General Relativity notes

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1 Special relativity

Definition 1.1. *An inertial frame is one in which Newton's laws hold: a free body moves with acceleration* $a^i = 0$.

Newton's first law establishes the *existence* of inertial frames.

Proposition 1.1. *The frames* O *and* O' *are both inertial frames iff* O' *moves with constant velocity wrt* O.

Proposition 1.2. *Coordinate transformations between inertial frames are Lorentz boosts, which in some coordinate frame can be written as*

$$t' = \gamma_v \left(t - \frac{vx}{c^2} \right) \tag{1a}$$

$$x' = \gamma_v (x - vt) \tag{1b}$$

$$y' = y \tag{1c}$$

$$z' = z\,, \tag{1d}$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$.

If $v \ll c$, so $v/c \sim 0$, they simplify to the identity for t, y, z and $x' = x - vt$: these are Galilean transformations.

If we have two events, x^μ and y^μ, they occur with some time and space separation $\Delta x^\mu = x^\mu - y^\mu$. We can compute $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$, where

$$\eta_{\mu\nu} = \operatorname{diag}\left(-c^2, 1, 1, 1 \right). \tag{2}$$

Proposition 1.3. *Under Lorentz transformations* Δs^2 *is invariant.*

We can classify separations between events as

- time-like when $\Delta s^2 < 0$;
- null-like when $\Delta s^2 = 0$;
- space-like when $\Delta s^2 > 0$.

We can draw spacetime diagrams. A light cone is the set of points which are null-like separated from a select point. Things can be only causally related to events inside the light-cone, with $\Delta s^2 \leq 0$.

1.1 Time dilation

Take two events which occur at the same location for O'. In the primed frame they will have coordinates $x^\mu = (t_0, x_0)$ and $y^\mu = (t_1, x_0)$.

Definition 1.2. *The* proper time *between these two events is* $t_1 - t_0 \stackrel{def}{=} \Delta\tau$.

We now see that $\Delta s'^2 = -c^2 \Delta\tau^2$. Then, any other observer will see the same $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = \Delta s'^2$.

This directly implies that $\Delta\tau \leq \Delta t$ for any observer, since $\Delta\tau^2 = \Delta t^2 - \Delta x^2/c^2$. This effect is called *time dilation*.

By how much exacly is time dilated? Of course $\Delta x = v\Delta t$, therefore $\Delta t = \gamma_v \Delta\tau$.

This effect explains a peculiar phenomenons: certain particles in the upper atmosphere decay into muons, which have a very short half-life. So short, in fact, that if we did not account for special relativity we'd expect to see next to none at the surface, since by the time they got here they would have alredy gone through several halving times. However, we must apply the rule of relativistic time dilation: the muons are travelling very fast towards the ground, therefore in the ground's frame of reference their time passes slower, allowing them to decay slower. So, a significant fraction of them arrives at the ground.

Inverse Lorentz transformation have the same expression as direct ones, but with $v \to -v$. This can be proved both mathematically by solving the equations and phisically by reasoning about their meaning. There is no preferential inertial frame.

A Lorentz transformation can be written in matrix form in the (ct, x) plane as:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \tag{3}$$

where we introduced the notation $\beta = v/c$.

The second equation is justified by the fact that there is an angle θ such that $\gamma = \cosh\theta$ and $\gamma\beta = \sinh\theta$: the angle θ will be $\theta = \tanh^{-1}(v/c)$. This is true because $\gamma^2 - \beta^2\gamma^2 = 1$, which is the same law that the hyperbolic functions obey: $\cosh^2 x - \sinh^2 x = 1$ holds for any x.

After a boost the ct' and x' axes are rotated into, respectively, the lines $ct = x/\beta$ and $ct = \beta x$: this comes directly from the transformation law. The ct' axis is defined by the equation $x' = 0$, which in the transformed coordinates ct and x reads $\gamma(x - \beta ct) = 0$, or $ct = x/\beta$.

Similarly $ct' = 0$ in the new coordinates reads $\gamma(ct - \beta x) = 0$, or $ct = \beta x$.

The axes are rotated by an angle which can approach $\pi/4$ but never reach it, since its tangent is defined by β, which can never reach 1.

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Last lecture we saw the fact that the ct' and x' axes are rotated by equal angles from the ct and x axes towards the $ct = x$ axis.

1.2 Relativity of simultaneity

Consider two events which are simultaneous in the O' frame. Their times in this frame are $t'_A = t'_B$.

In the O frame, instead, we have

$$ct'_{A,B} = \gamma_v \left(ct_{A,B} - \frac{v}{c} x_{A,B} \right) \tag{4}$$

$$ct_{A,B} = \underbrace{\frac{v}{c} x_{A,B}}_{\text{variable}} + \underbrace{\sqrt{1 - \frac{v^2}{c^2}} ct'_{A,B}}_{\text{constant}}, \tag{5}$$

where we can notice that the second term is the same if we switch from A to B, while the first one changes (the objects have different positions). So, the events are not simultaneous in the O frame.

1.3 Length contraction

If in the O frame, A occurs at $t = 0, x = 0$ while B occurs at $t = 0$, $x = L$, then L is the measured length of their spatial interval by O. We assume that this is the frame in which the object is moving, and we transform into a frame in which it is stationary: O'.

In the primed frame their coordinates will be:

$$x'_A = \gamma_v \left(x_A - \frac{v}{c} ct_A \right) \tag{6a}$$

$$x'_B = \gamma_v \left(x_B - \frac{v}{c} ct_B \right), \tag{6b}$$

therefore $x'_B - x'_A = \gamma_v (x_B - x_A)$: the length is contracted in the O frame, since $\gamma \geq 1$.

1.4 Addition of velocities

Two observers see an object moving with $v' = \mathrm{d}x'/\mathrm{d}t'$ and $v = \mathrm{d}x/\mathrm{d}t$ respectively. Their relative velocity is u. Differentiating we get:

$$v' = \frac{\gamma(\mathrm{d}x - u\,\mathrm{d}t)}{\gamma\left(\mathrm{d}t - \frac{u\mathrm{d}x}{c^2}\right)} = \frac{v - u}{1 - \frac{uv}{c^2}}. \tag{7}$$

We divide through by $\mathrm{d}t$

Two interesting limits of this formula are: $v' = v - u$ if $u \ll c$ or $v \ll c$; and $v' = c$ if $v = c$ for whatever u.

1.5 Tensor notation

The position four-vector is $x^\mu = (ct, x, y, z)$. The Euclidean scalar product is given by $x \cdot y = \delta_{\mu\nu} x^\mu x^\nu$. If we substitute the identity $\delta_{\mu\nu}$ with another metric we can find a more general metric space.

The Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The separation 4-vector is $\mathrm{d}x^\mu = (c\,\mathrm{d}t\,, \mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z)$.

Using Einstein summation notation, we can write the spacetime interval as $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$.

Specifically for the Minkowski metric we have the relation $\eta_{\mu\nu} = \eta^{\mu\nu}$: it is its own inverse. For a general metric $g_{\mu\nu}$ this will not hold.

How do we express the Lorentz boosts? They are linear transformations, therefore they look like $x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu$, with $\Lambda_\mu{}^\nu$ being constant $(1,1)$ tensors. Also, they preserve the spacetime interval, therefore they satisfy $\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$. This is called the *pseudo-orthogonality* relation.

The metric allows us to raise and lower indices. Raising an index in the pseudo-orthogonality relation gives us: $\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta \eta^{\beta\sigma} = \delta_\alpha{}^\sigma$, therefore $\eta_{\mu\nu}\Lambda^\nu{}_\beta \eta^{\beta\sigma}$ is the inverse of a Lorentz transformation.

Notice what this does to the Lorentz transformation matrix: it swaps a sign if *one* of the indices is spatial and one is temporal, but not if they are both spatial or both temporal; also, it transposes the matrix.

So, consider a Lorentz boost along the x axis: it is transformed as

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

so as we would expect the velocity β gets mapped to its opposite. On the other hand, a spatial rotation around a unit vector $\hat{v}$ of angle α looks like

$$\begin{bmatrix} 1 & 0 \\ 0 & R(\hat{v}, \alpha) \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & R(\hat{v}, \alpha)^\top \end{bmatrix}, \tag{9}$$

which makes sense: rotation matrices satisfy $R^\top R = \mathbb{1}$.

In fact, the two types of matrices written here, together with the matrices T and P which, respectively, swap the sign of the spatial and temporal coordinates, can be composed to generate *any* Lorentz transformation.

If we do not include the parities P and T, we have what is called the proper, orthochronous Lorentz group, which is the subset of the whole Lorentz group which can be reached starting with the identity and varying the parameters of the Lorentz transformation continuously.

Four-vectors can also have their indices down, and they will transform according to the inverse of Lorentz transformations:

$$
\begin{aligned}
(\eta_{\alpha\mu} x^\mu)' &= \eta_{\alpha\mu} \Lambda^\mu{}_\nu x^\nu && \text{(10a)} \\
&= \Lambda_{\alpha\sigma} \delta^\sigma{}_\nu x^\nu && \text{(10b)} \\
&= \Lambda_{\alpha\sigma} \eta^{\sigma\beta} \eta_{\beta\nu} x^\nu && \text{(10c)} \\
&= \Lambda_\alpha{}^\beta x_\beta \,. && \text{(10d)}
\end{aligned}
$$

Indices which are up are called "contravariant", indices which are down are called "covariant".

An intuitive explanation of covariance and contravariance

This is most easily illustrated by considering scaling transformations in a vector space, that is, linear transformations which look like $v^i \to S v^i$, where S is a scalar and v^i are the components of a vector

These can be thought of as "changing the measuring stick we use to figure out how long the components of a vector are"; where by "measuring stick" what we really mean is basis vector for our vector space. The basis vectors e_i allow us to write the full, geometric vector as $\vec{v} = v^i e_i$: we are summing the components times the corresponding basis vector.

If we, for example, increase the length of the basis vectors by 2, so we map $e_i \to e'_i = 2e_i$ for all i, then for the vector $\vec{v}$ to remain the same its components must change: specifically, they must be multiplied by $1/2$. This is where the word *contravariant* comes from.

Now, for covariant vectors: they are vectors in the *dual* vector space, that is, they represent linear applications from the vector space to a number. They are expressed with respect to the *dual basis*, e^i, as: $\vec{w} = w_i e^i$. We can choose this basis so that $e^i e_j$ (the i-th basis *linear transformation* applied to the j-th basis vector) is equal to δ^i_j.

Then, we can calculate what the transformation $\vec{w}$ applied to the vector $\vec{v}$ looks like: it is

$$
w_i e^i v^j e_j = w_i v^j \delta^i_j = w_i v^i \,. \tag{11}
$$

Note that all of this is basis-independent, and also independent of the metric: we are *not* calculating a scalar product, but instead applying a linear transformation to a vector.

Recall: we are transforming the basis vectors e_i by rescaling them by a factor 2:

so, what must the transformation law for the basis dual vectors e^i be if we want to preserve the property $e_j e^i = \delta^i_j$? The twos must simplify, so the transformation law we need is $e^i \to e^i/2$. Therefore, since we want the covector to stay the same, that is

$$\vec{w} = w_i e^i \stackrel{!}{=} w'_i e'^i \,, \tag{12}$$

the transformed components w'_i of the dual vector must be $w'_i = 2w_i$.

This is the meaning of the terms co- and contra-variant: the co-variant vectors (or dual vectors) transform *like* the basis vectors e_i, while the contra-variant vectors trasform *in the opposite way* as the basis vectors e_i.

We will write our laws as tensorial equations, which are covariant (which, in this context, means that the law has the exact same form in every reference frame).

By pseudo-orthogonality, the scalar product $A_\mu B^\mu = A^\nu g_{\mu\nu} B^\mu$ is a covariant (that is, invariant) scalar. The metric can act on either side, so it is equal to $A^\mu B_\mu$.

Definition 1.3 (Tensor)**.** *A* (p, q) tensor *is an object* $M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p}$ *with many components indexed by* $p + q$ *indices, which transforms as:*

$$M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p} \to \Lambda_{\mu_1}{}^{\mu'_1} \dots \Lambda_{\mu_p}{}^{\mu'_p} \Lambda^{\nu_1}{}_{\nu'_1} \dots \Lambda^{\nu_q}{}_{\nu'_q} M^{\nu'_1 \dots \nu'_q}_{\mu'_1 \dots \mu'_p} \tag{13}$$

under Lorentz transformations $\Lambda_\mu{}^\nu$*.*

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Last lecture we introduced tensors.

An example of those is the EM tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_x & B_y \\ -E_y/c & B_x & 0 & -B_z \\ -E_z/c & -B_y & B_z & 0 \end{bmatrix} , \tag{14}$$

which, it can be checked, transforms as a $(0, 2)$ tensor. Also, we can define the current vector $j^\mu = (c\rho, j^i)$. Then, the Maxwell equations read:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \qquad \text{and} \qquad \partial_{[\mu} F_{\nu\rho]} \,. \tag{15}$$

They are covariant!

1.6 Particles in motion

In Newtonian mechanics, the motion of a particle is described by a function of time $x^i = x^i(t)$.

In special relativity, we introduce the concept of *worldline* of a particle: the one-dimensional set of spacetime events in which the particle is found. It must be parametrized with respect to some parameter λ, such that $x^\mu = x^\mu(\lambda)$. A preferred choice for λ is the proper time of the particle, $\lambda = \tau$, which is defined by $\mathrm{d}s^2 = -\,\mathrm{d}\tau^2$.

We then define the 4-velocity:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,. \tag{16}$$

It is a tensor since it is the product of a scalar and a tensor.

Multiplying $u^\mu u_\mu$ we always get $-c^2$, since:

$$u^\mu u_\mu = \frac{\mathrm{d}x^\mu\,\mathrm{d}x_\mu}{\mathrm{d}\tau^2} = -c^2\frac{\mathrm{d}s^2}{\mathrm{d}s^2} \tag{17}$$

We can make the expression of the 4-velocity explicit using $\mathrm{d}\tau = \gamma\,\mathrm{d}t$, which gives us $u^\mu = (\gamma c, \gamma v^i)$. In the frame of the particle the particle is not moving, therefore $u^\mu = (c, 0)$.

The *four-momentum* of a massive particle with mass m is defined as:

$$p^\mu = m u^\mu = (m\gamma c, m\gamma v^i)\,. \tag{18}$$

The component p^0 is equal to mc at $v = 0$. What does it look like in the nonrelativistic approximation? we can expand it for small v/c:

$$\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \sim mc\left(1 + \frac{v^2}{2c^2}\right) = mc + \frac{1}{c}\frac{mv^2}{2}\,. \tag{19}$$

We get the mass, plus a kinetic energy term: more explicitly, $cp^0 = mc^2 + {}^1\!/\!{}_2 mv^2$.

We can rewrite Newton's first law in SR:

Proposition 1.4 (Newton I). *A free particle moves with constant u^μ, or*

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \tag{20}$$

To express this in an easier way we introduce the 4-acceleration:

$$a^\mu \stackrel{\text{def}}{=} \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} \tag{21}$$

We now wish to introduce the concept of a path minimizing proper time. Recall Snell's law, which allows us to relate the angles of incidence of light when it passes between one medium to another, if they have different indices of refraction:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \qquad \text{or} \qquad \frac{\sin(\theta_2)}{\sin(\theta_1)} = \frac{n_1}{n_2} = \frac{v_2}{v_1} \,. \tag{22}$$

It can be shown that a beam of light following this law is equivalent to the beam minimizing the time it takes to move from a point in one medium to a point in the other.

Analogously, saying that a massive particle travels along the worldline which minimizes τ is equivalent to Newton's first principle.

We want to perturb a generic worldline x^μ with some $\mathrm{d}x^\mu$, and consider the proper time functional τ which gives the proper time of a generic trajectory: we impose

$$\frac{\tau[x^\mu + \varepsilon^\mu] - \tau[x^\mu]}{|\varepsilon^\mu|} = \frac{\delta \tau}{\delta x^\mu} \stackrel{!}{=} 0 \,, \tag{23}$$

where a limit $|\varepsilon^\mu| \to 0$ is implied, and only the linear terms are considered.

The proper time functional for paths between A and B is given $\tau = \int_A^B \mathrm{d}\tau$. We can rewrite it as:

$$\tau = \int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = \int_A^B \mathrm{d}\tau \, \frac{-\eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu}{\mathrm{d}\tau^2} \,. \tag{24}$$

We now consider a perturbation $\varepsilon^\mu = \delta^\mu_1 \delta x$:

$$\tau_{AB}[x + \varepsilon] = \int_A^B \mathrm{d}\tau \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 - \frac{1}{c^2} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\mathrm{d}\delta x}{\mathrm{d}\tau} \right)^2 - \frac{1}{c^2} \left(\frac{\mathrm{d}y}{\mathrm{d}\tau} \right)^2 - \frac{1}{c^2} \left(\frac{\mathrm{d}z}{\mathrm{d}\tau} \right)^2 \right] . \tag{25}$$

We can discard a second order term $\left(\mathrm{d}\delta x / \mathrm{d}\tau \right)^2$, and subtract off $\tau_{AB}[x]$: we are left with

$$\delta \tau = -\frac{2}{c^2} \int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}x}{\mathrm{d}\tau} \frac{\mathrm{d}\delta x}{\mathrm{d}\tau} \tag{26}$$

Now, we integrate by parts, disregard the boundary terms since the endpoints of the path cannot be deformed, and get:

$$\frac{\delta \tau_{AB}}{\delta x} = + \frac{2}{c^2} \int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} \,, \tag{27}$$

which proves the equivalence for this type of perturbation, the others are analogous.

Let us write a more general sequence of equations, somewhat informally:

$$\delta\left(\int_A^B \mathrm{d}\tau\right) = -\delta \int \frac{\mathrm{d}x^\mu \,\mathrm{d}x_\mu}{\mathrm{d}\tau^2}\,\mathrm{d}\tau^2 \tag{28}$$

Expand $\mathrm{d}s^2$

$$= -\int \frac{\left(\mathrm{d}x_\mu + \mathrm{d}\epsilon_\mu\right)\left(\mathrm{d}x^\mu + \mathrm{d}\epsilon^\mu\right)}{\mathrm{d}\tau^2}\,\mathrm{d}\tau \tag{29}$$

Perturb

$$= -2\int \frac{\mathrm{d}\epsilon_\mu \,\mathrm{d}x^\mu}{\mathrm{d}\tau^2}\,\mathrm{d}\tau \tag{30}$$

Discard terms of order different than 1 in the perturbation

$$= +2\int \epsilon_\mu \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}\,\mathrm{d}\tau \overset{!}{=} 0\,, \tag{31}$$

Integrate by parts

which must hold for any ϵ_μ: so, by the fundamental lemma of variational calculus, we must have

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0\,. \tag{32}$$

The generalization of Newton's second law, which at low speeds is $F^i = ma^i$, can be similarly restated as $\delta S = 0$, for the action $S = \int L\,\mathrm{d}\tau$, where L is the Lagrangian for the system.

1.7 Motion of light rays

For light we cannot compute u^μ with the previous definition, since its proper time is always zero.

Instead, we *define* u^μ to be a normalized null-like vector, such that locally $x^\mu = \lambda u^\mu$ for some λ.

We know from quantum mechanics that $E = \hbar\omega$, where $\hbar = h/(2\pi)$ and $\omega = 2\pi/T = 2\pi f$.

The momentum is proportional to the wavevector k^i: $p^i = \hbar k^i/c$. The relativistic generalization of this fact is

$$p^\mu = \left(\frac{\hbar\omega}{c}, \frac{\hbar k^i}{c}\right) = \frac{\hbar k^\mu}{c}\,. \tag{33}$$

Since the momentum of light must be null-like ($p_\mu p^\mu = 0$) we have that necessarily $\omega = |k|$.

1.8 Doppler effect

We take a special case: radiation goes in the same direction as the observer. In the O frame we have $k^\mu = (\omega, \omega, 0, 0)$.

The observer, moving with velocity v, measures k'^{μ}. This can be easily computed with a Lorentz transformation: $k'^{\mu} = \Lambda^{\mu}{}_{\nu} k^{\nu}$.

We are mostly interested in $k'^0 = \omega'$: it comes out to be $\omega' = \gamma\omega + (-\gamma\beta)\omega = (1 - v/c)\gamma\omega$.

Some notes: at slow speeds $\omega' \approx (1 - v/c)\omega$; we have $f' < f$ when source and observer are moving away from each other. This can be readily proven by considering either $k^{\mu} = (\omega, -\omega, 0, 0)$ or $\beta \to -\beta$.

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1.9 Bases

In Euclidean 2D geometry we can choose, for example, the basis $e_1 = (1, 0)^{\top}$ and $e_2 = (0, 1)^{\top}$. This basis is orthonormal with respect to the scalar product $g_{\mu\nu} = \delta_{\mu\nu}$: $e_{(\alpha)} \cdot e_{(\beta)} = e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} g_{\mu\nu} = g_{(\alpha)(\beta)}$.

I use parentheses around indices to denote the fact that they are not tensorial indices, but instead denote which basis vector we are considering. We express our vectors in components with respect to this basis.

In SR, we can do the same: our coordinate basis can be given by $e^{\mu}_{(\alpha)} = \delta^{\mu}_{(\alpha)}$. Now, the orthonormality $e_{(\alpha)} \cdot e_{(\beta)} = g_{(\alpha)(\beta)}$ holds with respect to $g_{\mu\nu} = \eta_{\mu\nu}$.

1.10 Observers & observations

Every observer will be characterized by their trajectory $x^{\mu}(\tau)$. We can associate a coordinate system with the observer: the one in which the observer's own 4-velocity u^{μ} is the time-like unit vector (rescaled by a factor of c: $u^{\mu} = ce_{(0)}$).

When the observer sees a particle with $p^{\mu} = (E_p/c, p^i)$ they measure the energy of the particle to be $p^0 c$: in this frame this is $E_{\text{measured}} = -e^{\mu}_{(0)} p_{\mu} c = -u^{\mu} p_{\mu} c$. Do note that this is a covariant expression, while p^0 is not: the energy of a particle with 4-momentum p^{μ} measured by an observer with 4-velocity u^{μ} is an invariant.

In the rest frame of the observer, their own 4-velocity is $(c, \vec{0}) = c(1, \vec{0})$. In the rest frame of the particle, its own energy is measured to be mc^2. The measured energy by an observer such that the product of the 4-velocities of the particle and of the observer is $-\gamma c^2$ is $m\gamma c^2$.

The Earth moves with speed $10^{-4}c$ around the Sun.

Now we can start using $c = 1$. We can put the c back whenever we want with dimensional analysis.

2 Newtonian Gravity

2.1 The Equivalence Principle

Just like Newton supposedly thought about universal gravity when, while looking at the sky, an apple fell on his head; Einstein supposedly thought up the equivalence principle when he saw a man falling from a rooftop.

Proposition 2.1 (Equivalence principle)**.** *Experiments in a small free falling system over a short amount of time give the same result as experiments in an inertial frame in empty space.*

Why "small"? The gravitational field is not really homogeneous. The idea is that gravity can only be removed *locally*, if we consider an extended system there are *tidal effects*.

If we were to see that objects fall differently even in the same neighbouhood then we would lose the EP.

Definition 2.1. *The* inertial *mass is an object's resistance to motion:* $m_{inertial} = F^i / a^i$.

Definition 2.2. *The* gravitational *mass is the one which defines the gravitational force on an object:* $m_{gravitational} = |F| r^2 / (GM)$.

These are *a priori* different, but experimentally equal: in general the gravitational acceleration is given by

$$a^i = \frac{GMr^i}{r^3} \frac{m_{\text{gravitational}}}{m_{\text{inertial}}} \tag{34}$$

If the ratio of masses depended on the material, this could vary.

We can do a torsion pendulum experiment: the torsion applied by the Coriolis effect on a pendulum depends on the inertial mass, while its restoring force depends on the gravitational mass. Experimentally we have measured them to be equal with an accuracy of 10^{-12}.

A person on a rocket accelarating at g experiences the same acceleration as a person standing on Earth.

2.2 Gravitational redshift

We treat it now in a weak field approximation.

Alice sends radiation to Bob from a higher altitude on Earth. Alice sends it with frequency f, Bob receives it with f'. They are at rest with respect to one another: there is no kinematic Doppler effect here.

We do this by applying the equivalence principle! We imagine A and B to be standing in a rocket which is accelerating at g: there is no more gravity.

Bob will receive a greater frequency: $f' > f$. This can be seen by imagining two consecutive wavefronts as two particles. Alice sends them $\Delta t_A = 1/f$ apart, Bob receives them as $\Delta t_B = 1/f'$ apart.

If the rocket is at rest, the time for the radiation to reach B is h/c; if the rocket is moving then the time is $< h/c$.

When the second wavefront starts moving the rocket is already going: the second wave starts later but it has less distance to travel. Therefore $\Delta t_A > \Delta t_B$, which implies $f' > f$.

Claim 2.1. *The first terms in the expansion are:*

$$f' = f\left(1 + \frac{gh}{c^2} + O\left(\left(\frac{gh}{c^2}\right)^2\right)\right). \tag{35}$$

2.3 Potentials

In electromagnetism, the potential energy between a charge Q and a test charge q is $U = kQq/r$: then we define the electromagnetic potential $V = U/q$ which has the advantage of being test-charge independent.

Similarly, we define the gravitational potential $\Phi = U/m \approx gh$ in the regime in which we are far enough from the object that the gravitational field can be well approximated as a constant vector field.

Then, the second order term in the formula for the redshift becomes $\Delta\Phi c^{-2}$: now we can properly say that this *weak field* means $\Delta\Phi c^{-2} \ll 1$.

This is surely the case for the cases we can treat concretely. If two people are separated by 1 km of difference in altitude, they have $\Delta\Phi c^{-2} \approx 10^{-13}$: the difference they will experience is one second in a million years.

Our expression from Φ in the newtonian approximation is $\Phi = GM/r$.

We can say even now by dimensional analysis that $GM/(rc^2)$ is the parameter which tells us how relevant the gravitational effects are: if it is similar to 1 we must consider GR, if it is much smaller than 1 the GR effects will be negligible.

This is very close to the expression for the Schwarzschild radius: it is $r = GM$ in units $c = 1$, while the correct expression is $r = 2GM$: that one can actually be recovered exactly if we calculate the radius at which the escape velocity is equal to c.

Thu Oct 17 2019

3 The mathematical description of curved spacetime

The metric describes the spacetime. It depends on the coordinates.

Euclidean metric In 2D, it is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. We can express it as

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \, \mathrm{d}x_\mu = \mathrm{d}x^\mu \, \mathrm{d}x^\mu \, g_{\mu\nu} \,, \tag{36}$$

computed with $g_{\mu\nu} = \delta_{\mu\nu}$.

Polar coordinates Let us see how this changes when we change coordinates: for example, we can go to polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x) \end{cases} \,. \tag{37}$$

Let us compute the differentials $\mathrm{d}x$ and $\mathrm{d}y$:

$$\mathrm{d}x = \mathrm{d}r\cos(\theta) - r\sin(\theta)\,\mathrm{d}\theta \;, \tag{38}$$

and

$$\mathrm{d}y = \mathrm{d}r\sin(\theta) + r\cos(\theta)\,\mathrm{d}\theta \;. \tag{39}$$

Plugging these into the metric and simplifying we get

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \;. \tag{40}$$

It is not clear to see that these are equivalent. We then want to compute scalar quantities to characterize them.

Another issue is the fact that the polar metric is singular at the origin: the metric should always have the same signature, and if it is nondegenerate (no zero eigenvalues) we should always be able to invert it: in this case however we'd also have to invert

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{41}$$

at $r = 0$: we cannot do it! This is a *coordinate singularity*. There is nothing wrong with the space; $\mathbb{R}^2$ is perfectly regular at $(0,0)$, but our coordinate description fails; besides, what value of θ should we assign to that point?

An *arbitrarily large* change in θ can result in an arbitrarily small distance travelled if we move close enough.

Spherical coordinates In $\mathbb{R}^3$ we have the same issue. We use:

$$\begin{cases} x = \cos(\theta)\cos(\varphi) \\ y = \cos(\theta)\sin(\varphi) \\ z = \sin(\theta) \end{cases} \,, \tag{42}$$

and it is a simple computation as before to see that

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin(\theta)^2 \end{bmatrix}, \tag{43}$$

therefore these coordinates are not defined on the *whole z axis*!

