta\,\mathrm{d}\theta^2 \tag{1.3a}$$

$$\implies \mathrm{d}\theta^2 = \frac{\mathrm{d}r^2}{\cos^2\theta} = \frac{\mathrm{d}r^2}{1-r^2}\,, \tag{1.3b}$$

and similarly for the hyperboloid, except for the fact that in that case $\cosh^2\theta = 1 + \sinh^2\theta = 1 + r^2$.

So, the line elements become respectively:

$$\mathrm{d}l^2{}_{\text{sphere}} = a^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\,\mathrm{d}\varphi^2\right) \tag{1.4a}$$

$$\mathrm{d}l^2{}_{\text{hyperboloid}} = a^2\left(\frac{\mathrm{d}r^2}{1+r^2} + r^2\,\mathrm{d}\varphi^2\right). \tag{1.4b}$$

We have a striking similarity to the Robertson-Walker metric: we only need to make the substitution $\mathrm{d}\varphi \to \mathrm{d}\Omega$ in order to recover it.

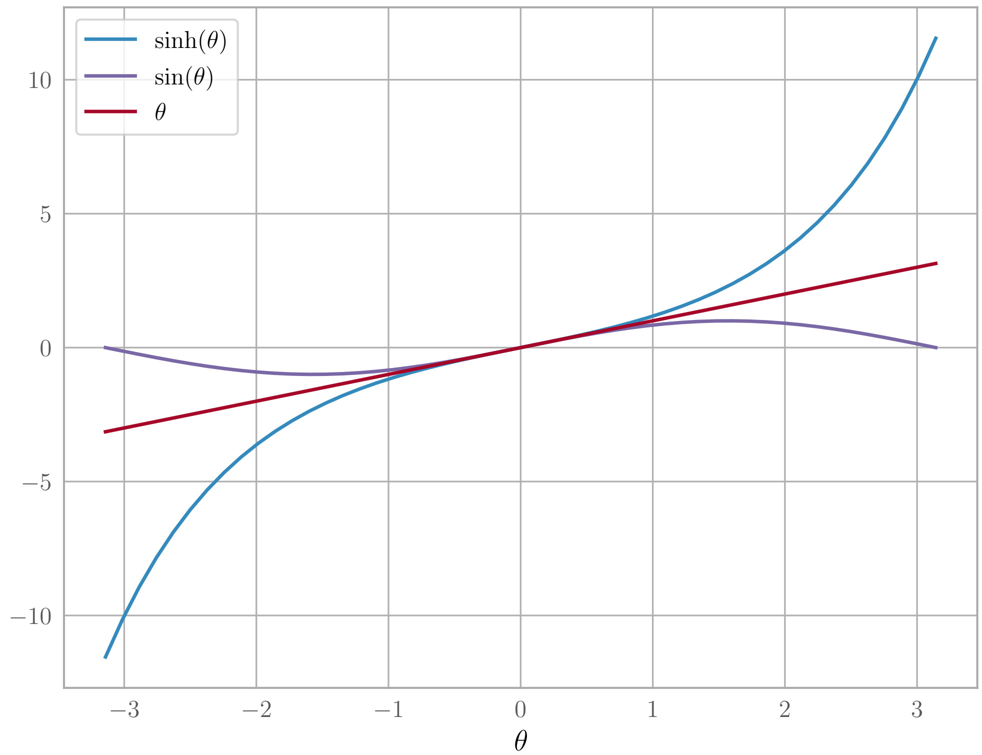

Figure 1.1: Plot of the functions $\sin(\theta)$ and $\sinh(\theta)$ in the interval $-\pi \leq \theta \leq \pi$.

We can also work backwards and rewrite the RW line element in sphere- or hyperboloid-like coordinates:

$$\mathrm{d}l^2 = c^2\,\mathrm{d}t^2 - a^2 \begin{cases} \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega^2 \\ \mathrm{d}\chi^2 + \chi^2\,\mathrm{d}\Omega^2 \\ \mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\Omega^2 \end{cases} \tag{1.5}$$

where we introduce a variable χ defined so that if $k = +1$ then $r = \sin\chi$, if $k = 0$ then $r = \chi$, and if $k = -1$ then $r = \sinh\chi$.

The properties of the sphere and of the hyperboloid actually carry over to the 3D case: a spacetime with positive curvature is bounded, while if it is flat or hyperbolic it is unbounded.

1.2.2 Other forms of the RW metric

If we wish to use Cartesian coordinates the RW metric takes the following expression:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(1 + \frac{k|x|^2}{4}\right)^{-2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{1.6}$$

Universes in which a is a constant are called *Einstein spaces*.

We can also change time variable: the *conformal time* η is such that $\mathrm{d}t = a(\eta)\,\mathrm{d}\eta$, where $a(\eta) \stackrel{\text{def}}{=} a(t(\eta))$: so, we will have

$$\mathrm{d}s^2 = a^2(\eta)\left(c^2\,\mathrm{d}\eta^2 - \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega\right)\right). \tag{1.7}$$

Clarify distinction on different types of conformal transformations and dependence on spatial curvature.

Weyl transformations are defined to be those which preserve angles locally; since angles are defined through the metric but the angle between two vectors does not change if they are rescaled, this can be translated into the condition that the metric is rescaled by a generic function:

$$g_{ab} \to a^2(x^i) g_{ab}\,. \tag{1.8}$$

If two metric are mapped into each other by a Weyl transformation, they are said to be in the same *conformal class*.

In our case, the dependence on the point in spacetime is reduced to a dependence on the cosmic time only since we have symmetry with respect to spatial translations.

The conformal time is called that because if we use it we can map our spacetime at any time to spacetime at another time using a Weyl transformation. If there is no spatial curvature, we can also map it to flat Minkowski spacetime.

The universe we inhabit does not have conformal symmetry: a generic massive particle in it has a Compton wavelength $\lambda = h/(mc)$ which defines its interaction cross section, so if the spacetime expands an ensemble of these particles will have different dynamics.

However, conformal geometry is useful for the description of particles which have no characteristic length, such as photons. Particles with no characteristic length are insensitive to dilation, since they do not have a "meter" to probe the expansion of spacetime. The photons of the CMB look like they are thermal: they were thermal in the early universe, since they were in thermal equilibrium with matter (photons and matter particles were constantly Compton-scattering off each other); when matter and radiation decoupled the photons scattered for the last time and then kept travelling. The universe has since expanded by a factor ≈ 1090, but because of the fact that the photons do not have a characteristic length their distribution can still be modelled as a blackbody distribution for an appropriately rescaled temperature. However, strictly speaking, we are not allowed to say that they are thermal, since keeping thermal equilibrium implies that interactions are occurring.

Notice that we have made no use of dynamics so far: we wrote the line element, the solution of the Einstein equations, without the Einstein equations themselves! Of course we can obtain the Robertson-Walker metric starting from the field equations as well, but here we have only based our considerations on geometrical assumptions. This approach is called **Cosmography**.

1.3 The energy budget of the universe

Up until now we did not consider any dynamics in our spacetime. We will discuss this topic in more detail in later sections, but for now we give the result: the dynamics of the universe are described by the Friedmann equations:

$$\dot{a}^2 = \frac{8\pi G_N}{3}\rho a^2 - kc^2 \tag{1.9a}$$

$$\ddot{a} = -\frac{4\pi G_N}{3} a\left(\rho + \frac{3P}{c^2}\right) \tag{1.9b}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \tag{1.9c}$$

where dots denote differentiation with respect to the *cosmic time* t, G_N is Newton's gravitational constant, $\rho = \rho(t)$ is the energy density, and $P = P(t)$ is the isotropic pressure.

The curvature k appears in the first equation: so we can try to measure it by comparing the other two terms in the equations. This way, we can determine whether the universe is flat or curved.

In order to discuss this problem, let us establish some notation: an important parameter is $H(t) \stackrel{\text{def}}{=} \dot{a}/a$, the *Hubble parameter*. We can write an equation for it from the first Friedmann one:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{1.10}$$

If $k = 0$, then there we must have a critical energy density

$$\rho_C(t) = 3H^2(t)/(8\pi G)\,, \tag{1.11}$$

and we define $\Omega(t) = \rho(t)/\rho_C(t)$. For a flat universe, $\Omega = 1$, and so we can determine $k = \operatorname{sign}(\Omega - 1)$, since:

$$1 = \frac{8\pi G}{3}\frac{\rho}{H^2} - k\frac{c^2}{a^2H^2} \tag{1.12}$$

Dividing equation (1.10) through by H^2

$$\frac{\rho}{\rho_c} - 1 = k\frac{c^2}{a^2H^2} \tag{1.13}$$

$$\operatorname{sign}(\Omega - 1) = \operatorname{sign}\left(k\frac{c^2}{a^2H^2}\right) = \operatorname{sign}(k) = k\,. \tag{1.14}$$

Took sign of both sides

This is a promising way to measure the curvature of the universe. As we will see, we can infer the densities of the various constituents of the universe through their dynamics. Notice that the measurement of the energy density is a "Newtonian" measurement, while that of the geometry of spacetime is a General Relativity one.
The alternative is trying to measure the curvature geometrically, however we should look at really large scales in order to see any effects: we are thus drawn to the CMB. Unfortunately, the geometrical effects of spatial curvature on the CMB power spectrum[4] are highest for

[4] The power spectrum is, roughly speaking, the set of the square moduli of the coefficients in the spherical-harmonics decomposition, classified according to the coefficient l; the higher l, the smaller the angular scale we are considering.

the lowest multipoles, for which we have the largest variance. This means that the direct geometric effects of curvature cannot be discerned in the CMB power spectrum, however the dynamical effects can.

Further, we define the **Hubble constant**:

$$H_0 = H(t_0) = 100h \times \mathrm{km\,s^{-1}\,Mpc^{-1}} \approx 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \tag{1.15}$$

where $h \approx 0.7$ is a number, and t_0 just means *now*.[5]

The reason for this peculiar way to write the constant is that historically it has been difficult to determine the value of H_0 precisely, and it affects many astronomical conversions: keeping it indeterminate in this way allows us to quickly update our old estimates if we measure H_0 more precisely later. Historically, in the American school, the pupils of Hubble thought $h \sim 0.5$, while the French school thought $h \sim 1$. Now, a great issue in cosmology is the disagreement between the measurements of H_0 obtained from the cosmic distance ladder and those obtained from the CMB [Won+19].

1.3.1 Energy density

So, in order to determine the spatial curvature of the universe we need to look at $\Omega \propto \rho$: so, we need to measure the energy density of the universe.

How do we do it?

Let us start by considering the energy density *today* (index 0) due to *galaxies* (index g): ρ_{0g}. We do not directly observe the mass[6] of galaxies: we can only measure their luminosities.

So, we do the following: we compute the mean value of ρ, the mass per unit volume, with the aid of the galaxy luminosity per unit volume ℓ: the mean density is given by the mean luminosity times the average ratio of mass to luminosity of galaxies:[7]

$$\langle \rho \rangle \sim \langle \ell \rangle \left\langle \frac{M}{L} \right\rangle , \tag{1.18}$$

where $\langle M/L \rangle$ is the average ratio of mass over luminosity per galaxy: we had a ratio of densities, but since we are considering averages we can integrate above and below with respect to the spatial volume.

[5] Also, pc means "parsec":

$$1\,\mathrm{pc} = \frac{1\,\mathrm{AU}}{1\,\mathrm{arcsec}} \approx 3.085 \times 10^{16}\,\mathrm{m} \approx 3.26\,\mathrm{ly}\,, \tag{1.16}$$

where the angle is to be interpreted as dimensionless (in radians), and AU is an astronomical unit, the Earth-Sun average distance. The definition is as such because of a way we have to measure the distances to nearby objects, by measuring their parallax between winter and summer: if they are close enough, they will have apparently moved with respect to further objects, because of the movement of the Earth around the Sun.

[6] Here the terms mass and energy are used equivalently: the velocity of galaxies with respect to the CMB is nonrelativistic, so we approximate $E = \gamma mc^2 \approx mc^2$.

[7] Formally, the steps are

$$\langle \rho \rangle = \left\langle \ell \frac{\rho}{\ell} \right\rangle \approx \langle \ell \rangle \left\langle \frac{\rho}{\ell} \right\rangle = \langle \ell \rangle \left\langle \frac{M}{L} \right\rangle , \tag{1.17}$$

where we made the assumption of the ratio ρ/ℓ being *uncorrelated* to the luminosity ℓ. This is not precisely verified, but we are giving order-of-magnitude estimates so this is close enough for our purposes.

It is measured in units of solar mass over solar luminosity: $M_\odot / L_\odot$. Reference values for these are $M_\odot \sim 1.99 \times 10^{33}\,\mathrm{g}$, while $L_\odot \sim 3.9 \times 10^{33}\,\mathrm{erg\,s^{-1}}$.

We denote $\langle \ell \rangle \overset{\text{def}}{=} \mathscr{L}_g$: it is the mean (intrinsic, bolometric[8]) luminosity of galaxies per unit volume.

By definition, it is given by

$$\mathscr{L}_g = \int_0^\infty L\Phi(L)\,\mathrm{d}L\,, \tag{1.19}$$

where $\Phi(L)$ is the number density of galaxies per unit volume and unit luminosity: the *luminosity function*.

Friday 2019-10-4, compiled 2020-09-14

The Schechter function is an empirical estimate for the shape of this distribution:

$$\Phi(L) = \frac{\Phi^*}{L^*}\left(\frac{L}{L^*}\right)^{-\alpha} \exp\left(-\frac{L}{L^*}\right), \tag{1.20}$$

where Φ_*, L_* and α are parameters, with dimensions of respectively a number density, a luminosity and a pure number.

These can be fit by observation: we find $\Phi^* \approx 10^{-2} h^3 \mathrm{Mpc}^{-3}$, $L^* \approx 10^{10} h^{-2} L_\odot$ (a typical galaxy contains roughly ten billion Suns) and $\alpha \approx 1$.

The integral for $\mathscr{L}_g$ converges despite the divergence of $\Phi(L)$ as $L \to 0$, since it is multiplied by L: so we do not need to really worry about the low-luminosity divergence of the distribution.

The result of the integral for a generic value of α is $\mathscr{L}_g = \Phi^* L^* \Gamma(2-\alpha)$, where Γ is the Euler gamma function; for the $\alpha = 1$ case we get a factor $\Gamma(2-1) = 1$.

Numerically, inserting reasonable estimates for the parameters, we get the following estimate for the mean luminosity $\mathscr{L}_g \approx (2.0 \pm 0.7) \times 10^8 h L_\odot \mathrm{Mpc}^{-3}$.

Now, we must estimate $\langle M/L \rangle$. The luminosity of galaxies can be measured readily, the great difficulty lies in estimating their mass.

1.3.2 Estimating the masses of galaxies

We must distinguish between the different shapes of the galaxies: spiral galaxies are characterized by rotation of the stars about the galactic center, while in elliptical galaxies the stars' motion is disordered.

Spiral galaxies

If we see a spiral galaxy edge-on, we will have a side of it coming in our direction, and the other side moving away from us (after correcting for other sources of Doppler shift, such as the velocity of the whole galaxy). So, using the Doppler effect we can measure the distribution of the velocity in the galaxy as a function of the radius.

In order to get a theoretical model, we can approximate the galaxy as a sphere: this is very rough (spiral galaxies are closer to being disk-like), but it gives the same qualitative result, so there is no need for a more precise model in this context.

[8] Bolometric means "total, over all wavelengths", as opposed to the luminosity in a certain wavelength band, which is easier to measure in astronomy.

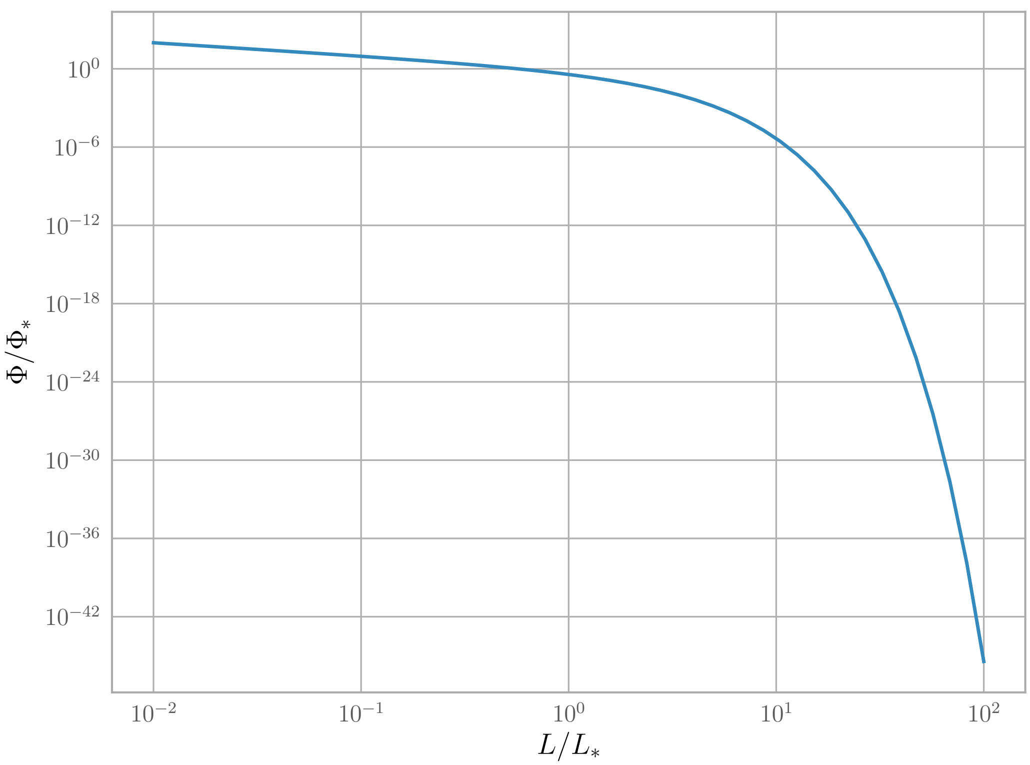

Figure 1.2: A rough plot of the Schechter function for $\alpha = 1$.

We model the galaxy velocity distribution using Newtonian mechanics: the GR corrections are negligible at these scales, galaxies are much larger than their Schwarzschild radii.

Equating the gravitational force GM/R^2 to the centripetal acceleration v^2/R we find:

$$v = \sqrt{\frac{GM}{R}}, \tag{1.21}$$

where v and M are functions of R: M is the mass contained in the spherical shell of radius R, and v is the orbital velocity at the boundary of the shell.

In the inside regions of the galaxy, where $M(R) \propto R^3$ since the density is approximately constant we will have $v \propto R$, while in the outskirts of the galaxy $M(R)$ will not change much, since all the mass is inside, so we will have $v \propto R^{-1/2}$.

Our prediction is then a roughly linear region, and then a region with $v \sim R^{-1/2}$. This is shown as the "Predicted" curve in figure 1.3.

Instead of this, when in the 1980s people started to be able to measure this curve accurately they saw that, after the linear region, $v(R)$ was approximately constant. So, is Newtonian gravity wrong?

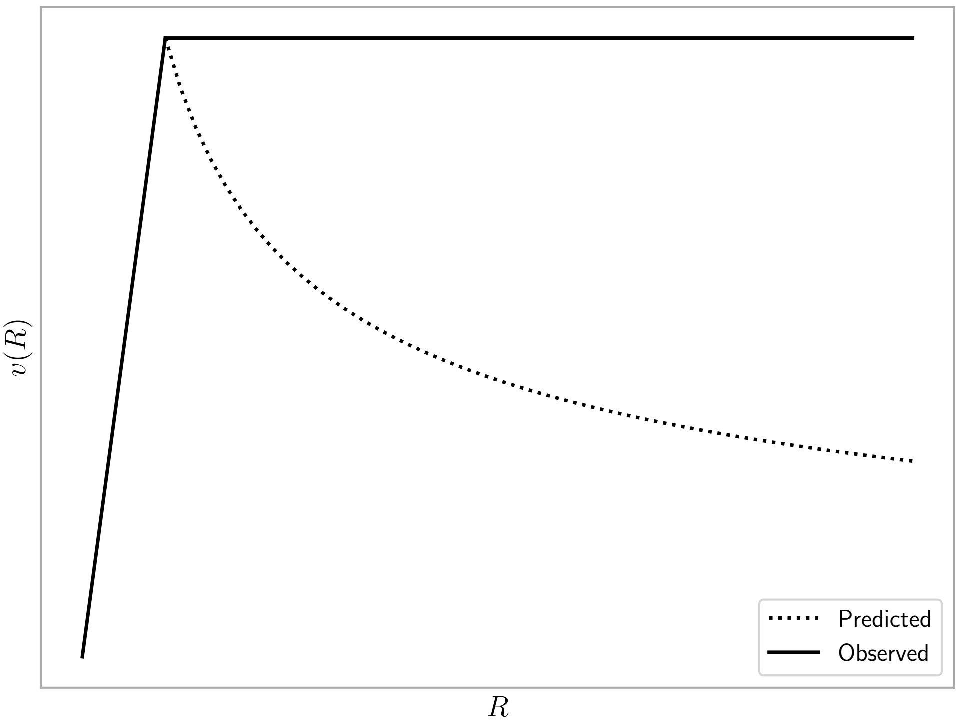

Figure 1.3: Predicted and observed velocity distribution for galaxies. The point at which they star to diverge is approximately the radius of the bulk of the galaxy. This plot is approximate, realistic models do not have such sharp corners, since there will not be a precise edge after which the density drops to zero immediately.

An option to solve this problem was proposed by Milgrom and collaborators: it is called MOdified Newtonian Dynamics, or MOND: they propose that gravity is not actually always described by a r^{-1} potential, but instead at low accelerations it behaves differently. Specifically, they posit that the gravitational acceleration g should be modulated by a factor $\mu(g/a_0)$, where $a_0 \approx 8 \times 10^{-10} h^2 \mathrm{m/s}^2$ is a characteristic acceleration while $\mu(x)$ is an dimensionless function such that $\mu \to 1$ for $x \gg 1$ and $\mu \to x$ for $x \ll 1$, such as $\mu = x/(1+x)$ [BM84].

This approach is Newtonian but there are also relativistic MOND variants. They do not match observation as well as the alternative approach does.[9] The heaviest thing weighing against MOND is the fact that even using it we still need dark matter in order to fully explain observations.

[9] MOND would be compatible with the speed of gravity being less than the speed of light, which is equivalent to the graviton being massive. Recent measurements of gravitational waves seem to agree with the general-relativistic prediction that it is massless.

Dark matter

The alternative option is that Newtonian mechanics describes galactic mechanics well, but the galaxy's matter distribution inferred from our observations is actually smaller than the real distribution, which extends outward further than the matter we can see: this is **dark matter**.

We'd need mass obeying $M(R) \propto R$ in order to have a constant value of v: since $M(R) = 4\pi \int_0^{R_{\max}} R^2 \rho(R) \, \mathrm{d}R$, we need the density profile to decay like $\rho(R) \propto R^{-2}$. This is called an *isothermal* density profile: we call it the *dark matter halo*, which surrounds all spiral galaxies.

We do not know what dark matter is: we can say that it interacts gravitationally but not electromagnetically. People tend to believe that it is made up of beyond-the-standard-model particles, like a *neutralino* or an *axion*. Historically, people thought the effect could be due to massive neutrinos; however their mass would need to be around 30 eV, and the analysis of the CMB data showed that the sum of the masses of the neutrino species must be $\sum m_\nu < 0.120\,\mathrm{eV}$ [Col18, Table 7].

The total density of matter (dark+regular) is ~ 6 times more than that of regular matter alone: this must be accounted for in our estimate of the M/L ratio for spiral galaxies (since we have additional mass but not additional luminosity): with this correction, we find that for spiral galaxies

$$\left\langle \frac{M}{L} \right\rangle \approx 300h \frac{M_\odot}{L_\odot}, \tag{1.22}$$

Historically, this was the first evidence for dark matter.

Elliptical galaxies

If galaxies are not spiral-shaped, we have to weigh them in a different way: the Doppler broadening of spectral lines gives us a measure of the root-mean-square velocity.

Later in the course we will obtain the (nonrelativistic) *virial theorem*, now we just state it: if T is the kinetic energy of a gravitationally bound system, U is the potential energy, then

$$2T + U = 0 \tag{1.23}$$

holds when the inertia tensor stabilizes, that is, when we have dynamical equilibrium.

The kinetic energy is $T = \frac{3}{2} M \left\langle v_r^2 \right\rangle$, where v_r is the radial[10] component of the velocity, which we expect to account for one third of total energy by the equipartition theorem. M is the total mass of the galaxy.

The potential energy, instead, is $U = -GM^2/R$. Substituting these expressions into the virial theorem we get

$$2 \times \frac{3}{2} M \left\langle v_r^2 \right\rangle - \frac{GM^2}{R} = 0 \tag{1.24}$$

[10] By "radial" we mean directed towards us, not towards the center of the galaxy.

$$M = \frac{3R}{G}\left\langle v_r^2 \right\rangle , \tag{1.25}$$

so if we can measure $\left\langle v_r^2 \right\rangle$ through Doppler broadening and we can give a reasonable estimate for the radius R of the galaxy we can give an estimate for M.

Global matter contributions

Accounting for the dark matter mass, we get $\langle M/L \rangle \approx 300 h M_\odot / L_\odot$.

The value of the critical energy density today, ρ_{0c}, is given by[11]

$$\rho_{0c} = \frac{3H^2}{8\pi G} \approx 1.88 \times 10^{-29} h^2 \mathrm{g}/\mathrm{cm}^3 , \tag{1.26}$$

so, in order to have $\Omega = 1$, we'd need $\langle M/L \rangle$ to be equal to:

$$\left\langle \frac{M}{L} \right\rangle = \frac{\rho_{0c}}{\mathscr{L}_g} \approx \frac{1.88 \times 10^{-29}\, h^2 \mathrm{g}/\mathrm{cm}^3}{2 \times 10^8\, h L_\odot / \mathrm{Mpc}^3} \approx 1390 h M_\odot / L_\odot . \tag{1.27}$$

We can define quantities of the form $\Omega_{0i} = \rho_{0i}/\rho_{0c}$, where i is a type of matter, such as baryonic matter, dark matter, dark energy, radiation and so on, whose density is represented as ρ_{0i}.

These variables quantify how much, at the present, time, of the cosmic energy budget is accounted for by that type of matter.

So, only $\Omega_{0b} \approx 5\%$ of the energy budget is given by baryonic matter (not all of which is visible), while around $\Omega_{0DM} \approx 27\%$ is dark matter. Together, they are just denoted as "matter", and $\Omega_{0m} \approx 30\%$.

We can ask ourselves: is dark matter actually baryonic matter which for some reason we cannot see, such as black holes or brown dwarfs?

This cannot be the case: our observations, combined with models for primordial nucleosynthesis, gives the following bounds for the baryonic energy density:

$$0.013 \leq \Omega_{0\mathrm{b}} h^2 \leq 0.025 . \tag{1.28}$$

The upper bound for Ω_{0b} is around $2.5\%/h^2 \approx 5.4\,\%$.

This would seem to indicate that $\Omega_0 \approx 0.3 \ll 1$: however we are failing to consider a crucial contribution. Consider the second Friedmann equation (1.9b): in the Newtonian limit $P \sim 0$ while $\rho > 0$, so we get $\ddot{a} < 0$: the universe contracts. This is not what is observed: we actually see it in accelerated expansion.

Dark energy

The measurements leading to this conclusion are performed by estimating the distance and redshift of far-away objects whose intrinsic luminosity is well known, called *standard candles*: the most commonly used are type Ia supernovae and Cepheid variables.

[11] Do note that the numerical figure, 1.88×10^{-29}, is approximate but its value is known to at least four significant digits, since the only source of uncertainty in it lies in the uncertainty in our measurement of G — all the uncertainty in H_0 is expressed in variable form, with the parameter h.

So, if the expansion is accelerated then $\ddot{a} > 0$: this means, again from the second Friedmann equation (1.9b), that $P < -\rho c^2/3$. This is commonly expressed by defining $w = P/\rho c^2$. In order to have accelerated expansion we need $w < -1/3$; what is observed is closer to $w \sim -1$.

This negative pressure has the effect of a *tension*, pulling the universe apart. As we will discuss in section 2.5, a candidate for a cosmological fluid with negative pressure (specifically, $w = -1$) is a cosmological constant term, called Λ, which can be inserted in the Einstein Field Equations. It is not the possible one: dark energy is defined to be what causes the expansion we see, so it could be constituted by any kind of fluid which is uniformly distributed in space and which has negative pressure.

We cannot see directly neither dark matter nor dark energy: how do we distinguish the two? Dark matter tends to cluster, while dark energy is uniformly distributed; furthermore, dark matter has negative pressure.

From observations of both the anisotropies in the CMB and the distribution of galaxies we can determine that $\Omega_\Lambda \approx 0.7$.

Radiation

We still need to compute the contribution of the energy of electromagnetic and neutrino radiation to the total energy balance. Let us start with radiation: the greatest fraction of the radiation energy density is contained in the CMB, which is extremely close to a Planckian distribution:

Add footnote

$$B(\nu, T) = \frac{2h}{c^2} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}, \tag{1.29}$$

with $T = T_{0\gamma} \approx (2.725 \pm 0.001)\,\mathrm{K}$. B is a measure of spectral intensity: it is measured in units of energy per unit second, area, solid angle and frequency. This is a well-known distribution, whose integral is given by

$$\rho_{0\gamma} = \frac{\sigma_r T_{0\gamma}^4}{c^2} = 4.6 \times 10^{-34}\,\mathrm{g\,cm^{-3}} \tag{1.30}$$

where $\sigma_r = \pi^2 k_B^4/(15\hbar^3 c^3)$, while $\sigma_{SB} = \sigma_r c/4$.

So, we have that the radiation contribution to the global energy balance is $\Omega_{0\gamma} \approx 2.5 \times 10^{-5}\,h^{-2} < 0.01\,\%$, definitely negligible.

We are going to show in later sections that if neutrinos were massless, their temperature would be $T_\nu = (4/11)^{1/3} T_\gamma < T_\gamma$.

They might not be massless, and if they are not the main contribution to the energy density they will give will be from their masses. However, recent observations (for example, by the Planck satellite) are bounding the mass of the neutrinos, $\sum m_\nu \leq 0.12\,\mathrm{eV}$: we have

$$\rho_\nu = 3N_\nu \frac{\langle m_\nu \rangle}{10\,\mathrm{eV}} 10^{-30}\,\mathrm{g\,cm^{-3}}, \tag{1.31}$$

where N_ν is the number of neutrino species. Even if we assume the upper bound, $N_\nu \langle m_\nu \rangle = 0.12\,\text{eV}$, we get $\Omega_{0\nu} < 0.5\,\%$.

Conclusions

If we add up all the contributions to $\Omega = \Omega_b + \Omega_{DM} + \Omega_\Lambda$ (neglecting, as we said, the contributions by EM radiation and neutrinos) we find experimentally $\Omega_k \overset{\text{def}}{=} 1 - \Omega \approx (5^{+38}_{-40}) \times 10^{-4}$ [Pla+19, Table A.2].

So, with the observational uncertainties we have currently we cannot determine the sign of the universe's spatial curvature: the value of Ω_k is very much compatible with 0, and even though one is drawn to say that this means that an open universe is more likely, no particular meaning should be drawn from the fact that the nominal value of Ω_k is slightly above 0.

1.3.3 The Hubble law

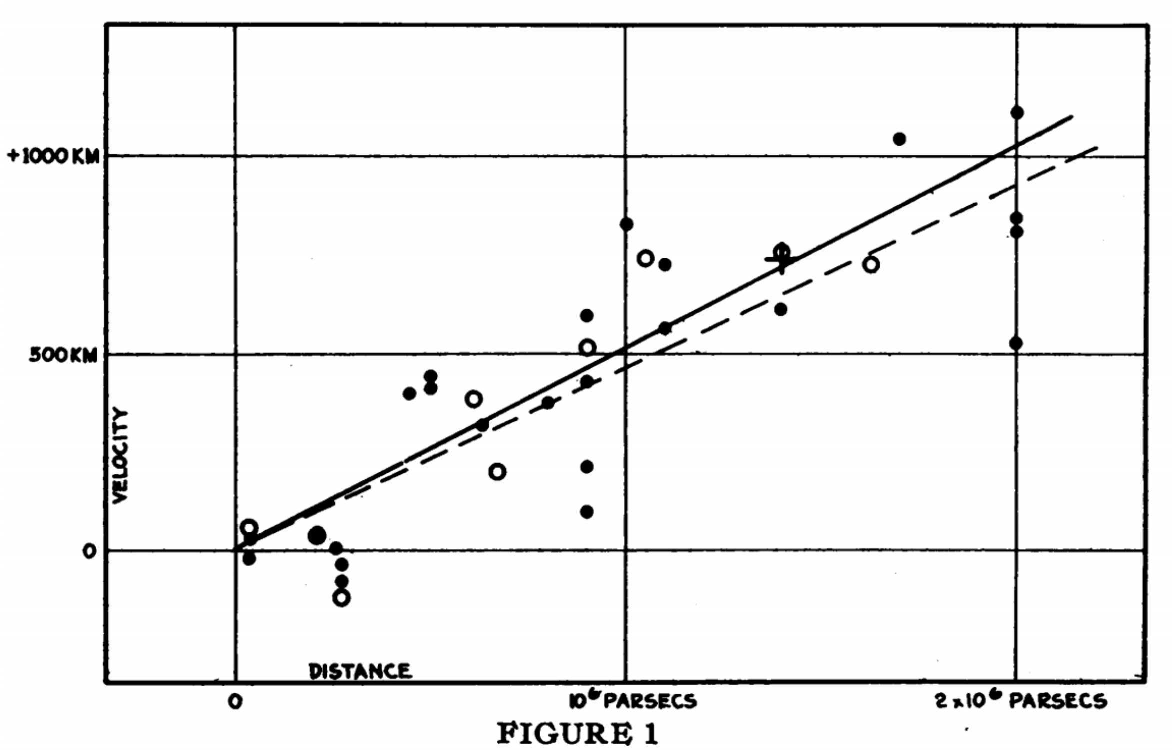

Figure 1.4: Original velocity versus distance data from Edwin Hubble's paper in 1929 [Hub29].

At the end of the 1920s Edwin Hubble compared the estimates for the distances of far-away galaxies (obtained through standard candles and other types of estimates) to their velocities relative to us, as measured through their redshift. His results are shown in figure

1.4: he obtained a roughly linear relation in the form:

$$v = H_0 d\,, \tag{1.32}$$

where v is the velocity of the galaxies, and d is their distance from us, while H_0 is a constant of proportionality. Hubble's measurements suggested $H_0 \sim 500\,\mathrm{km/s/Mpc}$, more refined modern ones using techniques similar to those used by Hubble yield a value of $(73.24 \pm 1.74)\,\mathrm{km/s/Mpc}$ [Rie+16].

Measurements of H_0 through analysis of the CMB gives an incompatible value: $H_0 = (67.8 \pm 0.9)\,\mathrm{km/s/Mpc}$ [Col16].

Let us now show that this H_0 is actually the same one we defined before, $H_0 = \dot{a}/a$. H_0 is called the **Hubble constant**, since it is constant with respect to the direction we look in the sky.

We are considering the distance connecting us (the center of our reference frame) to a distance galaxy: so we drop the angular part in the flat ($k = 0$) Friedmann-Lemaître-Robertson-Walker (FLRW) line element:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2 \tag{1.33}$$

So, at a fixed time the physical distance is given by $d = a(t)r$: therefore $v = \dot{d} = \dot{a}r = \frac{\dot{a}}{a}d = H_0 d$.

Do note that we neglected the temporal part of the metric: this is equivalent to assuming that photons travel instantaneously. Assuming the universe is spatially flat is correct up to second order: for a general value of k we have

$$d = a \int_0^r \frac{\mathrm{d}\widetilde{r}}{\sqrt{1 - k\widetilde{r}^2}} = a\Big(r + \mathcal{O}(r^3)\Big)\,, \tag{1.34}$$

since the integral gives either r, $\arcsin(r)$ or $\operatorname{arcsinh}(r)$, depending on k and all three of these equal r up to second order.

So, this is Newtonian and rough, but it gives us the correct intuitive idea. We now wish to make this reasoning more precise.

The first step, since we want to discuss our observations of light, is to define the redshift.

Definition 1.3.1 (Redshift)**.** *The redshift z of a photon is defined by*

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1 = \frac{\nu_e}{\nu_0} - 1\,, \tag{1.35}$$

where λ_0 and λ_e are the observed and emission wavelengths respectively, while ν are frequencies with the same notation.

We will show that the redshift can be found from the ratio of the scale factors now and at emission: $1 + z = a_0/a_e$. Therefore, $\nu_o/\nu_e = a_e/a_0$.

We wish to study the distribution of light from an astronomical source: in Minkowski spacetime the apparent luminosity ℓ decreases like r^{-2} if r is the distance from the object. In a generic spacetime this will not be the case: however, we can define a measure of spatial distance d_L such that $\ell = L/(4\pi d_L^2)$, where L is the intrinsic luminosity of the object.

Do note that ℓ is dimensionally a luminosity flux: it is measured in units of energy per unit time per unit area.

Definition 1.3.2. *The luminosity distance d_L is defined as:*

$$d_L = \sqrt{\frac{L}{4\pi\ell}}\,. \tag{1.36}$$

Since $L \propto \ell$, this is a well-defined measure of distance between two generic points in spacetime, regardless of the presence of a source of light there.

How do we relate the luminosity distance and the scale factor? The radiation from our source is spread on a sphere: we integrate the angular part of the FLRW metric over a sphere of fixed comoving radius r to find its area.

The metric restricted to the angular coordinates at fixed r is given by

$$\mathrm{d}s^2 = a^2r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{1.37}$$

so the area form on the surface of the sphere is

$$\mathrm{d}A = \sqrt{\det g}\,\mathrm{d}\theta \wedge \mathrm{d}\varphi\ , \tag{1.38}$$

where $\sqrt{\det g} = \sqrt{a^4r^4\sin^2\theta} = a^2r^2\sin\theta$. So,

$$A = \int_{S^2}\mathrm{d}A = a^2r^2\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\sin\theta = 4\pi a^2r^2\,, \tag{1.39}$$

where we substituted the wedge product for the regular tensor product of the differentials since our axes are orthogonal. Now, at which time do we compute the scale factor? We are measuring the flux at the surface of the sphere, at the time at which we are observing: therefore we need to compute it at observation time. This gives us $A = 4\pi r^2a_0^2$.

Now, the emitted luminosity is in the form:

$$L = \frac{\mathrm{d}N}{\mathrm{d}t_e}\left\langle h\nu_e\right\rangle\ , \tag{1.40}$$

where $\mathrm{d}N/\mathrm{d}t$ is the number of photons emitted per unit time, whose average energy is $\langle h\nu\rangle$. From the point of view of the observer, the number of photons is the same, while the frequency of the observed photon and the time interval $\mathrm{d}t_e$ change: specifically, $\nu_e = \nu_0 a_0/a_e$ and $\mathrm{d}t_e = \mathrm{d}t_0\,a_e/a_0$.[12] Therefore, the observed absolute luminosity obeys the relation $L = L_0(a_0/a_e)^2$.

So, putting everything together we get:

$$\ell = \frac{L_0}{A} = \frac{L}{4\pi r^2a_0^2}\left(\frac{a_e}{a_0}\right)^2, \tag{1.41}$$

note that the value of k does not enter into the equation.

Therefore:

$$d_L = \sqrt{\frac{L}{4\pi\ell}} = \sqrt{\frac{4\pi r^2a_0^2}{4\pi}\left(\frac{a_0}{a_e}\right)^2} = \frac{a_0^2}{a_e}r = a_0(1+z)r\,. \tag{1.42}$$

Another way of defining a distance is the one we get by directly integrating the radial part of the line element: this way, we are effectively using a space-like measuring stick, working at fixed cosmic time and fixing both of the angles θ and φ. Thursday 2019-10-10, compiled 2020-09-14

Definition 1.3.3 (Proper distance)**.** *The proper distance d_P at a fixed cosmic time t to an object at a comoving radial coordinate r is given by:*

$$d_P = a(t) \int_0^r \frac{\mathrm{d}\widetilde{r}}{\sqrt{1 - k\widetilde{r}^2}} \,. \tag{1.43}$$

A derivation of the Hubble law

We want to derive the Hubble law ($v = H_0 d$) mathematically. It can also be restated as $cz = H_0 d$: the observed velocity of recession of the objects is measured through redshift, which is described (in the nonrelativistic limit[13]) by the formula

$$\lambda_0 = \lambda_e \left(1 \pm \frac{v}{c} \right) , \tag{1.45}$$

where the sign in the $\pm$ is a plus if the object is receding from us. We also defined z through $\lambda_0 / \lambda_e = 1 + z$: this means that we can identify $z = v/c$.

