General Relativity exercises

Jacopo Tissino, Giorgio Mentasti, Alessandro Lovo

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We set $c = 1$.

Here we will often use (anti)symmetrization of indices, which makes some calculations much easier. The idea of symmetrization is to sum over all permutation of the selected indices, with a minus sign for the odd permutation if the case of anti symmetrization. So, for instance, $F_{\mu\nu}$ can be antisymmetrized into $F_{[\mu\nu]} = {}^1\!/\!{}_2\Big(F_{\mu\nu} - F_{\nu\mu}\Big)$ and symmetrized into $F_{(\mu\nu)} = {}^1\!/\!{}_2\Big(F_{\mu\nu} + F_{\nu\mu}\Big)$.

The factor ${}^1\!/\!{}_2$ is in general $1/n!$, where n is the number of antisymmetrized indices. This is included because in general we will be summing $n!$ terms, and we want to write things like: "$F_{\mu\nu}$ is antisymmetric means $F_{\mu\nu} = F_{[\mu\nu]}$", so we need to rescale the sum to make it into an average.

The general formulas are then:

$$F_{[\mu_1 \dots \mu_n]} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) F_{\sigma(\mu_1) \dots \sigma(\mu_n)} \tag{0.1a}$$

$$F_{(\mu_1 \dots \mu_n)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} F_{\sigma(\mu_1) \dots \sigma(\mu_n)} \,, \tag{0.1b}$$

where $\mathfrak{S}_n$ is the *symmetric group* of permutations of n elements, and the sign of a permutation $\sigma \in \mathfrak{S}_n$ is ± 1, depending on the parity of pair swaps that are needed to get that configuration (we fix $(\operatorname{sign}(\mathbb{1}) = 1)$).

If we want to symmetrize indices which are not next to each other, we will denote the end of the (anti)symmetrized indices by a vertical bar.

Sheet 1

1.1 Lorentz transformations

1.1.1 Inverses

We can consider a Lorentz boost with velocity v in the x direction, and we look at its representation in the (t, x) plane (since the y and z directions are unchanged). Its matrix expression looks like:

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix}, \tag{1.1}$$

where $\gamma = 1/\sqrt{1-v^2}$. The inverse of this matrix can be computed using the general formula for a 2x2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.2}$$

The determinant of Λ is equal to $\gamma^2(1-v^2) = 1$, therefore the inverse matrix is:

$$\Lambda = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix}. \tag{1.3}$$

1.1.2 Invariance of the spacetime interval

Our Lorentz transformation is

$$\begin{aligned} \mathrm{d}t' &= \gamma(\mathrm{d}t - v\,\mathrm{d}x) && (1.4a) \\ \mathrm{d}x' &= \gamma(-v\,\mathrm{d}t + \mathrm{d}x) && (1.4b) \\ \mathrm{d}y' &= \mathrm{d}y && (1.4c) \\ \mathrm{d}z' &= \mathrm{d}z && (1.4d) \end{aligned}$$

and we wish to prove that the spacetime interval, defined by $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is preserved: $\mathrm{d}s'^2 = \mathrm{d}s^2$. Let us write the claimed equality explicitly:

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{1.5a}$$

1.1.3 Tensor notation pseudo-orthogonality

The invariance of the spacetime interval $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$ can be also written as $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x'^{\,\mu}\,\mathrm{d}x'^{\,\nu}$. By making the primed differentials explicit we have:

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho\,\mathrm{d}x^\rho\,\Lambda^\nu{}_\sigma\,\mathrm{d}x^\sigma \;, \tag{1.6}$$

but the dummy indices on the LHS can be changed to ρ and σ, so that both sides are proportional to $\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$. Doing this we get:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = (\Lambda^\top)_\rho{}^\mu\eta_{\mu\nu}\Lambda^\nu{}_\sigma\,, \tag{1.7}$$

or, in matrix form, $\eta = \Lambda^\top\eta\Lambda$.

1.1.4 Explicit pseudo-orthogonality

For simplicity but WLOG we consider a boost in the x direction with velocity v and Lorentz factor γ. The matrix expression to verify is:

$$\begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \overset{?}{=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.8a}$$

$$\begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} -\gamma & v\gamma \\ -v\gamma & \gamma \end{bmatrix} \overset{?}{=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.8b}$$

$$\begin{bmatrix} -\gamma^2 + \gamma^2 v^2 & v\gamma^2 - v\gamma^2 \\ v\gamma^2 - v\gamma^2 & -v\gamma^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.8c}$$

which by $\gamma^2 = 1/(1-v^2)$ confirms the validity of the expression.

1.2 Muons

1.2.1 Nonrelativistic approximation

The survival probability is given by $\mathbb{P}(t) = \exp\Big(-t/2.2\times10^{-6}\,\mathrm{s}\Big)$. If the ground is $h = 15\,\mathrm{km}$ away, then the muon will reach it in $t = h/v = 15\,\mathrm{km}/(0.995c) \approx 5.03\times10^{-5}\,\mathrm{s}$, therefore $\mathbb{P}(t) \approx 1.2\times10^{-10}$.

1.2.2 Relativistic effects: ground perspective

The observer on the ground will see the muon having to traverse the whole $h = 15\,\mathrm{km}$, but the muon's time will be dilated for them by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\Big(-t/(\gamma_v \times 2.2\times10^{-6}\,\mathrm{s})\Big) \approx 0.1$.

1.2.3 Relativistic effects: muons perspective

The muons in their system will observe lenght contraction, with respect to Lorentz boost, by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\Big(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\Big) \approx 0.1$. This result is the same of the one predicted by ground observer, with respect to relativity principle.

1.3 Radiation

1.3.1 New angle

In the source frame the radiation velocity components are $u'_x = \cos\theta', u'_y = \sin\theta'$. From the composition of velocities we obtain:

$$u_y = \sin\theta = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\sin\theta'}{\gamma_v(1 + v\cos\theta')} \tag{1.9a}$$

$$u_x = \cos\theta = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma_v(\mathrm{d}x' + v\,\mathrm{d}t')}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\cos\theta' + v}{1 + v\cos\theta'}\,, \tag{1.9b}$$

hence:

$$\frac{1}{\tan\theta} = \frac{\gamma_v}{\tan\theta'} + \frac{\gamma_v v}{\sin\theta'}\,. \tag{1.10}$$

1.3.2 Angle plot and relevant limits

See the jupyter notebook in the `python` folder for plots. For $v = 0$ we have $\theta = \theta'$ as we expected, while for $v = 1$, $\theta = 0$.

1.3.3 Radiation speed invariance

Are the components of the velocity, which we called $\sin\theta$ and $\cos\theta$, actually normalized? Let us check:

$$\sin^2\theta + \cos^2\theta = \frac{(\frac{\sin\theta'}{\gamma_v})^2 + (\cos\theta' + v)^2}{(1 + v\cos\theta')^2} \tag{1.11a}$$

$$= \frac{(1 - v^2)\sin^2\theta' + \cos^2\theta' + v^2 + 2v\cos\theta'}{(1 + v\cos\theta')^2} \tag{1.11b}$$

$$= \frac{1 + v^2(1 - \sin\theta') + 2v\cos\theta'}{(1 + v\cos\theta')^2} = 1\,, \tag{1.11c}$$

therefore the square modulus of the speed of the radiation is still c, as we could have assumed earlier.

1.3.4 Isotropic emission

Since the angular distribution of emission varies when changing inertial reference, we might suppose that every system in relative motion respect to O with $v \neq 0$ observes nonisotropic emission.

This can be seen by noticing that for $v \simeq 1$ we have that in the observer system there is almost only emission at an angle $\theta = 0$. In general, since there is a Lorentz γ factor multiplying a function of the angle in the radiation emission frame O', the cotangent of the angle in the observation frame O must get larger and larger as the relative velocity v increases, therefore the radiation gets compressed towards angles with large cotangents: $\theta \sim 0$.

See the jupyter notebook in the `python` folder for interactive plots :)

Sheet 2

2.1 Constant acceleration

2.1.1 Coordinate velocity

We are given the position as a function of time,

$$x(t) = \frac{\sqrt{1+\kappa^2 t^2} - 1}{\kappa}\,, \tag{2.1}$$

and we can directly compute its derivative

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\kappa t}{\sqrt{\kappa^2 t^2 + 1}} = \frac{1}{\sqrt{\frac{1}{\kappa^2 t^2} + 1}}\,. \tag{2.2}$$

It is clear from the expression that $|v| < 1$ for all times, while v approaches 1 at positive temporal infinity and -1 at negative temporal infinity.

