General Relativity exercises

Jacopo Tissino, Giorgio Mentasti, Alessandro Lovo

December 16, 2019

Contents

We set $c = 1$.

Here we will often use (anti)symmetrization of indices, which makes some calculations much easier. The idea of symmetrization is to sum over all permutation of the selected indices, with a minus sign for the odd permutation if the case of anti symmetrization. So, for instance, $F_{\mu\nu}$ can be antisymmetrized into $F_{[\mu\nu]} = {}^1\!/_2\Big(F_{\mu\nu} - F_{\nu\mu}\Big)$ and symmetrized into $F_{(\mu\nu)} = {}^1\!/_2\Big(F_{\mu\nu} + F_{\nu\mu}\Big)$.

The factor ${}^1\!/_2$ is in general $1/n!$, where n is the number of antisymmetrized indices. This is included because in general we will be summing $n!$ terms, and we want to write things like: "$F_{\mu\nu}$ is antisymmetric means $F_{\mu\nu} = F_{[\mu\nu]}$", so we need to rescale the sum to make it into an average.

The general formulas are then:

$$F_{[\mu_1 \dots \mu_n]} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}\sigma F_{\sigma(\mu_1)\dots\sigma(\mu_n)} \tag{0.1a}$$

$$F_{(\mu_1 \dots \mu_n)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} F_{\sigma(\mu_1)\dots\sigma(\mu_n)}\,, \tag{0.1b}$$

where $\mathfrak{S}_n$ is the *symmetric group* of permutations of n elements, and the sign of a permutation $\sigma \in \mathfrak{S}_n$ is ± 1, depending on the parity of pair swaps that are needed to get that configuration (we fix ($\operatorname{sign}\mathbb{1} = 1$)).

If we want to symmetrize indices which are not next to each other, we will denote the end of the (anti)symmetrized indices by a vertical bar.

A useful mnemonic for the Riemann tensor: we'd like to write the formula $R = \partial\Gamma + \Gamma\Gamma$ keeping the indices in the same order on either side: the way to do it is this:

$$R^{\mu}_{\alpha\beta\gamma} = -2\left(\Gamma^{\mu}_{\alpha[\beta,\gamma]} + \Gamma^{\sigma}_{\alpha[\beta}\Gamma^{\mu}_{\gamma]\sigma}\right), \tag{0.2}$$

which is much easier to remember: one writes down the indices in the same order, adds a dummy index in the second Christoffel symbol on the last term, which must be contracted with the upper index on the other symbol.

Then, since the Riemann tensor is antisymmetric in the last two indices, we antisymmetrize in those.

Another useful relation is given by

$$R_{\mu\nu} = \partial_\gamma \Gamma^{\gamma}_{\mu\nu} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - \nabla_\mu\left(\partial_\nu \log\sqrt{|g|}\right), \tag{0.3}$$

where g is the determinant of the metric.

Sheet 1

1.1 Lorentz transformations

1.1.1 Inverses

We can consider a Lorentz boost with velocity v in the x direction, and we look at its representation in the (t, x) plane (since the y and z directions are unchanged). Its matrix expression looks like:

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix}, \tag{1.1}$$

where $\gamma = 1/\sqrt{1-v^2}$. The inverse of this matrix can be computed using the general formula for a 2x2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.2}$$

The determinant of Λ is equal to $\gamma^2(1-v^2) = 1$, therefore the inverse matrix is:

$$\Lambda = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix}. \tag{1.3}$$

1.1.2 Invariance of the spacetime interval

Our Lorentz transformation is

$$\mathrm{d}t' = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{1.4a}$$
$$\mathrm{d}x' = \gamma(-v\,\mathrm{d}t + \mathrm{d}x) \tag{1.4b}$$
$$\mathrm{d}y' = \mathrm{d}y \tag{1.4c}$$
$$\mathrm{d}z' = \mathrm{d}z \tag{1.4d}$$

and we wish to prove that the spacetime interval, defined by $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is preserved: $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$. Let us write the claimed equality explicitly:

$$-\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{1.5a}$$

1.1.3 Tensor notation pseudo-orthogonality

The invariance of the spacetime interval $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$ can be also written as $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x'^{\,\mu}\,\mathrm{d}x'^{\,\nu}$. By making the primed differentials explicit we have:

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho\,\mathrm{d}x^\rho\,\Lambda^\nu{}_\sigma\,\mathrm{d}x^\sigma\ , \tag{1.6}$$

but the dummy indices on the LHS can be changed to ρ and σ, so that both sides are proportional to $\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$. Doing this we get:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = (\Lambda^\top)_\rho{}^\mu\eta_{\mu\nu}\Lambda^\nu{}_\sigma\,, \tag{1.7}$$

or, in matrix form, $\eta = \Lambda^\top\eta\Lambda$.

1.1.4 Explicit pseudo-orthogonality

For simplicity but WLOG we consider a boost in the x direction with velocity v and Lorentz factor γ. The matrix expression to verify is:

$$\begin{bmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \tag{1.8a}$$
$$\begin{bmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-\gamma & v\gamma\\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \tag{1.8b}$$
$$\begin{bmatrix}-\gamma^2 + \gamma^2 v^2 & v\gamma^2 - v\gamma^2\\ v\gamma^2 - v\gamma^2 & -v\gamma^2 + \gamma^2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\,, \tag{1.8c}$$

which by $\gamma^2 = 1/(1-v^2)$ confirms the validity of the expression.

1.2 Muons

1.2.1 Nonrelativistic approximation

The survival probability is given by $\mathbb{P}(t) = \exp\left(-t/2.2 \times 10^{-6}\,\mathrm{s}\right)$. If the ground is $h = 15\,\mathrm{km}$ away, then the muon will reach it in $t = h/v = 15\,\mathrm{km}/(0.995c) \approx 5.03 \times 10^{-5}\,\mathrm{s}$, therefore $\mathbb{P}(t) \approx 1.2 \times 10^{-10}$.

1.2.2 Relativistic effects: ground perspective

The observer on the ground will see the muon having to traverse the whole $h = 15\,\mathrm{km}$, but the muon's time will be dilated for them by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$.

1.2.3 Relativistic effects: muons perspective

The muons in their system will observe lenght contraction, with respect to Lorentz boost, by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$. This result is the same of the one predicted by ground observer, with respect to relativity principle.

1.3 Radiation

1.3.1 New angle

In the source frame the radiation velocity components are $u'_x = \cos\theta', u'_y = \sin\theta'$. From the composition of velocities we obtain:

$$u_y = \sin\theta = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\sin\theta'}{\gamma_v(1 + v\cos\theta')} \tag{1.9a}$$

$$u_x = \cos\theta = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma_v(\mathrm{d}x' + v\,\mathrm{d}t')}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\cos\theta' + v}{1 + v\cos\theta'}\,, \tag{1.9b}$$

hence:

$$\frac{1}{\tan\theta} = \frac{\gamma_v}{\tan\theta'} + \frac{\gamma_v v}{\sin\theta'}\,. \tag{1.10}$$

1.3.2 Angle plot and relevant limits

See the jupyter notebook in the `python` folder for plots. For $v = 0$ we have $\theta = \theta'$ as we expected, while for $v = 1$, $\theta = 0$.

1.3.3 Radiation speed invariance

Are the components of the velocity, which we called $\sin\theta$ and $\cos\theta$, actually normalized? Let us check:

$$\sin^2\theta + \cos^2\theta = \frac{(\frac{\sin\theta'}{\gamma_v})^2 + (\cos\theta' + v)^2}{(1 + v\cos\theta')^2} \tag{1.11a}$$

$$= \frac{(1 - v^2)\sin^2\theta' + \cos^2\theta' + v^2 + 2v\cos\theta'}{(1 + v\cos\theta')^2} \tag{1.11b}$$

$$= \frac{1 + v^2(1 - \sin\theta') + 2v\cos\theta'}{(1 + v\cos\theta')^2} = 1\,, \tag{1.11c}$$

therefore the square modulus of the speed of the radiation is still c, as we could have assumed earlier.

1.3.4 Isotropic emission

Since the angular distribution of emission varies when changing inertial reference, we might suppose that every system in relative motion respect to O with $v \neq 0$ observes nonisotropic emission.

This can be seen by noticing that for $v \simeq 1$ we have that in the observer system there is almost only emission at an angle $\theta = 0$. In general, since there is a Lorentz γ factor multiplying a function of the angle in the radiation emission frame O', the cotangent of the angle in the observation frame O must get larger and larger as the relative velocity v increases, therefore the radiation gets compressed towards angles with large cotangents: $\theta \sim 0$.

See the jupyter notebook in the `python` folder for interactive plots :)

Sheet 2

2.1 Constant acceleration

2.1.1 Coordinate velocity

We are given the position as a function of time,

$$x(t) = \frac{\sqrt{1 + \kappa^2 t^2} - 1}{\kappa}\,, \tag{2.1}$$

and we can directly compute its derivative

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\kappa t}{\sqrt{\kappa^2 t^2 + 1}} = \frac{1}{\sqrt{\frac{1}{\kappa^2 t^2} + 1}}\,. \tag{2.2}$$

It is clear from the expression that $|v| < 1$ for all times, while v approaches 1 at positive temporal infinity and -1 at negative temporal infinity.

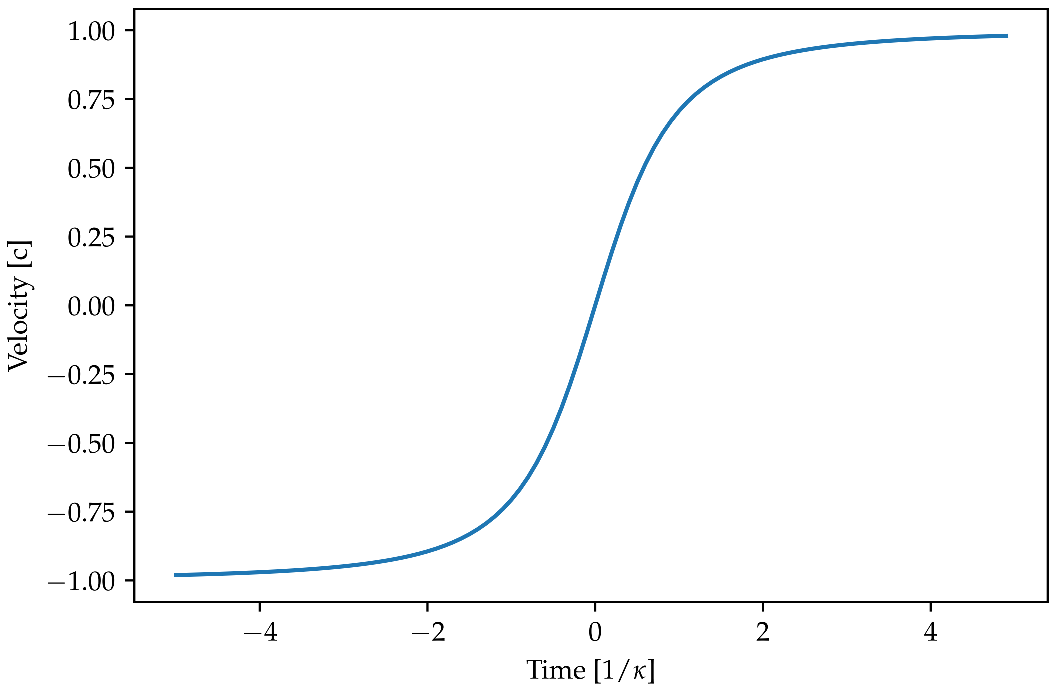

Figure 1: Velocity as a function of coordinate time t

2.1.2 Components of the 4-velocity

The Lorentz factor γ is given by

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{\kappa^2 t^2}{\kappa^2 t^2+1}}} = \sqrt{\kappa^2 t^2 + 1}, \tag{2.3}$$

therefore the four-velocity is given by:

$$u^\mu = \begin{bmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix}. \tag{2.4}$$

2.1.3 Proper time

The relation between coordinate and proper time is given by the definition of the first component of the four-velocity: $u^0 = \mathrm{d}t/\mathrm{d}\tau = \gamma$, therefore $\mathrm{d}\tau = \mathrm{d}t\,/\gamma$. Integrating this relation we get:

$$\tau = \int_0^\tau \mathrm{d}\tau' = \int_0^t \frac{\mathrm{d}t'}{\gamma(t')} = \frac{\operatorname{arcsinh}(\kappa t)}{\kappa}\,, \tag{2.5}$$

where the constant of integration is selected by imposing $t = 0 \iff \tau = 0$. Notice that, as we would expect, when expanding up to first order near $t = \tau = 0$ we have $t \sim \tau$, since in that region the velocity is much less than unity.

The inverse relation is given by $t = \sinh(\kappa\tau)/\kappa$. Using this, we can write:

$$x(t(\tau)) = \frac{\cosh(\kappa\tau) - 1}{\kappa}\,. \tag{2.6}$$

2.1.4 Four-acceleration

Now, we wish to compute the four-acceleration. There are many ways to approach this: an easy one is to simply find the explicit expression $u^\mu(\tau)$ and to differentiate it. The expression we get is:

$$a^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}u^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \sqrt{\sinh^2(\kappa\tau) + 1} \\ \frac{\sqrt{\kappa^2 t^2 + 1}\sinh(\kappa\tau)}{\sqrt{\sinh^2(\kappa\tau)+1}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\kappa\sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa\cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix}, \tag{2.7}$$

which is a bit unwieldy but it can be used to check two important facts: $a^\mu a_\mu = \text{const}$ and $a^\mu u_\mu = 0$. The first of the two is:

$$a^\mu a_\mu = -(a_0)^2 + (a_1)^2 = \kappa^2\cosh^2(\kappa\tau) - \frac{\kappa^2\sinh^2(2\kappa\tau)}{2\left(\cosh(2\kappa\tau) + 1\right)} = \kappa^2\,, \tag{2.8}$$

which tells us that the constant acceleration $\sqrt{a^\mu a_\mu} = \kappa$.

Also, we verify the orthogonality to the four-velocity:

$$a^\mu u_\mu = -\frac{\sqrt{2}\kappa\sqrt{\sinh^2(\kappa\tau) + 1}\,\sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau) + 1}} + \kappa\sinh(\kappa\tau)\cosh(\kappa\tau) = 0\,. \tag{2.9}$$

2.1.5 Local velocity & acceleration

We can apply a Lorentz boost corresponding to this velocity: it will be given by the matrix:

$$\begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.10}$$

where v and γ are those found before. Without doing any calculations we could already say that the transformed velocity will be equal to the time-like unit vector, while the acceleration will be equal to κ times the unit x-directed vector.

The velocity becomes:

$$(u^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.11}$$

as we expected.

The acceleration instead becomes:

$$(a^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}\kappa \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa \cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \kappa \\ 0 \\ 0 \end{bmatrix}, \tag{2.12}$$

At small speeds the Lorentz boost matrix reduces to the identity matrix: this implies $kt \simeq k\tau \simeq 0$. In this case we obtain the same results of the rest frame of the particle for both acceleration and speed.

2.2 Fixed target collision

2.2.1 Center of mass momenta

In the CoM frame, the momenta of the two protons are respectively $(E_p, \pm p, 0, 0)^\top = m_p(\gamma, \pm v, 0, 0)$, where $E_p^2 = m_p^2 + p^2$. The total CoM energy is $-(p_A^\mu + p_B^\mu)^2 = 2m_p^2$.

2.2.2 Center of mass velocity

The momentum of particle B will be given by $p^\mu = m_p u^\mu = (m_p\gamma, m_p\gamma v, 0, 0)^\top$. Therefore, $\gamma v = p/m_p$. Solving this we get:

$$v = \frac{p}{m_p}\sqrt{\frac{1}{(p/m_p)^2 + 1}} = \frac{p}{E_p}, \tag{2.13}$$

2.2.3 Lab frame momenta

The momentum of particle B in its own rest frame will just be $(m_p, 0, 0, 0)^\top$. The momentum of particle A instead will be given by a boost in the x direction with velocity $-v$:

$$(p_A^\mu)_{\text{lab}} = \begin{bmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_p \\ p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma E_p + v\gamma p \\ v\gamma E_p + \gamma p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_p \gamma^2 (1 + v^2) \\ 2\gamma p \\ 0 \\ 0 \end{bmatrix}, \tag{2.14}$$

2.3 Weak field gravitational time dilation

2.3.1 Time dilation expression

It is more intuitive geometrically to deal with a pulse sent from A to B, for which we expect the time dilation to work in the opposite sense:

$$\Delta t_B = \Delta t_A \big(1 + gh\big)\,, \tag{2.15}$$

up to first order in gh and $g\Delta t_A$, since $(1 + gh)(1 - gh) = 1 - (gh)^2 = 1$ to first order in gh. Alternatively, one can just map $g \to -g$ to recover the time contraction for pulses sent in the other direction.

We know that the paths of the observers are two curves of constant acceleration: we know their explicit expression from equation (**??**), and additionally we assume that they are separated by a space interval h:

$$x_A(t) = \frac{\sqrt{1 + (gt)^2} - 1}{g} \tag{2.16a}$$

$$x_B(t) = \frac{\sqrt{1 + (gt)^2} - 1}{g} + h\,. \tag{2.16b}$$

At $t = 0$ Alice sends a pulse, which then reaches Bob at a time t_1. After a time Δt_A, she sends another, which then reaches Bob at a time t_2. Right now, we are referring to all times as measured in the rest frame of Alice at $t = 0$. These times can be found by imposing that the space and time separation between the events of the pulse being sent and received are equal, since it travels at light speed: the equations which represent this are $x_B(t_1) = t_1$ and $x_B(t_2) - x_A(\Delta t_A) = t_2 - \Delta t_A$. Substituting the expressions for the positions:

$$t_1 = \frac{\sqrt{1+(gt_1)^2}-1}{g} + h \tag{2.17a}$$

$$t_2 - \Delta t_A = \frac{\sqrt{1+(gt_2)^2}-1}{g} + h - \left(\frac{\sqrt{1+(g\Delta t_A)^2}-1}{g}\right). \tag{2.17b}$$

Now, it is just a matter of calculation to solve these equations, expand up to first order in the adimensional parameters gh and $g\Delta t_A$ and one recovers the desidered expression for $\Delta t_B = t_2 - t_1$.

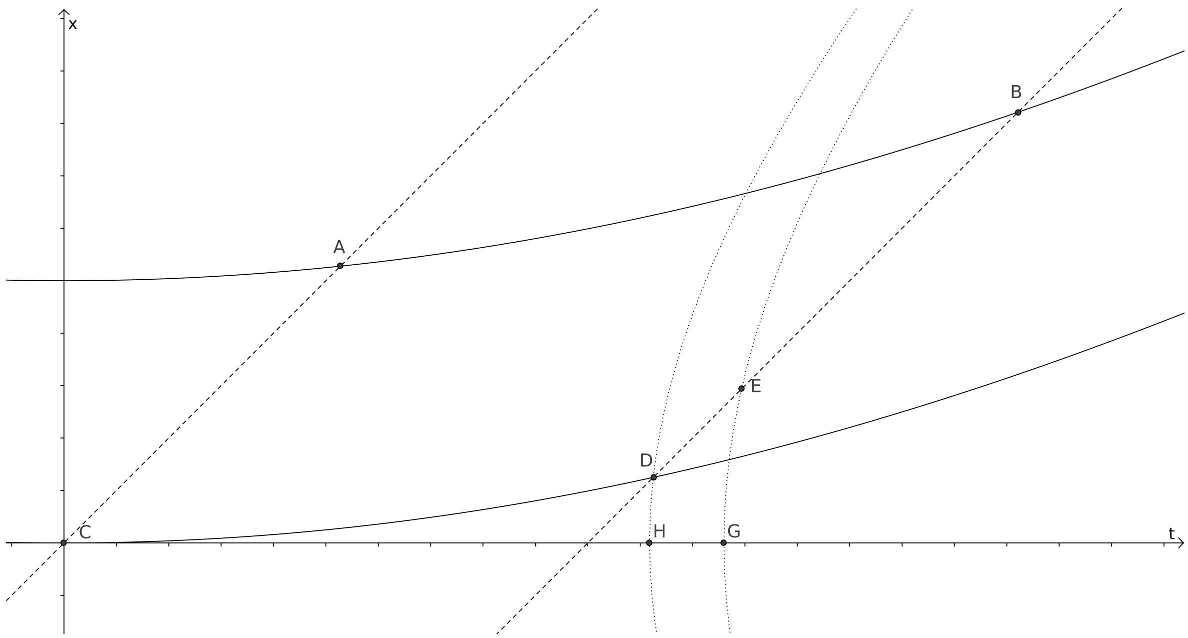

Figure 2: Visualization of the beams, in the frame where the rocket is stationary as the first beam is being sent. The two curves intersecting the space axis are the tip and tail of the spaceship; the beams being sent from the tail are events C and D, while their reception at the tip are events A and B. Event E is just calculated as $B - A$, to make comparisons with D easier. H and G are computed by tracing the dotted line of points which have the same spacetime interval from the origin as points D and E respectively, and selecting its intersection with the temporal axis: this effectively means finding the proper time separation between the two beams being sent/received.

There is one more consideration to make though: what about the Lorentz time dilation for Bob? This it actually a *second order effect*.