Now, we are going to "move away from the current epoch by Taylor expanding": the scale factor at a time t, $a(t)$, can be written as

$$a(t) = a_0 + \dot{a}_0 (t - t_0) + \frac{1}{2} \ddot{a}_0 (t - t_0)^2 + \mathcal{O}\left(\left| t - t_0 \right|^3 \right) \tag{1.46a}$$

We drop the error term

$$\approx a_0 \left(1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 \right) , \tag{1.46b}$$

Substituted $H_0 = \dot{a}_0 / a_0$.

where $q_0 \overset{\text{def}}{=} -\ddot{a}_0 a_0 / (\dot{a}_0)^2$ is called the *deceleration parameter* by historical reasons: people thought they would see deceleration $\ddot{a} < 0$ when first writing this, so a positive q_0, but the deceleration parameter is instead measured to be negative.

Now, $1 + z = a_0 / a$ can be expressed as:

$$1 + z \simeq \left(1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 \right)^{-1} . \tag{1.47}$$

[12] This is the case even though the temporal component of the metric does not change, since we are not considering a fixed instance of cosmic time (which is unphysical): instead, we are considering the time intervals that the emitter and observed measure between the crests of a light wave sent from one to the other.

[13] The relativistic expression is

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \gtrsim 1 + \frac{v}{c} \,. \tag{1.44}$$

This will not be relevant for our discussion, however, since before the special-relativistic corrections become relevant we will need to use general-relativistic corrections: roughly speaking, the effects of spacetime expanding while the photon travels from its source to us.

Do note that this is derived starting with a formula which is correct up to second order in the time interval $\Delta t = t - t_0$: when expanding we cannot trust terms of order higher than second. Expanding with this in mind we get:[14]

$$1 + z \simeq 1 - H_0 \Delta t + \frac{q_0}{2} H_0^2 \Delta t^2 + H_0^2 \Delta t^2 \,, \tag{1.52}$$

therefore

$$\begin{aligned} z &= H_0 (t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t)^2 & (1.53) \\ &= (t_0 - t) \left[H_0 + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t) z \right]. & (1.54) \end{aligned}$$

Changed the time intervals to $t_0 - t = -\Delta t$, the square of which is the same as before.

Bringing the bracket to the other side we get

$$\begin{aligned} t_0 - t &= z \left[H_0 + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t) \right]^{-1} & (1.55) \\ &= z \left[H_0 + \left(1 + \frac{q_0}{2}\right) H_0 z \right]^{-1} & (1.56) \\ &= \frac{z}{H_0} \left[1 + \left(1 + \frac{q_0}{2}\right) z \right]^{-1} , & (1.57) \end{aligned}$$

where we substituted the first order expression $t_0 - t = z / H_0$: we are allowed to make this substitution since the expression is multiplied by z, which has the same asymptotic order as $t_0 - t$ (since H_0 is finite): working to first order inside the brackets is equivalent to working to second order in the global expression.

By the same reasoning, we can expand the inverse bracket to first order:

$$t_0 - t = \frac{z}{H_0} \left[1 - \left(1 + \frac{q_0}{2} z\right) \right] = \frac{z}{H_0} - \left(1 + \frac{q_0}{2}\right) \frac{z^2}{H_0} \,, \tag{1.58}$$

[14] We need to compute the first and second derivatives of

$$\left(1 + H_0 \Delta t - \frac{q_0}{2} H_0^2 \Delta t\right)^{-1} = (1 + a\Delta t + b\Delta t)^{-1} = f(\Delta t) \ : \tag{1.48}$$

they are

$$\begin{aligned} \frac{\mathrm{d} f}{\mathrm{d}\Delta t} &= -(1 + a\Delta t + b\Delta t)^{-2} (a + 2b\Delta t) & (1.49) \\ \frac{\mathrm{d}^2 f}{\mathrm{d}\Delta t^2} &= 2(a + 2b\Delta t)^2 (1 + a\Delta t + b\Delta t) - (1 + a\Delta t + b\Delta t)^{-2} (2b) \,, & (1.50) \end{aligned}$$

so we have

$$\left.\frac{\mathrm{d} f}{\mathrm{d}\Delta t}\right|_{\Delta t = 0} = x \qquad \text{and} \qquad \frac{1}{2!} \left.\frac{\mathrm{d}^2 d}{\mathrm{d}\Delta t^2}\right|_{\Delta t = 0} = \frac{1}{2!} \left(2a^2 - 2b\right) = a^2 - b \,. \tag{1.51}$$

We would like the time interval to disappear: we want a distance, not a time, so we should seek an expression for r instead of Δt. We are observing photons, for which $\mathrm{d}s^2 = 0$, which is equivalent $c^2\,\mathrm{d}t^2 = a^2(t)\,\mathrm{d}r^2\,/(1 - kr^2)$. Taking a square root and integrating we get:

$$\int_t^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \pm \int_r^0 \frac{\mathrm{d}\widetilde{r}}{\sqrt{1 - k\widetilde{r}^2}}\,, \tag{1.59}$$

where we should select the negative sign since we want positive quantities on both sides. The other choice would correspond to the photon being emitted from the Earth and received at the comoving radius of the source.

The integral on the right hand side can be solved analytically: it is

$$-\int_r^0 \frac{\mathrm{d}\widetilde{r}}{\sqrt{1 - k\widetilde{r}^2}} = \begin{cases} \arcsin r = r + \mathcal{O}(r^3) & k = 1 \\ r & k = 0 \\ \operatorname{arcsinh} r = r + \mathcal{O}(r^3) & k = -1 \end{cases} \tag{1.60}$$

in all cases, it is just r up to *second* order (since the next term in the expansion of an arcsine or hyperbolic arcsine is of third order).

On the left hand side, we can substitute in $a(t)$ from equation (1.46b):

$$\int_t^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \frac{c}{a_0} \int_t^{t_0} \mathrm{d}\widetilde{t} \left[1 + H_0(\widetilde{t} - t_0) + \mathcal{O}(\Delta \widetilde{t}^2)\right]^{-1} \tag{1.61}$$

$$= \frac{c}{a_0} \left[t_0 - t + \frac{H_0}{2}(t - t_0)^2\right] + \mathcal{O}(\Delta t^3) \tag{1.62}$$

where we used the expression for the scale factor to first order only since the integration raised the order of the estimate by one. we have:

$$\frac{c}{a_0}\left((t_0 - t) + \frac{1}{2}H_0(t_0 - t)^2 + \mathcal{O}(\Delta t^3)\right) = r\,, \tag{1.63}$$

since the term proportional to q_0 only gives a third order contribution. We can now substitute the expression for the time difference with respect to the redshift (1.58), only computed to second order:

$$r = \frac{c}{a_0}\left[\frac{z}{H_0}\left(1 - \left(1 + \frac{q_0}{2}\right)z\right) + \frac{H_0}{2}\left(\frac{z}{H_0}\left(1 - \left(1 + \frac{q_0}{2}\right)z\right)\right)^2\right] \tag{1.64}$$

$$= \frac{c}{a_0}\left[\frac{z}{H_0} - \left(1 + \frac{q_0}{2}\right)\frac{z^2}{H_0} + \frac{H_0}{2}\frac{z^2}{H_0^2}\right] \tag{1.65}$$

Ignored the third and higher order terms in the square.

$$= \frac{c}{a_0 H_0}\left[z - \frac{1}{2}(1 + q_0)z\right]. \tag{1.66}$$

Now, we can insert this expression for r into the formula for the luminosity distance (1.42):

$$d_L = a_0^2 \frac{r}{a} = a_0(1 + z)r \tag{1.67}$$

$$= a_0(1+z)\frac{c}{a_0 H_0}\left(z - \frac{1}{2}(1+q_0)z^2\right). \tag{1.68}$$

As we expect, the term a_0 disappears: it is a bookkeeping parameter, the physical properties of a universe described by a FLRW metric are invariant under a global rescaling of the scale factor.

Our expression also contains cubic terms in z: removing these to get back to second order we find

$$d_L = \frac{cz}{H_0}(1+z)\left(1 - \frac{1}{2}(1+q_0)z\right) \tag{1.69}$$

$$= \frac{cz}{H_0}\left(1 + \left(1 - \frac{1}{2} - \frac{1}{2}q_0\right)z\right) \tag{1.70}$$

$$= \frac{cz}{H_0}\left(1 + \frac{1}{2}(1-q_0)z\right). \tag{1.71}$$

We can turn this into a relation for cz in terms of d_L by substituting in the first order expression $d_L H_0 = cz$ into the second order term:

$$cz = d_L H_0 + \frac{q_0 - 1}{2} cz^2 \tag{1.72}$$

$$cz = H_0\left(d_L + \frac{1}{2}(q_0 - 1)\frac{H_0}{c} d_L^2\right), \tag{1.73}$$

and we can notice that the relation is approximately linear and independent of acceleration for low redshift, but we can detect the acceleration at higher redshift. Typically we need to measure galaxies at least 10 Mpc away in order to detect these second order effects. As we mentioned in the beginning, the data show the parameter q_0 to be negative.

This effect is similar to a Doppler effect, but the analogy is not perfect: the redshift is caused by the expansion of space itself, and the apparent velocities of the galaxies at high redshift would be superluminal.

Interpretation of superluminal recession velocities

What follows is a synthesis of the enlightening article by Davis and Lineweaver [DL04], which should be referred to for clarification.

The proper way to define velocities is directly through the metric: so, we will need to differentiate the relation $d = a\chi$, where χ is the distance in comoving coordinates and d is the physical distance, with respect to the cosmic time. This yields $\dot{d} = \dot{a}\chi = Hd$. Note that we are assuming that the object is stationary with respect to the comoving coordinates ($\dot{\chi} = 0$), so we are ignoring what is called the *peculiar velocity*.

This $\dot{d}$ is precisely the recession velocity, and this definition can be extended to any redshift. As written it is quite implicit, since we do not know what the distance in comoving coordinates is to an object we observe with redshift z. This will be treated in section 2.4, but for now the result is (see also the aforementioned paper [DL04, eq. 1], and an older paper by Harrison [Har93, eq. 13]) that the velocity now of an object observed with redshift z is

$$v_{\text{rec}} = H_0 d_C(z) = c H_0 \int_0^z \frac{\mathrm{d}\widetilde{z}}{H(\widetilde{z})}\,, \tag{1.74}$$

which can be greater than c: doing the computation allows us to see that $v > c$ for $z \gtrsim 1.5$.[15]

So, does this result contradict General, or even Special Relativity? It does not! In General Relativity we cannot directly compare velocities of objects which are far away from each other: they lie in different vector spaces, and there can be no local inertial frame extending that far. So, it is consistent to have superluminal recession velocities, while observers near the emission, as well as observers near Earth, always measure velocities locally to be $\leq c$.

So, now we can clear some common misconceptions:

1. recession velocities can indeed exceed the speed of light;
2. they can do so in periods of "regular" expansion of the universe: we did not consider inflation, which surely did not occur for $z \sim 1.5$, which is the point at which the recession velocities start being superluminal;
3. we can indeed see the light from objects which are currently receding superluminally: the formula for the comoving distance is derived by integrating a photon's trajectory.

Definition 1.3.4 (Angular diameter distance)**.** *The angular diameter distance is defined as the ratio of the object's physical transverse size L to its angular size in radians $\Delta\theta$:*

$$d_A = \frac{L}{\Delta\theta} = a(t) r\,, \tag{1.75}$$

which is peculiar in that it is not monotonic in z [Hog00]: at $z \gtrsim 1$ it starts decreasing.

Redshift-scale factor relation

Let us prove the statement from before, $\lambda_0/\lambda = a_0/a$: photons are emitted with a certain wavelength λ_e, at a comoving radius r from us, and detected at λ_o.

The line element for the photon is $\mathrm{d}s^2 = 0$, therefore $c\,\mathrm{d}t\,/a(t) = \pm\,\mathrm{d}r\,/\sqrt{1-kr^2}$.

As before, we can integrate this relation from the emission to the absorption: we call it $f(r)$ (it can be any of the functions shown in equation (1.60)):

$$\int_t^{t_0} = \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})} = f(r) \tag{1.76}$$

If we map $t \to t + \delta t$ and $t_0 \to \delta t_0$ in the integration limits, the integral must be constant since it only depends on r — do note that all the expansion of the universe is accounted for by the increasing scale factor, objects are stationary with respect to the comoving radial

[15] An interesting (although meaningless) fact: the current recession velocity of the CMB ($z \approx 1090$) is around $3.14c \approx \pi c$, according to the latest Planck data [Col16]. For more details on how the recession velocity scales with the redshift, see figure 2 in Davis & Lineweaver [DL04].

coordinate. We are computing the integral for two successive wavefront of the light. Then, we equate the two:

$$f(r) = \int_t^{t_0} \frac{c\,\mathrm{d}\tilde{t}}{a(\tilde{t})} = \int_{t+\delta t}^{t_0+\delta t_0} \frac{c\,\mathrm{d}\tilde{t}}{a(\tilde{t})}\,, \tag{1.77}$$

which we can split into:

$$\left[\int_{t+\delta t}^{t} + \int_t^{t_0} + \int_{t_0}^{t_0+\delta t_0} - \int_t^{t_0}\right] \frac{c\,\mathrm{d}\tilde{t}}{a(\tilde{t})} = 0\,, \tag{1.78}$$

where, since all the integrals have the same argument, we collect it at the end for clarity. We simplify the original integral and swap the integration limits to get:

$$\int_t^{t+\delta t} \frac{c\,\mathrm{d}\tilde{t}}{a(\tilde{t})} = \int_{t_0}^{t_0+\delta t_0} \frac{c\,\mathrm{d}\tilde{t}}{a(\tilde{t})}\,, \tag{1.79}$$

which can we approximate by

$$\frac{c\delta t}{a(t)} = \frac{c\delta t_0}{a(t_0)}\,, \tag{1.80}$$

since the periods of the photons we are considering are generally much smaller than the cosmic timescales.

Since the frequency of the emitted and observed photons must be proportional to the inverse of the time intervals δt or δt_0, we have

$$\nu_e a(t_e) = \nu_o a(t_o)\,, \tag{1.81}$$

therefore

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a_0}{a}\,. \tag{1.82}$$

Chapter 2

Friedmann models

The Friedmann equations describe the dynamical evolution of the universe, as opposed to the static description given by the FLRW metric.

2.1 A Newtonian derivation of the Friedmann equations

The Friedmann equations are derived starting from the Einstein Field Equations for the FLRW metric, however we can derive them using an almost purely Newtonian argument.

We will not be able to recover the full equations, since a Newtonian fluid's pressure is $P \ll \rho c^2$: its contribution to the stress-energy-momentum tensor is negligible [TM20, eqs. 441–443]. So, through our argument we will recover the equations we would have with $P = 0$.

The only non-Newtonian step in our derivation is the justification of the Newtonian approximation: we wish to make use of the theorem attributed to Birkoff, but first derived by the Norwegian physicist J.T. Jebsen [JR05].

Proposition 2.1.1 (Jebsen-Birkhoff)**.** *The only solution to the vacuum Einstein Field Equations which is spherically symmetric is given by the Schwarzschild metric [MTW73, sec. 32.2]:*[1]

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \, dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} - r^2 \, d\Omega^2 \, . \tag{2.1}$$

With this in mind, let us take a spacetime with uniform density ρ. We consider a sphere, and imagine taking all the mass inside the sphere away.

By the Jebsen-Birkhoff theorem, the internal geometry of this shell is only determined by the mass distribution inside the shell: since we took all the mass inside the sphere away, the inside spacetime is Minkowski, that is, Schwarzschild spacetime with $M = 0$.[2]

[1] This is not relevant for what we will discuss here, but do note that there is no requirement of the generating (internal) mass distribution to be static: this is, in fact, the reason why spherically symmetric collapsing or pulsating stars cannot emit gravitational waves.

[2] There is actually some nuance to this: as is expounded upon in an article by Zhuang and Yi [ZY12], in general relativity the geometry inside the shell is actually influenced by the presence of the shell, in that its time coordinate is different to the one which would be measured by an outside observer. This is relevant, for

We describe this system through a mass coordinate: $M(\ell)$ is the mass enclosed by a spherical shell of radius ℓ.

The mass taken away will be $M(\ell) = \frac{4\pi}{3}\rho\ell^3$, where $\vec{l} = a(t)\vec{r}$ is the radius of the sphere, whose norm is $\ell = \left|\vec{l}\right|$, and ρ, as mentioned before, is the constant density.

We suppose that the gravitational field is *weak*: this is quantitatively expressed using the relation $\ell \ll r_g$, where r_g is the Schwarzschild radius of the system: this relation becomes

$$\frac{GM(\ell)}{\ell c^2} \ll 1\,, \tag{2.2}$$

and is a necessary assumption in order to apply a Newtonian approximation.

Now, we "put back" the mass which was removed, in order to restore the initial situation.

We put a test mass on the surface of the sphere. What is the motion of the mass due to the gravitational field from the center? It will surely be radial, and since as we said we are in the Newtonian approximation we can calculate it using Newton's equation:

$$\ddot{\ell} = -\frac{GM(\ell)}{\ell^2} = -\frac{4\pi G}{3}\rho\ell\,. \tag{2.3}$$

This seems to give us a net force even though we expect everything to be stationary because of isotropy. This is because we are not working in comoving coordinates: the radius of the sphere, ℓ, can change with the scale factor, even when the comoving vector $\vec{r}$ is constant.

Our final result will not depend on the unit vector we choose.

Plugging in $\ell = ar$ we find

$$\ddot{a}r = -\frac{4\pi G}{3}\rho a r \tag{2.4}$$

$$\ddot{a} = -\frac{4\pi G}{3}\rho a\,. \tag{2.5}$$

This is the second Friedmann equation, (1.9b), without the pressure term for the reasons mentioned before.

Starting from the acceleration equation (2.3) we can get

$$\dot{\ell}\ddot{\ell} = -\frac{GM}{\ell^2}\dot{\ell}\,, \tag{2.6}$$

Multiplied by $\dot{\ell}$

and identifying $2\dot{\ell}\ddot{\ell} = \mathrm{d}(\dot{\ell})^2 \big/ \mathrm{d}t$ we find the conservation of energy equation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\ell}\right)^2 = -\frac{GM\dot{\ell}}{\ell^2} \tag{2.7}$$

$$\frac{1}{2}\left(\dot{\ell}\right)^2 = -GM\int \frac{1}{\ell^2}\frac{\mathrm{d}\ell}{\mathrm{d}t}\,\mathrm{d}t \tag{2.8}$$

We integrate on both sides in $\mathrm{d}t$

instance, if we wish to compute the Shapiro delay of a ray of light passing through the shell and coming back out. This is not an issue for us, since the geometry on the inside is locally indistinguishable from a pure vacuum Minkowski spacetime if we measure only inside the shell.

$$\frac{1}{2}\dot{\ell}^2 = \frac{GM}{\ell} + C = \frac{4\pi}{3} G\rho\ell^2 + C\,, \tag{2.9}$$

Used $M = 4\pi\rho\ell^3/3$.

where C is an arbitrary constant, which we can express in terms of the scale factor:

$$\dot{a}^2 r^2 = \frac{8\pi G}{3}\rho a^2 r^2 + C \tag{2.10}$$

or, removing the r^2 term, which is a constant (since it is a comoving radius, with respect to which objects are stationary),

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{C}{r^2}\,. \tag{2.11}$$

Now, the dimensions of this constant are those of a speed, therefore we can express it as $C/r^2 = -k_N = -kc^2$, where k is dimensionless — we are allowed to do this since k_N, the Newtonian curvature constant, has the dimensions of an energy per unit mass, or equivalently a velocity squared.

This clarifies the statement that the magnitude of k is arbitrary: we get it by dividing by the dimensionless comoving radius, whose magnitude is indeed arbitrary, therefore we can normalize it however we wish: so we choose $|k| = 1$ or 0.

The equation we found is of the form:

$$E_{\text{kin}} + E_{\text{grav}} = -kc^2\,, \tag{2.12}$$

where E_{kin} and E_{grav} are the energies per unit mass in the form of either kinetic or gravitational energy. So we can directly see that k, in a sense, describes the *intrinsic* energy of a free, stationary (with respect to the comoving coordinates) test mass.

If it is positive ($k < 0$) then the particles have an intrinsic positive energy, which causes expansion, while if it is negative ($k > 0$) it is as if they were intrinsically gravitationally bound, which causes contraction.

Friday 2019-10-11, compiled 2020-09-14

We can also recover the third Friedmann equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \approx -3\frac{\dot{a}}{a}\rho\,, \tag{2.13}$$

where the approximation as before is the Newtonian one, $P \ll \rho c^2$. In this case, however, we will be able to also recover the nonrelativistic pressure term if we account for conservation of energy and not just mass.

When deriving these equations relativistically the third one comes from the conservation of the temporal component of the stress-energy tensor ($\nabla_\mu T^{\mu 0} = 0$), i.e. the "conservation of energy",[3] so to find it in our Newtonian calculation we will need to consider the nonrelativistic equivalent of that equation, which is the first law of thermodynamics.

[3] This is actually *not* a conservation law as stated, it is not covariant: in order for it to be we would need to project it along a temporal Killing vector, which does not exist in cosmology. However, if we did have a temporal Killing vector ξ_μ the projection $\xi_\nu \nabla_\mu T^{\mu\nu} = 0$ would indeed be the conservation of energy.

Indeed, it is more correct to say that this equation is not a conservation law at all, but is instead just an expression of the geometric Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$, which are written in terms of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$; if the Einstein equations hold then $G_{\mu\nu}$ is proportional to $T_{\mu\nu}$, therefore the two statements are equivalent.

We will consider *ideal fluids*. The first law, assuming adiabaticity — which must be present, since net heat transfer in the universe would violate isotropy — states:

$$\mathrm{d}E + P\,\mathrm{d}V = 0\,. \tag{2.14}$$

We can write the total energy as the product of the energy density times the volume: $E = \frac{4\pi}{3}\rho c^2 a^3$, since the volume is $V = \frac{4\pi}{3}a^3$.

So, the first law reads:

$$0 = \frac{4\pi}{3}\left[\mathrm{d}\left(\rho c^2 a^3\right) + P\,\mathrm{d}(a^3)\right] \tag{2.15}$$

$$= c^2\rho\,\mathrm{d}(a^3) + c^2 a^3\,\mathrm{d}\rho + P\,\mathrm{d}(a^3) \tag{2.16}$$

Divided through by $4\pi/3$

$$= 3\left(\rho + \frac{P}{c^2}\right)\frac{\mathrm{d}a}{a} + \mathrm{d}\rho\,, \tag{2.17}$$

Divided through by c^2a^3, collected terms.

which is the third Friedmann equation, (1.9c): the only manipulation left to do is to apply the differentials, which are covectors, to the temporal vector $\mathrm{d}/\mathrm{d}t$ in order to turn them into time derivatives.

Why were we able to recover the relativistic term this time? The completely non-relativistic approach to this would be to write $M = \frac{4\pi}{3}\rho a^3$, and to write down the equation for the conservation of mass alone. Indeed, this would yield the Friedmann equation without the P term.

The three Friedman equations are not independent: for example, the second one (1.9b) can be derived from the first and third.

This means that we can derive the full relativistic equations in this Newtonian context, using the derivations we have shown for the first and third equation, and then combining these to find the second.

Let us do this derivation explicitly: we differentiate the first Friedmann equation

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2 \tag{2.18}$$

with respect to time to find

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\dot{\rho}a^2 + \frac{16\pi G}{3}\rho\dot{a}a \tag{2.19}$$

We then substitute in the expression we have for $\dot{\rho}$ from the third equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right), \tag{2.20}$$

which gives us

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left[-3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)\right]a^2 + \frac{16\pi G}{3}\rho\dot{a}a \tag{2.21}$$

$$\ddot{a} = -4\pi G\left(\rho + \frac{P}{c^2}\right)a + \frac{8\pi G}{3}\rho a \tag{2.22}$$

Dividing through by $2\dot{a}$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\frac{3}{2}\left(\rho + \frac{P}{c^2}\right) - \rho\right] \tag{2.23}$$

Dividing through by a

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right), \tag{2.24}$$

which is precisely equation (1.9b).

2.2 The equation of state

So, the equation system is underdetermined: we do not in fact have three independent Friedmann equations, but just two. The variables we want to find, however, are three: $P(t)$, $\rho(t)$, $a(t)$.

So, we have to make an assumption: we will assume our fluid is a *barotropic* perfect fluid, that is, one for which the pressure only depends on the density: $P \stackrel{!}{=} P(\rho)$.

Very often this equation of state will be linear: $P = w\rho c^2$, with a dimensionless constant w.[4] We will assume this relation to be true.

2.2.1 Common equations of state

A thing we will compute for the different equations of state is the adiabatic[5] speed of sound:

$$c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\mathrm{d}P}{\mathrm{d}\rho} = wc^2\,. \tag{2.25}$$

We also will be able to tell what the evolution of the energy density is for a varying scale factor: this can be derived from the third Friedmann equation (1.9c): in this case it reads

$$\frac{\dot{\rho}}{\rho} + 3\frac{P}{\rho c^2}\frac{\dot{a}}{a} + 3\frac{\dot{a}}{a} = 0 \tag{2.26}$$

$$\frac{\dot{\rho}}{\rho} + 3(1+w)\frac{\dot{a}}{a} = 0 \tag{2.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\rho a^{3(1+w)}\right) = 0 \implies \rho a^{3(1+w)} = \text{const}\,, \tag{2.28}$$

so $\rho \propto a^{-3(1+w)}$.

1. $w = 0$ is equivalent to $P \equiv 0$: this is what we get in the nonrelativistic limit, for $P \ll \rho c^2$, since there is no pressure this can be interpreted as a *dust*. In this case $\rho \propto a^{-3}$; also, we have $c_s^2 \ll c^2$.

[4] This is a latin w, not a greek ω: students historically call it "omega" for some reason.

[5] The speed of sound is usually computed for adiabatic transformations, since the transmission of sound is usually close to an adiabatic process. In our case, adiabaticity is embedded in the hypotheses made in the derivation of the Friedmann equations. So, we can calculate the derivative without worrying about the adiabaticity condition being respected since for the solutions we will consider it always will be.

2. $w = 1/3$ is what we get if we seek the pressure of radiation.[6] In this case we have $c_s = c/\sqrt{3}$, while the energy density goes like $\rho \propto a^{-4}$, since we get a factor a^{-3} from the volume expansion and another a^{-1} from the decrease of the energy of each photon due to redshift. Alternatively, from what was derived before we can see that the exponent in the powerlaw must be $-3(1+1/3) = -4$.

   So, for a radiation-dominated universe the total energy $E \propto \rho a^3 \propto a^{-1}$ is not conserved.

3. $w = 1$ is called *stiff matter*: it has $P = \rho c^2$ and $c_s = c$. This is an incompressible fluid: it is so difficult to set this matter in motion that once one does it travels at the speed of light. Now, $\rho \propto P \propto a^{-6}$.

4. $w = -1$ means that $P = -\rho c^2$. We cannot compute a speed of sound (it would be imaginary). Now ρ and p are constants, since they are proportional to a^0. This is the case of dark energy: we will show in section 2.5 that the effect of inserting a cosmological constant Λ into the Einstein equations has precisely this effect.[7]

So, we replace the third Friedmann Equation with $w = \text{const}$ and

$$\rho(t) = \rho_*\big(a(t)/a_*\big)^{-3(1+w)}, \tag{2.29}$$

where a_* and ρ_* are the scale factor and density at some chosen time.

If we substitute this expression into the second FE we get that gravity is attractive ($\ddot{a} < 0$) if and only if $w > -1/3$.

Throughout this section we worked as if we had a single type of cosmic fluid in the universe: this is not really the case, we have many of them, and they will be interacting, but it is a good first approximation to consider them as separate.

In this plot, a can be interpreted as the time, since their relation is monotonic. We could insert the spatial curvature k in the plot: it decreases, but slower than matter, since it appears in the first Friedmann equation with an exponent a^{-2}. We can find an effective $\rho(a)$ law for the curvature by defining an effective ρ_k for curvature with $H^2 = \frac{8\pi G}{3}(\rho + \rho_k)$, which implies $\rho_k = -3kc^2/(8\pi G a^2)$.

We can also express this in units of the critical energy density $\rho_c = 3H^2/(8\pi G)$: we find

$$\frac{\rho_k}{\rho_c} = \Omega_k = -\frac{kc^2}{H^2 a^2}\,. \tag{2.30}$$

Now, the dark energy in the universe is the most important component. It it dominant over matter, radiation, and also dominant over spatial curvature.

[6] This expression can be derived in different ways, one of which is to start from the fact that the stress energy tensor must be traceless since it is of the form $T_{\mu\nu} \sim \sum_i \rho u_\mu^{(i)} u_\nu^{(i)}$, where $u_\mu^{(i)}$ are the four-velocities of photons: their norm is zero, so we must have $g^{\mu\nu}T_{\mu\nu} = 0$, but also for a perfect fluid $T = T_\mu^\mu = \rho - 3P/c^2$. Another, perhaps more illustrative derivation was given in the General Relativity course [TM20, pag. 86-87].

[7] This is shown by interpreting the additional term in the EFE as an addition to the stress-energy tensor and interpreting it as a perfect-fluid tensor [TM20, eqs. 434-438].

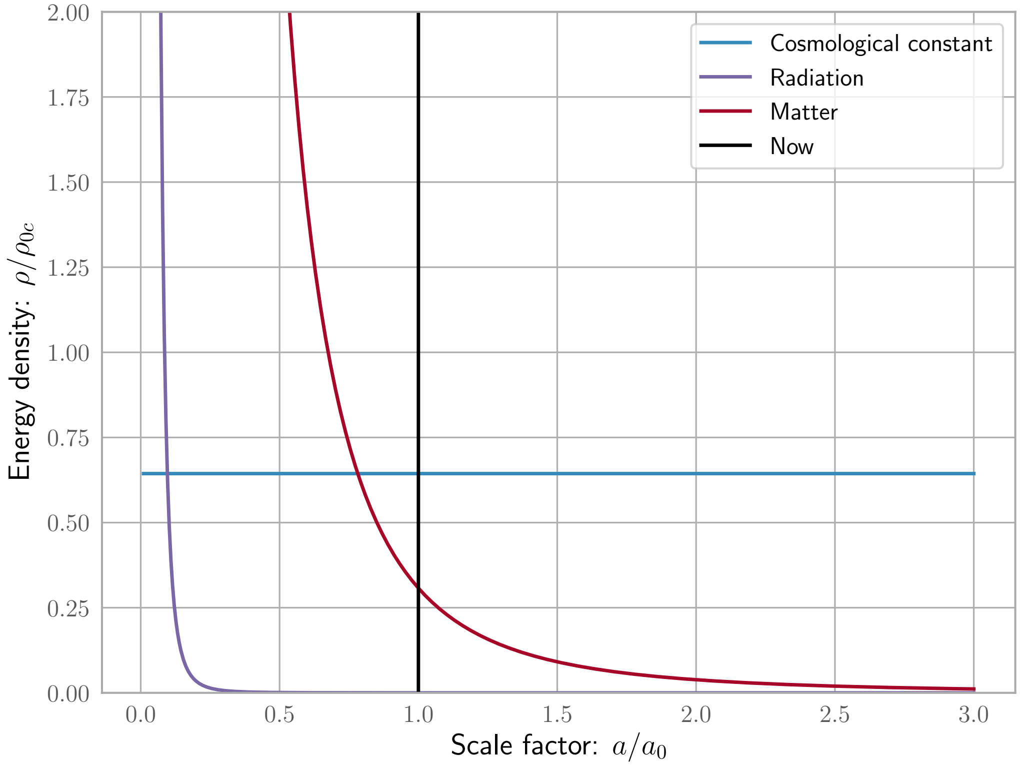

Figure 2.1: Contributions to the energy density varying with the scale factor. They are normalized to the current critical energy density, using data from the 2015 Planck mission [Col16]. The increase of the radiation energy density with an increasing scale factor is sharp, but the crossover point with the matter density is at $a = \rho_{\text{rad}}/\rho_{\text{m}} \approx 10^{-4}$, at which point we have $\rho \approx 10^{12}\rho_{0c}$.

2.3 Solutions of the Friedmann Equations

We want to solve the equation system

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a} \tag{2.31}$$

$$\rho(t) = \rho_* \left(\frac{a(t)}{a_*}\right)^{-3(1+w)}, \tag{2.32}$$

which encompasses all of the physical content of the FE, since the second equation can be derived from these two.

Inserting (2.32) into (2.31) we find:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_* \left(\frac{a}{a_*}\right)^{-3(1+w)} - \frac{kc^2}{a^2}. \tag{2.33}$$

2.3.1 Einstein-De Sitter models

As we discussed before, if we had zero spatial curvature ($k = 0$) then we would find $\rho = \rho_c = 3H^2/(8\pi G)$: so, we define the parameter $\Omega = \frac{8\pi G \rho}{3H^2} = \rho/\rho_C$ which quantifies how close this is to being true: experimentally this is compatible with 1.

The Einstein-de Sitter model is one where we take $\Omega \equiv 1$: negligible spatial curvature, which is equivalent to setting $k = 0$. So, the equation becomes:

$$\dot{a}^2 = \underbrace{\frac{8\pi G}{3}\rho_* a_*^{3(1+w)}}_{A^2} a^{-(1+3w)}, \tag{2.34}$$

Set $k = 0$, multiplied by a^2, defined A.

therefore $\dot{a} = \pm A a^{-\frac{1+3w}{2}}$, or $a^{\frac{1+3w}{2}}\,\mathrm{d}a = \pm A\,\mathrm{d}t$. We choose the positive sign, since we observe the universe to be expanding. This can be integrated directly: the equation is

$$\int_{a_*}^{a} \widetilde{a}^{\frac{1+3w}{2}}\,\mathrm{d}\widetilde{a} = A \int_{t_*}^{t} \mathrm{d}\widetilde{t} = A(t - t_*), \tag{2.35}$$

but we must distinguish two cases: either $(1 + 3w)/2 = -1$, which is equivalent to $w = -1$, or not. Let us first assume that $w \neq -1$. Then, we get:

A solution is:

$$\frac{2}{3+3w}\, a^{\frac{3+3w}{2}}\Big|_{a_*}^{a} = A(t - t_*) \tag{2.36}$$

$$a^{\frac{3+3w}{2}} - a_*^{\frac{3+3w}{2}} = \frac{3(1+w)}{2}\underbrace{\sqrt{\frac{8\pi G}{3}\rho_*}}_{H_*}\, a_*^{\frac{3+3w}{2}}(t - t_*) \tag{2.37}$$

Since $k = 0$ we have $H_*^2 = \frac{8\pi G}{3}\rho_*$

$$a^{\frac{3+3w}{2}} = a_*^{\frac{3+3w}{2}}\left(1 + \frac{3}{2}(1+w)H_*(t - t_*)\right) \tag{2.38}$$

$$a(t) = a_*\left(1 + \frac{3}{2}(1+w)H_*(t - t_*)\right)^{\frac{2}{3(1+w)}}, \tag{2.39}$$

which we can couple to the equation for the evolution of the density, by plugging this expression for the scale factor directly into (2.32):

$$\rho(t) = \rho_*\left(1 + \frac{3}{2}(1+w)H_*(t - t_*)\right)^{-2}, \tag{2.40}$$

and also the Hubble parameter $H \propto \sqrt{\rho}$:

$$H(t) = H_*\left(1 + \frac{3}{2}(1+w)H_*(t - t_*)\right)^{-1}. \tag{2.41}$$

There is a time where the bracket in $a(t)$ is zero, which means $a = 0$: this corresponds to the Big Bang, so we call it t_{BB}, defined by

$$1 + \frac{3}{2}(1+w)H_*(t_{\mathrm{BB}} - t_*) = 0. \tag{2.42}$$

Since the curvature scalar is $R \propto H^2 \propto \rho$,[8] at $t_{\rm BB}$ the curvature is diverges.
This time can be expressed by inverting the equation:

$$t_{\rm BB} = t_* - \frac{2}{3(1+w)H_*} \,. \tag{2.44}$$

Hakwing & Ellis proved that if $w > -1/3$ we unavoidably must have a Big Bang.
We can define a new time variable by

$$t_{\rm new} \equiv t - t_{\rm BB} = (t - t_*) + \frac{2}{3H_*(1+w)} \,. \tag{2.45}$$

Using this new variable the t_* simplifies, and we can just write:

$$a \propto {t_{\rm new}}^{\frac{2}{3(1+w)}} \,. \tag{2.46}$$

Inserting this new time variable (which we will just call t), we get that the Hubble parameter is:

$$H(t) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{3(1+w)} t^{\frac{2}{3(1+w)}-1} t^{-\frac{2}{3(1+w)}} \tag{2.47}$$

$$H(t) = \frac{2}{3(1+w)t} \,, \tag{2.48}$$

so we can compute the density:

$$\rho(t) = \frac{3H^2}{8\pi G} = \frac{3}{8\pi G}\frac{4}{9(1+w)^2 t^2} \tag{2.49}$$

$$= \frac{1}{6(1+w)^2 \pi G t^2} \,. \tag{2.50}$$

Let us now revisit the cases from before:

1. $w = 0$ is nonrelativistic matter: it has $a \propto t^{2/3}$, $\rho = 1/(6\pi G t^2)$ and $H = 2/(3t)$.

   This yields a prediction for the age of the universe of $t \approx 9.6\,\text{Gyr}$ (using the Planck data [Col16]): this is not correct, since the actual value is more like $t \approx 13.8\,\text{Gyr}$, but it has the right order of magnitude; the discrepancy is due to the fact that the assumption of the universe being dominated by nonrelativistic matter is wrong.

2. $w = 1/3$ is radiation: it has $a \propto t^{1/2}$, $\rho \propto 3/(32\pi G t^2)$ and $H = 1/(2t)$;

[8] This can be shown by a simple argument: we take the trace of the EFE, to get

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) = 8\pi G g^{\mu\nu} T_{\mu\nu} \implies -R = 8\pi G T \,, \tag{2.43}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ and $T = g^{\mu\nu}T_{\mu\nu}$ are the traces of the Ricci tensor and of the stress-energy tensor. We've shown that $R \propto T$: so, since $T \propto \rho$, we have $R \propto \rho$, but if we also assume that there is no spatial curvature then $H^2 \propto \rho$, therefore $R \propto H^2$.

3. $w = -1$ is dark energy: as we mentioned before, this is the case which must be treated separately. The integral yields:

$$\log\left(\frac{a}{a_*}\right) = A(t - t_*)\,, \tag{2.51}$$

which means

$$a(t) = a_* \exp\left(H_* a_*^{\frac{3(1+w)}{2}}(t - t_*)\right), \tag{2.52}$$

while, as we saw, ρ and P (and, therefore, H) are constant.

From the redshift we can trace back the time of emission of the photon: for a matter-dominated universe, for example, we have:

$$1 + z = \frac{a_0}{a} = \left(\frac{t_0}{t}\right)^{2/3} = \left(\frac{2}{3H_0 t}\right)^{2/3}. \tag{2.53}$$

We can calculate the deceleration parameter for this case, to check that it is indeed positive: we need to differentiate the expression

$$a(t) = a_0\left(\frac{3H_0 t}{2}\right)^{2/3}, \tag{2.54}$$

and we find

$$q_0 = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2} = -\frac{t_0^{-4/3} t_0^{2/3}}{(t_0^{-1/3})^2} \times \frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{\frac{2}{3}\frac{2}{3}} = \frac{1}{2}\,. \tag{2.55}$$

The $a_0(3H_0/2)^{2/3}$ terms all simplify.

This is a special case of the fact that [LC02, eq. 2.2.4b]

$$q \equiv q_0 = \frac{1 + 3w}{2}\,. \tag{2.56}$$

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2.4 Measuring distances

We want to be able to compute the comoving radius, given our knowledge of the evolution of the distribution of energy density in time.

We have shown that the luminosity distance is given by:

$$d_L \equiv \sqrt{\frac{L}{4\pi\ell}} = a_0(1 + z)r(z)\,. \tag{2.57}$$

Also recall *conformal time* η, which is defined by its relation to cosmic time, $a(\eta)\,\mathrm{d}\eta = \mathrm{d}t$: it allows us to write the FLRW metric as

$$\mathrm{d}s^2 = a^2(\eta)\left(c^2\,\mathrm{d}\eta^2 - \frac{\mathrm{d}r^2}{1 - kr^2} - r^2\,\mathrm{d}\Omega^2\right). \tag{2.58}$$

This is very important when we talk about zero-mass particles, with no intrinsic length scale: the photon, which is our primary tool for astrophysical observations, is one of these. This can be written in terms of the variable χ:

$$\mathrm{d}s^2 = a^2(\eta)\left(c^2\,\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - f_k^2(\chi)\,\mathrm{d}\Omega^2\right), \tag{2.59}$$

where $f_k(\chi) = r$ is equal to $\sin(\chi)$, χ or $\sinh(\chi)$ if k is equal to 1, 0 or -1; in other words we either have $\chi = \arcsin(r)$, $\chi = r$ or $\chi = \operatorname{arcsinh}(r)$.