2.1.2 Components of the 4-velocity

The Lorentz factor γ is given by

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1 - \frac{\kappa^2 t^2}{\kappa^2 t^2 + 1}}} = \sqrt{\kappa^2 t^2 + 1}\,, \tag{2.3}$$

therefore the four-velocity is given by:

$$u^\mu = \begin{bmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix}. \tag{2.4}$$

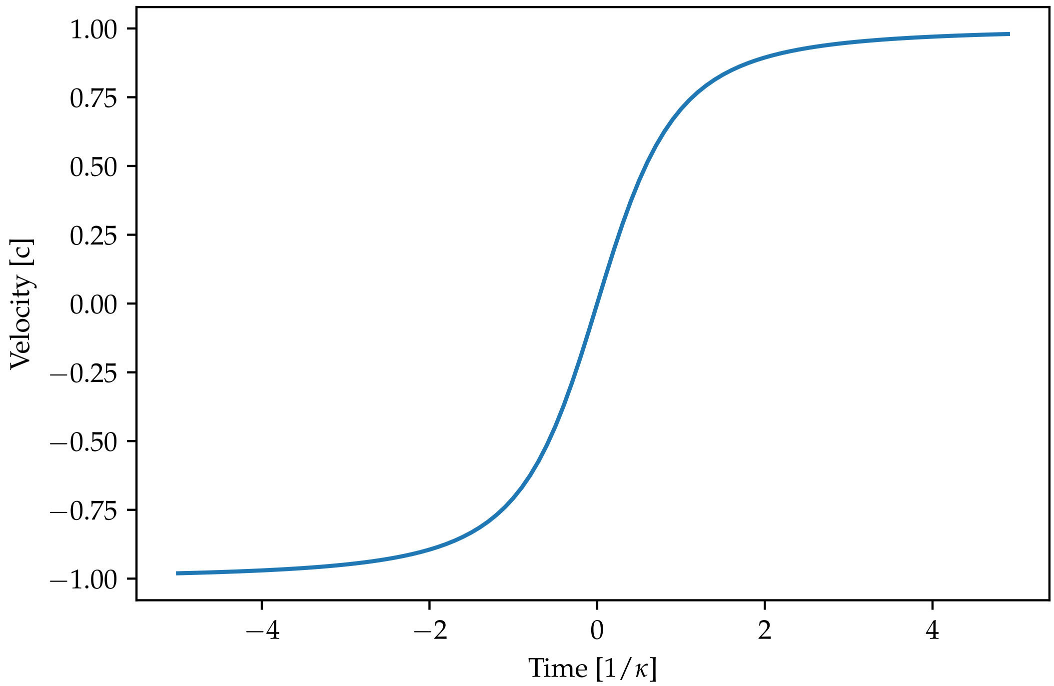

Figure 1: Velocity as a function of coordinate time t

2.1.3 Proper time

The relation between coordinate and proper time is given by the definition of the first component of the four-velocity: $u^0 = \mathrm{d}t/\mathrm{d}\tau = \gamma$, therefore $\mathrm{d}\tau = \mathrm{d}t\,/\gamma$. Integrating this relation we get:

$$\tau = \int_0^\tau \mathrm{d}\tau' = \int_0^t \frac{\mathrm{d}t'}{\gamma(t')} = \frac{\operatorname{arcsinh}(\kappa t)}{\kappa}\,, \tag{2.5}$$

where the constant of integration is selected by imposing $t = 0 \iff \tau = 0$. Notice that, as we would expect, when expanding up to first order near $t = \tau = 0$ we have $t \sim \tau$, since in that region the velocity is much less than unity.

The inverse relation is given by $t = \sinh(\kappa\tau)/\kappa$. Using this, we can write:

$$x(t(\tau)) = \frac{\cosh(\kappa\tau) - 1}{\kappa}\,. \tag{2.6}$$

2.1.4 Four-acceleration

Now, we wish to compute the four-acceleration. There are many ways to approach this: an easy one is to simply find the explicit expression $u^\mu(\tau)$ and to differentiate it. The expression we get is:

$$a^{\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau} u^{\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau} \begin{bmatrix} \sqrt{\sinh^2(\kappa\tau) + 1} \\ \frac{\sqrt{\kappa^2 t^2 + 1}\sinh(\kappa\tau)}{\sqrt{\sinh^2(\kappa\tau)+1}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\kappa \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa\cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix}, \tag{2.7}$$

which is a bit unwieldy but it can be used to check two important facts: $a^{\mu} a_{\mu} = \text{const}$ and $a^{\mu} u_{\mu} = 0$. The first of the two is:

$$a^{\mu} a_{\mu} = -(a_0)^2 + (a_1)^2 = \kappa^2 \cosh^2(\kappa\tau) - \frac{\kappa^2 \sinh^2(2\kappa\tau)}{2\left(\cosh(2\kappa\tau) + 1\right)} = \kappa^2, \tag{2.8}$$

which tells us that the constant acceleration $\sqrt{a^{\mu} a_{\mu}} = \kappa$.

Also, we verify the orthogonality to the four-velocity:

$$a^{\mu} u_{\mu} = -\frac{\sqrt{2}\kappa\sqrt{\sinh^2(\kappa\tau) + 1}\sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau) + 1}} + \kappa\sinh(\kappa\tau)\cosh(\kappa\tau) = 0. \tag{2.9}$$

2.1.5 Local velocity & acceleration

We can apply a Lorentz boost corresponding to this velocity: it will be given by the matrix:

$$\begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.10}$$

where v and γ are those found before. Without doing any calculations we could already say that the transformed velocity will be equal to the time-like unit vector, while the acceleration will be equal to κ times the unit x-directed vector.

The velocity becomes:

$$\left(u^{\mu}\right)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.11}$$

as we expected.

The acceleration instead becomes:

$$\left(a^{\mu}\right)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}\kappa \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa\cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \kappa \\ 0 \\ 0 \end{bmatrix}, \tag{2.12}$$

At small speeds the Lorentz boost matrix reduces to the identity matrix: this implies $kt \simeq k\tau \simeq 0$. In this case we obtain the same results of the rest frame of the particle for both acceleration and speed.

2.2 Fixed target collision

2.2.1 Center of mass momenta

In the CoM frame, the momenta of the two protons are respectively $(E_p, \pm p, 0, 0)^\top = m_p(\gamma, \pm v, 0, 0)$, where $E_p^2 = m_p^2 + p^2$. The total CoM energy is $-(p_A^\mu + p_B^\mu)^2 = 2m_p^2$.

2.2.2 Center of mass velocity

The momentum of particle B will be given by $p^\mu = m_p u^\mu = (m_p\gamma, m_p\gamma v, 0, 0)^\top$. Therefore, $\gamma v = p/m_p$. Solving this we get:

$$v = \frac{p}{m_p}\sqrt{\frac{1}{(p/m_p)^2 + 1}} = \frac{p}{E_p}\,, \tag{2.13}$$

2.2.3 Lab frame momenta

The momentum of particle B in its own rest frame will just be $(m_p, 0, 0, 0)^\top$. The momentum of particle A instead will be given by a boost in the x direction with velocity $-v$:

$$(p_A^\mu)_{\text{lab}} = \begin{bmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_p \\ p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma E_p + v\gamma p \\ v\gamma E_p + \gamma p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_p\gamma^2(1+v^2) \\ 2\gamma p \\ 0 \\ 0 \end{bmatrix}, \tag{2.14}$$

2.3 Weak field gravitational time dilation

2.3.1 Time dilation expression

It is more intuitive geometrically to deal with a pulse sent from A to B, for which we expect the time dilation to work in the opposite sense:

$$\Delta t_B = \Delta t_A\big(1 + gh\big)\,, \tag{2.15}$$

up to first order in gh and $g\Delta t_A$, since $(1+gh)(1-gh) = 1 - (gh)^2 = 1$ to first order in gh. Alternatively, one can just map $g \to -g$ to recover the time contraction for pulses sent in the other direction.

We know that the paths of the observers are two curves of constant acceleration: we know their explicit expression from equation (2.1), and additionally we assume that they are separated by a space interval h:

$$x_A(t) = \frac{\sqrt{1+(gt)^2}-1}{g} \tag{2.16a}$$

$$x_B(t) = \frac{\sqrt{1+(gt)^2}-1}{g} + h\,. \tag{2.16b}$$

At $t = 0$ Alice sends a pulse, which then reaches Bob at a time t_1. After a time Δt_A, she sends another, which then reaches Bob at a time t_2. Right now, we are referring to all times as measured in the rest frame of Alice at $t = 0$. These times can be found by imposing that the space and time separation between the events of the pulse being sent and received are equal, since it travels at light speed: the equations which represent this are $x_B(t_1) = t_1$ and $x_B(t_2) - x_A(\Delta t_A) = t_2 - \Delta t_A$. Substituting the expressions for the positions:

$$t_1 = \frac{\sqrt{1+(gt_1)^2}-1}{g} + h \tag{2.17a}$$

$$t_2 - \Delta t_A = \frac{\sqrt{1+(gt_2)^2}-1}{g} + h - \left(\frac{\sqrt{1+(g\Delta t_A)^2}-1}{g}\right). \tag{2.17b}$$

Now, it is just a matter of calculation to solve these equations, expand up to first order in the adimensional parameters gh and $g\Delta t_A$ and one recovers the desidered expression for $\Delta t_B = t_2 - t_1$.

There is one more consideration to make though: what about the Lorentz time dilation for Bob? This it actually a *second order effect*.

Claim 2.1. *The time interval measured by Bob in his frame at $t \sim t_1$ is the same as the one measured in the rest frame of Alice at $t = 0$ up to first order in gh and $g\Delta t_B$.*

Proof. We perform a Lorentz boost to the velocity of Bob at $t = t_1$: this is given by equation (2.2), and is equal to:

$$v = \frac{gt}{\sqrt{(gt)^2+1}}, \tag{2.18}$$

with a Lorentz factor of $\gamma = \sqrt{(gt)^2+1}$ (see equation (2.3)).