Claim 2.1. *The time interval measured by Bob in his frame at $t \sim t_1$ is the same as the one measured in the rest frame of Alice at $t = 0$ up to first order in gh and $g\Delta t_B$.*

Proof. We perform a Lorentz boost to the velocity of Bob at $t = t_1$: this is given by equation (**??**), and is equal to:

$$v = \frac{gt}{\sqrt{(gt)^2 + 1}}, \tag{2.18}$$

with a Lorentz factor of $\gamma = \sqrt{(gt)^2 + 1}$ (see equation (**??**)).

The temporal separation between the two events is Δt_B, while the spatial separation is $\Delta x_B \approx v \Delta t_B$ to first order. The boost, in the (t, x) plane, looks like:

$$\begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix}' = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix} = \begin{bmatrix} \Delta t_B \left(\sqrt{(gt)^2 + 1} - (gt)^2 / \sqrt{(gt)^2 + 1} \right) \\ -gt\Delta t_B + \sqrt{(gt)^2 + 1} gt\Delta t / \sqrt{(gt)^2 + 1} \end{bmatrix}, \tag{2.19}$$

therefore as we would expect the spatial separation is eliminated, while expanding the factor multiplying the temporal one near $gt = 0$ we get:

$$\sqrt{(gt)^2 + 1} - (gt)^2 / \sqrt{(gt)^2 + 1} = 1 + O((gt)^2), \tag{2.20}$$

which proves our result. □

2.3.2 Gravitational time dilation

By the equivalence principle, the effects measured in a uniformly accelerating frame at g are the same as those measured in a gravitational field with constant acceleration g. The gravitational field in such a frame is given by $\Phi = gh$, where h is the height (with arbitrary zero point): the result follows.

2.3.3 Twins and gravitation

The gravitational time dilation difference, in absolute value, is given by:

$$\Delta t = t_{\text{elapsed}} \frac{g\Delta h}{c^2} \approx 1\,\text{yr} \frac{10\,\text{m/s}^2 \times 100\,\text{m}}{(3 \times 10^8\,\text{m/s})^2} \approx 3.5 \times 10^{-7}\,\text{s}. \tag{2.21}$$

We are asked what is the age of the twin on the ground as measured by the twin who is higher up: this is analogous to the situation considered in the first section of this problem; the twin higher up will measure the twin lower down to be older, specifically if $\text{age}_{\text{up}} = 1\,\text{yr}$, then the observer up in the palace will measure the age of the twin at ground level as:

$$\text{age}_{\text{down}} = 1\,\text{yr} + 3.5 \times 10^{-7}\,\text{s} \approx (1 + 1 \times 10^{-14})\text{age}_{\text{up}}. \tag{2.22}$$

Sheet 3

3.1 Changes of coordinate system

We denote by $x^\mu = (x, y)$ and $x'^\mu = (r, \theta)$. Then, we have the following Jacobian matrices:

$$\frac{\partial x^\mu}{\partial x'^\nu} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \tag{3.1a}$$

$$\frac{\partial x'^\nu}{\partial x^\rho} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\dfrac{\sin(\theta)}{r} & \dfrac{\cos(\theta)}{r} \end{bmatrix}, \tag{3.1b}$$

which can be found by plain differentiation of the change of coordinates, recalling $(\arctan x)' = 1/(1+x^2)$. Then, we can compute the product of these two matrices: it comes out to be

$$\frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\rho} = \delta^\mu_\nu\,, \tag{3.2}$$

since on the diagonal we get $\cos^2(\theta) + \sin^2(\theta) = 1$, while on the off-diagonal terms we get a multiple of $\sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) = 0$.

Note that relation (**??**) is just the chain rule written in more generality: substituting the explicit coordinates for x^μ and x'^μ we get the desired expression.

3.2 Properties of covariant differentiation

3.2.1 Metric compatibility of the connection

We wish to show that $\nabla_\alpha g_{\mu\nu} = 0$. A very simple way to prove this is by going in the LIF: there, the equation reads $\partial_\alpha \eta_{\mu\nu} = 0$, which is immediately satisfied since the components of the Minkowski metric are constants. Then, since the equation is tensorial, the result extends to any frame.

This was not the spirit of the exercise, however: let us prove it in a different generic frame. To this end, we define the *Christoffel symbols of the first kind* (while the regular ones are of the second kind):

$$\Gamma_{\mu\nu\rho} = g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} = \frac{1}{2}\Big(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu}\Big)\,. \tag{3.3}$$

These are useful since, as all their indices are down, it is easier to study their symmetry properties.

Note that if we antisymmetrize the first and last index, we get $\Gamma_{[\mu|\nu|\rho]} = {}^1\!/\!{}_2 g_{\mu\rho,\nu}$ since the first and last terms in the sum cancel (in the latter we must invert the indices ν and ρ in order to see this, but this can always be done by the symmetry of the metric).

Then, we write the expression for the covariant derivative of the metric:

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\rho_{\mu\alpha} g_{\rho\nu} - \Gamma^\rho_{\nu\alpha} g_{\mu\rho} = g_{\mu\nu,\alpha} - \Gamma_{\nu\alpha\mu} - \Gamma_{\mu\alpha\nu}\,, \tag{3.4}$$

which is just $g_{\mu\nu,\alpha} - 2\Gamma_{[\nu|\alpha|\mu]} = g_{\mu\nu,\alpha} - g_{\nu\mu,\alpha} = 0$, again by the symmetry of the metric.

One could argue that this is the opposite way round: we should *assume* $\nabla_\mu g_{\rho\sigma} = 0$ and derive from it the formula that was given for the Christoffel symbols in terms of the partial derivatives of the metric.

3.2.2 Leibniz rule

As before, this can be proved in the LIF from the Leibniz rule of regular partial derivatives.

As before, we like to calculate therefore we show this explicitly in any frame.

The derivative of the tensor product looks like:

$$\nabla_\mu \left(A_{\nu\lambda} B_\rho \right) = \partial_\mu \left(A_{\nu\lambda} B_\rho \right) - \Gamma^\sigma_{\mu\nu} A_{\sigma\lambda} B_\rho - \Gamma^\sigma_{\mu\lambda} A_{\nu\sigma} B_\rho - \Gamma^\sigma_{\mu\rho} A_{\nu\lambda} B_\sigma\,, \tag{3.5}$$

while the sum of derivatives looks like:

$$\begin{aligned} B_\rho \nabla_\mu A_{\nu\lambda} + A_{\nu\lambda} \nabla_\mu B_\rho = & B_\rho \partial_\mu A_{\nu\lambda} - \Gamma^\sigma_{\mu\nu} A_{\sigma\lambda} B_\rho - \Gamma^\sigma_{\mu\lambda} A_{\nu\sigma} B_\rho \\ & + A_{\nu\lambda} \partial_\mu B_\rho - \Gamma^\sigma_{\mu\rho} A_{\nu\lambda} B_\sigma \end{aligned}\,, \tag{3.6}$$

so we can see that the Christoffel terms are equal, and the partial derivative terms also are since we have the Leibniz rule for partial derivatives.

3.3 2D Christoffel symbols

3.3.1 Polar coordinates

The metric and inverse metric are respectively given by $g_{\mu\nu} = \mathrm{diag}\left(1, r^2\right)$ and $g^{\mu\nu} = \mathrm{diag}\left(1, r^{-2}\right)$. We only care about the partial derivatives of the lower-indices one, and the only nonvanishing derivative is $g_{11,0} = 2r$, where we mean $(x^0, x^1) = (r, \theta)$.

Then it is tedious but straightforward to perform the direct computation. Things that make it faster are discarding immediately terms which cannot contribute (such as $g_{\alpha\beta,\gamma}$ where at least one of α and β is not 1 or γ is not 0, and only looking at the six independent symbols instead of the eight total ones (since $\Gamma^\alpha_{01} = \Gamma^\alpha_{10}$ for any α).

Then one can see that the nonvanishing symbols are

$$\Gamma^0_{11} = \frac{1}{2} g^{0\alpha}\left(g_{\alpha 1,1} + g_{\alpha 1,1} - g_{11,\alpha}\right) = \frac{1}{2} g^{00}(-g_{11,0}) = -\frac{2r}{2} = -r \tag{3.7a}$$

$$\Gamma^1_{01} = \frac{1}{2} g^{1\alpha}\left(g_{\alpha 0,1} + g_{\alpha 1,0} - g_{01,\alpha}\right) = \frac{1}{2} g^{11} g_{11,0} = \frac{1}{2}\frac{1}{r^2} 2r = \frac{1}{r}\,. \tag{3.7b}$$

3.3.2 Spherical surface

Now we have the following metric and inverse metric:

$$g_{\mu\nu} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2(\theta) \end{bmatrix}, \qquad g^{\mu\nu} = \begin{bmatrix} R^{-2} & 0 \\ 0 & R^{-2} \sin^{-2}(\theta) \end{bmatrix}, \tag{3.8}$$

but do note that R is a constant: given $(x^0, x^1) = (\theta, \varphi)$, we have as before that the only nontrivial derivative is $g_{11,0} = 2R^2 \sin(\theta)\cos(\theta)$.

Then this case is exactly analogous to the previous one: the same symbols are zero, so we can skip almost all of the computation and jump straight to:

$$\Gamma^0_{11} = -\frac{1}{2} g^{00} g_{11,0} = -\sin(\theta)\cos(\theta) \tag{3.9a}$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11} g_{11,0} = \frac{\cos(\theta)}{\sin(\theta)}\,. \tag{3.9b}$$

3.4 Parallel transport

We know from the last exercise the metric and Christoffel symbols of 2D space and of the surface of a sphere.

The equations of parallel transport are in general $u^\mu \nabla_\mu V^\nu = 0$.

3.4.1 Flat space

We want to determine the behaviour of a vector field $V^\mu(\theta)$ defined on a curve $x^\mu(\theta) = (R, \theta)$ with fixed R, such that $V^\mu(\theta = 0) = (0, 1/R)$ (a unit vector: $V^\mu(0) V^\nu(0) g_{\mu\nu}(0) = 1$).

In our case the tangent vector of the curve is $u^\mu = (0, 1)$. Therefore the equations simplify to:

$$\nabla_1 V^\mu = \partial_1 V^\mu + \Gamma^\mu_{1\alpha} V^\alpha = 0\,, \tag{3.10}$$

which are two coupled differential equations; we can make them explicit and substitute the Christoffel symbols found earlier.

$$\nabla_1 V^0 = \partial_1 V^0 + \cancel{\Gamma^0_{10} V^0} + \Gamma^0_{11} V^1 = \partial_1 V^0 - r V^1 \tag{3.11a}$$

$$\nabla_1 V^1 = \partial_1 V^1 + \Gamma^1_{10} V^0 + \cancel{\Gamma^1_{11} V^1} = \partial_1 V^1 + \frac{1}{r} V^0\,, \tag{3.11b}$$

so we can write this linear system like:

$$\partial_1 \begin{bmatrix} V^0 \\ V^1 \end{bmatrix} = \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix} \begin{bmatrix} V^0 \\ V^1 \end{bmatrix} . \tag{3.12}$$

The eigenvalues of this matrix are $\pm i$, so its exponential is a pure rotation matrix:

$$\exp\left(\theta \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix} \right) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R(-\theta) , \tag{3.13}$$

so the solution is

$$V^\mu(\theta) = R(-\theta) V^\mu(0) . \tag{3.14}$$

This means that our vector is rotating clockwise with unit angular velocity in the (r, θ) plane, just as the point it is defined at moves counterclockwise with unit angular velocity: therefore, if we look at the vector in Cartesian coordinates, we will see it always aligned with its initial direction and the same modulus.

Specifically, when $\theta = \pi/2$ we get $V^\mu = (1, 0)$: this has the same modulus as $(0, 1/R)$ since we compute lengths with respect to the metric of our space, as $\|V\|^2 = V^\mu V^\nu g_{\mu\nu}$.

Another way to solve (**??**) is to explicit from the second equation $V^0 = -r\partial_1 V^1$ and plug it into the second, so that it becomes

$$\partial_1(-r\partial_1 V^1) = V^1, \tag{3.15}$$

which leads to the same result.

3.4.2 Curved space: spherical surface

Now our curve is $x^\mu(\theta) = (\theta, 0)$ in the spherical coordinates $x^\mu = (\theta, \varphi)$. The metric is the one given in (**??**), the Christoffel symbols are the ones given in (**??**).

The parallel transport equations are now:

$$\nabla_1 V^0 = \partial_0 V^0 + \cancel{\Gamma^0_{00} V^0} + \cancel{\Gamma^0_{01} V^1} \tag{3.16a}$$

$$\nabla_1 V^1 = \partial_0 V^1 + \cancel{\Gamma^1_{00} V^0} + \Gamma^1_{01} V^1 = \partial_0 V^1 + \frac{V^1}{\tan(\theta)} . \tag{3.16b}$$

The first equation gives us $V^1 = \text{const}$, while we do not need to actually solve the second one: our initial condition is $V^\mu(\theta = 0) = (R^{-1}, 0)$, and $V^1 \equiv 0$ is a solution to that first-order equation, so by the uniqueness we have found the whole solution.

Therefore the vector is constant in these coordinates.

Specifically, at $\theta = \pi/2$ we get $V^\mu = (1/R, 0)$.

Sheet 4

4.1 Riemann tensor computations

4.1.1 LIF form

In the LIF, we have $g_{\mu\nu} = \eta_{\mu\nu}$ and $g_{\mu\nu,\alpha} = 0$. Therefore, the Christoffel symbols are all zero. In the explicit expression of the Riemann tensor, which looks like $R = \partial\Gamma + \Gamma\Gamma$ we can drop the second term and keep only the derivatives of the Christoffel symbols, which are nonzero since they depend on the second derivatives of the metric.

If the Christoffel symbols are zero, the computation of the curvature tensor is significantly easier, since we do not have the $\Gamma\Gamma$: we just need to account for the terms $\partial\Gamma$. The computation gives:

$$R^{\mu}_{\nu\rho\sigma} = 2\partial_{[\rho}\Gamma^{\mu}_{\sigma]\nu} \tag{4.1a}$$

$$= 2\partial_{[\rho}\left(\frac{1}{2}g^{\mu\alpha}\left(g_{\alpha|\sigma],\nu} + g_{\alpha\nu,|\sigma]} - g_{\nu|\sigma],\alpha}\right)\right) \tag{4.1b}$$

$$= g^{\mu\alpha}\left(g_{\alpha[\sigma|,\nu|\rho]} + \cancel{g_{\alpha\nu,[\sigma\rho]}} - g_{\nu[\sigma|,\alpha|\rho]}\right), \tag{4.1c}$$

where a term was cancelled since it contained the antisymmetrization of partial derivatives, which commute.

So, if we lower the index of the (1, 3) Riemann tensor we get the desired expression for the all-lower (0, 4) Riemann tensor:

$$R_{\mu\nu\rho\sigma} = g_{\mu[\sigma|,\nu|\rho]} - g_{\nu[\sigma|,\mu|\rho]}\,. \tag{4.2}$$

4.1.2 Ricci tensor and scalar

The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = g^{\alpha\beta}\left(g_{\alpha[\nu|,\mu|\beta]} - g_{\mu[\nu|,\alpha|\beta]}\right). \tag{4.3}$$

So we have

$$R_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\beta,\mu\nu} - g_{\mu\nu,\alpha\beta} + g_{\mu\beta,\alpha\nu}\right) \tag{4.4a}$$

$$= \frac{1}{2}g^{\alpha\beta}\left(2g_{\alpha(\nu,\mu)\beta} - g_{\mu\nu,\alpha\beta} - g_{\alpha\beta,\mu\nu}\right) \tag{4.4b}$$

$$= g_{\alpha(\nu,\mu)}{}^{\alpha} - \frac{1}{2}\Box g_{\mu\nu} - \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu\nu}\,, \tag{4.4c}$$

where[1] the square $\Box$ denotes the D'alambert operator, $\Box = \partial_\beta\partial^\beta = g^{\alpha\beta}\partial_\alpha\partial_\beta$.

[1] All the indices after the comma are derivatives, even when they become upper.

The Ricci scalar, on the other hand, is given by

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} {g_{\alpha(\nu,\mu)}}^{\alpha} - g^{\mu\nu} \Box g_{\mu\nu} - g^{\alpha\beta} \Box g_{\alpha\beta} = \left({g_{\alpha\nu}}^{,\alpha\nu} - g^{\mu\nu} \Box g_{\mu\nu} \right), \tag{4.5}$$

where we identified the two terms containing the metric contracted with its dalambertian, since they are equal up to a relabeling of indices.

We neglected the symmetrization in the first term since when contracting with the metric the indices $\mu\nu$ are automatically symmetrized.

4.1.3 LIF identities

$$\nabla_\alpha \left(R^\alpha_\beta - \frac{R}{2} \delta^\alpha_\beta \right) = \partial_\alpha \left(g^{\alpha\lambda} R_{\lambda\beta} - \frac{R}{2} \delta^\alpha_\beta \right) \tag{4.6a}$$

$$= \eta^{\alpha\lambda} R_{\lambda\beta,\alpha} - \frac{1}{2} R_{,\beta}, \tag{4.6b}$$

since in the LIF the metric is the Minkowski one, and the Christoffel symbols are zero therefore $\nabla_\alpha = \partial_\alpha$.

4.1.4 Contracted Bianchi identities

The first term is

$$\eta^{\beta\mu} R_{\mu\nu,\beta} = \eta^{\beta\mu} \left(2 {g_{\alpha(\nu,\mu)}}^{,\alpha} - \Box g_{\mu\nu} - g^{\alpha\beta} g_{\alpha\beta,\mu\nu} \right)_{,\beta}, \tag{4.7}$$

which we can expand out: the Minkowski metric raises the derivative with respect to x^β and turns it into a derivative with respect to x_μ, and we can also expand the Dalambertian and the symmetrization: we get

$$\eta^{\beta\mu} R_{\mu\nu,\beta} = {g_{\alpha\nu,\mu}}^{\alpha\mu} + {g_{\alpha\mu,\nu}}^{\alpha\mu} - {g_{\mu\nu,\alpha}}^{\alpha\mu} - g^{\alpha\beta} {g_{\alpha\beta,\mu\nu}}^{\mu} \tag{4.8a}$$

$$= + {g_{\alpha\mu,\nu}}^{\alpha\mu} - \left(g^{\alpha\beta} \Box g_{\alpha\beta} \right)_{,\nu} \tag{4.8b}$$

$$= \partial_\nu \left({g_{\mu\alpha}}^{,\mu\alpha} - g^{\alpha\beta} \Box g_{\alpha\beta} \right), \tag{4.8c}$$

since, up to a relabeling of indices, the first and third term in (**??**) are equal.

The derivative $\partial_\nu R$, on the other hand, is twice that:

$$\partial_\nu R = \partial_\nu \left(2 {g_{\mu\alpha}}^{,\mu\alpha} - 2 g^{\alpha\beta} \Box g_{\alpha\beta} \right), \tag{4.9}$$

as can be directly gathered from (**??**).

Therefore, the quantity in (**??**) is zero.

4.1.5 Alternative expression

Since the metric is covariantly constant, we can just bring it inside the derivative:

$$0 = g^{\beta\mu}\nabla_\alpha\left(R^\alpha{}_\mu - \frac{1}{2}R\delta^\alpha_\mu\right) = \nabla_\alpha\left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right). \tag{4.10}$$

These are called the *Contracted Bianchi identities*, and can be alternatively derived from the Bianchi identities of the Riemann tensor, $R_{\mu\nu[\rho\sigma;\alpha]} = 0$.

4.2 Properties of the Ricci tensor

4.2.1 Symmetry generality

The fact that $T^{\mu\nu} = T^{\nu\mu}$ is frame invariant could be derived plainly from the fact that it is a tensor equation; to be more explicit we can say that under a change of coordinates with Jacobian matrix Λ we have $\widetilde{T}^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta}$; when written in components like this the Jacobians commute, therefore we can just interchange them for both $\widetilde{T}^{\mu\nu}$ and $\widetilde{T}^{\nu\mu}$ to recover their equality.

4.2.2 Symmetry of the Ricci tensor.

The expression (**??**) is manifestly symmetric: part of it is explicitly symmetrized, part is proportional to the metric, which is symmetric.

Therefore, the Ricci tensor is symmetric in any frame.

4.3 Weak field Einstein equations

We consider the *weak field* case, when $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\left|h_{\mu\nu}\right| \ll 1$. We will then neglect second and higher order terms in $h_{\mu\nu}$.

4.3.1 The gravitational potential

We want to work towards Newton's equation, which is $\nabla^2\Phi = 4\pi G_N\rho$, where Φ is the gravitational potential. What defines the gravitational potential is[2] its effect on test masses: they accelerate with $\vec{a} = -\vec{\nabla}\Phi$.