If we look at photons moving radially we do not need to account for the angular part, and we find

$$\mathrm{d}s^2 = 0 = a^2(\eta)\left(c^2\,\mathrm{d}\eta^2 - \mathrm{d}\chi^2\right), \tag{2.60}$$

therefore $c^2\,\mathrm{d}\eta^2 = \mathrm{d}\chi^2$: we get $c\big(\eta(t_0) - \eta(t_e)\big) = \chi(r_e) - \chi(r_0)$, where a subscript e means "emission", while a subscript 0 means detection. We are choosing the negative sign when simplifying the square, since the problem we are considering is that of radiation starting from an astrophysical source and coming towards us: its radial coordinate χ decreases when the temporal coordinate η increases.

This means that we can find out the comoving distance $\Delta\chi$ between two events by calculating the difference between their comoving times $\Delta\eta$. This is what was meant by the fact that this expression of the metric is useful for massless particles: the scale factor gets factored out, we can write the expression in a very simple way.

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a} = \frac{\mathrm{d}a}{a\dot{a}}, \tag{2.61}$$

and now recall $(1+z) = a_0/a$: we differentiate this with respect to time to find

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{a_0}{a^2}\dot{a} = -\frac{a_0 H(z)}{a}, \tag{2.62}$$

which means

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a} = -\frac{\mathrm{d}z}{a_0 H(z)}, \tag{2.63}$$

Took the inverse of the equation, split the differentials, used the definition of η

so we get our final expression:

$$\mathrm{d}\chi = \frac{c\,\mathrm{d}z}{a_0 H(z)}. \tag{2.64}$$

Used the fact that $\mathrm{d}\chi = -c\,\mathrm{d}\eta$.

So, if we can find a way to parametrize the Hubble parameter $H(z)$ in terms of the redshift we will be able to measure distances.

The Hubble parameter is given by

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{2.65}$$

where the density comes from several components: $\rho(t) = \rho_r(t) + \rho_m(t) + \rho_\Lambda$, where the first term is the density of radiation and scales like a^{-4}, the second is the density of matter and scales like a^{-3}, the third is the density of dark energy and is constant.

In terms of the redshift, they scale like $(1+z)^4$, $(1+z)^3$ (and $(1+z)^0$) respectively.

We express the Hubble parameter as a multiple of its value now: $H(z) = H_0 E(z)$, where $E(z)$ is a dimensionless function.

Recall the definition of $\Omega(t)$: it describes the ratio of the density of a certain type of fluid to the critical density. We can look at the $\Omega_i(t)$ for i corresponding to matter, radiation and so on:

$$\Omega_i(z) = \frac{8\pi G \rho_i(z)}{3H^2(z)} = \frac{8\pi G \rho_i(z=0)}{3H_0^2} \times \frac{\rho_i(z)/\rho_i(z=0)}{E^2(z)} = \Omega_{i,0} \frac{(1+z)^\alpha}{E^2(z)}, \tag{2.66}$$

where α is the exponent of the scaling of the fluid: $\alpha = 4$ for radiation, $\alpha = 3$ for matter, $\alpha = 0$ for the cosmological constant Λ, while for spatial curvature $\alpha = 2$.

For the Ω corresponding to the curvature we define: $\Omega_k = -kc^2/(a^2H^2)$ (see equation (2.30)).

We must have

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k\,. \tag{2.67}$$

We can write an expression for $E^2(z)$ by taking the ratio of the densities at emission versus now:

$$E^2(z) = \frac{H^2}{H_0^2} = \Omega_{\Lambda,0} + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4\,, \tag{2.68}$$

and to get E we just take the square root.

Now we can finally compute our integral

$$\chi(z) = \frac{c}{a_0 H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')}\,, \tag{2.69}$$

therefore

$$r = f_k\left(\frac{c}{a_0 H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')} \right). \tag{2.70}$$

This does depend on k, but the differences between positive and negative curvature are only relevant starting from third order. If the curvature is zero, we get the comoving distance:

$$d_C = r a_0 = \frac{c}{H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')}\,. \tag{2.71}$$

If the curvature is not zero, we can still define a useful distance: the *transverse comoving distance*,

$$d_M = a_0 r = a_0 f_k\left(\frac{c}{a_0 H_0} \int_0^z \frac{\mathrm{d}z'}{E(z')} \right); \tag{2.72}$$

for $k = 0$ these two coincide.

Now, suppose we are looking at a certain far-away object with angular size $\Delta\theta$ and linear size *at emission* of Δx: then the *angular diameter distance* is given, in the small-angle approximation, by

$$d_A = \frac{\Delta x}{\Delta \theta} = a(t_e) r = \frac{a_0 r_z}{1+z} = \frac{d_M}{1+z}\,. \tag{2.73}$$

Distance name	Formula	Description
Comoving distance	$d_C = ra_0$ $= \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}$	Distance in comoving coordinates multiplied by the current scale factor: if the expansion of the universe froze during our measurement, this is the distance we would measure between the two events. Assumes $k = 0$.
Transverse comoving distance	$d_M = ra_0$ $= a_0 f_k\left(\frac{c}{H_0 a_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right)$	Generalization of the comoving distance to $k \neq 0$.
Luminosity distance	$d_L = d_M(1+z)$ $= \sqrt{L/(4\pi\ell)}$	Distance defined so that the radiative intensity we measure follows the inverse square law.
Angular diameter distance	$d_A = d_M(1+z)^{-1}$ $= \Delta x/\Delta\theta$	Distance defined by the ratio of a far-away object's size (measured using the scale factor at the time of the emission of the radiation we observe now) to its angular size.

Figure 2.2: A summary of the cosmological distances we defined, drawing on the summary by Hogg [Hog00].

Since the luminosity distance is given by

$$d_L = a_0(1+z)r = d_M(1+z) \tag{2.74}$$

their ratio is

$$\frac{d_L}{d_A} = (1+z)^2 . \tag{2.75}$$

2.5 The cosmological constant

Einstein thought that the universe had to be static: it was a common notion at the time that it should be, almost a philosophical principle.[9] Now we know that the universe is neither static nor stationary.[10]

So, he sought static solutions ($a = \text{const}$) for matter ($P = 0$) to the Friedmann equations (1.9): if we set $\dot{a} = \ddot{a} = 0$ the third equation becomes $\dot{\rho} = 0$, the second equation gives us

[9] An interesting historical fact: this was corroborated by a calculation error on Einstein's part, which was later pointed out by Friedmann. Einstein thought [Ein22] that $\nabla_\mu T^{\mu\nu} = 0$ implied $\partial_t \rho = 0$, while Friedmann pointed out [Fri22] that the correct equation reads $\partial_t\left(\sqrt{-g}\rho\right) = 0$: the density of the universe is not forced to be time independent if the determinant of the metric changes accordingly. Even the best make mistakes.

[10] The distinction between static and stationary is subtle but significant [Lud99]: *stationarity* is about the existence of a timelike Killing vector, while *staticity* is about the timelike Killing vector being orthogonal to spacelike submanifolds. A concrete example: Schwarzschild geometry is both static and stationary, Kerr geometry is stationary but not static, FLRW geometry is neither, since there is no timelike Killing vector field.

$\rho \equiv 0$, and from the first we must also have $k = 0$: the only way to have a static matter-filled universe is for the density of matter to be zero, and for the spatial curvature to be also zero.

In order to satisfy what he thought was an empirical fact, Einstein modified his equations in order to get a static non-empty solution.

The Einstein equations read

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{2.76}$$

when $c = 1$, where the Einstein tensor $G_{\mu\nu}$ can be defined in terms of the Ricci curvature tensor $R_{\mu\nu}$ and the scalar curvature R as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,. \tag{2.77}$$

This peculiar construction is the only one which can be made in terms of the curvature tensor and which is covariantly constant: $\nabla_\mu G^{\mu\nu} = 0$. This is a necessary condition since $\nabla_\mu T^{\mu\nu} = 0$: the Einstein equations state that they are proportional, so if we take the covariant derivative of the equations we must get the identity $0 = 0$.

Einstein added a term $-\Lambda g_{\mu\nu}$ to the LHS of the Einstein equations, with Λ a constant scalar. This is allowed since

1. it is tensorial (since it is a scalar multiple of the metric, which is a tensor);
2. it is symmetric;
3. it has zero covariant divergence, since Λ is constant and the metric is covariantly constant $\nabla_\mu g^{\mu\nu} = 0$.

Then, we can rewrite the EE in two equivalent ways: either

$$\widetilde{G}_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad \text{with} \qquad \widetilde{G}_{\mu\nu} = G_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2.78}$$

$$G_{\mu\nu} = 8\pi G \widetilde{T}_{\mu\nu} \qquad \text{with} \qquad \widetilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{\Lambda g_{\mu\nu}}{8\pi G}\,. \tag{2.79}$$

In the first interpretation, the cosmological constant is an intrinsic geometric property of spacetime; in the second interpretation cosmological constant is a particular kind of fluid, with the property of its contribution to the stress-energy tensor always being a constant multiple of the metric.

In order to find out what the properties of this fluid are, w compare its stress-energy tensor to a generic ideal fluid tensor:

$$T^{\text{(generic)}}_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \qquad T^{(\Lambda)}_{\mu\nu} = \frac{\Lambda g_{\mu\nu}}{8\pi G} = \frac{\Lambda}{8\pi G} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{2.80}$$

so the corrections to the stress energy tensor must be $\rho \to \rho + \Lambda/8\pi G$ and $P \to P - \Lambda/8\pi G$, or, in other words, the density and pressure of the "cosmological constant fluid" are $\rho_\Lambda =$

$-P_\Lambda = \Lambda/8\pi G$. This proves that the equation of state of the cosmological constant is $w = -1$.

Inserting this into the Friedmann equations we get:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{2.81}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \Lambda \tag{2.82}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\widetilde{\rho} + \widetilde{P}\right) = -3\frac{\dot{a}}{a}\left(\rho + P\right), \tag{2.83}$$

and we can see that in the third equation, the effect of the source is encompassed in a term $\widetilde{\rho} + \widetilde{P}$: the two Λ terms cancel, since they are opposite. For a cosmological constant-dominated universe — that is, for a universe in which the only fluid behaves like the cosmological constant – we have $\dot{\rho} = \dot{P} = 0$.

So, proceeding with the derivation by Einstein, we set $\dot{a} = \ddot{a} = 0$: for the first Friedmann equation we get

$$\frac{8\pi G}{3}\rho + \frac{\Lambda}{3} = \frac{k}{a^2}, \tag{2.84}$$

and for the second:

$$4\pi G\rho = \Lambda. \tag{2.85}$$

So, we substitute the expression for $4\pi G\rho$ into the first Friedmann equation:

$$\frac{2}{3}(4\pi G\rho) + \frac{\Lambda}{3} = \Lambda\left(\frac{1}{3} + \frac{2}{3}\right) = \Lambda = \frac{k}{a^2}. \tag{2.86}$$

What are the physical conclusions to draw? Since we want matter in the universe we must have $\rho > 0$, which implies $\Lambda > 0$, which implies $k = 1$: so the universe must be closed.

Friedmann studied perturbations around this solution and found it to be unstable: so, it is not suitable as a description of the universe. This, combined with the observations by Hubble of an expanding universe, prompted the scientific community to discard the idea of a stationary universe in favor of an expanding one.

Einstein probably [Aut18] called the introduction of the cosmological constant into the equation his "greatest blunder"; however in modern cosmology the idea of a cosmological constant has gained new vigor: we observe the universe's expansion to be accelerated, that is $\ddot{a} > 0$, and the only way for this to be the case if $\rho > 0$ is if $\Lambda > 0$ as well. It is the only kind of fluid which has a repulsive gravitational effect.

As opposed to the approach by Einstein, in which the cosmological constant was inserted to stationarize the universe, we make it a measurable parameter of our theory.

A candidate for the cosmological constant term, which is a kind of intrinsic energy of space, is the vacuum energy in QFT: however the estimate we get when trying to make this quantitative is around 10^{120} times the measured value of Λ.

Friday 2010-10-18, compiled 2020-09-14

2.5.1 Evolution of a dark energy dominated universe

In order to find out how this parameter affects the universe's expansion, we consider a universe in which the only fluid behaves like the cosmological constant. So, we take the first Friedmann equation (2.81) in the absence of ordinary matter ($\rho = 0$) and with negligible spatial curvature ($k = 0$). This yields:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}. \tag{2.87}$$

This is actually a good approximation for the asymptotic state of the universe, since the cosmological constant term is the only one which does not decay with the scale factor (and so, with time).

The solution to this differential equation is, as we mentioned in section 2.3:

$$a(t) = a_* \exp\left(\sqrt{\frac{\Lambda}{3}}(t - t_*)\right), \tag{2.88}$$

which can also be written as $a \propto e^{Ht}$, since $H = \dot{a}/a = \sqrt{\Lambda/3}$. This is called a steady-state solution, since the Hubble parameter is constant. It is also called a *de Sitter* solution: it belongs to the maximally symmetric 4D spacetime solutions to the Einstein Equations: Minkowski, de Sitter and Anti de Sitter: the latter has $\Lambda < 0$, the former has $\Lambda > 0$.

This actually seems to model the observed expansion of the universe well, and until recently it competed with the Big Bang theory.

The fraction of the cosmic fluid which behaves like dark energy is bound to increase with time, since as we saw it is the only component which does not decrease in density over time.

This is expressed formally using the so-called *no-hair cosmic theorem*, which is actually a conjecture if it is meant to describe the universe: it states that asymptotically only the dark energy contribution is relevant: all the matter and everything else is forgotten. In order to interpolate between the current — matter dominated, or in which at least matter has a sizeable contribution — universe and the asymptotic one we can use a solution in the form

$$a \propto \left(\sinh(At)\right)^{2/3}, \tag{2.89}$$

where we define $2A/3 = \sqrt{\Lambda/3}$, since the hyperbolic sine is asymptotically close to an exponential.

2.6 Curved models

We seek solutions to the Friedmann equations for nonzero spatial curvature k, for a universe containing nonrelativistic matter ($w = 0$) without dark energy. We make these assumptions since with them we can find an analytic solution.

We can rewrite the two independent Friedmann equations as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k \tag{2.90}$$

$$\rho = \rho_0 \left(\frac{a}{a_0} \right)^{-3} , \tag{2.91}$$

and now we will solve them with $k = \pm 1$.

Solutions to parametric ODEs In general, for an ODE like $y = f(y')$ for the function $y = y(x)$ with f' continuous we introduce $y' \equiv p$, assuming $p \neq 0$: then $y = f(p)$, which implies

$$y' = \frac{\mathrm{d}f}{\mathrm{d}p} p' , \tag{2.92}$$

Differentiated both sides of $y = f(p)$

which we can manipulate to get

$$p = \frac{\mathrm{d}f}{\mathrm{d}p} p' \implies \frac{\mathrm{d}x}{\mathrm{d}p} = \frac{1}{p} \frac{\mathrm{d}f}{\mathrm{d}p} , \tag{2.93}$$

so we can get the solution by integration: we get an expression for x in terms of p, which we will be able to invert since by assumption $p' \neq 0$: so, we get

$$x = \int \frac{1}{p} \frac{\mathrm{d}f}{\mathrm{d}p} \, \mathrm{d}p \qquad \text{and} \qquad y = f(p) . \tag{2.94}$$

We use this for our problem: our differential equation looks like

$$\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - k \tag{2.95}$$

$$\dot{a}^2 = \frac{8\pi G}{3} \rho_0 \frac{a_0^3}{a^3} a^2 - k \tag{2.96}$$

$$\dot{a}^2 = A a^{-1} - k , \tag{2.97}$$

Substituted $\rho = \rho_0 a_0^3 / a^3$ from the third Friedmann equation.

where we defined $A \equiv 8\pi G a_0^3 \rho_0 / 3$. We can rewrite this as

$$a = \frac{A}{p^2 + k} = f(p) \qquad \text{where} \qquad p = \dot{a} . \tag{2.98}$$

Then, using the general formula we get:

$$t = \int \frac{1}{p} \frac{\mathrm{d}f}{\mathrm{d}p} \, \mathrm{d}p \qquad \text{where} \qquad \frac{\mathrm{d}f}{\mathrm{d}p} = -\frac{2Ap}{(p^2 + k)^2} \tag{2.99}$$

$$= \int \frac{-2A}{(p^2 + k)^2} \, \mathrm{d}p \, . \tag{2.100}$$

2.6.1 Positive curvature: a closed universe

If $k = +1$, then we can make the substitution $p = \tan(\theta)$, which is helpful since $1 + p^2 = \sec^2 \theta$; for the change of variable we have $\mathrm{d}p = \mathrm{d}\theta \sec^2 \theta$. So, for the time we find:

$$t = -2A \int \frac{\sec^2 \theta \, \mathrm{d}\theta}{\sec^4 \theta} \tag{2.101}$$

$$= -2A \int \cos^2(\theta)\, \mathrm{d}\theta = -A\big(\theta + \sin(\theta)\cos(\theta)\big) + \text{const}\,, \tag{2.102}$$

and we can apply the trigonometric identity $\sin(\theta)\cos(\theta) = \sin(2\theta)$:

$$t = -\frac{A}{2}\big(2\theta + \sin(2\theta)\big) + \text{const}\,. \tag{2.103}$$

Now we can define $2\theta = \pi - \alpha$, which allows for the simplification $\sin(2\theta) = \sin(\alpha)$; also, we can express $p = \tan\theta$ in terms of α. This gives us

$$t = \frac{A}{2}\big(\alpha - \sin(\alpha)\big) + \text{const} \qquad \text{and} \qquad p = \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right). \tag{2.104}$$

Absorbed factor $-A\pi/2$ into the constant

We almost have our solution: inserting $p(\alpha)$ into the main equation for a (2.98) we get

$$a = \frac{A}{1 + \tan^2(\pi/2 - \alpha/2)} = A\cos^2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \tag{2.105}$$

$$= \frac{A}{2}\big(1 + \cos(\pi - \alpha)\big) = \frac{A}{2}\big(1 - \cos(\alpha)\big)\,, \tag{2.106}$$

Used $1 + \tan^2 x = 1/\cos^2 x$.

Used $\cos^2(x/2) = (1 + \cos x)/2$ and $\cos x = -\cos(\pi - x)$.

which should be complemented with the equation we found for t: in the end, our solution looks like

$$t = \frac{A}{2}(\alpha - \sin\alpha) + \text{const} \tag{2.107}$$

$$a = \frac{A}{2}(1 - \cos\alpha)\,, \tag{2.108}$$

so, in order to interpret this physically we fix $t = 0 \iff \alpha = 0$, which sets "const" to zero, and we reinsert the constants. In order to do so, we wish to express the constant $A/2$ in term of observables such as H_0 and $\Omega_0 = \rho_0/\rho_{0c}$. We have:

$$\Omega_0 = \frac{8\pi G\rho_0}{3H_0^2} \qquad \Longrightarrow \qquad A = \frac{8\pi G a_0^3 \rho_0}{3} = \Omega_0 H_0^2 a_0^3\,, \tag{2.109}$$

which we can simplify by making use of the first Friedmann equation, which reads:

$$H_0^2 = \frac{8\pi G}{3}\rho_0 - \frac{k}{a_0^2} \qquad \Longrightarrow \qquad 1 = \Omega_0 - \frac{1}{a_0^2 H_0^2} \qquad \Longrightarrow \qquad a_0^2 H_0^2 = \frac{1}{1 - \Omega_0}\,, \tag{2.110}$$

We are treating the case $k = 1$

so we can write $A/2$ in two different ways:

$$\frac{A}{2} = \frac{a_0}{2}\frac{\Omega_0}{1 - \Omega_0} = \frac{1}{2H_0}\frac{\Omega_0}{(1 - \Omega_0)^{3/2}}\,. \tag{2.111}$$

We use one of these for a and the other for t: this is done because it makes the prefactor of the expression manifestly dimensionally consistent with the quantity we are expressing — this is not always the case when working with $c = 1$. Then, the expressions for a and t become:

$$a = a_0 \frac{\Omega_0}{2(\Omega_0 - 1)}\big(1 - \cos(\alpha)\big) = \tilde{a}_0 \frac{1 - \cos\alpha}{2} \tag{2.112}$$

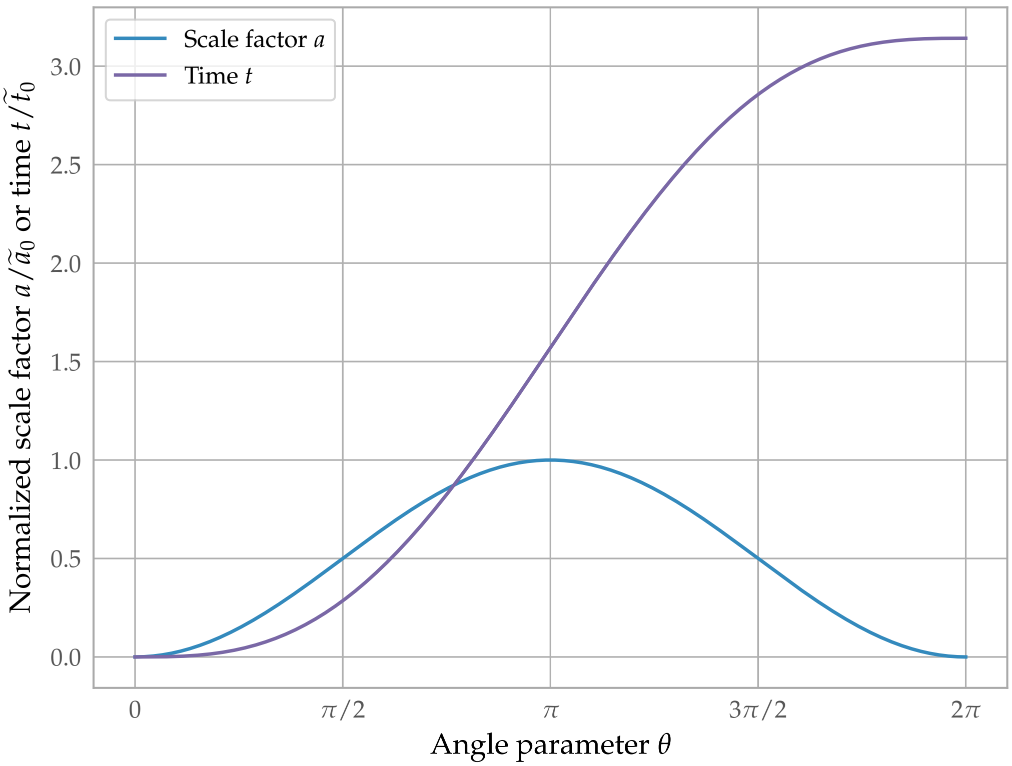

Figure 2.3: A plot of $a(\theta)$ and $t(\theta)$.

$$t = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} (\alpha - \sin(\alpha)) = \widetilde{t}_0 \frac{\alpha - \sin\alpha}{2} . \tag{2.113}$$

For the discussion of these results we rename the angle variable from α to θ for historical reasons.

We have $\dot{a} > 0$ when $0 \leq \theta \leq \theta_m = \pi$, while $\dot{a} < 0$ when $\theta_m \leq \theta \leq 2\pi$: so, we call θ_m the *turn-around* angle. The angles 0 and 2π correspond to the Big Bang and the Big Crunch.

At θ_m we have:

$$a_m = \widetilde{a}_0 = a_0 \frac{\Omega_0}{\Omega_0 - 1} \tag{2.114}$$

$$t_m = \frac{\pi}{2} \widetilde{t}_0 = \frac{\pi}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} . \tag{2.115}$$

The age of a closed universe

The total lifetime of the universe, in this scenario, is equal to $2t_m = \pi\widetilde{t}_0$. How does this compare to the result we found for a flat universe, namely $t_0 = 2/(3H_0)$ (equation (2.48) with $w = 0$)?

We set $a(t) = a_0$, which means we are normalizing the scale factor to the current one: this yields

$$1 = \frac{\Omega_0}{1-\Omega_0} \frac{1-\cos\theta}{2} \qquad \Longrightarrow \qquad \cos\theta = 1 - \frac{2(\Omega_0 - 1)}{\Omega_0} = \frac{2}{\Omega_0} - 1\,, \tag{2.116}$$

so we can invert the cosine (assuming we are in the expanding phase: it is not invertible globally) and insert our expression for θ into the expression for the time, to get[11]

$$t_0 = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \left(\arccos\left(\frac{2}{\Omega_0} - 1 \right) - \frac{2}{\Omega_0} \sqrt{\Omega_0 - 1} \right). \tag{2.118}$$

As we can see in figure 2.4, this means that the estimated age of the universe is lower than $2/(3H_0) \approx 9.6\,\text{Gyr}$, while the measured age of the universe is around $14\,\text{Gyr}$.

2.6.2 Negative curvature: an open universe

For $k = -1$ we do exactly the same steps with hyperbolic functions instead of trigonometric ones, calling the argument of these functions ψ instead of θ: we get

$$a(\psi) = a \frac{\Omega_0}{2(1-\Omega_0)} (\cosh\psi - 1) \tag{2.119}$$

$$t(\psi) = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} (\sinh\psi - \psi)\,, \tag{2.120}$$

and as before we can calculate the independent variable with $\cosh\psi = 2/\Omega_0 - 1$.[12]

An analogous reasoning to the one before gives us

$$t_0 = \frac{1}{2H_0} \frac{\Omega_0}{(1-\Omega_0)^{3/2}} \left(\frac{2}{\Omega_0} \sqrt{1-\Omega_0} - \operatorname{arccosh}\left(\frac{2}{\Omega_0} - 1 \right) \right), \tag{2.121}$$

Now $\sinh(\operatorname{arccossh}(x)) = \sqrt{x^2 - 1}$

which is plotted, again, in figure 2.4: in this case $t_0 > 2/(3H_0)$! This is then more attractive.

[11] We need to use the expression $\sin(\arccos(x)) = \sqrt{1 - \cos^2(\arccos x)} = \sqrt{1 - x^2}$ with $x = 2/\Omega_0 - 1$, and then the following manipulation:

$$\sqrt{1 - \left(\frac{2}{\Omega_0} - 1 \right)^2} = \sqrt{1 - \frac{4}{\Omega_0^2} + \frac{4}{\Omega_0} - 1} = \frac{2}{\Omega_0} \sqrt{\Omega_0 - 1}\,. \tag{2.117}$$

[12] A doubt one might have: where does the sign change from $\theta - \sin(\theta)$ to $\sinh(\psi) - \psi$?

The difference between the calculations with the trigonometric functions and the hyperbolic functions lies in the substitution $2\theta = \pi - \alpha$: in the hyperbolic case we cannot do it this way, since the hyperbolic functions do not have any periodicity like this. Instead, the right substitution looks like $2\psi = i\pi + u$, since $\sinh(i\pi + u) = -\sinh(u)$.

Then, we have the same expressions as before, but their sign is flipped.

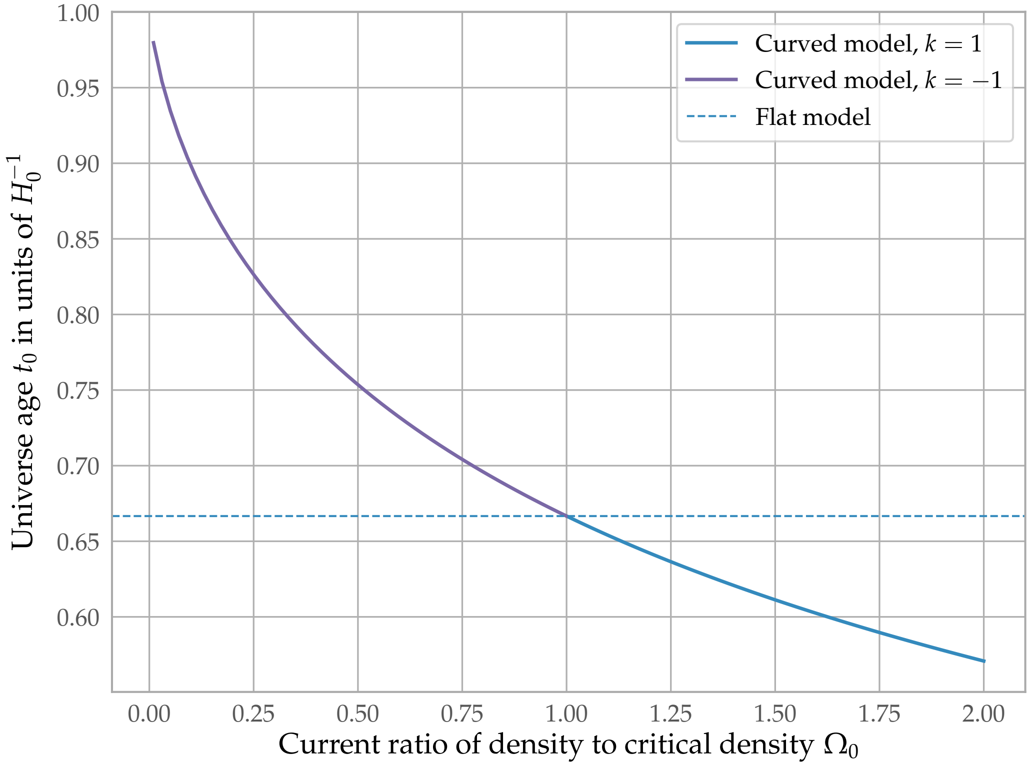

Figure 2.4: Universe age at the current time as a function of Ω_0. For $\Omega_0 > 1$ we use the positive curvature model in equation (2.118): the age is lower than $2/(3H_0)$; for $\Omega_0 < 1$ we use the negative curvature model (2.121): the age is greater than $2/(3H_0)$. The flat model is plotted with a horizontal line for clarity, but if the universe is spatially flat then we must have $\Omega_0 = 1$.

2.6.3 Considerations on curvature

The experimental fact that $t_0 > 2/(3H_0)$ seems to favour an open universe. However, the age of the universe is $t_0 \approx 0.96 H_0^{-1}$: looking at figure 2.4 it is clear that in order to account for it with spatial curvature only we would need $\Omega_0 \ll 1$, and actually $\Omega_0 < \Omega_{0m}$, where Ω_{0m} is the current measured ratio of the density of matter to the critical density.

In fact, in the current ΛCDM model of cosmology this is accounted for using dark energy, which means a positive cosmological constant.

From the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1+3w) \tag{2.122}$$

we know that $\ddot{a} < 0$, that is, the expansion of the universe decelerates if $w > -1/3$. A singular instant at which $a = 0$ must be reached if $w > -1/3$, while this is not necessarily the case for $w < -1/3$. As we will discuss in the next chapter, this will be one of the motivations behind the theory of inflation, which does not require the presence of an initial singularity.

Chapter 3

The thermal history of the universe

In the study of the early stages of the universe the variation of the temperature, which determines the distribution of the energies of the collisions of the particles, plays a central role. It makes sense to talk about temperature when the particles are actually in thermal equilibrium, as they were in the early universe: photons and electrons were continuously Compton-scattering off each other. After the particles stop continuously interacting we say that they *decoupled*, and each component evolves independently.

The fact that the universe's temperature was much higher in the past is need to explain *primordial nucleosynthesis*: Helium-4 is the outcome of Hydrogen burning but in stellar evolution it is burned into heavier elements after it is formed, so we would expect to see small amounts of it. Instead, we see a relatively large amount of Helium-4: it makes up about a quarter of the universe by mass. The primordial universe being very hot helps account for this. In fact, this was first predicted in 1948, in the notorious $\alpha\beta\gamma$ paper [ABG48].

3.1 Radiation energy density and the equality redshift

In section 2.2.1 we discussed the evolution of the energy density of matter and radiation, showing that for radiation $\rho_r(z) = \rho_{0r}(1+z)^4$ while for matter $\rho_m(z) = \rho_{0m}(1+z)^3$. We discussed this in the context of electromagnetic radiation, but it describes well the behavior of very relativistic particles, such as neutrinos.

We can define a moment called the *equality redshift* z_{eq}. This is when the energy density of radiation and that of matter were equal: $\rho_r(z_{\text{eq}}) = \rho_m(z_{\text{eq}})$. This means that

$$\rho_{0,r}(1+z_{\text{eq}})^4 = \rho_{0,m}(1+z_{\text{eq}})^3 \implies (1+z_{\text{eq}}) = \frac{\rho_{0,m}}{\rho_{0,r}} = \frac{\Omega_{0,m}}{\Omega_{0,r}}, \tag{3.1}$$

where we divided and multiplied by the critical density today.

We know that $\Omega_{0,m}$ is around 0.3, while for the radiation we can deduce the density from the spectrum of the CMB.

Accounting for everything, we think that

$$1 + z_{\text{eq}} \simeq 2.3 \times 10^4 \Omega_{0,m} h^2 \approx 3370\,. \tag{3.2}$$

The value is that obtained from the Planck Collaboration [Col16].

This means that the recombination of electrons and protons into Hydrogen, which occurred around redshift $z_{\text{CMB}} \approx 1090$, happened when the universe was already *matter dominated* — specifically, the density of matter was ≈ 3 times that of radiation.

Another interesting time is z_Λ, when the energy density due to the cosmological constant equalled that of matter: $\rho_m(z_\Lambda) = \rho_\Lambda(z_\Lambda)$, which is calculated with the same reasoning as z_{eq}, recalling that ρ_Λ is a constant with respect to the redshift:

$$1 + z_\Lambda = \left(\frac{\rho_{0,\Lambda}}{\rho_{0,m}}\right)^{1/3} \simeq \left(\frac{0.7}{0.3}\right)^{1/3} \approx 0.33\,. \tag{3.3}$$

This is relatively close, in cosmological terms: the comoving distance corresponding to this redshift is around 1350 Mpc, less than 10 % of the comoving distance to the CMB.

What is the temperature of a radiation-dominated universe? From the Stefan-Boltzmann law we know that $\rho_r \propto T^4$, while as we have discussed previously $\rho_r \propto a^{-4}$. Therefore, we expect $T \propto 1/a$ to hold: this is known as *Tolman's law*. In this chapter we will discuss how this does approximately hold, but we need to make some corrections due to the annihilation of ultrarelativistic particles.

We know that in a radiation dominated universe $a \propto t^{1/2}$, which means that $T \propto t^{-1/2}$.

We shall describe the pressure P, number density n and energy density ρ in the universe, as functions of the chemical potentials μ and of the temperature T. We will use natural units, so that $c = \hbar = k_B = 1$: so, temperatures and masses will be measured in electronVolts.

This is very convenient, since it allows us to make the following consideration: when the temperature will be of the order of the mass of a certain elementary particle, then statistically that type of particle will usually be ultra-relativistic.

This section will mostly follow Weinberg's book [Wei72, page 538, section 15.6].

3.2 Thermodynamics in the early universe

We will express the quantities mentioned above: P, ρ and n in terms of the distribution of particles in phase space: in general the phase space for a single particle in 3D is six-dimensional, but we operate under the assumption that the cosmological principle holds, so by homogeneity the spatial dependence of the distribution function can be neglected. Thus, we can talk of densities,[1] neglecting the spatial position, and integrating over momentum space to gather all the information there is to know about the particles in that position.

[1] Also, as a general rule, one should avoid talking about "global quantities": the universe is, in principle, infinite, so we should refer at most to what is or could be inside our light cone. It is better to work in terms of densities.

3.2.1 Number density, energy density and pressure

Number density

If $f(\vec{q})$ is the distribution density of particles with three-dimensional momentum $\vec{q}$, the number density of particles is given by:

$$n = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, f(\vec{q}, T, \mu) \,, \tag{3.4}$$

where the parameter g is the number of helicity states: it is the number of particles we can have with different quantum numbers, after fixing momentum and position.

This essentially is our choice for the normalization of the distribution function. We include the factor $(2\pi)^3$ in order to normalize the integral: the number of particles, a pure number, is given by

$$N \propto \int \mathrm{d}^3 q \, \mathrm{d}^3 x \, f(\vec{q}, \vec{x}) \,, \tag{3.5}$$

so the right hand side's differentials have the dimensions of an action cubed: we need to normalize them, and the conventional action used to do so is $h = 2\pi\hbar$. So, when we set $\hbar = 1$ we get a factor $(2\pi)^3$ on the denominator. Do note that the $\mathrm{d}^3 x$ integral, giving a volume, is brought to the left in our expression to give a number density.

The number of helicity states

The only quantum number which can vary after fixing those if we are considering an elementary particle is the spin component s_z; therefore if the total spin is s we should have $2s + 1$ possible spin states. For example electrons, which have spin $1/2$, will have $g = 2s + 1 = 2$.

Things, however, are more complicated than this: for photons, we only have two spin states ($g = 2$) even though they have $s = 1$, since $s_z = 0$ is unphysical for a photon. Gravitons also have $g = 2$ even though their total spin is $s = 2$: this is because $|s_z| \leq 1$ is unphysical for a graviton. In general for massless particles Lorentz invariance guarantees the fact that transverse modes cannot exist, since we cannot go in the rest frame of the particle.[2]

Even for massive particles we do not always have $g = 2s + 1$: g accounts for all internal degrees of freedom, and as the temperature drops below a certain value we need to consider composite particles as well: for atoms we also have vibration, rotation and such. These all contribute to g.

[2] Some more details on this: if we measure the spin component s_z to be equal to some value in a certain reference frame, then this will mean: (1) if $s_z \neq 0$ we need a rotation of at least $2\pi/s_z$ around the z axis in order to recover the system we started with, while (2) if $s_z = 0$ the system is symmetric with respect to rotations around the z axis.

So, for a photon to have $s_z = 0$ we would need to be in a frame in which its wavefunction was cylindrically symmetric. This cannot be the case if the photon is travelling in the z direction, so we must be in the rest frame of the photon, which does not exist.

Similarly, for gravitons the argument as to why $s_z \neq 0$ still applies, and we can exclude the spins $|s_z| = 1$ by the following argument: in full generality we can remove all the gauge freedom in a gravitational wave by going to TT gauge, and we can show that in TT gauge the wave is symmetric under rotations of angle π about the z axis. Therefore, the spin of the graviton must be at least 2 in magnitude.

Energy density

The energy density is given by

$$\rho = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, E(q) f(\vec{q}, T, \mu) \,, \tag{3.6}$$

where $E^2 = q^2 + m^2$[3]. For photons $E = q$, for nonrelativistic particles $E \approx m + q^2/2m$. Here we are denoting the modulus of the momentum vector as $q = |\vec{q}|$.

This formula is a weighted average of the energies on the distribution function on the momenta.

Pressure

The adiabatic pressure is

$$P = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 q \, \frac{q^2 f(\vec{q}, T, \mu)}{3E(q)} \,. \tag{3.7}$$

This comes from a consideration of the diagonal components of the stress energy tensor of an ideal fluid: we know that

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} = \frac{g}{(2\pi)^3} \left\langle \int \frac{\mathrm{d}^3 q}{E(\vec{q})} f(\vec{q}) p^\mu(\vec{q}) p^\nu(\vec{q}) \right\rangle \,, \tag{3.8}$$

where $p^\mu = (E(q), \vec{q})$ is the momentum vector. Although it may not look like it, this formula is covariant.[a] The average sign is meant to indicate a spatial average, across volumes which are wide enough for homogeneity to hold.

This formula reproduces equation (3.6) for $\mu = \nu = 0$: indeed, in this case $p^0(q) = E(q)$. The off-diagonal components are zero by isotropy: if they were not, we would see heat and particle flow in specific directions.

So, for the diagonal components spatial components the integrand looks like $q^i q^j / E(q)$.

The formula for the pressure then follows by isotropy: the total force per unit area to go around is q^2/E, and it must be distributed equally in the three spatial directions, so if we want to switch from the directional integral $T^{ii} = P \propto \int q^i q^i / E$ (not summed over i) to an integral of the modulus of the momentum, $P \propto \int q^2/E$ we must divide by 3.

[a] we are integrating a tensorial expression ($f p^\mu p^\nu$ is a tensor) with respect to a covariant integration element: $\mathrm{d}^3 q / E(q)$ is a scalar with respect to Lorentz transformations, since it can be obtained as

$$\mathrm{d}^4 q \, \delta(E^2 - p^2 - m^2) = \frac{\mathrm{d}^3 q}{2E} \delta(E - \sqrt{m^2 + p^2}) \,. \tag{3.9}$$

[3] Particles are on-shell, that is they obey the equations of motion (which is not mandatory, and this is what Quantum Mechanics is all about).

This definition gives us $P = \rho/3$ for photons directly, which can be seen by substituting $E = q$.

The distribution function

If the particles are in thermal equilibrium, the distribution in momentum space will be given by the following expression:

$$f(\vec{q}) = \left(\exp\left(\frac{E(q) - \mu}{T} \right) \pm 1 \right)^{-1} , \tag{3.10}$$

where we have a plus for fermions, and a minus for bosons. Here, $\mu = \partial\rho/\partial n$ is the chemical potential, the derivative of the energy (density) with respect to a change in the number (density) of particles: is becomes relevant when the gas becomes hot and dense, if it is sparse then adding particles does not affect the energy.

The Planck distribution, which describes the statistics of photons, is consistent with this, since it is given by:

$$f_k(\vec{q}) = \left(\exp\left(\frac{q}{T} \right) - 1 \right)^{-1} , \tag{3.11}$$

since they are bosons with no chemical potential.[4] The fact that the distribution of photons is indeed described by this distribution with $\mu = 0$ is a way to experimentally determine the fact that the chemical potential of photons is indeed zero. If we observed the distribution for physical blackbodies to have $\mu \neq 0$ this would be called a *spectral distortion*. The CMB is wonderfully consistent with $\mu = 0$, it is actually the best Planckian in Nature.