The temporal separation between the two events is Δt_B, while the spatial separation is $\Delta x_B \approx v\Delta t_B$ to first order. The boost, in the (t, x) plane, looks like:

$$\begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix}' = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix} = \begin{bmatrix} \Delta t_B\left(\sqrt{(gt)^2+1} - (gt)^2/\sqrt{(gt)^2+1}\right) \\ -gt\Delta t_B + \sqrt{(gt)^2+1}gt\Delta t/\sqrt{(gt)^2+1} \end{bmatrix}, \tag{2.19}$$

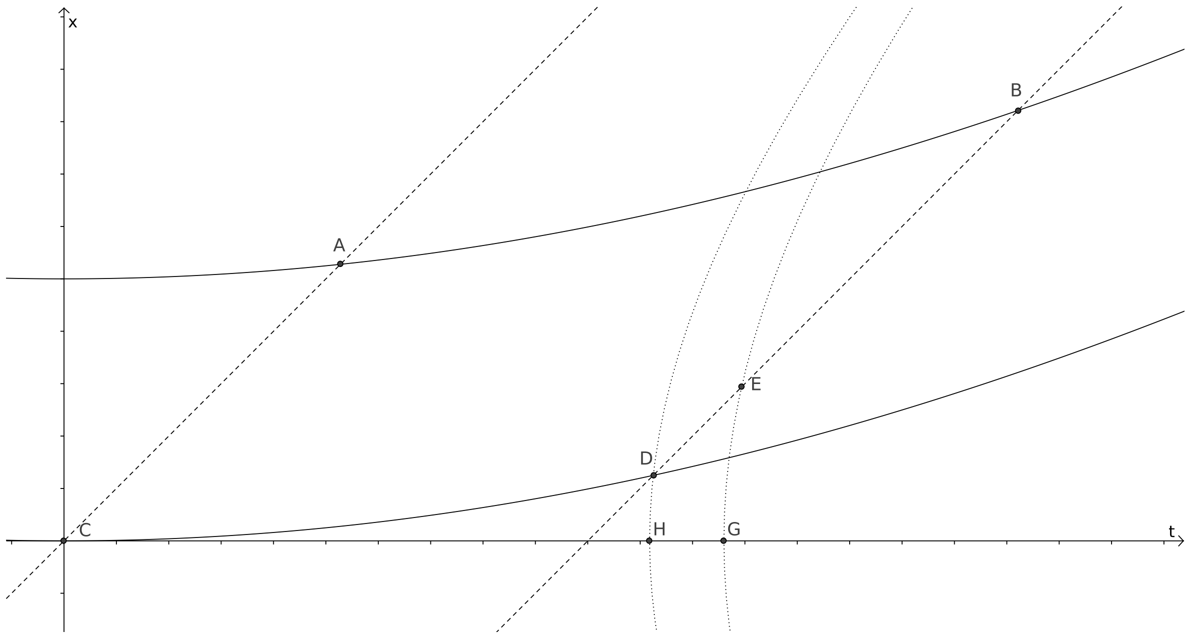

Figure 2: Visualization of the beams, in the frame where the rocket is stationary as the first beam is being sent. The two curves intersecting the space axis are the tip and tail of the spaceship; the beams being sent from the tail are events C and D, while their reception at the tip are events A and B. Event E is just calculated as $B - A$, to make comparisons with D easier. H and G are computed by tracing the dotted line of points which have the same spacetime interval from the origin as points D and E respectively, and selecting its intersection with the temporal axis: this effectively means finding the proper time separation between the two beams being sent/received.

therefore as we would expect the spatial separation is eliminated, while expanding the factor multiplying the temporal one near $gt = 0$ we get:

$$\sqrt{(gt)^2 + 1} - (gt)^2 / \sqrt{(gt)^2 + 1} = 1 + O((gt)^2) \,, \tag{2.20}$$

which proves our result. $\square$

2.3.2 Gravitational time dilation

By the equivalence principle, the effects measured in a uniformly accelerating frame at g are the same as those measured in a gravitational field with constant acceleration g. The gravitational field in such a frame is given by $\Phi = gh$, where h is the height (with arbitrary zero point): the result follows.

2.3.3 Twins and gravitation

The gravitational time dilation difference, in absolute value, is given by:

$$\Delta t = t_{\text{elapsed}} \frac{g \Delta h}{c^2} \approx 1\,\text{yr} \frac{10\,\text{m/s}^2 \times 100\,\text{m}}{(3 \times 10^8\,\text{m/s})^2} \approx 3.5 \times 10^{-7}\,\text{s}\,. \tag{2.21}$$

We are asked what is the age of the twin on the ground as measured by the twin who is higher up: this is analogous to the situation considered in the first section of this problem; the twin higher up will measure the twin lower down to be older, specifically if $\text{age}_{\text{up}} = 1\,\text{yr}$, then the observer up in the palace will measure the age of the twin at ground level as:

$$\text{age}_{\text{down}} = 1\,\text{yr} + 3.5 \times 10^{-7}\,\text{s} \approx (1 + 1 \times 10^{-14})\text{age}_{\text{up}}\,. \tag{2.22}$$

Sheet 3

3.1 Changes of coordinate system

We denote by $x^\mu = (x, y)$ and $x'^\mu = (r, \theta)$. Then, we have the following Jacobian matrices:

$$\frac{\partial x^\mu}{\partial x'^\nu} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \tag{3.1a}$$

$$\frac{\partial x'^\nu}{\partial x^\rho} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\dfrac{\sin(\theta)}{r} & \dfrac{\cos(\theta)}{r} \end{bmatrix}, \tag{3.1b}$$

which can be found by plain differentiation of the change of coordinates, recalling $(\arctan x)' = 1/(1+x^2)$. Then, we can compute the product of these two matrices: it comes out to be

$$\frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\rho} = \delta^\mu_\nu\,, \tag{3.2}$$

since on the diagonal we get $\cos^2(\theta) + \sin^2(\theta) = 1$, while on the off-diagonal terms we get a multiple of $\sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) = 0$.

Note that relation (3.2) is just the chain rule written in more generality: substituting the explicit coordinates for x^μ and x'^μ we get the desired expression.

3.2 Properties of covariant differentiation

3.2.1 Metric compatibility of the connection

We wish to show that $\nabla_\alpha g_{\mu\nu} = 0$. A very simple way to prove this is by going in the LIF: there, the equation reads $\partial_\alpha \eta_{\mu\nu} = 0$, which is immediately satisfied since the components of the Minkowski metric are constants. Then, since the equation is tensorial, the result extends to any frame.

This was not the spirit of the exercise, however: let us prove it in a different generic frame. To this end, we define the *Christoffel symbols of the first kind* (while the regular ones are of the second kind):

$$\Gamma_{\mu\nu\rho} = g_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} = \frac{1}{2}\Big(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu}\Big) . \tag{3.3}$$

These are useful since, as all their indices are down, it is easier to study their symmetry properties.

Note that if we antisymmetrize the first and last index, we get $\Gamma_{[\mu|\nu|\rho]} = {}^{1}/{}_{2} g_{\mu\rho,\nu}$ since the first and last terms in the sum cancel (in the latter we must invert the indices ν and ρ in order to see this, but this can always be done by the symmetry of the metric).

Then, we write the expression for the covariant derivative of the metric:

$$\nabla_{\alpha} g_{\mu\nu} = \partial_{\alpha} g_{\mu\nu} - \Gamma^{\rho}_{\mu\alpha} g_{\rho\nu} - \Gamma^{\rho}_{\nu\alpha} g_{\mu\rho} = g_{\mu\nu,\alpha} - \Gamma_{\nu\alpha\mu} - \Gamma_{\mu\alpha\nu} , \tag{3.4}$$

which is just $g_{\mu\nu,\alpha} - 2\Gamma_{[\nu|\alpha|\mu]} = g_{\mu\nu,\alpha} - g_{\nu\mu,\alpha} = 0$, again by the symmetry of the metric.

One could argue that this is the opposite way round: we should *assume* $\nabla_{\mu} g_{\rho\sigma} = 0$ and derive from it the formula that was given for the Christoffel symbols in terms of the partial derivatives of the metric.

3.2.2 Leibniz rule

As before, this can be proved in the LIF from the Leibniz rule of regular partial derivatives.

As before, we like to calculate therefore we show this explicitly in any frame.

The derivative of the tensor product looks like:

$$\nabla_{\mu}\Big(A_{\nu\lambda}B_{\rho}\Big) = \partial_{\mu}\Big(A_{\nu\lambda}B_{\rho}\Big) - \Gamma^{\sigma}_{\mu\nu}A_{\sigma\lambda}B_{\rho} - \Gamma^{\sigma}_{\mu\lambda}A_{\nu\sigma}B_{\rho} - \Gamma^{\sigma}_{\mu\rho}A_{\nu\lambda}B_{\sigma} , \tag{3.5}$$

while the sum of derivatives looks like:

$$\begin{aligned} B_{\rho}\nabla_{\mu}A_{\nu\lambda} + A_{\nu\lambda}\nabla_{\mu}B_{\rho} = & B_{\rho}\partial_{\mu}A_{\nu\lambda} - \Gamma^{\sigma}_{\mu\nu}A_{\sigma\lambda}B_{\rho} - \Gamma^{\sigma}_{\mu\lambda}A_{\nu\sigma}B_{\rho} \\ & + A_{\nu\lambda}\partial_{\mu}B_{\rho} - \Gamma^{\sigma}_{\mu\rho}A_{\nu\lambda}B_{\sigma} \end{aligned} , \tag{3.6}$$

so we can see that the Christoffel terms are equal, and the partial derivative terms also are since we have the Leibniz rule for partial derivatives.

