

# General Relativity exercises

Jacopo Tissino, Giorgio Mentasti, Alessandro Lovo

October 30, 2019

We set  $c = 1$ .

Here we will often use (anti)symmetrization of indices, which makes some calculations much easier. The idea of symmetrization is to sum over all permutation of the selected indices, with a minus sign for the odd permutation if the case of anti symmetrization. So, for instance,  $F_{\mu\nu}$  can be antisymmetrized into  $F_{[\mu\nu]} = 1/2(F_{\mu\nu} - F_{\nu\mu})$  and symmetrized into  $F_{(\mu\nu)} = 1/2(F_{\mu\nu} + F_{\nu\mu})$ .

The factor  $1/2$  is in general  $1/n!$ , where  $n$  is the number of antisymmetrized indices. This is included because in general we will be summing  $n!$  terms, and we want to write things like: “ $F_{\mu\nu}$  is antisymmetric means  $F_{\mu\nu} = F_{[\mu\nu]}$ ”, so we need to rescale the sum to make it into an average.

The general formulas are then:

$$F_{[\mu_1 \dots \mu_n]} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) F_{\sigma(\mu_1) \dots \sigma(\mu_n)} \quad (0.1a)$$

$$F_{(\mu_1 \dots \mu_n)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} F_{\sigma(\mu_1) \dots \sigma(\mu_n)} , \quad (0.1b)$$

where  $\mathfrak{S}_n$  is the *symmetric group* of permutations of  $n$  elements, and the sign of a permutation  $\sigma \in \mathfrak{S}_n$  is  $\pm 1$ , depending on the parity of pair swaps that are needed to get that configuration (we fix  $(\text{sign}(\mathbb{1}) = 1)$ ).

If we want to symmetrize indices which are not next to each other, we will denote the end of the (anti)symmetrized indices by a vertical bar.

# Sheet 1

## 1.1 Lorentz transformations

### 1.1.1 Inverses

We can consider a Lorentz boost with velocity  $v$  in the  $x$  direction, and we look at its representation in the  $(t, x)$  plane (since the  $y$  and  $z$  directions are unchanged). Its matrix expression looks like:

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix}, \quad (1.1)$$

where  $\gamma = 1/\sqrt{1-v^2}$ . The inverse of this matrix can be computed using the general formula for a 2x2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (1.2)$$

The determinant of  $\Lambda$  is equal to  $\gamma^2(1-v^2) = 1$ , therefore the inverse matrix is:

$$\Lambda = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix}. \quad (1.3)$$

### 1.1.2 Invariance of the spacetime interval

Our Lorentz transformation is

$$dt' = \gamma(dt - v dx) \quad (1.4a)$$

$$dx' = \gamma(-v dt + dx) \quad (1.4b)$$

$$dy' = dy \quad (1.4c)$$

$$dz' = dz \quad (1.4d)$$

and we wish to prove that the spacetime interval, defined by  $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$  is preserved:  $ds'^2 = ds^2$ . Let us write the claimed equality explicitly:

$$-dt^2 + dx^2 + dy^2 + dz^2 = \gamma^2(dt - v dx)^2 + \gamma^2(-v dt + dx)^2 + dy^2 + dz^2 \quad (1.5a)$$

### 1.1.3 Tensor notation pseudo-orthogonality

The invariance of the spacetime interval  $ds'^2 = ds^2$  can be also written as  $\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx'^\mu dx'^\nu$ . By making the primed differentials explicit we have:

$$\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \Lambda^\mu_\rho dx^\rho \Lambda^\nu_\sigma dx^\sigma, \quad (1.6)$$

but the dummy indices on the LHS can be changed to  $\rho$  and  $\sigma$ , so that both sides are proportional to  $dx^\rho dx^\sigma$ . Doing this we get:

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = (\Lambda^\top)_\rho^\mu \eta_{\mu\nu} \Lambda^\nu_\sigma, \quad (1.7)$$

or, in matrix form,  $\eta = \Lambda^\top \eta \Lambda$ .

### 1.1.4 Explicit pseudo-orthogonality

For simplicity but WLOG we consider a boost in the  $x$  direction with velocity  $v$  and Lorentz factor  $\gamma$ . The matrix expression to verify is:

$$\begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (1.8a)$$

$$\begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} -\gamma & v\gamma \\ -v\gamma & \gamma \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (1.8b)$$

$$\begin{bmatrix} -\gamma^2 + \gamma^2 v^2 & v\gamma^2 - v\gamma^2 \\ v\gamma^2 - v\gamma^2 & -v\gamma^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (1.8c)$$

which by  $\gamma^2 = 1/(1 - v^2)$  confirms the validity of the expression.

## 1.2 Muons

### 1.2.1 Nonrelativistic approximation

The survival probability is given by  $\mathbb{P}(t) = \exp(-t/2.2 \times 10^{-6} \text{ s})$ . If the ground is  $h = 15 \text{ km}$  away, then the muon will reach it in  $t = h/v = 15 \text{ km}/(0.995c) \approx 5.03 \times 10^{-5} \text{ s}$ , therefore  $\mathbb{P}(t) \approx 1.2 \times 10^{-10}$ .

### 1.2.2 Relativistic effects: ground perspective

The observer on the ground will see the muon having to traverse the whole  $h = 15 \text{ km}$ , but the muon's time will be dilated for them by a factor  $\gamma_v \approx 10$ : therefore the survival probability will be  $\mathbb{P}(t) = \exp(-t/(\gamma_v \times 2.2 \times 10^{-6} \text{ s})) \approx 0.1$ .