What is the relativistic equivalent of this? A particle in GR follows a geodesic, a curve without proper acceleration. Proper acceleration is a four-vector defined by $a^\mu = u^\nu\nabla_\nu u^\mu$. If the four-velocity of a particle is u^μ, then for it the equation of geodesic motion reads:

$$0 = a^\mu = u^\nu\partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha}u^\nu u^\alpha\,, \tag{4.11}$$

[2] In the lecture notes this is solved differently: instead of finding the gravitational potential from the equation of geodesic motion, there it is found by identification of the gravitational redshift.

which can be written in terms of derivatives with respect to proper time:

$$0 = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\alpha} u^\nu u^\alpha \,. \tag{4.12}$$

If the speeds are much less than that of light, this can be approximated: $s \approx t$, and the only terms which contributes to order 0 in v in the Christoffel symbol sum is the one with $u^0 u^0 \approx 1$: in the end then we get for the spatial components:

$$0 = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{00} \,, \tag{4.13}$$

so we can see that the main contribution in the low-speed limit to the (coordinate!) acceleration of the particle are the symbols Γ^i_{00}. The expression for these in terms of the perturbed metric is:

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\alpha}\left(2h_{\alpha 0,0} - h_{00,\alpha}\right) = -\frac{1}{2}h_{00}{}^{,i} = -\frac{1}{2}\vec{\nabla}^i h_{00} \,, \tag{4.14}$$

if we assume stationarity of the metric (which is justified if we are treating a problem such as the gravitational pull on a body on the surface of the Earth) at least up to first order in h.

Putting everything together: we found that

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\vec{\nabla}^i h_{00} \,, \tag{4.15}$$

but the equation which defines the gravitational field is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\vec{\nabla}^i \Phi \,, \tag{4.16}$$

therefore, in the low-gravitational field and low-speed limit, we must identify $h_{00} = -2\Phi$.

4.3.2 Reframing the EFE

We can contract the EFE with the inverse metric, recalling that $g^{\mu\nu} g_{\mu\nu} = 4$, to get $-R = 8\pi G_N T$. Therefore, they can be reframed by substituting the curvature scalar term with the trace of the stress-energy tensor:

$$R_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) . \tag{4.17}$$

This will aid us by reducing the number of difficult curvature tensors to compute.

In the low-speed, weak-field limit matter has negligible pressure and is mostly just travelling in the time direction. Therefore, we approximate the stress energy tensor as that of noninteracting dust: $T^{\mu\nu} = \rho u^\mu u^\nu$. In our frame matter is almost stationary, therefore we have that $T_{00} \approx T^{00}\eta_{00}\eta_{00} \approx \rho$. The trace of the curvature tensor is instead approximately $T \approx T^{00}\eta_{00} \approx -\rho$.

In the end, the 00 component of the equations reads:

$$R_{00} = 8\pi G_N \left(\rho - \frac{1}{2}(-\rho)\eta_{00} \right) = 4\pi G_N \rho \,. \tag{4.18}$$

Now we only need to show that $R_{00} = \nabla^2\Phi$.

4.3.3 The derivatives in the curvature tensor

The component we need to calculate is $R_{00} = R^\mu_{0\mu 0} = R^i_{0i0}$ by antisymmmetry.

Let us recall here the general expression for the Riemann tensor:

$$R^\mu_{\nu\rho\sigma} = 2\Gamma^\mu_{[\nu|\sigma,|\rho]} + 2\Gamma^\alpha_{\sigma[\nu}\Gamma^\mu_{\rho]\alpha} \,. \tag{4.19}$$

In our case, this simplifies to

$$R^i_{0i0} = \Gamma^i_{00,i} - \Gamma^i_{i0,0} + \Gamma^\alpha_{00}\Gamma^i_{i\alpha} - \Gamma^\alpha_{0i}\Gamma^i_{0\alpha} \,. \tag{4.20}$$

The second term contains time derivatives, which we decided to neglect by dealing with the stationary case. The third and fourth term are of second order in h.

The only remaining term is the first one: using the expression (**??**) and the identification $h_{00} = -2\Phi$, it is

$$R_{00} = \Gamma^i_{00,i} = \left(-\frac{1}{2}\partial^i h_{00} \right)_{,i} = \partial_i \partial^i \Phi \,, \tag{4.21}$$

which is what we wanted to prove.

Sheet 5

5.1 Alternative derivation of the contracted Bianchi identities

The form of the Riemann tensor in a LIF was derived in section **??**.

5.1.1 Bianchi identities of the Riemann tensor

In the LIF, given that $R_{\mu\nu\rho\sigma} = g_{\mu[\sigma|,\nu|\rho]} - g_{\nu[\sigma|,\mu|\rho]}$, we want to show that $R_{\mu\nu[\rho\sigma;\alpha]} = R_{\mu\nu[\rho\sigma,\alpha]} = 0$.

This is equivalent to the formulation of the Bianchi identities given in the problem sheet, because of the antisymmetry of the Riemann tensor in its last two indices:

there are six terms in the antisymmetrization of $R_{\mu\nu[\rho\sigma,\alpha]}$, but they are pairwise equal: the term $R_{\mu\nu\rho\sigma,\alpha}$ is equal to $-R_{\mu\nu\sigma\rho,\alpha}$ by antisymmetry, and these are exactly the pairs of terms which appear in the three-index antisymmetrization.

What we need to do is to take the derivative of the Riemann tensor in the LIF:

$$R_{\mu\nu\rho\sigma,\alpha} = g_{\mu[\sigma|,\nu|\rho]\alpha} - g_{\nu[\sigma|,\mu|\rho]\alpha}\,, \tag{5.1}$$

and permute the three indices $\rho\sigma\alpha$ cyclically. Writing all the terms out we get (up to a factor 2, which is irrelevant since we will find that all the terms cancel and everything is equal to 0):

$$\begin{aligned} &+g_{\mu\sigma,\nu\rho\alpha} - g_{\nu\sigma,\mu\rho\alpha} - g_{\mu\rho,\nu\sigma\alpha} + g_{\nu\rho,\mu\sigma\alpha} + \\ &+g_{\mu\alpha,\nu\sigma\rho} - g_{\nu\alpha,\mu\sigma\rho} - g_{\mu\sigma,\nu\alpha\rho} + g_{\nu\sigma,\mu\alpha\rho} + \\ &+g_{\mu\rho,\nu\alpha\sigma} - g_{\nu\rho,\mu\alpha\sigma} - g_{\mu\alpha,\nu\rho\sigma} + g_{\nu\alpha,\mu\rho\sigma}\,, \end{aligned} \tag{5.2}$$

so we have 6 terms with a + sign, and 6 with a $-$ sign: they cancel pairwise, since the partial derivatives commute.

5.1.2 Contracting the identities

We start by contracting $2R_{\mu\nu[\rho\sigma;\alpha]}$ with $g^{\mu\rho}$: we get

$$0 = g^{\mu\rho}\left(R_{\mu\nu\rho\sigma;\alpha} + R_{\mu\nu\alpha\rho;\sigma} + R_{\mu\nu\sigma\alpha;\rho}\right) = R_{\nu\sigma;\alpha} - R_{\nu\alpha;\sigma} + g^{\mu\rho}R_{\mu\nu\sigma\alpha;\rho}\,, \tag{5.3}$$

where, in the second term, we used the antisymmetry of the first two indices of the Riemann tensor in order to get the form which allowed us to use the definition of the Ricci tensor $R_{\mu\nu} = g^{\rho\sigma}R_{\rho\mu\sigma\nu}$. Also, we brought the metric inside the covariant derivatives since it is covariantly constant. Then, we contract the expression we found with $g^{\nu\sigma}$:

$$\begin{aligned} 0 = g^{\nu\sigma}\left(R_{\nu\sigma;\alpha} - R_{\nu\alpha;\sigma} + g^{\mu\rho}R_{\mu\nu\sigma\alpha;\rho}\right) &= R_{;\alpha} - R^{\sigma}{}_{\alpha;\sigma} - g^{\mu\rho}R_{\mu\alpha;\rho} & (5.4)\\ &= R_{;\alpha} - R^{\sigma}{}_{\alpha;\sigma} - R^{\sigma}{}_{\alpha;\sigma}\,, & (5.5) \end{aligned}$$

where we used the same properties as before and the definition of the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$. So, we have the contracted Bianchi identities $0 = R_{;\alpha} - 2R^{\sigma}{}_{\alpha;\sigma}$. Raising an index with the inverse metric $g^{\alpha\beta}$ and relabeling σ to α in the second term (after having raised the index), these can be written as

$$\nabla_\alpha\left(Rg^{\alpha\beta} - 2R^{\alpha\beta}\right). \tag{5.6}$$

5.2 Weak-field geodesic equation

This was already treated in section **??**.

5.3 Hyperbolic plane geodesics

Our coordinates are (x, y), and our metric is $g_{ij} = y^{-2}\delta_{ij}$, with inverse $g^{ij} = y^2\delta^{ij}$.

So, we can calculate the Christoffel symbols as:

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \Big(g_{mj,k} + g_{mk,j} - g_{jk,m} \Big) , \tag{5.7}$$

this calculation is simplified by the fact that the only nonvanishing derivatives of the metric are $g_{00,1} = g_{11,1} = -2y^{-3}$. If the index i in Γ^i_{jk} is zero, then the last term in the sum vanishes since it corresponds to a derivative with respect to x. With these we get:

$$\Gamma^0_{00} = \frac{1}{2} y^2 \big(2g_{00,0} \big) = 0 \tag{5.8a}$$

$$\Gamma^0_{01} = \frac{1}{2} y^2 \big(g_{00,1} \big) = -\frac{1}{y} \tag{5.8b}$$

$$\Gamma^0_{11} = \frac{1}{2} y^2 \big(g_{01,1} + g_{01,1} \big) = 0 \tag{5.8c}$$

$$\Gamma^1_{00} = \frac{1}{2} y^2 \big(-g_{00,1} \big) = \frac{1}{y} \tag{5.8d}$$

$$\Gamma^1_{01} = \frac{1}{2} y^2 \big(g_{10,0} + g_{11,0} - g_{01,1} \big) = 0 \tag{5.8e}$$

$$\Gamma^1_{11} = \frac{1}{2} y^2 \big(g_{11,1} + g_{11,1} - g_{11,1} \big) = -\frac{1}{y} , \tag{5.8f}$$

and the geodesic equation $u^\mu \nabla_\mu u^\nu = 0$ is written with respect to these.

5.3.1 Vertical lines

First of all we want to parametrize these vertical lines: we choose our parameter so that the length of the velocity vector is everywhere equal to one.

Since the lines are vertical, we want the position x^i with respect to the parameter s to look something like $x^i(s) = (x_0, y(s))$.

We use the arclength parameter:

$$s = \int \sqrt{g_{ij} u^i u^j} \, \mathrm{d}\lambda \ , \tag{5.9}$$

where $u^i = \mathrm{d}x^i \Big/ \mathrm{d}\lambda$ and λ is an arbitrary parameter.

We can rewrite this integral with respect to the Euclidean norm $\|u\|_E^2 = \delta_{ij} u^i u^j$: we get

$$s = \int \frac{1}{y} \|u\|_E \, \mathrm{d}\lambda \ , \tag{5.10}$$

so we can see that we get $s = \int \mathrm{d}\lambda$, or $s = \lambda$, iff $\|u\|_E = y$: so, let us drop the distinction between s and λ and apply this condition. The velocity vector is

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}s} = \left(0, \frac{\mathrm{d}y}{\mathrm{d}s}\right), \tag{5.11}$$

whose Euclidean norm is just (the absolute value of) $\mathrm{d}y/\mathrm{d}s$. So, we must impose the condition $y = \mathrm{d}y/\mathrm{d}s$, which can be solved by separation of variables to yield $s = \log y$, or $y = e^s$.

So our parametrization for the curve is $s \to x^i = (x_0, e^s)$, the velocity is $u^i = (0, e^s)$ and the derivative of velocity with respect to s is again

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} = (0, e^s). \tag{5.12}$$

Now we can plug these into our geodesic equation, which is simplified by the fact that $u^0 = 0$, therefore there is only one relevant term in the Christoffel sum:

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{11} u^1 u^1 = 0, \tag{5.13}$$

whose components are

$$\underbrace{\frac{\mathrm{d}u^0}{\mathrm{d}s}}_{0} + \underbrace{\Gamma^0_{11}}_{0} y^2 = 0 \tag{5.14}$$

and

$$\frac{\mathrm{d}u^1}{\mathrm{d}s} + \Gamma^1_{11} u^1 u^1 = 0 \tag{5.15a}$$

$$y + \left(-\frac{1}{y}\right) y^2 = 0, \tag{5.15b}$$

which are identities, therefore vertical lines are indeed geodesics in this hyperbolic plane.

5.3.2 More solutions

We have the Killing vector field $\xi = (1, 0)$ corresponding to the symmetry with respect to translations along the x axis: then, the quantity $\vec{\xi} \cdot \vec{u}$ is conserved: so $\xi^\mu u^\nu g_{\mu\nu} = \dot{x} g_{00} = \dot{x} y^{-2}$ is constant along the trajectory (here we denote derivatives with respect to the parameter s by a dot).

We also know that, if we parametrize with arclength, $\|u\|_E / y \equiv 1$, which means

$$y = \|u\|_E = \sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x}\sqrt{\left(y'\right)^2 + 1}, \tag{5.16}$$

where $y' = \mathrm{d}y/\mathrm{d}x = \dot{y}/\dot{x}$.

Then, the conserved quantity can be rewritten as

$$\dot{x}y^{-2} = y^{-2}\frac{y}{\sqrt{y'^2+1}} = \frac{1}{y\sqrt{y'^2+1}}, \tag{5.17}$$

which the exercise sheet calls A but I will call it $1/R$: so $R = y\sqrt{y'^2+1}$. We can square this to write the conservation equation

$$y^2\left(y'^2+1\right) = R^2 = \text{const}. \tag{5.18}$$

Alternatively, once one has found the equations of motion

$$y\ddot{y} = \dot{y}^2 - \dot{x}^2 \tag{5.19a}$$
$$y\ddot{x} = 2\dot{x}\dot{y}, \tag{5.19b}$$

to prove the constancy of $\dot{x}y^{-2}$ one can observe that

$$y^2\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{x}}{y^2}\right) = \ddot{x} - 2\frac{\dot{x}\dot{y}}{y}, \tag{5.20}$$

so $\dot{x}y^{-2}$ being a constant is equivalent to the second equation of motion holding.

5.3.3 Solving the equation

The equation $y^2(y'^2+1) = R^2$ can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\sqrt{\frac{R^2}{y^2}-1}, \tag{5.21}$$

which means that if we fix R the value of the derivative of the curve can only attain two opposite values. Do note that we can go from one branch to the other with the transformation $x \to -x$, a mirror symmetry around some center. Then, we can just make the gauge choice $y' > 0$ and integrate by separation of varibles:

$$\int \frac{y\,\mathrm{d}y}{\sqrt{R^2-y^2}} = \int \mathrm{d}x, \tag{5.22}$$

which can be solved with the substitution $y = R\sin(\theta)$, with $\mathrm{d}y = R\cos(\theta)\,\mathrm{d}\theta$. Inserting this, we find:

$$x - x_0 = \int \frac{R\sin\theta R\cos\theta\,\mathrm{d}\theta}{R\sqrt{1-\sin^2\theta}} = R\int \sin\theta\,\mathrm{d}\theta = -R\cos\theta, \tag{5.23}$$

which can be squared to find $(x - x_0)^2 = R^2(1 - \sin^2\theta) = R^2 - y^2$, or

$$R^2 = (x - x_0)^2 + y^2 \,, \tag{5.24}$$

the equation of a circle. We can then confirm that this solution also holds in the other branch, up to a change of integration constant, by rewriting $(x - x_0)^2 = ((-x) - x_1)^2$: this is solved by $x_0 = -x_1$.

Geometrically, we are looking at circles with origins on the x axis and radius R. If y and R are fixed, then there are only two possible circles, which can be found by connecting a certain point at height y to the x axis with a segment of length R. One can then see that the right halves of the circles we found can be found from the left halves by symmetry.

Sheet 6

6.1 Sphere pole Riemann coordinates

Recall from section **??** the spherical surface metric in the coordinates (θ, φ):

$$g_{\mu\nu} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix} , \tag{6.1}$$

and the Christoffel symbols, which are all zero except for $\Gamma^0_{11} = -\sin\theta\cos\theta$ and $\Gamma^1_{01} = \Gamma^1_{10} = 1/\tan\theta$.

We can plug these into the geodesic equation $u^\mu \nabla_\mu u^\nu$, which comes out to be

$$\ddot{\theta} = +\dot{\varphi}^2 \sin\theta\cos\theta \tag{6.2a}$$

$$\ddot{\varphi} = -2\frac{\dot{\varphi}\dot{\theta}}{\tan(\theta)} , \tag{6.2b}$$

for a trajectory $(\theta(s), \varphi(s))$ with velocity $(\dot{\theta}, \dot{\varphi})$: dots denote derivatives with respect to s.

Now, we want to check whether parallels and meridians are geodesics. First of all, we want to choose a parameter such that the velocity is of constant norm 1: the equation to satisfy is

$$R^2 \begin{bmatrix} \dot{\theta} & \dot{\varphi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} \equiv 1 , \tag{6.3}$$

or $\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta = R^{-2}$. Meridians have constant φ: for them, then, $\dot{\theta}^2 = R^{-2}$, so an appropriate parametrization is $(\theta(s), \varphi(s)) = (s/R, \varphi_0)$. Parallels have constant θ: by an analogous line of reasoning, we parametrize them as $(\theta(s), \varphi(s)) = (\theta_0, s/R\sin\theta)$.

One can readily check that for meridians both of the geodesic equations are identities, while for parallels the second one is an identity but the first reads $0 = \cos\theta / \sin\theta$: it can only be satisfied if $\theta = \pi/2$. This makes sense: geodesics on a sphere are great circles, and the only parallel which is a great circle is the equator.

6.1.1 Riemann coordinates

The coordinates we wish to use are

$$x = R\theta \cos\varphi \tag{6.4a}$$

$$y = R\theta \sin\varphi \,. \tag{6.4b}$$

They are in the form $x^\alpha = \theta n^\alpha$, for vectors $n^\alpha = R(\cos\varphi, \sin\varphi)$. An orthonormal basis for these vectors can be found by selecting $\varphi = 0, \pi/2$.

As we have shown before, the coordinates x^α describe geodesics if we consider them for fixed φ and with parameter θ, since they are meridians.

6.1.2 Metric computation

The metric transforms as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \,, \tag{6.5}$$

so we need the inverse Jacobian, which is expressed in terms of the new coordinates $x'^\mu = (x, y)$ and the old ones $x^\mu = (\theta, \varphi)$:

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \begin{bmatrix} \frac{1}{R}\frac{x}{\sqrt{x^2+y^2}} & \frac{1}{R}\frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y/x^2}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2} \end{bmatrix} , \tag{6.6}$$

so we get, using the expansion $\sin^2\theta \sim \theta^2 - \theta^4/3 + O(\theta^6)$ and the identification $R^2\theta^2 = x^2 + y^2$:

$$g'_{00} = R^2 \left(\frac{\partial x^0}{\partial x'^0} \right)^2 + R^2 \sin^2\theta \left(\frac{\partial x^1}{\partial x'^0} \right)^2 \tag{6.7a}$$

$$= \frac{x^2}{x^2+y^2} + R^2 \sin^2\theta \frac{y^2}{x^4 \big(1 + (y/x)^2\big)^2} \tag{6.7b}$$

$$= \frac{1}{x^2+y^2} \left(x^2 + \frac{R^2 \sin^2\theta y^2}{x^2+y^2} \right) \tag{6.7c}$$

$$= 1 - \frac{y^2}{3R^2} + O((x^2+y^2)y^2) \,, \tag{6.7d}$$

while for the off-diagonal elements $g'_{01} = g'_{10}$:

$$
\begin{aligned}
g'_{01} &= R^2 \frac{\partial x^0}{\partial x'^0}\frac{\partial x^0}{\partial x'^1} + R^2 \sin^2\theta \frac{\partial x^1}{\partial x'^0}\frac{\partial x^1}{\partial x'^1} && (6.8a)\\
&= \frac{xy}{x^2+y^2} + R^2\sin^2\theta\left(-\frac{xy}{(x^2+y^2)^2}\right) && (6.8b)\\
&= \frac{xy}{x^2+y^2}\left(1 - \frac{R^2\sin^2\theta}{x^2+y^2}\right) && (6.8c)\\
&= \frac{xy}{x^2+y^2}\left(\frac{(x^2+y^2)^2}{3R^2(x^2+y^2)}\right) + O(x^2+y^2) && (6.8d)\\
&= \frac{xy}{3R^2} + O(x^2+y^2)\,, && (6.8e)
\end{aligned}
$$

and lastly for the element g'_{11}:

$$
\begin{aligned}
g'_{11} &= R^2\left(\frac{\partial x^0}{\partial x'^1}\right)^2 + R^2\sin^2\theta\left(\frac{\partial x^1}{\partial x'^1}\right)^2 && (6.9a)\\
&= \frac{y^2}{x^2+y^2} + R^2\sin^2\theta\frac{1}{x^2\big(1+(y/x)^2\big)^2} && (6.9b)\\
&= \frac{1}{x^2+y^2}\left(y^2 + R^2\sin^2\theta\frac{x^2}{x^2+y^2}\right) && (6.9c)\\
&= 1 - \frac{x^2}{3R^2} + O((x^2+y^2)x^2)\,. && (6.9d)
\end{aligned}
$$

At the north pole $x = y = 0$, so there $g'_{\mu\nu} = \delta_{\mu\nu}$, and all the first derivatives calculated there vanish since there are no first order terms.

6.1.3 Scalar curvature calculation

The expression we have for the scalar curvature in a LIF is given in equation (**??**).