3.3 2D Christoffel symbols

3.3.1 Polar coordinates

The metric and inverse metric are respectively given by $g_{\mu\nu} = \mathrm{diag}\left(1, r^2\right)$ and $g^{\mu\nu} = \mathrm{diag}\left(1, r^{-2}\right)$. We only care about the partial derivatives of the lower-indices one, and the only nonvanishing derivative is $g_{11,0} = 2r$, where we mean $(x^0, x^1) = (r, \theta)$.

Then it is tedious but straightforward to perform the direct computation. Things that make it faster are discarding immediately terms which cannot contribute (such as $g_{\alpha\beta,\gamma}$ where at least one of α and β is not 1 or γ is not 0, and only looking at the six independent symbols instead of the eight total ones (since $\Gamma^\alpha_{01} = \Gamma^\alpha_{10}$ for any α).

Then one can see that the nonvanishing symbols are

$$\Gamma^0_{11} = \frac{1}{2} g^{0\alpha} \left(g_{\alpha 1,1} + g_{\alpha 1,1} - g_{11,\alpha}\right) = \frac{1}{2} g^{00} (-g_{11,0}) = -\frac{2r}{2} = -r \tag{3.7a}$$

$$\Gamma^1_{01} = \frac{1}{2} g^{1\alpha} \left(g_{\alpha 0,1} + g_{\alpha 1,0} - g_{01,\alpha}\right) = \frac{1}{2} g^{11} g_{11,0} = \frac{1}{2}\frac{1}{r^2} 2r = \frac{1}{r}\,. \tag{3.7b}$$

3.3.2 Spherical surface

Now we have the following metric and inverse metric:

$$g_{\mu\nu} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2(\theta) \end{bmatrix}, \qquad g^{\mu\nu} = \begin{bmatrix} R^{-2} & 0 \\ 0 & R^{-2} \sin^{-2}(\theta) \end{bmatrix}, \tag{3.8}$$

but do note that R is a constant: given $(x^0, x^1) = (\theta, \varphi)$, we have as before that the only nontrivial derivative is $g_{11,0} = 2R^2 \sin(\theta)\cos(\theta)$.

Then this case is exactly analogous to the previous one: the same symbols are zero, so we can skip almost all of the computation and jump straight to:

$$\Gamma^0_{11} = -\frac{1}{2} g^{00} g_{11,0} = -\sin(\theta)\cos(\theta) \tag{3.9a}$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11} g_{11,0} = \frac{\cos(\theta)}{\sin(\theta)}\,. \tag{3.9b}$$

3.4 Parallel transport

We know from the last exercise the metric and Christoffel symbols of 2D space and of the surface of a sphere.

The equations of parallel transport are in general $u^\mu \nabla_\mu V^\nu = 0$.

3.4.1 Flat space

We want to determine the behaviour of a vector field $V^\mu(\theta)$ defined on a curve $x^\mu(\theta) = (R, \theta)$ with fixed R, such that $V^\mu(\theta = 0) = (0, 1/R)$ (a unit vector: $V^\mu(0)V^\nu(0)g_{\mu\nu}(0) = 1$).

In our case the tangent vector of the curve is $u^\mu = (0, 1)$. Therefore the equations simplify to:

$$\nabla_1 V^\mu = \partial_1 V^\mu + \Gamma^\mu_{1\alpha} V^\alpha = 0\,, \tag{3.10}$$

which are two coupled differential equations; we can make them explicit and substitute the Christoffel symbols found earlier.

$$\nabla_1 V^0 = \partial_1 V^0 + \cancel{\Gamma^0_{10} V^0} + \Gamma^0_{11} V^1 = \partial_1 V^0 - r V^1 \tag{3.11a}$$

$$\nabla_1 V^1 = \partial_1 V^1 + \Gamma^1_{10} V^0 + \cancel{\Gamma^1_{11} V^1} = \partial_1 V^1 + \frac{1}{r} V^0\,, \tag{3.11b}$$

so we can write this linear system like:

$$\partial_1 \begin{bmatrix} V^0 \\ V^1 \end{bmatrix} = \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix} \begin{bmatrix} V^0 \\ V^1 \end{bmatrix}. \tag{3.12}$$

The eigenvalues of this matrix are $\pm i$, so its exponential is a pure rotation matrix:

$$\exp\left(\theta \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R(-\theta)\,, \tag{3.13}$$

so the solution is

$$V^\mu(\theta) = R(-\theta) V^\mu(0)\,. \tag{3.14}$$

This means that our vector is rotating clockwise with unit angular velocity in the (r, θ) plane, just as the point it is defined at moves counterclockwise with unit angular velocity: therefore, if we look at the vector in Cartesian coordinates, we will see it always aligned with its initial direction and the same modulus.

Specifically, when $\theta = \pi/2$ we get $V^\mu = (1, 0)$: this has the same modulus as $(0, 1/R)$ since we compute lengths with respect to the metric of our space, as $\|V\|^2 = V^\mu V^\nu g_{\mu\nu}$.

3.4.2 Curved space: spherical surface

Now our curve is $x^\mu(\theta) = (\theta, 0)$ in the spherical coordinates $x^\mu = (\theta, \varphi)$. The metric is the one given in (3.8), the Christoffel symbols are the ones given in (3.9).

The parallel transport equations are now:

$$\nabla_1 V^0 = \partial_0 V^0 + \cancel{\Gamma^0_{00} V^0} + \cancel{\Gamma^0_{01} V^1} \tag{3.15a}$$

$$\nabla_1 V^1 = \partial_0 V^1 + \cancel{\Gamma^1_{00} V^0} + \Gamma^1_{01} V^1 = \partial_0 V^1 + \frac{V^1}{\tan(\theta)} \,. \tag{3.15b}$$

The first equation gives us $V^1 = \text{const}$, while we do not need to actually solve the second one: our initial condition is $V^\mu(\theta = 0) = (R^{-1}, 0)$, and $V^1 \equiv 0$ is a solution to that first-order equation, so by the uniqueness we have found the whole solution.

Therefore the vector is constant in these coordinates.

Specifically, at $\theta = \pi/2$ we get $V^\mu = (1/R, 0)$.

Sheet 4

4.1 Riemann tensor computations

4.1.1 LIF form

In the LIF, we have $g_{\mu\nu} = \eta_{\mu\nu}$ and $g_{\mu\nu,\alpha} = 0$. Therefore, the Christoffel symbols are all zero. In the explicit expression of the Riemann tensor, which looks like $R = \partial\Gamma + \Gamma\Gamma$ we can drop the second term and keep only the derivatives of the Christoffel symbols, which are nonzero since they depend on the second derivatives of the metric.

If the Christoffel symbols are zero, the computation of the curvature tensor is significantly easier, since we do not have the $\Gamma\Gamma$: we just need to account for the terms $\partial\Gamma$. The computation gives:

$$R^\mu_{\nu\rho\sigma} = 2\partial_{[\rho}\Gamma^\mu_{\sigma]\nu} \tag{4.1a}$$

$$= 2\partial_{[\rho}\left(\frac{1}{2} g^{\mu\alpha}\left(g_{\alpha|\sigma],\nu} + g_{\alpha\nu,|\sigma]} - g_{\nu|\sigma],\alpha}\right)\right) \tag{4.1b}$$

$$= g^{\mu\alpha}\left(g_{\alpha[\sigma|,\nu|\rho]} + \cancel{g_{\alpha\nu,[\sigma\rho]}} - g_{\nu[\sigma|,\alpha|\rho]}\right) , \tag{4.1c}$$

where a term was cancelled since it contained the antisymmetrization of partial derivatives, which commute.

So, if we lower the index of the (1, 3) Riemann tensor we get the desired expression for the all-lower (0, 4) Riemann tensor:

$$R_{\mu\nu\rho\sigma} = g_{\mu[\sigma|,\nu|\rho]} - g_{\nu[\sigma|,\mu|\rho]} \,. \tag{4.2}$$

4.1.2 Ricci tensor and scalar

The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = g^{\alpha\beta} \left(g_{\alpha[\nu|,\mu|\beta]} - g_{\mu[\nu|,\alpha|\beta]} \right) . \tag{4.3}$$

So we have

$$R_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\alpha\nu,\mu\beta} - g_{\alpha\beta,\mu\nu} - g_{\mu\nu,\alpha\beta} + g_{\mu\beta,\alpha\nu} \right) \tag{4.4a}$$

$$= \frac{1}{2} g^{\alpha\beta} \left(2 g_{\alpha(\nu,\mu)\beta} - g_{\mu\nu,\alpha\beta} - g_{\alpha\beta,\mu\nu} \right) \tag{4.4b}$$

$$= g_{\alpha(\nu,\mu)}{}^{\alpha} - \frac{1}{2} \Box g_{\mu\nu} - \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\mu\nu} , \tag{4.4c}$$

where[1] the square $\Box$ denotes the D'alambert operator, $\Box = \partial_\beta \partial^\beta = g^{\alpha\beta} \partial_\alpha \partial_\beta$.