General coordinate transformations Now we consider a general transformation of space, which we denote by $x'^{\mu}(x^{\mu})$: we see how the metric should change in order for the spacetime distance to be invariant: we define $g'_{\mu\nu}$ by

$$g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} \stackrel{!}{=} g'_{\mu\nu}\,\mathrm{d}x'^{\mu}\,\mathrm{d}x'^{\nu}\,. \tag{44}$$

This means that the metric changes as a tensor:

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}\,, \tag{45}$$

and we can see that it transforms as a $(0,2)$ tensor since the primes are in the denominator: it transforms with the *inverse* of the Jacobian matrix.

The inverse metric transforms as:

$$g^{\mu\nu} \to g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}\,, \tag{46}$$

which we can check by proving that $g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho}$ is conserved when we transform both the metric and the inverse metric. We get:

$$g'^{\mu\nu}g'_{\sigma\tau}\delta^{\sigma}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}\delta^{\sigma}_{\nu}\frac{\partial x^{\pi}}{\partial x'^{\sigma}}\frac{\partial x^{\lambda}}{\partial x'^{\tau}} g_{\pi\lambda}\,, \tag{47}$$

which can be simplified using the relations between the partial derivatives:

$$\frac{\partial x'^{\nu}}{\partial x^{\beta}}\delta^{\sigma}_{\nu}\frac{\partial x^{\pi}}{\partial x'^{\sigma}} = \frac{\partial x^{\pi}}{\partial x^{\beta}} = \delta^{\pi}_{\beta}\,, \tag{48}$$

since we are multiplying the Jacobian with its inverse, thus we get the identity. One can make this reasoning more explicit by seeing it as an application of the chain rule: x^{μ} is a function of x'^{μ} which is a function of x^{μ}. Plugging this in we get

$$g'^{\mu\nu}g'_{\sigma\tau}\delta^{\sigma}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} g^{\alpha\beta}\delta^{\pi}_{\beta}\frac{\partial x^{\lambda}}{\partial x'^{\tau}} g_{\pi\lambda}\,, \tag{49}$$

and now we can apply our hypothesis that $g^{\alpha\beta}g_{\beta\lambda} = \delta^{\alpha}_{\lambda}$ to contract some more indices, get one more multiplication of a Jacobian with its inverse, which we can simplify in the same way as before to finally get the identity. This proves that the inverse metric is a $(2,0)$ tensor.

3.1 Lengths, areas, volumes and so on

In 4D space, if we fix x^0 and x^3 and we seek the shape of the differential area element — the *volume form* on the surface $\left\{x^0 = x^3 = 0\right\}$ — it will look like $\mathrm{d}A = \sqrt{g_{11}}\,\mathrm{d}x^1\,\sqrt{g_{22}}\,\mathrm{d}x^2$ if the metric is diagonal. The square roots come from the fact that the diagonal components tell us how to measure length along a certain direction: if we are moving along the x axis from point A to point B, then

$$\Delta s^1\Big|_A^B = \int_A^B \mathrm{d}s = \int_A^B \sqrt{g_{11}}\,\mathrm{d}x^1 = \sqrt{g_{11}}\,\Delta x^1\Big|_A^B\,, \tag{50}$$

since we have the relation $\mathrm{d}s^2 = g_{11}(\mathrm{d}x^1)^2$. So, a length measured along the x^1 direction is given by $\sqrt{g_{11}}$ times the coordinate distance. Of course, in general we will have to integrate if the metric depends on the point.

Similarly, the 4-volume is just

$$\mathrm{d}v = \sqrt{-g_{00}g_{11}g_{22}g_{33}}\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = \sqrt{-g}\,\mathrm{d}^4x\,, \tag{51}$$

where $g = \det g_{\mu\nu}$. We have shown it for a diagonal metric, but it can be shown that it holds for any general metric. The minus sign comes from the fact that our metric has signature -1 (and, as we will see, this holds in general): the determinant is then negative, and to take the square root we need a positive number.

This is not put in artificially, but it is a consequence of the fact that we measure positive times for curves with positive *proper* time $\mathrm{d}\tau^2 = -\,\mathrm{d}s^2$.

Claim 3.1 (Unproven)**.** *A metric can be always diagonalized, at least locally.*

When the metric is diagonal, then $\mathrm{d}v = \sqrt{-g}\,\mathrm{d}^4x$; then this always holds, since [claim!] $\sqrt{-g}\,\mathrm{d}^4x$ is a scalar.

Intuitively, the claim follows from the equivalence principle: we put a free-falling observer in our space. They will perceive spacetime as being flat.

Under a diffeomorphism with the determinant of the jacobian equal to J we have the following transformation law for the determinant:

$$g' = J^{-2}g \qquad \Longrightarrow \qquad \sqrt{-g'} = J^{-1}\sqrt{-g}\,, \tag{52}$$

which follows from the following properties of determinants:

$$\det(ABC) = \det A \det B \det C \qquad \text{and} \qquad \det\left(A^{-1}\right) = 1/\det A\,, \tag{53}$$

applied to the transformation equation for the metric tensor:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}\,. \tag{54}$$

This can be expressed by saying that $\sqrt{-g}$ is a *tensor density* of weight -1.

Definition 3.1. *A tensor density of weight w transforms just like a tensor, except we need to multply by a factor J^w in the transformation, where J is the determinant of the Jacobian of the transformation.*

A (p, q) tensor density *of weight w is an object $M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p}$ with many components indexed by $p + q$ indices, which transforms as:*

$$M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p} \to J^w \Lambda_{\mu_1}{}^{\mu_1'} \dots \Lambda_{\mu_p}{}^{\mu_p'} \Lambda^{\nu_1}{}_{\nu_1'} \dots \Lambda^{\nu_q}{}_{\nu_q'} M^{\nu_1' \dots \nu_q'}_{\mu_1' \dots \mu_p'} \tag{55}$$

under diffeomorphisms with Jacobian matrix $\Lambda_\mu{}^\nu$, with determinant J.

When we transform the 4-volume element $\mathrm{d}^4 x' = J \, \mathrm{d}^4 x$, we get a Jacobian: the coordinate volume element $\mathrm{d}^4 x$ is a *tensor density of weight 1*.

Then we see that the volume element $\sqrt{-g} \, \mathrm{d}^4 x$ is a *scalar*, since when multiplying tensor densities their weights are added (or, more simply put, the Js cancel out).

3.2 Vectors in curved spacetime

They are not objects *in spacetime*: instead they belong to the *tangent space*.

For each point x in the manifold we define a basis at that point: $\left\{ e^\mu_{(\alpha)} \right\} (x)$, where μ is a vector index while (α) denotes which vector we are considering.

Any vector $a^\mu(x)$ can be then decomposed as

$$a^\mu(x) = a^{(\alpha)}(x) e^\mu_{(\alpha)}(x) \,. \tag{56}$$

Since spacetime is not flat, the dependence on x is not trivial.

The scalar product between the vectors can be expressed with respect to the basis:

$$a(x) \cdot b(x) = a^{(\alpha)} b^{(\beta)} e_{(\alpha)} \cdot e_{(\beta)} \,, \tag{57}$$

and we can select our basis so that at every point it is orthonormal:

$$e_{(\alpha)} \cdot e_{(\beta)} = \eta_{(\alpha)(\beta)} \,. \tag{58}$$

We *cannot sum vectors in different tangent spaces*. But we want to: we need to define derivatives! To solve this issue, we will introduce the notion of *parallel transport*.

Fri Oct 18 2019

3.3 Covariant differentiation

We want to do derivatives, as in:

$$f(x + \mathrm{d}x) = f(x) + \frac{\partial f}{\partial x} \mathrm{d}x \,, \tag{59}$$

which allow us to “move around”. In more dimensions, the rule will be

$$f(x + \mathrm{d}x) - f(x) = \mathrm{d}x^{\mu}\, \partial_{\mu} f\,. \tag{60}$$

Now, we want to prove that, in Minkowski spacetime, $\partial_{\mu} f$ is a rank (0,1) tensor if f is a function.

Under a change of variables $x^{\mu} \to x'^{\mu}$ we have

$$\frac{\partial f(x)}{\partial x^{\mu}} \to \frac{\partial f(x')}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}} f(x)\,, \tag{61}$$

Since x' and x represent the same point, $f(x') = f(x)$.

which is the transformation law of a covariant vector, or (0,1) tensor.

However, for a vector $\mathrm{d}x^{\nu}\, \partial_{\nu} A^{\mu}$ is *not* a tensor!

Under a change of coordinates,

$$\begin{aligned}
\frac{\partial A^{\mu}(x)}{\partial x^{\nu}} &= \frac{\partial}{\partial x'^{\nu}} A^{\mu}(x') &\text{(62a)}\\
&= \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial}{\partial x^{\alpha}} \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} A^{\beta}(x) \right) &\text{(62b)}\\
&= \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} + \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\beta}} \frac{\partial A^{\beta}}{\partial x^{\alpha}}\,, &\text{(62c)}
\end{aligned}$$

and we can see that the second term is the transformation we want, but the first term spoils the transformation.

This is not an issue in SR: there, the second derivative vanishes:

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} = \frac{\partial}{\partial x^{\alpha}} \Lambda^{\mu}_{\beta} = 0\,, \tag{63}$$

since Lorentz matrices are constant.

So, we construct a *Covariant Derivative* which transforms as a tensor under diffeomorphisms.

We denote it as $\nabla_{\nu} A^{\mu}$. For any tensor T of arbitrary rank (p, q) we request $\nabla_{\nu} T$ to be a tensor of rank $(p, q+1)$. Also, we request $\nabla_{\mu} \to \partial_{\mu}$ for flat spacetime.

We define the Christoffel symbols:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \Big(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} \Big)\,, \tag{64}$$

where we introduced comma notation for partial non-covariant differentiation: the full notation is

$$\partial_{\mu} x_A \overset{\text{def}}{=} x_{A,\mu} \qquad \text{and} \qquad \nabla_{\mu} x_A \overset{\text{def}}{=} x_{A;\mu}\,. \tag{65}$$

They are symmetric in the two lower indices and they are not tensors. Their transformation law is:

$$\Gamma^{\mu}_{\nu\kappa} \to \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu} \partial x'^{\kappa}}\,, \tag{66}$$

which we note is *not tensorial*!

We define

$$\nabla_{\nu} V_{\mu} = \partial_{\nu} V_{\mu} - \Gamma^{\alpha}_{\nu\mu} V_{\alpha}\,, \tag{67}$$

and

$$\nabla_{\nu} V^{\mu} = \partial_{\nu} V_{\mu} + \Gamma^{\mu}_{\nu\alpha} V^{\alpha}\,. \tag{68}$$

How does it transform? For the covariant derivative, we have:

$$\begin{aligned}
\nabla'_{\nu} V'_{\kappa} &= \frac{\partial}{\partial x'^{\nu}} V'_{\kappa} - \Gamma'^{\mu}_{\nu\kappa} V'_{\mu} && \text{(69a)} \\
&= \frac{\partial x^{\mu}}{\partial x'^{\nu}} \partial_{\mu} \left(\frac{\partial x^{\lambda}}{\partial x'^{\kappa}} V_{\lambda} \right) - \left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x'^{\mu}}{\partial x'^{\nu} \partial x'^{\kappa}} \right) \left(\frac{\partial x^{\lambda}}{\partial x'^{\mu}} V_{\lambda} \right) && \text{(69b)} \\
&= \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\mu}} V_{\lambda} + \frac{\partial x^{\lambda}}{\partial x'^{\mu}} \frac{\partial V_{\lambda}}{\partial x'^{\nu}} - \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \left(\frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\kappa}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\kappa}} \right) \frac{\partial x^{\lambda}}{\partial x'^{\mu}} V_{\lambda} && \text{(69c)} \\
&= \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\mu}} V_{\lambda} + \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial V_{\lambda}}{\partial x^{\alpha}} - \left(\frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\kappa}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\kappa}} \right) \delta^{\lambda}_{\alpha} V_{\lambda} && \text{(69d)} \\
&= \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\kappa}} V_{\lambda} + \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial V_{\lambda}}{\partial x^{\alpha}} - \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\kappa}} \Gamma^{\alpha}_{\beta\gamma} V_{\alpha} - \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\kappa}} V_{\lambda} && \text{(69e)} \\
&= \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \left(\frac{\partial V_{\lambda}}{\partial x^{\alpha}} - \Gamma^{\sigma}_{\alpha\lambda} V_{\sigma} \right) && \text{(69f)} \\
&= \frac{\partial x^{\lambda}}{\partial x'^{\kappa}} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \nabla_{\alpha} V_{\lambda}\,, && \text{(69g)}
\end{aligned}$$

where we used: relabeling of indices, contraction of the Jacobian matrix with its inverse, the chain rule, the product rule, the transformation law of the Christoffel symbols (some steps are wrong... oh well, there's more interesting stuff to do).

The derivative of a contravariant tensor is a tensor: this can be proven by noticing that $\nabla_{\mu}(A^{\alpha} B_{\alpha}) = \partial_{\mu}(A^{\alpha} B_{\alpha}) = B_{\alpha} \nabla_{\mu} A^{\alpha} + A^{\alpha} \nabla_{\mu} B_{\alpha}$. Otherwise, we can compute away:

$$\nabla'_\nu V'^\mu = \frac{\partial V'^\mu}{\partial x'^\nu} + \Gamma'^\mu_{\nu\kappa} V'^\kappa \tag{70a}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial x'^\mu}{\partial x^\alpha}V^\alpha\right) + \left(\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^\gamma}{\partial x'^\mu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial^2 x'^\mu}{\partial x'^\nu \partial x'^\kappa}\right)\left(\frac{\partial x'^\kappa}{\partial x^\sigma}V^\sigma\right) \tag{70b}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha}V^\alpha + \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial V^\alpha}{\partial x^\lambda} + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu}\delta^\gamma_\sigma \Gamma^\alpha_{\beta\gamma}V^\gamma + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial^2 x'^\alpha}{\partial x^\nu \partial x^\mu}\frac{\partial x'^\kappa}{\partial x^\sigma}V^\sigma \tag{70c}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha}V^\alpha + \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\left(\frac{\partial V^\alpha}{\partial x^\lambda} + \Gamma^\alpha_{\lambda\gamma}V^\gamma\right) \tag{70d}$$

$$= \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\nabla_\lambda V^\alpha + \frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha}V^\alpha + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\kappa}\frac{\partial x'^\kappa}{\partial x^\sigma}V^\sigma\,, \tag{70e}$$

and we would like to see that the two last terms cancel: is

$$\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha}V^\alpha + \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\kappa}\frac{\partial x'^\kappa}{\partial x^\sigma}V^\sigma \stackrel{?}{=} 0\,, \tag{71}$$

for all V^μ? Let us factor the vector, changing the indices:

$$\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\alpha} + \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x'^\alpha \partial x'^\kappa}\frac{\partial x'^\kappa}{\partial x^\alpha} \stackrel{?}{=} 0\,, \tag{72}$$

we can rewrite it as

$$\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\frac{\partial x'^\mu}{\partial x^\lambda} + \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial}{\partial x'^\kappa}\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\alpha} \stackrel{?}{=} 0\,, \tag{73}$$

which can be recombined into:

$$\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\frac{\partial x'^\mu}{\partial x^\lambda} + \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial}{\partial x^\alpha}\frac{\partial x^\lambda}{\partial x'^\nu} \stackrel{?}{=} 0\,, \tag{74}$$

and becomes

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\lambda}\right) = \frac{\partial}{\partial x^\alpha}\delta^\mu_\nu = 0\,. \tag{75}$$

For any order tensor, we add a Christoffel symbol for every index, such as in:

$$\nabla_\mu V_{\alpha\beta} = \partial_\mu V_{\alpha\beta} - \Gamma^\lambda_{\mu\alpha}V_{\lambda\beta} - \Gamma^\lambda_{\mu\beta}V_{\alpha\lambda}\,, \tag{76}$$

or

$$\nabla_\mu V^\beta_\alpha = \partial_\mu V^\beta_\alpha - \Gamma^\lambda_{\mu\alpha}V^\beta_\lambda + \Gamma^\beta_{\mu\lambda}V^\lambda_\alpha\,. \tag{77}$$

The general formula reads:

$$\nabla_\pi T^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_p} = \partial_\pi T^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_p} + \sum_{i=1}^{n} \Gamma^{\mu_i}_{\pi\sigma} T^{\mu_1\dots\mu_{i-1}\sigma\mu_{i+1}\dots\mu_n}_{\nu_1\dots\nu_p} - \sum_{j=1}^{p} \Gamma^{\sigma}_{\pi\nu_j} T^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_{i-1}\sigma\nu_{i+1}\dots\nu_p}\,, \tag{78}$$

which can be generalized to tensor *densities* beyond regular tensors: any tensor tensity $\mathfrak{T}$ (with any amount of indices, omitted for simplicity) of weight w (that is, which transforms almost like a tensor, with an extra factor of the determinant of the metric raised to the wth power) can be written as $\mathfrak{T} = \sqrt{-g}^{w} T$, where g is the determinant of the metric while T is a regular tensor, since as we saw $\sqrt{-g}$ is a density of weight +1.

Then, we can use this to calculate the covariant derivative of a tensor density: we can express $T = \sqrt{-g}^{-w}\mathfrak{T}$, so the rule will be

$$\nabla_\mu \mathfrak{T} = \nabla_\mu\left(\sqrt{-g}^{w}\sqrt{-g}^{-w}\mathfrak{T}\right) = \sqrt{-g}^{w}\nabla_\mu\left(\sqrt{-g}^{-w}\mathfrak{T}\right), \tag{79}$$

where we used the fact that

$$\nabla_\mu \sqrt{-g} = 0\,, \tag{80}$$

which follows from the fact that the metric is covariantly constant: if we compute it in a LIF, then we get

$$\nabla_\mu\left(\sqrt{-g}\right)^{w} = \partial_\mu \sqrt{-g}^{w} - w\Gamma^{\alpha}_{\alpha\mu}\sqrt{-g}\,, \tag{81}$$

which is zero:[a] there, the coordinate derivatives of the metric are zero, and therefore the Christoffel symbols are also zero.

[a]For a derivation of this expression, see https://yarikraak.nl/tensor_densities.pdf

3.4 Properties of the covariant derivative

- The covariant derivative of a tensor is a tensor;
- the covariant derivative obeys the Leibniz rule: $\nabla_\mu(AB) = B\nabla_\mu A + A\nabla_\mu B$;
- the metric is covariantly constant: $\nabla_\mu g_{\alpha\beta} = 0$.

The fact that the metric is covariantly constant could actually be assumed instead of the explicit expression of the Christoffel symbols in terms of derivatives of the metric. These two are equivalent.

Notice that covariant derivatives do *not* commute!

We can check that $\partial_\mu (A^\alpha B_\alpha) = \nabla_\mu (A^\alpha B_\alpha)$. It is

$$\left(\partial_\mu A_\alpha - \Gamma^\lambda_{\mu\alpha} A_\lambda\right) B^\alpha + A_\alpha \left(\partial_\mu B^\alpha + \Gamma^\alpha_{\mu\lambda} B^\lambda\right), \tag{82}$$

expanding and relabeling indices we get the desired cancellation. Now, for parallel transport:

3.5 Parallel transport

Take a curve $x^\alpha(\lambda)$ and a vector V^μ defined at a certain point along the curve. For infinitesimal displacement we will have $\mathrm{d}V = \mathrm{d}x \cdot \nabla V$: in components $\mathrm{d}V^\mu = \mathrm{d}x^\alpha \, \nabla_\alpha V^\mu$.

Parallel transport means that the vector does not change when it is transported: $\nabla_t V^\mu = 0$ where the index t indicates derivation along the curve's tangent vector $t^\alpha = \mathrm{d}x^\alpha / \mathrm{d}\lambda$: more explicitly, V^μ is parallel-transported along the curve parametrized as $x^\alpha(\lambda)$ if

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\big(V^\mu(x^\alpha(\lambda))\big) = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \nabla_\alpha V^\mu = 0\,. \tag{83}$$

Thu Oct 24 2019

Last time we looked at the derivative along a curve.

Now, we are going to talk about the

4 Einstein equations

They look like

$$\text{curvature} \propto \text{energy}\,, \tag{84}$$

we will make this more formal in this lecture.

We take a vector and parallel transport it along a closed path on a curved manifold, such as a sphere [see homework sheet #3]: the vector does *not* come back to its original position. In a flat manifold this is not the case. Do note that flat manifolds can *look* curved: an infinite cylinder's surface is flat.

Therefore we can *define* a "curved manifold" by:

$$\nabla_\mu \nabla_\nu V^\alpha \neq \nabla_\nu \nabla_\mu V^\alpha\,, \tag{85}$$

for at least some directions. To quantify this noncommutativity, let us then look at the commutator:

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu] V^\alpha &= \nabla_\mu \big(\nabla_\nu V^\alpha \big) - (\mu \leftrightarrow \nu) && (86a) \\
&= \partial_\mu \big(\nabla_\nu V^\alpha \big) \cancel{- \Gamma^\lambda_{\mu\nu} \big(\nabla_\lambda V^\alpha \big)} + \Gamma^\alpha_{\mu\lambda} \Big(\nabla_\nu V^\lambda \Big) - (\mu \leftrightarrow \nu) && (86b) \\
&= \partial_\mu \Big(\cancel{\partial_\nu V^\alpha} + \Gamma^\alpha_{\nu\lambda} V^\lambda \Big) + \Gamma^\alpha_{\mu\lambda} \Big(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\sigma} V^\sigma \Big) - (\mu \leftrightarrow \nu) && (86c) \\
&= \partial_\mu \Gamma^\alpha_{\nu\sigma} V^\sigma + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} V^\sigma - (\mu \leftrightarrow \nu) && (86d) \\
&= \Big(\partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \Big) V^\sigma && (86e) \\
&\stackrel{\text{def}}{=} R^\alpha_{\sigma\mu\nu} V^\sigma \,. && (86f)
\end{aligned}
$$

Symmetric in $\mu\nu$

We have cancelled many terms which are symmetric with respect to $(\mu \leftrightarrow \nu)$. The (3, 1) tensor we defined is called the *Riemann tensor*.

4.1 Local Inertial Frame

Claim 4.1. *For any spacetime endowed with a metric and for any point P in it we can choose a ("prime") coordinate system for which $g'_{\mu\nu}(x'_p) = \eta_{\mu\nu}$ and $\partial'_\lambda g'_{\mu\nu}(x'_p) = 0$.*

We cannot, however, set all the second coordinate derivatives to 0.

We want to compute the Riemann tensor in the LIF. In the LIF, the Christoffel symbols are all zero since they are linear combinations of the coordinate derivatives of the metric. The *derivatives* of the CS, however, are not zero!

$$
\partial_\sigma \left(\frac{1}{2} g^{\alpha\lambda} \Big(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} \Big) \right) = \frac{g^{\alpha\lambda}}{2} \Big(g_{\lambda\mu,\nu\sigma} + g_{\lambda\nu,\mu\sigma} - g_{\mu\nu,\lambda\sigma} \Big) \,. \tag{87}
$$

Therefore, we have:

$$
\begin{aligned}
R^\alpha_{\sigma\mu\nu} &= \frac{g^{\alpha\lambda}}{2} \Big(g_{\lambda\nu,\sigma\mu} + \cancelto{0}{g_{\sigma\lambda,\mu\nu}} - g_{\sigma\nu,\lambda\mu} \Big) - (\mu \leftrightarrow \nu) && (88a) \\
&= \frac{g^{\alpha\lambda}}{2} \Big(g_{\lambda\nu,\sigma\mu} - g_{\lambda\mu,\sigma\nu} - g_{\sigma\nu,\lambda\mu} + g_{\sigma\mu,\lambda\nu} \Big) \,. && (88b)
\end{aligned}
$$

We lower an index of the Riemann tensor with the metric and get:

$$
R_{\gamma\sigma\mu\nu} = \frac{1}{2} \Big(g_{\gamma\nu,\sigma\mu} - g_{\gamma\mu,\sigma\nu} - g_{\sigma\nu,\gamma\mu} + g_{\sigma\mu,\gamma\nu} \Big) \,, \tag{89}
$$

which is a reasonably simple expression, however it is only true at a single point.

We can derive some symmetry properties:

$$
\begin{aligned}
R_{\gamma\sigma\mu\nu} &= -R_{\sigma\gamma\mu\nu} && \text{(90a)} \\
R_{\gamma\sigma\mu\nu} &= -R_{\gamma\sigma\nu\mu} && \text{(90b)} \\
R_{\gamma\sigma\mu\nu} &= R_{\mu\nu\gamma\sigma} && \text{(90c)} \\
R_{\gamma\sigma\mu\nu} + R_{\gamma\mu\nu\sigma} + R_{\gamma\nu\sigma\mu} &= 0\,, && \text{(90d)}
\end{aligned}
$$

these can be checked in the LIF, and since they are tensorial expressions they will hold in any frame.

Definition 4.1 (Ricci tensor and scalar)**.** *The Ricci tensor is the trace of the Riemann tensor:*

$$
R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}\,, \tag{91}
$$

while the Ricci scalar, or scalar curvature, is the trace of the Ricci tensor:

$$
R = g^{\mu\nu} R_{\mu\nu}\,. \tag{92}
$$

4.2 The stress-energy tensor

All of the objects we defined are defined locally in the tangent bundle of the manifold.

To discuss energy, we also need a local object: an energy *density*.

This will not be a scalar: it is frame dependent, since it is energy over volume, but volume changes if we change frame.

The number of particles N is a scalar (not a scalar field!). The number density is *not* a scalar field: a moving observer with velocity v will se the volume as being *smaller*. The average number density as measured in LIF will be

$$
n_* = \frac{N}{\mathrm{Vol}_*}\,, \tag{93}
$$

where Vol_* is the spatial volume of the box in its own rest frame. For another observer moving at $v = 1 - 1/\gamma^2$, we will have

$$
n = \frac{N}{\mathrm{Vol}} = \gamma n_* \geq n_*\,, \tag{94}
$$

since $\mathrm{Vol} = \mathrm{Vol}_*/\gamma$.

The 4-velocity of the box for the observer is $u^\mu = (\gamma, \gamma\vec{v})$. Therefore, if we define $n^\alpha = n_* u^\alpha$ we will have $n = n^0$ for any observer. It is not a scalar because it cannot be, but the whole density vector transforms as a proper vector.

What are the spatial components of this vector? n^i is called the *number current density*.

We imagine an area which is fixed with respect to the moving observer. How many particles cross the area $\mathrm{d}A$ in a time $\mathrm{d}t$, given that locally near the area the 3-velocity of the particles is $\vec{v}$?

It will be the density times the volume:

$$\frac{n_*}{\sqrt{1-v^2}} \mathrm{d}t \, \vec{v} \cdot \mathrm{d}\vec{A} = n^i \, \mathrm{d}A_i \ . \tag{95}$$

We have a scalar product because the area can be at an angle with respect to the velocity, and we need to compute the flux.

As an example, the electromagnetic current for electrons with number density n^μ is simply $j^\mu = -en^\mu$.

Now, we will discuss the *net flux* in or out of a certain region. If we imagine a certain region, the net flux will equal the variation of the particle number in the region: this gives us a conservation equation

$$\int_{\partial V} \mathrm{d}\vec{A} \cdot \vec{n} + \partial_t \int_V \mathrm{d}^3x \, n = \int_V \mathrm{d}^3x \, (\nabla \cdot \vec{n} + \partial_t n) = 0 \, , \tag{96}$$

where V is our volume, and its boundary is ∂V. If T is the time for which we consider the problem, this can be restated by integrating over time as well:

$$\int_{V \times T} \partial_\mu n^\mu \, \mathrm{d}^4x = 0 \, . \tag{97}$$

We used the divergence theorem, which states that the flux going out of the boundary of a volume is equal to the integral of the divergence over the volume (for any vector field).

This holds for any volume and for any time: therefore the integrand must be identically null, $\partial_\mu n^\mu \equiv 0$.

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The flux of particles through a surface with normal unit vector n^μ and ΔV is is given by $\Delta N = N^\mu n_\mu \Delta V$, where N^μ is the number density current while ΔV is a 3-volume element of the 3-surface orthogonal to n^μ.