It is a fact that the chemical potential μ can be neglected when dealing with the early universe. Let us justify this.

[4] This may seem weird at first: is the Planck function not

$$B_\nu(T) = \frac{2\nu^3}{e^{2\pi\nu/T} - 1} \tag{3.12}$$

in natural units? Well, these two are actually equivalent formulations. To see this, recall that in natural units $q = E = \omega = 2\pi\nu$ for a photon. First of all, they describe different physical quantities: the Planck function describes the *spectral radiance*, $\mathrm{d}E = B_\nu(T)\,\mathrm{d}t\,\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\Omega$, while the distribution $f(q)$ describes the number density of particles per unit momentum volume: $\mathrm{d}N = f(q)\,\mathrm{d}^3q$.

To check their equivalence, let us compute the energy density with both:

$$\rho = \frac{g}{(2\pi)^3} \int \frac{q}{e^{q/T} - 1}\,\mathrm{d}\Omega\, q^2\,\mathrm{d}q = \frac{2}{(2\pi)^3} \int \frac{\omega^3}{e^{q/T} - 1}\,\mathrm{d}\Omega\,\mathrm{d}q = \int \frac{2\nu^3}{e^{2\pi\nu/T}}\,\mathrm{d}\Omega\,\mathrm{d}q \tag{3.13}$$

$$\text{but also} \quad \rho = \frac{\mathrm{d}E}{\mathrm{d}V} = \int \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A\,\mathrm{d}q\,\mathrm{d}\Omega}\,\mathrm{d}q\,\mathrm{d}\Omega = \int B_q(T)\,\mathrm{d}\Omega\,\mathrm{d}q\ , \tag{3.14}$$

where we used the fact that, in natural units $\mathrm{d}V = \mathrm{d}t\,\mathrm{d}A$.

In general, we can say that if for some chemical species we have the reaction $i + j \leftrightarrow k + l$, and we reach chemical equilibrium, then the chemical potentials of the species will be connected by $\mu_i + \mu_j = \mu_k + \mu_l$: this is called the **Saha equation**.

Assuming that we are in thermal equilibrium is not in general valid, we will do so in our discussion for simplicity, but non-equilibrium dynamics must be considered when dealing with CMB anisotropies. Although the assumption does not perfectly hold, this is quite instructive: the CMB spectrum is very close to an equilibrium blackbody spectrum, the deviations from equilibrium are small.

By enumerating all the possible chemical reactions between the various particle types we will get a system of equations for their chemical potentials, complemented with some known facts, such as the fact that photons have $\mu_\gamma = 0$, which is the case since they do not interact with each other.

For example, from the annihilation of electron and positron $e^+ + e^- \leftrightarrow 2\gamma$ we can derive a relation between the chemical potentials of e^+ and e^-: $\mu_{e^+} = -\mu_{e^-}$.

Friday 2019-10-25, compiled 2020-09-14

We can relate some chemical potentials by reactions, but not all of them: our system of equations will be degenerate, with degeneracy corresponding precisely to the globally conserved quantities (electric charge, lepton number, baryon number) which follow from the symmetry group of our theory. These can have any value and are conserved in any reaction,[5] so they cannot be fixed by the system.

If there was a global electric charge, we'd expect global magnetic fields, but we only see them with magnitudes of the order of the 1 nT, which gives an upper bound on the global charge of the universe. So, any global electric charge would be quite small — we will assume it is exactly zero.

We can estimate the orders of magnitude for the abundances of the various particle species in the universe. The baryon number is very small when compared to the number of photons in the universe, roughly speaking $n_\gamma / n_b \sim 10^{10}$.

The lepton number is harder to estimate, but it is reasonable to assume that it is quite small as well. For slightly more detailed discussion, see the book by Weinberg [Wei72, before eq. 15.6.5].

In the end, we can say that in the early universe $\mu / T \ll 1$, so we can assume $\mu \approx 0$. This is just a reasonable simplification, which we make in order to get analytic results.

Under this assumption the quantities characterizing the matter distribution in the universe only depend on the temperature: so, we will just write $n(T)$, $\rho(T)$ and $P(T)$.

In general, when dealing with thermodynamic problems in an expanding spacetime there is a complication: in Minkowski spacetime we have symmetry under time translations, thus it makes sense to talk about stationarity. In an expanding universe, instead, we have no Killing vector with respect to time. There is a competition between two evolutions, the thermodynamic evolution of the system and the expansion of the universe: we cannot truly have equilibrium!

[5] This holds as long as the temperature is low enough: we are considering the reactions which are allowed by the Standard Model of interactions, with its symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y$, but it is not currently known whether at higher temperatures (i. e. earlier times) this is the most general symmetry group which is spontaneously broken to the SM group. So, the statements we make only apply at relatively late times.

The way to deal with this problem is: we assume that the first evolution is much faster than the other, that is, we reach thermal equilibrium on timescales that are short if compared to the expansion. This way, we can neglect the expansion of the universe while our system reaches equilibrium.

So our problem is oversimplified: we assume thermodynamic equilibrium, which makes sense in certain periods of the life of the universe, and that allows us to embed a thermal situation into a universe which evolves in time.

3.2.2 Entropy

From the second principle of thermodynamics we know that the entropy in a certain volume V at temperature T, denoted $S(V,T)$ is given by:

$$\mathrm{d}S = \frac{1}{T}\left(\underbrace{\mathrm{d}\big(\rho(T)V\big)}_{\mathrm{d}E} + P(T)\,\mathrm{d}V\right) = \frac{1}{T}\big(V\,\mathrm{d}\rho(T) + (P(T)+\rho(T))\,\mathrm{d}V\big)\,, \tag{3.15}$$

since in order to get the total energy we must multiply the constant energy by the volume: $E = \rho(T)V$.

Then we can read off the partial derivatives of the entropy:

$$\frac{\partial S}{\partial V} = \frac{1}{T}\big(\rho(T) + P(T)\big) \qquad \text{and} \qquad \frac{\partial S}{\partial T} = \frac{V}{T}\frac{\mathrm{d}\rho(T)}{\mathrm{d}T}\,. \tag{3.16}$$

In order for the differential to be exact it needs to be closed, which means that the second partial derivatives need to commute (these are known as the *Pfaff relations*):[6]

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \tag{3.17}$$

$$\frac{\partial}{\partial T}\left(\frac{1}{T}\big(\rho(T)+P(T)\big)\right) = \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{\mathrm{d}\rho(T)}{\mathrm{d}T}\right) \tag{3.18}$$

$$-\frac{1}{T^2}(\rho+P) + \frac{1}{T}\left(\frac{\mathrm{d}\rho}{\mathrm{d}T} + \frac{\mathrm{d}P}{\mathrm{d}T}\right) = \frac{1}{T}\frac{\mathrm{d}\rho}{\mathrm{d}T} \tag{3.19}$$

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{1}{T}(\rho+P)\,. \tag{3.20}$$

Simplified $T^{-1}(\,\mathrm{d}\rho/\mathrm{d}T\,)$, multiplied by T and brought $T^{-1}(\rho+P)$ to the other side

Cosmology has not entered into the picture yet, but it can by the third Friedmann equation, which can be rewritten as

$$\dot\rho = -3\frac{\dot a}{a}(\rho+P) \tag{3.21}$$

Multiplied by $-a^3$.

$$0 = 3\dot a a^2(\rho+P) + a^3\dot\rho \tag{3.22}$$

Added $a^3\dot P$ on both sides.

$$a^3\dot P = 3\dot a a^2(\rho+P) + a^3\dot\rho + a^3\dot P \tag{3.23}$$

$$a^3\dot P = \frac{\mathrm{d}(a^3)}{\mathrm{d}t} + a^3\frac{\mathrm{d}(\rho+P)}{\mathrm{d}t} \tag{3.24}$$

[6] They only hold in a simply connected space.

$$a^3 \dot{P} = \frac{\mathrm{d}}{\mathrm{d}t} \Big(a^3 (\rho + P) \Big) , \tag{3.25}$$

and these two, when put together, are equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{a^3}{T} (\rho(T) + P(T)) \right) = 0 , \tag{3.26}$$

therefore this quantity is a constant of motion. Let us verify this statement: expanding the derivative we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{a^3}{T} (\rho + P) \right) &= \frac{1}{T} \frac{\mathrm{d}}{\mathrm{d}t} \Big(a^3 (\rho + P) \Big) + a^3 (\rho + P) \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{T} \right) && (3.27) \\
&= \frac{1}{T} a^3 \dot{P} - a^3 (\rho + P) \frac{\dot{T}}{T^2} && (3.28) \\
&= \frac{a^3}{T} \frac{\mathrm{d}P}{\mathrm{d}T} \dot{T} - a^3 (\rho + P) \frac{\dot{T}}{T} && (3.29) \\
&= \frac{a^3}{T} \frac{(\rho + P)}{T} \dot{T} - a^3 (\rho + P) \frac{\dot{T}}{T} = 0 . && (3.30)
\end{aligned}$$

Used equation (3.20).

For the RW line element, the square root of the determinant is given by $\sqrt{-g} = a^3$, so the conserved quantity can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sqrt{-g} \frac{\rho + P}{T} \right) = 0 . \tag{3.31}$$

This is relevant because the volume of any given spatial region scales with $\sqrt{-g}$ as the universe expands.

So the quantity which is differentiated is constant. If we plug this back into the differential expression for the entropy, we get:[7]

$$\mathrm{d}S = \mathrm{d}\left(\frac{(\rho + P)V}{T}\right) , \tag{3.37}$$

therefore the differentiated quantities are equal up to an additive constant; from the conserved quantity we found and the fact that $V \propto a^3$ we now get that the **entropy is constant** in a comoving volume in thermal equilibrium:

$$S \equiv S(a^3, T) = \frac{a^3}{T}(\rho + P) = \text{const} . \tag{3.38}$$

Let us see what this entails: if we take photons, for example, we have $\rho \propto P \propto a^{-4}$: if we substitute this in we find that $a^{-4+3}/T = \frac{1}{aT}$ must be a constant, therefore $T \propto a^{-1}$. This is known as **Tolman's law**.

Did we derive it before?

We only consider photons since they have a much larger number density.

3.2.3 Explicit expressions for the thermodynamic quantities

Let us give explicit expressions for the number density, energy density and pressure as a function of time. We are always assuming isotropy, so in all cases we will be able to simplify the angular part of the triple integral in $\mathrm{d}^3 q$ as

$$\int \mathrm{d}^3\vec{q} = 4\pi \int_0^\infty \mathrm{d}q\, q^2 , \tag{3.39}$$

so the three expressions will read

$$n(T) = \frac{g}{2\pi^2}\int \mathrm{d}q\, q^2 f(q) \tag{3.40a}$$

[7] The procedure to prove this result is as follows: the expression we want to show is equal to $\mathrm{d}S$ can be written like

$$\mathrm{d}S \stackrel{?}{=} \mathrm{d}\left(\frac{(\rho+P)V}{T}\right) = \left(-\frac{(\rho+P)V}{T^2} + \frac{V}{T}\left(\frac{\mathrm{d}\rho}{\mathrm{d}T} + \frac{\mathrm{d}P}{\mathrm{d}T}\right)\right)\mathrm{d}T + \frac{\rho+P}{T}\,\mathrm{d}V , \tag{3.32}$$

while the definition of $\mathrm{d}S$ (3.15) can be written as

$$\mathrm{d}S = \frac{V}{T}\frac{\mathrm{d}\rho}{\mathrm{d}T}\,\mathrm{d}T + \frac{\rho+P}{T}\,\mathrm{d}V , \tag{3.33}$$

so we can see that, since the term proportional to $\mathrm{d}V$ is the same in both cases, we only need to show that the coefficients of $\mathrm{d}T$ are equal, so what we need to prove is

$$-\frac{(\rho+P)V}{T^2} + \frac{V}{T}\left(\frac{\mathrm{d}\rho}{\mathrm{d}T} + \frac{\mathrm{d}P}{\mathrm{d}T}\right) \stackrel{?}{=} \frac{V}{T}\frac{\mathrm{d}\rho}{\mathrm{d}T} \tag{3.34}$$

$$-\frac{(\rho+P)V}{T^2} + \frac{V}{T}\frac{\mathrm{d}P}{\mathrm{d}T} \stackrel{?}{=} 0 \tag{3.35}$$

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{\rho+P}{T} , \tag{3.36}$$

which is precisely the statement we found to be equivalent to the Pfaff relations (3.20).

$$\rho(T) = \frac{g}{2\pi^2} \int \mathrm{d}q\, q^2 f(q) E(q) \tag{3.40b}$$

$$P(T) = \frac{g}{6\pi^2} \int \mathrm{d}q\, q^2 f(q) \frac{q^2}{E(q)}\,. \tag{3.40c}$$

In general these do not have analytic solutions, however if we only consider the ultrarelativistic and nonrelativistic limiting cases we can do the calculation.

Ultrarelativistic limit A particle being ultrarelativistic means that its momentum is much greater than its rest energy, $q \gg m$.

In our case we do not really care about any single particle being ultrarelativistic, rather, we ask that the temperature is high enough that the bulk of the particles is ultrarelativistic.

The momentum of any single particles will not always be large — in fact the distribution has its maximum at $q = 0$ — but the regions in which it is large give a much greater contribution than those in which it is small, as long as the temperature is large.

We define the rescaled momentum $x = q/T$, so that then the term appearing in the exponential is $E(q)/T = \sqrt{x^2 + m^2/T^2} \approx x = q/T$ under the assumption that $m/T \ll 1$.

With this assumption we get:

$$n(T) = \frac{g}{2\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^2 \Big(\exp\big(q/T\big) \mp 1 \Big)^{-1} \tag{3.41}$$

$$\rho(T) = \frac{g}{2\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^3 \Big(\exp\big(q/T\big) \mp 1 \Big)^{-1} \tag{3.42}$$

$$P(T) = \frac{g}{6\pi^2} \int_{\mathbb{R}^+} \mathrm{d}q\, q^3 \Big(\exp\big(q/T\big) \mp 1 \Big)^{-1}, \tag{3.43}$$

so we can see that in this approximation, which is equivalent to $m \approx 0$, we get matter behaving like radiation: $P = \rho/3$.

The result of the integrals depends on the statistics of the particles (which determine the $\pm$ sign in the distribution), and it is given by the following expressions:

$$n(T) = \begin{cases} \dfrac{\zeta(3)}{\pi^2} g T^3 & \text{Bose-Einstein} \\[2ex] \dfrac{3}{4} \dfrac{\zeta(3)}{\pi^2} g T^3 & \text{Fermi-Dirac}\,, \end{cases} \tag{3.44}$$

where $\zeta(3)$ is the Riemann zeta function calculated at 3, giving

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202\,. \tag{3.45}$$

and we get the proportionality to T^3. For the energy density:

$$\rho(T) = \begin{cases} \dfrac{\pi^2}{30} g T^4 & \text{Bose-Einstein} \\[2ex] \dfrac{7}{8} \dfrac{\pi^2}{30} g T^4 & \text{Fermi-Dirac}\,, \end{cases} \tag{3.46}$$

while to get the result for the pressure $P(T) = \rho(T)/3$ we just divide by 3.

We note that in natural units the Stefan-Boltzmann constant is $\sigma_{\text{SB}} = \pi^2/15$; the result we have found coincides with the Stefan-Boltzmann law for photons, which obey Bose-Einstein statistics and have two helicity states: $g = 2$, therefore

$$\rho(T) = 2\frac{\pi^2}{30}T^4 = \sigma_{\text{SB}}T^4 \,. \tag{3.47}$$

Nonrelativistic limit Now we work in the opposite limit, $m \gg T$. We can then expand the energy in powers of q/m (or T/m: as before, the point is that the typical value of q is T, so we can do it either way):

$$E = m\sqrt{1 + \frac{q^2}{m^2}} \approx m + \frac{q^2}{2m} + \mathcal{O}\left(\frac{q^2}{m^2}\right) . \tag{3.48}$$

The first temptation one might have is to work at the lowest possible order, approximating $E \approx m$. The exponential $\exp(E/T)$ will be very large compared to 1, so we can neglect the ± 1 in the denominator (which also means that the difference between bosons and fermions becomes negligible).

So, to zeroth order in (q/m) we get

$$f \approx \exp\left(-\frac{m-\mu}{T}\right) , \tag{3.49}$$

therefore the number density will be given by

$$n = \frac{g}{2\pi^2}\exp\left(-\frac{m-\mu}{T}\right)\int_{\mathbb{R}^+} \mathrm{d}q\, q^2 \,, \tag{3.50}$$

which diverges.

This is called the ultraviolet catastrophe: it is due to the fact that, while we are assuming q is small, we are not enforcing this in any way, and approximating all states as having the same energy regardless of their momentum. If, instead, we go to first order in q/m then we find

$$n = \frac{g}{2\pi^2}\exp\left(-\frac{m-\mu}{T}\right)\int_{\mathbb{R}^+} \mathrm{d}q\, q^2 \exp\left(-\frac{q^2}{2mT}\right) = g\left(\frac{mT}{2\pi}\right)^{3/2}\exp\left(\frac{\mu-m}{T}\right) , \tag{3.51}$$

where we applied the identity

$$\int_{\mathbb{R}} \mathrm{d}x\, x^2 \exp\left(-\alpha x^2\right) = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \,. \tag{3.52}$$

The exponential factor $\exp(-m/T)$ is known as the Boltzmann suppression factor, which tells us that as long as relativistic and nonrelativistic particles are in thermal equilibrium there will be a much smaller number of the latter.

The energy density can be easily recovered from the number density if neglect higher order terms:

$$\rho(T) = \frac{g}{2\pi^2} \int \mathrm{d}q\, q^2 E(q) f(q) \approx \frac{g}{2\pi^2} \int \mathrm{d}q \left(m + \frac{q^2}{2m} \right) q^2 f(q) \approx m \underbrace{\frac{g}{2\pi^2} \int \mathrm{d}q\, q^2 f(q)}_{n(T)} . \tag{3.53}$$

For the pressure, on the other hand, we have

$$P(T) = \frac{g}{6\pi^2} \int \mathrm{d}q\, q^4 \frac{f(q)}{m + \frac{q^2}{2m}} \approx \frac{g}{6\pi^2} e^{-m/T} \int \mathrm{d}q\, \frac{q^4}{m} \exp\left(-\frac{q^2}{2mT} \right), \tag{3.54}$$

and now we can apply the Gaussian integral identity [WA03, special case of eq. 10.1.11 (b)]:

$$\int_{\mathbb{R}} \mathrm{d}x\, x^4 \exp\left(-\alpha x^2 \right) = \frac{3}{8\alpha^2} \sqrt{\frac{\pi}{\alpha}}, \tag{3.55}$$

where for us $\alpha = 1/2mT$, which gives us

$$P(T) = \frac{g}{6\pi^2} \frac{e^{-m/T}}{m} \frac{3(2mT)^2}{8} \sqrt{2mT\pi} = g \frac{m^{3/2} T^{5/2}}{\pi^{3/2} 2^{3/2}} e^{-m/T} = g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T} \times T \tag{3.56}$$

$$= n(T) T . \tag{3.57}$$

Therefore, $P = Tn = (T/m)\rho$, which tells us that the pressure of the nonrelativistic particles is much smaller than their energy density, since $T/m \ll 1$: we characterize them as *noninteracting dust*. The result we found, $P = nT$, is just the ideal gas law.

If we compare relativistic particles to nonrelativistic ones, the former dominate the latter in terms of all of these three quantities.

The physical context in which this becomes relevant, in the early universe, is that whenever the temperature drops below the mass of a certain particle, that particle starts to become nonrelativistic and its density drops exponentially, due to the Boltzmann suppression.

The main way for the particle to do so is generally to annihilate with its own antiparticle, thus producing radiation.

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Effective degrees of freedom We have been discussing the behavior of a single particle species with g degrees of freedom; however we know that there were many types of particles in the early universe, so we need a way to generalize these results. We do so by defining the number of effective degrees of freedom:

$$g_*(T) = \sum_{i \in \mathrm{BE}} g_i \left(\frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i \in \mathrm{FD}} g_i \left(\frac{T_i}{T} \right)^4, \tag{3.58}$$

where the index i labels all the particle species in our model, running over all of those which are relativistic at temperature T (that is, as a first approximation, we only count those with masses $m_i < T$). The equilibrium temperature is T, while T_i are the temperatures of the various particle species, which we allow to be different from T — we will elaborate on this point in a moment. We distinguish two different terms in the sum, depending on whether the particle species obey Bose-Einstein or Fermi-Dirac statistics, since as we have seen the latter have a prefactor of 7/8 in the expression for the energy density.

This definition is constructed so that we can write the compact relation

$$\rho(T) = g_*(T)\frac{\pi^2}{30}T^4 \,. \tag{3.59}$$

Of course, considering particles completely when $m_i < T$ and not at all when $m_i > T$ is a simplification: in the region in which the temperature is of the order of the mass of the particle there will be a transition, which can be calculated properly by doing the integrals numerically. The results are shown in figure 3 of a paper by Husdal [Hus16], which can also be referred to for many more details on effective degrees of freedom. Figure 1 of the same paper shows how g_* decreases while the temperature of the universe decreases and more and more particle species become nonrelativistic.

Why do we consider the possibility of the temperature of a particle species being different from the equilibrium temperature?

Each process involving particles, be it decay or scattering, is characterized by a certain timescale. If the timescale of a certain interaction is larger than the cosmological timescale (the age of the universe), then that interaction statistically will not happen. Particles which cannot reach thermal equilibrium because of this are called *decoupled*, ones for which this is not the case are called *coupled*.

Although they may not interact, as long as they are relativistic decoupled particles can still affect the energy density of the universe, so we need to count them.

The time-temperature relation We want to find a relation between time and temperature in the early universe. Let us consider ultrarelativistic particles which are coupled, in the early universe which is radiation dominated (here "radiation" refers to all kinds of ultrarelativistic particles).

We start from the third Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \,, \tag{3.60}$$

neglect the curvature term[8] and use the facts that for a radiation-dominated universe $\rho \propto a^{-4}$ while $a \propto t^{1/2}$, meaning that $H = \dot{a}/a = 1/(2t)$.

Substituting these, as well as the expression we have found for the energy density in terms of the effective degrees of freedom, we get

$$\frac{1}{4t^2} = \frac{8\pi G}{3} g_*(T)\frac{\pi^2}{30}T^4 \,. \tag{3.61}$$

[8] We can do so since we know that right now the contribution to the global Ω of curvature is small (we have not been able to distinguish it from zero) and while this term scales as a^{-2} the matter term scales as a^{-3}. Since the matter term is dominant over the curvature term now, it was even more so earlier.

Then we have a formula for temperature in terms of time:

$$\frac{1}{2t} = \left(\frac{8\pi G}{3}\right)^{1/2} g_*^{1/2} \left(\frac{\pi^2}{30}\right)^{1/2} T^2 \tag{3.62}$$

$$t \approx \underbrace{\frac{1}{2\sqrt{\frac{8\pi\pi^2}{3\times 30}}}}_{\approx 0.301} g_*^{-1/2} \frac{m_P}{T^2} \approx \left(\frac{T}{\mathrm{MeV}}\right)^{-2} \mathrm{s}\,, \tag{3.63}$$

where $m_P = G^{-1/2} \approx 1.2 \times 10^{19}\,\mathrm{GeV}$ is the Planck mass. Beware: there are different conventions for this, sometimes the definition is chosen as $m_P = (8\pi G)^{-1/2}$, which simplifies the Friedmann and Einstein equations somewhat. This mass corresponds to the energy scale at which quantum gravitational effects cannot be neglected.

The last approximation in (3.63) is quite rough, as it neglects the variation of the effective number of degrees of freedom completely: however, the factor $g_*^{-1/2}$ is of order 1 around $T \approx 1\,\mathrm{MeV}$, which is the region in which we will apply our formula, so this is fine for our purposes.

Entropy effective degrees of freedom Entropy density is defined as entropy per unit volume, $s = S/V = (P + \rho)/T$. Since the total entropy in a comoving region is conserved (if there is thermal equilibrium) the quantity sa^3, proportional to S, is conserved.

If we only have relativistic particles (which satisfy $P = \rho/3$), the entropy density can be expressed as

$$s = (P + \rho)/T = \frac{4}{3}\frac{\rho}{T} = (2\pi^2/45) g_{*s} T^3\,; \tag{3.64}$$

where we defined a new number of effective degrees of freedom, g_{*s}, whose definition is slightly different from that of the one used for the energy, $s \propto T^3$ as opposed to $\rho \propto T^4$:

$$g_{*s} \equiv \sum_{i\in BE} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i\in FD} g_i \left(\frac{T_i}{T}\right)^3 . \tag{3.65}$$

The expression for $s \propto g_{*s}T^3$ is more general than simply $s \propto T^3$, and in fact with this new one we can **update Tolman's law**: taking the cube root of the conserved quantity sa^3 we find $Tag_{*s}^{1/3} = \mathrm{const}$.

3.2.4 Decoupling and radiation temperature

The temperature $1\,\mathrm{MeV}$ occurs when the age of the universe is approximately $1\,\mathrm{s}$, and this is the point at which the weak interactions involving neutrinos stop occurring.[9]

[9] This point is also relevant for another process: the weak interaction mediated processes are also what allows there to be an equilibrium between protons and neutrons, so when those reactions stop they become independent, and they evolve differently, since protons are stable while neutrons are not. This is crucial when discussing nucleosynthesis, the formation of nuclei.

When this happens, neutrinos decouple, so they stop interacting: they start evolving as any relativistic particle species would (they are relativistic since their mass is much lower than the MeV).

At this point, the temperatures of neutrinos and photons are "disconnected", there is no mechanism to equalize them. However, while the neutrinos are freely floating by, their energy density scaling like $\rho \propto a^{-4}$, the photons will be active for a few more seconds.

The mass of electrons and positrons is around 0.5 MeV, so until about 4 s into the life of the universe, 3 s from the decoupling of the neutrinos, the reaction $e^+ + e^- \leftrightarrow 2\gamma$ is still in equilibrium. After this, the populations of electrons and positrons annihilate, and since they have less effective degrees of freedom in which to deposit their energy than before (since the neutrinos are not coupled anymore) they dump it all into photons, thus increasing their temperature.

After this occurs, the photons keep evolving like any other relativistic particle species, with $\rho \propto a^{-4}$ but their temperature is higher than that of the neutrinos. Since they evolve in the same way, the ratio of the temperatures is constant. Now we will calculate this ratio.

We impose continuity of the entropy in a comoving volume across the transition which happens at 0.5 MeV between the stage in which electrons, positrons and photons are in equilibrium and the stage in which they decouple, since the electrons and positrons are not relativistic anymore and thus annihilate.

This transition only affects the temperature of the photons, while the neutrinos decoupled three seconds earlier; so, before the transition the temperatures of photons and neutrinos are equal, after it the photons' temperature increases.

Let us denote with an index $>$ the quantities pertaining to an earlier time, $t_{\text{transition}} > t$, while an index $<$ will denote the quantities pertaining to a later time, $t_{\text{transition}} < t$.

The "updated" version of Tolman's law reads $Tag_{*s}^{1/3} = \text{const}$, and if the transition is fast enough the scale factor can be taken to be equal on both sides of it. Therefore, we have

$$T_< = \left(\frac{g_{*s>}}{g_{*s<}}\right)^{1/3} T_> \,, \tag{3.66}$$

so we can compute the temperature after the transition, $T_<$, if we find the effective degrees of freedom before and after: before the transition we have photons, electrons and positrons. Photons have two polarization, and so do both electrons and positrons; also, the latter are fermions, so we find

$$g_{*s>} = 2 + \frac{7}{8}(4) = \frac{11}{2} \,, \tag{3.67}$$

while after the transition only photons are relativistic, so we have

$$g_{*s<} = 2 \,. \tag{3.68}$$

This means that the temperature of the photons increases by a factor

$$T_< = \left(\frac{11}{4}\right)^{1/3} T_> \approx 1.4 T_> \,, \tag{3.69}$$

which allows us to compute the neutrino temperature at any time, since $T_</T_> = T_\gamma/T_\nu$, and because they scale in the same way:

$$T_\nu = T_\gamma \left(\frac{4}{11}\right)^{1/3} . \tag{3.70}$$

Right now, the temperature will be around $T_{0\nu} \approx (4/11)^{1/3} T_{0\gamma} \approx 1.94\,\mathrm{K}$, where $T_{0\gamma}$ is the current CMB temperature.

As an exercise, let us compute the number of effective degrees of freedom some time after the decoupling of electrons, say at $T = 0.1\,\mathrm{MeV}$. The global temperature T we are referring to is the one of the photons: so, applying the definition we find

$$g_* = \sum_{i \in BE} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i \in FD} g_i \left(\frac{T_i}{T}\right)^4 \tag{3.71a}$$

$$= 2 + \frac{7}{8} \left(3 \times 2 \left(\frac{T_\nu}{T_\gamma}\right)^4\right) \tag{3.71b}$$

$$= 2 + \frac{21}{4} \left(\frac{4}{11}\right)^{4/3} \approx 3.36\,, \tag{3.71c}$$

since we need to consider neutrinos (of which there are three flavors, each having two polarization states), which contribute to the total energy density, but not electrons which are not relativistic anymore.

With this result, we can find the energy density of radiation at that time according to (3.59): we get

$$\rho_r(0.1\,\mathrm{MeV}) \approx 1.1 \times 10^{-4}\,\mathrm{MeV}^4 = 25.7\,\mathrm{g cm^{-3}}\,. \tag{3.72}$$

Multiplied by $\hbar^{-3} c^{-5}$ to get the CGS units.

3.3 Problems with the Hot Big Bang model

Around the 1960s, cosmologies were trying to piece together a description of the early universe in terms of particle physics, as we discussed in this chapter up to here. However, soon it became apparent that the standard cosmological model in use has some inconsistencies. Let us now explore these.

3.3.1 The cosmological horizon problem

This was noticed as early as 1956. Let us consider radial null geodesics in a universe described by a FLRW metric. These are the worldlines of photons we can detect with telescopes. Imposing $\mathrm{d}s^2 = 0$ we find:

$$c^2\,\mathrm{d}t^2 = a^2(t) \frac{1}{1 - kr^2}\,\mathrm{d}r^2 \ , \tag{3.73}$$

which we can integrate (taking one of the two solutions for simplicity, choosing one over the other just amounts to parametrizing time in the opposite direction) to find

$$\int_0^t \frac{c\,\mathrm{d}t}{a(t)} = \int_0^r \frac{\mathrm{d}\widetilde{r}}{\sqrt{1-k\widetilde{r}^2}} = f(r)\,. \tag{3.74}$$

The function $f(r)$ gives us the proper distance between emission and detection of a photon, but it does so in terms of the adimensional coordinate r: in order to get something which has the dimensions of a length we need to multiply by a calculated at a certain time,[10]

$$d_{\mathrm{hor}}(t) = a(t) \int_0^t \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})}\,. \tag{3.75}$$

If this integral is convergent, we should be worried: let us see why.

If we integrate from the beginning of time to now, we get the spatial (current) comoving distance elapsed by a photon which started moving at the start of time. This is the radius of the largest region we could in principle observe. It is of the order of 3 Gpc. Since the integral is convergent this is finite, and it is increasing as time passes. So, ever-further regions are "coming into view" (at least in principle). Roughly speaking, the issue is that the regions which we start seeing at the edges should be causally disconnected from the ones already in view, so we would not expect them to exhibit the same properties — but *they do*. This is the basic idea, let us formalize it slightly and connect it to observations.

We cannot actually see light coming from the very edge of the in-principle-observable universe, since for redshifts larger than $z_{LS} \approx 1100$ the universe was opaque to electromagnetic radiation. The surface of points at this redshift is called the *Last Scattering* surface. So, we refer our expectations to the CMB, which was emitted as the primordial plasma became transparent.

The CMB was emitted at a cosmic time of $t \approx 3.8 \times 10^5$ yr after the Big Bang. It is observed to be very close to being uniform, with $\Delta T/T \sim 10^{-5}$ after correcting for the Doppler dipole modulation: it looks like a distribution emitted by matter in thermal equilibrium. Crucially, this holds at any angular scale we choose: the equilibrium is there across the whole sphere.

Recall that the *angular diameter distance* d_A is defined so that if an object with linear size at emission Δx spans an angle $\Delta\theta$ then we have (in the small-angle approximation)

$$d_A = \frac{\Delta x}{\Delta\theta}\,. \tag{3.76}$$

The angular diameter distance to the last scattering surface is approximately $d_A(z_{LS}) \approx 12.8$ Mpc. On the other hand, the scale of the particle horizon at that redshift can be calculated by taking the difference of comoving distances to us, $d_C(z_\infty) - d_C(z_{LS}) \approx 281$ Mpc and multiplying it by the scale factor, $a(z_{LS}) \approx 9.2 \times 10^{-4}$, which yields a horizon scale of

[10] Note that this choice is arbitrary: we are computing the comoving distance *as measured at the cosmic time of detection*.

approximately $r_H \approx 260\,\mathrm{kpc}$ at that time.[11]

We can then say that $\Delta x \sim r_H$, therefore the angular scale at which we expect to be able to observe correlations since there can be causal connections is around

$$\Delta\theta \approx \frac{\Delta x}{d_A} \approx 0.02\,\mathrm{rad} \approx 1.2^\circ\,. \tag{3.77}$$

A similar calculation [Toj, eqs. 8–12] yields $\Delta\theta \sim (1 + z_{LS})^{-1/2} \approx 1.7^\circ$, using the (reasonable) assumption of matter dominance in the epoch of recombination. Some steps there are not really clear to me, so I'm not sure whether my line of reasoning is equivalent (and valid) besides the assumption.

This is in stark opposition with the scale of observed correlations, which span the whole sky!

Mention in the lecture of the Mixmaster Universe by Misner (and the Bianchi classification of Lie Algebras for context) as an alternative to inflation — is this relevant here?

Cosmic inflation Now, if the quantity $d_{\mathrm{Hor}}(t)$ were to diverge this would mean that we could have a causal connection with any point in the universe, provided we went far enough back in time.

We can approximate

$$d_{\mathrm{Hor}}(t) = a(t) \int_0^t \frac{c\,\mathrm{d}\widetilde{t}}{a(\widetilde{t})} \approx ct \sim \frac{c}{H} \equiv d_{\mathrm{H}}\,, \tag{3.78}$$

where we defined the new *Hubble distance*, $d_H = c/H$. This is a physical distance, but we can also define the corresponding dimensionless comoving Hubble radius: $r_H = c/(Ha) = c/\dot{a}$, which satisfies $d_H = a r_H$.

We can hypothesize that there was a period in the early universe when the comoving radius r_H was decreasing with time: if this is the case, the regions we are observing today as "coming into view" could actually have been in causal contact in the early universe.

For the comoving radius to be decreasing ($\dot{r}_H < 0$), the condition is (neglecting factors of c, or working in natural units):

$$\dot{r}_H = -\frac{\ddot{a}}{\dot{a}^2} < 0\,, \tag{3.79}$$

[11] All the calculations were made automatically using the `astropy` package, using a flat ΛCDM model with parameters obtained from the Planck mission [Col16].

```python
from astropy.cosmology import Planck15 as cosmo
import numpy as np
import astropy.units as u
z_LS = 1089
dx = (cosmo.comoving_distance(np.inf) - cosmo.comoving_distance(z_LS)) * cosmo
    .scale_factor(z_LS)
dA = cosmo.angular_diameter_distance(z_LS)
(dx / dA).to(u.degree, equivalencies=u.dimensionless_angles())
```

therefore we need $\ddot{a} > 0$ for at least some time.

The second Friedmann equation (in natural units) tells us that

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3P), \tag{3.80}$$

therefore the condition we need to have is $\rho + 3P < 0$. So, since the energy density is positive, the condition is $P < -\rho/3$.

Another way to express the parameter $\ddot{a}$ is as the derivative of $\dot{a} = Ha$:

$$\ddot{a} = \dot{a}H + a\dot{H} = a(H^2 + \dot{H}) > 0, \tag{3.81}$$

so the condition can also be expressed as $H^2 + \dot{H} > 0$. We can also characterize the solutions based on the sign of $\dot{H}$, which determines whether $P/\rho = w$ is larger or smaller than -1: the possibilities are

1. $\dot{H} < 0$ while $\dot{H} + H^2 > 0$: this corresponds to $-1 < w < -1/3$;
2. $\dot{H} = 0$: this is a De Sitter, dark-energy dominated universe, whose scale factor evolves like $a(t) = \exp(Ht)$, corresponding to $w = -1$;
3. $\dot{H} > 0$ (and so also $\dot{H} + H^2 > 0$): here the solution looks like

   $$a(t) = a_* \left(1 + \frac{3}{2}\left((1+w)H_*(t - t_*)\right)\right)^{2/(3(1+w))}, \tag{3.82}$$

   which corresponds to $w < -1$ and means $a(t) \propto t^p$ for some $p > 1$. This is called power-law inflation.

The boundary at $w = -1$ is called the *phantom divide*.

By how much does the early universe need to inflate? The condition we need to impose is that the comoving radius of the universe at some early time, $r_H(t_0)$, should be larger than the current one, $r_H(t_0)$. Since $r_H = d_H/a$, we can write this inequality as

$$\frac{d_H(t_{\text{in}})}{a(t_{\text{in}})}a(t_f) \geq \frac{d_H(t_0)}{a(t_0)}a(t_f), \tag{3.83}$$

where we multiplied both sides by the scale factor calculated at t_f, a time corresponding to the end of inflation, a minimum for the scale factor. Let us define $Z_{\text{min}} = a(t_f)/a(t_{\text{in}})$. This will be $\gg 1$, and it will describe by how much the universe inflated.

In our rough approximation $d_H \sim H^{-1}$, so we can say that the boundary of the inequality, the minimum inflationary expansion, will be

$$Z_{\text{min}} = \frac{d_H(t_0)}{d_H(t_{\text{in}})}\frac{a(t_f)}{a(t_0)} \tag{3.84}$$

$$= \frac{H(t_{\text{in}})}{H(t_0)}\frac{a(t_f)}{a(t_0)} \tag{3.85}$$

$$= \frac{H(t_{\text{in}})}{H(t_f)}\frac{H(t_f)}{H(t_0)}\frac{a(t_f)}{a(t_0)} \tag{3.86}$$

$$Z_{\text{min}} \frac{H_f}{H_{\text{in}}} = \frac{H_f}{H_0} \frac{a_f}{a_0}. \tag{3.87}$$

Denoting $H(t_i) \equiv H_i$ and similarly for a.

Now, we want to put some numbers into this expression: we know from the first Friedmann equation that (as long as there is no spatial curvature) $H^2 \propto \rho$, while the third tells us that $\rho \propto a^{-3(1+w)}$: therefore, $H \propto a^{-3\frac{1+w}{2}}$. Since we are working with ratios, proportionality is all we need. Now, what w should we use? At any stage in the evolution of the universe there are several fluids, but in order to simplify the calculation we will only consider the dominant one and neglect dark energy in the current phase of the evolution of the universe. In the inflationary phase we will have an undetermined $w = w_{\text{inf}}$, so that

$$\frac{H_f}{H_{\text{in}}} = \left(\frac{a_f}{a_{\text{in}}}\right)^{-3\frac{1+w}{2}}, \tag{3.88}$$

so the left-hand side of the equation reads

$$Z_{\text{min}} \frac{H_f}{H_{\text{in}}} = Z_{\text{min}}^{1-3\frac{1+w_{\text{inf}}}{2}} = Z_{\text{min}}^{\frac{-1-3w_{\text{inf}}}{2}} = Z_{\text{min}}^{\left|\frac{1+3w_{\text{inf}}}{2}\right|}, \tag{3.89}$$

since $w_{\text{inf}} > -1/3$ means $1 + 3w_{\text{inf}} > 0$. The right-hand side has precisely the same form, so we can express it as

$$\frac{H_f}{H_0} \frac{a_f}{a_0} = \left(\frac{a_f}{a_0}\right)^{-\frac{1+3w}{2}}, \tag{3.90}$$

where w is that of the dominant fluid from the end of inflation to now. The issue is, there is not a single one! In the early stages radiation was dominant, then matter started dominating (now dark energy is dominant, but we shall not worry about it). So, we split the term in two, with the radiation-matter equality being the breaking point. The earlier radiation-dominated phase is characterized by $w = 1/3$, while the latter matter-dominated phase is characterized by $w = 0$, so we can compactly write the term as

$$\frac{H_f}{H_0} \frac{a_f}{a_0} = \left(\frac{a_f}{a_{\text{eq}}}\right)^{-\frac{1+1}{2}} \left(\frac{a_{\text{eq}}}{a_0}\right)^{-\frac{1+0}{2}} = \left(\frac{a_{\text{eq}}}{a_f}\right) \left(\frac{a_0}{a_{\text{eq}}}\right)^{1/2}. \tag{3.91}$$

With this result, we now have an almost explicit expression for Z_{min}:

$$Z_{\text{min}} = \left[\left(\frac{a_{\text{eq}}}{a_f}\right) \left(\frac{a_0}{a_{\text{eq}}}\right)^{1/2}\right]^{\left|\frac{2}{1+3w_{\text{inf}}}\right|} = \left[\left(\frac{a_0}{a_f}\right) \left(\frac{a_0}{a_{\text{eq}}}\right)^{-1/2}\right]^{\left|\frac{2}{1+3w_{\text{inf}}}\right|}. \tag{3.92}$$

The ratio a_0/a_{eq} can be written as $1 + z_{\text{eq}} \approx 10^4$ (very roughly). The other rough estimate we make is to apply Tolman's law, so that

$$\frac{a_0}{a_f} \approx \frac{T_f}{T_0} = \frac{T_f}{m_p} \underbrace{\frac{m_p}{T_0}}_{\sim 10^{32}}; \tag{3.93}$$

properly speaking this only holds when there is radiation dominance so it does not apply for the whole range in which we are applying it (up to today), but we are estimating the order of magnitude *of an exponent*, so even an order-of-magnitude error is not an issue. We normalized by the Planck mass (or temperature, since we are using natural units) since the temperature T_f at the end of inflation probably was of that order of magnitude. The final estimate we get is

$$Z_{\text{min}} \approx \left[10^{30}\frac{T_f}{m_p}\right]^{\left|\frac{2}{1+3w_{\text{inf}}}\right|}. \tag{3.94}$$

We do not know what w_{inf} is besides it being smaller than $-1/3$; let us say that it is of the order of -1 like the current dark-energy dominated phase. If we further assume that $T_f/m_p \sim 1$, we find $Z_{\text{min}} \sim 10^{30} \approx \exp(70)$.