The Ricci scalar, on the other hand, is given by

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g_{\alpha(\nu,\mu)}{}^{\alpha} - g^{\mu\nu} \Box g_{\mu\nu} - g^{\alpha\beta} \Box g_{\alpha\beta} = \left(g_{\alpha\nu}{}^{,\alpha\nu} - g^{\mu\nu} \Box g_{\mu\nu} \right) , \tag{4.5}$$

where we identified the two terms containing the metric contracted with its dalambertian, since they are equal up to a relabeling of indices.

We neglected the symmetrization in the first term since when contracting with the metric the indices $\mu\nu$ are automatically symmetrized.

4.1.3 LIF identities

$$\nabla_\alpha \left(R^{\alpha}_{\beta} - \frac{R}{2} \delta^{\alpha}_{\beta} \right) = \partial_\alpha \left(g^{\alpha\lambda} R_{\lambda\beta} - \frac{R}{2} \delta^{\alpha}_{\beta} \right) \tag{4.6a}$$

$$= \eta^{\alpha\lambda} R_{\lambda\beta,\alpha} - \frac{1}{2} R_{,\beta} , \tag{4.6b}$$

since in the LIF the metric is the Minkowski one, and the Christoffel symbols are zero therefore $\nabla_\alpha = \partial_\alpha$.

4.1.4 Contracted Bianchi identities

The first term is

$$\eta^{\beta\mu} R_{\mu\nu,\beta} = \eta^{\beta\mu} \left(2 g_{\alpha(\nu,\mu)}{}^{,\alpha} - \Box g_{\mu\nu} - g^{\alpha\beta} g_{\alpha\beta,\mu\nu} \right)_{,\beta} , \tag{4.7}$$

[1] All the indices after the comma are derivatives, even when they become upper.

which we can expand out: the Minkowski metric raises the derivative with respect to x^β and turns it into a derivative with respect to x_μ, and we can also expand the Dalambertian and the symmetrization: we get

$$\eta^{\beta\mu} R_{\mu\nu,\beta} = g_{\alpha\nu,\mu}{}^{\alpha\mu} + g_{\alpha\mu,\nu}{}^{\alpha\mu} - g_{\mu\nu,\alpha}{}^{\alpha\mu} - g^{\alpha\beta} g_{\alpha\beta,\mu\nu}{}^{\mu} \tag{4.8a}$$

$$= +g_{\alpha\mu,\nu}{}^{\alpha\mu} - \left(g^{\alpha\beta} \Box g_{\alpha\beta}\right)_{,\nu} \tag{4.8b}$$

$$= \partial_\nu \left(g_{\mu\alpha}{}^{,\mu\alpha} - g^{\alpha\beta} \Box g_{\alpha\beta}\right), \tag{4.8c}$$

since, up to a relabeling of indices, the first and third term in (4.8a) are equal.

The derivative $\partial_\nu R$, on the other hand, is twice that:

$$\partial_\nu R = \partial_\nu \left(2 g_{\mu\alpha}{}^{,\mu\alpha} - 2 g^{\alpha\beta} \Box g_{\alpha\beta}\right), \tag{4.9}$$

as can be directly gathered from (4.5).

Therefore, the quantity in (4.6b) is zero.

4.1.5 Alternative expression

Since the metric is covariantly constant, we can just bring it inside the derivative:

$$0 = g^{\beta\mu} \nabla_\alpha \left(R^\alpha{}_\mu - \frac{1}{2} R \delta^\alpha_\mu\right) = \nabla_\alpha \left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}\right). \tag{4.10}$$

These are called the *Contracted Bianchi identities*, and can be alternatively derived from the Bianchi identities of the Riemann tensor, $R_{\mu\nu[\rho\sigma;\alpha]} = 0$.

4.2 Properties of the Ricci tensor

4.2.1 Symmetry generality

The fact that $T^{\mu\nu} = T^{\nu\mu}$ is frame invariant could be derived plainly from the fact that it is a tensor equation; to be more explicit we can say that under a change of coordinates with Jacobian matrix Λ we have $\widetilde{T}^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$; when written in components like this the Jacobians commute, therefore we can just interchange them for both $\widetilde{T}^{\mu\nu}$ and $\widetilde{T}^{\nu\mu}$ to recover their equality.

4.2.2 Symmetry of the Ricci tensor.

The expression (4.4c) is manifestly symmetric: part of it is explicitly symmetrized, part is proportional to the metric, which is symmetric.

Therefore, the Ricci tensor is symmetric in any frame.

4.3 Weak field Einstein equations

We consider the *weak field* case, when $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\left|h_{\mu\nu}\right| \ll 1$. We will then neglect second and higher order terms in $h_{\mu\nu}$.

4.3.1 The gravitational potential

We want to work towards Newton's equation, which is $\nabla^2\Phi = 4\pi G_N \rho$, where Φ is the gravitational potential. What defines the gravitational potential is[2] its effect on test masses: they accelerate with $\vec{a} = -\vec{\nabla}\Phi$.

What is the relativistic equivalent of this? A particle in GR follows a geodesic, a curve without proper acceleration. Proper acceleration is a four-vector defined by $a^\mu = u^\nu \nabla_\nu u^\mu$. If the four-velocity of a particle is u^μ, then for it the equation of geodesic motion reads:

$$0 = a^\mu = u^\nu \partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha} u^\nu u^\alpha\,, \tag{4.11}$$

which can be written in terms of derivatives with respect to proper time:

$$0 = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\alpha} u^\nu u^\alpha\,. \tag{4.12}$$

If the speeds are much less than that of light, this can be approximated: $s \approx t$, and the only terms which contributes to order 0 in v in the Christoffel symbol sum is the one with $u^0 u^0 \approx 1$: in the end then we get for the spatial components:

$$0 = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{00}\,, \tag{4.13}$$

so we can see that the main contribution in the low-speed limit to the (coordinate!) acceleration of the particle are the symbols Γ^i_{00}. The expression for these in terms of the perturbed metric is:

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\alpha}\left(2h_{\alpha 0,0} - h_{00,\alpha}\right) = -\frac{1}{2}{h_{00}}^{,i} = -\frac{1}{2}\vec{\nabla}^i h_{00}\,, \tag{4.14}$$

if we assume stationarity of the metric (which is justified if we are treating a problem such as the gravitational pull on a body on the surface of the Earth) at least up to first order in h.

Putting everything together: we found that

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\vec{\nabla}^i h_{00}\,, \tag{4.15}$$

[2] In the lecture notes this is solved differently: instead of finding the gravitational potential from the equation of geodesic motion, there it is found by identification of the gravitational redshift.

but the equation which defines the gravitational field is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\vec{\nabla}^i \Phi \,, \tag{4.16}$$

therefore, in the low-gravitational field and low-speed limit, we must identify $h_{00} = -2\Phi$.

4.3.2 Reframing the EFE

We can contract the EFE with the inverse metric, recalling that $g^{\mu\nu} g_{\mu\nu} = 4$, to get $-R = 8\pi G_N T$. Therefore, they can be reframed by substituting the curvature scalar term with the trace of the stress-energy tensor:

$$R_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) . \tag{4.17}$$

This will aid us by reducing the number of difficult curvature tensors to compute.

In the low-speed, weak-field limit matter has negligible pressure and is mostly just travelling in the time direction. Therefore, we approximate the stress energy tensor as that of noninteracting dust: $T^{\mu\nu} = \rho u^\mu u^\nu$. In our frame matter is almost stationary, therefore we have that $T_{00} \approx T^{00} \eta_{00} \eta_{00} \approx \rho$. The trace of the curvature tensor is instead approximately $T \approx T^{00} \eta_{00} \approx -\rho$.

In the end, the 00 component of the equations reads:

$$R_{00} = 8\pi G_N \left(\rho - \frac{1}{2} (-\rho) \eta_{00} \right) = 4\pi G_N \rho \,. \tag{4.18}$$

Now we only need to show that $R_{00} = \nabla^2 \Phi$.

4.3.3 The derivatives in the curvature tensor

The component we need to calculate is $R_{00} = R^\mu_{0\mu 0} = R^i_{0i0}$ by antisymmmetry.

Let us recall here the general expression for the Riemann tensor:

$$R^\mu_{\nu\rho\sigma} = 2\Gamma^\mu_{[\nu|\sigma,|\rho]} + 2\Gamma^\alpha_{\sigma[\nu}\Gamma^\mu_{\rho]\alpha} \,. \tag{4.19}$$

In our case, this simplifies to

$$R^i_{0i0} = \Gamma^i_{00,i} - \Gamma^i_{i0,0} + \Gamma^\alpha_{00}\Gamma^i_{i\alpha} - \Gamma^\alpha_{0i}\Gamma^i_{0\alpha} \,. \tag{4.20}$$

The second term contains time derivatives, which we decided to neglect by dealing with the stationary case. The third and fourth term are of second order in h.