### 1.2.3 Relativistic effects: muons perspective

The muons in their system will observe length contraction, with respect to Lorentz boost, by a factor  $\gamma_v \approx 10$ : therefore the survival probability will be  $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6} \text{ s})\right) \approx 0.1$ . This result is the same of the one predicted by ground observer, with respect to relativity principle.

## 1.3 Radiation

### 1.3.1 New angle

In the source frame the radiation velocity components are  $u'_x = \cos \theta'$ ,  $u'_y = \sin \theta'$ . From the composition of velocities we obtain:

$$u_y = \sin \theta = \frac{dy}{dt} = \frac{dy'}{\gamma_v(dt' + v dx')} = \frac{\sin \theta'}{\gamma_v(1 + v \cos \theta')} \quad (1.9a)$$

$$u_x = \cos \theta = \frac{dx}{dt} = \frac{\gamma_v(dx' + v dt')}{\gamma_v(dt' + v dx')} = \frac{\cos \theta' + v}{1 + v \cos \theta'} \quad (1.9b)$$

hence:

$$\frac{1}{\tan \theta} = \frac{\gamma_v}{\tan \theta'} + \frac{\gamma_v v}{\sin \theta'}. \quad (1.10)$$

### 1.3.2 Angle plot and relevant limits

See the jupyter notebook in the python folder for plots. For  $v = 0$  we have  $\theta = \theta'$  as we expected, while for  $v = 1$ ,  $\theta = 0$ .

### 1.3.3 Radiation speed invariance

Are the components of the velocity, which we called  $\sin \theta$  and  $\cos \theta$ , actually normalized? Let us check:

$$\sin^2 \theta + \cos^2 \theta = \frac{\left(\frac{\sin \theta'}{\gamma_v}\right)^2 + (\cos \theta' + v)^2}{(1 + v \cos \theta')^2} \quad (1.11a)$$

$$= \frac{(1 - v^2) \sin^2 \theta' + \cos^2 \theta' + v^2 + 2v \cos \theta'}{(1 + v \cos \theta')^2} \quad (1.11b)$$

$$= \frac{1 + v^2(1 - \sin^2 \theta') + 2v \cos \theta'}{(1 + v \cos \theta')^2} = 1, \quad (1.11c)$$

therefore the square modulus of the speed of the radiation is still  $c$ , as we could have assumed earlier.

### 1.3.4 Isotropic emission

Since the angular distribution of emission varies when changing inertial reference, we might suppose that every system in relative motion respect to  $O$  with  $v \neq 0$  observes nonisotropic emission.

This can be seen by noticing that for  $v \simeq 1$  we have that in the observer system there is almost only emission at an angle  $\theta = 0$ . In general, since there is a Lorentz  $\gamma$  factor multiplying a function of the angle in the radiation emission frame  $O'$ , the cotangent of the angle in the observation frame  $O$  must get larger and larger as the relative velocity  $v$  increases, therefore the radiation gets compressed towards angles with large cotangents:  $\theta \sim 0$ .

See the jupyter notebook in the python folder for interactive plots :)

## Sheet 2

### 2.1 Constant acceleration

#### 2.1.1 Coordinate velocity

We are given the position as a function of time,

$$x(t) = \frac{\sqrt{1 + \kappa^2 t^2} - 1}{\kappa}, \quad (2.1)$$

and we can directly compute its derivative

$$v(t) = \frac{dx}{dt} = \frac{\kappa t}{\sqrt{\kappa^2 t^2 + 1}} = \frac{1}{\sqrt{\frac{1}{\kappa^2 t^2} + 1}}. \quad (2.2)$$

It is clear from the expression that  $|v| < 1$  for all times, while  $v$  approaches 1 at positive temporal infinity and  $-1$  at negative temporal infinity.

#### 2.1.2 Components of the 4-velocity

The Lorentz factor  $\gamma$  is given by

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{1 - \frac{\kappa^2 t^2}{\kappa^2 t^2 + 1}}} = \sqrt{\kappa^2 t^2 + 1}, \quad (2.3)$$

therefore the four-velocity is given by:

$$u^\mu = \begin{bmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix}. \quad (2.4)$$



Figure 1: Velocity as a function of coordinate time  $t$

### 2.1.3 Proper time

The relation between coordinate and proper time is given by the definition of the first component of the four-velocity:  $u^0 = dt/d\tau = \gamma$ , therefore  $d\tau = dt/\gamma$ . Integrating this relation we get:

$$\tau = \int_0^\tau d\tau' = \int_0^t \frac{dt'}{\gamma(t')} = \frac{\text{arcsinh}(\kappa t)}{\kappa}, \quad (2.5)$$

where the constant of integration is selected by imposing  $t = 0 \iff \tau = 0$ . Notice that, as we would expect, when expanding up to first order near  $t = \tau = 0$  we have  $t \sim \tau$ , since in that region the velocity is much less than unity.