This is often written as "70 e-foldings", meaning 70 e-fold increases in size.

3.3.2 The flatness problem

Now, let us consider the *flatness problem*, which was first proposed by Dick and Peebles in 1986.

The parameter $\Omega = \frac{8\pi G\rho(z)}{3H^2(z)}$ diverges from 1 as time increases. Measurements of Ω_{tot} gave approximately 0.1.

How is it possible that the universe is still so flat, even when the universe is so old? Oldness and flatness seem incompatible.

This is a type of *fine-tuning* problem. Typically, if there is a fine-tuning problem then it signals that we should improve our theory.

Let us assume that w is constant. Then, $\rho(z) = \rho_0(1+z)^{3(1+w)}$. Recall that

$$H^2(z) = \frac{8\pi G}{3}\rho(z) - \frac{k}{a^2} \tag{3.95a}$$

$$H_0^2 = \frac{8\pi G}{3}\rho_0 - \frac{k}{a_0^2}, \tag{3.95b}$$

the latter of which implies $1 = \Omega - k/(a_0^2 H_0^2)$. So,

$$H^2(z) = H_0^2\left(\frac{8\pi G}{3H_0^2}\rho(z) - \frac{k}{a^2H_0^2}\right) \tag{3.96a}$$

$$= H_0^2\left(\frac{\rho(t)}{\rho_0}\Omega_0\frac{a_0^2}{a}(1-\Omega_0)\right) \tag{3.96b}$$

$$= H^2(z) = H_0^2(1+z)^2\Big(\Omega_0(1+z)^{1+3w} + (1-\Omega_0)\Big), \tag{3.96c}$$

so in the end

$$\Omega(z) = \frac{8\pi G\rho_0}{3H_0^2}\frac{\rho(z)}{\rho_0}\frac{H_0^2}{H^2(z)} \tag{3.97a}$$

$$= \Omega_0 (1+z)^{1+3w} \Big(1 - \Omega_0 + \Omega_0 (1+z)^{1+3w} \Big) \,, \tag{3.97b}$$

and since $\rho(z) = \rho_0 (1+z)^{3(1+w)}$ we get

$$\Omega^{-1}(z) - 1 = \frac{\Omega_0 \Big((1-\Omega_0) + \Omega_0 (1+z)^{1+3w} \Big) - (1+z)^{1+3w}}{(1+z)^{1+3w}} \tag{3.98a}$$

$$= (\Omega_0^{-1} - 1)(1+z)^{-(1+3w)} \,. \tag{3.98b}$$

If we assume $w = 1/3$ for all times (which is false, but we do it to get a result that is close enough) we get

$$\Omega^{-1}(z) - 1 = (\Omega_0^{-1} - 1) \left(\frac{T_0}{T(z)} \right)^2 \,. \tag{3.99}$$

If we compute T_{Planck}/T_0 we get approximately 4.5×10^{32}. Then when squaring we get 10^{-64}, without the approximation $w = 1/3$ we get 10^{-60}.

Then, we see that there is something deeply unnatural in the Friedmann model.

Thu Nov 07 2019

We talk about inflation again. The comoving horizon increases with time:

$$d_H(t) = a(t) \int_0^t \frac{c \, \mathrm{d}\widetilde{t}}{a(\widetilde{t})} \sim ct \sim \frac{c}{H} \equiv \text{Hubble horizon} \,, \tag{3.100}$$

it is also called the *past event horizon*. The reason why we can plot the history of the universe in a single spacetime diagram is because there is a well-defined transformation which brings an infinite interval to a finite one, using a hyperbolic arctangent: this gives us a *Penrose diagram*.

The comoving Hubble radius is $r_H = c/aH = c/\dot{a}$. This can grow.

If we have positive pressure $p > 0$, than the scale factor goes like $a \propto t^{2/(3(1+w))}$ with $w > 0$, then the comoving radius increases with time.

We can actually still get increasing comoving radii with a weaker condition: $p > -\frac{1}{3}\rho c^2$. This is the actual boundary (it can be checked looking at the derivative of a).

This is directly connected to the sign of the acceleration in the Friedmann equation.

If these conditions are always met, Hawking and Ellis proved that a Big Bang is inevitable.

The inflation hypothesis is that, even though now we have $\ddot{a} < 0$ now (or, it was so for some time: now the expansion seems to be accelerating), there was a period in which we had $\ddot{a} > 0$.

These are drawn as straight lines, but it is only qualitative: we are looking at the sign of the slope.

Then, there was a time in the past at which the comoving horizon was as large as it is now: now we will see how much inflation there must have been in order to solve the horizon problem up to now, ignoring the fact that now the universe's expansion is accelerating.

The inequality we want to impose is

$$r_H(t_i) \geq r_H(t_0)\,, \tag{3.101}$$

where t_0 is now while t_i is the beginning of inflation.

A sphere with comoving radius $d_H(t_i)$ will expand after inflation up to

$$d_H(t_i)\frac{a(t_f)}{a(t_i)}\,, \tag{3.102}$$

where t_f is the time of the end of inflation.

$$d_H(t_i)\frac{a(t_f)}{a(t_i)} \geq d_H(t_0)\frac{a(t_f)}{a(t_0)}\,. \tag{3.103}$$

We want to see what the limiting condition is.

$$Z_{\text{min}} = \frac{d_H(t_0)}{d_H(t_i)}\frac{a_f}{a_0} = \frac{H_i}{H_0}\frac{a_f}{a_0}\,, \tag{3.104}$$

is Z a redshift?

$$z_{\text{min}}\frac{H_i}{H_0}\frac{a_f}{a_0} = \frac{H_i}{H_f}\frac{H_f}{H_0}\frac{a_f}{a_0}\,, \tag{3.105}$$

or

$$\frac{H_f}{H_i}z_{\text{min}} = \frac{H_f}{H_0}\frac{a_f}{a_0}\,, \tag{3.106}$$

in which we can insert our solution to the Friedmann equations, for the scale factor and Hubble parameter in function of time:

$$H(t) = H_*\left(\frac{a(t)}{a_*}\right)^{-\frac{3(1+w)}{2}}\,. \tag{3.107}$$

This can be found using the results we found some time ago: the expressions for a and H were equal up to a different thing multiplying the parenthesis, and a different exponent.

This is of course an approximation, but it works. A better number can be found by integrating numerically over different more realistic equations of state. We find:

$$z = \frac{a_f}{a_i} = \,, \tag{3.108}$$

Put earlier

$$\frac{H_f}{H_i} = z_{\text{min}}^{-\frac{3(1+w)}{2}}\,, \tag{3.109}$$

therefore we get

$$z_{\min} = \left(\frac{H_f}{H_0} \frac{a_f}{a_0} \right)^{-\frac{2}{(1+3w_{\text{inf}})}}, \tag{3.110}$$

where w_{inf} is calculated at the time of matter-radiation equality.

So we get:

$$\frac{H_f}{H_0} = \frac{H_f}{H_{\text{eq}}} \frac{H_{\text{eq}}}{H_0} = \left(\frac{a_f}{a_{\text{eq}}} \right)^{-2} \left(\frac{a_{\text{eq}}}{a_0} \right)^{-3/2} = \left(\frac{a_f}{a_0} \right)^{-2} \left(\frac{a_0}{a_{\text{eq}}} \right)^{-1/2}, \tag{3.111}$$

which means that the minimum inflation redshift must be

$$z_{\min} = \left(\left(\frac{a_f}{a_0} \right)^{-1} \left(\frac{a_0}{a_{\text{eq}}} \right)^{1/2} \right)^{\frac{-2}{1+3w_{\text{inf}}}}, \tag{3.112}$$

so the result can be expressed in terms of temperatures:

$$\frac{a_0}{a_f} = \frac{T_f}{T_0} = \frac{T_f}{T_{\text{pl}}} \frac{T_{\text{pl}}}{T_0}, \tag{3.113}$$

where the T_{pl} is the Planck temperature, and $a_0/a_{\text{eq}} = 1 + z_{\text{eq}}$: in the end our result is

$$z_{\min} = \left(\frac{T_{\text{pl}}}{T_0} (1 + z_{\text{eq}}) \frac{T_f}{T_{\text{pl}}} \right)^{-\frac{2}{1+3w_{\text{inf}}}}. \tag{3.114}$$

Recall that $1\,\text{GeV}$ is equal to $10^{13}\,\text{K}$, and $T_{\text{pl}} = 10^{19}\,\text{GeV}$. Also, $1 + z_{\text{eq}} = 2.3 \times 10^4 \Omega h^2$

What are these units?

We get

$$z_{\min} \approx 10^{30} \frac{T_f}{T_{\text{pl}}}, \tag{3.115}$$

but what is the early universe temperature at the end of inflation? It must allow baryiogenesis, but will still be less than one, but there is an upper bound based on the fact that we have not observed primordial gravitational waves from this time: it must be at most 10^{-3}, so we find that the minimum redshift is of the order $z_{\min} \lesssim 10^{30} \sim e^{60}$, or 60 e-folds.

This is an order of order of magnitude estimate.

From the Friedmann equation we get

$$1 = \Omega(t) - \frac{kc^2}{a^2 H^2}, \tag{3.116}$$

so $\Omega(t) - 1 = kr_H^2$. Now, consider the Ω_i of inflation: we get

$$\frac{\Omega - 1}{\Omega_i - 1} = \left(\frac{r_{H0}}{r_{Hi}} \right)^2 < 1. \tag{3.117}$$

What we discuss now might be outside of our possibilities of comprehension.

Inflation is equivalent to $\ddot{a} > 0$, which is equivalent to $p < -\frac{1}{3}\rho c^2$:

$$\ddot{a} = -\frac{8\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a\,. \tag{3.118}$$

A quantum Hamiltonian for a harmonic oscillator is

$$H = \frac{1}{2}\sum \omega\left(a^\dagger a + a a^\dagger\right), \tag{3.119}$$

but this might give infinite energy for the ground state. In nonrelativistic QM we know that the energy is defined up to a constant, but in GR this is not the case: this energy gravitates!

Another way this comes up is the Casimir effect: we have virtual particles, which can pop up for times satisfying $\Delta E \Delta t \sim \hbar$. If we put two metallic plates close to each other, we get a force.

In QFT, we either have scalars, vectors or spinors. Can a scalar field have a nonzero expectation value, while respecting the Robertson-Walker symmetries? Yes, we just take a function of time.

For a vector, we cannot have nonzero expectation: a nonzero expectation value gives us a preferred direction. For a spinor, the same holds.

However, an object like $\overline{\psi}\psi$ behaves like a scalar, even though it comes from a vector.

There are almost no scalar particles in nature! The only one is the Higgs field.

The action for GR is given by

$$S = S_\Phi + S_{g_{\mu\nu}} + S_{\text{world}}\,, \tag{3.120}$$

where

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\mathscr{L}\,, \tag{3.121}$$

and the gravitational Lagrangian is $\mathscr{L}_g = R/16\pi G$. A kinetic Lagrangian is

$$\mathscr{L} = \frac{m}{2}\dot{q}^2 - V(q)\,, \tag{3.122}$$

for a scalar field its equivalent would be

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 - V(\Phi)\,, \tag{3.123}$$

in Minkowski spacetime. It then becomes:

$$\mathscr{L} = \frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - V(\Phi)\,, \tag{3.124}$$

but the covariant derivative of a scalar is just its partial derivative.

If we add a massive term, proportional to $R\Phi^2$, we get that adding it to the global action looks like gravity.

Fri Nov 08 2019

We continue the discussion from yesterday on the dynamics of inflation.

The Lagrangian for a scalar field in GR is

$$\mathscr{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) \,. \tag{3.125}$$

The "contravariant derivative" does not exist.

Why?

We can add a term $\xi R \Phi^2$, which has the right dimensions. Actions are dimensionless since $\hbar = 1$, and since $\mathrm{d}s^2 = g_{\mu\nu} \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu$ the metric is also dimensionless. Therefore the dimensional analysis of $\int \mathrm{d}^4 x \sqrt{-g} \mathscr{L}$ gives us that $\mathscr{L}$ must have dimensions of m^{-4}.

The field Φ has the dimensions of a mass, which is an inverse length. The coupling constants are conventionally taken to be dimensionless: therefore if we are to add a term to the Lagrangian, it must be $\xi \Phi^2$ times an inverse square length: so we can insert R.

The value of ξ is undetermined: $\frac{[}{i}]16$ gives us conformal symmetry, while in other cases we get $\frac{[}{i}]14$. A Weyl transformation allows us to remove the additional term: we move from the Jordan frame (where we *do* have coupling between our scalar field and the curvature) to the Einstein frame, in which we do not.

This is a prototype for modified GR theories.

Varying the Einstein-Hilbert action with respect to the metric gives the LHS of the Einstein equations. If we vary with respect to something else we get the equations of motion of that other thing.

This means that the stress-energy tensor is just the functional variation of everything but the action of the metric in the global acction, with respect to the metric.

We get for our scalar field:

$$T_{\mu\nu}(\Phi) = \Phi_{,\mu} \Phi_{,\nu} - g_{\mu\nu} \left(\frac{1}{2} g^{\rho\sigma} \Phi_{,\rho} \Phi_{,\sigma} - V(\Phi) \right) , \tag{3.126}$$

and then $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.

We can get an explicit solution by using the symmetries of our spacetime: we assume that $\Phi(x^\mu) = \varphi(t)$. This however in QFT is an operator, but we cannot have an operator on the RHS of the EFE: so we do a semiclassical theory, an equivalent of Hartree-Fock: we take an average in the vacuum state, and our equations become

$$G_{\mu\nu} = 8\pi G \left\langle \hat{T}_{\mu\nu} \right\rangle_0 , \tag{3.127}$$

where we define the ground state as that one with the most symmetry allowed.

The symmetries we must consider are only rotations and translations. There are no issues of commutation, since we do not quantize space unlike the quantum loop gravity people.

If we perturb, we get $\Phi = \varphi + \delta\Phi$: so $\left\langle \Phi^2 \right\rangle = \varphi^2 + 2 \left\langle \varphi \delta\Phi \right\rangle + \left\langle \delta\Phi^2 \right\rangle$, but the second term is zero since $\langle \delta\Phi \rangle = 0$ and $\varphi$is constant. The last term in these diverges. We do not know how to deal with it. We therefore assume that it is small.

So: when computing the stress energy tensor we get only diagonal terms: a perfect fluid! The energy density is the Hamiltonian:

$$\rho = T_{00} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = H\,, \tag{3.128}$$

while the pressure is the Lagrangian:

$$P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) = \mathscr{L}\,. \tag{3.129}$$

We call the stuff in the universe which is not our field "radiation", with energy density ρ_r.

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V + \rho_r\right) \tag{3.130a}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\varphi}^2 - V + \rho_r\right) \tag{3.130b}$$

$$\dot{\rho}_{\text{tot}} = -3\frac{\dot{a}}{a}\left(\rho_{\text{tot}} + P_{\text{tot}}\right)\,, \tag{3.130c}$$

but in the continuity equation we can split the contributions by inserting an unknown factor Γ, the transfer of energy between the field and radiation.

$$\dot{\rho}_\varphi = -3\frac{\dot{a}}{a}\dot{\varphi}^2 + \Gamma \tag{3.131a}$$

$$\dot{\rho}_r = -4\frac{\dot{a}}{a}\rho_r - \Gamma\,, \tag{3.131b}$$

which, denoting $' = \partial_\varphi$:

$$\dot{\rho}_\varphi = \dot{\varphi}\ddot{\varphi} + V'\dot{\varphi} \tag{3.132a}$$

$$\ddot{\varphi}\dot{\varphi} + V'\dot{\varphi} = -3\frac{\dot{a}}{a} + \Gamma\,, \tag{3.132b}$$

but we drop Γ since we assume there is little radiation.

One solution is $\dot{\varphi} = 0$, if not:

$$\ddot{\varphi} + 3\frac{\dot{a}}{a} = -V'\,, \tag{3.133}$$

recall the definition of

$$w = -\frac{1}{3} = \frac{P}{\rho} = \frac{\frac{1}{2}\dot{\varphi}^2 - V}{\frac{1}{2}\dot{\varphi}^2 + V}\,, \tag{3.134}$$

so one possibility we have is

$$\dot{\varphi}^2 \gg 2|V| \implies w = 1\,, \tag{3.135}$$

or else

$$\dot{\varphi}^2 \ll 2|V| \implies w = -1\,. \tag{3.136}$$

The continuity equation gives us the Klein-Gordon equation again: it is tautological.

$\varphi = $ const was one of the first solutions proposed. This model seems so fit the data.

Several proposals were made in the late seventies, early eighties.

A very simple model for a symmetry-breaking potential is the Ginzburg-Landau:

$$V \propto \left(\Phi^2 - \sigma^2\right)^2 , \tag{3.137}$$

which gives a seeming "mass term" $-2\varphi^2\sigma^2$, which has the wrong sign: it is "tachyonic"!

The configuration at $\Phi = 0$ is unstable. The one at $|\Phi| = \sigma$ is not symmetric under $\Phi \to -\Phi$.

People realized that QFT is a subcase of a condensed matter approach in which we have a thermal bath, an *environment*. This is *finite temperature QFT*.

We consider then an *effective potential* for the temperature: $V_T(\Phi) = V(\Phi) +$ functions of T. This might be $V(\Phi) + \alpha\varphi^2 T^2 + \gamma T^4$, with positive α. The quadratic term then gives us a *positive* mass term: at temperatures larger than some critical temperature we get stability at $\Phi = 0$, but what happens if we lower the temperature?

Then, there is symmetry breaking.

Let us see how our Friedmann equations account for this situation.

The temperature of radiation is $\rho_r = \frac{\pi^2}{30} g_*(r) T^4$. If we start with a universe which is radiation dominate, then it ends up to be De Sitter.

This is a consequence of the *No Hair Cosmic Theorem*.

There is a potential barrier between the metastable "$\Phi = 0$" state, and the symmetry breaking other ones. (even though it does not show in the fourth degree potential model).

This can happen through quantum tunneling, but there is a delay: a *first order phase transition with supercooling* (by "super" what is meant is just that the temperature goes below T_C even though we still are in the center symmetric state).

We get bubbles of symmetry broken by fluctuation, expanding through the universe but never meeting because of the expansion.

This is the "old inflation model".

A new inflation model involves "slow rolling".

The equation $\ddot{\varphi} + 3H\dot{\varphi} = -V'$ looks like a regular equation of motion: after a time $1/H$ the "friction" velocity-dependent term dominates.

Then we get a slow-roll regime: $H^2 = \frac{8\pi G}{3} V$ and $\dot{\varphi} \approx -V'/3H$.

We exploit the flatness of the potential. There are quantum fluctuations during inflation.

The solution is *chaotic inflation*, by Linde 1984: Since $\Delta E \Delta t \approx \hbar$, we do not know at which state we are actually. As time passes, the energy uncertainty decreases. The initial condition for the distribution of the universe is then detemined by the uncertainty principle.

An alternative is *eternal* chaotic inflation. If a fluctuation increases the potential universe, then H^2 increases, then the region feels a larger volume. The case where the field goes towards the minimum is unlikely. Why did it happen? This can only be answered with the anthropic principle.

Thu Nov 14 2019

During inflation, the comoving Hubble radius ($r_H = 1/\dot{a}$) decreases.

t_i is the beginning time of inflation, t_f is its end, t_Λ is the time when the cosmological constant became dominant, and then we get to now: t_0.

What happened before inflation?

The cosmic no-hair conjecture is what allows inflation to delete inhomogeneties. So, there might have been perturbations before inflation: we cannot know. Up to which scale? Perturbations on scales larger than the cosmological horizon are not perceivable as perturbations: we only perceive our local mean value.

Below the largest inflation scale, the perturbations are erased by inflation: we see perturbations on these scales which are produced during inflation.

The energy density of radiation scales as $\rho_r \propto a^{-4}$, while the one of matter instead it scales as $\rho_m \propto a^{-3} \propto e^{-3Ht}$ since $a \propto e^{Ht}$.

The maximum temperature of radiation after inflation is given by the one which corresponds to the maximum latent energy released by our scalar field due to its coupling to the rest of the universe, which acts as a sort of viscous force.

So we get that this latent energy ΔV is of the order of T^4_{rad}.

What is the typical temperature needed to produce baryon symmetry?

What is baryon symmetry?

These questions are hard to answer without a grand unification theory.

How much antimatter is there in the universe? A long time ago, it was thought that there might have been regions in the universe which were filled with antimatter by looking for a γ-ray background, but this was not found. We did not find them.

The baryon number is the difference between the number of baryons and antibaryons: $(n_b - n_a)/(n_b + n_a)$. This is actually computed with quark numbers, so it can be computed even when protons and neutrons have not yet formed.

It seems like the only antimatter known is the one which was formed by us, or cosmic rays.

The value $\Omega_{0b} \approx 0.04$ is defined as

$$\rho_{0b} = \Omega_{0b}\rho_c \,, \tag{3.138}$$

where $\rho_c = 3H_0^2/(8\pi G)$.

When we define the number of protons in the universe we also fix the number of neutrons, since the universe is globally neutral. So we can estimate:

$$n_{0b} = \frac{\rho_{0b}}{m_p} \approx 1.12 \times 10^{-5}\,\text{cm}^{-3} \times h^2\Omega_{0b} \,, \tag{3.139}$$

where $h \sim 0.7$, and similarly we can compute

$$n_{0\gamma} \approx 420\,\text{cm}^{-3} \,, \tag{3.140}$$

so we can look at the baryon to photon number ratio:

$$\eta_0 = \frac{n_{0b}}{n_{0\gamma}} \approx 3 \times 10^{-8}\Omega_{0b}h^2 \,, \tag{3.141}$$

why does this number have this value?

The denominator in $(n_b - n_a)/(n_b + n_a)$ is approximately $2n_\gamma$ then, and both the difference in the numerator and the numerator scale like a^{-3}, so this value is a constant. We do not see antimatter, therefore $n_a = 0$. So, we get

$$\frac{n_b - n_a}{n_b + n_a} \approx \frac{n_b}{2n_\gamma} = \frac{1}{2}\eta_0 \,. \tag{3.142}$$

This must then have been the case also in the matter dominated epoch, during which there was a very slight imbalance in baryons vs antibaryons.

In order to generate a baryon-antibaryon asymmetry we need

1. ?
2. C and CP violation (while CPT symmetry must hold for any well-behaved QFT);
3. out-of-equilibrium processes.

These were proposed by a famous Soviet scientist.

We define the interaction rate Γ: it is the number of interactions per unit time.

The Hubble rate $H = \dot{a}/a$ is also an inverse time: baryons and antibaryons are practically speaking *decoupled* if $\Gamma \leq H$: this is equivalent to saying that the time of interaction is larger than the age of the universe.

When are particles actually coupled or decoupled? Γ can be calculated as $\Gamma = n \langle \sigma v \rangle$, where n is the number density, v is the velocity of the particles, and σ is the cross section of the interaction.

We need to distinguish the types of interactions we are dealing with. In general interactions are carried by gauge bosons. Either they are massless (like the photon) or they are massive (like the the $W^\pm$ and Z bosons, as long we are below the scale of electroweak symmetry breaking $\sim 10^2\,\text{GeV}$).

At larger energies than those, the weak interaction also becomes long-range.

In the massless case, the cross-section is $\sigma \sim \alpha^2/T^2$, where $g = \sqrt{4\pi\alpha}$.

In the massive case, for temperatures $T \leq m_x$, the cross section is of the order $\sigma \sim G_x^2 T^2$.

There is an inversion in the T dependence between long and short range interactions.

Typically $G_x = \alpha/m_x^2$. Then, $\sigma \sim \alpha^2 T^2/m_x^4$.

This difference is because in general the formula is like

$$\sigma \sim \frac{T^2}{E^4} \,, \tag{3.143}$$

and we either have $E \sim m$ in the low-speed case, or $E \sim T$ in the high-speed case.

So, we know that $\Gamma = n \langle \sigma v \rangle$. In the massless case this is something like $\Gamma \sim T^3 \sigma \sim T^3 \sim \alpha^2 T$ since $T \sim 1/a$.

Then, we get

$$H = \sqrt{\frac{8\pi G}{3}} g_*^{1/2} \left(\frac{\pi^2}{30}\right)^{1/2} T^2 \sim \frac{T^2}{m_{\text{pl}}} \,, \tag{3.144}$$

since $G \sim 1/m_{\text{pl}}^2$. Then,

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 T m_{\text{pl}}}{T^2} \sim \alpha^2 m_{\text{pl}} \frac{1}{T}, \tag{3.145}$$

so we have decoupling when $T > \alpha^2 m_{\text{pl}}$. Essentially, at temperatures larger than $T = 10^{16}\,\text{GeV}$ this massless photon is decoupled.

"Above the Planck epoch, even gravitational interactions are decoupled".

What about the massive interactions? We have $\Gamma \sim T^2 G_x^2 T^2$, to compare with $H \sim T^2/m_{\text{pl}}$: we get

$$\frac{\Gamma}{H} \sim \frac{T^3 G_x^2}{T^2/m_{\text{pl}}} \sim G_x^2 m_{\text{pl}} T^3 \leq 1, \tag{3.146}$$

equivalently, $T < m_{\text{pl}}^{-1/3} G_x^{-2/3}$.

Suppose that we are considering the gravitational interaction: in that case, we get $T < m_{\text{pl}}$ since G_x is related to G_N

What is the relation?

For the weak interaction, we have

$$T < \left(\frac{m_x}{100\,\text{GeV}}\right)^{4/3} \text{MeV}, \tag{3.147}$$

which is why below 1 MeV neutrinos are decoupled.

Let us now consider the consequences of these decoupling conditions. First of all we look at the recombination of hydrogen. At very high temperatures, there are free electrons and free protons. Protons first appeared in the universe as non-relativistic, at $T \sim 1\,\text{MeV}$ while $m_p \sim 1\,\text{GeV}$.

At a certain point, it becomes possible to create neutral H atoms from these free particles. We will use a special case of the Boltzmann formula, which governs this and many other phenomena: the Saha equation.

The reaction is $e + p \leftrightarrow H + \gamma$. We want to look at a density in phase space. We'd need all the scattering matrices, and all the phase space densities of the particles. The Saha equation is basically an ansatz at thermal and chemical equilibrium: $\mu_e + \mu_p = \mu_H + \mu_\gamma$. The chemical potential μ entered in the exponent of the FD and BE expressions.

We know that $\mu_\gamma = 0$. At thermal equilibrium the number density of the electrons is:

$$n_e = g_e \left(\frac{m_e T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_e - m_e}{T}\right), \tag{3.148}$$

and an exactly analogous formula holds for n_p and n_H: for protons we have

$$n_p = g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_p - m_p}{T}\right). \tag{3.149}$$

Degeneracy is not an issue, since we are talking about cosmology.

The number density of photons is given by

$$n_\gamma = \frac{2\zeta(3)T^3}{\pi^2}\,, \tag{3.150}$$

The binding energy of the hydrogen is $B = m_p + m_e - m_H = 13.6\,\text{eV}$. Instead of m_H we write $+m_p + m_e - B$. So the number density of hydrogen atoms is given by:

$$n_H = g_H\left(\frac{m_H T}{2\pi}\right)^{3/2}\exp\left(\frac{\mu_p - \mu_e - m_p - m_e + B}{T}\right), \tag{3.151}$$

and we can simplify things since $m_p, m_H \gg m_e, B$, and substitute in the number densities for electrons and protons. We get approximately

$$n_H = \frac{g_H}{g_e g_p} n_e n_p \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp(B/T)\,, \tag{3.152}$$

but the universe is locally and globally neutral: $n_e = n_p$, and $n_b = n_p + n_H$.

We will see that most of the hydrogen we produce will not be when the temperature is of the order of the binding energy, but much later.

Fri Nov 15 2019

We start again from where we left off, with hydrogen recombination.

We want to estimate the moment at which hydrogen first formed, which marks the point at which electrons and photons interact efficently: before, they interacted with Compton scattering which is very efficient; after they interact with hydrogen atoms in a way that is very inefficient.

After this, then, we say that photons and matter are *decoupled*.

The scattering cross section (for Compton?) goes like the inverse square of the mass.

This means that the universe is not only *globally*, but also *locally* neutral.

This decoupling is what allows for star formation. Also, this is when the CMB starts. It is made of microwaves now, but it was higher earlier.

The phase space distribution of photons is scale-invariant since they have zero mass: so we can say that the photons' distribution *looks* thermal, but it is actually not technically since there are no interactions anymore.

However the photons travel freely and are perceived as thermal, and they give an almost perfect blackbody! The errorbars in a plot for it must be magnified by 10^4 in order to be seen.

We have $\Gamma_\gamma = n_e \sigma_T$, where σ_T is the Thompson cross section. Neutrality implies $n_e = n_p$. In principle we should account for Helium: a couple minutes after the BB He-4 nuclei started to form, but they made up only something like 25% of the mass, which means 6% of the number density: so we say that the number density of baryons is

$$n_b = n_p + n_H\,. \tag{3.153}$$

Our ansatz for the Boltzmann equation is $\mu_e + \mu_p = \mu_H$ since photons have no chemical potential.

i denotes a generic one in e, p and H. Then

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T} \right), \tag{3.154}$$

we need to account for the chemical potential since it is the driver of this process. How much is n_e/n_b? the same as n_p/n_b. We call this quantity X_e, the ionization number. We expect $X_e = 1$ in the early universe, and at the end of the process it will diminish: naively we'd expect it to get to 0, but actually there remains some residual ionization, some free protons and electrons. A proper calculation would account for the non-equilibrium contributions. However, we estimate the process as being in equilibrium: this will underestimate the number of electrons.

Today, most of the hydrogen is ionized (there is a ***-Peterson effect which shows this): this means that matter and radiation interact again.

This is the second important time in the history of the universe.

Can we see the early stars? Not really, we see galaxies only up to $z \sim 10$, these stars would be at something like $z \sim 30$... There might be more to this.

Let us come back to the calculation:

$$\begin{aligned}
n_H &= g_H \left(\frac{m_H T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_H - m_H}{T} \right) && \text{(3.155a)} \\
&= g_H \left(\frac{m_H T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_e + \mu_p - m_e - m_p + B}{T} \right) && \text{(3.155b)} \\
&= \frac{g_H}{g_e g_p} \cancel{\left(\frac{m_H T}{2\pi} \right)^{3/2}} \left(\frac{m_e T}{2\pi} \right)^{-3/2} \cancel{\left(\frac{m_p T}{2\pi} \right)^{-3/2}} n_e n_p \exp\left(\frac{B}{T} \right) && \text{(3.155c)} \\
\frac{n_H}{n_e n_p} &= \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) && \text{(3.155d)} \\
\frac{n_b - n_p}{n_p^2} &= \left(\frac{m_H T}{2\pi} \right)^{3/2} \exp\left(\frac{B}{T} \right), && \text{(3.155e)}
\end{aligned}$$

which we can manipulate, using the following identity:

$$\frac{n_b - n_p}{n_p^2} = \frac{n_b \left(1 - n_p/n_b \right)}{n_b^2 X_e^2} = \frac{1}{n_b} \frac{1 - X_e}{X_e}, \tag{3.156}$$

where we use: $n_e = n_p$, and the definition of $X_e = n_p/n_b$. Then, we bring the n_b to the other side of the equation: we get

$$\frac{1 - X_e}{X_e^2} = \underbrace{\frac{n_b}{n_\gamma}}_{\eta_p} \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) n_\gamma \tag{3.157a}$$

$$
= \frac{4\sqrt{2}\zeta(3)T^3}{\pi^2} \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) \eta_0 \tag{3.157b}
$$

$$
= \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta_0 \left(\frac{T}{m_e} \right)^{3/2} \exp\left(\frac{B}{T} \right), \tag{3.157c}
$$

which means $T \sim 0.3\,\text{eV}$, much lower than the ionization energy of Hydrogen.

This does depend on the value we assign to Ω_0 and h.

Approximately, it occurred somewhere around $z \sim 1100$ (we conventionally say that recombination happened when $X_e = 0.1$).

After recombination, we have the *last scattering*: the moment at which the CMB was formed.

One Nobel prize this year was awarded to Jim Peebles, a friend of Sabino's: together with his PhD supervisor Dicke, he was the first to calculate this stuff.

Peebles in 1964 (?) did this calculation both in GR and in Brahms-Dicke theory, a modified gravity theory.

Let us describe the early universe, before the first nucleosynthesis. Important papers in this topic are by G. Gamow, and by Alpher, Bethe and Gamow.

Our hypotheses are:

1. the universe passed through a very high temperature phase, with $T > 10^{12}\,\text{K}$;
2. the universe is described by GR and SM;
3. the chemical potentials for the neutrinos μ_ν have certain upper bounds;
4. there is no matter-antimatter separation (as in, "bubbles");
5. there are no strong magnetic fields;
6. the number of exotic particles has a certain upper bound.

There are magnetic fields in the universe, but they are not homogeneous and relatively weak. Exotic particles are predicted by certain unification theories, they are generically defined as ones which we have not observed yet.

We have to explain the fact that we observe an excess of He-4 in the early universe: we define the yield

$$
y \equiv \frac{m_{\text{He}-4}}{m_b} > 0.25\,. \tag{3.158}
$$

In terms of particle number, the ratio is more like 0.06. We do not produce Carbon or anything higher than it: the process which forms it is inefficient at high temperature, low density like the early universe. Higher Z elements are only produced in stars.

Helium-4 is produced but also destroyed by stars.

The main channels in the early universe are:

1. $n + p \leftrightarrow d + \gamma$ (d denotes deuterium);

2. $d + d \leftrightarrow {}^3\mathrm{He} + n$;

3. ${}^3\mathrm{He} + d \leftrightarrow {}^4\mathrm{He} + p$.

There is no weak interaction here, unlike what happens in stars. In stars, there are no free neutrons so this process is not possible.

We cannot treat this properly, we only give a story. The slowest process of the three is the first, since it is heavily affected by photons: they destroy deuterium.

The binding energy of deuterium is around 2.2 MeV. After we have produced deuterium, Helium-4 is readily produced.

First of all, we need the neutron to proton ratio. We are working at energies of around 1 MeV, so protons and neutrons are not relativistic anymore. This process takes place around three minutes after the beginning. For $i = n, p$:

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i m_i}{T} \right), \tag{3.159}$$

so their number ratio is around:

$$\frac{n}{p} \sim \exp\left(\frac{m_p - m_n}{T} \right), \tag{3.160}$$

where $m_p - m_n \approx 1.3\,\mathrm{MeV} \approx 1.5 \times 10^{10}\,\mathrm{K}$.

We have the processes

1. $n + \gamma_e \leftrightarrow p + e^-$;

2. $n + e^+ \leftrightarrow p + \overline{\nu_e}$;

3. $n \to p + e^- + \overline{\nu_e}$.

We can replace the temperature in the exponential by $T \to T_{d_\nu}$, the decoupling temperature of the neutrinos, since that is the moment around which this happpens.

So, we get around $\exp(-1.5)$. We define:

$$X_n(t) \equiv \frac{n}{n+p} \sim 0.17\,. \tag{3.161}$$

Later it will change because of β decay, going like:

$$X_n(t) = X_n(t_{d_\nu}) \exp\left(-\frac{t - t_{d_\nu}}{\tau_n} \right), \tag{3.162}$$

where $\tau_n = \log 2 \tau_{1/2}$, and this half-life is $\tau_{1/2} \approx 10.5 \pm 0.2$.

The binding energy of deuterium is around 2.2 MeV. Then, we proceed exactly like we did with hydrogen, and finally get

$$X_d = \frac{3}{4} n_b X_n X_p \left(\frac{m_d}{m_n m_p} \right)^{3/2} \left(\frac{T}{2\pi} \right)^{-3/2} \exp\left(\frac{B}{T} \right) \tag{3.163a}$$

$$= \frac{3}{4}\eta_0 X_n X_p \left(\frac{m_d}{m_n m_p}\right) \frac{2\zeta(3)}{\pi^2}(2\pi)^{3/2} T^{3/2} \exp\left(\frac{B}{T}\right) \tag{3.163b}$$

$$\approx \frac{3}{4}\eta_0 X_n (1 - X_n) \left(\frac{m_d}{m_n m_p}\right)^{3/2} \frac{2}{\pi^2}(2\pi)^{3/2}\zeta(3) T^{3/2} \exp\left(\frac{B_d}{T}\right), \tag{3.163c}$$

Check calculation.

which describes the *deuterium bottleneck*, which is what impedes this process until photons are very diluted. As soon as photons are diluted enough, they stop bottlenecking.

Thu Nov 21 2019

We exponentiate the equation from before: we get

$$X_d = X_n X_p \exp\left(-29.33 + \frac{25.82}{T_\rho} - \frac{3}{2}\log T_\rho + \log(\Omega_0 h)\right), \tag{3.164}$$

where $T_\rho =$

What is going on? What is T_ρ?

We want to understand why there is so much He-4 in the universe, since it is destroyed in stars!

This model fits observation as long as $0.011h^{-2} \leq \Omega_0 \leq 0.25h^{-2}$. Most people agree that we are around the upper bound.

This is indirect evidence for dark energy: why?

The lifetime of the neutron, $\tau_{1/2}$ is something that is also relevant, since it affects the baryon ratios.

The Gamow factor Γ is proportional to the Fermi coupling constant G_F^2, which is connected to $\tau_{1/2}$.

Let us suppose we increase the lifetime of neutrons, $\tau_{1/2}$. This changes the moment at which we reach equation $\Gamma \sim H$.

Increasing the lifetime of neutrons decreases the amount of He-4 in the universe: less is produced.

We know that

$$H^2 = \frac{8\pi G}{3}\rho_r, \tag{3.165}$$

where $G = 1/m_p^2$ and $\rho = \frac{\pi^2}{30} g_*(T) T^4$.

If we fix the temperature, and change g_* (by adding degrees of freedom), then we get more He-4.

This gives us observational constraints on the additions of exotic particles to our theory, since that would change g_*. This bounds the number of neutrino families by 3.0 something, so there cannot be more than 3 families of light neutrinos.

If gravitons are thermal, then they also contribute to radiation.

There is also another parameter in the Friedmann equation; it is m_P: modified gravity theories often predict variations of the gravitational constant with time.

Dark matter has no relevant electromagnetic interactions: it only interacts gravitationally, and is able to cluster; dark energy, instead, is uniformly distributed.

We divide it into Hot and Cold dark matter: HDM and CDM. There is also something called *Warm* dark matter, which has intermediate properties.

In HDM, particles have very high thermal motion. They move fast, and tend to destroy gravitational potential wells in which they might settle by moving out of them, and thus decreasing the quantity of matter there.

They do this on scales comparable to the maximum distance travelled by them: this is calculated as vt, where v is their average thermal velocity, (and t is the age of the universe?).

The structures formed by these are of scales similar to or larger than $10^{15} M_\odot$, but we observe smaller structures also! They were formed later, by fragmentation: this is the top-down approach.

We also have a bottom-up approach, which is compatible with CDM.

Neutrinos were thought to be Dark Matter, and would have been hot.

The top-down approach, however, is falsified by the observation of high-redshift quasars combined with the scale of the anisotropies of the CMB: in order to account fo high-redshift small-scale structures (we have seen stars at $z \sim 20$!) we would have to increase the amplitude of the anisotropies to a scale which is not compatible to the anisotropies we see in the CMB.

Now, neutrinos are not useful for cosmology.

We have $\Gamma \sim T^5$, and $\Gamma = H$: $T^5 / \tau_{1/2} \sim T^2$ implies that $\tau_{1/2} \sim T^3$.