The only remaining term is the first one: using the expression (4.14) and the identification $h_{00} = -2\Phi$, it is

$$R_{00} = \Gamma^i_{00,i} = \left(-\frac{1}{2} \partial^i h_{00} \right)_{,i} = \partial_i \partial^i \Phi \,, \tag{4.21}$$

which is what we wanted to prove.

Sheet 5

5.1 Alternative derivation of the contracted Bianchi identities

The form of the Riemann tensor in a LIF was derived in section 4.1.1.

5.1.1 Bianchi identities of the Riemann tensor

In the LIF, given that $R_{\mu\nu\rho\sigma} = g_{\mu[\sigma|,\nu|\rho]} - g_{\nu[\sigma|,\mu|\rho]}$, we want to show that $R_{\mu\nu[\rho\sigma;\alpha]} = R_{\mu\nu[\rho\sigma,\alpha]} = 0$.

This is equivalent to the formulation of the Bianchi identities given in the problem sheet, because of the antisymmetry of the Riemann tensor in its last two indices: there are six terms in the antisymmetrization of $R_{\mu\nu[\rho\sigma,\alpha]}$, but they are pairwise equal: the term $R_{\mu\nu\rho\sigma,\alpha}$ is equal to $-R_{\mu\nu\sigma\rho,\alpha}$ by antisymmetry, and these are exactly the pairs of terms which appear in the three-index antisymmetrization.

What we need to do is to take the derivative of the Riemann tensor in the LIF:

$$R_{\mu\nu\rho\sigma,\alpha} = g_{\mu[\sigma|,\nu|\rho]\alpha} - g_{\nu[\sigma|,\mu|\rho]\alpha}\,, \tag{5.1}$$

and permute the three indices $\rho\sigma\alpha$ cyclically. Writing all the terms out we get (up to a factor 2, which is irrelevant since we will find that all the terms cancel and everything is equal to 0):

$$\begin{aligned}
&+g_{\mu\sigma,\nu\rho\alpha} - g_{\nu\sigma,\mu\rho\alpha} - g_{\mu\rho,\nu\sigma\alpha} + g_{\nu\rho,\mu\sigma\alpha} + \\
&+g_{\mu\alpha,\nu\sigma\rho} - g_{\nu\alpha,\mu\sigma\rho} - g_{\mu\sigma,\nu\alpha\rho} + g_{\nu\sigma,\mu\alpha\rho} + \\
&+g_{\mu\rho,\nu\alpha\sigma} - g_{\nu\rho,\mu\alpha\sigma} - g_{\mu\alpha,\nu\rho\sigma} + g_{\nu\alpha,\mu\rho\sigma}\,,
\end{aligned} \tag{5.2}$$

so we have 6 terms with a + sign, and 6 with a $-$ sign: they cancel pairwise, since the partial derivatives commute.

5.1.2 Contracting the identities

We start by contracting $2R_{\mu\nu[\rho\sigma;\alpha]}$ with $g^{\mu\rho}$: we get

$$0 = g^{\mu\rho}\Big(R_{\mu\nu\rho\sigma;\alpha} + R_{\mu\nu\alpha\rho;\sigma} + R_{\mu\nu\sigma\alpha;\rho}\Big) = R_{\nu\sigma;\alpha} - R_{\nu\alpha;\sigma} + g^{\mu\rho}R_{\mu\nu\sigma\alpha;\rho}\,, \tag{5.3}$$

where, in the second term, we used the antisymmetry of the first two indices of the Riemann tensor in order to get the form which allowed us to use the definition of the Ricci tensor $R_{\mu\nu} = g^{\rho\sigma}R_{\rho\mu\sigma\nu}$. Also, we brought the metric inside the covariant derivatives since it is covariantly constant. Then, we contract the expression we found with $g^{\nu\sigma}$:

$$0 = g^{\nu\sigma}\Big(R_{\nu\sigma;\alpha} - R_{\nu\alpha;\sigma} + g^{\mu\rho}R_{\mu\nu\sigma\alpha;\rho}\Big) = R_{;\alpha} - R^{\sigma}{}_{\alpha;\sigma} - g^{\mu\rho}R_{\mu\alpha;\rho} \tag{5.4}$$

$$= R_{;\alpha} - R^{\sigma}{}_{\alpha;\sigma} - R^{\sigma}{}_{\alpha;\sigma}\,, \tag{5.5}$$

where we used the same properties as before and the definition of the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$. So, we have the contracted Bianchi identities $0 = R_{;\alpha} - 2R^{\sigma}{}_{\alpha;\sigma}$. Raising an index with the inverse metric $g^{\alpha\beta}$ and relabeling σ to α in the second term (after having raised the index), these can be written as

$$\nabla_\alpha \left(R g^{\alpha\beta} - 2R^{\alpha\beta} \right) . \tag{5.6}$$

5.2 Weak-field geodesic equation

This was already treated in section 4.3.1.

5.3 Hyperbolic plane geodesics

Our coordinates are (x, y), and our metric is $g_{ij} = y^{-2}\delta_{ij}$, with inverse $g^{ij} = y^2 \delta^{ij}$.

So, we can calculate the Christoffel symbols as:

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(g_{mj,k} + g_{mk,j} - g_{jk,m} \right) , \tag{5.7}$$

this calculation is simplified by the fact that the only nonvanishing derivatives of the metric are $g_{00,1} = g_{11,1} = -2y^{-3}$. If the index i in Γ^i_{jk} is zero, then the last term in the sum vanishes since it corresponds to a derivative with respect to x. With these we get:

$$\Gamma^0_{00} = \frac{1}{2} y^2 \left(2g_{00,0} \right) = 0 \tag{5.8a}$$

$$\Gamma^0_{01} = \frac{1}{2} y^2 \left(g_{00,1} \right) = -\frac{1}{y} \tag{5.8b}$$

$$\Gamma^0_{11} = \frac{1}{2} y^2 \left(g_{01,1} + g_{01,1} \right) = 0 \tag{5.8c}$$

$$\Gamma^1_{00} = \frac{1}{2} y^2 \left(-g_{00,1} \right) = \frac{1}{y} \tag{5.8d}$$

$$\Gamma^1_{01} = \frac{1}{2} y^2 \left(g_{10,0} + g_{11,0} - g_{01,1} \right) = 0 \tag{5.8e}$$

$$\Gamma^1_{11} = \frac{1}{2} y^2 \left(g_{11,1} + g_{11,1} - g_{11,1} \right) = -\frac{1}{y} , \tag{5.8f}$$

and the geodesic equation $u^\mu \nabla_\mu u^\nu = 0$ is written with respect to these.

5.3.1 Vertical lines

First of all we want to parametrize these vertical lines: we choose our parameter so that the length of the velocity vector is everywhere equal to one.

Since the lines are vertical, we want the position x^i with respect to the parameter s to look something like $x^i(s) = (x_0, y(s))$.

We use the arclength parameter:

$$s = \int \sqrt{g_{ij}u^i u^j}\, \mathrm{d}\lambda \,, \tag{5.9}$$

where $u^i = \mathrm{d}x^i \Big/ \mathrm{d}\lambda$ and λ is an arbitrary parameter.

We can rewrite this integral with respect to the Euclidean norm $\|u\|_E^2 = \delta_{ij}u^i u^j$: we get

$$s = \int \frac{1}{y}\|u\|_E \, \mathrm{d}\lambda \,, \tag{5.10}$$

so we can see that we get $s = \int \mathrm{d}\lambda$, or $s = \lambda$, iff $\|u\|_E = y$: so, let us drop the distinction between s and λ and apply this condition. The velocity vector is

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}s} = \left(0, \frac{\mathrm{d}y}{\mathrm{d}s}\right), \tag{5.11}$$

whose Euclidean norm is just (the absolute value of) $\mathrm{d}y/\mathrm{d}s$. So, we must impose the condition $y = \mathrm{d}y/\mathrm{d}s$, which can be solved by separation of variables to yield $s = \log y$, or $y = e^s$.

So our parametrization for the curve is $s \to x^i = (x_0, e^s)$, the velocity is $u^i = (0, e^s)$ and the derivative of velocity with respect to s is again

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} = (0, e^s)\,. \tag{5.12}$$

Now we can plug these into our geodesic equation, which is simplified by the fact that $u^0 = 0$, therefore there is only one relevant term in the Christoffel sum:

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{11}u^1 u^1 = 0\,, \tag{5.13}$$

whose components are

$$\underbrace{\frac{\mathrm{d}u^0}{\mathrm{d}s}}_{0} + \underbrace{\Gamma^0_{11}}_{0} y^2 = 0 \tag{5.14}$$

and

$$\frac{\mathrm{d}u^1}{\mathrm{d}s} + \Gamma^1_{11}u^1 u^1 = 0 \tag{5.15a}$$

$$y + \left(-\frac{1}{y}\right) y^2 = 0\,, \tag{5.15b}$$

which are identities, therefore vertical lines are indeed geodesics in this hyperbolic plane.

5.3.2 More solutions

I find the solution which follows quite ugly and unjustified, I have yet to find a geometric justification for these manipulations, which were derived by reverse-engineering the first integral. Anyhow, these manipulations work...