Now, we define the energy-momentum-stress tensor $T^{\mu\nu}$ by imposing the condition $\Delta p^\alpha = T^{\alpha\beta} n_\beta \Delta V$, where Δp^μ is the momentum in the volume.

Let us consider an inertial frame in which ΔV is at rest, and take n_μ to be its 4-velocity.

Then the energy density is given by:

$$\epsilon = \frac{\Delta p^0}{\Delta V} = T^{00} \, , \tag{98}$$

while the momentum density is:

$$\Pi^i = \frac{\Delta p^i}{\Delta V} = T^{i0} \,. \tag{99}$$

Now let us consider a frame moving with velocity $\vec{v}$, in this frame each particle has energy $m\gamma$ and momentum $m\gamma\vec{v}$, and the energy density is $n_*\gamma$.

The moving observer sees exacly the energy density $\epsilon = m\gamma n_* \gamma = T^{00}$, and the momentum density $T^{i0} = m\gamma \vec{v}^i n_* \gamma$.

In general, for particles which do not interact with each other, we have

$$T^{\alpha\beta} = n_* m u^\alpha u^\beta \,. \tag{100}$$

So we see that in this case the stress-energy tensor is symmetric. This actually holds in general.

What is T^{0i}? We have $\Delta p^\alpha = T^{\alpha 1} \Delta y \Delta z \Delta t$ (if we select the normal n_μ parallel to the x axis).

For $\alpha = 0$, this is the flux of energy in time (power) along the x direction.

For $\alpha = i$, we can use the same relation:

$$T^{i1} = \frac{\Delta p^i}{\Delta y \Delta z \Delta t} = \frac{F^i}{\Delta y \Delta z} = \frac{F^i}{\text{Area}} \,. \tag{101}$$

Do note that i can be either x, y or z: we can consider the *pressure*, which is the force along the x axis across the x axis, but also the *deviatoric stresses* along the y or z axes but across the x axis.

Do also note that the stress tensor from fluid dynamics, σ^{ij}, is not equal to T^{ij}: it is its opposite.

The energy density measured by an observer with 4-velocity u^μ is $T_{\mu\nu} u^\mu u^\nu$.

Claim 4.2. *Energy momentum is conserved.*

Proof. In the LIF $\partial_\beta T^{\alpha\beta} = 0$ (in perfect analogy to the current number density) therefore in any frame $\nabla_\beta T^{\alpha\beta} = 0$. □

We will see that this also follows from the Bianchi identities of the Riemann tensor if we assume the Einstein equations, which can hold also in the absence of translational symmetry (this is not just theoretical wandering: as far as we can tell, the universe is symmetric with respect to spatial translations but not temporal ones).

However, translational symmetries give us Killing vectors (vectors corresponding to translational symmetries of the metric) along which we can project the equation $\nabla_\mu T^{\mu\nu} = 0$ to find conserved quantities in the geodesic motion of a particle. We are then able to do this for spatial translations but not for temporal ones:

energy is *not* conserved in the universe.

Definition 4.2 (Perfect fluid). *A fluid is perfect if it has no dissipative effects: heat conduction, viscosity.*

Claim 4.3. *The stress-energy tensor of a perfect fluid is*

$$T^{\alpha\beta} = diag(\rho, p, p, p) . \tag{102}$$

in its own rest frame; in any frame $T^{\alpha\beta} = \rho u^\alpha u^\beta + P h^{\alpha\beta}$, *where* $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$.

Definition 4.3 (Equation of state). *It is* $P = \omega\rho$.

4.3 The Einstein equations

We require them to be *mathematically consistent* and *physically correct*.

We have the principle of covariance: this tells us that the law should be a tensorial expression.

We know that energy is described by the tensor $T_{\mu\nu}$, while for the curvature part we have the various contractions of the metric and the Riemann tensor.

Our ansatz is

$$R_{\mu\nu} + c_1 R g_{\mu\nu} = c_2 T_{\mu\nu} . \tag{103}$$

Symmetry is all right: $R_{\mu\nu}$ is symmetric, and so is the metric. The stress energy tensor is also conserved: therefore we impose

$$\nabla^\mu \Big(R_{\mu\nu} + c_1 R g_{\mu\nu} \Big) = 0 . \tag{104}$$

By the contracted Bianchi identies, this implies $c_1 = -1/2$.

To get c_2, we can look at the Newtonian limit: they should simplify to $\Box\phi \propto \rho$. This will imply $c_2 = 8\pi G$.

4.3.1 The Newtonian limit

We start by tracing the equations: we get $R - 2R = c_2 T$, where $g^{\mu\nu} T_{\mu\nu} = T$. Note that $g^{\mu\nu} g_{\mu\nu} = 4$: it is not the trace of the metric, but its Frobenius norm.

This means that $R = -c_2 T$: we can then rewrite the EFE as

$$R_{\mu\nu} = c_2 \left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) . \tag{105}$$

Let us consider a slowly moving perfect fluid: $p \ll \rho$, and $v \approx 0$, so $u^\mu = (1, \vec{0})$.

We consider a weak gravitational field: $g_{\mu\nu} \approx \eta_{\mu\nu}$.

Our stress energy tensor is approximately $\rho u^\mu u^\nu$. We have $T = -\rho$.

Then, we will get:

$$R_{00} \approx c_2 \left(\rho - \frac{\rho}{2} (-)^2 \right) . \tag{106}$$

Therefore we get $R_{00} \approx c_2 \rho / 2$.

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We are trying to fix the last coefficient in the Einstein equations:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = c_2 T_{\mu\nu} \,. \tag{107}$$

The $1/2$ on the LHS is fixed by the conservation of the stress-energy tensor.

We look at a low-energy scenario: $\rho \ll P$, $u^\mu = (1, \vec{0})^\top$. Then the EFE become $R_{00} = c_2 \rho / 2$.

We consider a stationary metric $g_{\mu\nu}(\vec{x})$, which does not depend on time. This is consistent with the stuff we usually see: planetary dynamics are quasi-static with respect to the speed of light.

In a LIF we have shown that the Riemann tensor is:

$$R_{\gamma\sigma\mu\nu} = \frac{1}{2} \Big(g_{\gamma\nu,\sigma\mu} - g_{\sigma\nu,\gamma\mu} - g_{\gamma\mu,\sigma\nu} + g_{\sigma\mu,\gamma\nu} \Big) \,. \tag{108}$$

To get the Ricci tensor we need $R^\alpha_{\sigma\mu\nu} = \eta^{\alpha\gamma} R_{\gamma\sigma\mu\nu}$ since we are in the LIF.

We get the Ricci tensor:

$$R_{\sigma\nu} = \eta^{\alpha\gamma} \left(g_{\gamma\nu,\sigma\alpha} - g_{\sigma\nu,\gamma\alpha} - g_{\gamma\alpha,\sigma\nu} + g_{\sigma\alpha,\gamma\nu} \right) \,, \tag{109}$$

from which we can calculate R_{00}:

$$R_{00} = \frac{1}{2} \eta^{\alpha\gamma} \left(g_{\gamma 0,0\alpha} - g_{00,\gamma\alpha} - g_{\gamma\alpha,00} + g_{0\alpha,\gamma 0} \right) \tag{110a}$$

$$= -\frac{1}{2} \eta^{ij} g_{00,ij} = -\frac{1}{2} \sum_i g_{00,ii} = -\frac{1}{2} \nabla^2 g_{00} \,, \tag{110b}$$

where only the second term survives since the metric is time-independent, its time derivatives all vanish.

Therefore, our equation becomes $\nabla^2 g_{00} = c_2 \rho$. We just need to find out what the meaning of g_{00} is, how it is related to the gravitational potential. We do it with gravitational redshift: we found that

$$\Delta\tau_A \approx \Delta\tau_B (1 - \Phi_A + \Phi_B) \,, \tag{111}$$

if Alice, on the top of a building, is sending photons to Bob who is on the ground. To first order in the field, the expression is equivalent to

$$\Delta\tau_A \approx \Delta\tau_B (1 - \Phi_A)(1 + \Phi_B) \approx \Delta\tau_B \frac{1 + \Phi_B}{1 + \Phi_A} \,, \tag{112}$$

which is equivalent to the constancy of

$$\frac{\Delta\tau}{1 + \Phi} \,. \tag{113}$$

The $\Delta\tau$ are proper times measured by the observers A and B at rest. For an observer at rest, the spacetime interval is

$$-\,\mathrm{d}\tau^2 = \mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 \;, \tag{114}$$

since the $\mathrm{d}x^i$ are null. Therefore, $\mathrm{d}\tau = \sqrt{-g_{00}}\,\mathrm{d}t$.

We have

$$\Delta\tau_A = \sqrt{-g_{00}(A)}\Delta t \qquad \text{and} \qquad \Delta\tau_B = \sqrt{-g_{00}(B)}\Delta t\,, \tag{115}$$

but the Δt are the same, because the metric is constant with respect to time! so we get that $\Delta t = \text{const}$ and specifically it is equal to

$$\frac{\Delta\tau}{\sqrt{-g_{00}}}\,, \tag{116}$$

so we can finally identify $\sqrt{-g_{00}} \approx 1+\Phi$, or $g_{00} = -(1+\Phi)^2 = -(1+2\Phi)$.

We can now bring the Laplacian inside the g_{00}: we get

$$\nabla^2\Phi = \frac{c_2}{2}\rho\,, \tag{117}$$

Poisson's equation for the gravitational potential.

We know the gravitational potential to be defined by $-\nabla\Phi = \vec{F_G}$.

We can find Gauss' law for the gravitational field just like we did for the electromagnetic field, substituting $Q \to M$ and $1/(4\pi\epsilon_0) \to -G$. I will also denote the electric and gravitational charges with subscripts E and G.

The integral form of this equation is

$$\int_{\partial V} \mathrm{d}\vec{A}\cdot\vec{E} = \int_V \mathrm{d}^3x\,\rho_E\,, \tag{118}$$

but it can be expressed differentially with the divergence theorem as

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_E}{\epsilon_0}\,. \tag{119}$$

For the gravitational field $\vec{F_G}$ then we can just substitute:

$$\vec{\nabla}\cdot\vec{F_G} = -4\pi G\rho_G\,, \tag{120}$$

and then we can substutite the gravitational potential:

$$\vec{\nabla}\cdot(-\vec{\nabla}\Phi) = -\nabla^2\Phi = -4\pi G\rho_G\,, \tag{121}$$

or $\nabla^2\Phi = 4\pi G\rho_G$. So this is what we found before, with $c_2/2 = 4\pi G$. So, our constant is $c_2 = 8\pi G$. So we get Einstein's equations:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{122}$$

The LHS of these is called the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - {}^R\!/\!{}_2 g_{\mu\nu}$.

Why do we not write an equation with more derivatives, more indices? We may, but we'd detect these only in the regime of very strong curvature: the relativistic effects are already hard to detect as is! For now the vanilla EFE have always agreed with experiment.

5 Geodesics

In flat spacetime, we know that the curve with $\mathrm{d}^2x^\mu \big/ \mathrm{d}\tau^2 = 0$ stationarizes $\tau = \int \mathrm{d}\tau$.

Now we will do the exact same thing, except that $\mathrm{d}\tau$ will be calculated with $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$.

We perturb our path $x^\mu(\lambda)$ as $x^\mu + \delta x^\mu$, and we want to set to zero the first functional derivative $\delta\tau_{AB} \big/ \delta x^\mu$. This means that to first order

$$\delta\left(\int_0^1 \mathrm{d}\sigma \sqrt{-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}}\right) = 0\,, \tag{123}$$

where we performed a reparametrization of the curve. Computing away, with the notation $u^\mu = \mathrm{d}x^\mu \big/ \mathrm{d}\sigma$:

$$\begin{aligned}
\delta\tau_{AB} &= \int_0^1 \mathrm{d}\sigma \left(\frac{-\delta g_{\alpha\beta}u^\alpha u^\beta}{2\sqrt{-u^2}} + \frac{-g_{\alpha\beta}u^\alpha \frac{\mathrm{d}\delta x^\beta}{\mathrm{d}\sigma}}{\sqrt{-u^2}}\right) && \text{(124a)}\\
&= -\frac{1}{2}\int_0^1 \mathrm{d}\sigma \left(\delta g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + 2g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}\delta x^\beta}{\mathrm{d}\sigma}\right) && \text{(124b)}\\
&= -\frac{1}{2}\int_0^1 \mathrm{d}\sigma \left(\partial_\gamma g_{\alpha\beta}\delta x^\gamma \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} - 2\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\right)\delta x^\beta\right) && \text{(124c)}\\
&= -\frac{1}{2}\int_0^1 \mathrm{d}\tau \left(\partial_\gamma g_{\alpha\beta}\delta x^\gamma \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\delta x^\beta\right)\right) && \text{(124d)}\\
&\int \mathrm{d}\tau \left(-\frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\gamma}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\right)\delta x^\gamma\right)\,, && \text{(124e)}
\end{aligned}$$

where we applied the product rule, identified two symmetric terms in the velocity, used the identity

$$\frac{1}{\sqrt{-u^2}} \frac{\mathrm{d}}{\mathrm{d}\sigma} = \frac{\mathrm{d}}{\mathrm{d}\tau}\,, \tag{125}$$

which follows from

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \implies u^2 = \frac{\mathrm{d}x^\alpha\,\mathrm{d}x_\alpha}{\mathrm{d}\sigma^2} = -\frac{\mathrm{d}\tau^2}{\mathrm{d}\sigma^2}\,. \tag{126}$$

Also, we expanded the metric using:

$$\delta g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} \delta x^\gamma\,, \tag{127}$$

where we did not need to introduce a covariant derivative since we are just Taylor expanding.

Also, we integrated by parts (without boundary terms since the path variation vanishes at the path boundary), changed variables, and finally gotten an expression which must vanish for any path, therefore we get that

$$-\frac{1}{2}\partial_\gamma g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\gamma}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\right) = 0\,, \tag{128}$$

so we can expand the derivative: we get

$$0 = -\frac{1}{2}\partial_\gamma g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + \partial_\beta g_{\alpha\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} + g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2}\,, \tag{129}$$

where the second term can be symmetrized in $\alpha\beta$:

$$0 = g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \left(\frac{1}{2}\partial_\beta g_{\alpha\gamma} + \frac{1}{2}\partial_\alpha g_{\beta\gamma}\right) - \frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\,, \tag{130}$$

so we get

$$0 = g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \frac{1}{2}\Big(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}\Big)\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\,, \tag{131}$$

which, raising an index and identifying the Christoffel symbols, gives us

$$0 = \frac{\mathrm{d}^2x^\gamma}{\mathrm{d}\tau^2} + \Gamma^\gamma_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\,, \tag{132}$$

the *geodesic equation*.

It can be written also as $u^\mu \nabla_\mu u^\nu = a^\nu = 0$.

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It can be shown that the equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0\,, \tag{133}$$

in the low-field limit gives the regular acceleration in a gravitational field.

If we define u^μ as $\mathrm{d}x^\mu / \mathrm{d}\tau$, we get that the equation is equivalent to

$$u^\alpha \left(\frac{\partial u^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} u^\beta \right) = u^\alpha \nabla_\alpha u^\mu = a^\mu = 0\,. \tag{134}$$

The acceleration we feel corresponds to the difference between our motion and geodesic motion.

The four-velocity has constant square modulus: either 0, or ± 1, if we choose an appropriate parametrization. Therefore we can classify geodesics.

Timelike geodesics have $u^\mu u_\mu = -1$, and are related to the motion of a particle. They minimize the proper time $\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2}$, which is real in this case. We parametrize them by τ.

Spacelike geodesics have $u^\mu u_\mu = +1$ and can be seen as the shortest path between two points: for them, the integral of $\mathrm{d}s$ is stationary. We parametrize them by s.

Null geodesics have $u^\mu u_\mu = 0$ are characterized by $\mathrm{d}s = 0$. We parametrize them with some parameter of our choosing, λ, which must be independent of proper time or space.

5.1 Solutions of the geodesic equation

We treat the problem in the case of a two-dimensional Euclidean plane, using polar coordinates: we know we should find straight lines, but in these coordinates the problem is nontrivial.

The metric is $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$, and the nonzero Christoffel symbols are (see exercise 3.3):

$$\Gamma^r_{\theta\theta} = -r \qquad \text{and} \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}\,. \tag{135}$$

Our equation for the r coordinate is then:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = 0\,, \tag{136}$$

while for the θ coordinate by symmetry we can identify the two terms:

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0\,. \tag{137}$$

In general, in an n dimensional space, motion is defined by n scalar functions, which can be determined by our n differential equations with $2n$ initial conditions.

It is in general useful to find *first integrals*, quantities which are constant along the geodesic.

It can be shown that the second equation can be written as

$$\frac{1}{r^2}\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = 0\,, \tag{138}$$

which gives us the first integral $A = r^2\, d\theta/ds$.

An easy way to see this: the equation can be written, denoting derivatives with respect to s with a dot, as

$$\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} = 0\,, \tag{139}$$

which we can rearrange as

$$\frac{\ddot{\theta}}{\dot{\theta}} + 2\frac{\dot{r}}{r} = 0\,, \tag{140}$$

or

$$\frac{d}{ds}\left(\log\dot{\theta} + 2\log r\right) = \frac{d}{ds}\log\left(\dot{\theta}r^2\right) = 0\,, \tag{141}$$

so we have found our integral: the derivative of the logarithm of something is constant iff the thing is constant.

We can always also use the definition of the differential:

$$ds^2 = dr^2 + r^2\, d\theta^2\ , \tag{142}$$

so we can insert our integral:

$$ds^2 = dr^2 + r^2\frac{A^2}{r^4}\, ds^2\ , \tag{143}$$

so

$$ds^2\left(1 - \frac{A^2}{r^2}\right) = dr^2\ . \tag{144}$$

This has two solutions, but it can be shown that they give the same result in the end.

We want the trajectory: the locus of the points the geodesic passes through, $r(\theta)$ or $\theta(r)$.

We do:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{A}{r^2}\frac{1}{\sqrt{1-\frac{A^2}{r^2}}}\,, \tag{145}$$

so we can integrate this:

$$\theta = \int \mathrm{d}\theta = \int \frac{A}{r^2}\left(1-\frac{A^2}{r^2}\right)^{-1/2} \mathrm{d}r\,, \tag{146}$$

which comes out to be $\Delta\theta = \arccos(A/r)$, which can be inverted to find $r\cos(\Delta\theta) = A$: using the trigonometric relation

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)\,, \tag{147}$$

we find that this is equivalent to

$$r\cos(\theta)\cos(\theta_0) + r\sin(\theta)\sin(\theta_0) = A\,, \tag{148}$$

therefore this can be written as $y = \alpha x + \beta$.

5.2 Euler-Lagrange equations

The time interval can be written as

$$\tau_{AB} = \int \mathrm{d}\tau = \int \sqrt{-\,\mathrm{d}s^2} \tag{149a}$$

$$= \int \mathrm{d}\sigma\, \mathscr{L}\left(x^\alpha, \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\right), \tag{149b}$$

so under a perturbation we get:

$$0 = \delta\tau \tag{150a}$$

$$= \int \mathrm{d}\sigma \left(\frac{\partial\mathscr{L}}{\partial x^\alpha}\delta x^\alpha + \frac{\partial\mathscr{L}}{\partial\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}}\frac{\mathrm{d}\delta x^\alpha}{\mathrm{d}\sigma}\right) \tag{150b}$$

$$= \int \mathrm{d}\sigma \left(\frac{\partial\mathscr{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathscr{L}}{\partial\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}}\right)\right)\delta x^\alpha = 0\,, \tag{150c}$$

therefore the integrand must vanish identically: this gives us the Euler-Lagrange equations

$$\frac{\partial\mathscr{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial\mathscr{L}}{\partial\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}} = 0 \tag{151}$$

5.3 Killing vectors

Symmetries of the metric correspond to conserved quantities if our Lagrangian only depends on the metric. If the metric does not depend on a coordinate, then the unit vector in that direction is called a Killing vector field.

If we have a Killing vector field, then the momentum along the Killing vector is conserved.

If the Killing coordinate is x^1, it can be expressed as

$$\frac{\partial \mathscr{L}}{\partial \frac{\mathrm{d}x^1}{\mathrm{d}\sigma}} = \frac{1}{2\mathscr{L}} \left(-2 g_{1\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma} \right) = -g_{1\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} , \tag{152}$$

where we performed a change of variable from the derivation with respect to σ to one with respect to τ.

This holds because:

$$\mathscr{L} = \sqrt{-g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}} = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} . \tag{153}$$

This is usually written as $\xi^\mu u_\mu = \text{const}$, which is actually more general. We can write it with respect to the momentum: $p^\mu \xi_\mu$ since we are considering a constant-mass particle.

We can apply this to our 2D example: the metric does not depend on θ, therefore $\xi = (0, 1)$ is a Killing vector, so $g_{\theta\mu} u^\mu = r^2 \,\mathrm{d}\theta / \mathrm{d}s = \text{const}$.

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5.4 Riemann normal coordiates

We want to actually build a LIF, in which $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $g_{\mu\nu,\rho}(P) = 0$.

We can choose a set of vectors $e_{(\mu)}(P)$ which are orthonormal: $e_{(\mu)} \cdot e_{(\nu)} = \eta_{(\mu)(\nu)}$.

If we choose this *tetrad* as a basis, then the metric becomes the Minkowski one at that point.

Take a vector n^μ at P, and consider all possible geodesics which start from P with initial tangent vector n^μ.

Then, the coordinates of a point Q we get by moving for a time τ along this geodesic are $x^\alpha = \tau n^\alpha$ if n^α is timelike, $x^\alpha = s n^\alpha$ if it is spacelike, where

$$\tau = \int_P^Q \mathrm{d}\tau . \tag{154}$$

Since these lines are geodesics, they satisfy the geodesic equation:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0\,, \tag{155}$$

but we can insert the relation $x^\alpha = \tau n^\alpha$ here, to get $\Gamma^\alpha_{\beta\gamma} n^\beta n^\gamma = 0$: but this can be written for any n^α, therefore we immediately get that all the Christoffel symbols vanish: $\Gamma^\alpha_{\beta\gamma} \equiv 0$.

Therefore, we have 4^3 equations of sums of derivatives of the metric which vanish: but the gradient of the metric only has 4^3 independent components, therefore the solution $g_{\mu\nu,\alpha} \equiv 0$ is the only one, as long as $\Gamma^\alpha_{\beta\gamma}(g_{\alpha\beta,\gamma})$ is an invertible system, which is nontrivial to show. This invertibility is equivalent to the linear system $\Gamma_{\alpha\beta\gamma}(g_{\alpha\beta,\gamma}) = M^{\mu\nu\rho}_{\alpha\beta\gamma} g_{\mu\nu,\rho}$ being invertible. The following code shows this:

```python
import numpy as np

# linear transformation between
# metric derivatives and Christoffel symbols
M = np.zeros((4**3, 4**3))

# three indices run from 0 to 3:
# we incorporate them into one
# from 0 to 4**3-1
d = lambda i,j,k: 4**2*i+4*j+k

# we populate the matrix with the relevant coefficients,
# starting from the formula for the metric
# in terms of the Christoffel symbols
for i in range(4):
    for j in range(4):
        for k in range(4):
            M[d(i, j, k), d(i, j ,k)] += 1
            M[d(i, j, k), d(i, k, j)] += 1
            M[d(i, j, k), d(k, j, i)] += -1

print(np.linalg.det(M))
```

Can the geodesics cross each other far from the starting point? *Yes*, so the Riemann normal coordinates are only defined in a neighbourhood.

Let us consider an example: Riemann normal coordinates around the north pole of a sphere.

We can map every point on the sphere but the south pole by specifying the meridian and the distance to travel along the meridian.

So in our case, if R is the radius of the sphere, we get $x^\alpha = (R\theta\cos(\phi), R\theta\sin(\phi))$, where then $R\theta = \tau$ and $n^\alpha = (\cos(\theta), \sin(\theta))$. The angles ϕ and θ are the usual spherical coordinate angles: ϕ specifies the meridian, while $R\theta$ gives us the distance travelled away from the north pole (since θ is in radians).

The second order expansion of the metric is

$$g_{ij} = \begin{bmatrix} 1 - \frac{2y^2}{3R^2} & \frac{2xy}{3R^2} \\ \frac{2xy}{3R^2} & 1 - \frac{2x^2}{3R^2} \end{bmatrix}, \tag{156a}$$

where $x^\alpha = (x, y)$.

We can see that the metric is δ_{ij} at the north pole, and its derivatives are $g_{ij,k} = 0$ there.

6 The Schwarzschild solution

It descrives the geometry outside a stationary, spherically symmetric object which is not rotating and not electrically charged, such as a star, planet or BH.

In general $ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + C(r)^2\left(d\theta^2 + \sin^2\theta\,d\phi\right)$ is our line element.

We can define $\widetilde{r} = C(r)$, and then express A, B with respect to to this, and recalling

$$dr^2 = \frac{d\widetilde{r}^2}{(dC/dr)^2}, \tag{157}$$

which is what multiplies B, so we redefine $B(\widetilde{r})$ as $B(\widetilde{r})/(dC/dr)^2$.

So, we can just relabel B as this: the expression

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \tag{158}$$

is fully general.

Another way to see this is to define the radial coordinate by imposing the condition that the area of a sphere at radius r be $4\pi r^2$:

$$\text{Area}(r) = \int C^2(r)\,d\Omega = 4\pi C^2(r) \overset{!}{=} 4\pi r^2 \implies C(r) = r, \tag{159}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$.

So far we have used the hypothesis of stationarity by writing everything only as a function of r.

Recall the inverted Einstein equations:

$$\frac{1}{8\pi G}\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right) = R_{\mu\nu}. \tag{160}$$

We want to solve these outside of our source: we look for *vacuum solutions*. Then the equations are just $R_{\mu\nu} = 0$. A trivial solution to these is $g_{\mu\nu} = \eta_{\mu\nu}$, but we will show that it is not the only one! As a matter of fact, we know that the solution to a differential equation is determined by the boundary conditions: in our case, the mass of the object which sits at the origin. The Minkowski metric respects these boundary conditions if $M = 0$. In general, it does not.

A homework exercise will be to show that, denoting $\mathrm{d}/\mathrm{d}r$ with a prime:

$$R_{00} = \frac{A''}{2B} - \frac{A'B'}{4B^2} - \frac{A'^2}{4AB} + \frac{A'}{rB} \tag{161a}$$

$$R_{11} = -\frac{A''}{2A} + \frac{A'B'}{4B^2} - \frac{A'^2}{4A^2} + \frac{B'}{rB} \tag{161b}$$

$$R_{22} = 1 - \frac{1}{B} - \frac{rA'}{2AB} + \frac{rB'}{2B^2} \tag{161c}$$

$$R_{33} = \sin^2\theta R_{22} \tag{161d}$$

$$R_{ij} = 0 \qquad \text{if } i \neq j\,. \tag{161e}$$

If we compute $BR_{00} + AR_{11}$ we get many simplifications:

$$0 = A'B + AB' = (AB)'\,, \tag{162}$$

therefore AB is a constant with respect to r.

Also, we can write $A'/A = -B'/B$: we substitute it into R_{22}: we get

$$0 = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{B'}{B}\right) + \frac{rB'}{2B^2}\,, \tag{163}$$

so

$$0 = 1 - \frac{1}{B} + \frac{rB'}{B^2}\,, \tag{164}$$

which is first order in terms of B. We can solve it by separating the variables. We get

$$\frac{\mathrm{d}r}{r} = \frac{\mathrm{d}B}{B(1-B)}\,, \tag{165}$$

from which we find that, if at r_* the variable $B = B_*$, then

$$\int_{r_*}^{r} \frac{\mathrm{d}r}{r} = \int_{B_*}^{B} \frac{\mathrm{d}B}{B(1-B)}\,, \tag{166}$$

where we can rewrite the term

$$\frac{1}{B(1-B)} = \frac{1}{B} + \frac{1}{1-B}\,. \tag{167}$$

So,

$$\log \frac{r}{r_*} = \log \frac{B}{B_*} - \log \frac{1-B}{1-B_*}\,, \tag{168}$$

or, exponentiating:

$$\frac{r}{r_*} = \frac{B}{1-B}\frac{1-B_*}{B_*}\,, \tag{169}$$

from which we can find $B(r)$: first we write

$$\frac{1-B}{B} = \frac{1-B_*}{B_*}\frac{r_*}{r} = \frac{\gamma}{r}\,, \tag{170}$$

where we collected the integration constants into γ.