The decouplig temperature for HDM is larger than the temperature at which they become nonrelativistic, which is of the order of the mass.

For CDM, instead, the decoupling temperature is *smaller* than the temperature at which they become relativistic.

Now, we discuss the Boltzmann equation: there is an operator acting in phase space, the Liouville operator, which is equal to the collision operator.

$$\mathbb{L}[f] = \mathbb{C}[f] \,, \tag{3.166}$$

where all the scattering operators live on the RHS.

We start with a Newtonian description: the phase space has position, momentum and time as coordinates, and on it we define a density function $f(\vec{q}, \vec{p}, t)$.

The Einstein equations are blind to the momentum distribution, since $T_{\mu\nu}$ does not depend on the momentum. We can say that all of the momentum has been marginalized.

We define the operator

$$\hat{\mathbb{L}} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \nabla_x + \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \cdot \nabla_p \tag{3.167a}$$

$$= \frac{\partial}{\partial t} + \vec{v} \cdot \nabla_x + \frac{\vec{F}}{m} \cdot \nabla_p \,. \tag{3.167b}$$

In GR, we geometrize the gravitational force: it is included in the inertial motion of the particles.

The relativistic version of this is

$$\hat{\mathbb{L}} = p^\alpha \partial_\alpha - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}\,. \tag{3.168}$$

This is because $p^\alpha = \mathrm{d}x^\alpha / \mathrm{d}\lambda$ satisfies the geodesic equation:

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma = 0\,, \tag{3.169}$$

or $\frac{Dp^\alpha}{Dt} = 0$. We can then write down the Christoffel bit of the geodesic equation instead of the derivative $\mathrm{d}p^\alpha / \mathrm{d}\lambda$. This is because, *on shell*:

$$g^{\alpha\beta} p_\alpha p_\beta = m^2\,. \tag{3.170}$$

We want to describe the *abundance* of dark matter, in phase space: the number of DM particle per EM-interacting particle.

We then make the Liouville operator explicit, with the Christoffel symbols of the RW metric. Acting on f, we get

$$\hat{\mathbb{L}} = p^0 \partial_t f - \frac{\dot{a}}{a} \left|\vec{p}\right|^2 \frac{\partial f}{\partial E}\,, \tag{3.171}$$

which we will not prove.

The derivative with respect to the momentum must be ∂_{p^0},

because of isotropy?

the Christoffel symbols of RW are

$$\Gamma^0_{\beta\gamma} = \frac{\dot{a}}{a} \delta_{\beta\gamma}\,. \tag{3.172}$$

Recall the definition of the number density

$$n(t) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3\vec{p}\, f(\left|\vec{p}\right|, t)\,, \tag{3.173}$$

so in the end the equation becomes:

$$\hat{\mathbb{L}} = \dot{n} + 3\frac{\dot{a}}{a} n\,. \tag{3.174}$$

Conventionally we divide by p^0: we get for the Boltzmann equation:

$$\frac{\partial f}{\partial t} - \frac{\dot{a}}{a} \frac{p^2}{E} \frac{\partial f}{\partial E} = \frac{1}{E} \hat{\mathbb{C}}[f]\,. \tag{3.175}$$

Now we integrate in $\mathrm{d}^3\vec{p}$. We get

$$\frac{\partial n}{\partial t} - \frac{\dot{a}}{a} \int \mathrm{d}^3\vec{p}\, \frac{p^2}{E} \frac{\mathrm{d}f}{\mathrm{d}E} = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, \frac{1}{E} \mathbb{C}[f]\,. \tag{3.176}$$

More properly, we are doing $\mathbb{L}\left\langle f\right\rangle$.

We manipulate:

$$\int \mathrm{d}^3 p \, \frac{p^2}{E} \frac{\partial f}{\partial E} = 2 \int \mathrm{d}^3 p \, p^2 \frac{\partial f}{\partial E^2} \tag{3.177a}$$

$$= 2 \int \mathrm{d}^3 p \, p^2 \frac{\partial f}{\partial p^2} = \int \mathrm{d}^3 p \, p \frac{\partial f}{\partial p} \tag{3.177b}$$

$$= 2\pi \int_0^\infty \mathrm{d}p \, p^3 \frac{\partial f}{\partial p} = -3 \int \mathrm{d}^3 p \, f \,. \tag{3.177c}$$

Here, we integrated by parts in the second to last step, and set to zero the boudary term $4\pi p^3 f$, calculated from 0 to infinity, since at 0 we have $p = 0$, and at (momentum) infinity we have $f = 0$.

So, we can see that the LHS is equal to $\dot{n} + 3\dot{a}n/a$: so we find

$$\dot{n} + 3\frac{\dot{a}}{a} n = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p \, \frac{1}{E} \hat{\mathbb{C}}[f] \,. \tag{3.178}$$

This is the cosmological version of the Boltzmann equation.

We model the RHS as something like

$$\hat{\mathbb{C}}[f] = \Psi - \left\langle \sigma v \right\rangle n^2 \,, \tag{3.179}$$

where we have $\Gamma_A = \left\langle \sigma v \right\rangle n$ times n, where Γ is the rate of annihilation.

At equilibrium the LHS is equal to zero: so we can write $\Psi = \left\langle \sigma v \right\rangle n_{\mathrm{eq}}^2$, and the RHS becomes $\left\langle \sigma v \right\rangle \left(n_{\mathrm{eq}}^2 - n^2 \right)$.

Fri Nov 22 2019

Yesterday we arrived at the following equation:

$$\dot{n} + 3\frac{\dot{a}}{a} n = \Psi - \left\langle \sigma_A v \right\rangle n^2 \,, \tag{3.180}$$

where the term Ψ is the source of new particles, while the next term accounts for the annihilation of particles.

Under decoupling, the equation reduces to

$$\dot{n} + 3\frac{\dot{a}}{a} n = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(n a^3 \right) = 0 \,. \tag{3.181}$$

Under equilibrium, $\Gamma > H$.

Therefore $\Psi = \left\langle \sigma_A v \right\rangle n_{\mathrm{eq}}^2$. Recall that the Γ of annihilation is equal to $\Gamma = \left\langle \sigma_A v \right\rangle n$.

If the timescale of the collisions is longer than the age of the universe, $\Gamma < H$, then once again we have $a^3 n = \text{const}$.

We can see thermal *relics*, thermal distributions of objects which are not coupled anymore.

We define

$$n_C = n\left(\frac{a}{a_0}\right)^3, \tag{3.182}$$

so we can simplify the expression:

$$\dot{n} + 3\frac{\dot{a}}{a}n = \dot{n}\left(\frac{a_0}{a}\right)^3. \tag{3.183}$$

We want to factor our a parameter H. We get:

$$\dot{n}_C = -\left(\frac{a}{a_0}\right)^3 \langle\sigma_A v\rangle \left(\frac{a_0}{a}\right)^6 \left(n_C^2 - n_{\text{eq}}^2\right), \tag{3.184}$$

which can be written as

$$\frac{a}{n_{\text{C, eq}}}\frac{\mathrm{d}n_C}{\mathrm{d}a} = -\frac{\langle\sigma_A v\rangle\, n_{\text{eq}}}{\dot{a}/a}\left(\left(\frac{n}{n_{\text{eq}}}\right)^2 - 1\right), \tag{3.185}$$

so, if we define τ_{coll} as $1/(\langle\sigma_A v\rangle\, n_{\text{eq}})$ and $\tau_{\text{exp}} = 1/H$, we get

$$n_{\text{C, eq}}a\frac{\mathrm{d}n_C}{\mathrm{d}a} = -\frac{\tau_{\text{exp}}}{\tau_{\text{coll}}}\left(\left(\frac{n}{n_{\text{eq}}}\right)^2 - 1\right), \tag{3.186}$$

so if $\Gamma \gg H$, then $\tau_{\text{exp}}/\tau_{\text{coll}} \gg 1$, therefore $n = n_{\text{eq}}$, which also implies $n_{\text{C}} = n_{\text{C, eq}}$. This is the equilibrium case.

On the other hand, if $\Gamma \ll H$, then $\tau_{\text{exp}}/\tau_{\text{coll}} \ll 1$ we have decoupling, therefore $n_C =$ const.

We know that at temperatures below 1.5 MeV (after decoupling), the following holds:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma, \tag{3.187}$$

and we want to do the same for dark matter.

Neutrinos are nonrelativistic today, but they became so at a relatively low redshift, a short time ago. We can parametrize the number density of neutrinos by the temperature.

The formula for the temperature of neutrinos is a special case of the following formula:

$$T_{0\nu} = \left(\frac{g_{*0}}{g_{*d}}\right)^{1/3} T_{0\gamma}, \tag{3.188}$$

where 0 means *now*, while d means *decoupling*. This can be applied to any species.

Let us compute this for a generic species x: we find

$$n_{0x} = Bg_*\frac{\zeta(3)}{\pi^2}T_{0x}^3, \tag{3.189}$$

where the factor B accounts for the statistics: it is 1 for bosons, 3/4 for fermions. It is important to note that the temperature is a parameter, but these particles are *not thermal* anymore!

So, we get, using equation (3.150):

$$n_{0x} = \frac{B}{2} n_{0\gamma} g_x \frac{g_{*0}}{g_{*dx}} . \tag{3.190}$$

The energy density in general is given by $\rho_{0x} = m_x n_{0x}$. We get:

$$\rho_{0x} = \frac{B}{2} m_x n_{0\gamma} g_x \frac{g_{0x}}{g_{*gx}} , \tag{3.191}$$

therefore

$$\Omega_{0x} h^2 = \frac{m_x n_{0x}}{\rho_{0x}} h^2 = 2B g_x \frac{g_{*0}}{g_{*dx}} \frac{m_x}{10^2\,\mathrm{eV}} . \tag{3.192}$$

CDM is made of particles which were already nonrelativistic when they decoupled.

Then, in the formula

$$n_x(T_{dx}) = g_x \left(\frac{m_x T_{dx}}{2\pi} \right)^{3/2} \exp\left(-\frac{m_x}{T_{dx}} \right) , \tag{3.193}$$

we can assume that $T_{dx} \ll m_x$. So,

$$n_{0x} = n_x(T_{dx}) \left(\frac{a(T_{dx})}{a_0} \right)^3 = n(T_{dx}) \frac{g_{*0}}{g_{*x}} \left(\frac{T_{0\gamma}}{T_{dx}} \right)^3 , \tag{3.194}$$

but it is difficult to compute T_{dx}, which is the solution to the equation $\Gamma(T_{dx}) = H(T_{dx})$.

We know that

$$H^2(T_{dx}) = \frac{8\pi G}{3} g_{*x} \frac{\pi^2}{30} T_{dx}^4 , \tag{3.195}$$

or, in terms of the quantity $\tau_{\mathrm{exp}} = 1/H$:

$$\tau_{\mathrm{exp}} = 0.3 g_*^{-1/2} \frac{m_{\mathrm{pl}}}{T_{dx}^2} . \tag{3.196}$$

On the other hand, $\Gamma = n \langle \sigma_A v \rangle$, and $\tau_{\mathrm{coll}}(T_{dx})$, and in terms of τ_{coll} we get

$$\tau_{\mathrm{coll}}(T_{dx}) = \left(n(T_{dx}) \sigma_0 \left(\frac{T_{dx}}{m_x} \right)^N \right) , \tag{3.197}$$

where $N = 0, 1$: it is a fact from particle physics that the average has this kind of dependence:

$$\langle \sigma_A v \rangle = \sigma_0 \left(\frac{T}{m_x} \right)^N . \tag{3.198}$$

So, equaling the two τ we get:

$$\left(n(T_{dx})\sigma_0\left(\frac{T_{dx}}{m_x}\right)^N\right) = 0.3g_*^{-1/2}\frac{m_{\rm pl}}{T_{dx}^2}\,, \tag{3.199}$$

which can be solved iteratively. We solve it in terms of the parameter $x_{dx} = m_x/T_{dx}$, which must be much larger than one: this allows us to select the physical solution to the equation among the nonphysical ones.

The solution is found to be something like:

$$x_{dx} = \log\left(0.038\frac{g_x}{g_{*xd}^{1/2}}m_{\rm pl}m_x\sigma_0\right) - \left(N - \frac{1}{2}\right)\log\log\left(\dots\right). \tag{3.200}$$

Chapter 4

Stellar Astrophysics

Dark matter is collisionless, however it is not *really*: the censorship theorem says that it cannot actually collapse into a naked singularity.

A star is a sphere of matter, usually baryonic matter, characterized by a density $\rho(r)$: the mass enclosed in a radius r is

$$m(r) = \int_0^r 4\pi\widetilde{r}^2\rho(\widetilde{r})\,\mathrm{d}\widetilde{r}\,. \tag{4.1}$$

The gravitational acceleration can be calculated from Gauss' theorem:

$$g(r) = \frac{Gm(r)}{r^2}\,. \tag{4.2}$$

Let us consider a small volume, with its enclosed mass $\Delta M = \rho(r)\Delta A\Delta r$. It can contrast the inward force due to gravity if there is a differential pressure.

Let us denote P as the pressure at the inner surface, and $P + \Delta P$ the pressure at the outer surface. Then,

$$(P+\Delta P)\Delta A - P\Delta A = \left(P(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\Delta r\right)\Delta A - P(r)\Delta A = \frac{\mathrm{d}P}{\mathrm{d}r}\Delta A\Delta r\,. \tag{4.3}$$

The minus sign in what follows is since the force is inward:

$$-\Delta M\ddot{r} = \Delta M g(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\Delta r\Delta A\,. \tag{4.4}$$

The equation of motion is

$$-\ddot{r} = g(r) + \frac{1}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\,, \tag{4.5}$$

so we see that the pressure gradient must have a minus sign.

If the internal energy is used up to do internal (chemical, nuclear) work, then it cannot support the star anymore, and it then collapses.

Usually, stars start from the Jeans phenomenon: dark matter and baryons collapse under their own weight.

Let us start at the decoupling of matter and radiation, something like $z = 1100$.

The clouds of matter collapse freely up to the moment at which their internal pressure starts to slow them down. Then, we get the necessary conditions for the fusion of Hydrogen.

During freefall, we do not have a pressure gradient, so we have

$$-\ddot{r} = g(r) \,. \tag{4.6}$$

It is not generally the case, but let us suppose that the collapse is *orderly*, the interior collapses before the outer layers. We have to account for energy conservation: the total energy when our test shell is at a radius r is conserved as r changes. If the energy (per unit mass) is zero at infinity we have

$$\cancel{E_{\text{tot}}(r_0)} - \frac{Gm_0}{r_0} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{Gm_0}{r} \,, \tag{4.7}$$

therefore

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = Gm_0\left(r^{-1} - r_0^{-1}\right) . \tag{4.8}$$

What is the freefall time? we take $\mathrm{d}t/\mathrm{d}r$ from the equation an integrate it from r_0 to 0

$$t_{\text{free fall}} = \int_{r_0}^{0} \mathrm{d}r \, \frac{\mathrm{d}t}{\mathrm{d}r} = -\int_{r_0}^{0} \mathrm{d}r \, \frac{1}{2GM}\left(\frac{1}{r} - \frac{1}{r_0}\right)^{-1/2} , \tag{4.9}$$

where we have a minus sign since $\mathrm{d}r/\mathrm{d}t < 0$. We get

$$t_{\text{free fall}} = \sqrt{\frac{r_0^3}{2Gm_0}} \int_0^1 \mathrm{d}x \, \sqrt{\frac{x}{1-x}} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2Gm_0}} \,, \tag{4.10}$$

and if we define the average density $\bar{\rho} = m_0/\left(4\pi r_0^3/3\right)$ we get the simpler expression

$$t_{\text{free fall}} = \sqrt{\frac{3\pi}{32G\bar{\rho}}} \,. \tag{4.11}$$

We might be tempted to ignore the expansion of the universe in these calculations: we know that

$$H^2 = \frac{8\pi G}{3}\bar{\rho} \,, \tag{4.12}$$

and in the purely Newtonian case we know that $a(t) \propto t^{2/3}$ and $H = 2/(3t)$.

Therefore

$$\frac{4}{3t^2} = \frac{8\pi G}{3}\bar{\rho} \implies t^2 = \frac{4}{9}\frac{3}{8\pi G}\bar{\rho}^{-1} \,, \tag{4.13}$$

so

$$t_{\text{exp}} \propto \bar{\rho}^{-1/2} \,. \tag{4.14}$$

what is the relation between the universe's density and the star's?
Is the idea: the matter in the universe does not disperse too fast, stars theoretically are allowed to form?

At equilibrium, $\ddot{r} = 0$: so

$$-\ddot{r} = 0 = g(r) + \frac{1}{\rho(r)} \frac{\mathrm{d}P}{\mathrm{d}r}, \tag{4.15}$$

so hydrostatic equilibrium implies that, locally,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G \frac{m(r)\rho(r)}{r^2}. \tag{4.16}$$

We multiply both sides by $4\pi r^3$ and integrate in $\mathrm{d}r$: we get

$$\int_0^R \mathrm{d}r \, 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r} = -G \int_0^R \frac{m(r)\rho(r) 4\pi r^2}{r} \, \mathrm{d}r \,, \tag{4.17}$$

and we can change variables: $\rho(r) 4\pi r^2 \, \mathrm{d}r = \mathrm{d}m$, so we can identify the LHS with the total gravitational energy:

$$E_{\mathrm{grav}} = -G \int \frac{m(G) \, \mathrm{d}m}{r}. \tag{4.18}$$

On the RHS, instead, we can integrate by parts:

$$\Big[P(r) 4\pi r^3 \Big]_0^R - 3 \int_0^R \mathrm{d}r \, 4\pi r^2 P(r) \,, \tag{4.19}$$

where the boundary term is zero: at the origin $r = 0$, at the surface (by definition) $P = 0$.

So, inserting the volume $V(R) \equiv 4\pi R^3/3$, we have

$$E_{\mathrm{grav}} = -3 \int_0^R \frac{\mathrm{d}r \, 4\pi r^2 P(r)}{V(R)} V(R) \,, \tag{4.20}$$

which gives us the virial theorem: we can interpret the integral as a weighted average, so we get

$$E_{\mathrm{grav}} = -3 \langle P \rangle V \qquad \text{or} \qquad \langle P \rangle = -\frac{1}{3} \frac{E_{\mathrm{grav}}}{V}. \tag{4.21}$$

This is Newtonian: above a certain limit, the relativistic corrections destabilize the system.

Thu Nov 28 2019

We derived

$$\langle T \rangle = -\frac{1}{3} \frac{E_{\mathrm{grav}}}{V}, \tag{4.22}$$

so now we proceed: we want a relation between kinetic and gravitational energy densities.

We use a statistical mechanics approach.

The rate of momentum transfer in the direction x is given by

$$\frac{N}{L^3} p_x v_x \,, \tag{4.23}$$

and this will be true for either direction by isotropicity: so

$$P = \frac{n}{3} \left\langle \vec{p} \cdot \vec{v} \right\rangle \,, \tag{4.24}$$

in full generality.

Let us consider the two cases of nonrelativistic and fully relativistic. In the nonrelativistic case we have

$$\epsilon_p = mc^2 + \frac{p^2}{2m} \,, \tag{4.25}$$

where $p = mv$. In the ultrarelativistic case we have

$$\epsilon_p = pc \,, \tag{4.26}$$

and the velocity is approximately the speed of light.

Then, for a gas of nonrelativistic particles, we have

$$P = \frac{1}{3} nmv^2 = \frac{2}{3} n \left\langle \frac{1}{2} mv^2 \right\rangle = \frac{2}{3} \times \text{translational KE density} \,, \tag{4.27}$$

while in the relativistic case we have:

$$P = \frac{1}{2} n \left\langle pc \right\rangle = \frac{1}{3} \times \text{translational KE density} \,. \tag{4.28}$$

We will show that, if a star is made of a gas of classical nonrelativistic particles it tends to be stable, if the particles are relativistic then it tends not to be stable.

The virial theorem tells us that

$$2E_\text{K} + E_\text{grav} = 0 \,, \tag{4.29}$$

in the nonrelativistic approximation.

We define: $\Delta E_\text{tot} = -\Delta E_\text{K} = \frac{1}{2} \Delta E_\text{grav}$.

We know that

$$\left\langle P \right\rangle = \frac{1}{3} \frac{E_\text{K}}{V} = -\frac{1}{3} \frac{E_\text{grav}}{V} \tag{4.30}$$

by the virial theorem: so the total binding energy is equal to zero, since this gives us

$$E_\text{grav} + E_\text{K} = E_\text{tot} = 0 \,. \tag{4.31}$$

Now we differentiate the law $\mathrm{d}(PV^\gamma) = 0$, since it is a constant for an adiabatic transformation: it gives us, using logarithmic derivatives,

$$\gamma \frac{\mathrm{d}V}{V} + \frac{\mathrm{d}P}{P} = 0\,, \tag{4.32}$$

so

$$\mathrm{d}(PV) = -(\gamma - 1) P \,\mathrm{d}V\,, \tag{4.33}$$

and we know that for an adiabatic transformation

$$\mathrm{d}E_{\text{in}} + P\,\mathrm{d}V = 0\,, \tag{4.34}$$

which implies

$$\mathrm{d}E_{\text{in}} = \frac{1}{\gamma - 1} \mathrm{d}(PV)\,, \tag{4.35}$$

and let us assume that γ is approximately constant in the transformation: this means

$$E_{\text{in}} = \frac{PV}{\gamma - 1}\,, \tag{4.36}$$

so

$$P = (\gamma - 1) \frac{E_{\text{in}}}{V}\,, \tag{4.37}$$

which justifies the relations we used in cosmology, $P = w\rho$ with $w = \gamma - 1$.

We can rewrite the equation from before as

$$3(\gamma - 1) E_{\text{in}} + E_{\text{gr}} = 0\,, \tag{4.38}$$

which, together with $E_{\text{tot}} = E_{\text{in}} + E_{\text{gr}}$, give us

$$E_{\text{tot}} = -(3\gamma - 4) E_{\text{in}}\,, \tag{4.39}$$

which means that $\gamma > 4/3$ characterizes a bound system, while $\gamma < 4/3$ characterizes a free system.

There are two dangers: one is the fight against the pressure forces, one is the fight against the quantum forces (the Pauli exclusion principle) which do not allow the compression to happen further.

Now we discuss Jeans instability:

$$E_{\text{grav}} = -f \frac{GM^2}{R}\,, \tag{4.40}$$

where f is a numerical factor of the order 1, depending on the mass distribution. If the distribution is uniform, it is $3/2$.

3/2?

The kinetic energy is

$$E_{\mathrm{K}} = \frac{3}{2} N k_B T \,, \tag{4.41}$$

and we want to impose the condition

$$f \frac{GM^2}{R} > \frac{3}{2} N k_B T \,, \tag{4.42}$$

and the Jeans criterion is this boundary of the stability region:

$$f \frac{gM_J^2}{R} = \frac{3}{2} \frac{M_J}{\bar{m}} k_B T \,, \tag{4.43}$$

where $\bar{m} = M/N$. The J denotes the fact that we are considering the specific boundary mass on both sides. Simplifying the formula we find:

$$M_J = \frac{3}{2} \frac{k_B T}{G\bar{m}} R \,, \tag{4.44}$$

and we can reframe this in terms of the density, which is defined by

$$M_J = \frac{4\pi}{3} \rho_J R^3 \,. \tag{4.45}$$

We cube and multiply on both sides:

$$\rho_J M_J^3 = \left(\frac{3k_B T}{2G\bar{m}} \right)^3 R^3 \rho_J \,, \tag{4.46}$$

so we get

$$\rho_J = \frac{1}{M_J^3} \left(\frac{3k_B T}{G\bar{m}} \right)^3 \left(\frac{4\pi}{3} \rho_J R^3 \right) \frac{1}{4\pi} \,, \tag{4.47}$$

so

$$\rho_J = \frac{3}{4\pi M_J^2} \left(\frac{3k_B T}{2G\bar{m}} \right)^3 \,, \tag{4.48}$$

and we will habe stability if the density is larger than this. So, if we want a collapse, we must decrease the mass…

When the last scattering happens, the pions are decoupled from the photons. Dark matter behaves differently from conventional matter.

We have

$$\dot{\rho}_r = -3H\big(\rho_r + P_r\big) \,, \tag{4.49}$$

and

$$\dot{\rho}_m = -3H\big(\rho_m + P_m\big) \,, \tag{4.50}$$

and $P_r = \rho_r/3$, which scale like a^{-4} and also as T^4, which means $T \sim 1/a$.

$$\mathrm{d}\left(\rho_m c^2 a^3\right) + P_m da^3 = 0\,, \tag{4.51}$$

where we usually approximate $\rho_m c^2 = m_p n_b c^2$, but we can include more terms:

$$\rho_m c^2 = m_p n_b c^2 \left(1 + (\gamma - 1)^{-1} \frac{k_B T}{m_p c^2}\right), \tag{4.52}$$

while the pressure is given by $P = n_b k_B T$: so in the end we find

$$\mathrm{d}\left(\left(m_p n_b c^2 + \frac{3}{2} m_p n_b \frac{k_B T}{m_p}\right) a^3\right) = -n_b k_B T \,\mathrm{d}a^3\,, \tag{4.53}$$

which after some computation gives us

$$\frac{1}{2}\,\mathrm{d}T = -T\frac{\mathrm{d}a}{a}\,, \tag{4.54}$$

which implies $T_m \propto a^{-2}$ after baryogenesis.

This is for monoatomic baryonic matter, right?

Let us start writing equations for the stellar interior. The continuity equation is

$$\partial_t \rho + \nabla \cdot \left(\rho \vec{v}\right) = 0\,, \tag{4.55}$$

and also we have the Euler equation

$$\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla} P - \vec{\nabla}\Phi\,, \tag{4.56}$$

which can be written using the convective time derivative:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + \vec{v} \cdot \nabla_x = u^\mu \partial_\mu\,, \tag{4.57}$$

which allows us to write

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho + \rho \nabla \cdot \vec{v} = 0\,, \tag{4.58}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}t}\vec{v} = -\frac{1}{\rho} - \nabla\Phi\,. \tag{4.59}$$

What happens to the entropy? we define the entropy density s by $S = s\rho$.
An isentropic process is one in which

$$\frac{\mathrm{D}S}{\mathrm{D}t} = 0\,, \tag{4.60}$$

which in terms of the entropy density is

$$\partial_t s + \vec{v} \cdot \vec{\nabla} s = 0 . \tag{4.61}$$

The external force is provided by the gravitational field:

$$\nabla^2 \Phi = 4\pi G \rho . \tag{4.62}$$

Jeans looked for a simple solution, an ansatz, called the background solution and then tried to perturb it: if it is stable than it was a good solution.

He started with $\rho =$ const, $\vec{v} = 0$, $s =$ const, $\Phi =$ const.

It is obviously wrong! It cannot satisfy the Poisson equation.

However, we start from it and add some $\delta\rho$, $\delta\vec{v}$ (which we just call $\vec{v}$), δs and $\delta\Phi$; then we only keep the linear terms in these perturbations.

We find:

$$\partial_t \delta\rho + \rho_0 \vec{\nabla} \cdot \vec{v} = 0 , \tag{4.63}$$

$$\partial_t \vec{v} = -\frac{1}{\rho_0} \nabla \delta P - \nabla \delta \Phi , \tag{4.64}$$

and

$$\nabla^2 \delta\Phi = 4\pi G \delta\rho , \tag{4.65}$$

and finally

$$\partial_t \delta s = 0 . \tag{4.66}$$

We can expand

$$\delta P = \frac{\partial P}{\partial \rho} \delta\rho + \frac{\partial P}{\partial s} \delta s , \tag{4.67}$$

and here we define

$$c_s^2 = \frac{\partial P}{\partial \rho} . \tag{4.68}$$

We will then consider an exponential solution:

$$\delta\rho = \delta\rho_0 \exp\left(i \left(\vec{k} \cdot \vec{x} - \omega r \right) \right) , \tag{4.69}$$

and similarly for $\vec{v}$, s, Φ.

We will see that we will need to stick to $\delta s = 0$, and find a dispersion relation with ω and $\vec{k}$: it will be

$$\omega^2 = c_s^2 \vec{k}^2 - 4\pi G \rho , \tag{4.70}$$

so if the wavenumber is small enough we will have an imaginary ω.

Fri Nov 29 2019

We want to derive the Jeans instability criterion, starting from the structure equations:

1. continuity;
2. momentum conservation;
3. Poisson for the gravitational field;
4. entropy conservation.

Yesterday there was a mistake: the entropy conservation is actually

$$\frac{\mathrm{D}s}{\mathrm{D}t} + s\nabla \cdot \vec{v} = 0\,. \tag{4.71}$$

There is no need to make an ansatz for the pressure, since we can compute $P = P(\rho, s)$: we have the relation seen last time, involving the speed of sound.

Our *ansatz* is

$$\delta x_i = x_{i0} \exp\left(i\vec{k} \cdot \vec{x} - i\omega t\right), \tag{4.72}$$

with $x_i = \rho, \vec{v}, \Phi, s$.

Our equations become:

$$i\omega\delta\rho_0 + \rho_0 i\vec{k} \cdot \vec{v}_0 = 0 \tag{4.73a}$$

$$i\omega\vec{v}_0 = \frac{1}{\rho_0} i\vec{k}\left(c_s^2\delta\rho_0 + \frac{\partial P}{\partial s}\delta s_0\right) - i\vec{k}\delta\Phi_0 \tag{4.73b}$$

$$k^2\delta\Phi = 4\pi G\delta\rho_0 \tag{4.73c}$$

$$\omega\delta s_0 = 0\,, \tag{4.73d}$$

so one class of solutions will have $\omega = 0$, that is, we consider time-independent ones:

$$\rho_0 i\vec{k} \cdot \vec{v}_0 = 0 \tag{4.74a}$$

$$0 = \frac{1}{\rho_0}\vec{k}\left(c_s^2\delta\rho_0 + \frac{\partial P}{\partial s}\delta s_0\right) - \vec{k}\delta\Phi_0 \tag{4.74b}$$

$$k^2\delta\Phi = 4\pi G\delta\rho_0\,, \tag{4.74c}$$

and we have a result from Helmholtz: the fact that every velocity field can be decomposed into $\vec{v} = \nabla\Psi + \vec{T}$ with $\nabla \cdot \vec{T} = 0$.

As soon as we ask the perturbation to be time independent we only find vortical motions, with $\vec{v} \perp \vec{k}$.

Now we consider the velocity to be irrotational we consider the term $\delta s_0 = 0$, since $\omega \neq 0$ in general.

$$\omega\delta\rho_0 + \rho_0\vec{k} \cdot \vec{v}_0 = 0 \tag{4.75a}$$

$$\omega \vec{v}_0 = \frac{1}{\rho_0} \vec{k} c_s^2 \delta\rho_0 - \vec{k}\delta\Phi_0 \tag{4.75b}$$

$$k^2 \delta\Phi = 4\pi G \delta\rho_0 \,, \tag{4.75c}$$

which we can write as a linear system for the vector $\delta\rho_0, v_0, \delta\Phi_0$.

The 5x5 coefficient matrix is:

$$\begin{bmatrix} \omega & \rho_0 \vec{k} & 0 \\ \frac{1}{\rho_0} \vec{k} c_s^2 & \omega & \vec{k} \\ 4\pi G & 0 & k^2 \end{bmatrix} , \tag{4.76a}$$

which has determinant

$$\omega k^2 - \rho_0 \vec{k} \cdot \left(\frac{1}{\rho_0} \vec{k} c_s^2 - 4\pi G \vec{k} \right) = 0 \,, \tag{4.77}$$

which gives the dispersion relation $\omega^2 = c_s^2 k^2 - \rho_0 4\pi G$.

A solution will be a combination of these. If $\omega^2 < 0$, which can happen because of the minus sign. If that happens, instead of a propagating wave we have a standing wave.

We have $\omega = (4\pi G \rho_0)^{1/2}$, so the characteristic time is

$$\tau = \frac{1}{\sqrt{4\pi G \rho_0}} \,, \tag{4.78}$$

The free fall time can be computed to be:

$$\tau_{\text{free fall}} = \left(\frac{3\pi}{32 G \rho_0} \right)^{1/2} . \tag{4.79}$$

We have a typical Jeans wavenumber:

$$k_J^2 = \frac{4\pi G \rho_0}{c_s^2} \,, \tag{4.80}$$

corresponding to when the frequency becomes imaginary.

We can compare this to a plasma of charged particles, with the electrostatic potential instead of the gravitational field. In that case we find

$$\omega^2 = c_s^2 k^2 + \frac{4\pi n_e e^2}{m_e} \,, \tag{4.81}$$

where n_e is the number density of electrons, m_e is the electron mass.

We have the following analogies:

$$n_e \to \rho_0 / m \tag{4.82a}$$

$$m_e \to m \tag{4.82b}$$

$$e^2 \to G m^2 \tag{4.82c}$$

$$
+ \to - ; \tag{4.82d}
$$

the first of these are due to the inertial mass being equal to the gravitational mass; the plus becoming a minus is due to the fact that we do not have negative charge in the gravitational setting, so there cannot be a screening effect.

We now come back to the RW line element:

$$
\mathrm{d}s^2 = c^2 \,\mathrm{d}t^2 - a^2(t) \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \,\mathrm{d}\Omega^2 \right), \tag{4.83}
$$

in which the most important thing is that a is a function of time.

We will now use a Newtonian approach: we ignore spatial curvature, that is, set $k = 0$: this is good for small enough scales. The thing is: the time scales of gravitational collapse are long, and we cannot ignore the effects of spacetime expansion.

We use a coordinate defined by $\vec{r} = a(t)\vec{x}$, so that the line element becomes

$$
\mathrm{d}s^2 = c^2 \,\mathrm{d}t^2 - \mathrm{d}|x|^2 - |x|^2 \,\mathrm{d}\Omega^2 \,. \tag{4.84}
$$

right?

What will be done now could have been done by Newton: he did not just because he thought the universe was static. We have (dropping the vector sign, but still impying it):

$$
u = \dot{r} = \dot{a}x + a\dot{x} = \frac{\dot{a}}{a} r + v \,, \tag{4.85}
$$

where $v = a\dot{x}$ is called the peculiar velocity which galaxies and such can have.

Let us forget about the pressure gradient and just look at how the instability evolves: this will also apply to dark matter which has no pressure. We look at

$$
\left[\frac{\partial \rho}{\partial t} \right]_{\vec{r}} + \nabla_{\vec{r}} (\rho \vec{u}) = 0 \,, \tag{4.86}
$$

the velocity equation is

$$
\left[\frac{\partial \vec{u}}{\partial r} \right]_{\vec{r}} + (\vec{u} \cdot \nabla_{\vec{r}}) \vec{u} = -\nabla_{\vec{r}} \Phi \,, \tag{4.87}
$$

and finally

$$
\nabla^2_{\vec{r}} = 4\pi G \rho \,. \tag{4.88}
$$

Our expression for $\vec{u}$ is $H\vec{r} + \vec{v}$, also we have

$$
\rho(\vec{r}, t) = \rho_b(t) + \delta\rho(\vec{r}, t) \,, \tag{4.89}
$$

a background plus a perturbation.

To find the background, we assume that the density of matter is space independent but time dependent: then we find

$$\Phi_b(\vec{r}, t) = \frac{2\pi G}{3} \rho_b(t) r^2 \,, \tag{4.90}$$

which has gradient

$$\nabla_{\vec{r}} \Phi_b = \frac{4\pi G}{3} \rho_b(t) \vec{r} \,, \tag{4.91}$$

and laplacian

$$\nabla^2_{\vec{r}} \Phi_b = 4\pi G \rho_b \,, \tag{4.92}$$

but this diverges at $r \to \infty$! this is not an issue with the solution, but with Newtonian mechanics.

This is an elliptic differential equation: hyperbolic ones look like $\Box \Phi = 0$, elliptic ones like $\nabla^2 \Phi = 0$, while parabolic ones (???)

We want to get equations in the comoving coordinates, not the local inertial ones. Take a generic function $f(\vec{r}, t)$, which can also be expressed with respect to $(\vec{x}, t)$.

Then, the difference between the derivatives is:

$$\left[\frac{\partial f}{\partial t}\right]_{\vec{x}} = \left[\frac{\partial f}{\partial t}\right]_{\vec{r}} + H(\vec{r} \cdot \nabla_{\vec{r}}) f \,. \tag{4.93}$$

Take the convective derivative

$$\frac{\mathrm{D} f}{\mathrm{D} t} = \frac{\partial f}{\partial t}_{\vec{r}} + \frac{\partial f}{\partial \vec{r}} \cdot \dot{\vec{r}} \,, \tag{4.94}$$

where $\dot{\vec{r}} = \vec{u} = H\vec{r} + \vec{v}$. From what we saw, this can be expressed as

$$\frac{\partial f}{\partial t}_{\vec{r}} + H(\vec{r} \cdot \nabla_{\vec{r}}) + (\vec{v} \cdot \nabla_{\vec{r}}) f \,, \tag{4.95}$$

but we can also do it the other way round, keeping $\vec{x}$ constant: that way, we find

$$\frac{\mathrm{D} f}{\mathrm{D} t} = \frac{\partial f}{\partial t}_{\vec{x}} + \frac{\partial f}{\partial \vec{x}} \,, \tag{4.96}$$

and identifying the terms we ge the desired relation, equation (4.93).

Skipping some passages, from the continuity equation we get:

$$\frac{\partial \rho}{\partial t}_{\vec{x}} + 3H\rho + \frac{1}{a} \nabla_{\vec{x}} (\rho \vec{v}) = 0 \,, \tag{4.97}$$

while for the Euler equation we have:

$$\frac{\partial H\vec{r}}{\partial t}_{\vec{r}} + H^2 (\vec{r} \cdot \nabla_{\vec{r}}) \vec{r} = -\nabla_{\vec{r}} \Phi_b \,. \tag{4.98}$$

The divergence term is just $r^j \partial_j r^i = r^i$. Inserting the expression we know for the gravitational potential, whose gradient is proportional to $\vec{r}$, we get:

$$\vec{r}\left(\dot{H} + H^2 = -\frac{4\pi G}{3}\rho_b\right), \tag{4.99}$$

which must hold for any $\vec{r}$, so we drop it and recover one of the Friedmann equations! Note that we did not use any GR, everything was Newtonian.

We will get another result:

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v} \cdot \nabla_{\vec{x}})\vec{v} = -\frac{1}{a}\nabla_{\vec{x}}\delta\Phi, \tag{4.100}$$

our new perturbed Euler equation; while the new Poisson equation is

$$\nabla^2 \delta\Phi = a^2 4\pi G \delta\rho. \tag{4.101}$$

Some cosmologists like to define

$$\delta(\vec{x}, t) = \frac{\delta\rho}{\rho_b} = \frac{\rho(\vec{x}, t) - \rho_b(t)}{\rho_b(t)}, \tag{4.102}$$

which can be anywhere from -1 to $+\infty$.

Then, we can interpret a negative $\delta\rho$ as a negative charge.

Why the Newtonian approach? Nobody knows how to write down the equations for a general relativistic self-gravitating fluid.

Is the Newtonian approximation a good one? In it, we have

$$\nabla^2 \delta\Phi = 4\pi G \rho_b a^2 \delta, \tag{4.103}$$

can we make the weak field approximation?

Typically

$$\delta\Phi \sim 4\pi G \rho_b a^2 \delta \lambda^2, \tag{4.104}$$

where λ is the variation scale, from the Laplacian.

From the Friedmann equation, $H^2 = \frac{8\pi G}{3}\rho_b$, then we have

$$\delta\Phi \sim \frac{3}{2}H^2 \delta a^2 \lambda^2 \sim \left(\frac{\lambda^2}{\lambda_{\text{hor}}^2}\right)\delta, \tag{4.105}$$

and $\lambda \sim$ Mpc, the galactic perturbation scale, while $\lambda_{\text{hor}} \sim$ Gpc.

As long as the perturbations are only galactic, the Newtonian approximation is good.

Thu Dec 05 2019

There will be a section on gravitational waves in cosmology in January.

We found the relation:

$$\vec{r} = a(\lambda)\vec{x}\,, \tag{4.106}$$

and

$$\dot{\vec{r}} = \vec{u} = \dot{a}\vec{x} + a\dot{\vec{x}} = H\vec{r} + \vec{v}\,, \tag{4.107}$$

since $H = \dot{a}/a$.

An issue we have is the *redshift space distortion*, caused by *peculiar velocities* which cause the light we see to have redshift beyond the one caused by cosmology alone.

In comoving coordinates, we can use the regular Newtonian fluid dynamics equations.

Since we know that the gravitational instability is relevant when the gravitational force overcomes the pressure forces, we set the pressure gradient to be equal to zero. We have

$$\frac{\partial \rho}{\partial t}|_{\vec{r}} + \nabla_{\vec{r}}(\rho\vec{u}) = 0 \tag{4.108a}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \left(\vec{u}\cdot\vec{\nabla}_{\vec{r}}\right)\vec{u} = 0 \tag{4.108b}$$

$$\nabla^2_{\vec{r}}\Phi = 4\pi G\rho\,, \tag{4.108c}$$

and we consider a background $\Phi_b(\vec{r}, t)$, and $\rho = \overline{\rho}(t)$: we have a peculiar gravitational field beyond the Robertson-Walker background: we define

$$\phi = \Phi - \Phi_b\,, \tag{4.109}$$

and analogously we define by

$$\delta\rho = \rho - \overline{\rho} = \overline{\rho}(1+\delta)\,, \tag{4.110}$$

with $-1 \leq \delta \leq \infty$.