The inverse relation is given by  $t = \sinh(\kappa\tau)/\kappa$ . Using this, we can write:

$$x(t(\tau)) = \frac{\cosh(\kappa\tau) - 1}{\kappa}. \quad (2.6)$$

### 2.1.4 Four-acceleration

Now, we wish to compute the four-acceleration. There are many ways to approach this: an easy one is to simply find the explicit expression  $u^\mu(\tau)$  and to differentiate

it. The expression we get is:

$$a^\mu = \frac{d}{d\tau} u^\mu = \frac{d}{d\tau} \begin{bmatrix} \frac{\sqrt{\sinh^2(\kappa\tau) + 1}}{\sqrt{\kappa^2 t^2 + 1} \sinh(\kappa\tau)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2\kappa} \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau) + 1}} \\ \kappa \cosh(\kappa\tau) \\ 0 \end{bmatrix}, \quad (2.7)$$

which is a bit unwieldy but it can be used to check two important facts:  $a^\mu a_\mu = \text{const}$  and  $a^\mu u_\mu = 0$ . The first of the two is:

$$a^\mu a_\mu = -(a_0)^2 + (a_1)^2 = \kappa^2 \cosh^2(\kappa\tau) - \frac{\kappa^2 \sinh^2(2\kappa\tau)}{2(\cosh(2\kappa\tau) + 1)} = \kappa^2, \quad (2.8)$$

which tells us that the constant acceleration  $\sqrt{a^\mu a_\mu} = \kappa$ .

Also, we verify the orthogonality to the four-velocity:

$$a^\mu u_\mu = -\frac{\sqrt{2\kappa} \sqrt{\sinh^2(\kappa\tau) + 1} \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau) + 1}} + \kappa \sinh(\kappa\tau) \cosh(\kappa\tau) = 0. \quad (2.9)$$

### 2.1.5 Local velocity & acceleration

We can apply a Lorentz boost corresponding to this velocity: it will be given by the matrix:

$$\begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.10)$$

where  $v$  and  $\gamma$  are those found before. Without doing any calculations we could already say that the transformed velocity will be equal to the time-like unit vector, while the acceleration will be equal to  $\kappa$  times the unit  $x$ -directed vector.

The velocity becomes:

$$(u^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (2.11)$$

as we expected.

The acceleration instead becomes:

$$(a^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2\kappa} \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau) + 1}} \\ \kappa \cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \kappa \\ 0 \\ 0 \end{bmatrix}, \quad (2.12)$$

At small speeds the Lorentz boost matrix reduces to the identity matrix: this implies  $kt \simeq k\tau \simeq 0$ . In this case we obtain the same results of the rest frame of the particle for both acceleration and speed.

## 2.2 Fixed target collision

### 2.2.1 Center of mass momenta

In the CoM frame, the momenta of the two protons are respectively  $(E_p, \pm p, 0, 0)^\top = m_p(\gamma, \pm v, 0, 0)$ , where  $E_p^2 = m_p^2 + p^2$ . The total CoM energy is  $-(p_A^\mu + p_B^\mu)^2 = 2m_p^2$ .

### 2.2.2 Center of mass velocity

The momentum of particle  $B$  will be given by  $p^\mu = m_p u^\mu = (m_p \gamma, m_p \gamma v, 0, 0)^\top$ . Therefore,  $\gamma v = p/m_p$ . Solving this we get:

$$v = \frac{p}{m_p} \sqrt{\frac{1}{(p/m_p)^2 + 1}} = \frac{p}{E_p}, \quad (2.13)$$

### 2.2.3 Lab frame momenta

The momentum of particle  $B$  in its own rest frame will just be  $(m_p, 0, 0, 0)^\top$ . The momentum of particle  $A$  instead will be given by a boost in the  $x$  direction with velocity  $-v$ :

$$(p_A^\mu)_{\text{lab}} = \begin{bmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_p \\ p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma E_p + v\gamma p \\ v\gamma E_p + \gamma p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_p \gamma^2 (1 + v^2) \\ 2\gamma p \\ 0 \\ 0 \end{bmatrix}, \quad (2.14)$$

## 2.3 Weak field gravitational time dilation

### 2.3.1 Time dilation expression

It is more intuitive geometrically to deal with a pulse sent from  $A$  to  $B$ , for which we expect the time dilation to work in the opposite sense:

$$\Delta t_B = \Delta t_A (1 + gh), \quad (2.15)$$

up to first order in  $gh$  and  $g\Delta t_A$ , since  $(1 + gh)(1 - gh) = 1 - (gh)^2 = 1$  to first order in  $gh$ . Alternatively, one can just map  $g \rightarrow -g$  to recover the time contraction for pulses sent in the other direction.



We know that the paths of the observers are two curves of constant acceleration: we know their explicit expression from equation (2.1), and additionally we assume that they are separated by a space interval  $h$ :

$$x_A(t) = \frac{\sqrt{1 + (gt)^2} - 1}{g} \quad (2.16a)$$

$$x_B(t) = \frac{\sqrt{1 + (gt)^2} - 1}{g} + h. \quad (2.16b)$$

At  $t = 0$  Alice sends a pulse, which then reaches Bob at a time  $t_1$ . After a time  $\Delta t_A$ , she sends another, which then reaches Bob at a time  $t_2$ . Right now, we are referring to all times as measured in the rest frame of Alice at  $t = 0$ . These times can be found by imposing that the space and time separation between the events of the pulse being sent and received are equal, since it travels at light speed: the equations which represent this are  $x_B(t_1) = t_1$  and  $x_B(t_2) - x_A(\Delta t_A) = t_2 - \Delta t_A$ . Substituting the expressions for the positions:

$$t_1 = \frac{\sqrt{1 + (gt_1)^2} - 1}{g} + h \quad (2.17a)$$

$$t_2 - \Delta t_A = \frac{\sqrt{1 + (gt_2)^2} - 1}{g} + h - \left( \frac{\sqrt{1 + (g\Delta t_A)^2} - 1}{g} \right). \quad (2.17b)$$

Now, it is just a matter of calculation to solve these equations, expand up to first order in the adimensional parameters  $gh$  and  $g\Delta t_A$  and one recovers the desired expression for  $\Delta t_B = t_2 - t_1$ .