So,

$$\frac{1}{B} = \frac{\gamma}{r} + 1 \implies B = \left(1 + \frac{\gamma}{r}\right)^{-1}. \tag{171}$$

and because of $AB = k$ we also get

$$A = k\left(1 - \frac{\gamma}{r}\right). \tag{172}$$

We are almost at the full solution: by continuity with the large-r limit, for which we have the Minkowski metric with $A = B = 1$, we have that $k = 1$. Now we have

$$\mathrm{d}s^2 = -\left(1 + \frac{\gamma}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{\gamma}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{173}$$

and γ can only be found by continuity with the weak-field approximation: else, every value for it solves the equations. We know that $g_{00} = -(1+2\Phi)$. In the weak-field limit we have $\Phi = -GM/r$, so we identify $\gamma = -2GM$.

6.1 Gravitational redshift

Now we can do the exact calculation for the gravitational redshift: Alice, at r_A, sends photons to Bob at r_B. The motion of the photons need not be radial. Alice and Bob are not following geodesics, we consider them to be stationary in this metric.

We know the metric to be independent of time: $\xi^\mu = (1, \vec{0})$ is a Killing vector field. Light has momentum p^μ and moves along a geodesic. We will use the relation $p^\mu \xi_\mu = \text{const}$ to solve our problem.

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We want to study the behaviour of light and particles in a Schwarzschild background, for $r > 2GM$. The metric is:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \,. \tag{174}$$

In general, the motion of light is nonradial. The metric is independent of time: our Killing vector field is $\xi^\alpha = (1, \vec{0})$, so $p \cdot \xi = \text{const}$.

From QM we know that $f_A = E_A/h$, where E_A is the energy measured by A.

The 4-velocity of A has spatial components equal to zero, since A does not move. Its norm must be $u^\mu u_\mu = -1$, therefore $u^\mu_A = (1/\sqrt{1 - 2GM/r_A}, \vec{0})$, or $u^\alpha_A = (1, \vec{0})/\sqrt{1 - 2GM/r_A} = \xi^\alpha/\sqrt{1 - 2GM/r_A}$.

For Bob, we have the exact same thing, except $A \to B$.

So, for $i = A, B$:

$$f_i = u^{(i)}_\mu p^\mu_{\text{photon}} = -\left(1 - \frac{2GM}{r_i}\right)^{-1/2} \frac{p_{\text{photon}} \cdot \xi}{h} \,, \tag{175}$$

since both of the 4-velocities are proportional to the Killing (unit) vector field, with *different proportionality constants*. The last part is exactly the same since the light moves along a geodesic: so their ratio is given by

$$\frac{f_B}{f_A} = \sqrt{\frac{1 - 2GM/r_A}{1 - 2GM/r_B}} \,, \tag{176}$$

or

$$f_{\text{obs}} = f_{\text{emit}} \frac{\sqrt{1 - \frac{2GM}{r_{\text{emit}}}}}{\sqrt{1 - \frac{2GM}{r_{\text{obs}}}}} \,. \tag{177}$$

Let us consider the nonrelativistic approximation: $r_A = R + h$, while $r_B = R$, where $R = R_{\text{earth}}$. If we Taylor expand (with $2GM \ll R$), we get:

$$f_{\text{obs}} = f_{\text{emit}} \left(1 - \frac{GM}{r_{\text{emit}}} + \frac{GM}{r_{\text{obs}}}\right) , \tag{178}$$

and then we expand in h/R: we get

$$f_{\text{obs}} = f_{\text{emit}} \left(1 - \frac{GM}{R}\left(1 - \frac{h}{R}\right) + \frac{GM}{R}\right) = f_{\text{emit}} \left(1 + \frac{GM}{R^2} h\right) = f_{\text{emit}} (1 + gh) \,, \tag{179}$$

if we want more precision then we can keep more orders.

6.1.1 Classical orbits

Kepler's laws are *wrong*! We will show this.

In classical circular orbits, for a planet with mass $m = 1$, we have the gravitational force GM/r^2 equalling v^2/r, or with respect to the angular momentum $l = vr$:

$$\frac{GM}{r^2} = \frac{l^2}{r^3}\,, \tag{180}$$

so we get $r = l^2/GM$. The energy is given by

$$\frac{v^2}{2} - \frac{GM}{r} = E\,, \tag{181}$$

and if we want to write these with respect to the velocity vector in polar coordinates, $v_r = \mathrm{d}v/\mathrm{d}t$ and $v_\theta = r\,\mathrm{d}\theta/\mathrm{d}t$ we have

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2, \tag{182}$$

while the angular momentum in general is $\vec{L} = \vec{r} \times \vec{v}$, whose modulus is $l = \left|\vec{L}\right| = rv_\theta = r^2\,\mathrm{d}\theta/\mathrm{d}t$. Therefore, $v_\theta = l/r$. So, the velocity is

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{l^2}{r^2}\,. \tag{183}$$

Then, the equation for the radial motion of the planet is

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} = E\,, \tag{184}$$

which for large r tends to 0 from below, while for small r tends to $+\infty$.

The circular orbit is the one which corresponds to the bottom of the potential.

6.1.2 Relativistic Schwarzschild orbits

Planets *do not* actually orbit in true ellipses, but this is not actually the case even in Newtonian mechanics, since there are other objects in the universe. The orbit of Mercury was expected to precede by $532''$ every 100 yr, but people observed an additional $43''$ every 100 yr. We will compute this.

For semplicity, we will say that in our spherical coordinates Mercury will always have $\theta = \pi/2$. The 4-velocity of the planet will be

$$u^\alpha = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}r}{\mathrm{d}\tau}, 0, \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right), \tag{185}$$

and we have two immediate Killing vectors: $\xi_t^\alpha = (1, \vec{0})$ and $\xi_\varphi^\alpha = (0,0,0,1)$ since the metric is independent of t and φ.

We call $e = -\xi_t \cdot u = (1 - 2GM/r)\frac{\mathrm{d}t}{\mathrm{d}\tau}$.

The other Killing vector is $l = \xi_\varphi \cdot u = r^2 \sin^2\theta \, \mathrm{d}\varphi/\mathrm{d}\tau$, but $\theta = \pi/2$ so $l = r^2 \, \mathrm{d}\varphi/\mathrm{d}\tau$.

Now we want to impose the condition $-1 = u \cdot u$:

$$-1 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2 \tag{186a}$$

$$= -\left(1 - \frac{2GM}{r}\right)\left(\frac{e}{1 - \frac{2GM}{r}}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\sin^2\theta\left(\frac{l^2}{r^2}\right)^2 \tag{186b}$$

$$0 = -e^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{l^2}{r^2} + 1\right)\left(1 - \frac{2GM}{r}\right), \tag{186c}$$

so

$$e^2 - 1 = \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{l^2}{r^2} + 1\right)\left(1 - \frac{2GM}{r}\right) - 1 \tag{187a}$$

$$E = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}, \tag{187b}$$

where

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}, \tag{188}$$

and

$$E = \frac{e^2 - 1}{2}, \tag{189}$$

so the GR effects are exactly contained in that last term GMl^2r^{-3}. This is very small: we consider it as a perturbation.

Are there circular orbits in this case?

This r^{-3} term means that for $r \to 0$ the effective potential goes to $-\infty$.

In general we'd expect two stationary points: one which is closer and unstable, and one which is further and stable.

If we differentiate $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ we get:

$$GMr^2 - l^2 r + 3GMl^2 = 0, \tag{190}$$

and we consider the positive solution:

$$r = \frac{l^2 \pm \sqrt{l^4 - 12G^2M^2l^2}}{2GM}\,, \tag{191}$$

and we take the plus sign since we want the orbit which is further out.

$$r = \frac{l^2}{2GM}\left(1 + \sqrt{1 - 12\frac{G^2M^2}{l^2}}\right), \tag{192}$$

which is the formula for the stable circular orbit. Expanding for small relativistic corrections we get

$$r_{\text{classical}} = \frac{l^2}{2GM}\left(1 + 1 - 6\frac{G^2M^2}{l^2}\right) = \frac{l^2}{GM} - 3GM\,, \tag{193}$$

which is the Newtonian orbit $r = l^2/GM$ with a correction.

Where is the boundary at which the solution disappears? it is where the square root vanishes:

$$l^2 = 12G^2M^2\,, \tag{194}$$

or $l = GM\sqrt{12}$. Solutions exist as long as the angular momentum is greater than $GM\sqrt{12}$.

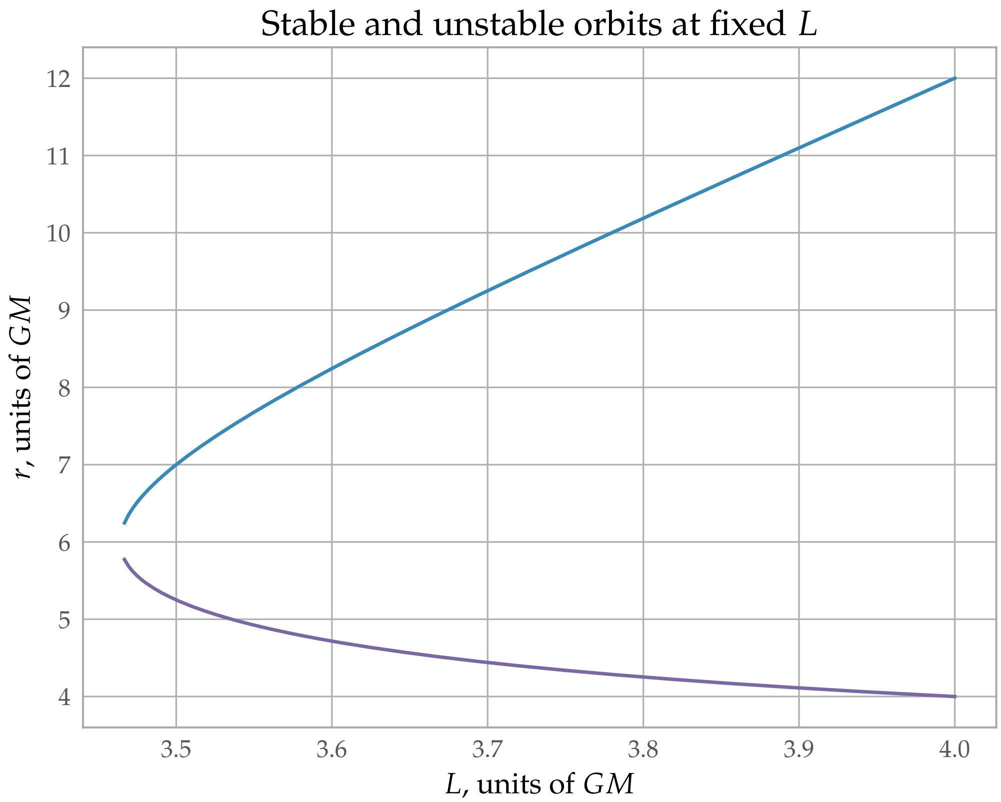

Figure 1: Allowed orbits at fixed L. The unstable branch approaches $r = 3GM$ asymptotically, the stable branch allows for arbitrarily large radii.

For l_{min} we have $r_{\text{min}} = 6GM$. This is called the ISCO: *innermost stable circular orbit*: it is 3 times the Schwazschild radius $2GM$.

6.1.3 Orbital precession

Now, we consider elliptical orbits.

The idea is: the angle between two consecutive perihelions is $2\pi + \delta\varphi_{\text{precession}}$.

In order to find the orbits, we want a relation between different coordinates during the orbit: we will use the equations

$$l = r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \tag{195a}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} = E\,, \tag{195b}$$

so $\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}$. Then:

$$\frac{l^2}{2r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - \frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} = E\,, \tag{196}$$

and it is convenient to solve for $u = r^{-1}$: we get

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}\,, \tag{197}$$

so

$$\frac{l^2}{2}u^4\frac{1}{u^4}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 - GMu + \frac{l^2u^2}{2} - GMl^2u^3 = E \tag{198a}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 - \frac{GM}{l}u + \frac{u^2}{2} - GMu^3 = \frac{E}{l^2}\,, \tag{198b}$$

and we want to remove E so we differentiate with respect to φ:

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} - \frac{GM}{l^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi} + u\frac{\mathrm{d}u}{\mathrm{d}\varphi} - 3GMu^2\frac{\mathrm{d}u}{\mathrm{d}\varphi} = 0\,, \tag{199}$$

and the orbit is monotonic so $\frac{\mathrm{d}u}{\mathrm{d}\varphi} \neq 0$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{GM}{l^2} + 3GMu^2\,, \tag{200}$$

where the term from GR is precisely $3GMu^2$, the rest is fully Newtonian.

This can be solved exactly with respect to complicated elliptic function, but we do it in a simpler way: a nearly circular orbit: $u = u_c(1 + w(\varphi))$, where $w \ll 1$.

To the order w^0: $u_c = \frac{GM}{l^2} + 3GMu_c^2$, since it is a circular orbit (u_c is a constant!)

To first order in w, instead, we get:

$$u_c\frac{\mathrm{d}^2w}{\mathrm{d}\varphi^2} + u_c(1+w) = \frac{GM}{l^2} + 3GMu_c^2(1+2w)\,, \tag{201}$$

since $w \ll 1$. But the terms without w simplify: they satisfy the zeroth order equation. So, we are left with

$$\frac{\mathrm{d}^2w}{\mathrm{d}\varphi^2} = (6GMu_c - 1)w\,, \tag{202}$$

which is in the form $\ddot{w} + \omega^2 w = 0$, since $u_c < 1/(6GM)$. If we look at unstable orbits with radii smaller than $6GM$, then this is exponentially diverging.

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Let's start the show.

The effective potential equation is

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V_{\text{eff}}(r) = E = \frac{e^2-1}{2}\,, \tag{203}$$

where

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3}\,, \tag{204}$$

and the timelike Killing vector gives us e, while the azimuthal Killing vector gives us l.

We can satisfy the equation with $V_{\text{eff}} \equiv E$ (eq1), and $\mathrm{d}r/\mathrm{d}t$ (eq2). These are circular orbits; the equations are, explicitly,

$$-\frac{GM}{l} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} = \frac{e^2-1}{2} \qquad \text{eq1} \tag{205}$$

and

$$\frac{GM}{r^2} - \frac{l^2}{r^3} + \frac{3GMl^2}{r^4} = 0 \qquad \text{eq2}\,. \tag{206}$$

Let us compute eq1 $+\, r(1 - r/2GM)$eq2: after some algebra, we get

$$\frac{l}{e} = \sqrt{GMr}\left(1 - \frac{2GM}{r}\right)^{-1}\,. \tag{207}$$

The angular velocity Ω is defined as a derivative with respect to coordinate time:

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\,, \tag{208}$$

so

$$\Omega = \frac{\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}}{\frac{\mathrm{d}t}{\mathrm{d}\tau}} = \frac{l/r^2}{e\left(1 - \frac{2GM}{r}\right)^{-1}} = \frac{\sqrt{GMr}\left(1 - \frac{2GM}{r}\right)^{-1}\frac{1}{r^2}}{\left(1 - \frac{2GM}{r}\right)^{-1}} = \frac{\sqrt{GMr}}{r^2}\,, \tag{209}$$

so $\Omega^2 = GM/r^3$ or $\Omega^2 r = GM/r^2$, a relation which is the same as in Newtonian physics.

Now, review of the equation for $u = r^{-1}$. When we perturbed it, we got a harmonic oscillator. Our equation then is solved by cosinusoidal waves $u = u_c(1 + \cos(\omega\varphi))$, with $\omega = \sqrt{1 - 6GMu_c}$:

$$r(\varphi) = \frac{r_C}{1 + A\cos\left(\sqrt{1 - \frac{6GM}{r_c}}\varphi\right)}, \tag{210}$$

so $\Delta\varphi$ in one orbit is

$$2\pi\left(1 + \frac{3GM}{r_c}\right), \tag{211}$$

and then $\delta\varphi = 6\pi GM/r_c$. But $r_c = l^2/GM$: so we get

$$\delta\varphi = 6\pi\left(\frac{GM}{l}\right)^2. \tag{212}$$

We can plug numbers into this: we know that $G = 6.674\,08 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$, and our l has units of $\mathrm{m^2s^{-1}}$, so we need to insert a c: the calculation then is

$$\delta\varphi = 6\pi \times \frac{180 \times 3600}{\pi}\left(\frac{GM_\odot}{r_{\mathrm{merc}}v_{\mathrm{merc}}c}\right) \times \frac{100\,\mathrm{yr}}{0.241\,\mathrm{yr}} \approx 43'', \tag{213}$$

where the radius and velocity of the orbit of Mercury must be *both* calculated at the perihelion, or at the aphelion, or one can take the average velocity and the semimajor axis of the ellipse. The factor $100\,\mathrm{yr}/0.241\,\mathrm{yr}$ was inserted to account for the number of orbits of Mercury in a century.

```python
from scipy.constants import G, c, pi

sun_mass = 2e30
mercury_orbital_velocity = 4.7e4
mercury_semimajor_axis = 57.9e9
mercury_angular_momentum = mercury_orbital_velocity *
    mercury_semimajor_axis
mercury_period_yr = .241
rad2arcsec = 3600*180/pi

delta_phi = 6*pi*(G*sun_mass/mercury_angular_momentum/c)**2

print(delta_phi * rad2arcsec * 100/mercury_period_yr)
```

6.2 Radial orbit

We treat motion with $l = 0$ of an object which is at rest at infinity.

The equation becomes

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r} = \frac{e^2-1}{2}\,, \tag{214}$$

and if the object is at rest at infinity we can calculate e for $r \to \infty$:

$$e = \left(1 - \frac{2GM}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1\,. \tag{215}$$

Then, our equation is

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{GM}{r} = 0\,. \tag{216}$$

The time component of the velocity is

$$u^t = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{e}{1-\frac{2GM}{r}} = \frac{1}{1-\frac{2GM}{r}}\,, \tag{217}$$

the radial component is given by our equation of motion:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\sqrt{\frac{2GM}{r}} = -\sqrt{\frac{2GM}{r}}\,, \tag{218}$$

where we choose the minus sign since we are falling in. The components v^θ and v^φ are zero since the motion is radial.

Let us compute $\vec{u}\cdot\vec{u}$: we get

$$g_{00}\left(u^0\right)^2 + g_{11}\left(u^1\right)^2 = -\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-2} + \left(1-\frac{2GM}{r}\right)^{-1}\frac{2GM}{r} \tag{219a}$$

$$= \left(1-\frac{2GM}{r}\right)^{-1}\left(-1+\frac{2GM}{r}\right) = -1\,. \tag{219b}$$

We can integrate

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2GM}{r}}\,, \tag{220}$$

we get

$$\int_0^r \sqrt{r}\,\mathrm{d}r = -\int_0^\tau \sqrt{2GM}\,\mathrm{d}\tau\,, \tag{221}$$

so

$$r(\tau) = \left(\frac{3}{2}\right)^{2/3} \sqrt[3]{2GM}\sqrt[3]{-\tau}\,. \tag{222}$$

Do keep in mind that we set $\tau = 0$ at $r = 0$. It seems like nothing bad really happens at $r = 2GM$. This solution only makes sense for negative τ.

At $r = 2GM$, a finite $\Delta\tau$ corresponds to an infinite Δt since $\mathrm{d}t/\mathrm{d}\tau$ diverges there.

So, the coordinates t and r seem like a bad choice at the horizon: they vary infinitely for a finite time as measured in the frame of the infalling particle.

The horizon is a *coordinate singularity*.

The 4-velocity for a particle going *outward* is:

$$u^\alpha = \begin{bmatrix} \left(1 - \frac{2GM}{r}\right)^{-1} \\ +\sqrt{\frac{2GM}{r}} \\ 0 \\ 0 \end{bmatrix}, \tag{223a}$$

if it reaches radial infinity at rest.

What is the escape velocity? What is the three-velocity as measured by an observer at rest at constant r?

The 4-momentum of the particle is $p^\alpha = mu^\alpha$, while the observer's 4-velocity will be

$$u^\alpha_{\text{obs}} = \begin{bmatrix} \frac{1}{\sqrt{-g_{00}}} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{224a}$$

The energy measured by the observer is $E = -\vec{p} \cdot \vec{u}_{\text{obs}}$. It is

$$E = -g_{00} u^0_{\text{obs}} p^0 = \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-1/2} m \left(1 - \frac{2GM}{r}\right)^{-1} = \frac{m}{\sqrt{1 - \frac{2GM}{r}}}\,. \tag{225}$$

However, any observer sees $E = m\gamma$, so they identify $\gamma = 1/\sqrt{1 - \frac{2GM}{r}}$.

Then, the escape velocity is $\sqrt{\frac{2GM}{r}}$, just like the Newtonian case.

Now, we will look at the motion of light: by how much is it deflected?

We still have the Killing vector $\xi^\mu = (1, \vec{0})$: so

$$-\xi \cdot u = e = \left(1 - \frac{2GM}{r}\right) \frac{\mathrm{d}t}{\mathrm{d}\lambda} \tag{226}$$

is constant, and from $\xi^\mu = (\vec{0}, 1)$ we have

$$\xi \cdot u = l = r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}, \tag{227}$$

where λ is the parameter of the light trajectory, and we set $\theta = \pi/2$.

The square modulus of the light's velocity is zero, so

$$0 = -\left(1 - \frac{2GM}{r}\right) \left(\frac{e}{1 - \frac{2GM}{r}}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + r^2 \left(\frac{l}{r^2}\right). \tag{228}$$

This can be rewritten as

$$0 = -\frac{e^2}{l^2} + \frac{1}{l^2} \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{1}{r^2} \left(1 - \frac{2GM}{r}\right), \tag{229}$$

or

$$\frac{1}{l^2} \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}}(r) = \frac{1}{b^2}, \tag{230}$$

where $b^2 = l^2/e^2$ and

$$W_{\text{eff}}(r) = \frac{1}{r^2} \left(1 - \frac{2GM}{r}\right). \tag{231}$$

A note: λ is a parameter, but we can do affine transformations to it: $\lambda \to a\lambda + b$. Actually we can do any $\lambda' = \lambda'(\lambda)$ which is monotonic. We can check that all physical quantities which appear in the equation are invariant, since all the factors $\mathrm{d}\lambda' / \mathrm{d}\lambda$ cancel.

Also, the equation is even in l, since it must be invariant under $\varphi \to -\varphi$ (which corresponds to P symmetry). Then, we choose l to be positive as our gauge.

At $r = 3GM$ we have a maximum of W_{eff}, which attains the value $1/27G^2M^2$ there. Since it is a maximum, the orbit is unstable.

If a photon approaches the BH with energy $1/b^2$ larger than $W_{\text{eff}}(3GM)$ then it "bounces back": r decreases, there is a minimum r and then r increases.

The critical equation is $b^{-2} > (27G^2M^2)$, which means $l > \sqrt{27}GMe$. It is actually a critical *angular momentum*.

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We found the equation

$$\frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}}(r) = \frac{1}{b^2}\,, \tag{232}$$

where

$$W_{\text{eff}} = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) \tag{233}$$

and

$$b^2 = \frac{l^2}{e^2}\,, \tag{234}$$

l and e being the integrals corresponding to the Killing vectors of time translations and azimuthal angle rotations.

Now, we want to give the interpretation of b as the impact parameter. We consider a BH at the origin of the x, y axes, and a photon approaching parallel to the x axis with impact parameter d. The impact parameter is the distance between the two lines: the trajectory of the photon far away from the BH and the line parallel to the trajectory and passing through the BH.

We can calculate

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \underbrace{\frac{\mathrm{d}r}{\mathrm{d}t}}_{-1}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\left(-\frac{d}{r^2}\right), \tag{235}$$

where we used a small angle approximation: $\varphi \approx d/r$, which can be differentiated with respect to r. So

$$b = \frac{l}{e} = \frac{r^2\,\mathrm{d}\varphi/\mathrm{d}\lambda}{\mathrm{d}t/\mathrm{d}\lambda} = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} = d\,, \tag{236}$$

which means that $b = d$: the ratio of l to e is the impact parameter.

So, the photon interacts with the BH and the angle of deflection of the path compared to a straight path is denote as $\delta\varphi$. The total deflection angle is $\Delta\varphi = \pi + \delta\varphi$.

The parameter l is

$$l = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\,, \tag{237}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{l}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\,. \tag{238}$$

This allows us to change variables in our 1D equation, and we get

$$\frac{1}{l^2}\frac{l^2}{r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) = \frac{1}{b^2}\,, \tag{239}$$

so if we change variables to $u = r^{-1}$, with

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}\,, \tag{240}$$

we get

$$u^4 u^{-4}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2(1 - 2GMu) = \frac{1}{b^2}\,, \tag{241}$$

and then, differentiating, we find

$$2\frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + 2u\frac{\mathrm{d}u}{\mathrm{d}\varphi} - 6GMu^2\frac{\mathrm{d}u}{\mathrm{d}\varphi} = 0\,, \tag{242}$$

and we can simplify as long as $\mathrm{d}u/\mathrm{d}\varphi \neq 0$, which only fails at one point. So in the end our equation is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = 3GMu^2\,. \tag{243}$$

We solve it perturbatively. If $GM = 0$, there is no BH and we have a straight line: the impact parameter is constant and equal to $b = r\sin(\varphi)$, therefore

$$u = \frac{1}{b}\sin(\varphi)\,, \tag{244}$$

is the most general solution to the harmonic oscillator which satisfies the boundary conditions. So, we hypothesize that our solution satisfies

$$u(\varphi) = \frac{1}{b}\Big(\sin(\varphi) + W(\varphi)\Big)\,, \tag{245}$$

where W is small.

Then, we insert this:

$$-\frac{1}{b}\sin(\varphi) + \frac{1}{b}\frac{\mathrm{d}^2W}{\mathrm{d}\varphi^2} + \frac{1}{b}W \approx 3GM\frac{\sin^2(\varphi)}{b}\,, \tag{246}$$

but since the zeroth order equation is satisfied we find:

$$\frac{\mathrm{d}^2W}{\mathrm{d}\varphi^2} + W \approx \frac{3GM}{b}\sin^2(\varphi)\,. \tag{247}$$

Our ansatz is then:

$$W = A + B\sin^2\varphi\,, \tag{248}$$

which looks like it might solve the equation. Its second derivative is

$$\frac{\mathrm{d}^2W}{\mathrm{d}\varphi^2} = 2B\Big(\cos^2\varphi - \sin^2\varphi\Big) = 2B - 4B\sin^2\varphi\,. \tag{249}$$

Inserting this we get:

$$2B - 4B\sin^2\varphi + A + B\sin^2\varphi = \frac{3GM}{b}\sin^2\varphi\,, \tag{250}$$

which implies $2B + A = 0$ and $-3B = 3GM/b$. Then,

$$W = \frac{2GM}{b} - \frac{GM}{b}\sin^2\varphi = \frac{2GM}{b}\left(1 - \frac{\sin^2\varphi}{2}\right) \tag{251}$$

solves the perturbed equation. We can see that our condition of $W \ll 1$ is actually the physically meaningful $GM \ll b$, or the impact parameter being much larger than the Schwarzschild radius.

Our complete solution is

$$u(\varphi) = \frac{1}{b}\left(\sin\varphi + \frac{2GM}{b}\left(1 - \frac{\sin^2\varphi}{2}\right)\right), \tag{252}$$

and we are interested in the asymptotic past and future, which correspond to $u = 0$. Now, $\varphi_{\text{in}} = 0$ and $\varphi_{\text{out}} = \pi$ will not be a solution anymore. However, the deflection is small so we write the solution as $\varphi_{\text{in}} = \epsilon_{\text{in}}$ and $\varphi_{\text{out}} = \pi + \epsilon_{\text{out}}$. We substitute these in to the equation $u = 0$:

$$0 = \sin(\epsilon_{\text{in}}) + \frac{2GM}{b}\,, \tag{253}$$

or $\epsilon_{\text{in}} \approx -2GM/b$, since the deflection is small.

For ϵ_{out} we will have

$$0 = \sin(\pi + \epsilon_{\text{out}}) + \frac{2GM}{b} \approx -\epsilon_{\text{out}} + \frac{2GM}{b}\,, \tag{254}$$

so $\delta\varphi = 2\varphi_{\text{in}} = 2\varphi_{\text{out}} = 4GM/b$.