The Laplace equation for ϕ is given by

$$\nabla^2_{\vec{x}}\phi(\vec{x}, t) = 4\pi G a^2(t)\overline{\rho}(t)\delta(\vec{x}, t)\,, \tag{4.111}$$

and we actually *can* have negative quantities on the RHS. Not repulsive gravity, but kind of.

The relation between the derivatives is

$$\frac{\partial}{\partial r} = \frac{1}{a}\frac{\partial}{\partial x}\,, \tag{4.112}$$

therefore the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_x(\rho\vec{v}) = 0\,, \tag{4.113}$$

we get that the density scales like a^{-3} if there is no velocity: then it becomes

$$\frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a} = 0\,. \tag{4.114}$$

For the momentum equation the calculation is longer, but in the end we find:

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla_x)\vec{v} = -\frac{1}{a}\nabla_x\phi\,, \tag{4.115}$$

and once again we remark that we are not using GR because, in the absence of symmetry, that is definitely too difficult.

If we perturb, we will find that plane waves are not solutions anymore, because we will have time-dependent coefficients.

The perturbation of the density is:

$$\rho(\vec{x},t) = \overline{\rho}(t)\big(1+\delta(\vec{x},t)\big)\,, \tag{4.116}$$

and for the field:

$$\phi(\vec{x},t) = \Phi - \Phi_b\,. \tag{4.117}$$

The derivative is:

$$\frac{\partial \rho}{\partial t} = \frac{\partial \overline{\rho}}{\partial t}(1+\delta) + \overline{\rho}\frac{\partial \delta}{\partial t}\,, \tag{4.118}$$

and we can discard higher-order terms: simplifying the zeroth-order equation we find for the continuity equation:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}\cdot\vec{v} = 0\,. \tag{4.119}$$

On the other hand, for the momentum equation we find:

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} = -\frac{1}{a}\nabla\phi\,, \tag{4.120}$$

and in order to solve these we can expand in Fourier space:

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\,\widetilde{\delta}(t)\exp\Big(i\vec{k}\cdot\vec{x}\Big)\,, \tag{4.121}$$

and similarly for $\vec{v}$ and ϕ.

The actual quantities must be real: therefore we know that $\widetilde{\delta}^*_{\vec{k}}(t) = \widetilde{\delta}_{-\vec{k}}(t)$.

Any vector field $\vec{V}$ can be decomposed by the Helmholtz theorem into a gradient and a divergence:

$$\vec{V} = \nabla\Psi + \vec{T}\,, \tag{4.122}$$

where $\nabla\cdot\vec{T} = 0$.

Using the convective time derivative, we have

$$\frac{\mathrm{D}\vec{v}}{\mathrm{d}t} + H\vec{v} = -\frac{1}{a}\nabla\phi\,. \tag{4.123}$$

Inserting here the Helmholtz decomposition applied to the velocity, we find

$$\frac{\mathrm{D}\vec{T}}{\mathrm{D}t} + H\vec{T} = 0\,, \tag{4.124}$$

which means that vorticity is conserved in fluid motion.

This means that any vorticity which might have been there at the beginning would have been diluted out over time.

We project the equations along the versor $\vec{u}_{\vec{k}} = \vec{k}/\left|\vec{k}\right|$.

Then the equation becomes

$$\dot{\delta}_{\vec{k}} + \frac{ik}{a}v_{\vec{k}} = 0\,, \tag{4.125}$$

and

$$\dot{v}_{\vec{k}} + Hv_{\vec{k}} = -\frac{i\vec{k}}{a}\phi_{\vec{k}}\,, \tag{4.126}$$

and finally

$$-k^2\phi_{\vec{k}} = 4\pi G a^2 \overline{\rho}\delta_{\vec{k}}\,. \tag{4.127}$$

Now, by differentiating in one single step we will get a linear second order differential equation for δ which is separated from the others.

Differentiating one more time and making the $\vec{k}$ implicit we get:

$$\ddot{\delta} + \frac{ik}{a}\dot{v} - \frac{ik}{a^2}\dot{a} = 0 \tag{4.128a}$$

$$\ddot{\delta} + \frac{ik}{a}\left(-Hv - \frac{ik}{a}\phi\right) - \frac{ik}{a}Hv = 0 \tag{4.128b}$$

$$\ddot{\delta} - \frac{2ik}{a}Hv + \frac{k^2\phi}{a^2} = 0\,, \tag{4.128c}$$

but the last term is $-4\pi G a^2\overline{\rho}\delta$: then in the end we find

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\overline{\rho}\delta = 0\,, \tag{4.129}$$

and we use the following solutions of the background equations:

$$a(t) \propto t^{2/3} \tag{4.130a}$$

$$H = \frac{2}{3t} \tag{4.130b}$$

$$\overline{\rho} = \left(6\pi G t^2\right)^{-1}\,, \tag{4.130c}$$

and plugging these in we find

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0\,, \tag{4.131}$$

which will be a power of t: plugging t^α we find the equation

$$\alpha(\alpha - 1) + \frac{4}{3}\alpha - \frac{2}{3} = 0\,, \tag{4.132}$$

which gives $\alpha = -1, 2/3$.

The solution with $\alpha = 2/3$ is called the growing mode, while the one with $\alpha = -1$ is called the decaying mode. When the inflaton field becomes classical we lose a degree of freedom: this removes the decaying solutions. Another way to see it is to see that the decaying mode explodes as $t \to 0$.

Thus, we usually remove the decaying solution.

An interesting observation is the fact that the growing mode grows just as fast as the scale factor.

Inserting this into the other equations we find $v \propto t^\beta$ with $\beta = 1/3$ or $-4/3$, while $\phi \propto t^\gamma$ with $\gamma = 0, -5/2$.

understand what the equation

$$\chi_J = c_s \sqrt{\frac{\pi}{4\overline{\rho}}}\,, \tag{4.133}$$

The Jeans density is given by

$$\rho_J = \frac{3}{4\pi M^2}\left(\frac{3k_B T}{2G\overline{m}}\right)^3\,. \tag{4.134}$$

$$\chi_J = c_s \left(\frac{\pi}{4\overline{\rho}}\right)^{1/2}\,, \tag{4.135}$$

$$M_J = \frac{4\pi}{3}\overline{\rho}\left(\frac{\lambda}{2}\right)^3\,, \tag{4.136}$$

$$c_s^2 \sim \frac{k_B T}{\overline{m}}\,, \tag{4.137}$$

and now we are assuming that $\rho > \rho_J$.

Stars are made of baryons.

Molecular hydrogen:

$$H + e \leftrightarrow H^- + \gamma \tag{4.138a}$$

$$H^- + H \leftrightarrow H_2 + e \tag{4.138b}$$

$$H + p \leftrightarrow H_2^+ + \gamma \tag{4.138c}$$

$$H_2^+ + H \leftrightarrow H_2 + p\,, \tag{4.138d}$$

and these processes are generally not very efficient: they start to be efficient at redshifts of about $z \sim 200$.

It is easier to get the Jeans density if the mass is high and the temperature is low.

The formation of stars is somewhat top-down: larger structures form first.

The critical energy density today is $\rho_{0c} \sim 10^{-29}\,\text{gcm}^{-3}$, while for baryons we have $\rho_{0b} \sim 1 \times 10^{-31}\,\text{gcm}^{-3}$.

On the other hand, at $z \sim 200$ we must multiply the density by a factor $(1+z)^3$ we get a density of $\rho_b(z \sim 200) \sim 10^{-22}\,\text{kgm}^{-3}$.

If we take $M = 1000 M_\odot$ and $T \approx 20\,\text{K}$, we have a Jeans critical density of around $10^{-20}\,\text{kgm}^{-3}$.

Is 20 K a typical temperature then?

The matter starts free-falling: then the kinetic energy increases. However, it is used in chemical processes. That way matter can become opaque.

We have the equation

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{Gm_0}{r} - \frac{Gm_0}{r_0}\,, \tag{4.139}$$

which gives us a typical free-fall time of $2 \times 10^4\,\text{yr}$.

The energy needed to ionize all the hydrogen is

$$E = \frac{M}{2m_H}\epsilon_D + \frac{M}{m_H}\epsilon_I\,, \tag{4.140}$$

where M is the mass of the star, m_H is the mass of a hydrogen atom, while $\epsilon_D \approx 4.5\,\text{eV}$ and $\epsilon_U \approx 13.6\,\text{eV}$.

We are going from a radius R_1 to a radius R_2: we get $R_1 \sim \sqrt[3]{3M/4\pi\rho_J}$, since that is the radius at which the collapse starts.

However, the R_1 term hardly contributes: the term R_2 is typically 4 orders of magnitude smaller.

Our objects must still become hotter and smaller in order to reach the $10^7\,\text{K}$ needed for ignition.

The final equation we get comes from the virial theorem:

$$2E_k + E_{\text{gr}} = 0\,, \tag{4.141}$$

and

$$E_k = \frac{3}{2}Nk_BT = \frac{3}{2}\frac{M}{\overline{m}}k_BT \approx \frac{3M}{m_H}k_BT\,, \tag{4.142}$$

since $\overline{m} = 0.5 m_H$: so in the end, equating the energies, we have

$$2 \times 3 k_B T \frac{M}{m_H} = \frac{M}{m_H} \left(\frac{\epsilon_D}{2} + \epsilon_I \right), \tag{4.143}$$

which means that $k_B T = \frac{1}{12}(\epsilon_D + 2\epsilon_I) \sim 2.6\,\text{eV}$, which is still very low compared to the temperature we need.

Fri Dec 06 2019

Today we will discuss another possible *caveat* for star formation: the case where the star collapses but does not ignite.

After recombination the baryons' temperature decays more rapidly than the radiation's, so we get a temperature low enough to satisfy the instability criterion.

The temperature stays low during the collapse: the thermal energy is used in order to break the bonds in H_2 and ionize the hydrogen.

Then we have a gas which is opaque to radiation: there is scattering, which means we lose energy through radiation very slowly. Then the Virial theorem is very close to being true.

At the end of the process, even the mass does not matter anymore: we got $2.6\,\text{eV}$ as the temperature regardless of the mass. This is equivalent to around $30 \times 10^3\,\text{K}$. We must compare this to the ignition temperature: that is in the order of keV ($15 \times 10^6\,\text{K}$ is equivalent to around $1.3\,\text{keV}$).

After free-fall the radius is on the order of $10^{10}\,\text{m}$ for a solar mass star, while the Sun's radius is smaller by two orders of magnitude.

Now we discuss the *conditions for stardom*: we need to account for the fermionic nature of protons and electrons, which will give us a maximum density due to the Pauli exclusion principle.

The way we will treat this today will be quite rough.

We know that the De Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}. \tag{4.144}$$

How do we calculate p? we assume that the particles are nonrelativistic and apply $E_k = p^2/2m_e$.

Objects which will not satisfy the conditions we talk about today become brown dwarves.

The kinetic energy is of the order $k_B T$, therefore

$$p \sim \sqrt{2 m_e k_B T}, \tag{4.145}$$

and the critical density is defined by

$$\rho_c = \frac{\overline{m}}{\lambda^3}, \tag{4.146}$$

where from the formula we found

$$\lambda = \frac{2\pi\hbar}{\sqrt{2m_e k_B T}}\,, \tag{4.147}$$

which gives approximately

$$\rho_c \sim \overline{m}\frac{(m_e k_B T)^{3/2}}{(2\pi\hbar)^3}\,, \tag{4.148}$$

and from the virial theorem $2E_k + E_{\mathrm{gr}} = 0$, with

$$E_k = \frac{3}{2}N k_B T = \frac{3}{2}\frac{M}{\overline{m}}k_B T\,, \tag{4.149}$$

while

$$E_{\mathrm{gr}} = -\frac{GM^2}{R}\,, \tag{4.150}$$

which means

$$3N k_B T = \frac{GM^2}{R}\,, \tag{4.151}$$

and we can rewrite this as

$$\frac{3k_B T}{\overline{m}} = \frac{GM}{R}\,, \tag{4.152}$$

and we can express the mass as

$$M = \frac{4}{3}\pi\overline{\rho}R^3\,, \tag{4.153}$$

so

$$\frac{1}{R} = \left(\frac{4\pi}{3}\frac{\overline{\rho}}{M}\right)^{1/3}\,, \tag{4.154}$$

which gives us the result

$$k_B T = \frac{GM\overline{m}}{3}\left(\frac{4\pi}{3}\frac{\overline{\rho}}{M}\right)^{1/3}\,, \tag{4.155}$$

and if we substitute the critical density in for $\overline{\rho}$ we will get the maximum possible temperature allowed at a given mass.

This yields

$$k_B T = \frac{GM\overline{m}}{3}\left(\frac{4\pi}{3M}\right)^{1/3}\frac{\overline{m}^{1/3}}{(2\pi\hbar)}(m_e k_B T)^{1/2}\,, \tag{4.156}$$

which it is convenient to square:

$$(k_B T)^2 = \frac{G^2 M^2 \overline{m}^2}{9} \left(\frac{4\pi}{3M} \right)^{2/3} \frac{\overline{m}^{2/3}}{(2\pi\hbar)^2} m_e k_B T \,, \tag{4.157}$$

so we can simplify, and up to an order-1 constant

$$k_B T = \frac{G^2 \overline{m}^{8/3} M^{4/3}}{(2\pi\hbar)^2} \,, \tag{4.158}$$

and inserting the ignition temperature of around 1 keV we get $M_{\text{min}} \sim 0.08 M_\odot$.

This is confirmed experimentally.

Let us consider the Sun. Its mass is $M_\odot \approx 1.99 \times 10^{30}$ kg, the radius is $R_\odot \approx 6.96 \times 10^8$ m, the electromagnetic luminosity is $L_\odot = 3.86 \times 10^{26}$ W.

The age of the Sun is around $t_\odot \approx 4.55 \times 10^9$ yr, which is comparable to the age of the Universe.

The central density is $\rho_c \approx 1.48 \times 10^5 \,\text{kgm}^{-3}$, while the central temperature is $T_c = 1.56 \times 10^7$ K, and the central pressure is around $P_c = 2.29 \times 10^{16}$ Pa.

The effective temperature is around $T_E \approx 5780$ K.

Definition?

What is the corresponding free fall time? It is much shorter than the age of the Sun: the Sun is not in free fall.

What is the number?

We know that

$$\langle P \rangle = -\frac{1}{3} \frac{E_{\text{gr}}}{V} \,, \tag{4.159}$$

where $E_{\text{gr}} = -GM^2/R$ while $V = 4\pi R^3/3$: plugging the Sun's numbers we get

$$\langle P \rangle = 10^{14} \,\text{Pa} \,, \tag{4.160}$$

100 times less than the central density.

The density of the Sun is actually very similar to the density of water.

Was it correct to use nonrelativistic equations? (???)

$$\langle P \rangle = \frac{\overline{\rho}}{\overline{m}} k_B T_I \,, \tag{4.161}$$

where T_I is the mean internal temperature of the Sun. The value of $\overline{m} \approx 0.61$ instead of 0.5 when considering the proper composition of the Sun.

We get

$$k_B T_I \approx \frac{G M_\odot \overline{m}}{3 R_\odot} \approx 1.5 \,\text{keV} \approx 6 \times 10^6 \,\text{K} \,. \tag{4.162}$$

We have evidence that the Sun is a blackbody, we can write

$$L_\odot = 4\pi R_\odot \sigma T_E^4 \,, \tag{4.163}$$

where σ is Stefan's constant: $\sigma \approx 5.67 \times 10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$.

In principle we could define another quantity:

$$L'_\odot = 4\pi R_\odot^2 \sigma T_I^4\,, \tag{4.164}$$

which does not fit the data. Why is this?

We must consider a photon which comes from the interior of the Sun and goes towards the outside: it will follow a random walk scattering many times. However, depending on the density of electron in the outer regions (electron scattering dominates in the outside), the last scattering is in the outermost regions of the star.

We need to deal with the Fourier equation. First of all, we need to write the Langevin equation: this is connected to the work by Einstein on Browinan motion.

These are processes in which there is a "random" force (due to the fact that our description of the microscopic state is probabilistic), and possibly deterministic forces.

What is the stuff about viscosity?

The equation in the end is like:

$$\dot{\vec{x}} = \vec{\eta}\,, \tag{4.165}$$

What?

with $\left\langle \eta_i(t)\eta_j(t')\right\rangle = 2D\delta_{ij}\delta(t-t')$: there is complete uncorrelation. This is the Markov property: the process has no memory.

This gives us a Gaussian distribution of the positions of the particles, and we get the equation:

$$\frac{\partial P}{\partial t} = D\nabla^2 P\,, \tag{4.166}$$

where $P = P(\vec{x}, t)$. This is a *parabolic equation*, the *Fokker-Planck* formula. We need to give it both initial conditions and boundary conditions.

What is P?

There are three kinds of boundary conditions:

1. nothing: free boundary, the solution can diverge;
2. a reflective boundary: particles "bounce back", in order to deal with this we use the images method, as in electromagnetism;
3. an absorbing boundary: particles disappear if they reach the boundary.

This is equivalent to the Fourier transport equation.

The solution without boundary is a Gaussian with variance $\left\langle x^2\right\rangle = 2Dt$ and centered around zero.

If $\sigma = \sqrt{\left\langle x^2\right\rangle}$ becomes greater than the boundary then most of the particles have escaped.

$\vec{D}$ is the displacement vector of the Sun: it is

$$\vec{D} = \sum_i \vec{l}_i \,, \tag{4.167}$$

where the $\vec{l}_i$ are the displacement vectors of its various steps.

$$\left\langle \vec{D}^2 \right\rangle = \sum_i \left\langle \vec{l}_i^2 \right\rangle + \sum_{i<j} \left\langle \vec{l}_i \cdot \vec{l}_j \right\rangle \,, \tag{4.168}$$

but if we have isotropy then the scalar products have mean zero. This might be unphysical since the steps are larger at the boundary than at the center...

Then we find:

$$\left\langle \vec{D}^2 \right\rangle = Nl^2 = R_\odot^2 \,, \tag{4.169}$$

where $N = R_\odot^2 / l^2$.

The time it takes for a photon to cover a distance l is $t = l/c$. Then we have $t_{RW} = Nt = R_\odot^2 l/(l^2 c)$, which means

$$t_{RW} = \frac{R_\odot^2}{cl} \,, \tag{4.170}$$

while in direct flight the photon would take $t_0 = R_\odot / c$: their ratio is

$$\frac{t_{RW}}{t_0} = \frac{R_\odot}{l} \,, \tag{4.171}$$

and then

$$L_\odot = L'_\odot \frac{l}{R_\odot} \,, \tag{4.172}$$

which means

$$T_E = \left(\frac{l}{R_\odot} \right)^{1/4} T_I \,, \tag{4.173}$$

so we can gather l by knowing the other three parameters: we get $l = 1\,\text{mm}$. This is actually an average of the mean free paths.

$$L = L' \frac{l}{R} = 4\pi R_\odot^2 \sigma T_I^4 \frac{l}{R} \,, \tag{4.174}$$

and we know that $k_B T_I = \frac{GM\overline{m}}{3\hbar}$: replacin this we find

$$L = 4\pi R_\odot^2 \sigma \left(\frac{GM\overline{m}}{3Rk_B} \right)^4 \frac{l}{R} = \frac{(4\pi)^2}{3^5} \frac{\sigma}{k_B^4} G^4 \overline{m}^4 \overline{\rho} l M^3 \,. \tag{4.175}$$

What is the typical lifespan of a star? This gives us $L \sim M^3$, which means $\tau \sim M/L \sim M^{-2}$. Observationally, this matches the data pretty well.

Now we discuss thermonuclear fusion. The reactions we need are

$$\mathrm{p} + \mathrm{p} \to \mathrm{d} + \mathrm{e}^+ + \nu_e \tag{4.176a}$$

$$\mathrm{p} + \mathrm{d} \to {}^3\mathrm{He} + \gamma \tag{4.176b}$$

$${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + \mathrm{p} + \mathrm{p}\,, \tag{4.176c}$$

which involves the weak interaction (for the first process: $\tau \sim 5 \times 10^9\,\mathrm{yr}$), EM interaction (for the second process: $\tau \sim 1\,\mathrm{s}$) and strong interaction (for the third process: $\tau \sim 3 \times 10^5\,\mathrm{yr}$). On the other hand, the process

$$\mathrm{n} + \mathrm{p} \to \mathrm{d} + \gamma \tag{4.177a}$$

$$\mathrm{d} + \mathrm{d} \to {}^3\mathrm{He} + \mathrm{n} \tag{4.177b}$$

$${}^3\mathrm{He} + \mathrm{d} \to {}^4\mathrm{He} + \mathrm{p}\,, \tag{4.177c}$$

which does not use the weak interactions. However, later there are no more free neutrons: this means that even if it is slower the first process is the only one which can happen.

The net balance is 4 protons in, 1 ^{4}He out.

Thu Dec 12 2019

Check room availability for the first week of January.

Tomorrow / next week we will discuss some black holes and neutron stars.

Last week we discussed the nuclear processes which occur in the center of the Sun when $T \sim 10^7\,\mathrm{K}$ is reached.

The minimum mass for a star in order to ignite fusion, as we found, is around $0.08 M_\odot$.

The most important difference between the processes outlined last week is in the first equation of each: there are very few free neutrons.

So, the weak interaction process dominates: since it is so slow, it has a very low power density: $P = 4 \times 10^{26}\,\mathrm{W}$, but we need to take its ratio to the volume of the Sun. This gives a density lower than that of a human.

For each ^{4}He nucleus we get $26\,\mathrm{MeV}$, and we need 4 protons to make it. Then, this gives us the number of protons per second the Sun uses in order to produce the power it does. We will use the following relation:

$$1\,\mathrm{MeV} \approx 1.78 \times 10^{-30}\,\mathrm{kg} \approx 1.6 \times 10^{-13}\,\mathrm{J}\,, \tag{4.178}$$

and then, in SI units: we get

$$\frac{4 \times 10^{26}}{2.6 \times 10^{-13}/4} \approx 4 \times 10^{38} \frac{\text{protons}}{\text{s}}\,. \tag{4.179}$$

For each process we also emit one electron neutrino, and we need to do the first two steps of the process twice for each ^{4}He nucleus, so we are producing 2×10^{38} neutrinos per second.

A proton's mass is around $m_p \approx 1\,\text{GeV} \approx 1.78 \times 10^{-27}\,\text{kg}$, so in the Sun there are around 10^{56} of them: this means that the typical lifetime of the Sun is around 10^{10} years.

What is the final state of the Sun? After hydrogen runs out, the Sun should start the next process: helium burning. The core contracts and heats. However we also have the degeneracy pressure.

There is a boundary, the Chandrasekar mass $M_C \approx 1.4 M_\odot$, between the final fate of the star being a white dwarf or a neutron star.

From the point of view of the state of matter, brown dwarves and white dwarves are very similar.

So the core becomes denser and hotter and starts burning helium. The external parts, by the virial theorem, must then expand.

So the possibilities, in order of mass, are white dwarf, neutron star, black hole.

The matter which is expelled can form a *planetary nebula*.

Process	Fuel	Products	T_{min}	M_{min}
Hydrogen burning	Hydrogen	Helium	$10^7\,\text{K}$	$0.08 M_\odot$
Helium burning	Helium	Carbon, Oxygen	$10^8\,\text{K}$	$0.5 M_\odot$
Carbon burning	Carbon	Oxygen, Neon, Sodium	$5 \times 10^8\,\text{K}$	$8 M_\odot$
Neon burning	Neon	Magnesium, Oxygen	$10^9\,\text{K}$	$9 M_\odot$
Oxygen burning	Oxygen	Magnesium to Sulphur	$2 \times 10^9\,\text{K}$	$10 M_\odot$
Silicon burning	Silicon	Iron and nearby elements	$3 \times 10^9\,\text{K}$	$11 M_\odot$

Figure 4.1: Solar processes.

A plot of the binding energy per nucleon $B = E - M_{\text{nucleons}}$ shows that it is maximum for iron.

Something about the possibility to have ^{8}Be be stable in order for potassium to be formed.

When the nucleus cannot reach the temperature needed for the next process, the collapse is stopped by electron degeneracy.

The final radius of the red giant phase of the Sun is around 70 times the radius of the Sun, while the white dwarf phase is 70 times smaller.

Is this correct? I couldn't quite hear.

Our equation of hydrostatic equilibrium is:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\,, \tag{4.180}$$

and the equation giving the variation of the mass is $\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r)$. So we get

$$\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r} = -Gm(r)\,, \tag{4.181}$$

which can be restated as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\frac{\mathrm{d}m}{\mathrm{d}r} = -4\pi G\rho(r)r^2\,, \tag{4.182}$$

more commonly stated as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho(r)\,, \tag{4.183}$$

which holds if we have hydrostatic equilibrium.

Commonly the equation of state used for this is called *polytropic*:

$$P = k\rho^{\frac{n+1}{n}}\,, \tag{4.184}$$

where $k = \text{const}$ and $n = 1/(\gamma - 1)$: so if $\gamma = 5/3$, which holds for a monoatomic gas, then $n = 3/2$ while if $\gamma = 4/3$, which holds for an ultrarelativistic gas, then $n = 3$.

Using this law, we can write this equation in terms of either only the density or only the pressure.

Let us kave only the density:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho^{\frac{n+1}{n}}\right)\right) = -4\pi G\rho(r)\,, \tag{4.185}$$

and we need two boundary conditions: we can fix the central density $\rho(r = 0) = \rho_c$ and the derivative of the density at the center:

$$\frac{\mathrm{d}\rho_c}{\mathrm{d}r} = 0\,. \tag{4.186}$$

This is because:

$$\rho^{1/n}\frac{\mathrm{d}\rho}{\mathrm{d}r} \propto -\frac{Gm(r)}{r^2}\rho(r)\,, \tag{4.187}$$

and the mass is given by $m(r) \sim \rho_c r^3$: therefore $\rho^{1/n}\,\mathrm{d}\rho/\mathrm{d}r \propto r$, so it makes sense to set the derivative to zero.

The radius of the star is defined as the one at which the density goes to 0: $\rho(R) = 0$, and the mass of the star is given by $m(R) = M$.

These equations can be solved numerically. Also, we can use an ansatz.

A model by Claytron 1986: $P_c = 2 \times 10^{16}\,\mathrm{Pa}$, and since

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\mathrm{gr}}}{V_\odot}\,, \tag{4.188}$$

where $E_{\mathrm{gr}} = GM_\odot^2/R_\odot$, and $V_\odot = \frac{[}{i}]43\pi R_\odot^3$: therefore the mean pressure is approximately $1/200$ times the pressure at the center.

We can write the equation of hydrostatic equilibrium as

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2} \approx -\frac{4\pi G}{3}\rho_c^2 r\,, \tag{4.189}$$

so the pressure gradient goes to zero linearly in r. So, we know that

$$\frac{\mathrm{d}P}{\mathrm{d}r} = 0\,, \tag{4.190}$$

also, as $r \to R$ we get $\mathrm{d}P/\mathrm{d}r \to 0$ as well, since it is proportional to $\rho(r)$ in that region.

The pressure gradient around the center follows the density. Its variation is much larger near the center than on the outside. The ansatz by Clayton is

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4\pi}{3}G\rho_c^2 r \exp\left(-\frac{r^2}{a^2}\right), \tag{4.191}$$

where the parameter a has the dimensions of a length, and we take it to be $a \ll R$. This model is quite accurate near the center, not so much near the surface!

Integrating we find:

$$P(r) = \frac{2\pi}{3}G\rho_c^2 a^2\left(\exp\left(-\frac{r^2}{a^2}\right) - \exp\left(-\frac{R^2}{a^2}\right)\right), \tag{4.192}$$

so that the pressure is exactly zero at the surface: $P(R) = 0$. We will check *a posteriori* that this model makes sense for a typical star.

Do we set the pressure to stay at zero for $r > R$? Maybe does not matter…

We have the relation

$$Gm(r)\,\mathrm{d}m = -4\pi r^4\,\mathrm{d}P\,, \tag{4.193}$$

which can be integrated in order to give the following result:

$$\frac{1}{2}Gm^2(r) = -4\pi\int_0^r \mathrm{d}\widetilde{r}\,\widetilde{r}^4\frac{\mathrm{d}P}{\mathrm{d}\widetilde{r}}\,. \tag{4.194}$$

We recover $m(r)$ by taking the square root:

$$m(r) = \frac{4\pi a^3}{3}\rho_c\Phi(x)\,, \tag{4.195}$$

where $x = r/a$ and $\Phi(x)$ is defined from the integral from before:

$$\Phi^2(x) = 6\int_0^x \mathrm{d}y\, y^5 e^{-y^2} = 6 - 3\left(x^4 + 2x^2 + 2\right)e^{-x^2}\,, \tag{4.196}$$

and the second term is almost zero near the surface if $a \ll R$: the power series starts from the sixth power (? what?).

The density profile then is given by:

$$\rho(r) = \frac{1}{4\pi r^2}\frac{\mathrm{d}m}{\mathrm{d}r} = \rho_c \left(\frac{x^3 e^{-x}}{\Phi(x)} \right), \tag{4.197}$$

and we can also recover the temperature profile from the ideal gas law:

$$T(r) = \frac{\overline{m}}{k_B}\frac{P(r)}{\rho(r)}, \tag{4.198}$$

and, when $x \ll 1$ we find

$$\Phi(x) \sim \left(x^6 - \frac{3}{4}x^8 + \frac{3}{10}x^{10} - \frac{1}{12}x^{12} + \dots \right)^{1/2}, \tag{4.199}$$

and inserting this we find

$$\rho(r) \approx \rho_c \left(1 - \frac{5}{8}\frac{r^2}{a^2} + \dots \right), \tag{4.200}$$

and

$$T(r) \approx T_c \left(1 - \frac{3}{8}\frac{r^2}{a^2} + \dots \right), \tag{4.201}$$

which allows us to get the total mass

$$M = m(R) = \frac{4\pi\rho_c a^3}{3}\Phi(R/a) \approx \frac{4\pi\rho_c a^3\sqrt{6}}{3}, \tag{4.202}$$

and lots of it are within a radius a:

$$\rho(a) = 0.53\rho_c \qquad \text{and} \qquad m(a) = 0.28 M_\odot\,. \tag{4.203}$$

We also have the relation $a = R_\odot/5.4$, and

$$\frac{\langle \rho \rangle}{\rho_c} \sim \left(\frac{a}{R} \right)^3 \sim 1\,. \tag{4.204}$$

Where does this come from?

We have the relations

$$m(r) = \frac{4\pi a^3}{3}\rho_c \Phi(x)\,, \tag{4.205}$$

and

$$P_c = \frac{2\pi}{3} G \rho_c^2 a^2\,. \tag{4.206}$$

Then we can combine these:

$$P_c = \left(\frac{\pi}{36}\right)^{1/3} GM^{2/3}\rho_c^{4/3}\,. \tag{4.207}$$

The factor $(\pi/36)^{1/3} \sim 0.44$. Changing γ we get: for $\gamma = 5/4$ the factor is 0.48.
We use the ideal gas relation: then we find

$$k_B T_c = \left(\frac{\pi}{36}\right)^{1/3} G\overline{m}M^{1/3}\rho_c^{1/3}\,. \tag{4.208}$$

Then, we can figure out which processes actually can happen.

Fri Dec 13 2019

We discuss the maximum mass of stars.
We were able to give meaning to the parameter a: the pressure is given by

$$P(r) = \frac{2\pi}{3}G\rho_c^2 a^2\left(\exp\left(-\frac{r^2}{a^2}\right) - \exp\left(-\frac{R^2}{a^2}\right)\right), \tag{4.209}$$

so we can see that

$$a = \left(\frac{3M}{4\pi\rho_c\sqrt{6}}\right)^{1/3}, \tag{4.210}$$

and then we got an expression for the central pressure P_c: the parameter multiplying it is approximately 0.44, while more accurate models give: if $\gamma = 5/3$ (ideal gas) we get 0.48 whiile if $\gamma = 4/3$ (ultrarelativistic) we get 0.36.
We have the relation

$$P_c = \frac{\rho_c}{\overline{m}}k_B T_c\,, \tag{4.211}$$

which we use to get the last relation from last time.
This allows us to get some figures for main sequence (hydrogen burning) stars. In the Herzsprung Russel diagram we plot $L/L_\odot$ versus T_{eff}, the latter increasing right to left.
The Main Sequence runs from the upper left to the lower right, we have Red Giants on the upper right and White Dwarves on the lower left. Most of the stars are on the Main Sequence: the hydrogen burning phase lasts a long time.
What is the maximum mass for Main Sequence stars? A star becomes unstabel when most of its material becomes ultrarelativistic: then, its total energy goes from a negative value to 0 and the adiabatic index approaches 4/3.
Suppose that the central energy is partly given by radiation and partly by matter. We write

$$P_c = P_\rho + P_r = \beta P_c + (1-\beta)P_c\,, \tag{4.212}$$

where the terms of the two sums exactly correspond to each other, and

$$\beta P_c = P_\rho = \frac{\rho_c k_B T_c}{\overline{m}} \tag{4.213}$$

and

$$(1-\beta)P_c = P_r = \frac{1}{3}a\,, \tag{4.214}$$

where

$$a = \frac{\pi^2 k_B^2}{15\hbar^3 c^3}\,. \tag{4.215}$$

So we have

$$\left(\beta P_c\right)^4 = \frac{\rho_c^4}{\overline{m}^4}(k_B T_c)^4\,, \tag{4.216}$$

and

$$\frac{1-\beta}{\beta^4} P_c^{-3} = \frac{a}{3}\left(\frac{k_B \rho_c}{\overline{m}}\right)^{-4}. \tag{4.217}$$

Inverting this we can eliminate the temperature dependence:

$$P_c = \left(\frac{3}{a}\frac{1-\beta}{\beta^4}\right)^{1/3}\left(\frac{k_B\beta}{\overline{m}}\right)^{4/3} = \left(\frac{\pi}{36}\right)^{1/3} G M^{1/3} \rho_c^{4/3}\,, \tag{4.218}$$

so

$$\left(\frac{\pi}{36}\right)^{1/3} G M^{2/3} = \left(\frac{3}{a}\frac{(1-\beta)}{\beta^4}\right)^{1/3}\left(\frac{k_B}{\overline{m}}\right)^{4/3}, \tag{4.219}$$

so as β decreases, M increases.

[Plot of $1-\beta$ versus $M/M_\odot$, showing this.]

Now we deal with the degenerate electron gas in stars.

The distribution function is approximately given by

$$f(p) \propto \frac{1}{\exp\left(\frac{\epsilon_p - \mu}{k_B T}\right) + 1}\,, \tag{4.220}$$

where $\epsilon_p = c\left(p^2 + m^2c^2\right)^{1/2}$. Then, as $T \to 0$ we get: $f(\epsilon_p) = 1$ if $\epsilon_p \leq \epsilon_F$ and $f(\epsilon_p) = 0$ if $\epsilon_p > \epsilon_F$; we can express this energy in terms of the momentum:

$$\epsilon_F^2 = c^2 p_F^2 + m^2 c^4\,. \tag{4.221}$$

The number of electrons is given by

$$n_e = 2\int_0^{p_F} \mathrm{d}p\, p^2 4\pi \frac{1}{h^3} = \frac{8\pi}{3}\left(\frac{p_F}{h}\right)^3, \tag{4.222}$$

where we have a factor of 2 to account for the spin-1/2 nature of the electrons. This means that

$$p_F = \left(\frac{3n}{8\pi} \right)^{1/3} h \,. \tag{4.223}$$

The density is given by

$$\rho = \frac{2}{h^3} \int_0^{p_F} \mathrm{d}p \, p^2 4\pi \epsilon_p \,, \tag{4.224}$$

and we can consider either the nonrelativistic or the ultrarelativistic limit. In the nonrelativistic limit we find

$$\epsilon_p = mc^2 + \frac{p^2}{2m} \,, \tag{4.225}$$

so the result becomes

$$\rho = n \left(mc^2 + \frac{3}{10} \frac{p_F^2}{m} \right) \,, \tag{4.226}$$

and we know that for a nonrelativistic gas the pressure is given by

$$P = \frac{2}{3} \frac{E_k}{V} \,, \tag{4.227}$$

where E/V is the kinetic energy density.

$$P = n \frac{p_F^2}{5m} \,, \tag{4.228}$$

where does this come from?

$$P = k_{NR} n^{5/3} \,, \tag{4.229}$$

where

$$k_{NR} = \frac{h^2}{5m} \left(\frac{3}{8\pi} \right)^{2/3} \,. \tag{4.230}$$

In the relativistic case, on the other hand, we get $\epsilon_p \approx cp$, and $\rho = \frac{3}{4} n \rho_F c$. In this case, we also know that the pressure becomes

$$P = \frac{1}{3} \frac{E_k}{V} \,. \tag{4.231}$$

So we find

$$P = K_{VR} n^{4/3} \,, \tag{4.232}$$

where

$$K_{VR} = \frac{hc}{4}\left(\frac{3}{8\pi}\right)^{1/3}. \tag{4.233}$$

We make a plot: on the x axis we have the number density in m^{-3}, on the y axis we have the temperature in K.

We divide the plot into:

1. Classical UR: $P \propto nk_BT$;
2. classical NR (like the Sun);
3. degenerate NR: $P = K_{NR}n^{4/3}$
4. degenerate UR.

Classical vs degenerate is marked by a line similar to $T \sim n$, while we have NR for both T and n lower than certain critical values (since a degenerate gas can become ultrarelativistic even at low temperatures! this is the point).

We have

$$P_c = \frac{\rho_C}{\overline{m}}k_BT_c\,, \tag{4.234}$$

and

$$k_BT_c = \left(\frac{\pi}{36}\right)^{1/3} G\overline{m}M^{2/3}\rho_c^{1/3}\,, \tag{4.235}$$

and it can be (easily?) shown that

$$\overline{m} = 2m_H \times \frac{1}{1+3x_1+0.5x_4}\,, \tag{4.236}$$

where $x_{1,4}$ are the concentrations of hydrogen and helium respectively.

To estimate the maximum achievable central temperature we do:

$$P_c = k_{NR}n_e^{5/3} + n_ik_BT_c\,, \tag{4.237}$$

where $n_e = n_i = \rho_c/\overline{m}_H$.

$$\left(\frac{\pi}{36}\right)^{1/3} GM^{2/3}\rho_c^{4/3} = k_{NR}\left(\frac{\rho_c}{m_H}\right)^{5/3} + \frac{\rho_c}{m_H}k_BT_c\,, \tag{4.238}$$

which implies

$$k_bT_c = \left(\frac{\pi}{36}\right)^{1/3} Gm_HM^{2/3}\rho_c^{1/3} - k_{NR}\left(\frac{\rho_c}{m_H}\right)^{2/3}, \tag{4.239}$$

which is in the shape $k_B T_c = A\rho_c^{1/3} - B\rho_c^{2/3}$: we can find a maximum of this function, which comes out to be at $\rho_c = (A/2B)^3$, where we have

$$k_B T_c = \frac{A^2}{2B} = \left(\frac{\pi}{36}\right)^{2/3} \frac{G^2 m_H^{8/3}}{4k_{NR}} M^{4/3} . \tag{4.240}$$

Now, we can set this temperature to be larger than the ignition temperature for any process we want, to see whether it will happen.

The minimum mass is

$$M_{\min} = \left(\frac{36}{\pi}\right)^{1/2} \left(\frac{4k_{NR}}{G^2 m_H^{8/3}}\right)^{3/4} \left(k_B T_{\text{ign}}\right)^{3/4} . \tag{4.241}$$

The potential energy between two hydrogen nuclei separated by a distance equal to their quantum wavelength is

$$E_g = -\frac{G m_H^2}{r} = -\frac{G m_H^3 c}{\hbar} , \tag{4.242}$$

where we inserted $r = \hbar / m_H c$. This corresponds to an energy $E = m_H c^2$, and we have a value

$$\alpha_G = \frac{E_g}{E} = \frac{G m_H^2}{\hbar c} \sim 5.9 \times 10^{-39} . \tag{4.243}$$

For electromagnetic interaction, we get

$$\alpha_{EM} = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137} , \tag{4.244}$$

which is *much greater*.

Then we find

$$M_{\min} \approx 16 \left(\frac{k_B T_{\text{ign}}}{m_e c^2}\right)^{3/4} \alpha_G^{-3/2} m_H . \tag{4.245}$$

If $T_{\text{ign}} \sim 1.5 \times 10^6$ K, one tenth of the temperature of the Sun, we find

$$M_{\min} \sim 0.03 \alpha_G^{-3/2} m_H , \tag{4.246}$$

while for the maximum mass, from before with $\beta = 0.5$ and $\overline{m} = 0.61 m_H$ we get

$$M_{\max} \approx 56 \alpha_G^{-3/2} m_H , \tag{4.247}$$

so this hints to the fact that $m_* = \alpha_G^{-3/2} m_H$ is an important characteristic mass. This is around $1.85 M_\odot$.