There is one more consideration to make though: what about the Lorentz time dilation for Bob? This is actually a *second order effect*.

**Claim 2.1.** *The time interval measured by Bob in his frame at  $t \sim t_1$  is the same as the one measured in the rest frame of Alice at  $t = 0$  up to first order in  $gh$  and  $g\Delta t_B$ .*

*Proof.* We perform a Lorentz boost to the velocity of Bob at  $t = t_1$ : this is given by equation (2.2), and is equal to:

$$v = \frac{gt}{\sqrt{(gt)^2 + 1}}, \quad (2.18)$$

with a Lorentz factor of  $\gamma = \sqrt{(gt)^2 + 1}$  (see equation (2.3)).

The temporal separation between the two events is  $\Delta t_B$ , while the spatial separation is  $\Delta x_B \approx v\Delta t_B$  to first order. The boost, in the  $(t, x)$  plane, looks like:

$$\begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix}' = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Delta t_B \\ \Delta x_B \end{bmatrix} = \begin{bmatrix} \Delta t_B \left( \sqrt{(gt)^2 + 1} - (gt)^2 / \sqrt{(gt)^2 + 1} \right) \\ -gt\Delta t_B + \sqrt{(gt)^2 + 1}gt\Delta t / \sqrt{(gt)^2 + 1} \end{bmatrix}, \quad (2.19)$$



Figure 2: Visualization of the beams, in the frame where the rocket is stationary as the first beam is being sent. The two curves intersecting the space axis are the tip and tail of the spaceship; the beams being sent from the tail are events  $C$  and  $D$ , while their reception at the tip are events  $A$  and  $B$ . Event  $E$  is just calculated as  $B - A$ , to make comparisons with  $D$  easier.  $H$  and  $G$  are computed by tracing the dotted line of points which have the same spacetime interval from the origin as points  $D$  and  $E$  respectively, and selecting its intersection with the temporal axis: this effectively means finding the proper time separation between the two beams being sent/received.

therefore as we would expect the spatial separation is eliminated, while expanding the factor multiplying the temporal one near  $gt = 0$  we get:

$$\sqrt{(gt)^2 + 1} - (gt)^2 / \sqrt{(gt)^2 + 1} = 1 + O((gt)^2), \quad (2.20)$$

which proves our result.  $\square$

### 2.3.2 Gravitational time dilation

By the equivalence principle, the effects measured in a uniformly accelerating frame at  $g$  are the same as those measured in a gravitational field with constant acceleration  $g$ . The gravitational field in such a frame is given by  $\Phi = gh$ , where  $h$  is the height (with arbitrary zero point): the result follows.

### 2.3.3 Twins and gravitation

The gravitational time dilation difference, in absolute value, is given by:

$$\Delta t = t_{\text{elapsed}} \frac{g\Delta h}{c^2} \approx 1 \text{ yr} \frac{10 \text{ m/s}^2 \times 100 \text{ m}}{(3 \times 10^8 \text{ m/s})^2} \approx 3.5 \times 10^{-7} \text{ s}. \quad (2.21)$$

We are asked what is the age of the twin on the ground as measured by the twin who is higher up: this is analogous to the situation considered in the first section of this problem; the twin higher up will measure the twin lower down to be older, specifically if  $\text{age}_{\text{up}} = 1 \text{ yr}$ , then the observer up in the palace will measure the age of the twin at ground level as:

$$\text{age}_{\text{down}} = 1 \text{ yr} + 3.5 \times 10^{-7} \text{ s} \approx (1 + 1 \times 10^{-14}) \text{age}_{\text{up}}. \quad (2.22)$$

## Sheet 3

### 3.1 Changes of coordinate system

We denote by  $x^\mu = (x, y)$  and  $x'^\mu = (r, \theta)$ . Then, we have the following Jacobian matrices:

$$\frac{\partial x^\mu}{\partial x'^\nu} = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{bmatrix} \quad (3.1a)$$

$$\frac{\partial x'^\nu}{\partial x^\rho} = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{bmatrix}, \quad (3.1b)$$

which can be found by plain differentiation of the change of coordinates, recalling  $(\arctan x)' = 1/(1 + x^2)$ . Then, we can compute the product of these two matrices: it comes out to be

$$\frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\rho} = \delta_\rho^\mu, \quad (3.2)$$

since on the diagonal we get  $\cos^2(\theta) + \sin^2(\theta) = 1$ , while on the off-diagonal terms we get a multiple of  $\sin(\theta) \cos(\theta) - \sin(\theta) \cos(\theta) = 0$ .

Note that relation (3.2) is just the chain rule written in more generality: substituting the explicit coordinates for  $x^\mu$  and  $x'^\mu$  we get the desired expression.

### 3.2 Properties of covariant differentiation

#### 3.2.1 Metric compatibility of the connection

We wish to show that  $\nabla_\alpha g_{\mu\nu} = 0$ . A very simple way to prove this is by going in the LIF: there, the equation reads  $\partial_\alpha \eta_{\mu\nu} = 0$ , which is immediately satisfied since

the components of the Minkowski metric are constants. Then, since the equation is tensorial, the result extends to any frame.