This was one of the first tests of GR by Sir Eddington in 1919: during an eclipse he saw a shift in the apparent position of the stars. reinserting c we find that we must divide $4GM/b$ by c^2. Our b is approximately the radius of the Sun: the calculation is

```python
from scipy.constants import c, G, pi
sun_mass = 2e30
sun_radius = 696e6
rad2arcsec = 3600 * 180 / pi
4*G*sun_mass/c**2 /sun_radius * rad2arcsec
```

7 Horizons and coordinate systems

For the rest of today, we will talk about the Schwarzschild horizon: recall the line element

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{255}$$

It is useful to plot light cones in order to understand the structure of the spacetime. We restrict ourselves to radial motion of light. So, we have

$$0 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2\,, \tag{256}$$

or

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}\,. \tag{257}$$

The light cones become slimmer and slimmer as we approach the horizon, they are straight lines at $r = 2GM$: the photon appears to cover less and less $\mathrm{d}r$ for a fixed $\mathrm{d}t$ as it approaches the horizon. Massive particles are even slower. Let us integrate this relation:

$$\int_0^t \mathrm{d}t = -\int_{r_0}^{r(t)} \frac{\mathrm{d}r}{1 - \frac{2GM}{r}}\,, \tag{258}$$

which comes out, by a separation of fractions, to be

$$t = r_0 - r(t) + 2GM\log\left(r_0 - 2GM\right) - 2GM\log\left(r(t) - 2GM\right)\,, \tag{259}$$

which diverges as $r(t)$ approaches $2GM$.

What do we really mean by t? An observer which is far away and at rest has $g_{\mu\nu} = \eta_{\mu\nu}$, and $t = \tau$: t is the proper time, as measured on the clock of an observer who is far away. They will measure the photon as going slower and slower, and becoming redder and redder.

Let us neglect Doppler redshift, which is due to motion. The formula for gravitational redshift is

$$f_{\text{obs}} = f_{\text{emit}} \sqrt{\frac{-g_{00}(\text{emit})}{-g_{00}(\text{obs})}} \,. \tag{260}$$

In our case we have

$$f_{\text{obs}} = f_{\text{emit}} \sqrt{1 - \frac{2GM}{r}} \,, \tag{261}$$

which approaches zero as $r \to 2GM$. We see the infalling observer becoming redder and ultimately freezing.

We should use different coordinates to describe the infalling observer who passes through the event horizon.

First of all, we discuss Minkowski spacetime as seen by an accelerating observer: Rindler spacetime and the Riddler horizon.

Recall the exercise in sheet 2, about an accelerating observer: we now consider an observer moving with the position law

$$x(t) = \frac{1}{\kappa} \sqrt{1 + \kappa^2 t^2} \,, \tag{262}$$

(as opposed to the homework, we remove the constant added to this position). Like in the homework we compute the proper time for the observer:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 \left(1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 \right) , \tag{263}$$

so

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\sqrt{1 + \kappa^2 t^2}} \,, \tag{264}$$

which means

$$t = \frac{1}{\kappa} \sinh(\kappa\tau) \qquad \tau = \frac{1}{\kappa} \operatorname{arcsinh}(\kappa\tau) \,, \tag{265}$$

therefore

$$x = \frac{1}{\kappa} \cosh(\kappa\tau) \,. \tag{266}$$

We want a coordinate system in which

1. the observer is at constant spatial position;

2. where, up to a constant, the time is equal to the proper time.

Our change of variable is

$$\begin{cases} t = \rho \sinh \eta \\ x = \rho \cosh \eta \end{cases} . \tag{267a}$$

the observer is at fixed spatial coordinate $\rho_* = 1/\kappa$, and the proper time measured is $\tau = \eta/\kappa = \eta \rho_*$.

Let us consider a family of observers at different spatial locations in the new frame: each has a constant acceleration, this means varying κ or ρ.

If instead we vary η we have:

$$\frac{t}{x} = \tanh \eta \implies \eta = \tanh^{-1}\left(\frac{t}{x}\right), \tag{268}$$

and we can see that since $\tanh 0 = 0$ we have that the $t = 0$ axis has $\eta = 0$, while the lightspeed observers are at $\eta = \pm\infty$.

These coordinates cover one quadrant of Minkowski spacetime. The line element in these new coordinates is

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 && (269a) \\ &= -\left(d\rho^2 \sinh \eta + \rho \cosh \eta \, d\eta\right)^2 + \left(d\rho^2 \cosh \eta + \rho \sinh \eta \, d\eta\right)^2 && (269b) \\ &= -\rho^2 \, d\eta^2 + d\rho^2 \, . && (269c) \end{aligned}$$

This is *Rindler geometry*.

If we have an observer staying at $x_0 > 0$, and there is a Rindler observer, then after a time x_0 the observer exits the quarter of the plane covered by the Rindler coordinates: if event A is at $(0, x_0)$, and B is at (x_0, x_0) then the η of event A is zero, while the η of event B is infinite.

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Now we talk about the *Shapiro time delay of light*. This will not be on the exam: only the idea of how to sketch the calculation might be asked.

We want to send signals from A to B, with a massive object very near the geodesic connecting A and B.

We have the conserved quantities

$$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda}, \tag{270}$$

and

$$l = r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} . \tag{271}$$

Since the photon path is a null geodesic, we also have the conserved quantity $u^2 = 0$. This gives us a relation for $\mathrm{d}r/\mathrm{d}\lambda$: so we find

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\mathrm{d}t/\mathrm{d}\lambda}{\mathrm{d}r/\mathrm{d}\lambda} . \tag{272}$$

Then, if d is the shortest distance attained by the photon, we have

$$t = 2\left(\int_{r_A}^{d} \frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r + \int_{d}^{r_B} \frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r \right) , \tag{273}$$

where we multiply by 2 since we want to compute the time to go from A to B and back again to A, and split the integrals since r first decreases and then increases. Then, we will have $t = 2|\vec{x}_A - \vec{x}_B| + O(GM)$, and we consider the first order correction. It comes out to be:

$$\Delta t = 4GM \log\left(\frac{4 r_A r_B}{d^2} \right) , \tag{274}$$

under an assumption of $d \ll r_{A,B}$.

Now, we come back to Schwarzschild.

We want to show that the Schwarzschild horizon looks like a Rindler horizon. We do the coordinate change: $r = 2GM + \widetilde{r}$ and ignore the angular part: we find

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{\widetilde{r} + 2GM}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{\widetilde{r} + 2GM}\right)^{-1} \mathrm{d}\widetilde{r} , \tag{275}$$

but in the regime of $\widetilde{r} \ll 2GM$ (which is true near the horizon), we find that

$$g_{00} = -\left(1 - \frac{2GM}{\widetilde{r} + 2GM}\right) \approx -\left(1 - \left(1 - \frac{\widetilde{r}}{2GM}\right)\right) , \tag{276}$$

so we find

$$\mathrm{d}s^2 = -\frac{\widetilde{r}}{2GM}\,\mathrm{d}t^2 + \frac{2GM}{\widetilde{r}}\,\mathrm{d}\widetilde{r}^2 . \tag{277}$$

Now we do $(t, \widetilde{r}) \to (t, \rho)$ with $\rho = \sqrt{8GM\widetilde{r}}$. The change of coordinates is

$$\mathrm{d}\rho = \sqrt{\frac{8GM}{\widetilde{r}}}\,\mathrm{d}\widetilde{r} , \tag{278}$$

which means that

$$
\mathrm{d}s^2 = -\frac{1}{2GM}\frac{\rho^2}{8GM}\,\mathrm{d}t^2 + \mathrm{d}\rho^2 \ , \tag{279}
$$

and now we just need to rescale $t = \eta \times 4GM$: then

$$
\mathrm{d}s^2 = -\rho^2\,\mathrm{d}\eta^2 + \mathrm{d}\rho^2 \ . \tag{280}
$$

The old coordinates (t, r, θ, φ) are adapted to a constantly accelerating observer in the Schwarzschild geometry; just like the η, ρ coordinates were adapted to a constantly accelerating observer in the Rindler geometry. We see this since the Rindler-like coordinates are constant if and only if the Schwarzschild coordinates are constant.

However, the horizon is a *horizon* and not a *singularity* since there are coordinates which make it disappear. These are very much non-obvious, however somebody has done it and here is the result: Kruskal-Szekeres coordinates: $(t, r) \rightarrow (U, V)$. They have a different form inside and outside the horizon, however the map *is* continuous across it.

For $r > 2GM$:

$$
\begin{cases}
U &= \cosh\left(\frac{t}{4GM}\right)\left[\frac{r}{2GM} - 1\right]^{1/2}\exp\left(\frac{r}{4GM}\right) \\
V &= \sinh\left(\frac{t}{4GM}\right)\left[\frac{r}{2GM} - 1\right]^{\frac{1}{2}}\exp\left(\frac{r}{4GM}\right)
\end{cases} , \tag{281a}
$$

while for $r < 2GM$:

$$
\begin{cases}
U &= \sinh\left(\frac{t}{4GM}\right)\left[1 - \frac{r}{2GM}\right]^{\frac{1}{2}}\exp\left(\frac{r}{4GM}\right) \\
V &= \cosh\left(\frac{t}{4GM}\right)\left[1 - \frac{r}{2GM}\right]^{\frac{1}{2}}\exp\left(\frac{r}{4GM}\right)
\end{cases} . \tag{282a}
$$

There is no continuity in t, but that is to be expected. We show continuity (and, more importantly, differentiability) in r:

$$
U^2 - V^2 = \left[\frac{r}{2GM} - 1\right]\exp\left(\frac{r}{2GM}\right) \tag{283}
$$

inside the horizon, but outside of it we have the opposite expression in square brackets: however the hyperbolic sine and cosine are swapped, so we have the same expression.

[plot of $U^2 - V^2$: it is 0 at $R = 1$]

From this expression, we can invert $r(U, V)$.

The line element is given by

$$\mathrm{d}s^2 = \frac{32G^3M^3}{r} \exp\left(-\frac{r}{2GM}\right)\left(-\mathrm{d}V^2 + \mathrm{d}U^2\right) + r^2\,\mathrm{d}\Omega^2 \,, \tag{284}$$

where by r we mean $r(U, V) = r(U^2 - V^2)$, just a shortcut for writing.

Everything is regular except for $r = 0$.

Now, we discuss *Kruskal diagrams*: the spacetime description in the U, V coordinates.

We plot null geodesics in the 2D plane U, V. Since $\mathrm{d}s^2 \propto -\mathrm{d}U^2 + \mathrm{d}V^2$ and everything multiplying it is positive, then light moves with $\mathrm{d}V/\mathrm{d}U = \pm 1$.

Outside the horizon, we have the relations: $U^2 - V^2 > 0$, and $U > 0$.

This defines something that looks like a light cone, going rightward from the origin.

Inside the horizon, we have $V > 0$ and $V^2 - U^2 > 0$.

The singularity is $r = 0$ which implies $U^2 - V^2 = -1$, or $V = \pm\sqrt{1 + U^2}$. "It's a parabola with a square root"

The regions above and below these hyperbolas have $r < 0$, so they do not exist.

One might think that the horizon is $U = V = 0$ by substituting in, but actually it is the whole of $U = V$ for infinite time t, however this is not a physical problem, it is a problem with the coordinate t being a bad one.

Observers moving with constant acceleration (stationary r in Schwarzschild, only moving throught t) are hyperbolas, like in Rindler. Lines of constant time instead have constant

$$\frac{V}{U} = \tanh\left(\frac{t}{4GM}\right) . \tag{285}$$

Inside the horizon, instead,

$$\frac{U}{V} = \tanh\left(\frac{t}{4GM}\right) \; : \tag{286}$$

now all the constant-t lines get to the singularity.

What about the regions 3 and 4? Light can come from them, through $t = -\infty$.

The lower bit of the singularity is called a *white hole*.

Regions I and IV are mathematically "connected", but physically disconnected: no light can travel between them.

Now: consider a spacetime at fixed time coordinate V, in "God mode". First, consider $V \equiv V_0 < 0$: we start in region IV, cross the horizon, reach the white hole, than exit it and move into region I: these two regions (before & after the WH) are completely disconnected.

If we restore the coordinate θ and draw the space in r, θ coordinates as U changes, then we get something which looks like a wormhole.

Every point of this is $(V_0, U_0, \theta_0, \varphi)$ with varying φ : a circle.

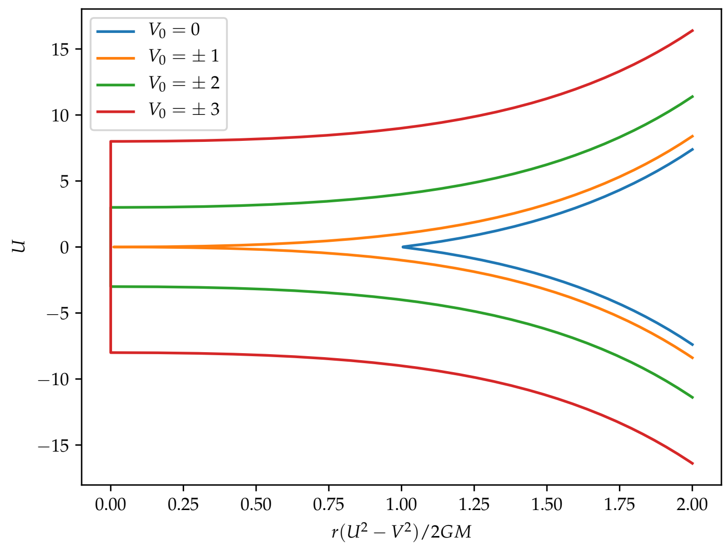

Figure 2: Kruskal U in terms of Schwarzschild r at constant V.

Fri Nov 29 2019

Yesterday we talked about Kruskal diagrams and coordinates.

There is a 1-to-1 map between $U^2 - V^2$ and $R = r/2GM$:

$$U^2 - V^2 = (R-1)\exp(R) , \tag{287}$$

and if we fix $V = V_0$, we can vary U, θ: each point in the manifold parametrized by U, θ is a circle of radius $r(U^2 - V^2)\sin\theta$, parametrized by the coordinate we left out, φ.

What if we vary V_0? there opens up a "breach", but it is only ever spacelike: no timelike curves can cross it.

White holes would have existed "before", but they appear as solutions to a stationary problem: real black holes form from the collapse of a star, there is a time after which the BH exists and before which it does not.

There is some more material at the end of the lecture notes on geodesics.

An observer in geodesic motion feels no acceleration: $a^\mu = 0$.

What is the acceleration felt by an observer, a^i, moving with 4-acceleration a^μ in general (non-geodesic) motion?

A noncontroversial fact is $-1 = \text{const}$: this directly implies $u^\mu a_\mu = 0$, since the metric is covariantly constant and $u^\mu u_\mu = 0$.

The acceleration felt by A is his acceleration measured in a LIF defined as having the same 4-velocity as A. There,

$$a^{\mu} = u^{\nu}\nabla_{\nu}u^{\mu} = \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} + \cancel{\Gamma^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta}}, \tag{288}$$

but $t = \tau$, and the 4-velocity in the LIF is $u^{\mu} = (1, 0, 0, 0)$: this implies that

$$a^{\mu} = \frac{\mathrm{d}u^{\mu}}{\mathrm{d}t} = \begin{bmatrix} 0 & a^{i} \end{bmatrix}^{\top}, \tag{289}$$

and the invariant quantity $a^{\mu}a_{\mu} = |\vec{a}|^2$ is the modulus of the 3-acceleration felt. Then, we are able to compute the norm of a^{i} in any frame using $\sqrt{a^{\mu}a_{\mu}}$.

Let us see this in some examples.

The uniformly accelerating observer in Minkowski spacetime has a law of motion which looks like

$$x(t) = \frac{\sqrt{1 + \kappa^2 t^2}}{\kappa}, \tag{290}$$

and we showed that $t = \sinh(\kappa\tau)/\kappa$ and $x = \cosh(\kappa\tau)/\kappa$.

The three-velocity is

$$u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \begin{bmatrix} \cosh(\kappa t) \\ \sinh(\kappa t) \end{bmatrix}, \tag{291a}$$

with the metric $\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, so we find $u^{\mu}u_{\mu} = -1$; also

$$a^{\mu} = \begin{bmatrix} \kappa \sinh(\kappa\tau) \\ \kappa \cosh(\kappa\tau) \end{bmatrix}, \tag{292a}$$

which means $a \cdot u = 0$. Also, we compute $a \cdot a = \kappa^2$.

We also consider an observer at rest in the Schwarzschild geometry: $x^{\mu} = (t, r_0, \theta_0, \varphi_0)$. The 4-velocity is $u^{\mu} = \left(1/\sqrt{-g_{00}}, 0, 0, 0\right)$.

The 4-acceleration is given by

$$a^{\mu} = u^{\nu}\left(\partial_{\nu}u^{\mu} + \Gamma^{\mu}_{\nu\beta}u^{\nu}u^{\beta}\right) = \Gamma^{\mu}_{tt}\left(u^{0}\right)^{2}, \tag{293}$$

and also $\Gamma^{\mu}_{tt} = 0$ for any $\mu \neq r$, while

$$\Gamma^{r}_{tt} = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)(-1)\frac{\partial}{\partial r}\left(-1 + \frac{2GM}{r}\right) = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)\frac{2GM}{r^2}. \tag{294}$$

So,

$$a^r = \left(1 - \frac{2GM}{r}\right)\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} = \frac{GM}{r^2}\,. \tag{295}$$

We have recovered the 4-acceleration felt by an observer: its modulus is

$$|\vec{a}| = \sqrt{g_{rr}a^r a^r} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1/2} \overset{r \gg 2GM}{\rightarrow} \frac{GM}{r^2}\,, \tag{296}$$

while if we are at $r = 2GM + \epsilon$, the acceleration looks like

$$|\vec{a}| \sim \frac{1}{\sqrt{\epsilon}}\,. \tag{297}$$

8 Rotation and the Kerr solution

The object $u^\nu \nabla_\nu u^\mu = a^\mu$ describes only the variation of the velocity *along* the four-momentum.

Let us consider a gyroscope: an object which has net angular momentum.

In the rest frame of the object, we introduce the spin four-vector $s^\mu = (0, \vec{s})$. In the rest frame $s \cdot u = 0$, so this holds in any frame.

A free object in Minkowski spacetime in its own rest frame has a constant $\vec{s}$: so $\frac{\mathrm{d}s^\mu}{\mathrm{d}t} = 0$. So in a LIF, for a moving object,

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\tau} = u^\nu \partial_\nu s^\mu = 0\,, \tag{298}$$

so in a general frame

$$u^\nu \nabla_\nu s^\mu = 0\,. \tag{299}$$

We can so the following computation: $u^\nu \nabla_\nu (s \cdot s) = 2 s_\mu u^\nu \nabla_\nu s^\mu = 0$.

Also, the product of two different spins is conserved by the same reasoning.

Now we consider three specific examples: first, a gyroscope moving around a Schwarzschild mass. This effect is called *geodetic precession*. The other effect we will treat is a gyroscope in a slowly rotating geometry: we will use the gyroscope as a probe for the rotating geometry. This effect is called *frame dragging* or *Lense Thirring precession*. Then, we will discuss a generic rotating black hole: the *Kerr* metric.

8.1 Geodesic precession

We consider a slice of Schwarzschild geometry: we will have r, φ coordinates for space and t coordinate for time, while $\theta = \pi/2$.

We will consider an observer moving along a geodesic orbit.

The 4-velocity is given by

$$u^{\mu} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\left(1, 0, 0, \frac{\mathrm{d}\varphi}{\mathrm{d}t}\right), \tag{300}$$

and

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega = \sqrt{\frac{GM}{r^3}}, \tag{301}$$

so we can write

$$-1 = u^{\mu}u_{\mu} = -\left(u^t\right)^2\left(g_{00} - r^2\Omega^2\right) = -\left(u^t\right)^2\left(1 - \frac{2GM}{r} - \frac{GM}{r}\right), \tag{302}$$

so we have

$$u^t = \frac{1}{\sqrt{1 - \frac{3GM}{r}}}. \tag{303}$$

Initially we have a vector $s^{\mu} = \left(s^t, s^r, s^{\theta}, s^{\varphi}\right)$. s^{θ} starts out zero in our coordinates, and it remains so by symmetry.

We know that $0 = g_{\mu\nu}s^{\mu}u^{\nu}$, but the metric is diagonal so this is

$$0 = g_{00}s^t u^t + g_{33}s^{\varphi}u^{\varphi}, \tag{304}$$

which is

$$0 = u^t\left(-\left(1 - \frac{2GM}{r}\right)s^t + r^2 s^{\varphi}\Omega\right), \tag{305}$$

so

$$s^t = \left(1 - \frac{2GM}{r}\right)^{-1} r^2\Omega s^{\varphi}. \tag{306}$$

Consistently with our assumptions, as $r \to \infty$ we have $s^t \to 0$.

The evolution of the spin is described by

$$\frac{\mathrm{d}s^{\alpha}}{\mathrm{d}\tau} + \Gamma^{\alpha}_{\beta\gamma}u^{\beta}s^{\gamma} = 0. \tag{307}$$

One can start by $\alpha = r$: then we must look at the symbols $\Gamma^r_{\beta\gamma}$: potentially, by the nonzero components of u and s, we should compute the symbols with $\beta = 0, 3$ and $\gamma = 0, 1, 3$.

Since the metric is diagonal, the Christoffel symbols with three different indices are automatically zero. Also, if two components are the same and one is different the different one has to be the one with respect to which one computes the derivatives.

So our equation is

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} + \Gamma^1_{00} u^0 s^0 + \Gamma^1_{33} u^3 s^3 = 0\,, \tag{308}$$

and the relavant Christoffel symbols are:

$$\Gamma^1_{00} = \left(1 - \frac{2GM}{r}\right)\frac{GM}{r^2} \tag{309}$$

and

$$\Gamma^1_{33} = -\left(1 - \frac{2GM}{r}\right) r\,. \tag{310}$$

We find that the equation is

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} + \Gamma^1_{00}\frac{\mathrm{d}t}{\mathrm{d}\tau} s^t + \Gamma^1_{33}\frac{\mathrm{d}t}{\mathrm{d}\tau}\Omega s^\varphi = 0\,, \tag{311}$$

which becomes, changing variable in the derivative from τ to t everywhere:

$$\frac{\mathrm{d}s^1}{\mathrm{d}t} + \Gamma^1_{00} s^t + \Gamma^1_{33}\Omega s^\varphi = 0\,, \tag{312}$$

which means

$$\frac{\mathrm{d}s^r}{\mathrm{d}t} + (3GM - r)\Omega s^\varphi = 0\,. \tag{313}$$

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We have the spin vector s^μ: it is orthogonal to the 4-velocity and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} s^\mu = 0\,. \tag{314}$$

By symmetry we also have $s^\theta = 0$.

By orthogonality to the 4-velocity in the Schwarzschild frame we have

$$s^t = \frac{R^2\Omega}{1 - \frac{2GM}{r}} s^\varphi\,, \tag{315}$$

and for geodesic circular orbit we have $\Omega^2 R^3 = GM$.

Let us also evolve s^φ:

$$\frac{\mathrm{d}s^\varphi}{\mathrm{d}\tau} + \Gamma^\varphi_{\beta\gamma} u^\beta s^\gamma = 0\,, \tag{316}$$

and the nonzero Christoffel symbols which can appear are the $\Gamma^3_{\beta\gamma}$ with $\beta = 0,3$ and $\gamma = 0,1,3$. The only one which is nonzero is Γ^3_{13}:

$$\Gamma^3_{13} = \frac{1}{2} g^{33} \left(g_{33,1} \right) = \frac{r^{-2}}{2} (2r) = \frac{1}{r}\,. \tag{317}$$

Then the evolution equation is

$$\frac{\mathrm{d}s^\varphi}{\mathrm{d}\tau} + \frac{1}{r} \underbrace{u^t \Omega}_{u^\varphi} s^r = 0\,, \tag{318}$$

and as before we can use the fact that $u^t = \mathrm{d}t/\mathrm{d}\tau$ in order to write the equation as

$$\frac{\mathrm{d}s^\varphi}{\mathrm{d}t} + \frac{\Omega}{r} s^r = 0\,, \tag{319}$$

which is coupled with the equation from before:

$$\frac{\mathrm{d}s^r}{\mathrm{d}t} + (3GM - r)\Omega s^\varphi = 0\,, \tag{320}$$

which seems to look like a harmonic oscillator: we just need to differentiate one of them, to get

$$\frac{\mathrm{d}^2 s^r}{\mathrm{d}t^2} + (3GM - r)\Omega \frac{\mathrm{d}s^\varphi}{\mathrm{d}t} = \frac{\mathrm{d}^2 s^r}{\mathrm{d}t^2} - \frac{3GM - r}{r} \Omega^2 s^r = 0\,, \tag{321}$$

so we found a harmonic oscillator with angular velocity

$$\overline{\Omega} = \Omega \sqrt{1 - \frac{3GM}{r}}\,, \tag{322}$$

and since s^φ satisfies the same equation up to a constant it is also a harmonic oscillator with thee same frequency.

We can relate them by the equation

$$-A\overline{\Omega} \sin(\Omega t) = R\Omega \left(1 - \frac{3GM}{r} \right) s^\varphi\,, \tag{323}$$

where A is the constant multiplying the solution for s^r, so in the end we have

$$s^\varphi = -\frac{A}{r} \frac{\Omega}{\overline{\overline{\Omega}}} \sin\left(\overline{\Omega} t \right). \tag{324}$$

Also, we can use

$$s^t = r^2\Omega\left(1 - \frac{2GM}{r}\right)^{-1} s^{\varphi}\,. \tag{325}$$

By the normalization $s^{\mu}s^{\nu}g_{\mu\nu} = 1$ we have $(s^1)^2 g_{11} = A^2(1 - 3GM/r)^{-1} = s_*^2 =$ const at $t = 0$: so

$$A = s_*\sqrt{1 - \frac{3GM}{r}}\,. \tag{326}$$

What does an observer see? What does somebody at infinity see?

We call $\Delta\varphi$ the angle between the spin vector and the radial direction: then

$$\cos\Delta\varphi = \frac{\text{radial component of the spin vector now}}{\text{radial component of the spin vector at } t = 0}\,, \tag{327}$$

which means

$$\cos(\Delta\varphi) = \frac{e_r \cdot s(t)}{e_r \cdot s(0)}\,, \tag{328}$$

where s is the spin vector and e_r is the radial unit vector: in Schwarzschild coordinates $e_r = (0, 1/\sqrt{g_{11}}, 0, 0)$.

This just simplifies to

$$\cos(\Delta\varphi) = \frac{s^r}{A} = \cos\left(\overline{\Omega}t\right)\,, \tag{329}$$

and we are allowed to do this calculation in the Schwarzschild frame since the spatial velocity is always orthogonal to the radial unit vector. If we were to use a direction different from the radial one we'd need to boost along it.

What does this solution mean? In the end our solution is

$$s^t = s_*\sqrt{1 - \frac{2GM}{r}}\cos\left(\overline{\Omega}t\right) \tag{330a}$$

$$s^{\varphi} = -s_*\sqrt{1 - \frac{2GM}{r}}\,\frac{\Omega}{\overline{\Omega}r}\sin\left(\overline{\Omega}t\right) \tag{330b}$$

$$s^t = r^2\Omega\left(1 - \frac{2GM}{r}\right)^{-1} s^{\varphi}\,, \tag{330c}$$

and the solution to $\cos(\Delta\varphi) = \cos\left(\overline{\Omega}t\right)$ is $\Delta\varphi = \pm\overline{\Omega}t$, and we choose the solution for continuity with the case $M = 0$.

If we are rotating around a BH counterclockwise (with $M \to 0$) then the spin must rotate clockwise in order to remain aligned with itself: therefore the right solution to choose is $\Delta\varphi = -\overline{\Omega}t$.

This is the reason why there is a minus sign in the expression for s^φ.

The basis is rotating, and the spin must rotate the other way to compensate and remain stationary.

Someone at infinity sees the spin to be rotating with angular velocity $\Omega - \overline{\Omega}$, where Ω is the $\mathrm{d}\varphi / \mathrm{d}t$ of the orbit and $\overline{\Omega}$ is the angular velocity of the spin in the coordinate system φ, r.