We can evaluate it for $g'_{\mu\nu}$. Do note that the non-differentiated metric can be identified with the identity, and derivatives with upper and lower indices are the same. So, we get:

$$
R_{\mathrm{Ric}} = g_{\alpha\nu,}{}^{\alpha\nu} - \delta^{\mu\nu}\Box g_{\mu\nu}\,, \tag{6.10}
$$

where the only nonvanishing terms are:

$$
g_{\alpha\nu,}{}^{\alpha\nu} = 2\frac{\partial^2}{\partial x \partial y}\left(\frac{xy}{3R^2}\right) = \frac{2}{3R^2} \tag{6.11}
$$

and what was denoted as the Dalambertian before is just the Laplacian: $\Box = \delta^{\mu\nu}\partial_\mu\partial_\nu = \partial^2_{xx} + \partial^2_{yy}$:

$$\delta^{\mu\nu}\Box g_{\mu\nu} = \partial^2_{xx} g_{11} + \partial^2_{yy} g_{00} = -2\frac{2}{3R^2}\,, \tag{6.12}$$

so in the end we get

$$R_{\text{Ric}} = \frac{2}{3R^2} + \frac{4}{3R^2} = \frac{2}{R^2}\,, \tag{6.13}$$

which means that the curvature decreases as the radius increases, as we might expect.

6.2 Schwarzschild metric curvature

6.2.1 Christoffel symbols

The computation is tedious and not particularly enlightening: we start from the metric [3]

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + r^2\Big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\Big)\,, \tag{6.14}$$

and compute the Christoffel symbols with the usual formula:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}\Big(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}\Big)\,, \tag{6.15}$$

where fortunately, since the metric is diagonal, we only need to compute one term in the sum (that is, $\mu \equiv \alpha$ always), and the two indices in the metric must be equal in order for the term to not vanish.

The metric only depends on θ and r, so any derivatives with respect to t and φ are to be discarded.

With this out of the way, we start computing the 40 independent symbols and find that the nonzero ones are (denoting differentiation with respect to r with a prime):

$$\Gamma^t_{rt} = \frac{A'}{2A} \qquad \Gamma^r_{rr} = \frac{B'}{2B} \tag{6.16a}$$

$$\Gamma^r_{tt} = \frac{A'}{2B} \qquad \Gamma^\theta_{r\theta} = \frac{1}{r} \tag{6.16b}$$

$$\Gamma^r_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \tag{6.16c}$$

$$\Gamma^r_{\varphi\varphi} = -\frac{r}{B}\sin^2\theta \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \tag{6.16d}$$

$$\Gamma^\varphi_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}\,. \tag{6.16e}$$

[3] There is a typo in the homework assignment: the coefficients are written as functions of time.

6.2.2 Ricci component

We want to compute $R^\mu_{t\mu t}$, and in order to do so we must find the three components R^i_{tit} with varying i and sum them (here $i = r, \theta, \varphi$), since R^t_{ttt} vanishes by antisymmetry.

In general we have:

$$R^i_{tit} = 2\left(\Gamma^i_{[t|t,|i]} + \Gamma^\alpha_{t[t}\Gamma^i_{i]\alpha}\right) \tag{6.17a}$$

$$= \Gamma^i_{tt,i} - \cancel{\Gamma^i_{it,t}} + \Gamma^\alpha_{tt}\Gamma^i_{i\alpha} - \Gamma^\alpha_{ti}\Gamma^i_{t\alpha}\,. \tag{6.17b}$$

Note that the index i is consider not to be summed here, we are writing a formula for the components of the Riemann tensor; although the expression holds when summing over i as well.

So, we can compute this for the specific values of i: for $i = r$ we have

$$R^r_{trt} = \Gamma^r_{tt,r} + \Gamma^\alpha_{tt}\Gamma^r_{r\alpha} - \Gamma^\alpha_{tr}\Gamma^r_{t\alpha} \tag{6.18a}$$

$$= \left(\frac{A'}{2B}\right)' + \frac{A'}{2B}\frac{B'}{2B} - \frac{A'}{2A}\frac{A'}{2B} \tag{6.18b}$$

$$= \frac{A''}{2B} - \frac{A'}{2B^2}B' + \frac{A'}{4B}\left(\frac{B'}{B} - \frac{A'}{A}\right) \tag{6.18c}$$

$$= \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right), \tag{6.18d}$$

for $i = \theta$ instead

$$R^\theta_{t\theta t} = \cancel{\Gamma^\theta_{tt,\theta}} + \Gamma^\alpha_{tt}\Gamma^\theta_{\theta\alpha} - \cancel{\Gamma^\alpha_{t\theta}\Gamma^i_{t\alpha}} \tag{6.19a}$$

$$= \frac{A'}{2B}\frac{1}{r}\,, \tag{6.19b}$$

and for $i = \varphi$:

$$R^\varphi_{t\varphi t} = \cancel{\Gamma^\varphi_{tt,\varphi}} + \Gamma^\alpha_{tt}\Gamma^\varphi_{\varphi\alpha} - \cancel{\Gamma^\alpha_{t\varphi}\Gamma^i_{t\alpha}} \tag{6.20a}$$

$$= \frac{A'}{2B}\frac{1}{r}\,, \tag{6.20b}$$

so our final solution is

$$R_{00} = \sum_i R^i_{0i0} = \frac{A''}{2B} - \frac{A'}{4B}\frac{(AB)'}{AB} + \frac{A'}{Br} \tag{6.21}$$

$$= \frac{A''}{2B} - \frac{A'^2}{4AB} - \frac{A'B'}{4B^2} + \frac{A'}{Br}\,. \tag{6.22}$$

6.3 Schwarzschild geometry orbits

The derivation up to the equation for the perturbed orbit equation is documented in the lecture notes, I might copy it here later, but for now one can find it there.

During the lecture we got up to the first order equation for the perturbation w for the orbit u, written in the form $u(\varphi) = u_c\big(1 + w(\varphi)\big)$:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} = (6GMu_c - 1)w\,, \tag{6.23}$$

which is in the form $\ddot{w} + \omega^2 w = 0$, for $\omega^2 = 1 - 6GMu_c$. Now, we know that the first order equation must be complemented by the zeroth order one:

$$u_c = \frac{GM}{l^2} + 3GMu_c^2\,, \tag{6.24}$$

which can be solved for u_c to yield:

$$u_c = \frac{1 \pm \sqrt{1 - 3 \times 4\frac{G^2M^2}{l^2}}}{6GM}\,, \tag{6.25}$$

therefore the square angular velocity of the perturbation's evolution is:

$$\omega^2 = 1 - \cancel{6GM}\left(\frac{1 \pm \sqrt{1 - 12\frac{G^2M^2}{l^2}}}{\cancel{6GM}}\right) = \pm\sqrt{1 - 12\frac{G^2M^2}{l^2}}\,. \tag{6.26}$$

The solution with the minus sign has no meaning for us, since the solution we want to consider must be stable, with positive ω^2. So, the angular velocity is

$$\omega = \left(1 - 12\frac{G^2M^2}{l^2}\right)^{1/4}\,, \tag{6.27}$$

and we know that angular velocity and period are related by $T = 2\pi/\omega$: therefore we get

$$T = 2\pi\left(1 - 12\frac{G^2M^2}{l^2}\right)^{-1/4}\,, \tag{6.28}$$

which we can Taylor expand: at $x = 0$ we have

$$(1 - 12x)^{-1/4} = 1 - \frac{1}{4}(1 - 12 \times 0)^{-5/4}(-12x) + O(x^2) = 1 + 3x + O(x^2)\,. \tag{6.29}$$

Therefore:

$$T = 2\pi \left(1 + 3 \left(\frac{GM}{l} \right)^2 \right) + O\left(\left(\frac{GM}{l} \right)^4 \right), \tag{6.30}$$

which is approximately 2π as we should expect: the Newtonian approximation is $l \gg GM$, and Newtonian orbits have a period of exactly 2π. Then we can read off the first-order correction directly from the first term in the expansion: it is

$$\delta\varphi = 6\pi \left(\frac{GM}{l} \right)^2 . \tag{6.31}$$

The M here is the mass of the central object, while l is the angular momentum of the orbit.

Sheet 7

7.1 Photons travelling in the Schwarzschild metric

7.1.1 A different proof for the conservation of the component of the velocity of a geodesic along a Killing vector field (complement)

Here I present a different proof to what was done in the lectures for the fact that the component of the 4-velocity along the Killing vector field is conserved. This is not necessary to know for the exam, do skip this section if it does not interest you.

If the metric does not depend on the coordinate $\widetilde{\alpha}$, then $\partial_{\widetilde{\alpha}} g_{\mu\nu} = 0$. So, let us differentiate covariantly the vector $\xi_\mu = g_{\mu\nu}\delta^\nu_{\widetilde{\alpha}}$. It will be apparent later that differentiating the lower-index vector field gives us the interesting property. We get

$$\nabla_\mu \xi_\nu = g_{\nu\sigma} \nabla_\mu \xi^\sigma = g_{\nu\sigma} \left(\cancel{\partial_\mu \xi^\sigma} + \Gamma^\sigma_{\mu\rho} \xi^\rho \right) = \Gamma_{\nu\mu\widetilde{\alpha}} , \tag{7.1}$$

since the only component which survives the contraction with ξ is the one along $\widetilde{\alpha}$; also, we lowered an index of the Christoffel symbols with the metric.

The explicit expression for the lower indices Christoffel symbols is

$$\Gamma_{\nu\mu\widetilde{\alpha}} = \frac{1}{2} \left(\cancel{g_{\mu\nu,\widetilde{\alpha}}} + g_{\mu\widetilde{\alpha},\nu} - g_{\mu\widetilde{\alpha},\nu} \right) , \tag{7.2}$$

since by hypothesis any derivative of the metric along $\widetilde{\alpha}$ is zero. So, we can directly see that the object $\nabla_\mu \xi_\nu$ is antisymmetric in its indices: this can be written as

$$\nabla_{(\mu} \xi_{\nu)} = 0 , \tag{7.3}$$

and is called *Killing's equation*. We have shown that is equivalent to the metric not depending on the coordinate $x^{\widetilde{\alpha}}$.

Now, we can quickly prove the conservation a component of the 4-velocity of a geodesic along the Killing vector field: we just need to differentiate $u^\mu \xi_\mu$ with respect to the arc parameter. Recall that geodesics are defined by the equation

$$a^\mu = \frac{\mathrm{d}}{\mathrm{d}s} u^\mu = u^\nu \nabla_\nu u^\mu = 0 \,. \tag{7.4}$$

We find

$$u^\nu \nabla_\nu \left(u^\mu \xi_\mu \right) = u^\nu u^\mu \nabla_\nu \xi_\mu + \xi_\mu u^\nu \nabla_\nu u^\mu = 0 \,, \tag{7.5}$$

where both terms are zero: the first because it is the contraction of an antisymmetric object with a symmetric one, and the second one because of the geodesic equation.

7.1.2 Conserved quantities in Schwarzschild motion

The Schwarzschild metric is given by

$$g_{\mu\nu} = \begin{bmatrix} -(1-\frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{2GM}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix} \tag{7.6}$$

in the coordinates (t, r, θ, φ). The vector fields $\xi^\mu_{(t)} = (1, \vec{0})$ and $\xi^\mu_{(\varphi)} = (0, 0, 0, 1)$ in these coordinates are Killing vector fields, since the metric does not depend on t or φ.

So, the following quantities are conserved in geodesic motion parametrized as $x^\mu(\lambda)$: [4]

$$e = -u^\mu g_{\mu\nu} \xi^\nu_{(t)} = -u^t g_{tt} \times 1 = \frac{\mathrm{d}t}{\mathrm{d}\lambda} \left(1 - \frac{2GM}{r} \right) \tag{7.7}$$

and

$$l = u^\mu g_{\mu\nu} \xi^\nu_{(\varphi)} = u^\varphi g_{\varphi\varphi} \times 1 = \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} r^2 \sin^2\theta \,, \tag{7.8}$$

which for motion on the xy plane, for which $\theta = \pi/2$, reduces to $l = r^2 \, \mathrm{d}\varphi / \mathrm{d}\lambda$.

[4] There is a typo in the exercise sheet: a G is missing in the definition of e.

7.1.3 Photons escaping a black hole

The equation of motion can be derived from the normalization of the photon's four velocity: The equation $u^\mu u_\mu = 0$ can be written as

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 g_{tt} + \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 g_{rr} + \left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 g_{\theta\theta} + \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\right)^2 g_{\varphi\varphi} = 0\,, \tag{7.9}$$

but the term $\mathrm{d}\theta/\mathrm{d}\lambda$ is zero if we assume the motion to be in the xy plane, while the velocity components along t and φ can be written in terms of the integrals of motion: $\mathrm{d}t/\mathrm{d}\lambda = e(1-2GM/r)^{-1}$ and $\mathrm{d}\varphi/\mathrm{d}\lambda = lr^{-2}$. So, we find

$$-\left(1-\frac{2GM}{r}\right)^{-2+1} e^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + l^2 r^{-4} r^2 = 0\,, \tag{7.10}$$

we divide through by $-g_{tt}$ and find:

$$-e^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{l^2}{r^2}\left(1-\frac{2GM}{r}\right) = 0\,, \tag{7.11}$$

or, dividing through by l:

$$-\frac{e^2}{l^2} + \frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{1}{r^2} - \frac{2GM}{r^3} = 0\,. \tag{7.12}$$

We can give names to the terms in this equation: we call

$$V_{\mathrm{eff}}(r) \equiv \frac{1}{r^2} - \frac{2GM}{r^3} \tag{7.13}$$

the *effective potential*, and

$$b^2 = \frac{l^2}{e^2} \tag{7.14}$$

the *impact parameter* (unjustified for now). Then, the equation is in the form

$$\frac{\dot{r}^2}{l^2} + V_{\mathrm{eff}}(r) = \frac{1}{b^2}\,, \tag{7.15}$$

where we denoted derivation with respect to λ with a dot. So, we can study the motion of the photon as if it were 1-dimensional.

To study the problem, it is convenient to use the rescaled adimensional radial coordinate $R = r/2GM$. In this variable, the effective potential (which I will denote as just V hereafter) looks like:

$$V(R) = (2GM)^{-2}\left(R^{-2} - R^{-3}\right)\,, \tag{7.16}$$

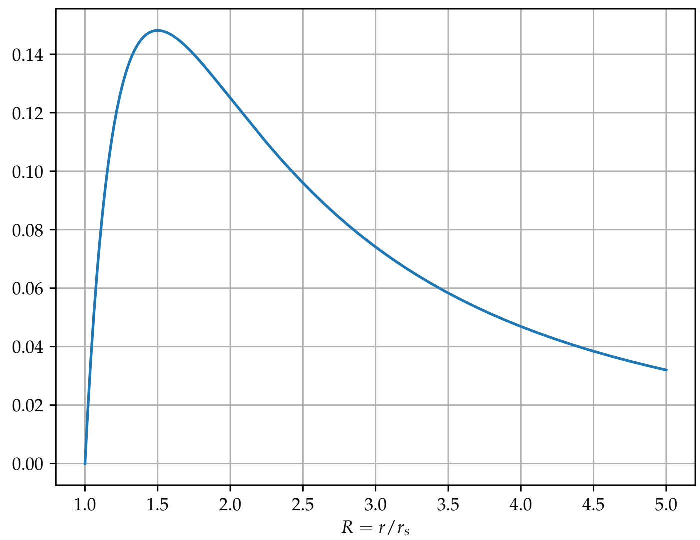

Figure 3: Plot of the function $R^{-2} - R^{-3}$.

so we can readily differentiate it to find its stationary points: there is only one, the equation is $V'(R) \propto -2R^{-1} + 3R^{-2} = 0$, which is satisfied by $R = 3/2$.

The term b^{-2} is the maximum value which can be attained by the LHS: we can call it the total energy, the kinetic and potential contributions must add up to it. Also, the kinetic term is always positive. So, if the total energy is less than the maximum of the potential, the photon is constrained to stay on either side of the potential barrier around $R = 3/2$. The potential there equals

$$V(3/2) = (2GM)^{-2}\Big((3/2)^{-2} - (3/2)^{-3}\Big) = \frac{4/27}{(2GM)^2} = \frac{1}{27(GM)^2}\,, \tag{7.17}$$

so the condition of the photon being above the potential barrier is

$$E_{\text{tot}} > V_{\text{max}} \tag{7.18}$$

$$\frac{e^2}{l^2} > \frac{1}{27G^2M^2} \tag{7.19}$$

$$\frac{l^2}{e^2} < 27G^2M^2\,, \tag{7.20}$$

and under this condition, if the photon initially has positive $\mathrm{d}r/\mathrm{d}\lambda$ it will remain as such, since everything is continuous and (if the strict inequality is satisfied) we can never have $\mathrm{d}r/\mathrm{d}\lambda = 0$.

If instead we had $l^2/e^2 < 27G^2M^2$ and the photon was initially travelling away from the black hole, there would come a point for which $\mathrm{d}r/\mathrm{d}\lambda$ would equal zero, and then it would become negative, since the photon could not go away from the center anymore.

This can be understood graphically by drawing horizontal lines of constant total energy in the potential diagram.

7.1.4 A basis for a stationary observer

If our observer's coordinates are $(t, r_*, \pi/2, \varphi_*)$, that is, it is at rest with respect to our spatial coordinates but it is not following geodesic motion, then it will have nonzero 4-acceleration. So, since we know that velocity and acceleration are orthogonal, we can form our coordinate system as $(u^\mu, a^\mu/\sqrt{a^\rho a_\rho}, e_\theta, e_\varphi)$, where the angular basis vectors are simply normalized vectors in the θ and φ directions.

This method has the advantage of being easily generalizable to find a comoving basis for any non-geodesic motion of our observer.

Their 4-velocity must be normalized so that $u^\mu u_\mu = -1$: so

$$u^\mu = \begin{bmatrix} \frac{\mathrm{d}t}{\mathrm{d}\tau} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{1 - \frac{2GM}{r}} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{7.21}$$

then we can compute the 4-acceleration: it will be

$$a^\mu = u^\nu \nabla_\nu u^\mu = u^\nu \left(\partial_\nu u^\mu + \Gamma^\mu_{\nu\rho} u^\rho \right) \tag{7.22}$$

$$= \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \left(\cancel{\partial_t u^\mu} + \Gamma^\mu_{tt} u^t \right) \tag{7.23}$$

$$= \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \frac{A'}{2B} \delta^\mu_r, \tag{7.24}$$

where $A = 1/B = (1 - 2GM/r)$ are the coefficients of the Schwarzschild metric, with respect to which the Christoffel symbols are expressed in (**??**). The δ^μ_r signifies that the only component which survives is the radial one. The two square root terms simplify the B, so we get that the only nonzero component of the 4-acceleration is:

$$a^r = \frac{A'}{2} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}r} \left(1 - \frac{2GM}{r} \right) = \frac{GM}{r^2}. \tag{7.25}$$

The actual value of this component is not actually useful to us, since we need a normalized basis. It can, however, be used to illustrate the equivalence principle: in the weak-field limit where the metric is approximately the Minkowski one the value of this component approaches the modulus of the 3-acceleration.

Therefore, we will use a vector parallel to a^μ but of length defined by the normalization $e_r \cdot e_r = 1$. So, we get for our basis:

$$\begin{cases} (e_t)^\mu = u^\mu = \left[\ (1-2GM/r)^{-1/2},\ \ 0,\ \ 0,\ \ 0\ \right]^\top \\ (e_r)^\mu = a^\mu/\sqrt{a_\rho a^\rho} = \left[\ 0,\ \ (1-2GM/r)^{1/2},\ \ 0,\ \ 0\ \right]^\top \\ (e_\theta)^\mu = \left[\ 0,\ \ 0,\ \ 1/r,\ \ 0\ \right]^\top \\ (e_\varphi)^\mu = \left[\ 0,\ \ 0,\ \ 0,\ \ 1/(r\sin\theta)\ \right]^\top \end{cases}, \tag{7.26}$$

where the last vector is written for a generic angle θ, but the sine is equal to one if $\theta = \pi/2$. This basis satisfies the property $e_\alpha \cdot e_\beta = \eta_{\alpha\beta}$.

We will denote vectors written with respect to this basis with a subscript c.

7.1.5 Critical angle of launch

In the flat frame, the 4-velocity of the photon launched at $\theta = \pi/2$ looks like:

$$u^\mu_{\text{ph}} = \begin{bmatrix} 1 \\ \cos\psi \\ 0 \\ \sin\psi \end{bmatrix}_c , \tag{7.27}$$

which means that in the Schwarzschild frame it looks like

$$u^\mu_{\text{ph}} = \begin{bmatrix} \frac{1}{\sqrt{1-\frac{2GM}{r}}} \\ \cos\psi\sqrt{1-\frac{2GM}{r}} \\ 0 \\ \frac{\sin\psi}{r} \end{bmatrix} . \tag{7.28}$$

This means that the parameter $l = r\sin\psi$, while $e = \sqrt{1-2GM/r}$.