This was not the spirit of the exercise, however: let us prove it in a different generic frame. To this end, we define the *Christoffel symbols of the first kind* (while the regular ones are of the second kind):

$$\Gamma_{\mu\nu\rho} = g_{\mu\sigma}\Gamma_{\nu\rho}^{\sigma} = \frac{1}{2}\left(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu}\right). \quad (3.3)$$

These are useful since, as all their indices are down, it is easier to study their symmetry properties.

Note that if we antisymmetrize the first and last index, we get  $\Gamma_{[\mu|\nu|\rho]} = 1/2g_{\mu\rho,\nu}$  since the first and last terms in the sum cancel (in the latter we must invert the indices  $\nu$  and  $\rho$  in order to see this, but this can always be done by the symmetry of the metric).

Then, we write the expression for the covariant derivative of the metric:

$$\nabla_{\alpha}g_{\mu\nu} = \partial_{\alpha}g_{\mu\nu} - \Gamma_{\mu\alpha}^{\rho}g_{\rho\nu} - \Gamma_{\nu\alpha}^{\rho}g_{\mu\rho} = g_{\mu\nu,\alpha} - \Gamma_{\nu\alpha\mu} - \Gamma_{\mu\alpha\nu}, \quad (3.4)$$

which is just  $g_{\mu\nu,\alpha} - 2\Gamma_{[\nu|\alpha|\mu]} = g_{\mu\nu,\alpha} - g_{\nu\mu,\alpha} = 0$ , again by the symmetry of the metric.

One could argue that this is the opposite way round: we should *assume*  $\nabla_{\mu}g_{\rho\sigma} = 0$  and derive from it the formula that was given for the Christoffel symbols in terms of the partial derivatives of the metric.

### 3.2.2 Leibniz rule

As before, this can be proved in the LIF from the Leibniz rule of regular partial derivatives.

As before, we like to calculate therefore we show this explicitly in any frame.

The derivative of the tensor product looks like:

$$\nabla_{\mu}\left(A_{\nu\lambda}B_{\rho}\right) = \partial_{\mu}\left(A_{\nu\lambda}B_{\rho}\right) - \Gamma_{\mu\nu}^{\sigma}A_{\sigma\lambda}B_{\rho} - \Gamma_{\mu\lambda}^{\sigma}A_{\nu\sigma}B_{\rho} - \Gamma_{\mu\rho}^{\sigma}A_{\nu\lambda}B_{\sigma}, \quad (3.5)$$

while the sum of derivatives looks like:

$$\begin{aligned} B_{\rho}\nabla_{\mu}A_{\nu\lambda} + A_{\nu\lambda}\nabla_{\mu}B_{\rho} = & B_{\rho}\partial_{\mu}A_{\nu\lambda} - \Gamma_{\mu\nu}^{\sigma}A_{\sigma\lambda}B_{\rho} - \Gamma_{\mu\lambda}^{\sigma}A_{\nu\sigma}B_{\rho} \\ & + A_{\nu\lambda}\partial_{\mu}B_{\rho} - \Gamma_{\mu\rho}^{\sigma}A_{\nu\lambda}B_{\sigma}, \end{aligned} \quad (3.6)$$

so we can see that the Christoffel terms are equal, and the partial derivative terms also are since we have the Leibniz rule for partial derivatives.

### 3.3 2D Christoffel symbols

#### 3.3.1 Polar coordinates

The metric and inverse metric are respectively given by  $g_{\mu\nu} = \text{diag}(1, r^2)$  and  $g^{\mu\nu} = \text{diag}(1, r^{-2})$ . We only care about the partial derivatives of the lower-indices one, and the only nonvanishing derivative is  $g_{11,0} = 2r$ , where we mean  $(x^0, x^1) = (r, \theta)$ .

Then it is tedious but straightforward to perform the direct computation. Things that make it faster are discarding immediately terms which cannot contribute (such as  $g_{\alpha\beta,\gamma}$  where at least one of  $\alpha$  and  $\beta$  is not 1 or  $\gamma$  is not 0, and only looking at the six independent symbols instead of the eight total ones (since  $\Gamma_{01}^\alpha = \Gamma_{10}^\alpha$  for any  $\alpha$ ).

Then one can see that the nonvanishing symbols are

$$\Gamma_{11}^0 = \frac{1}{2}g^{0\alpha}(g_{\alpha 1,1} + g_{\alpha 1,1} - g_{11,\alpha}) = \frac{1}{2}g^{00}(-g_{11,0}) = -\frac{2r}{2} = -r \quad (3.7a)$$

$$\Gamma_{01}^1 = \frac{1}{2}g^{1\alpha}(g_{\alpha 0,1} + g_{\alpha 1,0} - g_{01,\alpha}) = \frac{1}{2}g^{11}g_{11,0} = \frac{1}{2}\frac{1}{r^2}2r = \frac{1}{r}. \quad (3.7b)$$

#### 3.3.2 Spherical surface

Now we have the following metric and inverse metric:

$$g_{\mu\nu} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2(\theta) \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} R^{-2} & 0 \\ 0 & R^{-2} \sin^{-2}(\theta) \end{bmatrix}, \quad (3.8)$$

but do note that  $R$  is a constant: given  $(x^0, x^1) = (\theta, \varphi)$ , we have as before that the only nontrivial derivative is  $g_{11,0} = 2R^2 \sin(\theta) \cos(\theta)$ .