Over a turn, which takes a time $t = 2\pi/\Omega$, the total $\Delta\varphi$ is given by

$$\frac{2\pi}{\Omega}\left(\Omega - \overline{\Omega}\right) = 2\pi\left(1 - \sqrt{1 - \frac{3GM}{r}}\right) \approx \frac{3GM\pi}{r} \tag{331}$$

in the same direction as the orbit.

Let us look at the gyroscope in a slowly rotating geometry: we consider the metric

$$\mathrm{d}s^2 = \mathrm{d}s^2{}_{\mathrm{schw}} - \frac{4GJ}{r}\sin^2\theta \,\mathrm{d}t\,\mathrm{d}\varphi + O(J^2)\,, \tag{332}$$

which (see homework sheet 8) is a vacuum solution to the Einstein equations. We have a 4 since we have to account both for g_{03} and g_{30}.

Reinserting c, we get:

$$\mathrm{d}s^2 = \mathrm{d}s^2{}_{\mathrm{schw}} - \frac{4GJ}{c^3 r^2}\sin^2\theta \big(r\,\mathrm{d}\varphi\big)\left(c\,\mathrm{d}t\right)\,, \tag{333}$$

then the quantity $4GJ/c^2r^2$ must be adimensional: then the dimensions of J are those of c^3r^2/G, which are

$$\mathrm{m}^3\mathrm{s}^{-3} \times \mathrm{m}^2 \times \frac{1}{\mathrm{kg}\mathrm{m}^3\mathrm{s}^{-1}} = \mathrm{kg}\mathrm{m}^2\mathrm{s}^{-2}\,, \tag{334}$$

the dimensions of an angular momentum.

Essentially we are doing a boost along the φ direction.

Let us consider geodetic motion of the gyroscope along the rotation axis: the gyroscope is falling into the rotating BH along that axis.

The spin starts along the x axis, and we expect a change of the spin of the order

$$\frac{GJ}{c^3r^2}\,. \tag{335}$$

We might also have terms of order

$$O\left(\frac{GJ}{c^3r^2} \times \frac{GM}{rc^2}\right)\,, \tag{336}$$

but there cannot be terms of just order $O(GM/rc^2)$ since if there is no spin, even with positive BH mass the spin does not change.

This is because: the 4-velocity of the infalling observer looks like $u^\mu = (u^t, u^r, 0, 0,)$. The spin in the LIF of the infalling observer only ever has angular components, $s^\nu = (0, 0, s^\theta, 0)$ (let us neglect the singularity of the coordinates at the z axis, it is not relevant).

When we perform a boost to go to the Schwarzschild frame from the LIF, the transformation will mix the temporal and radial components of vectors, but it will leave the angular ones unchanged. Therefore, the spin vector will look like $(0, 0, s^\theta, 0)$ in the Schwarzschild frame as well.

The BH-mass contribution is small: then the mixed term is "second order", we can discard it and consider the $M = 0$ case.

Therefore we can simply use the metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - \frac{4GJ}{r}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi\ , \tag{337}$$

and only keep the $O(J)$ terms.

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From last time we have the metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - \frac{4GJ}{r}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi\ , \tag{338}$$

and since in polar coordinates the z axis is singular we use cartesian ones:

$$x = r\sin\theta\cos\varphi \tag{339a}$$
$$y = r\sin\theta\sin\varphi\,, \tag{339b}$$

which means $\varphi = \arctan(y/x)$. Then

$$\mathrm{d}\varphi = \frac{1}{1 + y^2/x^2}\,\mathrm{d}(y/x) = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{r^2\sin^2\theta}\,\mathrm{d}\varphi\ , \tag{340}$$

so we get

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - \frac{4GJ}{r^2}\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{r}\,\mathrm{d}t\ , \tag{341}$$

since the $\sin^2\theta$ simplified. So our metric is Minkowski plus a perturbation:

$$\delta g_{01} = \delta g_{10} = \frac{2GJy}{(x^2 + y^2 + z^2)^{3/2}}\,, \tag{342}$$

and

$$\delta g_{02} = \delta g_{20} = \frac{-2GJx}{(x^2+y^2+z^2)^{3/2}}\,, \tag{343}$$

which means that our Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\gamma}\Big(\delta g_{\gamma\beta,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma}\Big) + O(J^2)\,. \tag{344}$$

The 4-velocity is $u^{\alpha} = (u^t, 0, 0, u^z)$, and the spin vector initially is $s^{\alpha}_{\text{in}} = (0, s^x_{\text{in}}, 0, 0)$. We want to show that s^z is zero at all times.

The equation is

$$\frac{\mathrm{d}s^z}{\mathrm{d}\tau} + \Gamma^z_{\alpha\beta}u^{\alpha}s^{\beta} = 0\,, \tag{345}$$

but the velocity has only two components: then

$$\frac{\mathrm{d}s^z}{\mathrm{d}\tau} + \Gamma^3_{0\beta}u^0 s^{\beta} + \Gamma^3_{3\beta}u^3 s^{\beta} = 0\,, \tag{346}$$

but

$$\Gamma^3_{0\beta} = \frac{1}{2}\eta^{33}\Big(\delta g_{3\beta,0} + \delta g_{03,\beta} - \delta g_{0\beta,3}\Big)\,, \tag{347}$$

but the first two are zero since they contain an index 3. The last is also zero:

$$\delta g_{0\beta,3} = \frac{\partial}{\partial z}\left(\frac{2GJy}{r^3}\right) \quad \text{or} \quad \frac{\partial}{\partial z}\left(-\frac{2GJx}{r^3}\right)\,, \tag{348}$$

and in either case we need to calculate the derivative on the $x = y = 0$ axis, therefore since the terms are proportional to x or y they will vanish.

For the other term we also have

$$\Gamma^3_{3\beta} = \frac{1}{2}\eta^{33}\Big(\delta g_{3\beta,3} + \delta g_{33,\beta} - \delta g_{\beta 3,3}\Big) = 0\,, \tag{349}$$

so $s^z = 0$ for all times.

Now let us write the evolution for the s^t component.

$$\frac{\mathrm{d}s^t}{\mathrm{d}\tau} + \Gamma^0_{\alpha\beta}u^{\alpha}s^{\beta} = 0\,, \tag{350}$$

and then we must look at the Christoffels:

$$\Gamma^0_{0\beta} = \frac{1}{2}\eta^{00}\Big(\delta g_{00,\beta} + \delta g_{0\beta,0} - \delta g_{0\beta,0}\Big) = 0 \tag{351a}$$

$$\Gamma^0_{3\beta} = \frac{1}{2}\eta^{00}\Big(\delta g_{0\beta,3} + \delta g_{30,\beta} - \delta g_{3\beta,0}\Big) = 0\,, \tag{351b}$$

so the only components which evolve are

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} + \Gamma^1_{\alpha\beta} u^\alpha s^\beta = 0 \tag{352a}$$

$$\frac{\mathrm{d}s^2}{\mathrm{d}\tau} + \Gamma^2_{\alpha\beta} u^\alpha s^\beta = 0 \,, \tag{352b}$$

and now we notice this: since we need to form δg_{01} or δg_{02}, one of the lowe two indices in the Christoffels must be zero: but it cannot be β since $s^t = 0$: so we set $\alpha = 0$, and consider only $\Gamma^1_{0\beta} u^0 s^\beta$ and $\Gamma^2_{0\beta} u^0 s^\beta$. Also, β can only be 1 or 2. Then the equations become:

$$\frac{\mathrm{d}s^1}{\mathrm{d}\tau} + \Gamma^1_{01} u^0 s^1 + \Gamma^1_{02} u^0 s^2 = 0 \tag{353a}$$

$$\frac{\mathrm{d}s^2}{\mathrm{d}\tau} + \Gamma^2_{01} u^0 s^1 + \Gamma^2_{02} u^0 s^2 = 0 \,, \tag{353b}$$

and as yesterday we can use $u^0 = \mathrm{d}t / \mathrm{d}\tau$ to get derivatives with respect to t: we then get

$$\frac{\mathrm{d}s^1}{\mathrm{d}t} + \Gamma^1_{01} s^1 + \Gamma^1_{02} s^2 = 0 \tag{354a}$$

$$\frac{\mathrm{d}s^2}{\mathrm{d}t} + \Gamma^2_{01} s^1 + \Gamma^2_{02} s^2 = 0 \,. \tag{354b}$$

Now we can compute the Christoffels:

$$\Gamma^1_{01} = \frac{1}{2} \eta^{11} \left(\delta g_{11,0} + \delta g_{01,1} \right) = 0 \,, \tag{355}$$

and similarly for $\Gamma^2_{02} = 0$.

On the other hand:

$$\Gamma^1_{02} = \frac{1}{2} \eta^{11} \left(\delta g_{12,0} + \delta g_{01,2} - \delta g_{02,1} \right) \tag{356a}$$

$$\Gamma^2_{01} = \frac{1}{2} \eta^{22} \left(\delta g_{21,0} + \delta g_{02,1} - \delta g_{01,2} \right) = -\Gamma^1_{02} \,, \tag{356b}$$

and the time derivative vanishes: so in the end we get

$$\Gamma^1_{02} = \frac{1}{2} \left(\frac{\partial}{\partial y} \left(\frac{2GJy}{(x^2+y^2+z^2)^{3/2}} \right) - \frac{\partial}{\partial x} \left(-\frac{2GJx}{(x^2+y^2+z^2)^{3/2}} \right) \right) , \tag{357}$$

which we must calculate at $x = y = 0$: therefore the only relevant contribution to the derivative is the one in which the derivative acts on the numerator, since otherwise we get a term proportional to either x or y. So in the end we get:

$$\Gamma^1_{02} = -\Gamma^2_{01} = \frac{2GJ}{z^3} \,. \tag{358}$$

In the end then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} s^x \\ s^y \end{bmatrix} = \frac{2GJ}{z^3}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} s^x \\ s^y \end{bmatrix}, \tag{359a}$$

which, if z were constant, would be an angular precession with angular velocity $\Omega = 2GJ/z^3$, called the Lense-Thirring precession.

z is not actually a constant: however this describes the instantaneous angular velocity.

In general, one finds:

$$\vec{\Omega}_{LT} = \frac{GJ}{c^2 r^3}\Big(3(\vec{J}\cdot e^{\hat{r}})e^{\hat{r}} - \vec{J}\Big), \tag{360}$$

which notably has the samme factor as the electric field of a dipole. It reduces to our formula if $\vec{J}$ is parallel to $e^{\hat{r}}$.

Now we treat Kerr geometry: from 1963, the vacuum solution of a rotating spherical mass. It is:

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4GMar\sin^2\theta}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 \\ &+ \rho^2\,\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2GMra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2 \end{aligned}, \tag{361a}$$

where $a = J/M$, $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2GMr + a^2$.

Recall that in $c = 1$ velocities are dimensionless, so the units of angular momentum are mass times length, therefore the term $a = J/M$ is a length.

At $O(a^0)$ we recover the Schwarzschild metric.

At $O(a^1)$ we have the slowly rotating geometry.

At $r \gg GM$ we have

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \frac{4GMa}{r}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi \\ &+ \left(1 + \frac{2GM}{r}\right)\mathrm{d}r^2 + r^2\Big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi\Big) \end{aligned}, \tag{362a}$$

where we approximated $r^2/(r^2 - 2GMr) \sim 1 + \frac{2GM}{r}$.

Then, we can orbit far away with a gyroscope and measure M and J.

We still have the Killing vectors $\xi^\mu = (1, \vec{0})$ and $\xi^\mu = (0,0,0,1)$.

Notice that we have the symmetry $\theta \to \pi - \theta$. We have a real singularity at $\rho = 0$: this means $r = 0$ and $\cos\theta = 0$.

$r = 0$ is not a single point! This can be seen from the fact that to go from $(r,\theta) = (0,\theta_0)$ to $(0,\theta_1)$ we have a positive distance since the $\mathrm{d}\theta^2$ metric element is nonzero if $a^2\cos^2\theta \neq 0$.

We have a coordinate singularity, an horizon, when $\Delta = 0$: this means

$$r^2 - 2GMr + a^2 = 0 \implies r_\pm = GM \pm \sqrt{G^2M^2 - a^2}\,, \tag{363}$$

which means that we have an outer horizon and an inner horizon. As $a = 0$ we have $r_+ = 2GM$, which is fine. However if $a > GM$ we have a *naked singularity*: there is a postulate called the *cosmic censorship* postulate which states that naked singularities do not exist. This is not a theorem: it is just something which is suggested by black hole formation.

The case in which $a = GM$ is called the extreme Kerr solution.

We are going to focus our attention on r_+, the largest horizon.

First of all: *r_+ is a null surface*.

It is a boundary which separates the region where light can go to $r \to \infty$ from the one in which light eventually always goes to $r \to 0$.

Then light is “trapped inside this surface”.

The simplest case of a null surface is a light cone in Minkowski: in polar coordinates, the set of vectors in the form $x^\mu = (\alpha, \alpha, \beta, \gamma)$.

We can write this as

$$x^\mu = \alpha l^\mu + \beta m^\mu + \gamma n^\mu\,, \tag{364}$$

and then we can see that $l^\mu l_\mu = \alpha^2 - \alpha^2 = 0$. m^μ and n^μ are instead spacelike vectors.

The set (l, m, n) is a basis for the tangent space of the light cone. So, not all the vectors tangent to the surface are null.

For the Schwarzschild horizon instead we can select

$$l^\mu = (1, 0, 0, 0) \tag{365a}$$
$$m^\mu = (0, 0, 1, 0) \tag{365b}$$
$$n^\mu = (0, 0, 0, 1)\,, \tag{365c}$$

and as before m and n are spacelike, while

$$l \cdot l = -\left(1 - \frac{2GM}{r}\right)(l^0)^2 = 0\,, \tag{366}$$

since we are precisely at $r = 2GM$.

At the Kerr horizon instead we have:

$$l^\alpha = (1, 0, 0, \Omega_H)\,, \tag{367}$$

where $\Omega_H = a/(2GMr_+)$: light will rotate along the Kerr black hole, while in the Schwarzschild case the light is stationary with respect to the spatial coordinates.

Now let us take a picture of the horizon at fixed t: it does not matter which time by stationarity. It will be homework to show that in that case on the horizon the metric will be

$$ds^2 = \rho_+^2 \, d\theta^2 + \left(\frac{2GMr_+}{\rho_+} \right)^2 \sin\theta \, d\varphi \; . \tag{368}$$

This is not a spherical surface, since ρ_+ depends on θ.

It can be shown that it looks like an ellipsoid. The equator is larger than the corresponding equator for a sphere. To show this we will compute the length of the equator: $\theta = \pi/2$ and $\varphi \in [0, 2\pi]$, and also we will compute the length of a north-south-north path: $\varphi = \text{const}$ and $\theta \in [0, \pi]$ and then $\theta \in [\pi, 0]$.

For the equator we get

$$L_{\text{equator}} = \int_0^{2\pi} d\varphi \, \sqrt{g_{33}} \,, \tag{369}$$

calculated at $\theta = \pi/2$. This becomes

$$L_{\text{equator}} = \int_0^{2\pi} d\varphi \, 2GM = 4\pi GM \,, \tag{370}$$

since the $\sin\theta = 1$ while $\cos\theta = 0$. The harder one is the length of the circle: it comes out to be

$$L_{\text{NSN}} = 2 \int_0^{\pi} d\theta \, \sqrt{g_{\theta\theta}} = 2 \int_0^{\pi} d\theta \, \sqrt{r^2 + a^2 \cos^2\theta} \,, \tag{371}$$

which we will expand for small a since we do not like elliptic integrals. This becomes

$$L_{\text{NSN}} \approx 2 \int_0^{\pi} d\theta \left(2GM + \frac{a^2}{4GM} \left(-2 + \cos^2\theta \right) + O(a^2) \right) , \tag{372}$$

and since we are integrating over a period of $\cos^2\theta$ we can substitute its average of $1/2$: then the result is

$$L_{\text{NSN}} \approx 4\pi GM - \frac{3a^2\pi}{4GM} \,, \tag{373}$$

so the length is less than the equatorial one.

Complement: what does the region $r = 0$ look like?

Inserting $r = 0$ into the Kerr metric we find

$$ds^2 = - dt^2 + \cos^2\theta \, dr^2 + a^2 \left(\cos^2\theta \, d\theta^2 + \sin^2\theta \, d\varphi^2 \right) , \tag{374}$$

and we can do the change of variable $\psi = \sin\theta$: then the line element becomes

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \left(1 - \psi^2\right)\mathrm{d}r^2 + a^2\left(\mathrm{d}\psi^2 + \psi^2\,\mathrm{d}\varphi^2\right). \tag{375}$$

This is not a point anymore: it actually consists of two disjoint “emispheres”, and to see what the geometry of these looks like we can notice that at constant θ we have a circle of radius $\psi = \sin\theta$.

How does this surface actually look like? We can notice that the metric $\mathrm{d}s^2{}_{(2)} = a^2\left(\mathrm{d}\psi^2 + \psi^2\,\mathrm{d}\varphi^2\right)$ is, in polar coordinates, the metric for the plane, which is intrinsically flat: we do not need to actually compute the Riemann tensor to already know that all of its components are zero.

Then, we have a bounded surface, made up of circles, which is flat so it can be made by folding a piece of paper; also it is axisymmetric and symmetric with respect to $\theta \to -\theta$: the surface is made up of two cones joined at the equator.

In a paper[a] I found the result that the aperture angle of the cone is

$$2\arctan\frac{|a|}{|GM|}\,, \tag{376}$$

but I have not figured out how to prove it myself yet.

[a] https://arxiv.org/abs/1205.5848v6

Thu Dec 12 2019

We discuss orbits in Kerr geometry.

In general, orbits are nonplanar. Say we have an orbit which is not aligned with the rotation plane: frame dragging will change its spin and make it precess around, so it will span a bidimensional region.

An exceptional case is a planar orbit with $\theta \equiv \pi/2$. We will treat this case. Here, we have $\rho^2 = r^2 + a^2\cos^2\theta$ but $\cos\theta = 0$ and $\sin\theta = 1$: so $\rho^2 = r^2$. Then the line element becomes

$$\begin{aligned}\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \frac{4GMa}{r}\,\mathrm{d}t\,\mathrm{d}\varphi\, + \\ + \frac{r^2}{\Delta}\,\mathrm{d}r^2 + \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\mathrm{d}\varphi^2\,,\end{aligned} \tag{377a}$$

where $\Delta = r^2 - 2GMr + a^2$. We do not even write the $\mathrm{d}\theta^2$ term since θ is constant.

We only outline the steps: first of all we introduce a 4-velocity

$$
u^{\alpha} = \begin{bmatrix} u^t & u^r & 0 & u^{\varphi} \end{bmatrix}^{\top}. \tag{378}
$$

We have

$$
e = -\xi_t \cdot u = -g_{00} u^t - g_{03} u^{\varphi} , \tag{379}
$$

while

$$
l = \xi_{\varphi} \cdot u = g_{00} u^t + g_{30} u^{\varphi} . \tag{380}
$$

These are conserved in the motion, and they represent the energy and the angular momentum of the particle per unit mass as observed by a far away observer. We insert these into $u \cdot u = -1$: this gives us

$$
\frac{1}{2} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau} \right)^2 + V_{\mathrm{eff}}(r, e, l) = \frac{e^2 - 1}{2} , \tag{381}
$$

with

$$
V_{\mathrm{eff}}(r, e, l) = -\frac{GM}{r} + \frac{l^2 - a^2(e^2 - 1)}{2r^2} - \frac{GM(l - ae)^2}{r^3} , \tag{382}
$$

which, as we can see, reduces to Schwarzschild for $a = 0$.

Now we will consider circular orbits. These are characterized by $r = \mathrm{const}$: therefore $\mathrm{d}r \big/ \mathrm{d}\tau = 0$. So the equation reduces to

$$
-\frac{GM}{r} + \frac{l^2 - a^2(e^2 - 1)}{2r^2} - \frac{GM(l - ae)^2}{r^3} = \frac{e^2 - 1}{2} , \tag{383}
$$

which must certainly hold, but we also should require to be in an extremum of the potential: we impose

$$
\frac{\mathrm{d}V}{\mathrm{d}r} = 0 . \tag{384}
$$

This equation looks like:

$$
r^2 GM - r \Big(l^2 - a^2(e^2 - 1) \Big) + 3GM(l - ae)^2 = 0 , \tag{385}
$$

and we should look at the solution of this where $\mathrm{d}^2 V \big/ \mathrm{d}r^2 > 0$, so that our orbit is stable.

We are interested in the Kerr ISCO: the infimum of the set of rs defined by the conditions

$$\begin{cases} V_{\text{eff}}(r) & = \frac{e^2-1}{2} \\ \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}(r) & = 0 \\ \frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}r^2}(r) & \geq 0 \end{cases} , \tag{386a}$$

which is characterized by the second derivative actually being *equal* to 0. We solve these for the variables r, e, l. The algebra is extremely involved, and will not be an exam requirement. We plot the solutions in a plane R_{ISCO}/GM versus $a/GM \in [0,1]$. When $a \neq 0$ we actually have two separate solutions, for the different signs of l (the difference is really between the relative signs of l and a: we can alternatively write $a \in [-1,1]$ and $l \geq 0$).

As $a \to 1$ we get $r_{\text{ISCO}} \to GM$ if we are corotating, and $r_{\text{ISCO}} \to 9GM$ if we are counterrotating.

8.2 Ergosphere

As we get closer to the BH, we *must* spin in the same direction it is. This holds for *any* motion, not just geodesic motion. A stationary person has $u^\mu = (1, \vec{0})$; we can show that below a certain r this cannot have $u^\mu g_{\mu\nu} u^\nu = -1$. This is

$$u^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)(u^t)^2 = -1 \,, \tag{387}$$

which means that, since $\frac{\mathrm{d}t}{\mathrm{d}\tau} \geq 0$, we must have

$$1 - \frac{2GMr}{r^2 + a^2\cos^2\theta} \geq 0 \,, \tag{388}$$

which means

$$r^2 + a^2\cos^2\theta \geq 2GMr \,, \tag{389}$$

and unlike Schwarzschild this is not inside the horizon: the solutions are

$$r_{E\pm} = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta} \,, \tag{390}$$

and the sign is positive outside of the two solutions. Recall that the horizon is given by

$$r_{H\pm} = GM \pm \sqrt{(GM)^2 - a^2} \,, \tag{391}$$

so we can see that since $0 \leq \cos^2\theta \leq 1$ the horizon radii are *inner* with respect to the ergo radii: the inequality is

$$r_{E-} \leq r_{H-} \leq r_{H+} \leq r_{E+} \,. \tag{392}$$

so we have a region *outside the horizons*: $r_{H+} \leq r \leq r_{E+}$ in which one *cannot stay at rest*. The full inequalities defining the out-of-horizon ergoregion are:

$$GM + \sqrt{(GM)^2 - a^2} \leq r \leq GM + \sqrt{(GM)^2 - a^2\cos^2\theta} \,, \tag{393}$$

and to see what this looks like, we fix things: if $\theta = 0$ we have $r_{E+} = r_{H+}$, while on the equator $\theta = \pi/2$ we have $r_{E+} = 2GM$, while $r_{H+} = GM + \sqrt{(GM)^2 - a^2} < 2GM$. So, the maximum extension of the ergoregion is given by

$$\Delta r(\theta = \pi/2) = GM - \sqrt{(GM)^2 - a^2} = GM\left(1 - \sqrt{1 - \left(\frac{a}{GM}\right)^2}\right) \sim \frac{a^2}{2GM} \tag{394}$$

if $a \ll GM$, otherwise we must do the full calculation.

8.3 Penrose process

It is possible to extract energy and momentum from a black hole. We have a particle, called "in", which comes from infinity, reaches the ergosphere, goes inside of it, decays into a particle which we call "out" which goes to infinity plus a second particle which we call "BH" which goes inside the BH. All of these move with geodesic motion.

For simplicicty, we consider the process in the equatorial plane although this is not necessary.

In a LIF, energy and momentum are conserved on decay:

$$p^\mu_{\text{in}} = p^\mu_{\text{out}} + p^\mu_{\text{BH}} \,, \tag{395}$$

but since this is tensorial it holds in all frames.

A stationary observer at infinity observes $E_{\text{in}} = -p^0_{\text{in}}$ and $E_{\text{out}} = -p^0_{\text{out}}$, where the components of the momentum are written in the usual Schwarzschild coordinates.

Recall: $\xi^\alpha = (1, \vec{0})$ is a Killing vector of this geometry. Therefore, $E_{\text{in}} = -\xi \cdot p_{\text{in}}$ is conserved along the trajectory and the same holds for E_{out}.

Projecting the conservation of momentum along $-\xi$, we get

$$E_{\text{in}} = E_{\text{out}} - \xi \cdot p_{\text{BH}} \,, \tag{396}$$

which tells us how we can compare the infalling energy to the energy we get out. If the particle BH reached infinity, then, $-\xi \cdot p_{\mathrm{BH}}$ would be its energy as measured by the observer and it would need to be positive. However, it does not.

So, we can arrange our system so that $-\xi \cdot p_{\mathrm{BH}} < 0$: then we have $E_{\mathrm{out}} > E_{\mathrm{in}}$.

The ergoregion is precisely the one in which $g_{tt} > 0$ instead of $g_{tt} < 0$ as usual.

So if the decay happens inside the ergoregion, the projection of the conservation of momentum along $\xi = (1, \vec{0})$ is actually the conservation of a *spatial* component of the momentum, which can have any sign.

From the POV of an outside observer, this energy must come from the BH: so the BH must have lost energy.

We cannot actually model this directly: we consider the geometry as fixed, since it almost is. This is analogous to the angular momentum transferred in a gravitational slingshot.

We take an observer in the ergosphere at fixed r, θ with velocity $u^\alpha = u^t (1, 0, 0, \Omega)^\top$ with $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega > 0$.

For this observer the measured energy of the is

$$E_{\mathrm{obs}} = -u_{\mathrm{obs}} \cdot p_{\mathrm{BH}} > 0\,. \tag{397}$$

We have that

$$u^\alpha_{\mathrm{obs}} = u^t_{\mathrm{obs}} \xi_t + u^t_{\mathrm{obs}} \Omega_{\mathrm{obs}} \xi_\varphi\,, \tag{398}$$

so the measured energy is given by

$$E_{\mathrm{obs}} = -u^t_{\mathrm{obs}} \xi_t \cdot p_{\mathrm{BH}} - u^t_{\mathrm{obs}} \Omega_{\mathrm{obs}} \xi_\varphi \cdot p_{\mathrm{BH}} = +u^t_{\mathrm{obs}} (e_{\mathrm{BH}} - \Omega_{\mathrm{obs}} l_{\mathrm{BH}}) > 0\,, \tag{399}$$

but $e_{\mathrm{BH}} < 0$ so this means that we *must have* $l_{\mathrm{BH}} < 0$.

Fri Dec 13 2019

There are two topics left: cosmology and gravitational waves. Today and Thursday we do cosmology.

9 Cosmology

Now we introduce the Planck scale: the Compton wavelength is defined by

$$\lambda = \frac{\hbar c}{E}\,, \tag{400}$$

while the Schwarzschild radius is of the order of

$$r_s \sim \frac{GM}{c^2} = \frac{GE}{c^4} \,. \tag{401}$$

So, we have a quantum gravity regime when $\lambda < r_S$: when the particle is so localized that it will form a BH by itself. We get:

$$\frac{\hbar c}{E} \approx \frac{GE}{c^4} \implies E \approx \sqrt{\frac{\hbar c^5}{G}} \approx 1.96 \times 10^9 \,\mathrm{J} \approx 1.22 \times 10^{19} \,\mathrm{GeV} \,. \tag{402}$$

In $c = 1$ units, this is also the Planck mass. It is also useful to define the reduced Planck mass:

$$M_p = \frac{E_p}{\sqrt{8\pi}} = 2.43 \times 10^{18} \,\mathrm{GeV} \,. \tag{403}$$

Natural units are ones in which

1. $c = 1$: velocities are dimensionless: then the unit of length is equal to the unit of time;
2. $\hbar = 1$: then angular momenta are also dimensionless: then length and time have the dimensions of 1/mass or 1/energy.