The inequality to satisfy for the photon to be able to escape is:

$$\left[\frac{e^2}{l^2}\right]_{r_*} \geq \frac{1}{27G^2M^2}, \tag{7.29}$$

which translates to

$$\frac{1-\frac{2GM}{r_*}}{r_*^2\sin^2\psi} \geq \frac{1}{27G^2M^2}, \tag{7.30}$$

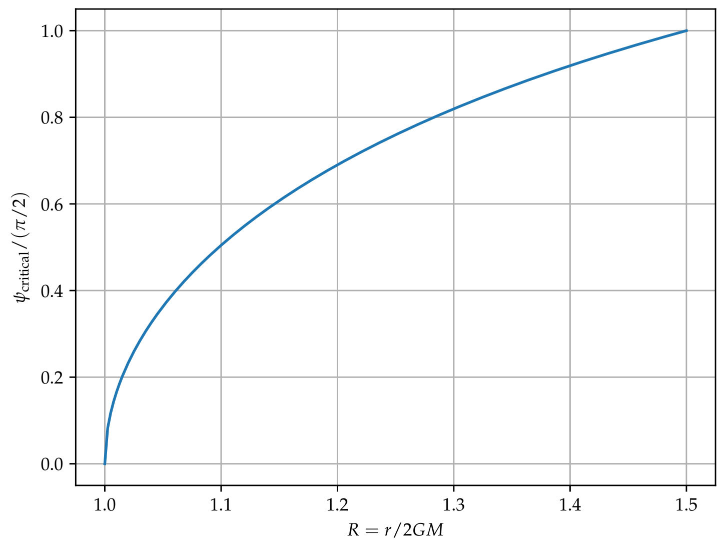

Figure 4: Critical angle ψ in terms of the adimensional radial coordinate $R = r/2GM$.

or

$$r_*^2 \sin^2 \psi \leq 27G^2M^2(1 - 2GM/r_*)\,, \tag{7.31}$$

which once again can be expressed in terms of the adimensional radial coordinate $R = r/2GM$: we find

$$\sin^2 \psi \leq \frac{27}{4R_*^2}\left(1 - \frac{1}{R_*}\right), \tag{7.32}$$

so the critical angle can be computed by making the relation explicitly in terms of ψ, which gives the plot shown in figure **??**, for the equation

$$\psi = \arcsin \sqrt{\frac{27}{4R_*^2}\left(1 - \frac{1}{R_*}\right)}\,. \tag{7.33}$$

So, as we move closer to the horizon, the range of angles at which we can throw our photon and still have it escape decreases, until finally we can only throw it straight forward otherwise it will fall back in. After $r = 2GM$, not even that is enough.

All communication with the outside world is lost.

7.2 Light motion in a Schwarzschild geometry

7.2.1 Equation of motion

This was done during the lectures, but I will recall it here.

We start from the equation of motion of a photon, a reframing of $u^2 = 0$:

$$\frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + V_{\text{eff}}(r) = \frac{1}{b^2} = \frac{e^2}{l^2}\,. \tag{7.34}$$

We want to write this as a differential equation for the radius in terms of the angle φ: so, we use the expression of the first integral

$$l = \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}r^2 \implies \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \frac{l}{r^2} \implies \frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\,. \tag{7.35}$$

So, we can rewrite

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{l^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2, \tag{7.36}$$

which allows us to write the equation as

$$\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + V_{\text{eff}}(r) = \frac{1}{b^2}\,, \tag{7.37}$$

since the l^2 simplifies.

Now, notice that if we define $u = 1/r$ we find

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}\,, \tag{7.38}$$

which conveniently simplifies the r^{-4}: expressing everything with respect to u we get

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + \left(u^2 - 2GMu^3\right) = \frac{1}{b^2}\,. \tag{7.39}$$

Now, we just need to differentiate everything to eliminate the constant term and find

$$2u'u'' + 2uu' - 6GMu^2u' = 0\,, \tag{7.40}$$

where we denoted derivatives with respect to φ with primes. One solution is $u' = 0$: a circular orbit, which we are not interested in now (and which is unstable. . .).

Otherwise, we can simplify a factor $2u'$: we find

$$u'' + u = 3GMu^2 \tag{7.41}$$

as we wanted to show.

7.2.2 Zero black hole mass solution

If $M = 0$, our diffential equation is simply $u'' + u = 0$, a harmonic oscillator. Our boundary condition is $r(\varphi = 0, \pi) \to \infty$, which means $u \to 0$ in those cases.

Also, one can geometrically see that at any radius $\sin\varphi = b/r$ where b is the impact parameter. Therefore, $bu = \sin\varphi$ or $u = \sin(\varphi)/b$. This actually is already a solution to our differential equation!

We are giving two boundary conditions, therefore the solution $u = \sin(\varphi)/b$ is unique.

7.2.3 Small mass deflection

We now insert a mass M, and consider a solution which is a perturbation to the sinusoidal one: our proposed solution is $u = b^{-1}(\sin\varphi + w)$, and we will only look at the first order in w. Substituting into $u'' + u = 3GMu^2$ we get

$$-\sin\varphi + w'' + \sin\varphi + w = \frac{3GM}{b}\left(\sin^2\varphi + \cancel{2w\sin\varphi} + \cancel{w^2}\right), \tag{7.42}$$

where we kept only terms which are of first order in either GM or w, since both are small relative to the scale of the problem. Also, we simplified a common factor of b. So the differential equation for w is

$$w'' + w = \frac{3GM}{b}\sin^2\varphi\,. \tag{7.43}$$

This can be solved with an *ansatz* in the form $w = A + B\sin^2\varphi$. Its derivatives are $w' = 2B\sin\varphi\cos\varphi = B\sin(2\varphi)$, and $w'' = 2B\cos(2\varphi) = 2B\left(1 - 2\sin^2\varphi\right)$.

Plugging this in we find

$$2B(1 - 2\sin^2\varphi) + A + B\sin^2\varphi = \frac{3GM}{b}\sin^2\varphi\,, \tag{7.44}$$

so equating the terms in $\sin^2\varphi$ and the constant ones we get

$$\begin{cases} 2B + A = 0 \\ -3B = 3GMb^{-1} \end{cases}, \tag{7.45}$$

so our perturbation is $w = 2GMb^{-1}(1 - \sin^2\varphi/2)$, therefore the full solution is

$$u(\varphi) = \frac{1}{b}\left(\sin\varphi + \frac{2GM}{b}\left(1 - \frac{\sin^2\varphi}{2}\right)\right). \tag{7.46}$$

What follows is an alternative derivation, more complicated than plainly discarding the second-order $\sin^2 \varphi$ term and expanding the sine to first order, which gives $\varphi \sim -2GM/b$ right away. I did it in this way mostly to check that it still works.

We want to solve the equation of the particle coming in from radial infinity at an angle φ_{in}, and leaving at an angle φ_{out} towards radial infinity. This means that we are seeking two solutions to $r = \infty \implies u = 0$, respectively near $\varphi = 0$ and $\varphi = \pi$. This can be written as a second degree equation in terms of the variable $x = \sin\varphi$, and the adimensionalized impact parameter $b/2GM = d$, by which we multiply everything to get:

$$x^2 - 2dx - 2 = 0\,, \tag{7.47}$$

which can be solved as

$$x = d \pm \sqrt{d^2 + 2} = d\left(1 \pm \sqrt{1 + \frac{2}{d^2}}\right), \tag{7.48}$$

so we have two solutions: one near $x = 0$, one near $x = 2d$. The second is meaningless, since $d \gg 1$ but $x \leq 1$. Expanding around $d = \infty$ we find:

$$x = d\left(1 - \left(1 + \frac{1}{2}\frac{2}{d^2}\right)\right) = -\frac{d}{2}\frac{2}{d^2} = -\frac{1}{d} = -\frac{2GM}{b} = \sin\varphi\,. \tag{7.49}$$

So, we want solutions to this near $\varphi = 0$ and $\varphi = \pi$, and we are working up to first order in φ. Then, we have

$$\varphi_{\text{in}} = -\arcsin\left(\frac{2GM}{b}\right) \sim -\frac{2GM}{b} \tag{7.50}$$

and

$$\varphi_{\text{out}} = \pi + \arcsin\left(\frac{2GM}{b}\right) \sim \pi + \frac{2GM}{b}\,. \tag{7.51}$$

The deflection can be calculated by assuming $\varphi_{\text{out}} - \varphi_{\text{in}} = \pi + \delta\varphi$, which gives

$$\delta\varphi \approx \frac{4GM}{b}\,. \tag{7.52}$$

7.2.4 Newtonian prediciton and Eddington observations (complement)

The Newtonian prediction, which is computed using the Keplerian formula for eccentricity (and, notably, simplifying the "mass" of a light particle) is instead:[5]

$$\delta\varphi \approx \frac{2GM}{b}\,. \tag{7.53}$$

[5] https://arxiv.org/pdf/physics/0508030.pdf

During the solar eclipse of 1919, Sir Eddington[6] made two observations: one gave

$$\delta\varphi = (4.5 \pm 0.3)\frac{GM}{b}, \tag{7.54}$$

and another gave

$$\delta\varphi = (3.7 \pm 0.7)\frac{GM}{b}. \tag{7.55}$$

7.3 Orbit at 7GM

7.3.1 Orbital radius

The radius of a $7GM_\odot$ orbit is equal to

$$7GM_\odot/c^2 \approx \tag{7.56}$$

$$7 \times 6.67 \times 10^{-11}\,\mathrm{kgm^3s^{-2}} \times 2 \times 10^{30}\,\mathrm{kg} \times \left(3 \times 10^8\,\mathrm{ms^{-1}}\right)^{-2} \approx \tag{7.57}$$

$$\approx 1.0 \times 10^4\,\mathrm{m}, \tag{7.58}$$

or around 10 km.

7.3.2 Proper period

The equation governing a circular orbit around a BH can be derived from the normalization of the 4-velocity $u \cdot u = -1$: in terms of $u = 1/r$ it is

$$u_c = \frac{GM}{l^2} + 3GMu_c^2, \tag{7.59}$$

and in our case we know that $u_c = 1/7GM$, substituting in we find

$$\frac{1}{7GM} = \frac{GM}{l^2} + \frac{3GM}{49(GM)^2}, \tag{7.60}$$

or

$$l^2 = (GM)^2\left(\frac{1}{7} - \frac{3}{49}\right)^{-1} = 7(GM)^2\left(1 - \frac{3}{7}\right)^{-1} = \frac{49}{4}(GM)^2, \tag{7.61}$$

so $l = 7GM/2$. Now, recall the definition of this first integral:

$$l = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}r^2, \tag{7.62}$$

[6] https://royalsocietypublishing.org/doi/abs/10.1098/rsta.1920.0009

into which we can substitute the expression we found:

$$\frac{7GM}{2}(7GM)^{-2} = \frac{1}{14GM} = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \omega_{\text{proper}}\,, \tag{7.63}$$

which gives us the angular velocity as measured by the orbiting observer. We actually have it in units of $1/\mathrm{m}^3\mathrm{s}^{-2}$, so we will need to multiply by c^3 to get inverse seconds.

This gives us a pulsation of

$$\omega_{\text{proper}} = \frac{c^3}{14GM_\odot} \approx 1.44 \times 10^4\,\mathrm{rad/s}\,, \tag{7.64}$$

which corresponds to a period of

$$T_{\text{proper}} = \frac{2\pi}{\omega_{\text{proper}}} \approx 4.36 \times 10^{-4}\,\mathrm{s} = 436\,\mu\mathrm{s}\,. \tag{7.65}$$

7.3.3 Period at infinity

Now, to compute the period for an observer at infinity: from the regular equation of motion of a massive observer we have

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \underbrace{\left(-\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}\right)}_{V_{\text{eff}}}, \tag{7.66}$$

but since the orbit is circular r is constant so the derivative vanishes, and we can substitute in our expressions for r and l to find out what e is: we get

$$\frac{e^2-1}{2} = -\frac{GM}{7GM} + \frac{(7GM/2)^2}{2\times(7GM)^2} - \frac{GM(7GM/2)^2}{(7GM)^3} \tag{7.67}$$

$$= -\frac{1}{7} + \left(\frac{7}{2}\right)^2\frac{1}{2\times 7^2} - \left(\frac{7}{2}\right)^2\frac{1}{7^3} \tag{7.68}$$

$$= -\frac{3}{56}, \tag{7.69}$$

therefore $e = 5\sqrt{7}/14$. Now recall the definition of e:

$$e = \frac{\mathrm{d}t}{\mathrm{d}\tau}\left(1 - \frac{2GM}{r}\right), \tag{7.70}$$

so

$$\frac{T_\infty}{T_{\text{proper}}} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{5\sqrt{7}}{14}\left(1 - \frac{2GM}{7GM}\right)^{-1} = \frac{\sqrt{7}}{2} \approx 1.323\,, \tag{7.71}$$

therefore the period as measured by the outside observer is

$$T_\infty \approx 577\,\mu\text{s}\,. \tag{7.72}$$

Do note that this is *not* the same as plainly applying the gravitational redshift formula as $T_1/T_2 = \sqrt{g^1_{tt}/g^2_{tt}}$: that gives a smaller ratio. The Doppler effects of the orbital speed do not average out over a period: this is treated in more depth in section **??**.

An alternative way to derive $\mathrm{d}t/\mathrm{d}\tau$ is to use Ω: we defined it as $\mathrm{d}\varphi/\mathrm{d}t$, and proved that it is given by

$$\Omega^2 = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = \frac{GM}{r^3}\,. \tag{7.73}$$

It allows us to write the 4-velocity as

$$u^\alpha = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, 0, 0, \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^\top = \frac{\mathrm{d}t}{\mathrm{d}\tau}(1, 0, 0, \Omega)^\top\,. \tag{7.74}$$

Now, we can write the proper time interval starting from the line element:

$$-\mathrm{d}s^2 = \mathrm{d}\tau^2 = g_{00}\,\mathrm{d}t^2 - g_{33}\,\mathrm{d}\varphi^2 = \mathrm{d}t^2\left(g_{00} - g_{33}\Omega^2\right), \tag{7.75}$$

which we can calculate, since we know all the parameters: we find

$$\frac{\mathrm{d}\tau^2}{\mathrm{d}t^2} = \left(1 - \frac{2GM}{r} - r^2\frac{GM}{r^3}\right) = \left(1 - \frac{3GM}{r}\right). \tag{7.76}$$

This gives

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{\sqrt{1 - \frac{3GM}{r}}} \sim \mathrm{d}\tau\left(1 + \frac{3GM}{2r}\right). \tag{7.77}$$

7.3.4 Proper acceleration

The orbit is a geodesic, so a pointlike observer feels no acceleration.

7.3.5 Adimensionalized parameters of motion and a general formula for time dilation (complement)

We can make the expressions above more clear in terms of adimensionalized coordinates: we will use $R = r/(2GM)$, $L = l/(2GM)$, while e is already adimensional.

Under the assumption of circularity for the orbit, the expression for L can be calculated to be:

$$L = \frac{R}{\sqrt{2R-3}}, \tag{7.78}$$

while the one for e is:

$$e^2 = \left(1 + \frac{L^2}{R^2}\right)\left(1 - \frac{1}{R}\right). \tag{7.79}$$

Also, the definition for e is

$$e = \frac{\mathrm{d}t}{\mathrm{d}\tau}\left(1 - \frac{1}{R}\right), \tag{7.80}$$

therefore we can write

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = e\left(1 - \frac{1}{R}\right)^{-1} = \left(1 - \frac{1}{R}\right)^{-1+1/2}\left(1 + \frac{1}{2R-3}\right)^{1/2} \tag{7.81}$$

$$= \frac{\sqrt{1 + \frac{1}{2R-3}}}{\sqrt{1 - \frac{1}{R}}} = \sqrt{\frac{2R-3+1}{2R-3}\frac{R}{R-1}} = \sqrt{\frac{R}{R-3/2}}. \tag{7.82}$$

With this, we can see that in the limit of $R \to \infty$ we have the following effect:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} \sim \sqrt{1 + \frac{1}{R} + \frac{1}{2R-3}} \sim 1 + \frac{1}{2}\Bigg(\underbrace{\frac{1}{2R}}_{\text{gravitational}} + \underbrace{\frac{1}{R}}_{\text{Doppler}}\Bigg) = 1 + \frac{3}{4R}, \tag{7.83}$$

where we have a contribution both from the angular momentum, and from the gravitational redshift; the gravitational contribution is larger than the Doppler one in the limit (specifically for any $R > 2$), and the ratio between the contributions approaches 2.

Also, we can see that we have a divergence of this time dilation at $R \to 3/2$: I suspect that this corresponds to the velocity needed to stay in orbit at constant R diverges towards 1 as $R \to 3/2$.

In our specific case, the formula comes out to be

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \underbrace{\sqrt{\frac{7}{5}}}_{\text{grav}}\underbrace{\sqrt{\frac{5}{4}}}_{\text{Dopp}} = \sqrt{\frac{7}{4}}. \tag{7.84}$$

7.3.6 Tidal acceleration (complement)

A pointlike particle would feel no acceleration, however it would be quite unconfortable to be in that orbit if one happens not to be pointlike. As for the first part of the exercise, this is not in the exercise sheet, so do skip it if you are in a hurry. I think it is quite interesing though.

Let us compute the tidal effects on an extended observer.

They are described by the equation

$$\frac{\mathrm{d}^2 \xi^\mu}{\mathrm{d}\tau^2} = R^\mu_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma \,, \tag{7.85}$$

where u^μ is the 4-velocity of a geodesic, ξ^μ is a small deviation in starting position for the geodesic such that the geodesic starting from there almost has the same tangent vector. For now we assume it to be true, in the next section we give justification for it.

We can restrict ourselves to radial geodesic deviation: then we only need to compute the $R^r_{\nu\rho r}$ components of the Riemann tensor. I might do the full computation, but I found a source[7] which gives the nonzero components:

$$R^r_{ttr} = \frac{2GM}{r^3}\left(1 - \frac{2GM}{r}\right) \tag{7.86}$$

$$R^r_{\theta\theta r} = \frac{GM}{r} \tag{7.87}$$

$$R^r_{\varphi\varphi r} = \frac{GM \sin^2\theta}{r} \,, \tag{7.88}$$

and since the orbital motion is assumed to happen on the $\theta = \pi/2$ plane, the formula simplifies to

$$\frac{\mathrm{d}^2 \xi^r}{\mathrm{d}\tau^2} = R^r_{ttr} u^t u^t \xi^r + R^r_{\varphi\varphi r} u^\varphi u^\varphi \xi^r \tag{7.89}$$

$$= \left(\frac{e^2}{(1 - 2GM/r)^2} \frac{2GM}{r^3}\left(1 - \frac{2GM}{r}\right) + \frac{l^2}{r^4}\frac{GM}{r} \right) \xi^r \tag{7.90}$$

$$= \left(\frac{e^2}{1 - 2GM/r} + \frac{l^2}{r^2} \right) \frac{2GM}{r^3} \xi^r \,, \tag{7.91}$$

[7] https://physics.stackexchange.com/questions/295814/non-zero-components-of-the-riemann-tensor-of-the-schwarzschild-metric

which holds in general: in our specific case, at $r = 7GM$ we find

$$\frac{\mathrm{d}^2\xi^r}{\mathrm{d}\tau^2} = \left(\frac{7}{5}\left(\frac{5\sqrt{7}}{14}\right)^2 + \left(\frac{7GM/2}{7GM}\right)^2\right)\frac{2GM}{(7GM)^3}\xi^r \tag{7.92}$$

$$= \frac{3}{343}\frac{1}{(GM)^2}\xi^r\,, \tag{7.93}$$

so the object multiplying ξ^r, in units of inverse square seconds, is

$$\frac{3}{343}\frac{c^6}{(GM)^2} \approx 356 \times 10^6\,\mathrm{s}^{-2} \approx (18.9\,\mathrm{kHz})^2\,. \tag{7.94}$$

The acceleration which would be experienced from one side of the other of a meter-long observer would then be of around $36\,\mathrm{M}g$.

7.3.7 A proof for the geodesic deviation formula (complement)

The starting separation between geodesics is not, properly speaking, a *vector* since it does not belong to the tangent space, however we can be sloppy and approximate it as a tangent vector. Then it is a fact from differential geometry that under an assumption of vanishing torsion (which we always make anyways in GR) and if the vector fields u^μ and ξ^μ commute (which they do) the following holds:

$$\xi^\mu \nabla_\mu u^\nu = u^\mu \nabla_\mu \xi^\nu\,. \tag{7.95}$$

So, we want to see by how much the two initially-close geodesics diverge: in order to do this, we compute the second derivative of ξ^μ with respect to proper time: recall that the derivative with respect to proper time is also the Lie derivative along the 4-velocity: $\mathrm{d}/\mathrm{d}\tau = u^\mu \nabla_\mu$.