We insert the Christoffel symbols into the geodesic equations and multiplying through by y we get the following system, in which we denote differentiation with respect to s by a dot:

$$y\ddot{y} = \dot{y}^2 - \dot{x}^2 \tag{5.16a}$$

$$y\ddot{x} = 2\dot{x}\dot{y}\,. \tag{5.16b}$$

Now, we will denote the derivative of y with respect to x as

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}}\,. \tag{5.17}$$

We want to find a first integral. We start with the definition of y':

$$\dot{y} - y'\dot{x} = 0\,, \tag{5.18}$$

and add and subtract the quantity $\dot{y}y'^2$:

$$\dot{y} + (\dot{y}y' - \dot{x})y' - \dot{y}y'^2 = 0\,, \tag{5.19}$$

which can also be written as

$$\dot{y} + (\dot{y}y' - \dot{x})y' + \dot{y}y'^2 - 2\dot{y}y'^2 = 0\,. \tag{5.20}$$

Now, we want to substitute in our equations of motion: we will need them in the form $y\ddot{y}/\dot{x} = \dot{y}y' - \dot{x}$ and $y\ddot{x}/\dot{x} = 2\dot{y}$. We recognize the LHS of both of these in (5.20) and plug them in: it becomes

$$\dot{y} + \frac{y\ddot{y}}{\dot{x}}y' + \dot{y}y'^2 - \frac{y\ddot{x}}{\dot{x}}y'^2 = 0\,. \tag{5.21}$$

We can recognize that the second derivative terms look similar to the derivative of y', which is:

$$\frac{\mathrm{d}}{\mathrm{d}s}y' = \frac{\ddot{y}}{\dot{x}} - \frac{\dot{y}}{\dot{x}}\ddot{x} = \frac{\ddot{y}}{\dot{x}} - y'\ddot{x}\,, \tag{5.22}$$

so equation (5.21) becomes:

$$\dot{y} + \dot{y}y'^2 + yy'\frac{\mathrm{d}y'}{\mathrm{d}s} = 0\,, \tag{5.23}$$

which is $1/(2y)$ times the derivative of $y^2(y'^2 + 1)$, which comes out to be:

$$\frac{1}{2y}\frac{\mathrm{d}}{\mathrm{d}s}\Big(y^2(y'^2+1)\Big) = \frac{1}{2y}\left(2y\dot{y}(y'^2+1) + y^2 2y'\frac{\mathrm{d}}{\mathrm{d}s}y'\right), \tag{5.24}$$

exactly what we had before.

So, $y^2(y'^2 + 1) = \text{const}$, and we can call this constant R^2 (which is $1/A^2$ in the homework notation: I find this notation to be more suggestive).

5.3.3 Solving the equation

The equation $y^2(y'^2 + 1) = R^2$ can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\sqrt{\frac{R^2}{y^2} - 1}\,, \tag{5.25}$$

which means that if we fix R the value of the derivative of the curve can only attain two opposite values. Do note that we can go from one branch to the other with the transformation $x \to -x$, a mirror symmetry around some center. Then, we can just make the gauge choice $y' > 0$ and integrate by separation of varibles:

$$\int \frac{y\,\mathrm{d}y}{\sqrt{R^2 - y^2}} = \int \mathrm{d}x\ , \tag{5.26}$$

which can be solved with the substitution $y = R\sin(\theta)$, with $\mathrm{d}y = R\cos(\theta)\,\mathrm{d}\theta$. Inserting this, we find:

$$x - x_0 = \int \frac{R\sin\theta R\cos\theta\,\mathrm{d}\theta}{R\sqrt{1-\sin^2\theta}} = R\int \sin\theta\,\mathrm{d}\theta = -R\cos\theta\,, \tag{5.27}$$

which can be squared to find $(x - x_0)^2 = R^2(1 - \sin^2\theta) = R^2 - y^2$, or

$$R^2 = (x - x_0)^2 + y^2\,, \tag{5.28}$$

the equation of a circle. We can then confirm that this solution also holds in the other branch, up to a change of integration constant, by rewriting $(x - x_0)^2 = ((-x) - x_1)^2$: this is solved by $x_0 = -x_1$.

Geometrically, we are looking at circles with origins on the x axis and radius R. If y and R are fixed, then there are only two possible circles, which can be found by connecting a certain point at height y to the x axis with a segment of length R. One can then see that the right halves of the circles we found can be found from the left halves by symmetry.

Sheet 6

6.1 Sphere pole Riemann coordinates

Recall from section 3.3.2 the spherical surface metric in the coordinates (θ, φ):

$$g_{\mu\nu} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix}\,, \tag{6.1}$$

and the Christoffel symbols, which are all zero except for $\Gamma^0_{11} = -\sin\theta\cos\theta$ and $\Gamma^1_{01} = \Gamma^1_{10} = 1/\tan\theta$.

We can plug these into the geodesic equation $u^\mu \nabla_\mu u^\nu$, which comes out to be

$$\ddot{\theta} = +\dot{\varphi}^2 \sin\theta\cos\theta \tag{6.2a}$$

$$\ddot{\varphi} = -2\frac{\dot{\varphi}\dot{\theta}}{\tan(\theta)}\,, \tag{6.2b}$$

for a trajectory $(\theta(s), \varphi(s))$ with velocity $(\dot{\theta}, \dot{\varphi})$: dots denote derivatives with respect to s.

Now, we want to check whether parallels and meridians are geodesics. First of all, we want to choose a parameter such that the velocity is of constant norm 1: the equation to satisfy is

$$R^2 \begin{bmatrix} \dot{\theta} & \dot{\varphi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} \equiv 1\,, \tag{6.3}$$

or $\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta = R^{-2}$. Meridians have constant φ: for them, then, $\dot{\theta}^2 = R^{-2}$, so an appropriate parametrization is $(\theta(s), \varphi(s)) = (s/R, \varphi_0)$. Parallels have constant θ: by an analogous line of reasoning, we parametrize them as $(\theta(s), \varphi(s)) = (\theta_0, s/R\sin\theta)$.

One can readily check that for meridians both of the geodesic equations are identities, while for parallels the second one is an identity but the first reads $0 = \cos\theta/\sin\theta$: it can only be satisfied if $\theta = \pi/2$. This makes sense: geodesics on a sphere are great circles, and the only parallel which is a great circle is the equator.

6.1.1 Riemann coordinates

The coordinates we wish to use are

$$x = R\theta\cos\varphi \tag{6.4a}$$

$$y = R\theta\sin\varphi\,. \tag{6.4b}$$

They are in the form $x^\alpha = \theta n^\alpha$, for vectors $n^\alpha = R(\cos\varphi, \sin\varphi)$. An orthonormal basis for these vectors can be found by selecting $\varphi = 0, \pi/2$.

As we have shown before, the coordinates x^α describe geodesics if we consider them for fixed φ and with parameter θ, since they are meridians.

6.1.2 Metric computation

The metric transforms as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}\,, \tag{6.5}$$

so we need the inverse Jacobian, which is expressed in terms of the new coordinates $x'^{\mu} = (x, y)$ and the old ones $x^{\mu} = (\theta, \varphi)$:

$$\frac{\partial x^{\alpha}}{\partial x'^{\mu}} = \begin{bmatrix} \frac{1}{R}\frac{x}{\sqrt{x^2+y^2}} & \frac{1}{R}\frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{bmatrix}, \tag{6.6}$$

so we get, using the expansion $\sin^2\theta \sim \theta^2 - \theta^4/3 + O(\theta^6)$ and the identification $R^2\theta^2 = x^2 + y^2$:

$$g'_{00} = R^2\left(\frac{\partial x^0}{\partial x'^0}\right)^2 + R^2\sin^2\theta\left(\frac{\partial x^1}{\partial x'^0}\right)^2 \tag{6.7a}$$

$$= \frac{x^2}{x^2+y^2} + R^2\sin^2\theta\frac{y^2}{x^4\big(1+(y/x)^2\big)} \tag{6.7b}$$

$$= \frac{1}{x^2+y^2}\left(x^2 + \frac{R^2\sin^2\theta y^2}{x^2+y^2}\right) \tag{6.7c}$$

$$= 1 - \frac{y^2}{3R^2} + O((x^2+y^2)y^2), \tag{6.7d}$$

while for the off-diagonal elements $g'_{01} = g'_{10}$:

$$g'_{01} = R^2\frac{\partial x^0}{\partial x'^0}\frac{\partial x^0}{\partial x'^1} + R^2\sin^2\theta\frac{\partial x^1}{\partial x'^0}\frac{\partial x^1}{\partial x'^1} \tag{6.8a}$$

$$= \frac{xy}{x^2+y^2} + R^2\sin^2\theta\left(-\frac{xy}{(x^2+y^2)^2}\right) \tag{6.8b}$$

$$= \frac{xy}{x^2+y^2}\left(1 - \frac{R^2\sin^2\theta}{x^2+y^2}\right) \tag{6.8c}$$

$$= \frac{xy}{x^2+y^2}\left(\frac{(x^2+y^2)^2}{3R^2(x^2+y^2)}\right) + O(x^2+y^2) \tag{6.8d}$$

$$= \frac{xy}{3R^2} + O(x^2+y^2), \tag{6.8e}$$

and lastly for the element g'_{11}:

$$g'_{11} = R^2 \left(\frac{\partial x^0}{\partial x'^1} \right)^2 + R^2 \sin^2\theta \left(\frac{\partial x^1}{\partial x'^1} \right)^2 \tag{6.9a}$$

$$= \frac{y^2}{x^2+y^2} + R^2 \sin^2\theta \frac{1}{x^2\left(1+(y/x)^2\right)^2} \tag{6.9b}$$

$$= \frac{1}{x^2+y^2} \left(y^2 + R^2 \sin^2\theta \frac{x^2}{x^2+y^2} \right) \tag{6.9c}$$

$$= 1 - \frac{x^2}{3R^2} + O((x^2+y^2)x^2)\,. \tag{6.9d}$$

At the north pole $x = y = 0$, so there $g'_{\mu\nu} = \delta_{\mu\nu}$, and all the first derivatives calculated there vanish since there are no first order terms.