In natural units the Einstein equations look like:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{404}$$

but $M_P = \frac{1}{\sqrt{8\pi G}}$: therefore $8\pi G = 1/M_P^2$, so

$$G_{\mu\nu} = \frac{1}{M_P^2} T_{\mu\nu} \,. \tag{405}$$

We are going to discuss the Friedmann-Lemaître-Robertson-Walker metric, which describes a homogeneous and isotropic universe.

Homogeneous means symmetry with respect to translations, isotropic means symmetry with respect to rotations.

Something which is *homogeneous but not isotropic* is, for example, the inside of a capacitor. Also, the surface of a cylinder can be an example.

We can have a space which is *isotropic but not homogeneous* only around one point, *global isotropy implies homogeneity*.

We will also require the condition that the universe be *spatially flat*. If a triangle has angles α, β, γ then $\mathrm{sign}(\alpha + \beta + \gamma - \pi) = k$ is a constant along the space and it measures the curvature of the space.

The smaller the curvature (and the larger the length scale of the curvature) the more difficult it is to measure what k is.

The line element in this kind of space is particularly simple, since we can :

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\vec{x}^2\ . \tag{406}$$

Since the metric is diagonal, the only nonvanishing Christoffel symbols are Γ^0_{ij} and Γ^i_{0j}, and both must be proportional to δ_{ij}. These symbols are

$$\Gamma^0_{ij} = \frac{1}{2} g^{00}\Big(-g_{ij,0}\Big) = \delta_{ij} a\dot{a}\,, \tag{407}$$

where $\dot{a}$ denotes the derivative of a with respect to coordinate time, while

$$\Gamma^i_{0j} = \frac{1}{2} g^{ik}\Big(g_{jk,0}\Big) = \delta_{ij} a\dot{a} \times \frac{1}{a^2} = \delta_{ij}\frac{\dot{a}}{a}\,. \tag{408}$$

The trace of the Christoffels are $\Gamma^0_{ii} = 3a\dot{a}$, and $\Gamma^i_{0i} = 3\dot{a}/a$. We want to calculate R_{00}, R_{0i} and R_{ij}. We have $R_{0i} = 0$ since nothing on the indices depends on space. We also have $R_{ij} \propto \delta_{ij}$, since all the Christoffels depend only on δ_{ij}.

We have

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\alpha_{\lambda\alpha} - \partial_\nu \Gamma^\alpha_{\mu\alpha} - \Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\nu\lambda}\,. \tag{409}$$

Specializing to the R_{00} case:

$$R_{00} = \partial_\alpha \Gamma^\alpha_{00} + \Gamma^\lambda_{00}\Gamma^\alpha_{\lambda\alpha} - \partial_0 \Gamma^\alpha_{0\alpha} - \Gamma^\lambda_{0\alpha}\Gamma^\alpha_{0\lambda}\,. \tag{410}$$

The Γ^α_{00} must vanish. So we get

$$\begin{aligned}
R_{00} &= -\partial_0 \Gamma^i_{0i} - \Gamma^i_{0j}\Gamma^j_{0i} &(411a)\\
&= -\partial_0\left(\frac{3\dot{a}}{a}\right) - \frac{\dot{a}}{a}\delta_{ij}\frac{\dot{a}}{a}\delta_{ij} &(411b)\\
&= -\frac{3\ddot{a}}{a} + \frac{3\dot{a}^2}{a^2} - 3\frac{\dot{a}^2}{a^2} = -3\frac{\ddot{a}}{a}\,. &(411c)
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
R_{ij} &= \partial_\alpha \Gamma^\alpha_{ij} + \Gamma^\lambda_{ij}\Gamma^\alpha_{\lambda\alpha} - \cancel{\partial_j \Gamma^\alpha_{i\alpha}} - \Gamma^\lambda_{i\alpha}\Gamma^\alpha_{j\lambda} &(412a)\\
&= \partial_0 \Gamma^0_{ij} + \Gamma^\lambda_{ij}\Gamma^\alpha_{\lambda\alpha} - \Gamma^\lambda_{i\alpha}\Gamma^\alpha_{j\lambda} &(412b)\\
&= \Big(a\ddot{a} - \dot{a}^2\Big)\delta_{ij} + 3a^2\delta_{ij} - \dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} &(412c)\\
&= a^2\delta_{ij}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)\,. &(412d)
\end{aligned}$$

So we have the whole of the Ricci tensor. The scalar curvature is given by

$$
\begin{aligned}
R &= g^{00}R_{00} + g^{ij}R_{ij} && \text{(413a)}\\
&= +3\frac{\ddot{a}}{a} + \delta_{ij}\frac{a^2}{a^2}\delta_{ij}\left(2\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\right) && \text{(413b)}\\
&= 6\frac{\dot{a}^2}{a^2} + \frac{6\ddot{a}}{a}\,, && \text{(413c)}
\end{aligned}
$$

and we have

$$
G_{00} = R_{00} - Rg_{00} = -3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} + 3\frac{\ddot{a}}{a} = \frac{3\dot{a}^2}{a^2}\,, \tag{414}
$$

while

$$
\begin{aligned}
G_{ij} &= R_{ij} - Rg_{ij} && \text{(415a)}\\
&= a^2\delta_{ij}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right) - \frac{3\dot{a}^2}{a^2}a^2 && \text{(415b)}\\
&= a^2\delta_{ij}\left(-\frac{\dot{a}^2}{a^2} - \frac{2\ddot{a}}{a}\right). && \text{(415c)}
\end{aligned}
$$

For the SEM tensor we choose a perfect fluid: $T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu}$, where $h^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ is the projector on the space orthogonal to the velocity.

We know that, in the expression of

$$
u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right)^\top, \tag{416}
$$

the compontent u^0 must be positive. We have

$$
0 = G_{0i} = \frac{T_{0i}}{M_P^2}\,, \tag{417}
$$

but this means that u^i must be zero. *The cosmic fluid is at rest*. This means that we are selecting a special frame: the rest frame of the cosmic fluid, the rest frame of the CMB.

When we look at the CMB we see a large dipolar contribution, due to the motion of the Earth with respect to the cosmic fluid. The theory is globally Lorenz-invariant, however its realization is not. This means that $u^0 = 1$, since $g_{00} = -1$.

This means that $h^{\mu\nu} = a^2\delta^{ij}$ (informal, I mean that the only nonzero components are the spatial ones.)

Then $T_{00} = \rho$, and $T_{ij} = a^2\delta_{ij}P$.

So the EFE are:

$$\frac{3\dot{a}^2}{a^2} = \frac{\rho}{M_P^2} \qquad \text{00 equation} \tag{418a}$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{P}{M_P^2} \qquad \text{ij equations}\,, \tag{418b}$$

where we factored out the $a^2\delta_{ij}$ in the ij equations. "Just for fun" we discuss the Bianchi identities: $\nabla_\mu G^{\mu\nu} = 0$. If $\nu = 0$ we have

$$\partial_\mu G^{\mu 0} + \Gamma^\mu_{\mu\lambda} G^{\lambda 0} + \Gamma^0_{\mu\lambda} G^{\mu\lambda} = 0\,, \tag{419}$$

but simplifying the indices we get

$$\partial_0 G^{00} + \Gamma^i_{i0} G^{00} + \Gamma^0_{ij} G^{ij} = 0\,, \tag{420}$$

and substituting in the expressions we have we get

$$\partial_0\left(\frac{3\dot{a}^2}{a^2}\right) + \frac{3\dot{a}}{a}\frac{3\dot{a}^2}{a^2} + a\dot{a}\delta_{ij}\frac{1}{a^2}\delta_{ij}\left(-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \tag{421a}$$

$$=6\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + 9\frac{\dot{a}^3}{a^3} - 6\frac{\dot{a}}{a}\frac{\ddot{a}}{a} - 3\frac{\ddot{a}^3}{a^3} = 0\,, \tag{421b}$$

which confirms what we already knew. Verifying $\nabla_\mu G^{\mu i} = 0$ is easier:

$$\partial_\mu G^{\mu i} + \Gamma^\mu_{\mu\lambda} G^{\lambda i} + \Gamma^i_{\mu\lambda} G^{\mu\lambda} = 0\,, \tag{422}$$

because all three of the terms vanish immediately.

Correspongingly, we have $\nabla_\mu T^{\mu\nu} = 0$. This is a local conservation law, not a global one generally since we do not have 4 Killing vectors. If $\nu = 0$ we have

$$\partial_\mu T^{\mu 0} + \Gamma^\mu_{\lambda\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} = \tag{423a}$$

$$=\partial_0 T^{00} + \Gamma^i_{i0} T^{i0} + \Gamma^0_{ij} T^{ij} \tag{423b}$$

$$=\partial_0\rho + 3\frac{\dot{a}}{a}\rho + \dot{a}a\delta_{ij}\frac{1}{a^2}\delta_{ij}P\,, \tag{423c}$$

which means

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,, \tag{424}$$

which we can add to the other equations.

It is easier to study the case $\nu = i$: we get

$$\partial_\mu T^{\mu i} + \Gamma^\mu_{\lambda\lambda} T^{\lambda i} + \Gamma^i_{\mu\lambda} T^{\mu\lambda} = 0\,, \tag{425}$$

since all three terms vanish immediately. Since the conservation equation $\nabla_\mu T^{\mu 0}$ comes from the Einstein equations, we should be able to derive it from the first two Friedmann equations: differentiating the 00 one we get

$$6\left(\frac{\dot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \dot{\rho} M_P^{-2}\,, \tag{426}$$

while for the second one, multiplying by $3\dot{a}/a$, we get

$$-6\frac{\dot{a}}{a}\frac{\ddot{a}}{a} - 3\frac{\dot{a}^3}{a^3} = \frac{3\dot{a}}{a} P M_P^{-2}\,. \tag{427}$$

Adding them together we find

$$-9\frac{\dot{a}^3}{a^3} = M_P^{-2}\left(\dot{\rho} + 3\frac{\dot{a}}{a}P\right)\,, \tag{428}$$

and from the first FE we have

$$9\frac{\dot{a}^3}{a^3} = 3\frac{\dot{a}}{a}\rho M_P^{-2}\,, \tag{429}$$

so we get

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0\,. \tag{430}$$

So, since the equations are not independent, we consider only two of the three. It is convenient to use the first and the third since they have no second derivatives.

What sources do we put for the equations? First of all, commonly we do $w = P/\rho$, and sometimes we do $w = -1$: this means $\rho = \text{const}$. This corresponds to *vacuum energy*, associated with the space itself: it does not scale inversely with the volume.

We can have vacuum energy: "vacuum" just means we are in a minimum of the potential. Mexican hats and stuff.

In principle, the EFE can be modified in a simple way:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{T_{\mu\nu}}{M_P^2}\,, \tag{431}$$

since the metric is covariantly constant. This can be interpreted as a *constant negative energy density*: It would look like

$$T_{\mu\nu} \to T_{\mu\nu} + \Lambda M_P^2 g_{\mu\nu}\,. \tag{432}$$

Then we have

$$G_{00} = \frac{1}{M_P^2} \left(\rho + \Lambda M_P^2 \right) , \tag{433}$$

and

$$G_{ij} = \frac{1}{M_P^2} \left(a^2 \delta_{ij} p - \Lambda M_P^2 a^2 \delta_{ij} \right) . \tag{434}$$

So, we have

$$\frac{P_\Lambda}{\rho_\Lambda} = w_\Lambda = -1 \,. \tag{435}$$

This ratio between pressure and density is characteristic of a cosmological constant.

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We found the equation

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \rho (1 + w) = 0 \,, \tag{436}$$

where $w = p/\rho$, and we found that for vacuum energy, with constant ρ, we get $w = -1$.

Now we will derive the fact that $w = 1/3$ for radiation. This can be derived from Maxwell's equations, but it also can come from a more illustrative argument: we consider photons in a box.

If they travel along the x axis, they have wavevectors $k^\mu = (\omega, \omega, 0, 0)$ and momenta $p^\mu = \omega\hbar(1, 1, 0, 0,)$.

In general they will travel in a different direction, of course, but we treat this case. Each photon hits a wall every $\Delta t = 2L$, and transfers momentum equal to $2\hbar\omega$. So, the average force exerted is $\Delta p / \Delta t = \hbar\omega / L$.

In the cavity there is an energy $\rho V = \rho AL$, where ρ is the photons' energy density, L is the length along x while A is the area normal to the x direction.

Each photon has energy $\hbar\omega$, so there are $\rho AL/\hbar\omega$. We assume for simplicity (but it gives the exact correct answer) that $1/3$ of the photons travel along each spatial direction.

So there are $\rho AL/3\hbar\omega$ photons travelling in the x direction. So the average force exerted on the wall is given by

$$F_x = \frac{\hbar\omega}{L} \times \frac{\rho AL}{3\hbar\omega} = \frac{\rho A}{3} \,, \tag{437}$$

so the average pressure is $\rho/3$, so $w = 1/3$.

Now let us consider nonrelativistic particles: if $v \ll 1$ then the 4-momentum looks like

$$p^\mu = (m, mv, 0, 0)\,, \tag{438}$$

so the force along the x axis is given by

$$F_{\text{one particle}} = \frac{\Delta p_x}{\Delta t} = \frac{2mv}{2L/v} = \frac{mv^2}{L}\,, \tag{439}$$

and as before the total energy is ρLA, so the number of particles travelling along the x direction is $\rho LA/3m$, since the momentum contribution to the particle's energy is negligible: therefore

$$F_{x,\text{tot}} = \frac{mv^2}{L} \times \frac{\rho LA}{3m} = \frac{\rho A v^2}{3} \approx 0\,, \tag{440}$$

which we can neglect since it is quadratic in v which is very small. So we say that for relativistic matter $w = 0$.

We have the differential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}(1+w)\rho = 0\,, \tag{441}$$

so we can integrate:

$$\int \frac{\mathrm{d}\rho}{\rho} = -3(1+w)\int \frac{\mathrm{d}a}{a}\,, \tag{442}$$

wihch is readily solved by integrating from $\rho = \rho_0$ and $a = a_0$:

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}\,. \tag{443}$$

From the square root of the 00 EFE we get:

$$\frac{\dot{a}}{a} = \frac{1}{\sqrt{3}M_P}\rho^{1/2}\,, \tag{444}$$

into which we can put the solution we found:

$$\frac{\dot{a}}{a} = \frac{1}{\sqrt{3}M_P}\rho_0^{1/2}\left(\frac{a_0}{a}\right)^{\frac{3(1+w)}{2}}\,, \tag{445}$$

and if we define $x = a/a_0$ this becomes:

$$\frac{\mathrm{d}x}{\mathrm{d}t} x^{\frac{3(1+w)}{2}-1} = \frac{\rho_0^{1/2}}{\sqrt{3} M_P}, \tag{446}$$

so we integrate:

$$\int_1^{a/a_0} \mathrm{d}x \, x^{\frac{3(1+w)}{2}} = \frac{\rho_0^{1/2}}{\sqrt{3} M_P} (t - t_0), \tag{447}$$

so we need to integrate $\int x^\alpha \, \mathrm{d}x$: this can be $\log x$ if $\alpha = -1$ or $x^{\alpha+1}/(\alpha+1)$ if $\alpha \neq -1$.

We get the logarithm when $w = -1$: then

$$\log \frac{a}{a_0} = \frac{\rho_0}{\sqrt{3} M_P} (t - t_0), \tag{448}$$

which means

$$a = a_0 \exp\left(\frac{\rho_0^{1/2}}{\sqrt{3} M_P} (t - t_0) \right), \tag{449}$$

which is called *De Sitter spacetime*. Notice that in this case $\rho \equiv \rho_0$ since the density of vacuum energy is constant. In all other cases

$$\frac{2}{3(1+w)} \left(\left(\frac{a}{a_0} \right)^{3\frac{1+w}{2}} - 1 \right) = \frac{r_0^{1/2}}{\sqrt{3} M_P} (t - t_0), \tag{450}$$

and notice that t_0 is an arbitrary choice: we usually choose

$$t_0 = \frac{2}{3(1+w)} \frac{\sqrt{3} M_P}{\rho_0^{1/2}}, \tag{451}$$

which then gives:

$$\left(\frac{a}{a_0} \right)^{\frac{3(1+w)}{2}} = \frac{3(1+w)}{2} \frac{\rho_0^2}{\sqrt{3} M_P} t = \frac{t}{t_0}. \tag{452}$$

Then we get an "easy" answer:

$$\frac{a}{a_0} = \left(\frac{t}{t_0} \right)^{\frac{2}{3(1+w)}}. \tag{453}$$

Some cases are:

1. Matter: $w = 0$ implies $\rho \sim a^{-3}$, $a \sim t^{2/3}$;

2. Radiation: $w = 1/3$ implies $\rho \sim a^{-4}$ and $a \sim t^{1/2}$;

3. Vacuum energy: $w = -1$ implies $\rho = \text{const}$ and $a \sim \exp t$.

If $a = \overline{a} \equiv \text{const}$ then we can rescale the spatial coordinated $\vec{x} \to \overline{a}\vec{x}$, so we recover Minkowski spacetime.

This implies that, in a flat Friedmann - Robertson - Lemaître - Walker universe, the normalization of the scale factor is unphysical.

This can be noticed from the equations: only the *logarithmic* derivative of the scale factor enters them.

If we have a universe which is *not* flat, we get an additional factor k/a^2, with $k = \pm 1$ in the Friedmann equations: then the normalization of the scale factor becomes physical.

Let us characterize expansion: let us consider two galaxies, one at the spatial coordinates $\vec{x}_1 = \vec{0}$ and the other at $\vec{x}_1 = (\overline{x}, 0, 0)$.

The distance between them is given by

$$d_P(t) = \int_0^{\overline{x}} \mathrm{d}x \, \sqrt{g_{11}} = a(t)\overline{x} \,. \tag{454}$$

How much does the distance change between two times t_1 and t_2? Their ratio is

$$\frac{d_P(t_1)}{d_P(t_2)} = \frac{\overline{x}a(t_1)}{\overline{x}a(t_2)} = \frac{a(t_1)}{a(t_2)} \,. \tag{455}$$

These are called *comoving coordinates*: the coordinates on a grid which is expanding.

So the distance can change both because of this comoving expansion, and because of *proper motion*: things actually moving through the grid.

We can write the FRLW metric with respect to spherical coordinates:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right) . \tag{456}$$

A photon moves with $\mathrm{d}s^2 = 0$, which implies

$$\mathrm{d}r = \frac{\mathrm{d}t}{a(t)} \,. \tag{457}$$

Say we have an emitter Alice and an observer Bob, both at fixed radial coordinates. Alice sends two photons (or wavecrests, whatever) at a time difference Δt_A apart, and Bob receives them at a time difference Δt_B apart.

In general $\Delta t_A \neq \Delta t_B$. We have

$$\int_{r_A}^{r_B} \mathrm{d}r = \int_{t_A}^{t_B} \frac{\mathrm{d}t}{a(t)} = \int_{t_A+\Delta t_A}^{t_B+\Delta t_B} \frac{\mathrm{d}t}{a(t)}\,. \tag{458}$$

This then implies that

$$\int_{t_A}^{t_A+\Delta t_A} \frac{\mathrm{d}t}{a(t)} = \int_{t_B}^{t_B+\Delta t_B} \frac{\mathrm{d}t}{a(t)}\,. \tag{459}$$

The Δt are *very* small with respect to the total times: then this means

$$\frac{\Delta t_A}{a(t_A)} \approx \frac{\Delta t_B}{a(t_B)}\,, \tag{460}$$

which can be written as

$$\Delta t_B = \Delta t_A \frac{a(t_B)}{a(t_A)}\,, \tag{461}$$

or, equivalently,

$$\omega_B = \omega_A \frac{a(A)}{a(B)}\,, \tag{462}$$

or

$$\lambda_B = \lambda_A \frac{a(B)}{a(A)}\,, \tag{463}$$

since $\omega\lambda = 2\pi$. Since $a(B) > a(A)$ we get $\lambda_B > \lambda_A$: this is a *redshift*.

Since $E_\gamma = \hbar\omega \propto a^{-1}$, while the volume density $N_\gamma \propto a^{-3}$, we get the global effect of $\rho_\gamma \propto a^{-4}$.

We define

$$z = \frac{\lambda_B}{\lambda_A} - 1 = \frac{a(t_B)}{a(t_A)} - 1\,, \tag{464}$$

fixing $t_B = \text{now} = t_0$.

Hubble's law is a relation between redshift and distance of nearby objects.

Since we are nearby, we Taylor expand the scale factor: we say

$$a(t) = a_0 + \dot{a}_0(t - t_0) + \mathcal{O}(t^2)\,, \tag{465}$$

where by $\dot{a}_0$ we mean the derivative of the scale factor computed at the present time.

We have

$$d_P(t) = a(t)\int_{t_A}^{t_B} \mathrm{d}t = a(t)\int_{t_A}^{t_B} \frac{\mathrm{d}\widetilde{t}}{a(\widetilde{t})}\,, \tag{466}$$

so this, at the time t_0 gives us

$$d_P(t_0) = a_0 \int \frac{\mathrm{d}t}{a_0} = t_B - t_A + \mathcal{O}(t^2)\,, \tag{467}$$

if we consider the lowest possible order (at which the universe is not expanding). We can identify $B = 0$, that is, we observe now.

We can write the first order expansion of the scale factor as

$$a_0 - a(t_A) = \dot{a}_0(t_0 - t_A) \implies t_0 - t_A = \frac{a_0 - a(t_A)}{\dot{a}_0}\,, \tag{468}$$

so we get

$$d_P = t_0 - t_A = \frac{a_0}{\dot{a}_0}\left(1 - \frac{a(t_A)}{a_0}\right) = \frac{a_0}{\dot{a}_0}\left(1 - \frac{1}{1+z}\right) \approx \frac{a_0}{\dot{a}_0} z + \mathcal{O}(z^2)\,, \tag{469}$$

since $z/(1+z) \sim z$ if $z \sim 0$.

The (inverse of the) constant multiplying z is called the *Hubble Constant*:

$$H_0 = \frac{\dot{a}_0}{a_0}\,. \tag{470}$$

There is also the fact we neglected: what is actually measurable is not the comoving distance but the luminosity distance. The value of this constant is around

$$H_0 \approx 70\,\mathrm{kms^{-1}Mpc^{-1}}\,. \tag{471}$$

There is a disagreement between the measurement of H_0 from the CMB and the one using standard candels.

We finish off the topic by discussing the relationship between time and energy. We discuss the present age of the universe and the time of the Big Bang nucleosynthesis.

As long as $w \neq 1$ we can immediately derive a relation for $\dot{a}/a$ by differentiating:

$$\frac{\dot{a}}{a} = \frac{2}{3(1+w)t}\,, \tag{472}$$

so the ratio does depend on w, but the t-dependence is always $H \sim 1/t$. To get an order of magnitude, we can use a matter-dominated universe, so the prefactor becomes $2/3$:

$$t_0 \sim \frac{1}{H_0} \approx 4.4 \times 10^{17}\,\mathrm{s} \sim 1.4 \times 10^{10}\,\mathrm{yr}\,. \tag{473}$$

At which time did the universe have a temperature of 1 MeV? We know that the energy density looks like

$$\rho_\gamma = c(k_B T)^4 \,, \tag{474}$$

which follows from integrating the Planck distribution.

In natural units ($c = \hbar = k_B = 1$), an energy density has dimensions of an energy to the fourth power, since it is energy divided by length cubed, but lengths are inverse energies. Then,

$$\rho_\gamma \propto T^4 \,, \tag{475}$$

and the proportionality constant is approximately equal to 1. We know that

$$H^2 = \frac{1}{3M_P^2}\rho \,, \tag{476}$$

which means that, neglecting the order-1 factor of 3:

$$H = \frac{T^2}{M_P} \,, \tag{477}$$

so

$$t \sim \frac{M_P}{T^2} \,, \tag{478}$$

which is the relation between time and temperature in a radiation-dominated universe. In natural units, $M_P \sim 2 \times 10^{18}$ GeV, so

$$t \sim \frac{2 \times 10^{18}\,\mathrm{GeV}}{(1\,\mathrm{MeV})^2} = 2 \times 10^{24}\,\mathrm{GeV}^{-1} \,, \tag{479}$$

so this can be transformed in seconds by multiplying by

$$\hbar = 6.6 \times 10^{25}\,\mathrm{GeVs} \,, \tag{480}$$

so $\mathrm{GeV}^{-1} = 6.6 \times 10^{-25}$ s: then

$$t = 2 \times 10^{24}\,\mathrm{GeV}^{-1} \times 6.6 \times 10^{-25}\,\mathrm{GeVs} \approx 1.2\,\mathrm{s} \,. \tag{481}$$

This holds for a radiation-dominated universe.

Fri Dec 20 2019

10 Gravitational waves

They are solutions of the linearized EFE on a given fixed background. Today we only treat GW on a Minkowski background, but we could do it for any number of other (cosmological) background.

So, we say:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{482}$$

with $h_{\mu\nu} \ll 1$, so we neglect anything which is $\mathcal{O}(h^2)$.

GW are very topical today: they could allow us to see beyond the last scattering surface, when the universe stopped being opaque to EM radiation: then, it had been transparent to gravitational radiation from a long time before.

We can do statistics to the stochastic GW background, for now however we have only detected GW from localized event.

We must start to write down the EFE from the Christoffel symbols: in general they are

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \Big(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda} \Big) \,, \tag{483}$$

but in the derivatives of the metric we only have $\mathcal{O}(h)$ terms, so in the inverse metric we neglect all the $\mathcal{O}(h)$ terms since their global contribution would be $\mathcal{O}(h^2)$: so we get

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} \eta^{\alpha\lambda} \Big(h_{\lambda\nu,\mu} + h_{\mu\lambda,\nu} - h_{\mu\nu,\lambda} \Big) \,, \tag{484}$$

and when we compute the Ricci tensor we will only keep the $\partial\Gamma$ terms, since the $\Gamma\Gamma$ terms are $\mathcal{O}(h^2)$. So our expression becomes

$$\begin{aligned}
R_{\mu\nu} &= \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} && (485\text{a}) \\
&= \frac{1}{2} \eta^{\alpha\lambda} \partial_\alpha \Big(h_{\lambda\nu,\mu} + \cancel{h_{\mu\lambda,\nu}} - h_{\mu\nu,\lambda} \Big) - \frac{1}{2} \eta^{\alpha\lambda} \partial_\nu \Big(h_{\lambda\alpha,\mu} + \cancel{h_{\mu\lambda,\alpha}} - h_{\mu\alpha,\lambda} \Big) && (485\text{b}) \\
&= \frac{1}{2} \eta^{\alpha\lambda} \left(h_{\lambda\nu,\mu\alpha} - h_{\mu\nu,\alpha\lambda} - \frac{1}{2} h_{\lambda\alpha,\mu\nu} - \frac{1}{2} h_{\lambda\alpha,\mu\nu} + h_{\mu\alpha,\lambda\nu} \right) \,, && (485\text{c})
\end{aligned}$$

where we split a term in two in order to collect, so we get

$$R_{\mu\nu} = \frac{1}{2} \eta^{\alpha\lambda} \partial_\mu \left(h_{\lambda\nu,\alpha} - \frac{1}{2} h_{\lambda\alpha,\nu} \right) + \frac{1}{2} \eta^{\alpha\lambda} \partial_\nu \left(h_{\mu\alpha,\lambda} - \frac{1}{2} h_{\lambda\alpha,\mu} \right) - \frac{1}{2} \eta^{\alpha\lambda} \partial_\alpha \partial_\lambda h_{\mu\nu} \,, \tag{486}$$

where we recognize the Dalambertian $\Box = \eta^{\alpha\lambda}\partial_\alpha\partial_\lambda$ and the trace of $h_{\mu\nu}$: $h \equiv h^\mu_\mu$. So we get

$$R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\partial_\mu\left(\partial_\lambda h^\lambda_\nu - \frac{1}{2}\partial_\nu h\right) + \frac{1}{2}\partial_\nu\left(\partial_\lambda h^\lambda_\mu - \frac{1}{2}\partial_\mu h\right), \tag{487}$$

so we need to solve $R_{\mu\nu} = 0$ with some initial conditions, since we need to solve the EFE $G_{\mu\nu} = M_P^{-2} T_{\mu\nu} = 0$, which implies $R_{\mu\nu} = 0$. The problem is that the solution is not unique: we have the freedom of choosing a gauge, since different metrics connected to each other through changes of coordinates represent the same physical scenario: $h_{\mu\nu}$ *has no direct physical meaning*.