Then this case is exactly analogous to the previous one: the same symbols are zero, so we can skip almost all of the computation and jump straight to:

$$\Gamma_{11}^0 = -\frac{1}{2}g^{00}g_{11,0} = -\sin(\theta) \cos(\theta) \quad (3.9a)$$

$$\Gamma_{01}^1 = \frac{1}{2}g^{11}g_{11,0} = \frac{\cos(\theta)}{\sin(\theta)}. \quad (3.9b)$$

### 3.4 Parallel transport

We know from the last exercise the metric and Christoffel symbols of 2D space and of the surface of a sphere.

The equations of parallel transport are in general  $u^\mu \nabla_\mu V^\nu = 0$ .

### 3.4.1 Flat space

We want to determine the behaviour of a vector field  $V^\mu(\theta)$  defined on a curve  $x^\mu(\theta) = (R, \theta)$  with fixed  $R$ , such that  $V^\mu(\theta = 0) = (0, 1/R)$  (a unit vector:  $V^\mu(0)V^\nu(0)g_{\mu\nu}(0) = 1$ ).

In our case the tangent vector of the curve is  $u^\mu = (0, 1)$ . Therefore the equations simplify to:

$$\nabla_1 V^\mu = \partial_1 V^\mu + \Gamma_{1\alpha}^\mu V^\alpha = 0, \quad (3.10)$$

which are two coupled differential equations; we can make them explicit and substitute the Christoffel symbols found earlier.

$$\nabla_1 V^0 = \partial_1 V^0 + \cancel{\Gamma_{10}^0 V^0} + \Gamma_{11}^0 V^1 = \partial_1 V^0 - r V^1 \quad (3.11a)$$

$$\nabla_1 V^1 = \partial_1 V^1 + \Gamma_{10}^1 V^0 + \cancel{\Gamma_{11}^1 V^1} = \partial_1 V^1 + \frac{1}{r} V^0, \quad (3.11b)$$

so we can write this linear system like:

$$\partial_1 \begin{bmatrix} V^0 \\ V^1 \end{bmatrix} = \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix} \begin{bmatrix} V^0 \\ V^1 \end{bmatrix}. \quad (3.12)$$

The eigenvalues of this matrix are  $\pm i$ , so its exponential is a pure rotation matrix:

$$\exp\left(\theta \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R(-\theta), \quad (3.13)$$

so the solution is

$$V^\mu(\theta) = R(-\theta)V^\mu(0). \quad (3.14)$$

This means that our vector is rotating clockwise with unit angular velocity in the  $(r, \theta)$  plane, just as the point it is defined at moves counterclockwise with unit angular velocity: therefore, if we look at the vector in Cartesian coordinates, we will see it always aligned with its initial direction and the same modulus.

Specifically, when  $\theta = \pi/2$  we get  $V^\mu = (1/R, 0)$ .

### 3.4.2 Curved space: spherical surface

Now our curve is  $x^\mu(\theta) = (\theta, 0)$  in the spherical coordinates  $x^\mu = (\theta, \varphi)$ . The metric is the one given in (3.8), the Christoffel symbols are the ones given in (3.9).

The parallel transport equations are now:

$$\nabla_1 V^0 = \partial_0 V^0 + \cancel{\Gamma_{00}^0 V^0} + \cancel{\Gamma_{01}^0 V^1} \quad (3.15a)$$

$$\nabla_1 V^1 = \partial_0 V^1 + \cancel{\Gamma_{00}^1 V^0} + \Gamma_{01}^1 V^1 = \partial_0 V^1 + \frac{V^1}{\tan(\theta)}. \quad (3.15b)$$

The first equation gives us  $V^1 = \text{const}$ , while we do not need to actually solve the second one: our initial condition is  $V^\mu(\theta = 0) = (R^{-1}, 0)$ , and  $V^1 \equiv 0$  is a solution to that first-order equation, so by the uniqueness we have found the whole solution.

Therefore the vector is constant in these coordinates.

Specifically, at  $\theta = \pi/2$  we get  $V^\mu = (1/R, 0)$ .

## Sheet 4

### 4.1 Riemann tensor computations

#### 4.1.1 LIF form

In the LIF, we have  $g_{\mu\nu} = \eta_{\mu\nu}$  and  $\Gamma_{\mu\nu,\alpha} = 0$ . Therefore, the Christoffel symbols are all zero. In the explicit expression of the Riemann tensor, which looks like  $R = \partial\Gamma + \Gamma\Gamma$  we can drop the second term and keep only the derivatives of the Christoffel symbols, which are nonzero since they depend on the second derivatives of the metric.

If the Christoffel symbols are zero, the computation of the curvature tensor is significantly easier:

$$R_{\nu\rho\sigma}^\mu = 2\partial_{[\rho}\Gamma_{\sigma]\nu}^\mu \quad (4.1a)$$

$$= 2\partial_{[\rho}\left(\frac{1}{2}g^{\mu\alpha}\left(g_{\alpha[\sigma],\nu]} + g_{\alpha\nu,[\sigma]} - g_{\nu[\sigma],\alpha]}\right)\right) \quad (4.1b)$$

$$= g^{\mu\alpha}\left(g_{\alpha[\sigma],\nu|\rho]} + g_{\alpha\nu,[\sigma|\rho]} - g_{\nu[\sigma],\alpha|\rho]}\right). \quad (4.1c)$$

So, if we lower the index of the (1, 3) Riemann tensor we get the desired expression for the all-lower (0, 4) Riemann tensor:

$$R_{\mu\nu\rho\sigma} = g_{\mu[\sigma],\nu|\rho]} - g_{\nu[\sigma],\mu|\rho]}. \quad (4.2)$$