6.1.3 Scalar curvature calculation

The expression we have for the scalar curvature in a LIF is given in equation (4.5).

We can evaluate it for $g'_{\mu\nu}$. Do note that the non-differentiated metric can be identified with the identity, and derivatives with upper and lower indices are the same. So, we get:

$$R_{\mathrm{Ric}} = g_{\alpha\nu,}{}^{\alpha\nu} - \delta^{\mu\nu}\Box g_{\mu\nu}\,, \tag{6.10}$$

where the only nonvanishing terms are:

$$g_{\alpha\nu,}{}^{\alpha\nu} = 2\frac{\partial^2}{\partial x \partial y}\left(\frac{xy}{3R^2}\right) = \frac{2}{3R^2} \tag{6.11}$$

and what was denoted as the Dalambertian before is just the Laplacian: $\Box = \delta^{\mu\nu}\partial_\mu\partial_\nu = \partial^2_{xx} + \partial^2_{yy}$:

$$\delta^{\mu\nu}\Box g_{\mu\nu} = \partial^2_{xx} g_{11} + \partial^2_{yy} g_{00} = -2\frac{2}{3R^2}\,, \tag{6.12}$$

so in the end we get

$$R_{\mathrm{Ric}} = \frac{2}{3R^2} + \frac{4}{3R^2} = \frac{2}{R^2}\,, \tag{6.13}$$

which means that the curvature decreases as the radius increases, as we might expect.

6.2 Schwarzschild metric curvature

6.2.1 Christoffel symbols

The computation is tedious and not particularly enlightening: we start from the metric [3]

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{6.14}$$

and compute the Christoffel symbols with the usual formula:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}\left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}\right), \tag{6.15}$$

where fortunately, since the metric is diagonal, we only need to compute one term in the sum (that is, $\mu \equiv \alpha$ always), and the two indices in the metric must be equal in order for the term to not vanish.

The metric only depends on θ and r, so any derivatives with respect to t and φ are to be discarded.

With this out of the way, we start computing the 40 independent symbols and find that the nonzero ones are (denoting differentiation with respect to r with a prime):

$$\Gamma^{t}_{rt} = \frac{A'}{2A} \qquad \Gamma^{r}_{rr} = \frac{B'}{2B} \tag{6.16a}$$

$$\Gamma^{r}_{tt} = \frac{A'}{2B} \qquad \Gamma^{\theta}_{r\theta} = \frac{1}{r} \tag{6.16b}$$

$$\Gamma^{r}_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^{\varphi}_{r\varphi} = \frac{1}{r} \tag{6.16c}$$

$$\Gamma^{r}_{\varphi\varphi} = -\frac{r}{B}\sin^2\theta \qquad \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta \tag{6.16d}$$

$$\Gamma^{\varphi}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}\,. \tag{6.16e}$$

6.2.2 Ricci component

We want to compute $R^{\mu}_{t\mu t}$, and in order to do so we must find the three components R^{i}_{tit} with varying i and sum them (here $i = r, \theta, \varphi$), since R^{t}_{ttt} vanishes by antisymmetry.

In general we have:

$$R^{i}_{tit} = 2\left(\Gamma^{i}_{[t|t,|i]} + \Gamma^{\alpha}_{t[t}\Gamma^{i}_{i]\alpha}\right) \tag{6.17a}$$

$$= \Gamma^{i}_{tt,i} - \cancel{\Gamma^{i}_{it,t}} + \Gamma^{\alpha}_{tt}\Gamma^{i}_{i\alpha} - \Gamma^{\alpha}_{ti}\Gamma^{i}_{t\alpha}\,. \tag{6.17b}$$

[3] There is a typo in the homework assignment: the coefficients are written as functions of time.

Note that the index i is consider not to be summed here, we are writing a formula for the components of the Riemann tensor; although the expression holds when summing over i as well.

So, we can compute this for the specific values of i: for $i = r$ we have

$$
\begin{aligned}
R^r_{trt} &= \Gamma^r_{tt,r} + \Gamma^\alpha_{tt}\Gamma^r_{r\alpha} - \Gamma^\alpha_{tr}\Gamma^r_{t\alpha} && \text{(6.18a)} \\
&= \left(\frac{A'}{2B}\right)' + \frac{A'}{2B}\frac{B'}{2B} - \frac{A'}{2A}\frac{A'}{2B} && \text{(6.18b)} \\
&= \frac{A'}{4B}\left(\frac{A'}{B} + \frac{B'}{B} + \frac{A'}{A}\right), && \text{(6.18c)}
\end{aligned}
$$

for $i = \theta$ instead

$$
\begin{aligned}
R^\theta_{t\theta t} &= \cancelto{}{\Gamma^\theta_{tt,\theta}} + \Gamma^\alpha_{tt}\Gamma^\theta_{\theta\alpha} - \cancelto{}{\Gamma^\alpha_{t\theta}\Gamma^i_{t\alpha}} && \text{(6.19a)} \\
&= \frac{A'}{2B}\frac{1}{r}, && \text{(6.19b)}
\end{aligned}
$$

and for $i = \varphi$:

$$
\begin{aligned}
R^\varphi_{t\varphi t} &= \cancelto{}{\Gamma^\varphi_{tt,\varphi}} + \Gamma^\alpha_{tt}\Gamma^\varphi_{\varphi\alpha} - \cancelto{}{\Gamma^\alpha_{t\varphi}\Gamma^i_{t\alpha}} && \text{(6.20a)} \\
&= \frac{A'}{2B}\frac{1}{r}, && \text{(6.20b)}
\end{aligned}
$$

so our final solution is

$$
R_{00} = R^i_{0i0} = \frac{A'}{2B}\left(\frac{A'}{B} + \frac{B'}{B} + \frac{A'}{A} + \frac{4}{r}\right). \tag{6.21}
$$

6.3 Schwarzschild geometry orbits

The derivation up to the equation for the perturbed orbit equation is documented in the lecture notes, I might copy it here later, but for now one can find it there.

During the lecture we got up to the first order equation for the perturbation w for the orbit u, written in the form $u(\varphi) = u_c\big(1 + w(\varphi)\big)$:

$$
\frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} = (6GMu_c - 1)w, \tag{6.22}
$$

which is in the form $\ddot{w} + \omega^2 w = 0$, for $\omega^2 = 1 - 6GMu_c$. Now, we know that the first order equation must be complemented by the zeroth order one:

$$
u_c = \frac{GM}{l^2} + 3GMu_c^2, \tag{6.23}
$$

which can be solved for u_c to yield:

$$u_c = \frac{1 \pm \sqrt{1 - 3 \times 4\frac{G^2M^2}{l^2}}}{6GM}\,, \tag{6.24}$$

therefore the square angular velocity of the perturbation's evolution is:

$$\omega^2 = 1 - \cancel{6GM}\left(\frac{1 \pm \sqrt{1 - 12\frac{G^2M^2}{l^2}}}{\cancel{6GM}}\right) = \pm\sqrt{1 - 12\frac{G^2M^2}{l^2}}\,. \tag{6.25}$$

The solution with the minus sign has no meaning for us, since the solution we want to consider must be stable, with positive ω^2. So, the angular velocity is

$$\omega = \left(1 - 12\frac{G^2M^2}{l^2}\right)^{1/4}\,, \tag{6.26}$$

and we know that angular velocity and period are related by $T = 2\pi/\omega$: therefore we get

$$T = 2\pi\left(1 - 12\frac{G^2M^2}{l^2}\right)^{-1/4}\,, \tag{6.27}$$

which we can Taylor expand: at $x = 0$ we have

$$(1-12x)^{-1/4} = 1 - {}^{1}\!/_{4}(1 - 12 \times 0)^{-5/4}(-12x) + O(x^2) = 1 + 3x + O(x^2)\,. \tag{6.28}$$

Therefore:

$$T = 2\pi\left(1 + 3\left(\frac{GM}{l}\right)^2\right) + O\left(\left(\frac{GM}{l}\right)^4\right)\,, \tag{6.29}$$

which is approximately 2π as we should expect: the Newtonian approximation is $l \gg GM$, and Newtonian orbits have a period of exactly 2π. Then we can read off the first-order correction directly from the first term in the expansion: it is

$$\delta\varphi = 6\pi\left(\frac{GM}{l}\right)^2\,. \tag{6.30}$$