So, this second derivative looks like

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}\tau^2} = u^\alpha \nabla_\alpha \left(u^\beta \nabla_\beta \xi^\mu\right) = u^\alpha \nabla_\alpha \left(\xi^\beta \nabla_\beta u^\mu\right) \tag{7.96}$$

by the formula shown above. Now, we can apply the Leibniz rule: we find

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}\tau^2} = u^\alpha \left(\left(\nabla_\alpha \xi^\beta\right)\left(\nabla_\beta u^\mu\right) + \xi^\beta \nabla_\alpha \nabla_\beta u^\mu\right), \tag{7.97}$$

and using the fact that[8] $[\nabla_\alpha, \nabla_\beta]V^\mu = R^\mu_{\gamma\alpha\beta}V^\gamma$ we can commute the covariant derivatives by inserting a Riemann tensor: we get

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}\tau^2} = u^\alpha \left(\left(\nabla_\alpha \xi^\beta\right)\left(\nabla_\beta u^\mu\right) + \xi^\beta \nabla_\beta \nabla_\alpha u^\mu + \xi^\beta R^\mu_{\gamma\alpha\beta}u^\gamma\right), \tag{7.98}$$

[8] A trick for remembering this: it would be intuitive to just write the lower indices as $\alpha\beta\gamma$, but the last two are antisymmetric and must correspond to the antisymmetrization of the covariant derivatives in the LHS.

and now we have gotten the term which will remain in the end: let us bring it to the front, then we will show that the rest of the expression is null. We get

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}\tau^2} = R^\mu_{\gamma\alpha\beta} u^\gamma u^\alpha \xi^\beta + u^\alpha \left(\left(\nabla_\alpha \xi^\beta\right)\left(\nabla_\beta u^\mu\right) + \xi^\beta \nabla_\beta \nabla_\alpha u^\mu \right). \tag{7.99}$$

To show that the rest of the expression is zero, the thing to remember is that we want to use the fact that our curve is a geodesic, which is expressed as

$$u^\alpha \nabla_\alpha u^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau} u^\mu = 0\,, \tag{7.100}$$

so we apply the Leibniz rule backward, to find:

$$u^\alpha \left(\left(\nabla_\alpha \xi^\beta\right)\left(\nabla_\beta u^\mu\right) + \xi^\beta \nabla_\beta \nabla_\alpha u^\mu \right) = \tag{7.101}$$

$$= u^\alpha \left(\nabla_\alpha \xi^\beta\right)\left(\nabla_\beta u^\mu\right) + \xi^\beta \nabla_\beta \cancel{\left(u^\alpha \nabla_\alpha u^\mu\right)} - \xi^\beta \left(\nabla_\alpha u^\mu\right)\left(\nabla_\beta u^\alpha\right) \tag{7.102}$$

$$= u^\alpha \left(\nabla_\alpha \xi^\beta\right)\left(\nabla_\beta u^\mu\right) - u^\beta \nabla_\beta \xi^\alpha \left(\nabla_\alpha u^\mu\right) = 0\,, \tag{7.103}$$

where in the last step we applied again the commutation relation (**??**), and finally recognised that the terms were equal up to a relabeling of indices.

Sheet 8

8.1 Acceleration in Rindler coordinates

Rindler coordinates for the Minkowski (1, 1) space, originally parametrized with (t, x), are defined by

$$\begin{cases} t = \rho \sinh \eta \\ x = \rho \cosh \eta \end{cases} . \tag{8.1}$$

The metric is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 = -\rho^2\,\mathrm{d}\eta^2 + \mathrm{d}\rho^2\ . \tag{8.2}$$

We defined Rindler coordinates so that our uniformly accelerated observer would have velocity only along η, so we can find the general expression of the 4-velocity by computing the normalization: since it will look like $u^\alpha = (N, 0)$ for some N and we want $u^\alpha u_\alpha = -1$, we get that $N^2(-\rho^2) = -1$, which means $N = 1/\rho$.

The only nonzero Christoffel symbols are $\Gamma^\eta_{\eta\rho} = 1/\rho$ and $\Gamma^\rho_{\eta\eta} = \rho$.

The computation of the acceleration then can be started:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u^\mu = u^\nu \nabla_\nu u^\mu \tag{8.3}$$

$$= u^\eta \left(\cancel{\partial_\eta u^\mu} + \Gamma^\mu_{\eta\nu} u^\nu \right) \tag{8.4}$$

$$= \frac{1}{\rho^2} \Gamma^\mu_{\eta\eta} , \tag{8.5}$$

so the only nonzero component is the one where $\mu = \rho$, which is $a^\rho = 1/\rho$, or in other words

$$a^\mu = \begin{bmatrix} 0 \\ \kappa \end{bmatrix} , \tag{8.6}$$

where $\kappa = 1/\rho$ is the modulus of the 4-acceleration. Indeed

$$a^\mu a^\nu g_{\mu\nu} = \kappa^2 g_{\rho\rho} = \kappa^2 . \tag{8.7}$$

8.2 Acceleration in Schwarzschild motion

8.2.1 Acceleration computation

The acceleration is given in general by

$$a^\mu = u^\nu \left(\partial_\nu u^\mu + \Gamma^\mu_{\nu\rho} u^\rho \right) , \tag{8.8}$$

and we can see that in this formula we can only have a nonzero result if $\mu = t, r, \varphi$, $\nu = t, \varphi$ and $\rho = t, \varphi$.

Now we use the following facts: the velocity is both stationary ($\partial_t u^\mu = 0$), rotationally symmetric ($\partial_\varphi u^\mu = 0$), and the the Christoffel symbols with the upper index different from r must have at least one r between the lower ones in order to be nonzero, therefore the terms containing them must be zero since they must then be contracted with $u^r = 0$. This gives us that the only nonzero component of the acceleration is

$$a^r = u^\nu u^\nu \Gamma^r_{\nu\nu} = \left(u^t\right)^2 \Gamma^r_{tt} + \left(u^\varphi\right)^2 \Gamma^r_{\varphi\varphi} . \tag{8.9}$$

This already confirms that $u^\mu a_\mu = 0$, since the Schwarzschild metric is diagonal. The symbols which appear are

$$\Gamma^r_{tt} = \frac{1}{2} g^{rr} \left(-g_{tt,r}\right) = \left(1 - \frac{2GM}{r_*}\right) \frac{GM}{r_*^2} \tag{8.10}$$

and

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2} g^{rr} \left(-g_{\varphi\varphi,r} \right) = -\left(1 - \frac{2GM}{r_*} \right) r_* \,. \tag{8.11}$$

What are the components of the 4−velocity? We can use the normalization $u^\mu u_\mu = -1$; inserting the variable name $u^\varphi = \omega$ this gives us

$$u^t u^t g_{tt} + \omega^2 g_{\varphi\varphi} = -1 \,, \tag{8.12}$$

which can be written as

$$u^t u^t = \frac{1 + \omega^2 r_*^2}{1 - \frac{2GM}{r_*}} \,. \tag{8.13}$$

Let us plug this expression into the acceleration one:

$$a^r = \frac{1 + \omega^2 r_*^2}{1 - \frac{2GM}{r_*}} \left(1 - \frac{2GM}{r_*} \right) \frac{GM}{r_*^2} - \omega^2 \left(1 - \frac{2GM}{r_*} \right) r_* \,, \tag{8.14}$$

which simplifies to

$$a^r = \frac{GM}{r_*^2} + \omega^2 GM - \omega^2 r_* + 2GM\omega^2 \tag{8.15}$$

$$= \frac{GM}{r_*^2} + 3GM\omega^2 - \omega^2 r_* \,. \tag{8.16}$$

If we set it to zero we find a relation which we can express in terms of $l = g_{\varphi\varphi} u^\varphi = r_*^2 \omega$. We get

$$0 = \frac{GM}{r_*^2} + 3GM \frac{l^2}{r^4} - \frac{l^2}{r_*^3} \tag{8.17}$$

$$= GMr_*^2 + 3GMl^2 - l^2 r_* \,, \tag{8.18}$$

which is the relation we know for the angular momentum in circular orbits.

8.2.2 Acceleration modulus limits

The modulus of the acceleration is given by $|\vec{a}|^2 = a^r a^r g_{rr}$, so we will have

$$|\vec{a}|^2 = \frac{1}{1 - \frac{2GM}{r_*}} \left(\frac{GM}{r_*^2} + 3GM\omega^2 - \omega^2 r_* \right)^2 , \tag{8.19}$$

which can be written in terms of the adimensionalized variables $R = r/2GM$ and $\widetilde{\omega} = 2GM\omega$:

$$|\vec{a}|^2 = \frac{1}{(4GM)^2} \frac{R_*}{R_* - 1} \left(\frac{1}{R_*^2} + \widetilde{\omega}^2 (3 - 2R_*) \right)^2 . \tag{8.20}$$

This diverges as $R_* \to 1$, as we might expect: a *stationary* observer with respect to the Schwarzschild radial coordinate must have ever more acceleration in order to stay on their non-geodesic path.

Of we set $\omega = 0$ it becomes

$$|\vec{a}|^2 = \frac{1}{(4GM)^2} \frac{R_*}{R_* - 1} \frac{1}{R_*^4}, \tag{8.21}$$

while for $R_* \gg 1$ it becomes

$$|\vec{a}|^2 \sim \frac{1}{(4GM)^2} \left(\frac{1}{R_*^2} - 2\widetilde{\omega}^2 R_* \right)^2, \tag{8.22}$$

which can become 0 (that is, we have an orbit, a geodesic) if we set $2\widetilde{\omega}^2 R_*^3 = 1$, or $\omega^2 r_*^3 = GM$, (the spherical orbit formulation of) Kepler's third law.

We neglected only the $3\widetilde{\omega}^2$ term since it is the smallest one: in terms of powers of (GM) and of c we have: $R_*^{-2} \sim (GM)^{-2}$, $\widetilde{\omega}^2 \sim (GM)^{-2} c^{-2}$ while $\widetilde{\omega}^2 R_*^2 \sim (GM) c^{-2}$.

8.2.3 Some comments on orbits (complement)

Another interesting thing to note is the fact that at $R_* = 3/2$ the acceleration becomes independent of ω, and at $R_* < 3/2$ the effect of the rotation is to *increase* the acceleration instead of decreasing it; this corresponds to the fact that there are no orbits (not even unstable ones) for $R_* < 3/2$.

An interesting fact which was not mentioned in class: there exist orbits for any R_* between 3/2 and 3, although they are unstable.

We found that the radius of a circular orbit is given by

$$R = L^2 \left(1 \pm \left(\sqrt{1 - \frac{3}{L^2}} \right) \right), \tag{8.23}$$

where $L = l/2GM$ and $R = r/2GM$.

The two branches of this expression correspond to stable and unstable orbits: in both cases we have $\sqrt{3} < L < \infty$, and for the stable (plus sign) branch we find $R > 3$ and $\mathrm{d}R/\mathrm{d}L > 0$ everywhere, while for the unstable branch we have $3/2 < R < 3$ and $\mathrm{d}R/\mathrm{d}L < 0$ everywhere.

8.3 Perturbed rotating metrics

We want to prove that, to first order in $\delta g_{\mu\nu}$, the inverse of $g_{\mu\nu} = \overline{g}_{\mu\nu} + \delta g_{\mu\nu}$ is

$$g^{\mu\nu} = \overline{g}^{\mu\nu} + \overline{g}^{\mu\alpha} \overline{g}^{\nu\beta} \delta g_{\alpha\beta} + O((\delta g_{\mu\nu})^2). \tag{8.24}$$

This can be proved directly by verifying $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu + O((\delta g)^2)$: it comes out to be

$$
\begin{aligned}
&\Big(\overline{g}_{\mu\nu} + \delta g_{\mu\nu}\Big)\Big(\overline{g}^{\nu\rho} - \overline{g}^{\nu\alpha}\overline{g}^{\rho\beta}\delta g_{\alpha\beta} + O((\delta g)^2)\Big) && (8.25)\\
&= \delta^\rho_\mu + \delta g_{\mu\nu}\overline{g}^{\nu\rho} - \overline{g}_{\mu\nu}\overline{g}^{\nu\alpha}\overline{g}^{\rho\beta}\delta g_{\alpha\beta} + O((\delta g)^2) && (8.26)\\
&= \delta^\rho_\mu + \delta g_{\mu\nu}\overline{g}^{\nu\rho} - \delta^\alpha_\mu \delta g_{\alpha\beta}\overline{g}^{\rho\beta} + O((\delta g)^2)\,, && (8.27)
\end{aligned}
$$

so we can notice that the first order terms cancel: we have the inverse, up to first order.

8.3.1 Inverse perturbed Schwarzschild

The Schwarzschild metric is

$$
\overline{g}_{\mu\nu} = \begin{bmatrix} -\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}, \tag{8.28}
$$

and its inverse is

$$
\overline{g}^{\mu\nu} = \begin{bmatrix} -\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{r}\right) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} \sin^{-2}\theta \end{bmatrix}. \tag{8.29}
$$

We want to compute $\overline{g}^{\mu\alpha}\overline{g}^{\nu\beta}\delta g_{\alpha\beta}$, and we know that $\delta g_{\alpha\beta}$ has only one independent component:

$$
\delta g_{t\varphi} = -\frac{2GJ\sin^2\theta}{r}\,. \tag{8.30}
$$

This, combined with the fact that the background metric is diagonal, gives us the result that we only have one entry in the sum:

$$
\overline{g}^{\mu\alpha}\overline{g}^{\nu\beta}\delta g_{\alpha\beta} = \overline{g}^{tt}\overline{g}^{\varphi\varphi}\delta g_{\varphi t} = (-)^2 \frac{2GJ\sin^2\theta/r}{(1-\frac{2GM}{r})r^2\sin^2\theta} = \frac{2GJ}{(r-2GM)r^2}\,, \tag{8.31}
$$

so the full inverse metric to first order in J is given by subtracting this off of the regular Schwarzschild inverse's $t\varphi$ components:

$$g^{\mu\nu} = \begin{bmatrix} -\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 & -\frac{2GJ}{(r-2GM)r^2} \\ 0 & \left(1-\frac{2GM}{r}\right) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ -\frac{2GJ}{(r-2GM)r^2} & 0 & 0 & r^{-2}\sin^{-2}\theta \end{bmatrix}. \tag{8.32}$$

8.3.2 Ricci component computation

We want to compute the 00 component of the Ricci tensor, $R_{00} = R^{\alpha}_{0\alpha 0}$. It is given by

$$R_{00} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{00} - \Gamma^{\alpha}_{0\lambda}\Gamma^{\lambda}_{0\alpha}\,, \tag{8.33}$$

and we will show that each of these 4 terms is either constant with respect to J or quadratic in J, when computed with respect to the metric

$$g_{\mu\nu} = \begin{bmatrix} -\left(1-\frac{2GM}{r}\right) & 0 & 0 & -\frac{2GJ}{r}\sin^2\theta \\ 0 & \left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\frac{2GJ}{r}\sin^2\theta & 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{8.34}$$

First of all, note that the only derivatives of the metric which can give J-dependent contributions are $g_{03,1}$, $g_{03,2}$ or these terms with 0 and 3 exchanged.

The sum $\Gamma^{\alpha}_{00,\alpha}$ is independent of J: the only terms which can contribute in the sum are $\alpha = r, \theta$ and then the three indices in the Christoffel symbol are either 001 or 002: in either case we cannot form the J-dependent metric component g_{03}.

More explicitly: the expression is

$$\Gamma^{\alpha}_{00} = \frac{1}{2} g^{\alpha\beta}\left(2g_{\beta 0,0} - g_{00,\beta}\right), \tag{8.35}$$

and the time derivatives vanish, terms with $\beta = 1, 2$ could contribute but in neither case would we have the possibility to form the g^{03} component.

The term $\partial_0 \Gamma^{\alpha}_{0\alpha}$ is zero by stationarity: the metric is t-independent.

In the term $\Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{00}$ we ask ourselves: where can the metric component g_{03} or g^{03} appear? As we saw above it cannot appear in the second symbol, while for it to appear in the first symbol we would need to have $\alpha, \lambda = 0, 3$ (in either order), but then the third index in that symbol would also be 0 or 3, so the metric component g_{03} would be differentiated with respect to t or φ, and it is independent of both. So, in order to have the symbol Γ^{λ}_{00} not be zero we need to have $\lambda = 1, 2$. Then, in the expression

$$\Gamma^{\alpha}_{\alpha\lambda} = \frac{1}{2} g^{\alpha\beta} \Big(g_{\beta\alpha,\lambda} + g_{\beta\lambda,\alpha} - g_{\alpha\lambda,\beta} \Big) \tag{8.36}$$

the last two terms of the sum cannot depend on J since they have an index which is neither 0 nor 3. Terms such as those with $\alpha, \beta = 0, 3$ or the inverse can contribute: however in these terms we have to multiply a g^{03}, which is linear in J, with a g_{03}, which also is. So in the end the term is quadratic in J.

The term $\Gamma^{\alpha}_{0\lambda}\Gamma^{\lambda}_{0\alpha}$ is also $O(J^2)$; to see this, let us write a symbol $\Gamma^{\alpha}_{0\lambda}$: it is

$$\Gamma^{\alpha}_{0\lambda} = \frac{1}{2} g^{\alpha\beta} \Big(g_{\beta 0,\lambda} + \cancel{g_{\beta\lambda,0}} - g_{0\lambda,\beta} \Big) \,, \tag{8.37}$$

and we ask ourselves: how can this depend on J? We could have $\alpha, \beta = 0, 3$: then the term $g_{0\lambda,\beta}$ vanishes and we have $g^{03} g_{30,\lambda}$ which is already quadratic in J.

We could have $\alpha, \beta = 3, 0$: then $g_{0\lambda,\beta}$ vanishes. We could have a nonzero term $g^{30} g_{00,\lambda}$ if $\lambda = 1, 2$: in this case the whole term would look like $\Gamma^{3}_{01}\Gamma^{1}_{03}$ or $\Gamma^{3}_{02}\Gamma^{2}_{03}$. In either case, in both symbols the sum of derivatives of the metric the only nonvanishing term will be necessarily linear in J since it will be differentiated with respect to the 1 or 2 index. Therefore, the product of the two Christoffels will be at least quadratic in J.

The final case for the single Christoffel is $\alpha = \beta$: in that case, to have J-dependence we need to have either $\beta = 3$ which means $\alpha = 3$ or $\lambda = 3$. When we set one of the two indices α, λ to 3 the other one must necessarily be 1 or 2, since otherwise the derivatives would vanish. Then we get back to the case in which only one term in the sum of the three derivatives of the metric survives, which means that the symbol is linear in J as a whole, but we have two symbols for which this holds multiplied together, so on the whole the term is quadratic in J.

In the end then the sum, when expressed as a function of J, looks like:

$$R_{00} = \text{const}(J) + O(J^2) \,, \tag{8.38}$$

and we know that if $J = 0$ then $R_{00} = 0$, so we are done: $R_{00} = O(J^2)$.

Sheet 9

9.1 Censorship principle justification

9.1.1 Boundary condition

We assume that the Kerr black hole starts off as *critical*:[9] so initially we have $a = GM$, and that the particle's addition makes it so that $a' \geq GM'$: this can be written by making the dependence on $J = aM$ explicit: when we add the particle we get $J' = J + lm$ and $M' = M + me$. So the condition reads:

$$J' \geq GM'^2 \tag{9.1a}$$

$$J + lm \geq G\Big(M^2 + 2mMe + (me)^2\Big) \tag{9.1b}$$

$$l \gtrsim 2GMe\,, \tag{9.1c}$$

where we used the fact that $J = GM^2$ and we ignored the $O(m^2)$ term, which is negligible (one could collect a M^2 and get a polynomial in m/M to consider up to linear order: it is the same thing in the end). Hereafter, we assume that we have exactly the boundary condition $l = 2GMe$.

9.1.2 Effective potential barrier

We write the potential with respect to the adimensional coordinates $R = r/2GM$ and $L = l/2GM = e$, and we use the fact that $a = GM$, which implies $a/2GM = 1/2$. With these substitutions, we find:

$$V_{\text{eff}}(R, e, L) = -\frac{1}{2R} + \frac{L^2 - (e^2-1)/4}{2R^2} - \frac{(L - e/2)^2}{2R^3} \tag{9.2}$$

$$= -\frac{1}{2R} + \frac{e^2 - (e^2-1)/4}{2R^2} - \frac{e^2}{8R^3} \tag{9.3}$$

$$= e^2\left(\frac{3}{8R^2} - \frac{1}{8R^3}\right) - \frac{1}{2R} + \frac{1}{8R^2}\,. \tag{9.4}$$

We want to see, then, whether the potential barrier is as high as $(e^2-1)/2$. Where is this potential barrier? The photon is absorbed when it gets inside the event horizon (which then should disappear), and we know that the locations of the horizons are given by $r_\pm = GM \pm \sqrt{(GM)^2 - a^2} = GM$, so in our case they are situated at $R = 1/2$.

[9] I initially thought this could work as well if we assumed the BH was initially subcritical and became critical: this is not the case, the calculation is much worse.

Plugging this into the formula we find

$$V_{\text{eff}} = e^2\left(\frac{3}{2} - 1\right) - 1 + \frac{1}{2} = \frac{e^2 - 1}{2}, \tag{9.5}$$

so at $R = 1/2$ we must have $\mathrm{d}r/\mathrm{d}\tau = 0$. This is actually enough for our purposes: it does not reall matter what the rest of the potential looks like, since at at least one point it is as large as the energy. However, it is interesting to see the global shape of the effective potential: it is plotted in figure **??** for different values of e.