#### 4.1.2 Ricci tensor and scalar

The Ricci tensor is given by:

$$R_{\mu\nu} = R_{\mu\alpha\nu}^\alpha = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = g^{\alpha\beta}\left(g_{\alpha[\nu],\mu|\beta]} - g_{\mu[\nu],\alpha|\beta]}\right). \quad (4.3)$$

So we have

$$R_{\mu\nu} = g^{\alpha\beta}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\beta,\mu\nu} - g_{\mu\nu,\alpha\beta} + g_{\mu\beta,\alpha\nu}\right) \quad (4.4a)$$

$$= g^{\alpha\beta}\left(2g_{\alpha(\nu,\mu)\beta} - g_{\mu\nu,\alpha\beta} - g_{\alpha\beta,\mu\nu}\right) \quad (4.4b)$$

$$= 2g_{\alpha(\nu,\mu)}^\alpha - \square g_{\mu\nu} - g^{\alpha\beta}g_{\alpha\beta,\mu\nu}, \quad (4.4c)$$

where<sup>1</sup> the square  $\square$  denotes the D'Alembert operator,  $\square = \partial_\beta \partial^\beta = g^{\alpha\beta} \partial_\alpha \partial_\beta$ .

The Ricci scalar, on the other hand, is given by

$$R = g^{\mu\nu} R_{\mu\nu} = 2g^{\mu\nu} g_{\alpha(\nu,\mu)}^{\alpha} - g^{\mu\nu} \square g_{\mu\nu} - g^{\alpha\beta} \square g_{\alpha\beta} = 2(g_{\alpha\nu}^{\alpha\nu} - g^{\mu\nu} \square g_{\mu\nu}), \quad (4.5)$$

where we identified the two terms containing the metric contracted with its d'Alembertian, since they are equal up to a relabeling of indices.

We neglected the symmetrization in the first term since when contracting with the metric the indices  $\mu\nu$  are automatically symmetrized.

#### 4.1.3 LIF identities

$$\nabla_\alpha \left( R_\beta^\alpha - \frac{R}{2} \delta_\beta^\alpha \right) = \partial_\alpha \left( g^{\alpha\lambda} R_{\lambda\beta} - \frac{R}{2} \delta_\beta^\alpha \right) \quad (4.6a)$$

$$= \eta^{\alpha\lambda} R_{\lambda\beta,\alpha} - \frac{1}{2} R_{,\beta}, \quad (4.6b)$$

since in the LIF the metric is the Minkowski one, and the Christoffel symbols are zero therefore  $\nabla_\alpha = \partial_\alpha$ .

#### 4.1.4 Contracted Bianchi identities

The first term is

$$\eta^{\beta\mu} R_{\mu\nu,\beta} = \eta^{\beta\mu} \left( 2g_{\alpha(\nu,\mu)}^{\alpha} - \square g_{\mu\nu} - g^{\alpha\beta} g_{\alpha\beta,\mu\nu} \right)_{,\beta}, \quad (4.7)$$

which we can expand out: the Minkowski metric raises the derivative with respect to  $x^\beta$  and turns it into a derivative with respect to  $x_\mu$ , and we can also expand the D'Alembertian and the symmetrization: we get

$$\eta^{\beta\mu} R_{\mu\nu,\beta} = g_{\alpha\nu,\mu}^{\alpha\mu} + g_{\alpha\mu,\nu}^{\alpha\mu} - g_{\mu\nu,\alpha}^{\alpha\mu} - g^{\alpha\beta} g_{\alpha\beta,\mu\nu}^{\mu} \quad (4.8a)$$

$$= +g_{\alpha\mu,\nu}^{\alpha\mu} - \left( g^{\alpha\beta} \square g_{\alpha\beta} \right)_{,\nu} \quad (4.8b)$$

$$= \partial_\nu \left( g_{\mu\alpha}^{\mu\alpha} - g^{\alpha\beta} \square g_{\alpha\beta} \right), \quad (4.8c)$$

since, up to a relabeling of indices, the first and third term in (4.8a) are equal.

The derivative  $\partial_\nu R$ , on the other hand, is twice that:

$$\partial_\nu R = \partial_\nu \left( 2g_{\mu\alpha}^{\mu\alpha} - 2g^{\alpha\beta} \square g_{\alpha\beta} \right), \quad (4.9)$$

as can be directly gathered from (4.5).

Therefore, the quantity in (4.6b) is zero.

---

<sup>1</sup>All the indices after the comma are derivatives, even when they become upper.



### 4.1.5 Alternative expression

Since the metric is covariantly constant, we can just bring it inside the derivative:

$$0 = g^{\beta\mu} \nabla_\alpha \left( R^\alpha_\mu - \frac{1}{2} R \delta^\alpha_\mu \right) = \nabla_\alpha \left( R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \right). \quad (4.10)$$

These are called the *Contracted Bianchi identities*, and can be alternatively derived from the Bianchi identities of the Riemann tensor,  $R_{\mu\nu[\rho\sigma;\alpha]} = 0$ .

## 4.2 Properties of the Ricci tensor

### 4.2.1 Symmetry generality

The fact that  $T^{\mu\nu} = T^{\nu\mu}$  is frame invariant could be derived plainly from the fact that it is a tensor equation; to be more explicit we can say that under a change of coordinates with Jacobian matrix  $\Lambda$  we have  $\tilde{T}^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$ ; when written in components like this the Jacobians commute, therefore we can just interchange them for both  $\tilde{T}^{\mu\nu}$  and  $\tilde{T}^{\nu\mu}$  to recover their equality.