This corresponds to a number of particles:

$$N_* = \frac{m_*}{m_H} \approx 2 \times 10^{52} . \tag{4.248}$$

Suppose the core of a star is held together by the pressure of degenerate electrons: we discuss *white dwarves*. We define

$$n_e = Y_e \frac{\rho_c}{m_H}, \tag{4.249}$$

where $Y_e = (1 + x_1)/2$. The pressure is

$$P = k_{NR} n_e^{5/3} = k_{NR} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3}, \tag{4.250}$$

which must compared to

$$P_c = \left(\frac{\pi}{36} \right)^{1/3} G M^{2/3} \rho_c^{4/3}, \tag{4.251}$$

and we assume that $P = P_c$: then we get

$$\rho_c \approx \frac{3.1}{Y_e^5} \frac{M}{m_*} \frac{m_H}{(h/m_e c^2)^3}. \tag{4.252}$$

Missing square on the M/m_*?

The pressure is

$$P = k_{UR} n_e^{4/3} = k_{UR} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3}, \tag{4.253}$$

so we get

$$k_{UR} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3} \approx \left(\frac{\pi}{36} \right)^{1/3} G M^{2/3} \rho_c^{4/3}, \tag{4.254}$$

so we can see that in this limit the expression becomes independent of ρ_c. This will give us a limit mass: the Chandrasekhar mass.

$$M_{CH} = \left(\frac{36}{\pi} \right)^{1/2} \left(\frac{Y_e}{m_H} \right)^2 \left(\frac{k_{UR}}{G} \right)^{3/2} \approx 2.3 Y_e^2 m_* \approx 4.3 Y_e^2 M_\odot \approx 1.4 M_\odot. \tag{4.255}$$

This is the maximum mass of a white dwarf to remain stable, held together by the degeneracy pressure of electrons. Above this, it becomes a neutron star.

Thu Dec 19 2019

The lessons on gravitational waves will be on the 8th and 9th of January, from 14:30 to 16:30, in rooms LUF2 and P2B respectively.

We want to find the Chandrasekhar limit in a more precise manner.

The number density of electrions is given by

$$n_e = Y_e \frac{\rho_c}{m_H}, \tag{4.256}$$

in the nonrelativistic case the pressure is given by

$$P_c = k_{\mathrm{NR}} n_e^{5/3} = k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} , \tag{4.257}$$

but we can also derive it by

$$P_c = k_{\mathrm{NR}} n_e^{5/3} = k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} , \tag{4.258}$$

and equating these we find:

$$k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} = k_{\mathrm{NR}} n_e^{5/3} = k_{\mathrm{NR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{5/3} , \tag{4.259}$$

which implies

$$\rho_c = \frac{3.1}{Y_e^5} \left(\frac{M}{M_*} \right)^2 \frac{m_H}{(h/m_e c^2)^3} , \tag{4.260}$$

where $\alpha_G = G m_H^2 / (\hbar c) \approx 5.9 \times 10^{-30}$, while

$$m_* = \alpha_G^{-3/2} m_H = 1.85 M_\odot , \tag{4.261}$$

while for the ultrarelativistic case we get

$$P_C = k_{\mathrm{UR}} n_e^{4/3} = k_{\mathrm{UR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3} , \tag{4.262}$$

so in this particular case the density ρ_c simplifies from the equations: we get a critical mass

$$M_{\mathrm{CHANDRA}} = \left(\frac{36}{\pi} \right)^{1/2} \left(\frac{Y_e}{m_H} \right)^2 \left(\frac{k_{\mathrm{UR}}}{G} \right)^{3/2} \approx 2.3 Y_e^2 m_* \approx 4.3 Y_e^2 M_\odot , \tag{4.263}$$

and we assume that we are in the fully degenerate case: in the integration over momenta of the phase space distribution we insert a cutoff at the Fermi energy.

We find:

$$P = \frac{4\pi}{3h^3} g_* \int_0^{p_F} \mathrm{d}p \, p^2 \frac{p^2 c^2}{\epsilon_p} , \tag{4.264}$$

where $\epsilon_p = \left(p^2 c^2 + m^2 c^4 \right)^{1/2}$. We integrate with the dimensionless variable $x = p/(m_e c)$: we get

$$P = \frac{8\pi}{3h^3} m_e^4 c^5 \int_0^{x_F} \frac{x^4}{(1+x^2)^{1/2}} \, \mathrm{d}x , \tag{4.265}$$

so we get $P = k_{\mathrm{UR}} n_e^{4/3} I(x_F)$, where we incorporated the integral in the term $I(x_F)$: this is given by

$$I_(x) = \frac{3}{2x^4} \left(x(1+x^2)^{1/2} \left(\frac{2x^2}{3} - 1 \right) + \log \left(x + (1+x^2)^{1/2} \right) \right), \tag{4.266}$$

and

$$x_F = \frac{p_F}{m_e c} = \left(\frac{3n_e}{8\pi} \right)^{1/3} \frac{h}{m_e c} = \left(\frac{3Y_e \rho_c}{8\pi m_H} \right)^{1/3} \frac{h}{m_e c}, \tag{4.267}$$

so if $x_F \gg 1$ we have $I(x_F) \sim 1$, the ultrarelativistic case, while if $x_F \ll 1$ we have $I(x_F) \sim 4x_F/5$. This then interpolates between our different cases.

$$k_{\mathrm{UR}} \left(\frac{Y_e \rho_c}{m_H} \right)^{4/3} I(x_F) \approx \left(\frac{\pi}{36} \right)^{1/3} G M^{4/3} \rho_c^{4/3}, \tag{4.268}$$

so we can extract the mass:

$$M = I(x_F)^{3/2} M_{\mathrm{CH}}, \tag{4.269}$$

and the important thing is that $x_F \propto n_e^{1/3} \propto \rho_c^{1/3}$.

[Graph: on the x axis M/M_{CHANDRA}, on the y axis ρ_c.]

If we increase the mass, the star is not able to support itself by the pressure due to being a gas of degenerate electrons.

This gives us

$$M_{\mathrm{CHANDRA}} = 3.1 Y_e^2 m_* = 5.8 Y_e^2 M_\odot = 1.4 M_\odot. \tag{4.270}$$

It can be shown that the mean density is around

$$\langle \rho \rangle = \frac{1}{6} \rho_c = \frac{0.51}{Y_e^2} \left(\frac{M}{m_*} \right)^2 \frac{m_H}{(h/m_e c)^3}, \tag{4.271}$$

so we can estimate the radius as

$$R = \left(\frac{3M}{4\pi \langle \rho \rangle} \right)^{1/3} \approx 0.77 Y_e^{5/3} \left(\frac{M}{m_*} \right)^{1/3} \alpha_G^{-1/2} \frac{h}{m_e c}, \tag{4.272}$$

and the object at the end is

$$\alpha_G^{-1/2} \frac{h}{m_e c} \approx 3 \times 10^7 \,\mathrm{m}, \tag{4.273}$$

so if we take $Y_e = 0.5$ we find:

$$R = \frac{R_\odot}{74} \left(\frac{M_\odot}{M} \right)^{1/3}. \tag{4.274}$$

The luminosity is given by

$$L = 4\pi R^2 \sigma T_E^4 = \frac{1}{74^2} \left(\frac{M_\odot}{M} \right)^{4/3} \left(\frac{T_E}{6000\,\mathrm{K}} \right) L_\odot \,, \tag{4.275}$$

so if we take a typical effective temperature of around $10^4\,\mathrm{K}$ (recall that neutron stars are in the blue part of the HR diagram), $M = 0.4 M_\odot$ we get $L \approx 3 \times 10^{-3} L_\odot$.

We deal with degenerate stars: the Pauli exclusion principle plays a critical role, and the process

$$\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \overline{\nu}_e \tag{4.276}$$

is inhibited, since there is no more room for electrons. On the other hand, the process

$$\mathrm{e}^- + \mathrm{p} \to \mathrm{n} + \nu_e \tag{4.277}$$

is favoured. We can look at the Saha formula to get numerical estimates for this. The chemical potential of neutrinos can be neglected, therefore we find

$$\mu_\mathrm{n} = \mu_\mathrm{p} + \mu_\mathrm{e} \,, \tag{4.278}$$

and the same equation holds for their Fermi energies.

This means that a certain point we will have only neutrons.

We need to exploit the Fermi exclusion principle. Why does the equation

$$\epsilon_{F,\mathrm{n}} = \epsilon_{F,\mathrm{p}} + \epsilon_{F,\mathrm{e}} \,, \tag{4.279}$$

favour neutrons? we have the constraint that the number of protons must equal the number of electrons, but the Fermi energy depends on the number density of these. typical numbers then become

$$n_\mathrm{p} = n_\mathrm{e} = \frac{n_\mathrm{n}}{200} \,. \tag{4.280}$$

We will have

$$n_\mathrm{n} = Y_n \frac{\rho_c}{m_\mathrm{n}} \approx \frac{\rho_c}{m_\mathrm{n}} \qquad \text{since} \qquad Y_n \approx 1 \,. \tag{4.281}$$

For a neutron star we have

$$\rho_c \approx 3.1 \left(\frac{M}{M_*} \right)^2 \frac{m_n}{(h/m_n c)^3} \,, \tag{4.282}$$

while for a white dwarf we have

$$\rho_c \approx \frac{3.1}{Y_e^5} \left(\frac{M}{M_*} \right)^2 \frac{m_H}{(h/m_e c^2)^3} \,, \tag{4.283}$$

the mass is given by $M_* = \alpha_G^{-3/5} m_n \approx 1.85 M_\odot$ and for the radius we get

$$R = 0.77 \left(\frac{M_*}{M} \right)^{1/3} \alpha_G^{-1/2} \frac{h}{m_n c}, \tag{4.284}$$

where the characteristic length is given by

$$L_n = \alpha_G^{-1/2} \frac{h}{m_n c} \approx 17\,\text{km} \approx \frac{1}{1200} L_e, \tag{4.285}$$

1200 times smaller than the corresponding length scale for electrons: L_n is the characteristic length scale for a NS, L_e is the characteristic length for a white dwarf. The maximum mass is $M_{\text{max}}^{\text{NS}} = 3.1 M_* = 5.8 M_\odot$. This is the mass of a star *remnant*, which encompasses mass from the core only: the initial star will be much larger.

Can this NS become a BH? we need to compute

$$\frac{GM}{Rc^2} \approx 0.2 \left(\frac{M}{M_*} \right)^{4/3}, \tag{4.286}$$

so if the mass is large enough we can reach the critical value of $GM/Rc^2 = 2$.

Neutron stars were first detected as rotating objects: *pulsars*. We have

$$\frac{GM}{R^2} \approx R \omega_{\text{max}}^2, \tag{4.287}$$

the maximum angular velocity which can be supported gravitationally: it comes out to be

$$\tau_{\text{min}} = \frac{2\pi}{\omega_{\text{max}}} \approx 2\pi \left(\frac{R^3}{GM} \right)^{1/2}, \tag{4.288}$$

which is of the order

$$\tau_{\text{min}} \approx 11 \left(\frac{M_*}{M} \right) \alpha_G^{-1/2} \frac{h}{m_n c^2} \approx 0.6 \frac{M_*}{M} \text{ms}. \tag{4.289}$$

This gives us a bound for gravitational waves of astrophysical origin.

no comment on the GR corrections to this formula: however I'd expect them to be significant

We move on to *the GR issue*.

In a star, the classical equation of hydrostatic balance is:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2}, \tag{4.290}$$

while the GR equations for this are the TOV equation: Tolman-Oppenheimer-Volkov:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2} \left(1 + \frac{P}{\rho c^2} \right) \left(1 + \frac{4\pi r^3 P}{mc^2} \right) \left(1 - \frac{2Gm}{rc^2} \right)^{-1}, \tag{4.291}$$

and the first correction is reminiscent of cosmology: *the pressure itself contributes to the inertia of the system*.

If we have constant density, in the Newtonian case we get

$$m(r) = \frac{4\pi}{3} \rho_0 r^3 \,, \tag{4.292}$$

so than we can integrate and get

$$P(r) = \frac{2\pi G}{3} \rho_0^2 \left(R^2 - r_0^2 \right) , \tag{4.293}$$

where we inserted the boundary condition $P(R) = 0$. In the first-order GR case, this can still be solved analytically! We get

$$P(r) = \rho_0 c^2 \left(\frac{(1 - 2GMr^2/R^3c^2)^{1/2} - (1 - 2GM/Rc^2)^{1/2}}{3\left(1 - 2GM/Rc^2\right)^{1/2} - \left(1 - 2GMr^2/R^3c^2\right)^{1/2}} \right) , \tag{4.294}$$

Check exponents

so we get

$$P_c = \frac{2\pi}{3} G \rho_0^2 R^2 = \left(\frac{\pi}{6} \right)^{1/3} G M^{2/3} \rho_c^{4/3} \,, \tag{4.295}$$

so we can look at what happens when we consider $r = 0$: we get

$$P_c = \rho_0 c^2 \left(\frac{1 - \sqrt{1 - 2GM/Rc^2}}{3\sqrt{1 - 2GM/Rc^2} - 1} \right) , \tag{4.296}$$

so we can see that the central pressure is finite as long as

$$\frac{GM}{Rc^2} < \frac{4}{9} \,, \tag{4.297}$$

which is not $1/2$ since we have made some approximations.

Then we have a bound

$$M_{\max} \approx \left(\frac{8\pi f}{9} \right)^{3/2} M_* \,, \tag{4.298}$$

with $f \sim 1$. This means than the objects becomes contained inside its Schwarzschild radius, $R = 2GM/c^2$.

Tomorrow we will speak of galaxy formation.

Fri Dec 20 2019

The lectures in january will be about *cosmological* gravitational waves mainly.

Write to him if you do not have access to the dropbox.

On the exam: a traditional oral exam, with questions on the main topics dealt with in class. The days on the calendar do not mean anything. The exams should be agreed upon by email.

Now we talk about the formation of dark matter halos. This is in Sabino's notes in the dropbox.

We say we have a spherical object in the universe, focus on it and apply Birkoff's theorem: we can study it independently of the surroundings.

We can consider halos of different densities to account for over and under densities. In the real world, we will not have spheres, but three-axial ellipsoids: it is known that if we have over-densities the non-sphericity will increase, if we have under-densities it will decrease.

However we treat spherical models because it is simple. Historically the Americans supported the spherical models, the Russians supported the "pancake model". Now we know that we have pancakes with spheres inside (?).

We consider a sphere. We define:

$$\delta(\vec{x}, t) = \frac{\rho(\vec{x}, t) - \overline{\rho}(t)}{\overline{\rho}(t)}, \tag{4.299}$$

and we assume $0 < \delta \ll 1$ (a *small over-density*), although in general we could have $-1 < \delta < +\infty$.

We found in previous lessons that we have a growing mode $\delta \propto t^{2/3}$ and a decaying mode $\delta \propto t^{-1}$, for which we had $v \propto t^{1/3}$ and $v \propto t^{-4/3}$ respectively.

So we choose a time t_i such that

$$\delta(t) = \delta_+(t_i)\left(\frac{t}{t_i}\right)^{2/3} + \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1}. \tag{4.300}$$

The linearized continuity equation was:

$$v = i\frac{\dot{\delta}}{k}a \propto \left(\frac{2}{3}\delta_+(t_i)\left(\frac{t}{t_i}\right)^{1/3} - \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-4/3}\right), \tag{4.301}$$

since $a \propto t^{2/3}$. This is in order to write explicitly the fact that we need two initial conditions.

We suppose that at $t = t_i$ we have *unperturbed Hubble flow*: $v(t_i) = 0$, which means

$$\delta_-(t_i) = \frac{2}{3}\delta_+(t_i), \tag{4.302}$$

and we can have a generic initial density, which we can express as:

$$\delta(t_i) = \delta_i = \delta_+ + \delta_- = \frac{5}{3}\delta_+. \tag{4.303}$$

Our perturbation can be dealt with cosmologically as being a *local FRLW closed universe*. Say our sphere has a radius R: then if we have $\delta = \delta_i$ then

$$\Omega(t_i) = 1 + \delta_i \,. \tag{4.304}$$

We have then, from the Friedmann equations:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k \,, \tag{4.305}$$

where $k = +1$ if $\delta_i > 0$, because of the fact that our sphere is locally a closed universe. This then means:

$$-k = (1 - \Omega) a^2 H^2 \,, \tag{4.306}$$

so

$$\frac{\dot{a}^2}{a_i^2} = H_i^2 \left(\Omega_p(t_i) \frac{a_i}{a} + (1 - \Omega_p(t_i)) \right) , \tag{4.307}$$

where the index p denotes the fact that we are talking about the perturbation. We have:

$$\rho_p(t) = \rho_p(t_i) \left(\frac{a_{pi}}{a_p} \right)^3 \tag{4.308a}$$

$$= \rho_c(t_i) \Omega_p(t_i) \left(\frac{a_{pi}}{a} \right)^3 . \tag{4.308b}$$

We want to derive a time for the *turnaround time* t_m:

$$\rho_p(t_m) = \rho_c(t_i) \Omega_p(t_i) \left(\frac{\Omega_p(t_i) - 1}{\Omega_p(t_i)} \right)^3 \tag{4.309a}$$

$$= \rho_c(t_i) \frac{\left(\Omega_p(t_i) - 1\right)^3}{\Omega_p(t_i)^2} \,, \tag{4.309b}$$

since at $t = t_m$ we have $\frac{\dot{a}^2}{a_i^2} = 0$.

since $\dot{a}(t_m) = 0$?

Some time ago we had found:

$$t(\theta_m = \pi) = t_m = \frac{\pi}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \,, \tag{4.310}$$

but calculating this *now* is arbitrary: the same formula holds replacing 0 with i.

Then we have

$$t_m = \frac{\pi}{2H_i} \left(\frac{\rho_c(t_i)}{\rho_p(t_m)} \right)^{3/2} , \tag{4.311}$$

since we have found that in the relation from a few lessons ago we have exactly the inverse of the relation we just derived connecting Ω and ρ.

It is convenient to consider unperturbed Hubble flow so that the Hubble parameter inside and outside is the same.

Then we have

$$H^2(t_i) = \frac{8\pi G}{3}\rho_c(t_i)\,, \tag{4.312}$$

so we have a cancellation: we find

$$t_m = \frac{\pi}{2H_i}\left(\frac{\rho_c(t_i)}{\rho_p(t_i)}\right)^{1/2} = \left(\frac{3\pi}{32G\rho_p(t_m)}\right)^{1/2}\,, \tag{4.313}$$

so we get

$$\rho_p(t_m) = \frac{3\pi}{32Gt_m^2}\,, \tag{4.314}$$

which holds inside the sphere, while the critical density *outside* is

$$\rho_c(t_m) = \frac{1}{6\pi G t_m^2}\,. \tag{4.315}$$

We are implicitly using the *synchronous gauge*, and we are finding an exact solution to the EFE: The Lemaître-Tolman-Bondi solution.

We can ask: at a certain given time, how much is the interior density larger than the exterior one? this is given by

$$1+\delta_p(t_m) = \chi(t_m) = \frac{\rho_p(t_m)}{\rho_c(t_m)} = \frac{3\pi}{32G}6\pi G = \left(\frac{3\pi}{4}\right)^2 \approx 5.6\,. \tag{4.316}$$

Then we have $\delta_p(t_m) \approx 4.6$. How does this compare to linear theory?

$$\delta_p(t_m) = \delta_t(t_i)\left(\frac{t_m}{t_i}\right)^{2/3} + \delta_-(t_i)\left(\frac{t_m}{t_i}\right)^{-1} \approx \frac{3}{5}\delta_p(t_i)\left(\frac{t_m}{t_i}\right)^{2/3}\,, \tag{4.317}$$

since the second term vanishes.

We know that $H_i = 2/(3t_i)$, so

$$t_m = \frac{3\pi t_i}{4} \times \left(\frac{1+\delta_i}{\delta_i^{3/2}}\right) \approx \frac{3\pi t_i}{4}\delta_i^{-3/2}\,, \tag{4.318}$$

which comes from the equation from the perturbation density at t_m, in which we substitute $\Omega_p - 1 = (\delta_p+1)-1 = \delta_p$, and we approximate $1+\delta \approx 1$ since δ is small.

So we have

$$\frac{t_m}{t_i} = \frac{3\pi}{4}\delta_i^{-3/2}\,, \tag{4.319}$$

which gives us a correction

$$\delta_p(t_m) = \frac{3}{5}\delta_i \left(\frac{3\pi}{4} \delta_i^{-3/2} \right)^{2/3} = \frac{3}{5} \left(\frac{3\pi}{4} \right)^{2/3} \approx 1.87 \,. \tag{4.320}$$

So linear theory would have given us a wrong answer, as we should expect: we are very far from the $\delta \ll 1$ regime in which we expect linearization to hold.

We speak of the virialization time (the time after which the VT starts applying?). It is hard to calculate, different books give different values.

$$t_{\text{vir}} \approx t_c = 2t_m \,. \tag{4.321}$$

Then we have

$$E_{\text{tot}} = T + E_{\text{gr}} = \frac{1}{2} E_{\text{gr}} = -T \,, \tag{4.322}$$

since $2T + E_{\text{gr}} = 0$.

The total energy at virialization is given by

$$E_{\text{eq}} = -\frac{1}{2}\frac{3}{5}\frac{GM^2}{R_{\text{eq}}} \,, \tag{4.323}$$

where the factor $3/5$ is given by the geometry of the system, we are assuming constant density for our sphere, while the $1/2$ comes from the VT. At the time of collapse instead the energy is given by

$$E_m = -\frac{3}{5}\frac{GM^2}{R_m} \,, \tag{4.324}$$

This then means that the radius at virialization, R_{eq}, is twice the radius at the start of the collapse, R_m:

$$2R_{\text{eq}} = R_m \,. \tag{4.325}$$

This also means, by mass conservation, that $\rho_{\text{vir}} = 8\rho_m$ since the volume shrinks by 2^3.

We want to know the density at collapse::

$$\frac{\rho_p(t_c)}{\rho_c(t_c)} = \underbrace{\frac{\rho_p(t_c)}{\rho_p(t_m)}}_{8} \underbrace{\frac{\rho_p(t_m)}{\rho_c(t_m)}}_{\chi \approx 5.6} \underbrace{\frac{\rho_c(t_m)}{\rho_c(t_c)}}_{2^2} \approx 180 \,. \tag{4.326}$$

This object is called $1 + \delta(t_c)$, so $\delta(t_c) \approx 179$.

What would happen if we were to use linear theory at this time? Then we would find

$$\delta_+(t_c) = \delta_+(t_m) \left(\frac{t_c}{t_m} \right)^{2/3} \,, \tag{4.327}$$

which gives us

$$\delta_+(t_c) \approx \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} 2^{2/3} \approx 1.686\,, \tag{4.328}$$

which is a kind of "clock": how long can we use linear theory for?

We deal with *** theory: we introduce

$$n(M) = \frac{\mathrm{d}N}{\mathrm{d}M} = \text{\# of objects per unit volume with mass in } [M, M + \mathrm{d}M]\,. \tag{4.329}$$

Let us consider linear perturbations $\delta(\vec{x}, t)$: perturbations in the matter density dealt with using linear theory only.

These tend to fluctuate a lot. We need a *filter*: we use a *low-pass* filter, ignoring the high-frequency modes. It is $W_R(\vec{x})$: R is a spatial radius, we ignore modes with spatial frequency larger than $1/R$.

We use a filter labelled by a mass M, which we find by assuming a certain density, and then using $M \propto R^3$.

It is generally a good idea to assume gaussianity. In the Planck data it was found by Sabino's team that the bounds on non-gaussianity are very low, the data are almost gaussian. So, we have

$$p(\delta_M)\,\mathrm{d}\delta_M = \frac{1}{\sqrt{2\pi\sigma_M^2}} \exp\left(-\frac{\delta_M^2}{2\sigma_M^2}\right) \mathrm{d}\delta_M\,, \tag{4.330}$$

where the variance is typically diverging if we do not apply the filter: we have

$$\sigma_M^2 = \left\langle \delta_M^2 \right\rangle \propto M^{-2\alpha}\,, \tag{4.331}$$

and typically $\alpha \sim 1/2$.

We define a threshold for the value of δ_M, and we want to compute the probability of the value becoming larger than it. We use for the critical value $\rho_c = 1.686$ from before. We have

$$\mathbb{P}_{>\delta_c}(M) = \int_{\delta_c}^{\infty} \mathrm{d}\delta_M\, p(\delta_M)\,. \tag{4.332}$$

so we have

$$n(M) M\,\mathrm{d}M = \rho_m \big(\mathbb{P}_{>\delta_c}(M) - \mathbb{P}_{>\delta_c}(M + \mathrm{d}M)\big) \tag{4.333a}$$

$$= \rho_m \left|\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}M}\right| \mathrm{d}M \tag{4.333b}$$

$$= \rho_m \left|\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}\sigma_M}\right| \left|\frac{\mathrm{d}\sigma_M}{\mathrm{d}M}\right| \mathrm{d}M\,. \tag{4.333c}$$

Integrating $\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}M}$ we expect to find the matter density again, but we find $1/2$ of it.

This comes from a miscount: as the mass we are considering shrinks, we might me already including smaller objects inside the gravitational influence of larger ones. Properly accounting for this one gets precisely a factor 2.

Integrating we get

$$n(M) = \frac{2}{\sqrt{\pi}} \frac{\rho_m}{M_*^2} \alpha \left(\frac{M}{M_*} \right)^{\alpha} \exp\left(- \left(\frac{M}{M_*} \right)^{2\alpha} \right), \tag{4.334}$$

where $M_* = (2/\delta_c)^{1/2\alpha} M_0$.

Accounting for non spherical collapse we get much better estimates.

Wed Jan 08 2020

4.1 Gravitational waves and interferometry

Guest lecture by Angelo Ricciardone.

An outline:

1. introduction about gravitational waves;
2. frequency bands $\iff$ sources of GW $\iff$ detectors;
3. GWs and observables;
4. stochastic background of GW: characterization, sources, detection (here we discuss *cosmological* sources).

References:

1. Book: "Gravitational Waves - theory and experiments" by Michele Maggiore (Oxford University Press, 2007);
2. Book: "Gravitational Waves - Astrophysical sources" (2018);
3. Paper: "The basics of gravitational wave theory", F. Flannagan (https://arxiv.org/abs/gr-qc/0501041);
4. Review: "Gravitational waves from inflation", (https://arxiv.org/abs/1605.01615);
5. Review: "Cosmological background of gravitational waves", (https://arxiv.org/abs/1801.04268);
6. Kind of unrelated review: "Inflation and the Theory of Cosmological Perturbations" https://arxiv.org/abs/hep-ph/0210162.

We start with some general facts.

Gravitational waves appear naturally in GR: they are propagating oscillations of the gravitational field.

The gravitational interaction is *weak*: this implies that GWs travel freely, but they are also hard to detect.

The frequencies we see for the EM spectrum are in the range $f_{EM} \sim 10^4\,\text{Hz} \div 10^{20}\,\text{Hz}$, radio waves to γ rays.

The typical frequencies of GWs are, instead, in the range $f_{GW} \sim 10^{-16}\,\text{Hz} \div 10^4\,\text{Hz}$, the lower end of this range is the frequency of the CMB GWs while the higher end is given off by astrophysical sources. Ground-based detectors can detect the higher end of this spectrum.

But the CMB is electromagnetic! Is there gravitational radiation corresponding to it?

The wavelength and frequency are typically comparable to the size of the object which is emitting the GWs.

Let us consider astrophysical sources:

$$T = 2\pi\sqrt{\frac{R^3}{GM}} \implies f = \sqrt{\frac{G\rho}{4\pi^2}}, \tag{4.335}$$

and since the mass is related to the density by $M = \rho V = \frac{4}{3}\pi R^3 \rho$, then we have

$$f_{GW} \approx \frac{1}{2\pi}\sqrt{\frac{3GM}{4\pi R^3}}. \tag{4.336}$$

We know that the radius of the object must be greated than the Schwarzschild radius $R_s = 2GM/c^2$: substituting this in we get

$$f_{GW} \approx \frac{1}{4\pi}\frac{c^3}{GM}, \tag{4.337}$$

which means, substituting the numbers, that

$$f_{GW} \approx 10^4\,\text{Hz}\frac{M_\odot}{M}, \tag{4.338}$$

Is the difference between the formulas coming from different geometries of the problem? Like, a rotating objects versus two inspiralling ones?

We make some estimates for neutron stars. Typically they have $M \sim 1.4M_\odot$ and $R \sim 10^4\,\text{m}$. So, we have $f \sim 10^4\,\text{Hz}$.

For small BHs, we have $M \sim 30M_\odot$, so $f_{GW} \sim 300\,\text{Hz}$. This is the band in which LIGO/VIRGO works: these interferometers cannot measure GWs with frequency smaller than $1\,\text{Hz}$.

If we increase the mass, if we use $M \sim 10^7 M_\odot$, we get $f_{GW} \sim 10^{-3}\,\text{Hz}$: this is the band in which LISA will work.

There are indirect evidences of GW emission.

Pulsars slow down: this is the Hulse-Taylor binary pulsar, two neutron stars which are rotating around each other emit gravitational waves. The GW emission back-reacts on the dynamics of the binary, on a timescale which is observable.

A reference for this is https://arxiv.org/abs/astro-ph/0407149.

We can make a plot: radius of the system vs mass of the system. Since $f \propto M^{1/2} R^{-3/2}$, constant frequency means a powerlaw in this plot, so a straight line in $\log R$ vs $\log M$.

The chirp of the inspiral is *also* a powerlaw. The GR predictions for the period decrease of the Hulse-Taylor binary NS match observations very precisely.

On the 14/09/2015, we had the first direct evidence of GWs with the first detection.

On 17/08/2017 we had the first detection of a NS/NS merger: this was the birth of multimessenger astronomy, since we saw the event with optical telescopes also. This meant that the signal got to us with a speed which is the same as the electromagnetic speed of light.

Now, there is an app: the "GW events app" https://apps.apple.com/us/app/gravitational-wave-events/id1441897107.

In GR, gravitational waves can be polarized in two different ways. These are called "plus" and "cross" polarizations.

Can one of these be rotated into the other? There might be some issue since one is a pseudotensor...

We distinguish: the High Frequency band goes from 10^4 Hz to 1 Hz; the Low Frequency band goes from 1 Hz to 10^{-4} Hz; the Very Low Frequency band goes from 10^{-7} Hz to 10^{-9} Hz; the Extremely Low frequency badn goes from 10^{-15} Hz to 10^{-18} Hz.

There are gaps since in certain frequency regions the methods we have on either side fail for different reasons leaving a gap.

Let us start from HF: it is the domain of Earth-based interferometers: LIGO (Livingstone & Hanford), which has 4 km arms, the two detectors are separated by 3000 km. Also, there is VIRGO near Pisa: an arm is 3 km long.

There is also Geo600 in Hannover, Germany: it has 600 m arms. In Japan there is Kagra: it has 4 km arms.

There are several reasons to have more than one detector: we can identify the position of the source, we can verify signals.

There will be the new Einstein Telescope, in a triangular configuration, in Italy or the Netherlands. There will be the Cosmic Explorer, with 40 km arms.

Now, let us discuss the main sources in the HF band. We have:

1. coalescence of stellar-mass BH binaries and NSs, for these we have an upper bound of $M \lesssim 10^3 M_\odot$;
2. rotation of neutron stars (pulsar);
3. stellar collapse: supernova to BH or NS.

In the LF band, the domain of space-based interferometers, we will have LISA, in a triangular shape with 2.5×10^6 km, and the Japanese DECIGO, with $L \sim 1000$ km.

In the LF band, the sources are:

1. white dwarves merging;
2. NS merging;

3. inspiral and coalescence of SMBH (masses from 100 to 10^8 $M_\odot$).

In the VLF band, we can use Pulsar-Time Array. The sources in this frequency range are:

1. GWs from SMBH with $M > 11M_\odot$, but it seems like there are no black holes this large;
2. GWs from cosmic strings & from phase transitions;

In the ELF band, we have cosmological sources:

1. primordial GWs: here we have $h \sim (E_{\text{infl}}/M_P)^2$.

Typically we have amplitudes increasing as the frequency decreases.
For scalar perturbations we have

$$\frac{\Delta T}{T} \propto \delta\phi \,. \tag{4.339}$$

Vector perturbations decay with the expansion of the universe. We do have tensor perturbations. B mode polarizations correspond to primordial gravitational waves.

What are B modes?

In a simple MM interferometry, $\Delta L \propto h$.

LISA will orbit the Sun at 20° from the Earth. The astrophysical targets for LISA are MBHBs, EMRIs and compact WDs.

Also, there are potential cosmological sources. We have first order phase transitions around the TeV, inflationary GWs, cosmic strings and using MBHBs as standard sirens.

We put powerlaw amplitude spectra in a graph. What is Ω?

We saw powerlaw spectra with Ω increasing with f: do they not have a UV problem since the energy density increased with the frequency? No, since because of the transfer function between the early universe and now at a certain point the powerlaw becomes decreasing.

Cosmological sources are stochastic: they give a background.

If we will have a detector with high sensitivity, we will see many events: they will form a stochastic background.

Can we not correlate the signals in such a way that we only look at GWs from a specific angular region?

A stochastic background of GW comes from a large number of independent uncorrelated sources that are not individually resolvable.

CSGWB is a candidate source for LISA.

For tomorrow, we can either give an overview of cosmological sources for GWs or we can derive the relation

$$\frac{\Delta L}{L} \propto h \,, \tag{4.340}$$

Thu Jan 09 2020

Many of these topics will be discussed in more details in the course by Nicola Bartolo on cosmological perturbations and in the course by Giacomo Ciani on gravitational waves.

Today we will focus on the LISA detector, which works for frequencies $f \sim 10^{-5} \div 10^{-1}$ Hz, and on the cosmological stochastic background of GWs (CSGWB).

We can make an analogy with the CMB: we expect to see a background of gravitational radiation coming from all directions.

The stochastic background can have different origins:

1. astrophysical origin: a coherent superposition of a large number of astrophysical sources, which are too weak to be detected separately;
2. cosmological origin: it is generated in the early universe by a variety of mechanisms:
   (a) amplification of primordial tensor fluctuations via inflation;
   (b) GWs from phase transition around the TeV scale (this is not the only energy scale at which they can be emitted, but it is the energy we'd need in order to detect them with LISA);
   (c) GWs from topological defects.

A stochastic background of cosmological origin is expected to be

1. isotropic;
2. stationary;
3. unpolarized, which means that the cross and plus polarizations will have the same amplitude.

Let us discuss the main properties of the frequency spectrum: it is characterized

1. in terms of (normalized) energy density per unit logarithmic interval of frequency: this is called $h_0^2 \Omega_{GW}(f)$;
2. in terms of the spectral density of the ensemble average of the Fourier component of the metric $S_h(f)$;
3. more on the experimental side: in terms of a characteristic amplitude of the stochastic background $h_c(f)$.

Let us define these three quantities. The energy density is given by

$$\Omega_{GW} = \frac{1}{\rho_c} \frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f}, \tag{4.341}$$

where ρ_{GW} is the energy density of SGWB, f is the frequency, while ρ_c is the present value of the critical energy density, defined as

$$\rho_c = \frac{3H_0^2}{8\pi G}, \tag{4.342}$$

where we usually write $H_0 = h_0 \times 100\,\mathrm{km}/(\mathrm{sMpc})$.

So, usually we plot $h_0^2\Omega_{GW}$ in order to ignore the uncertainties on the measurements of H_0.

We write stochastic GW at a given point $\vec{x} = 0$ in the transverse traceless (TT) gauge: $h_{ii} = 0$ and $\partial_i h^{ij}$: we get

$$h_{ab}(t) = \sum_{A=+,\times} \int_{\mathbb{R}} \mathrm{d}f \int_{S^2} \mathrm{d}\Omega\, \hat{h}_A(f,\Omega) \exp(-2\pi i f t) e^{A}_{ab}(\hat{\Omega}) \,, \tag{4.343}$$

where we must have the condition $\hat{h}_A(f,\Omega) = \hat{h}^*_A(-f,\Omega)$. Here $\hat{\Omega}$ is a unit vector representing the direction of propagation of the wave, and $\mathrm{d}\hat{\Omega} = \mathrm{d}\cos(\theta)\,\mathrm{d}\phi$, e^A_{ab} are the polarization tensors:

1. $e^{+}_{ab}(\hat{\Omega}) = 2\hat{m}_{[a}\hat{m}_{b]}$;
2. $e^{\times}_{ab}(\hat{\Omega}) = 2\hat{m}_{(a}\hat{m}_{b)}$;

where $\hat{m}_{a,b}$ are unit vectors orthogonal to each other and to the propagation direction.

We have the condition $e^{A}_{ab}e^{Bab} = 2\delta^{AB}$.

Then, assuming a SGWB which is isotropic, unpolarized and stationary we will have

$$\left\langle \hat{h}^*_A(f,\hat{\Omega})\hat{h}_{A'}(f',\hat{\Omega}') \right\rangle = \delta(f-f')\frac{1}{4\pi}\delta^{(2)}(\hat{\Omega},\hat{\Omega}')\delta_{AA'}\frac{1}{2}S_h(f) \,, \tag{4.344}$$

where

$$\delta^{(2)}(\hat{\Omega},\hat{\Omega}') = \delta(\phi-\phi')\delta(\cos(\theta)-\cos(\theta')) \,. \tag{4.345}$$

The factors of $1/4\pi$ and $1/2$ are for convention, for normalization, so that $\int S_h(f)\,\mathrm{d}f = 1$.

Then the remaining bit, $S_h(f)$, is called the spectral density.

So, using the equaitions we found so far, we get

$$\left\langle h_{ab}(t)h^{ab}(t) \right\rangle = 2\int_{\mathbb{R}} \mathrm{d}f\, S_h(f) \tag{4.346a}$$

$$= 4\int_{f=0}^{f=\infty} \mathrm{d}\log f\, f S_h(f) \,. \tag{4.346b}$$

We define the characteristic amplitude $h_c(f)$ as

$$\left\langle h_{ab}(t)h^{ab}(t) \right\rangle = 2\int_{f=0}^{f=\infty} \mathrm{d}\log f\, h_c^2(f) \,, \tag{4.347}$$

which means that $h_c^2(f) = 2fS_h(f)$.

The last step is to relate $h_c(f)$ and $h_0^2\Omega_{GW}(f)$.

The energy density is defined as:

$$\rho_{GW} = \frac{1}{32\pi G}\left\langle \dot{h}_{ab}\dot{h}^{ab} \right\rangle \,, \tag{4.348}$$

where the average is performed over a wavelength, but by the ergodic theorem it can also be performed over a period. So,

$$\rho_{GW} = \frac{4}{32\pi G} \int_{f=0}^{f=\infty} \mathrm{d}(\log f)\, f (2\pi f)^2 S_h(f)\,, \tag{4.349}$$

whicm means that

$$\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f} = \frac{\pi}{4G} f^2 h_c^2(f) \tag{4.350a}$$

$$= \frac{\pi}{2G} f^3 S_h(f)\,, \tag{4.350b}$$

so in the end we have

$$\Omega_{GW}(f) = \frac{2\pi^2}{3H_0^2} f^2 h_c^2(f)\,, \tag{4.351}$$

therefore

$$\Omega_{GW}(f) = \frac{4\pi^2}{3H_0^2} f^3 S_h(f)\,. \tag{4.352}$$

The *strain* is the quantity S_h.

An advantage of GWs is the fact that they decouple right after emission: they maintain the spectral shape.

If we start from the Einstein-Hilbert action

$$S = \int \mathrm{d}^4x\, Fg \frac{M_P^2}{2} R\,, \tag{4.353}$$

and plug in a tensor-perturbed FRLW metric:

$$\mathrm{d}s^2 = -a^2 \Big(\mathrm{d}\eta^2 + (\delta_{ij} + h_{ij})\, \mathrm{d}\vec{x}^2 \Big)\,, \tag{4.354}$$

we get the equations of motion for gravitational waves. The scenario of slow-roll inflation gives rise to an energy spectrum which is unobservable with LISA as well as ground-based detectors.

If we add an axion to the inflaton Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) + \underbrace{\frac{\phi}{\Lambda} F_{\mu\nu}\widetilde{F}^{\mu\nu}}_{\text{Additional term}}\,, \tag{4.355}$$

we can see enhanced gravitational waves, which we hope would be detectable with LISA! However the enhancement is very small, we cannot see it.

Chaotic preheating models predict higher amplitudes, however they are at very high frequencies.

We can get observable signals by fine-tuning the parameters, unlikely scenario.

What about phase transition, the collisions of primordial vacuum bubbles? We can get numerical estimates on the spectral shape of this signal, and they seem promising and detectable.

What about cosmic defects? We can have Domain Walls, Cosmic Strings and Cosmic Monopoles.

The prediction here is a flat spectrum at observable frequencies, at possibly observable amplitudes depending on the emission time.

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