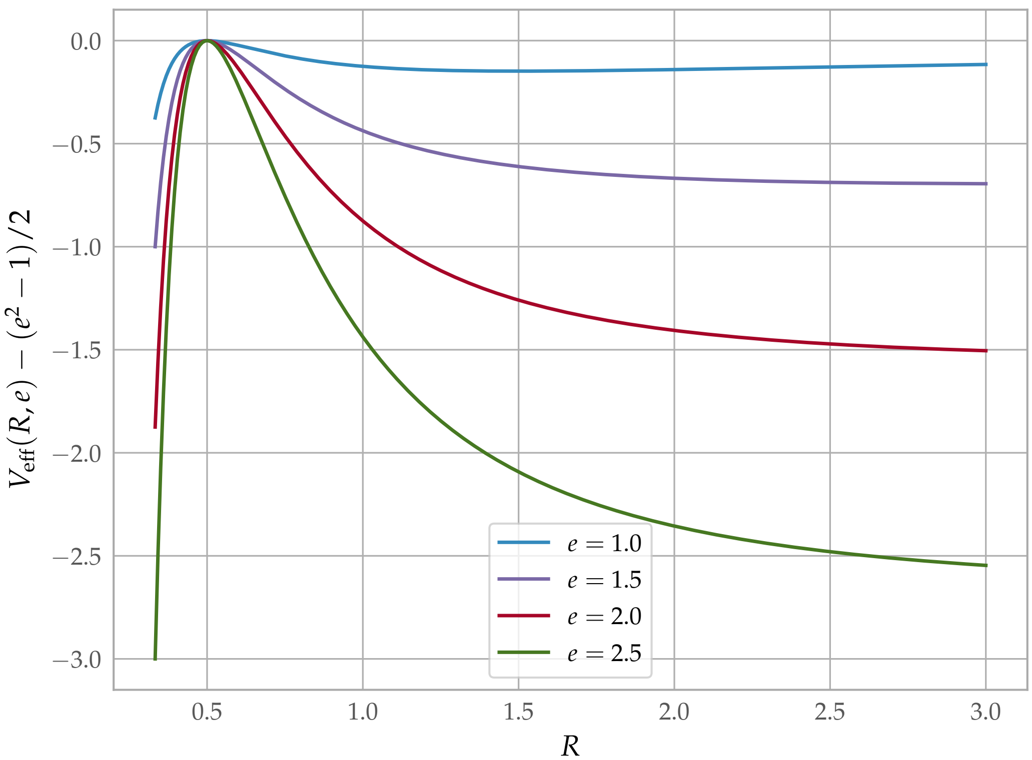

Figure 5: Effective potential $V_{\text{eff}}(R, e) - (e^2 - 1)/2$.

9.1.3 Comments on the possibility of undressing a singularity (complement)

We have shown that the potential barrier will prevent a particle with angular momentum l with respect to the BH from being absorbed. There are some things to consider: one of these is the fact that we assumed m^2 is negligible: what if we threw a *really massive* object into our Kerr black hole? In that case we would have

$L = e + me^2/2M > e$, and since $\mathrm{d}V/\mathrm{d}L > 0$ everywhere the potential barrier would be even higher. This is not the way to go.

However, we neglected *intrinsic* spin: what if we threw lots of, say, electrons aligned with the rotation axis down on the BH (on a trajectory with $\theta \equiv 0$)? They may have angular momentum with respect to the hole which allows them to go in, but increase its angular momentum by a little bit more through their intrinsic spin in order to make the singularity naked…

Say we did it with an electron: then we would have a spin of $j = \hbar/2$ and $m = m_e$. If its four-velocity while falling in looks like $u^\mu = (u^t, u^r, 0, 0)$ then we have

$$e = -g_{t\mu}u^\mu = \left(1 - \frac{R}{R^2 + 1/4}\right)u^t\,, \tag{9.6}$$

since $\rho^2 = r^2 + a^2$ there; it is convenient, in order to minimize the energy imput to the BH, to throw in the electron with asymptotically nonrelativistic velocities: then we will have $e \sim 1$. So our saturated inequality looks like

$$j = \frac{\hbar}{2} = 2GMm_e \implies M = \frac{\hbar c}{4Gm_e} \sim 10^{14}\,\mathrm{kg}\,, \tag{9.7}$$

where in the last equality we reinserted c; the mass of the black hole needs to be *lower* than this in order for singularity to become naked through absorption of electrons in this manner. If we substitute in neutrinos for electrons, which also have $j = \hbar/2$ but mass of at most $m_\nu \sim 0.12\,\mathrm{eV} \sim 2.14 \times 10^{-37}\,\mathrm{kg}$, we get $M_{\mathrm{min}} \sim 10^{21}\,\mathrm{kg}$.

These are quite small masses when compared to those of known black holes, the smallest of which are of the order of the solar mass, $M_\odot \sim 10^{30}\,\mathrm{kg}$. So we must conclude that throwing massive particles down on the axis of a real black hole is not the way to go. It seems like the censorship principle is sound.

This is treated more in depth in the wonderfully titled article *Return of the quantum cosmic censor*:[10] there they show that a proper analysis of the interaction of a fermionic field with a (Kerr-Newman: charged and rotating) BH cannot actually over-spin it; the probability of the modes which could to be absorbed is suppressed by a factor $\exp(-M/M_P)$.

[10] http://arxiv.org/abs/0810.0079v1

9.2 Kerr geometry null surfaces

9.2.1 Horizon metric

Let us compute the metric elements at the outer horizon: it is a straight calculation. We just substitute in $r = r_+ = GM + \sqrt{(GM)^2 - a^2}$ and $\rho^2 = \rho_+^2$:

$$g_{00} = -\left(1 - \frac{2GMr_+}{\rho_+^2}\right) \tag{9.8a}$$

$$= \frac{1}{\rho_+^2}\left(2GM\left(GM + \sqrt{(GM)^2 - a^2}\right) - r_+^2 - a^2\cos^2\theta\right) \tag{9.8b}$$

$$= \frac{1}{\rho_+^2}\left(2(GM)^2 + 2GM\sqrt{\dots} - \left((GM)^2 + (GM)^2 - a^2 + 2GM\sqrt{\dots}\right) - a^2\cos^2\theta\right) \tag{9.8c}$$

$$= \frac{a^2}{\rho_+^2}\left(1 - \cos^2\theta\right) = \frac{a^2\sin^2\theta}{\rho_+^2}, \tag{9.8d}$$

where we omitted the argument of the square root for brevity: it is the same on either side, and it simplifies. For the components g_{03} and g_{22} we cannot simplify anything. The only thing to recall is that the off-diagonal components of the metric when written in component form are half of what they are when written with respect to the differentials, since we usually do not bother to write two separate $\mathrm{d}t\,\mathrm{d}\varphi$ and $\mathrm{d}\varphi\,\mathrm{d}t$ terms: the diffentials commute, therefore we simply write one of the two combinations with double the value. So we have:

$$g_{03} = -\frac{2GMar_+\sin^2\theta}{\rho_+^2} \tag{9.9}$$

and

$$g_{22} = \rho_+^2\,. \tag{9.10}$$

The computation of the g_{33} component[11] is more interesting: we get

$$g_{33} = \left(r_+^2 + a^2 + \frac{2GMr_+a^2\sin^2\theta}{\rho_+^2}\right)\sin^2\theta\,, \tag{9.11}$$

and we recall the definition of $\Delta = r^2 + a^2 - 2GMr$: since we are dealing with the

[11] There is a typo in the assignment: this component is denoted as g_{03} again.

case in which $\Delta = 0$, so we have $r^2 + a^2 = 2GMr$, which we substitute in and collect:

$$g_{33} = 2GMr_+ \left(1 + \frac{a^2 \sin^2\theta}{\rho_+^2}\right) \sin^2\theta \tag{9.12}$$

$$= 2GMr_+ \frac{\sin^2\theta}{\rho_+^2} \left(r_+^2 + a^2\cos^2\theta + a^2\sin^2\theta\right), \tag{9.13}$$

and then we can notice again the combination $r_+^2 + a^2 = 2GMr_+$: so in the end we get

$$g_{33} = \left(\frac{2GMr_+ \sin\theta}{\rho_+}\right)^2 . \tag{9.14}$$

9.2.2 Uniqueness of the null vector

The metric we found for the 3D space $r \equiv r_+$ is:

$$\mathrm{d}s^2 = \frac{a^2\sin^2\theta}{\rho_+^2}\,\mathrm{d}t^2 - \frac{4GMar_+\sin^2\theta}{\rho_+^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \rho_+^2\,\mathrm{d}\theta^2 + \left(\frac{2GMr_+\sin\theta}{\rho_+}\right)^2 \mathrm{d}\varphi^2 , \tag{9.15}$$

and now we want to find a vector field $l^\mu = (l^t, l^\theta, l^\varphi)$ such that $l^2 = l^\mu g_{\mu\nu} l^\nu = 0$. Since $\rho_+^2 \neq 0$ we can multiply everything by it: we get

$$l^2 = a^2\sin^2\theta (l^t)^2 - 4GMar_+\sin^2\theta l^t l^\varphi + \rho_+^4 (l^\theta)^2 + (2GMr_+)^2 \sin^2\theta (l^\varphi)^2 , \tag{9.16}$$

and we notice that the tt, $\varphi\varphi$ and φt terms form a square: we find

$$l^2 = \left(a\sin\theta l^t - 2GMr_+\sin\theta l^\varphi\right)^2 + \rho_+^4 (l^\theta)^2 , \tag{9.17}$$

so if we want to set $l^2 = 0$ we need to have both square terms be separately equal to zero. This means that $l^\theta = 0$, and that

$$al^t = 2GMr_+ l^\varphi , \tag{9.18}$$

so if we choose a value for one of the components the other one is fixed. Therefore, the vector field only has a scalar degree of freedom: we can rescale it by a scalar field, but its direction is fixed. Specifically, we find that

$$\frac{l^\varphi}{l^t} = \frac{a}{2GMr_+} \tag{9.19}$$

is the only allowed ratio of these components. If we interpret the vector l as the 4-velocity of light trapped at the horizon, we have

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{a}{2GMr_+} . \tag{9.20}$$

9.2.3 A basis for the horizon

The $\varphi\varphi$ and $\theta\theta$ components of the metric are squares of real numbers, so they are positive, so the vectors $m^\alpha = [0, 0, 1, 0]^\top$ and $n^\alpha = [0, 0, 0, 1]^\top$ are spacelike.

They are orthogonal, since $m^\alpha g_{\alpha\beta} n^\beta = g_{23} = 0$. Let us compare them to the null vector we found earlier, for which we can use the normalization: $l^\alpha = [2GMr_+, 0, 0, a]^\top$. We have that $m^\alpha g_{\alpha\beta} l^\beta = g_{2\beta} l^\beta = 0$, since the only 2β component of the metric is g_{22} but $l^2 = 0$. On the other hand, we have

$$\begin{aligned} n^\alpha g_{\alpha\beta} l^\beta &= g_{3\beta} l^\beta && (9.21) \\ &= -\frac{2GMar_+ \sin^2\theta}{\rho_+^2} 2GMr_+ + \left(\frac{2GMr_+ \sin\theta}{\rho_+} \right)^2 a = 0\,. && (9.22) \end{aligned}$$

A basis is defined by linear independence and spanning, which are properties independent of the metric. For an n-dimensional set of vectors, each of these properties is equivalent to our set being a basis.

So, we prove linear independence: let us consider a combination of these vectors:

$$Al^\mu + Bm^\mu + Cn^\mu = 0\,, \tag{9.23}$$

and we need to show that this implies $A = B = C = 0$. In order for the $\mu = 0$ component of the result to be zero we need $A = 0$. In order for the $\mu = 2$ component to be zero we need $B = 0$. In order for the $\mu = 3$ component to be zero we need $C = 0$, so the result is proven.

9.3 Solar system Kerr

9.3.1 Earth's implausible Kerr description

We need to estimate the angular momentum of the Earth: for our order of magnitude calculation we will model it as a constant-density sphere, with $M = 6 \times 10^{24}\,\mathrm{kg}$ and $r = 6 \times 10^6\,\mathrm{m}$. The moment of inertia is then given by

$$I = \frac{2}{5} Mr^2 \approx 1 \times 10^{38}\,\mathrm{kgm^2}\,. \tag{9.24}$$

This is a slight over-estimate: more of the Earth's mass is concentrated around the center, so the actual moment of inertia is about 20% lower. This is close enough for our purposes.

Then, since the Earth rotates at a revolution per day so with $\omega = 2\pi / 1\,\mathrm{d}$, we have our estimate for the angular momentum:

$$J = I\omega \approx 7 \times 10^{33}\,\mathrm{kgm^2 s^{-1}}\,. \tag{9.25}$$

In order to find the length corresponding to the natural-units formula $a = J/M$ we need to divide by c: so we get

$$a = \frac{J}{Mc} \approx 4\,\mathrm{m}\,, \tag{9.26}$$

while we have $GM/c^2 \approx 4 \times 10^{-3}\,\mathrm{m}$: their ratio is of the order 10^3.

This of course does not mean that there is a naked singularity deep inside the Earth: we'd need all the mass to be concentrated around $r = 0$ for that to be the case, and instead the radius of the Earth is quite larger than 4 m.

So we have the quite interesting result that the general-relativistic description of the Earth corresponds to a Kerr metric with $a \gg GM$. It is not really clear to me what meaning should be drawn from this but it seems to be the case.

9.3.2 The Sun's Kerr metric

We can do the same calculation for the Sun, which has $m \approx 2 \times 10^{30}\,\mathrm{kg}$ and $r \approx 7 \times 10^{8}\,\mathrm{m}$; its moment of inertia factor (I/Mr^2) is quite badly approximated by 2/5: it is around 0.07.[12] The rotation of the Sun depends on the latitude of the surface since it is a fluid, but its period is on the order of 600 hr. So we have

$$J = I\omega \approx 1.7 \times 10^{41}\,\mathrm{kgm^2s^{-1}}\,, \tag{9.27}$$

which gives

$$a = \frac{JM}{c} \approx 330\,\mathrm{m}\,, \tag{9.28}$$

while

$$\frac{GM}{c^2} \approx 1500\,\mathrm{m}\,, \tag{9.29}$$

therefore the ratio a/GM is around 0.2. So the Sun's Kerr metric would have an inner and outer horizon.

9.3.3 Critical ω in general (complement)

We can compute the value of ω for which any object will have $a = GM$: if the moment of inertia is $I = \kappa M r^2$ in the end we find

$$\omega_{\mathrm{crit}} = \frac{GM}{\kappa r^2 c}\,, \tag{9.30}$$

so if we consider objects with a constant density $\rho \sim Mr^{-3}$ the critical angular velocity scales with r: it gets quite low for small objects! A carousel with $r \sim 2\,\mathrm{m}$ and $M \sim 100\,\mathrm{kg}$ (and $\kappa \sim 0.5$) will have a critical angular velocity of around $10^{-17}\,\mathrm{rad/s}$! It is *really easy* for a small object to have $a \gg GM$.

[12] https://web.archive.org/web/20191030204430/https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html

Sheet 10

10.1 Allowed Kerr orbits

10.1.1 Ω inequality

The condition we want to impose is $u^\mu g_{\mu\nu} u^\nu = -1$. Explicitly, since the four-velocity is $u^\mu = u^t(1, 0, 0, \Omega)$, this means that

$$(u^t)^2 \Big(g_{00} + 2g_{03}\Omega + g_{33}\Omega^2 \Big) = -1 \,. \tag{10.1}$$

We can work up to a positive multiplicative factor, at the price of weakening our equality to an equality: since $(u^t)^2 > 0$, we can also write

$$g_{00} + 2g_{03}\Omega + g_{33}\Omega^2 < 0 \,, \tag{10.2}$$

and since many of the terms contain divisions by ρ^2 we can also multiply by $\rho^2 > 0$ which will not change the sign; also we flip the sign of the inequality since we have more negative terms than positive ones (both of these really are for consistency with the assignment):

$$-\rho^2 \Big(g_{00} + 2g_{03}\Omega + g_{33}\Omega^2 \Big) > 0 \,. \tag{10.3}$$

Now we can substitute in the Kerr metric components (which we will not need to remember by heart):

$$\rho^2 - 2GMr + \Big(4GMra \sin^2\theta \Big)\Omega - \Big(\rho^2(r^2 + a^2) + 2GMra^2 \sin^2\theta \Big) \sin^2(\theta)\Omega^2 > 0 \,, \tag{10.4}$$

so we can read off the values of the coefficients c_n as everything multiplying Ω^n, for $n = 0, 1, 2$:

$$c_0 = \rho^2 - 2GMr \tag{10.5a}$$

$$c_1 = 4GMra \sin^2\theta \tag{10.5b}$$

$$c_2 = -\Big(\rho^2(r^2 + a^2) + 2GMra^2 \sin^2\theta \Big) \sin^2(\theta) \,. \tag{10.5c}$$

10.1.2 Ergosphere characterization

The ergosphere, as was discussed during the lectures, is defined as the region in which there are no timelike worldlines which are stationary with respects to the spatial coordinates, that is to say, with velocity parallel to the vector $(1, \vec{0})$. This is equivalent to saying that the metric component $g_{00} > 0$, which corresponds to a region which extends beyond the horizon, and is shaped like an ellipsoid for low a/GM and like a donut without the hole for high a/GM.[13]

Now, $c_0 = -\rho^2 g_{00}$, so $c_0 < 0$ is equivalent to $g_{00} > 0$. The characterization of the ergoregion is a special case of the prolem we are discussing now, where we fix $\Omega = 0$.

10.1.3 Inequality discriminant

We start by writing everything out the pieces separately:

$$\frac{c_1^2}{4} = \frac{16}{4}\Big(GMar\sin^2\theta\Big)^2\,, \tag{10.6}$$

while

$$\begin{aligned} -c_0c_2 = +\sin^2\theta\Big(\rho^4\Big(r^2+a^2\Big) + \rho^2 2GMa^2 r\sin^2\theta + \\ - 2GMr\rho^2(r^2+a^2) - 4(GMra\sin\theta)^2\Big)\,, \end{aligned} \tag{10.7}$$

so we can see that the last term in $-c_0c_2$ precisely cancels $c_1^2/4$: we are then left with

$$\frac{c_1^2}{4} - c_0c_2 = \sin^2\theta\left(\rho^4\Big(r^2+a^2\Big) + \rho^2 2GMa^2r\sin^2\theta - 2GMr\rho^2(r^2+a^2)\right) \tag{10.8}$$

$$= \rho^4\sin^2\theta\left(\Big(r^2+a^2\Big) - 2GMr\left(\frac{r^2+a^2-a^2\sin^2\theta}{\rho^2}\right)\right) \tag{10.9}$$

$$= \rho^4\sin^2\theta\Big(r^2+a^2-2GMr\Big)\,, \tag{10.10}$$

where in the last step we used the fact that $r^2 + a^2\Big(1-\sin^2\theta\Big) = r^2 + a^2\cos^2\theta = \rho^2$.

10.1.4 Allowed angular velocities

We have a second degree polynomial inequality, and the quadratic coefficient c_2 is negative: therefore as $|\Omega| \to \infty$ the parabola in Ω goes to $-\infty$, so if there are any solutions to the inequality they are an interval between the two solutions of the equation $\sum c_n\Omega^n = 0$.[14]

These are given by the quadratic formula

$$\Omega_{+,-} = \frac{-c_1/2 \pm \sqrt{c_1^2/4 - c_0c_2}}{c_2}\,, \tag{10.11}$$

[13] See the python folder (or, possibly, the notes later) for visualizations of this.

[14] This is just the way I like to remember which solutions to select, but in the end it's just a quadratic inequality, solve it however you like.

which gives:

$$\Omega_{-,+} = \frac{-2GMra\sin^2\theta \pm \sqrt{\rho^4 \sin^2\theta \big(r^2 + a^2 - 2GMr\big)}}{-\sin^2\theta\Big(\rho^2(r^2+a^2) + 2GMra^2\sin^2\theta\Big)} \tag{10.12}$$

$$= \frac{2GMra \mp \rho^2 \sqrt{\big(r^2 + a^2 - 2GMr\big)/\sin^2\theta}}{\rho^2(r^2+a^2) + 2GMra^2\sin^2\theta}\,, \tag{10.13}$$

which I do not think can be simplified further. Do note that we did not use the hypothesis of being in the ergosphere: the bounds for the angular velocity are general, it's just that in the ergosphere they preclude $\Omega = 0$.

Do these solutions actually exist? They do if $r^2 + a^2 - 2GMr = \Delta > 0$ (the Δ from the definition of the Kerr geometry). Recall that this Δ is positive outside the outer horizon and inside the inner horizon: in the region between the outer horizon and the ergosphere the solutions are well defined. This region is precisely the ergoregion.

10.1.5 Corotation necessity

We want to show that $\Omega > 0$ in the ergoregion: this is implied by $\Omega_- > 0$, because $\Omega_- < \Omega < \Omega_+$. The condition reads:

$$\frac{c_1}{2} - \sqrt{(c_1/2)^2 - c_0 c_2} > 0 \tag{10.14}$$

$$\left(\frac{c_1}{2}\right)^2 > \left(\frac{c_1}{2}\right)^2 - c_0 c_2 \tag{10.15}$$

$$0 > -c_0 c_2\,, \tag{10.16}$$

but we have shown before that $c_0 < 0$ in the ergoregion, while c_2 is always negative: so we have proven the statement.

10.1.6 Corotation necessity - the long way (complement)

This is the way I originally did it, I think it is quite illustrative so I will keep it here.