### 4.2.2 Symmetry of the Ricci tensor.

The expression (4.4c) is manifestly symmetric: part of it is explicitly symmetrized, part is proportional to the metric, which is symmetric.

Therefore, the Ricci tensor is symmetric in any frame.

## 4.3 Weak field Einstein equations

We consider the *weak field* case, when  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  with  $|h_{\mu\nu}| \ll 1$ . We will then neglect second and higher order terms in  $h_{\mu\nu}$ .

### 4.3.1 The gravitational potential

We want to work towards Newton's equation, which is  $\nabla^2 \Phi = 4\pi G_N \rho$ , where  $\Phi$  is the gravitational potential. What defines the gravitational potential is<sup>2</sup> its effect on test masses: they accelerate with  $\vec{a} = -\vec{\nabla} \Phi$ .

What is the relativistic equivalent of this? A particle in GR follows a geodesic, a curve without proper acceleration. Proper acceleration is a four-vector defined by  $a^\mu = u^\nu \nabla_\nu u^\mu$ . If the four-velocity of a particle is  $u^\mu$ , then for it the equation of geodesic motion reads:

$$0 = a^\mu = u^\nu \partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha} u^\nu u^\alpha, \quad (4.11)$$

---

<sup>2</sup>In the lecture notes this is solved differently: instead of finding the gravitational potential from the equation of geodesic motion, there it is found by identification of the gravitational redshift.

which can be written in terms of derivatives with respect to proper time:

$$0 = \frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\alpha}^\mu u^\nu u^\alpha. \quad (4.12)$$

If the speeds are much less than that of light, this can be approximated:  $s \approx t$ , and the only terms which contribute to order 0 in  $v$  in the Christoffel symbol sum is the one with  $u^0 u^0 \approx 1$ : in the end then we get for the spatial components:

$$0 = \frac{d^2 x^i}{dt^2} + \Gamma_{00}^i, \quad (4.13)$$

so we can see that the main contribution in the low-speed limit to the (coordinate!) acceleration of the particle are the symbols  $\Gamma_{00}^i$ . The expression for these in terms of the perturbed metric is:

$$\Gamma_{00}^i = \frac{1}{2} \eta^{i\alpha} (2h_{\alpha 0,0} - h_{00,\alpha}) = -\frac{1}{2} h_{00}{}^{,i} = -\frac{1}{2} \vec{\nabla}^i h_{00}, \quad (4.14)$$

if we assume stationarity of the metric (which is justified if we are treating a problem such as the gravitational pull on a body on the surface of the Earth) at least up to first order in  $h$ .

Putting everything together: we found that

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} \vec{\nabla}^i h_{00}, \quad (4.15)$$

but the equation which defines the gravitational field is

$$\frac{d^2 x^i}{dt^2} = -\vec{\nabla}^i \Phi, \quad (4.16)$$

therefore, in the low-gravitational field and low-speed limit, we must identify  $h_{00} = -2\Phi$ .

### 4.3.2 Reframing the EFE

We can contract the EFE with the inverse metric, recalling that  $g^{\mu\nu} g_{\mu\nu} = 4$ , to get  $-R = 8\pi G_N T$ . Therefore, they can be reframed by substituting the curvature scalar term with the trace of the stress-energy tensor:

$$R_{\mu\nu} = 8\pi G_N \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \quad (4.17)$$

This will aid us by reducing the number of difficult curvature tensors to compute.

In the low-speed, weak-field limit matter has negligible pressure and is mostly just travelling in the time direction. Therefore, we approximate the stress energy

tensor as that of noninteracting dust:  $T^{\mu\nu} = \rho u^\mu u^\nu$ . In our frame matter is almost stationary, therefore we have that  $T_{00} \approx T^{00} \eta_{00} \eta_{00} \approx \rho$ . The trace of the curvature tensor is instead approximately  $T \approx T^{00} \eta_{00} \approx -\rho$ .

In the end, the 00 component of the equations reads:

$$R_{00} = 8\pi G_N \left( \rho - \frac{1}{2}(-\rho)\eta_{00} \right) = 4\pi G_N \rho. \quad (4.18)$$

Now we only need to show that  $R_{00} = \nabla^2 \Phi$ .

### 4.3.3 The derivatives in the curvature tensor

The component we need to calculate is  $R_{00} = R_{0\mu 0}^\mu = R_{0i0}^i$  by antisymmetry.

Let us recall here the general expression for the Riemann tensor:

$$R_{\nu\rho\sigma}^\mu = 2\Gamma_{[\nu|\sigma,\rho]}^\mu + 2\Gamma_{\sigma[\nu}^\alpha \Gamma_{\rho]\alpha}^\mu. \quad (4.19)$$

In our case, this simplifies to

$$R_{0i0}^i = \Gamma_{00,i}^i - \Gamma_{i0,0}^i + \Gamma_{00}^\alpha \Gamma_{i\alpha}^i - \Gamma_{0i}^\alpha \Gamma_{0\alpha}^i. \quad (4.20)$$

The second term contains time derivatives, which we decided to neglect by dealing with the stationary case. The third and fourth term are of second order in  $h$ .

The only remaining term is the first one: using the expression (4.14) and the identification  $h_{00} = -2\Phi$ , it is

$$R_{00} = \Gamma_{00,i}^i = \left( -\frac{1}{2} \partial^i h_{00} \right)_{,i} = \partial_i \partial^i \Phi, \quad (4.21)$$

which is what we wanted to prove.