We want to show that in the ergoregion, which is defined by $\rho^2 < 2GMr$ and $\Delta > 0$, we have that $\Omega > 0$ for any solution for our inequality. This is equivalent to saying that the lower of the two solutions for the equality (let us say the lower one is Ω_-) must be > 0 in this case. The denominator is always positive, so we can discard it. Then we want to show that

$$2GMra - \rho^2\sqrt{(r^2 + a^2 - 2GMr)/\sin^2\theta} > 0\,, \tag{10.17}$$

which can be rephrased as

$$\big|\sin(\theta)\big| 2GMra > \rho^2 \sqrt{r^2 + a^2 - 2GMr}\,. \tag{10.18}$$

In adimensional units $A = a/GM$, $R = r/GM$ and $P^2 = R^2 + A^2 \cos^2\theta$ this becomes

$$\frac{2R}{R^2 + A^2\cos^2\theta} > \frac{\sqrt{R^2 - 2R + A^2}}{A|\sin\theta|}\,, \tag{10.19}$$

to be proven under the constraint that we are in the ergoregion, which is defined by:

$$R^2 - 2R + A^2 > 0 > R^2 - 2R + A^2\cos^2\theta\,. \tag{10.20}$$

The LHS of (**??**) is greater than 1 because of $P^2 < 2R$: we will prove that the RHS is always less than 1. To show this we can square it, since the property of a positive number being less or greater than 1 is preserved when we square it. Then we must prove:

$$\frac{R^2 - 2R + A^2}{A^2\sin^2\theta} < 1\,, \tag{10.21}$$

which can also be written as

$$R^2 - 2R + A^2(1 - \sin^2\theta) < 0\,, \tag{10.22}$$

which is precisely the condition of being inside the ergosphere.

10.1.7 Horizon angular velocity

If $\Delta = 0$, then the two solutions to the quadratic equation collapse: only one speed is allowed, and it is equal to

$$\Omega = \frac{2GMra}{\rho^2\big(r^2 + a^2\big) + 2GMra^2\sin^2\theta}\,, \tag{10.23}$$

which we can simplify using $r^2 + a^2 = 2GMr$ (which is precisely $\Delta = 0$):

$$\Omega = \frac{2GMra}{2GMr\Big(\rho^2 + a^2\sin^2\theta\Big)} = \frac{a}{r^2 + a^2\Big(\cos^2\theta + \sin^2\theta\Big)} = \frac{a}{2GMr} = \frac{a}{r_+^2 + a^2}\,, \tag{10.24}$$

where the radius is precisely that of the (outer) horizon, the greatest solution to $\Delta = 0$.

An observer reaching the horizon must spin with exactly this frequency: then we are justified in thinking that this is the *angular velocity of the horizon*.

10.1.8 Extreme Kerr equatorial orbit

We set $a = GM$ and $\theta = 0$. Then we have $\rho^2 = r^2$, $\cos\theta = 0$ and $\sin\theta = 1$, so

$$\Omega_{1,2} = \frac{2(GM)^2 r \mp r^2\sqrt{r^2 + (GM)^2 - 2GMr}}{r^2\big(r^2 + (GM)^2\big) + 2(GM)^3 r} \tag{10.25}$$

$$= \frac{2(GM)^2 \mp r|r - GM|}{r^3 + (GM)^2 r + 2(GM)^3} \tag{10.26}$$

$$O_{1,2} = \frac{2 \mp R(R-1)}{R^3 + R + 2}\,, \tag{10.27}$$

where the absolute value is redundant since we are working at $r > GM$. We introduced adimensionalized variables $O = GM\Omega$ and $R = r/GM$, which make the expressions way simpler. Using these is equivalent to setting $GM = 1$.

The upper bound is given by the solution with the plus sign:

$$O_+ = \frac{R^2 - R + 2}{R^3 + R + 2} = \frac{1}{1+R}\,; \tag{10.28}$$

to check that this is indeed the case one can simply multiply the polynomials together. How might one guess or calculate this? If we want to divide the denominator by the numerator, we could use the algorithm of polynomial division, but there is a faster way. By inspection of the exponents we might guess that the result is a first degree polynomial in R, and looking at the coefficients one can see that both the coefficient of R and the constant in this first order polynomial must be 1. This ansatz can then be checked by multiplication. Reinserting GM, we get

$$\Omega_+ = \frac{1}{GM + r}\,. \tag{10.29}$$

For the lower bound we get:

$$O_- = \frac{-R^2 + R + 2}{R^3 + R + 2} = -\frac{(R+1)(R-2)}{(R+1)(R^2 - R + 2)} = \frac{2 - R}{R^2 - R + 2}\,, \tag{10.30}$$

where we used the decomposition of the denominator we had derived in the last paragraph, and the regular quadratic decomposition for the numerator. Reinserting GM, this is

$$\Omega_- = \frac{2GM - r}{r^2 - rGM + 2(GM)^2}\,, \tag{10.31}$$

which I find to be simpler than the equivalent decomposition proposed in the assignment, which is

$$\frac{1}{GM}\left(1 - \frac{r^2}{2(GM)^2 - GMr + r^2}\right) = \frac{1}{GM}\left(\frac{2(GM)^2 - GMr + r^2 - r^2}{2(GM)^2 - GMr + r^2}\right) \tag{10.32}$$

$$= \frac{2GM - r}{2(GM)^2 - GMr + r^2}\,. \tag{10.33}$$

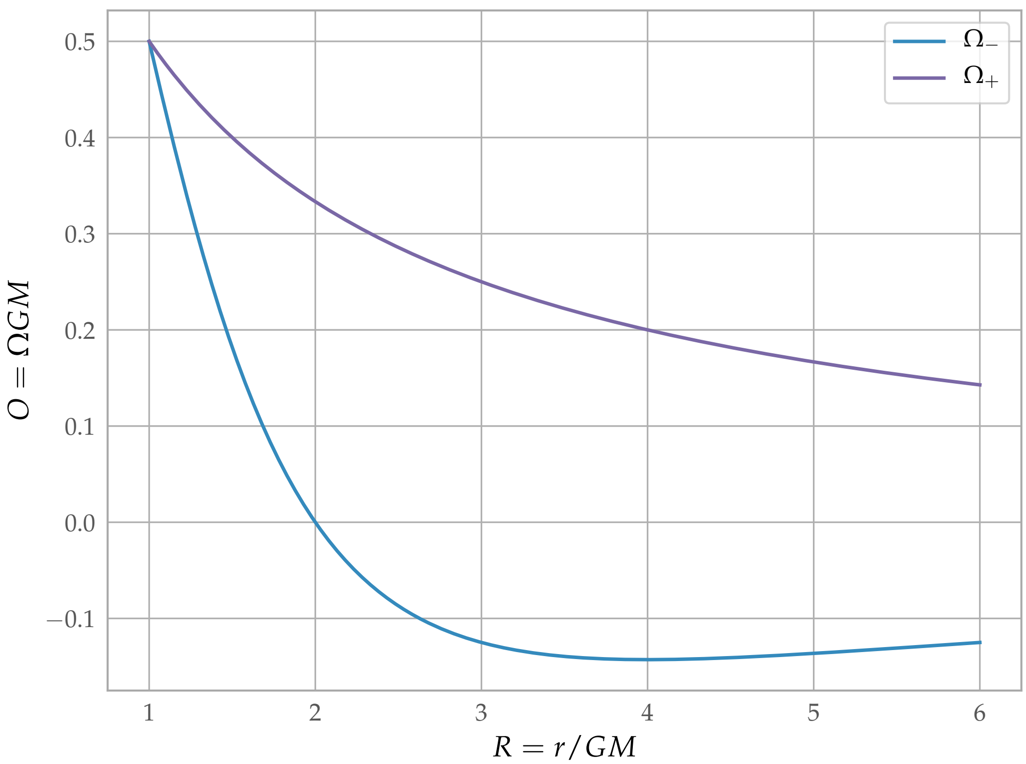

Figure 6: Allowed angular velocities.

A plot of these limiting velocities is shown in figure **??**: we can see that for $1 < R < 2$ (that is, inside the ergoregion) we must have a positive Ω.

It is interesting to note that both $\Omega_\pm$ tend to 0 as $R \to \infty$: this makes sense since the *tangential* velocity which corresponds to Ω is $r\Omega$, which approaches ± 1 for $\Omega_\pm$, as we'd expect.

10.2 Natural units

Let us denote the numeric value of the time in seconds by $\texttt{t}$, so that $t = \texttt{t}\text{s}$. It will be given by

$$\texttt{t} = \frac{1}{\text{GeVs}} \times \hbar^\alpha c^\beta = \frac{\text{J}}{\text{GeV}} \times \frac{1}{\text{Js}} \times \hbar^\alpha c^\beta \,, \tag{10.34}$$

where we inserted some generic powers of $\hbar$ and c, to be determined through dimensional analysis. We should write a system of linear equations for the exponents,

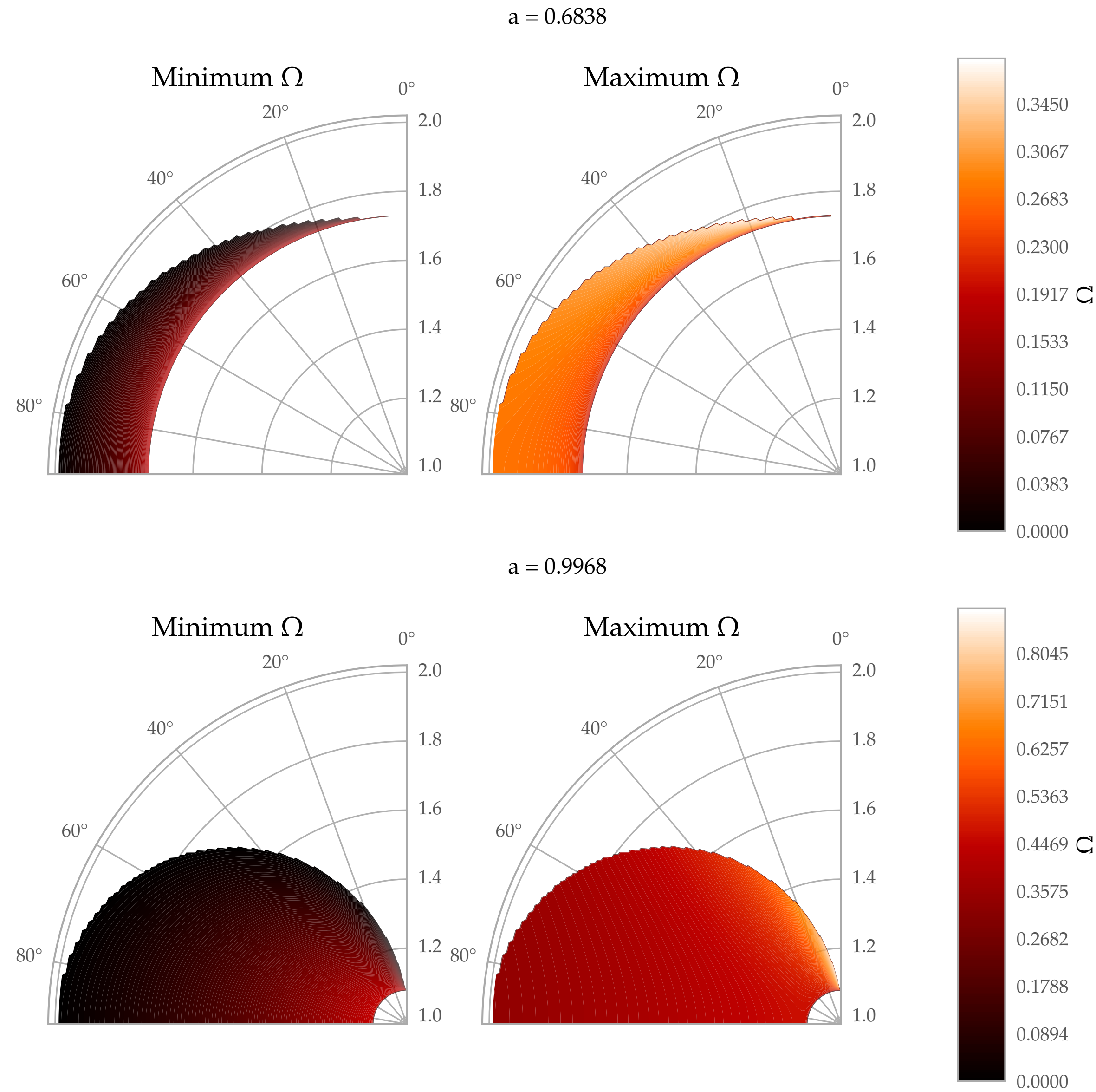

Figure 7: Allowed angular velocities - polar plot with varying θ, for $a/GM = 0.6838$ and $a/GM = 0.9968$.

but since the units of the reduced Planck constants are Js we can already say that the solution will be $\alpha = 1$, $\beta = 0$. So, the value will be

$$\mathtt{t} = \frac{\mathrm{J}}{\mathrm{GeV}} \times \frac{\hbar}{\mathrm{Js}} = 6.58 \times 10^{-25} \,. \tag{10.35}$$

We can use the exact same procedure to get the value of the length in meters, as defined by $L = \mathtt{L}\mathrm{m}$. We find $\alpha = \beta = 1$:

$$\mathtt{L} = \frac{\mathrm{J}}{\mathrm{GeV}} \times \frac{\hbar}{\mathrm{Js}} \times \frac{c}{\mathrm{ms}^{-1}} = 1.97 \times 10^{-16} . \tag{10.36}$$

If everything we have is in SI units, we can drop them so the computations become quite fast: up to SI units we have $t = \hbar/(10^9 e)$ and $L = \hbar c/(10^9 e)$.

Alternatively, we can adimensionalize any physical dimensional quantity by its Planck-units counterpart:

$$\frac{t}{t_P} = \frac{E_P}{\mathrm{GeV}} \qquad \text{and} \qquad \frac{L}{L_P} = \frac{E_P}{\mathrm{GeV}} , \tag{10.37}$$

and since the Planck units are formed from combinations of $\hbar, c, G, k_B$ and k_e we must have $t_P E_P = \hbar$ and $L_P E_P = \hbar c$.

The number of seconds in a year is approximately given by $3600 \times 24 \times 365.25 = 31557600 \approx \pi \times 10^7$.

10.3 Friedmann equations derivation

We start off by computing the Christoffel symbols for the FRLW line element $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\vec{x}^2$: for $\Gamma^\mu_{\nu\rho}$ to be nonzero it must have one temporal index and two spatial indices, since the only nonconstant element in the metric is a, which only depends on time. By symmetry we can write just two of the Christoffel symbols:

$$\Gamma^0_{ij} = \frac{1}{2} g^{00} \Big(\cancel{g_{0i,j}} + \cancel{g_{0j,i}} - g_{ij,0} \Big) = (-)^2 \frac{2}{2} \dot{a} a \delta_{ij} = \dot{a} a \delta_{ij} \tag{10.38}$$

and

$$\Gamma^i_{0j} = \frac{1}{2} g^{ik} \Big(\cancel{g_{k0,j}} + g_{kj,0} - \cancel{g_{0j,k}} \Big) = \frac{1}{2a^2} \delta_{jk} g_{jk,0} \delta_{ik} = \frac{\dot{a}}{a} \delta_{ij} . \tag{10.39}$$

The Ricci tensor is given by

$$R_{\mu\nu} = R^\beta_{\mu\beta\nu} = \Gamma^\beta_{\mu\nu,\beta} - \Gamma^\beta_{\mu\beta,\nu} + \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha} , \tag{10.40}$$

and we can see, term by term, that the nonzero Christoffels cannot contribute in the R_{0i} expression (for example, in the third term we have that if $\mu = 0$ and $\nu = i$ then α must be spatial, but then we have a symbol like $\Gamma^\beta_{j\beta}$, which is always zero).

So, we only compute R_{00} and R_{ij}.

We have:

$$R_{00} = R^{\beta}_{0\beta 0} = \cancel{\Gamma^{\beta}_{00,\beta}} - \Gamma^{i}_{0i,0} + \cancel{\Gamma^{\alpha}_{00}}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{i}_{0j}\Gamma^{j}_{0i} \tag{10.41}$$

$$= -3\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \tag{10.42}$$

$$= -3\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2}\right) \tag{10.43}$$

$$= -3\frac{\ddot{a}}{a} \tag{10.44}$$

while for the spatial components we have:

$$R_{ij} = R^{\beta}_{i\beta j} \tag{10.45}$$

$$= \Gamma^{0}_{ij,0} - \cancel{\Gamma^{\beta}_{i\beta,j}} + \Gamma^{0}_{ij}\Gamma^{k}_{0k} - \Gamma^{0}_{ik}\Gamma^{k}_{j0} - \Gamma^{k}_{i0}\Gamma^{0}_{jk}\,, \tag{10.46}$$

where we split the last term in two based on whether the upper index of the first symbol was 0 or spatial. Anytime two spatial indices appear we have a δ, so we bring it forward; also we have a term containing $\Gamma^{k}_{0k} = 3\dot{a}/a$. We find:

$$R_{ij} = \delta_{ij}\left(\partial_0(\dot{a}a) + a\dot{a}\times 3\frac{\dot{a}}{a} - 2\dot{a}^2\right) = \delta_{ij}\left(\dot{a}^2 + a\ddot{a} + 3\dot{a}^2 - 2\dot{a}^2\right) = \delta_{ij}\left(a\ddot{a} + 2\dot{a}^2\right), \tag{10.47}$$

so now we can compute

$$R = g^{\mu\nu}R_{\mu\nu} = -R_{00} + 3a^{-2}(R_{11}) = +3\frac{\ddot{a}}{a} + 3a^{-2}\left(a\ddot{a} + 2\dot{a}^2\right) \tag{10.48}$$

$$= 6\frac{\ddot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)^2, \tag{10.49}$$

which allows us to compute the full Einstein tensor, which is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\,, \tag{10.50}$$

so we have

$$G_{00} = -3\frac{\ddot{a}}{a} + (-)^2\frac{1}{2}\left(6\frac{\ddot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)^2\right) = 3\frac{\dot{a}^2}{a^2}\,, \tag{10.51}$$

while

$$G_{ij} = \delta_{ij}\left(a\ddot{a} + 2\dot{a}^2 - \frac{1}{2}a^2\left(6\frac{\ddot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)^2\right)\right) \tag{10.52}$$

$$= \delta_{ij}\left(a\ddot{a} + 2\dot{a}^2 - 3a\ddot{a} - 3\dot{a}^2\right) \tag{10.53}$$

$$= \delta_{ij}\left(-2a\ddot{a} - \dot{a}^2\right). \tag{10.54}$$

Now we can write the Einstein Field equations: we just need a stress-energy tensor. We choose a perfect fluid:

$$T^{\mu\nu} = (p+\rho)u^\mu u^\nu + pg^{\mu\nu} = \rho u^\mu u^\nu + ph^{\mu\nu}\,, \tag{10.55}$$

where we define the *projection tensor* onto the subspace orthogonal to the 4-velocity at each point: $h^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$. What should its velocity be? by isotropy it *must* be $u^\mu = \delta^\mu_0$ in the frame where we assumed isotropy and homogeneity.

Then, we can write the time and space components of the EFE:

$$G_{\mu\nu} = \frac{1}{M_P^2} T_{\mu\nu}\,, \tag{10.56}$$

for which we need the lower-index $T_{\mu\nu}$: it is defined by

$$T_{\mu\nu} = \rho u_\mu u_\nu + ph_{\mu\nu}\,, \tag{10.57}$$

and since $g_{00} = g^{00} = -1$ while $u_\mu = (1, \vec{0})^\top$ we have $h_{00} = h_{0i} = 0$ and $h_{ij} = g_{ij} = a^2\delta_{ij}$.

So we get

$$3\frac{\dot{a}^2}{a^2} = \frac{\rho}{M_P^2}\,, \tag{10.58}$$

and

$$-2a\ddot{a} - \dot{a}^2 = \frac{pa^2}{M_P^2} \qquad \Longrightarrow \qquad 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{p}{M_P^2}\,. \tag{10.59}$$

Now we need to treat the conservation of the stress-energy tensor: we start with the equations $\nabla_\mu T^{\mu i} = 0$. These, written explicitly, are:

$$\partial_\mu T^{\mu i} + \Gamma^\mu_{\mu\nu} T^{\nu i} + \Gamma^i_{\mu\nu} T^{\mu\nu} = 0\,, \tag{10.60}$$

but recall the fact that the only dependence is time-dependence: so $\partial_\mu T^{\mu i} = \partial_0 T^{0i} = 0$ always since $T^{0i} = 0$. Similarly, in the second term we must have $\mu = i$ and $\nu = 0$ for the symbol to be nonzero, but then it vanishes, and in the third term we would need one of μ and ν to be 0 ant the other to be spatial. So, the whole equation is zero with no conditions, since the stress-energy tensor is diagonal.[15]

So we only need to treat

$$0 = \nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\nu} T^{\nu 0} + \Gamma^0_{\mu\nu} T^{\mu\nu}\,, \tag{10.61}$$

[15] A comment: we assumed the velocity field to be uniformly timelike, so it should come as no surprise that the conservation of the stress energy tensor along the spatial directions is an identity.

and like before we have some semplifications: $\partial_\mu T^{\mu 0} = \partial_0 T^{00} = \dot{\rho}$; while in the second term we must have $\nu = 0$ and $\mu = i$, so we get a term depending on $\rho = T^{00}$ and the contraction of the Christoffel symbols $\Gamma^i_{i0} = 3\dot{a}/a$. The last term must have $\mu = \nu = j$ in order to be nonzero, and it comes out to be $\Gamma^0_{jj} T^{jj} = 3\dot{a}ap/a^2 = 3\dot{a}p/a$. This is because $T^{ij} = h^{ij}p$, but $h^{ij} = g^{ij} = a^{-2}$.

In the end, we have

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0\,. \tag{10.62}$$