In general we have a change of coordinates $x \to \widetilde{x}$, and if we do this

$$g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \widetilde{x}^\mu}\frac{\partial x^\beta}{\partial \widetilde{x}^\nu} g_{\alpha\beta}, \tag{488}$$

but we can only do changes of coordinates which leave the Minkowski part of the metric invariant, that is $\widetilde{g}_{\mu\nu} = \eta_{\mu\nu} + \widetilde{h}_{\mu\nu}$. We can actually restrict ourselves to infinitesimal changes of coordinates: $x \to \widetilde{x} = x + \epsilon$. So we do

$$\frac{\partial \widetilde{x}^\mu}{\partial x^\alpha} = \partial_\alpha\left(x^\mu + \epsilon^\mu\right) = \delta^\mu_\alpha + \frac{\partial \epsilon^\mu}{\partial x^\alpha} = \delta^\alpha_\mu + \partial_\alpha \epsilon^\mu. \tag{489}$$

Inserting this inside the inverse change of coordinates formula for the metric (the one for $g_{\mu\nu}$ in terms of $\widetilde{g}_{\mu\nu}$, the inverse of the one we wrote down before) we get

$$\left(\delta^\mu_\alpha + \partial_\alpha \epsilon^\mu\right)\left(\partial^\nu_\beta + \partial_\beta \epsilon^\nu\right)\left(\eta_{\mu\nu} + \widetilde{h}_{\mu\nu}\right) = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{490a}$$

$$\left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\alpha \partial_\beta \epsilon^\nu + \delta^\nu_\beta \partial_\alpha \epsilon^\mu + \mathcal{O}(\epsilon^2)\right)\left(\eta_{\mu\nu} + \widetilde{h}_{\mu\nu}\right) = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{490b}$$

$$\widetilde{h}_{\alpha\beta} + \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha = h_{\alpha\beta}. \tag{490c}$$

This freedom of changing $h_{\mu\nu}$ is called gauge freedom: if $h_{\alpha\beta}$ solves the linearized EFE, then $\widetilde{h}_{\alpha\beta} = h_{\alpha\beta} + \partial_{(\alpha}\epsilon_{\beta)}$ also does (there is a factor 2 missing, but it does not matter since ϵ is generic and still infinitesimal).

We can solve $R_{\mu\nu}$ together with some extra conditions: this means *choosing a gauge*. We can do this as long as, starting from a generic $h_{\mu\nu}$, we can find a transformation ϵ_μ which leaves $\eta_{\mu\nu}$ invariant and for which $\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_{(\mu}\epsilon_{\nu)}$.

For example, a condition which cannot be imposed is $h_{\mu\nu} = 0$: if we could impose it, then it would be equivalent to have a wave or not to have it. This can be shown from degrees of freedom: we only have 4 degrees of gauge freedom.

So when we set a condition we must prove that we *can actually* do so.

Then, any gauge we are actually allowed to impose will give the same results for any physical experiment.

An electromagnetic analogy: since $\vec{\nabla} \cdot \vec{B} = 0$, we know that we can locally choose a potential $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$. This has no intrinsic physical meaning, as any $\vec{A'} = \vec{A} + \vec{\nabla}\lambda$ for a scalar function λ also gives the exact same magnetic field. Some gauge fixing choices are $\vec{\nabla} \cdot \vec{A} = 0$ (Coulomb), $A_3 = 0$ (axial).

We do, however, use potentials since they are convenient. Analogously, $h_{\mu\nu}$ is convenient to use but it is not measurable nor unique.

Is there a gauge-independent tensorial quantity we can define starting from $h_{\mu\nu}$, a gravitational "field strength"?

The last parentheses in the expression for $R_{\mu\nu}$ is annoying: so we choose the gauge

$$\partial_\lambda h^\lambda_\nu - \frac{1}{2}\partial_\nu h = 0\,, \tag{491}$$

and then in this gauge we have $R_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}$. There are two questions we must ask: can we actually impose this condition, and why is it called harmonic?

Let us prove the first one:

$$0 \overset{?}{=} \partial_\lambda \widetilde{h}^\lambda_\nu - \frac{1}{2}\partial_\nu \widetilde{h} = \partial_\lambda \left(h^\lambda_\nu - \partial^\lambda \epsilon_\nu - \cancel{\partial^\nu \epsilon_\lambda} \right) - \frac{1}{2}\partial_\nu \left(h - 2\cancel{\partial_\lambda \epsilon^\lambda} \right) \tag{492a}$$

$$= \partial_\lambda h^\lambda_\nu - \frac{1}{2}\partial_\nu h - \Box \epsilon_\nu\,, \tag{492b}$$

which can be solved by setting an ϵ_ν such that:

$$\Box \epsilon_\nu = \partial_\lambda h^\lambda_\nu - \frac{1}{2}\partial_\nu h\,. \tag{493}$$

Notice that in this gauge $\Box \widetilde{h}_{\mu\nu} = 0$: the wave equation, for a wave which propagates with velocity 1.

We have residual gauge freedom! As long as $\Box \widetilde{\epsilon}_\mu = 0$, we can still change $h_{\mu\nu}$. So the solution is not unique. The gauge is called harmonic because we can do harmonic residual gauge fixing. We have 4 more degrees of freedom: we choose $h_{0\mu} \equiv 0$. Now we prove that this can be done.

We do not prove this in general (although it can be done), because we want to move towards a more specific example.

We then consider plane wave solutions: waves that only depend on one spatial coordinate, and are constant with respect to both other spatial coordinates.

If the direction of motion is the z axis, then the solution looks like

$$h_{\mu\nu}(t, x, y, z) = C_{\mu\nu} \exp\bigl(-i(kt - kz)\bigr)\,. \tag{494}$$

For a wave propagating in a generic direction we will have

$$h_{\mu\nu}(t, \vec{x}) = C_{\mu\nu} \exp\bigl(ik_\alpha x^\alpha\bigr)\,, \tag{495}$$

where $k^\alpha k_\alpha = 0$ (which comes from the Fourier expression of the Dalambertian: $\square = -k^\alpha k_\alpha$) is a constant vector. The scalar product in the exponent must be computed with the Minkowski metric since we are working at first order in h and $C_{\mu\nu}$ is already first order.

$$\begin{aligned}
\square h_{\mu\nu} &= \eta^{\alpha\beta}\partial_\alpha\partial_\beta\Big(C_{\mu\nu}\exp\big(ik_\alpha x^\alpha\big)\Big) && \text{(496a)}\\
&= \eta^{\alpha\beta}C_{\mu\nu}\partial_\alpha\Big(\exp\big(ik_\alpha x^\alpha\big)\partial_\beta\big(ik_\alpha x^\alpha\big)\Big) && \text{(496b)}\\
&= \eta^{\alpha\beta}C_{\mu\nu}ik_\beta\partial_\alpha\big(\exp(ik\cdot x)\big) && \text{(496c)}\\
&= \eta^{\alpha\beta}(ik_\beta)(ik_\alpha)C_{\mu\nu}\exp(ik\cdot x) = -k^2 C_{\mu\nu}\exp(ik\cdot x)\,, && \text{(496d)}
\end{aligned}$$

then we see that this is always zero as long as $k^2 = 0$, which is what we impsed for k^α.

What is the particle analog of a gravitational wave? it is a so-called *graviton*. For it we will have a momentum operator $\vec{p} = -i\hbar\vec{\nabla}$ and an energy operator $E = i\hbar\partial_t$. We can apply both of these to our wave to find that it is an eigenvector of both:

$$Eh_{\mu\nu} = i\hbar\partial_t\Big(C_{\mu\nu}\exp(ik\cdot x)\Big) = \hbar k^t h_{\mu\nu}\,, \tag{497}$$

and we know that $k^t = \left|\vec{k}\right|$.

If we compute the momentum, analogously we find $\vec{p}h_{\mu\nu} = \hbar\vec{k}$.

This means that $p\cdot p = 0$, which means that the mass of the graviton is equal to zero.

Thu Jan 09 2020

In the last four hours we will finish the discussion on GWs.

We solved the vacuum equations $R_{\mu\nu} = 0$ for a perturbed Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for small $h_{\mu\nu}$.

We found that these are equivalent to

$$R_{\mu\nu} = -\frac{1}{2}\square h_{\mu\nu} + \partial_{(\mu}\left(h^\lambda_{\nu),\lambda} - \frac{1}{2}\partial_{\nu)}h\right) = 0\,, \tag{498}$$

which can become the wave equation $\square h_{\mu\nu} = 0$ in the harmonic gauge

$$\partial_\lambda h^\lambda_\nu - \frac{1}{2}\partial_\nu h = 0\,. \tag{499}$$

Our gauge transformations are

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_{(\nu}\epsilon_{\mu)}\,, \tag{500}$$

where $x^\mu \to x^\mu + \epsilon^\mu(x)$

The solutions to the wave equation are

$$h_{\mu\nu} = C_{\mu\nu} \exp(ik \cdot x)\,, \tag{501}$$

where $k^2 = 0$, since $p^\mu = \hbar k^\mu$ and the wave travels at the speed of light $p^2 = 0$.

Claim 10.1. *If $h_{\mu\nu}$ satisfies the harmonic gauge and ϵ_μ satisfies $\Box\epsilon_\mu = 0$ then $\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_{(\mu}\epsilon_{\nu)}$ also satisfies the harmonic gauge.*

Proof. The equation we want to show is true is

$$0 = \partial_\lambda \widetilde{h}^\lambda_\mu - \frac{1}{2}\partial_\mu \widetilde{h}\,, \tag{502}$$

where $\widetilde{h} = \widetilde{h}^\mu_\mu$. We can expand it into

$$0 = \partial_\lambda \left(\cancel{h^\lambda_\mu} - \partial^\lambda \epsilon_\mu - \partial_\mu \epsilon^\lambda \right) - \frac{1}{2}\partial_\mu \left(\cancel{h} - 2\partial^\nu \epsilon_\nu \right) \tag{503a}$$

Harmonic gauge for $h_{\mu\nu}$

$$= -\Box\epsilon_\mu - \partial_\mu \left(\partial_\nu \epsilon^\nu \right) + \partial_\mu \left(\partial^\nu \epsilon_\nu \right) = 0\,. \tag{503b}$$

$\Box\epsilon_\mu = 0$ and commuting derivatives

□

Now, we want to impose 4 additional conditions that remove the residual freedom $x^\mu \to x^\mu + \epsilon^\mu$ with $\Box\epsilon^\mu = 0$.

This is a wave equation for ϵ^μ, so the solutions will look like

$$\epsilon_\mu = \gamma_\mu \exp(ik \cdot x)\,, \tag{504}$$

with γ_μ constant and $k^2 = 0$.

Under this transformation, the coefficient $C_{\mu\nu}$ changes into

$$\widetilde{C}_{\mu\nu} = C_{\mu\nu} - 2ik_{(\mu}\gamma_{\nu)}\,, \tag{505}$$

as can be seen by substitutin in our expression for γ_μ into the transformation law for $h_{\mu\nu}$. Do note that these correspond to the same physical fields.

We further require $\widetilde{C}_{00} = \widetilde{C}_{0i} = 0$. This fixes the residual gauge freedom completely.

We need to show that from a generic $C_{\mu\nu}$ we can find a γ_μ such that this condition holds.

We write out the conditions:

$$\widetilde{C}_{00} = C_{00} - 2ik_0\gamma_0\,, \tag{506}$$

so setting $\widetilde{C}_{00} = 0$ fixes

$$\gamma_0 = \frac{C_{00}}{2ik_0}\,, \tag{507}$$

while in order to set

$$\begin{aligned}
0 = \widetilde{C}_{0i} &= C_{0i} - ik_0\gamma_i - ik_i\gamma_0 && \text{(508a)} \\
&= C_{0i} - ik_0\gamma_i - ik_i\frac{C_{00}}{2ik_0} && \text{(508b)} \\
\gamma_i &= \frac{1}{ik_0}\left(C_{0i} - \frac{k_i C_{00}}{2k_0}\right)\,. && \text{(508c)}
\end{aligned}$$

With these choices for γ_μ we can impose the desired condition.

There is no issue if $k_0 = 0$, since that would correspond to a GW of zero frequency, which can be shown to be equivalent to flat spacetime.

In the end, we need to solve the system

$$\boxed{\begin{aligned}
\square h_{\mu\nu} &= 0 && \text{(509a)} \\
\partial_\lambda h^\lambda_\mu - \frac{1}{2}\partial_\mu h &= 0 && \text{(509b)} \\
h_{0\mu} &= 0\,, && \text{(509c)}
\end{aligned}}$$

since $C_{\mu\nu} = 0$ for some $\mu\nu$ implies $h_{\mu\nu} = 0$ for those indices.

A priori, we have 10 degrees of freedom for our metric $h_{\mu\nu}$ since it is symmetric. Equation (509b) eliminates 4 of them, and equation (509c) eliminates 4 more.

So, in the end we have 2 independent degrees of freedom.

Let us write the harmonic gauge condition explicitly: for $\mu = 0$ we get

$$\begin{aligned}
\partial_\lambda h^\lambda_0 - \frac{1}{2}\partial_0 h &= 0 && \text{(510a)} \\
-\partial_0 h_{00} + \partial_i h_{i0} - \frac{1}{2}\partial_0 h &= 0\,, && \text{(510b)}
\end{aligned}$$

but the first two vanish if $h_{0\mu} = 0$: so we get $\partial_0 h = 0$: this, when it is written explicitly, is

$$\partial_0\Big(C^\mu_\mu \exp(ik\cdot x)\Big) = 0\,, \tag{511}$$

which means that $C^\mu_\mu = C = 0$: so the gravitational wave is traceless, this can be written as $h_{ii} = 0$ since $h_{00} = 0$.

So, for $\mu = j$ we can write the harmonic gauge condition as

$$\partial_\lambda h^\lambda_j = 0 \tag{512a}$$
$$\partial_i h_{ij} = 0\,. \tag{512b}$$

$h = 0$

$\eta_{ij} = \delta_{ij}$ and $h_{0j} = 0$

This implies that the GW is *transverse*: it oscillates only in the directions perpendicular to the motion.

Then, our system becomes

$$\Box h_{\mu\nu} = 0 \tag{513a}$$
$$h_{0\mu} = 0 \tag{513b}$$
$$\partial_i h_{ij} = 0 \tag{513c}$$
$$h_{ii} = 0\,. \tag{513d}$$

In terms of the matrix $C_{\mu\nu}$ these conditions mean:

$$C_{00} = C_{0i} = C_{ii} = k_i C_{ij} = 0\,, \tag{514}$$

since the condition $\partial_i h_{ij} = 0$ means $ik_i C_{ij} \exp(ik \cdot x) = 0$.

Without loss of generality we can write a coordinate system in which our gravitational wave is travelling along the z direction: therefore $\vec{k} = [0, 0, k]^\top$. Then, we get the conditions: $C_{i3} = 0$, $C_{11} + C_{22} = 0$ and $C_{0\mu} = 0$.

So we have two conditions: $C_{12} = C_{21} \neq 0$ and the rest of the matrix is zero, and $C_{11} = -C_{22} \neq 0$ and the rest of the matrix is zero. Then, the basis tensors are

$$e_{ij,+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{515a}$$

and

$$e_{ij,\times} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{516a}$$

The Frobenius norm of these matrices is $e_{ij,+}e_{ij,+} = e_{ij,\times}e_{ij,\times} = 2$, which can also be written as $\operatorname{tr}(e_+ e_+) = \operatorname{tr}(e_\times e_\times) = 2$. Also, in this sense they are orthogonal: $\operatorname{tr}(e_+ e_\times) = 0$.

So, we can write $\operatorname{tr}(e_r e_s) = 2\delta_{rs}$, where r and s are either $+$ or $\times$.

Then, any GW h_{ij} can be written as

$$h_{ij}(x) = \Big(h_+ e_{ij,+} + h_\times e_{ij,\times}\Big) \exp(ik \cdot x)\,, \tag{517}$$

with $k^2 = 0$. Do note that h_+ and $h_\times$ are two independent complex numbers.

The solutions we want have real $h_{\mu\nu}$, so if $h_{\mu\nu} \in \mathbb{C}$ is a solution then $h^*_{\mu\nu}$ also is a solution. So, we are allowed to take the real part with $(h_{\mu\nu} + h^*_{\mu\nu})/2$. This will then give us

$$\mathrm{Re}(C_{ij} \exp(ik \cdot x)) = C_{ij} \cos\bigl(ik \cdot x + \varphi\bigr) , \tag{518}$$

for some real C_{ij}. So, in general we will have

$$h^k_{ij}(x) = \sum_{r=+,\times} h_r e_{ij,r} \cos\bigl(kt - kz + \varphi_r\bigr) , \tag{519}$$

do note that we changed the sign of the argument of the cosine, which we can do since the cosine is even.

Do also note that the phase can in principle depend on the polarization.

We can then recover any general solution travelling along the z direction by having h_r depend on k: then we will have

$$h_{ij} = \int \mathrm{d}k \sum_{r=+,\times} h_r(k) e_{ij,r} \cos\bigl(kt - kz + \varphi_r(k)\bigr) . \tag{520}$$

Also, note that the frequency of the gravitational waves can be found by $f = k/2\pi$; this can be seen from the periodicity of the waves.

What do these polarizations physically entail?

Recall that the distance along a certain direction, say x, is given by the integral $\int \mathrm{d}x \sqrt{g_{xx}}$.

So, if g_{xx} changes than the physical distance also changes.

Let us consider the point $z = 0$, at $t = 0$, with zero phase φ for the polarization $e_{ij,+}$: then the component g_{xx} has increased, since the cosine is equal to one, while the component g_{yy} has decreased

After half a wavelength, when the cosine is equal to -1, this effect reverses. Then, g_{xx} decreases while g_{yy} increases.

The wave is travelling along the z axis, but because of the transversality condition it stretches the x and y directions.

What happens if we perform a 45° rotation? the spatial rotation matrix is

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} , \tag{521a}$$

so, for example, $[1, 1, 0]^\top \to [\sqrt{2}, 0, 0]^\top$.

The transformation law with this matrix is $\widetilde{x}^i = R^i_j x^j$.

It is known that a rotation matrix satisfies $R^\top R = \mathbb{1}$.

The line element changes as

$$\mathrm{d}\ell^2 = \mathrm{d}x^i \, g_{ij} \, \mathrm{d}x^j = \mathrm{d}\widetilde{x}^i \, \widetilde{g}_{ij} \, \mathrm{d}\widetilde{x}^j \,, \tag{522}$$

which means that $g = R^\top \widetilde{g} R$, since $\mathrm{d}\widetilde{x} = R \, \mathrm{d}x$. This is coherent with the tensor transformation law. Inverting it, we find $\widetilde{g} = R g R^\top$.

How does the perturbed metric change?

$$\mathbb{1} + \widetilde{h} = R(\mathbb{1} + h)R^\top \tag{523a}$$

$$= R\mathbb{1}R^\top + RhR^\top \tag{523b}$$

$$= \mathbb{1} + RhR^\top \,, \tag{523c}$$

so $\widetilde{h} = RhR^\top$.

Let us suppose that $h = e_\times$: what is then $\widetilde{h}$? It can be shown straightforwardly that

$$\widetilde{h} = Re_\times R^\top = e_+ \,, \tag{524}$$

so the cross-polarization is equivalent to the plus-polarization, rotated by 45°. This is somewhat of an informal line of reasoning, since the basis polarization "tensors" are not actually tensors under generic transformation. However, the result is correct and the reasoning is illustrative so we stick with it.

Fri Jan 10 2020

Why does the photon have two degrees of freedom?

We know that we can express

$$\vec{B} = \nabla \times \vec{A} \qquad \text{and} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} \,, \tag{525}$$

and if we define the 4-vector $A^\mu = (\phi, \vec{A})$ then this potential is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \xi$ for any function ξ.

This is the defining property of electromagnetism! If we have a potential A_μ we can define the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$: then the Lagrangian is written as

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2 \phi^2 - F_{\mu\nu} F^{\mu\nu} \,, \tag{526}$$

and this field theory has $U(1)$ symmetry, $\phi \to e^{i\xi}\phi$, which corresponds to $A_\mu \to A_\mu + \partial_\mu \xi$! This is actually the defining property of electromagnetism in the QED formulation.

Similarly, GR follows from the invariance under a certain kind od gauge transformation: in this case, the gauge transformations are *diffeomorphisms* $x \to x'(x)$.

In electromagnetism, we have the 4 component for the potential, but we can fix the gauge by setting $\partial_\mu A^\mu = 0$ and the residual gauge by setting $A^0 = 0$: then we are left with 2 degrees of freedom.

This is completely analogous to the way we found the graviton to have 2 degrees of freedom, that is, two polarizations.

Now we discuss how we actually detected the presence of gravitons.

10.1 Interferometric GW detection

GWs are detected interferometrically since interferometers are our most accurate way to measure distances.

The setup is a Michelson-Morley interferometer. Call ΔL the difference in the two paths the light takes, if $\Delta L = n\lambda$ with $n \in \mathbb{N}$ and λ begin the wavelength of light, then we have perfectly constructive interference. If instead $\Delta L = (n + 1/2)\lambda$ we have perfectly destructive interference.

In actual GW experiments we will have $\Delta L \ll \lambda$, so we will not go from in-phase to counter-phase, but there will be a slight decrease of power, by which we will be able to detect the GW.

We take the mirrors to be suspended and free to move in the horizontal direction. So, for our purposes (to linear order in the displacement), the mirrors are free particles which move along geodesics.

So we have two mirrors at $\vec{x}_1$ and $\vec{x}_2$: let us say that the x axis is along their distance.

Then, their distance will be given by

$$d = \int_{x_1}^{x_2} \mathrm{d}x \sqrt{g_{11}} \,. \tag{527}$$

In principle, the GW can change d by changing x_1 and x_2 and *also* by changing the metric element $\sqrt{g_{11}}$ in the space between them.

Claim 10.2. *The effect of the change of the positions x_1 and x_2 is at most $\mathcal{O}(h^2)$.*

Proof. First, let us assume that there is no GW and we have just one mirror in a fixed position in Minkowski spacetime. Then, the initial 4-velocity of the mirror (before the arrival of the GW) is $u^\alpha_{\tau=0} = [1, \vec{0}]$.

Then, the GW arrives and the mirror will follow the geodesic equation:

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0 \,. \tag{528}$$

We want to consider this at first order in h. The Christoffel sybols at order h^0 are zero since in Minkowski spacetime there is no curvature.

So, if we want a nonzero first-order Christoffel term, we need to consider the zeroth order in the 4-velocity.

Then, we are left with $u^\alpha = u^\alpha_{\tau=0}$, or

$$\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = -\Gamma^i_{00}\,, \tag{529}$$

since the Γ^0_{00} is zero. Now we can compute

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\lambda}\left(h_{\lambda 0,0} + h_{\lambda 0,0} - h_{00,\lambda}\right) = 0\,, \tag{530}$$

because in our gauge $h_{0\mu} = 0$. □

A note: this is only true in our gauge. In other gauges, the result is the same but there is also a first-order contribution to the change in the distance between the mirrors from the change in their coordinate positions.

We set the mirrors at $\vec{x}_1$ and $\vec{x}_2 = \vec{x}_1 + T\hat{L}_{12}$, where $\hat{L}_{12}$ is a unit vector while T is the (unperturbed) travel time (or distance, since $c = 1$) between the two mirrors.

The worldline of the laser light is given by $t = t_1 + \lambda$ and $\vec{x} = \vec{x}_1\hat{L}_{12}\lambda$, for $\lambda \in [0, T]$.

The photons move with $\mathrm{d}s^2 = 0$, which means

$$\begin{aligned} 0 &= -\,\mathrm{d}t^2 + \left(\delta_{ij} + h_{ij}\right)\mathrm{d}x^i\,\mathrm{d}x^j && (531a)\\ &= -\,\mathrm{d}t^2 + \left(\delta_{ij} + h_{ij}\right)\hat{L}^i_{12}\hat{L}^j_{12}\,\mathrm{d}\lambda^2\,, && (531b)\end{aligned}$$

but $\hat{L}^i_{12}\hat{L}^j_{12}\delta_{ij} = 1$ since it is a unit vector, so we get

$$0 = -\,\mathrm{d}t^2 + \left(1 + \hat{L}^i_{12}\hat{L}^j_{12}h_{ij}\right)\mathrm{d}\lambda^2\,, \tag{532}$$

so we get

$$\begin{aligned} \mathrm{d}t &= \sqrt{1 + \hat{L}^i_{12}\hat{L}^j_{12}h_{ij}}\,\mathrm{d}\lambda && (533a)\\ &\approx \left(1 + \frac{1}{2}\hat{L}^i_{12}\hat{L}^j_{12}h_{ij}\right)\mathrm{d}\lambda\,. && (533b)\end{aligned}$$

So, we need to evaluate this along the trajectory: we find

$$T_{12} = \int_0^T \mathrm{d}\lambda\left(1 + \frac{1}{2}\hat{L}^i_{12}\hat{L}^j_{12}\sum_{r=+,\times} h_r e_{ij,r}\cos\Big(k(t_1 + \lambda t) - \vec{k}\cdot\left(\vec{x}_1 + \hat{L}_{12}\lambda\right)\Big)\right), \tag{534}$$

which just amounts to plugging in our expressions for t and $\vec{x}$ into the formula for $h_{ij}(t, \vec{x})$.

We set the phase to zero for brevity, a more general consideration will include it.

The first term is the unperturbed travel time T. Then, we can bring the constants outside the integral:

$$T_{12} = T + \frac{1}{2}\hat{L}^i_{12}\hat{L}^j_{12} \sum_{r=+,\times} h_r e_{ij,r} \int_0^T \mathrm{d}\lambda \cos\Big(k(t_1+\lambda) - \vec{k}\cdot\Big(\vec{x}_1 + \hat{L}_{12}\lambda\Big)\Big) . \tag{535}$$

Now, we can make an approximation which is applicable to ground-based interferometers: the *short-arm* approximation. The arguments of the cosine depend on λ, but we can assume $k\lambda < kT \ll 1$. This means that the travel time of the laser is much smaller than the period of the GW. Then, the cosine is approximately λ-independent and can be brought outside of the integral. So we find

$$T_{12} = T + \frac{1}{2}\hat{L}^i_{12}\hat{L}^j_{12} \sum_{r=+,\times} h_r e_{ij,r} \cos\Big(kt_1 - \vec{k}\cdot\vec{x}_1\Big) T . \tag{536}$$

Let us justify the approximation: $k \sim 100\,\mathrm{Hz}$, while $L \sim 4\,\mathrm{km}$. So, $kT \approx 2\pi \times 4\,\mathrm{km} \times 100\,\mathrm{km}/c \approx 8 \times 10^{-3}$, which is small.

We have three points in the interferometer: $\vec{x}_1$ is the beamsplitter, $\vec{x}_2$ and $\vec{x}_3$ are the ends of the arms. We need to compute the difference between the times $\vec{x}_1 \to \vec{x}_2 \to \vec{x}_1$ and $\vec{x}_1 \to \vec{x}_3 \to \vec{x}_1$, repeated a few hundred times (the number of bounces of the beam is called the *finesse factor*). Let us just compute $\Delta T = T_{12} - T_{13}$ for a single "bounce", to see what the effect looks like. We actually compute $\Delta T/T$, in order to see what fraction of difference of travel time we are looking at. It will be given by

$$\frac{\Delta T}{T} = \cos\Big(kt - \vec{k}\vec{x}\Big) \sum_{r=+,\times} h_r e_{ij,r} \frac{\hat{L}^i_{12}\hat{L}^j_{12} - \hat{L}^i_{13}\hat{L}^j_{13}}{2} . \tag{537}$$

Writing it this way is convenient, since we do not need to consider a single geometry: there are 90° interferometers such as LIGO, and 60° ones such as LISA, or the Einstein telescope.

Let us say that

$$\hat{L}_{12} = [\cos\alpha, \sin\alpha, 0]^\top \qquad \text{and} \qquad \hat{L}_{13} = [\cos(\alpha+\pi/2), \sin(\alpha+\pi/2), 0]^\top . \tag{538}$$

We need to compute the following products:

$$\hat{L}^\top_{12} e_+ \hat{L}_{12} = \cos^2\alpha - \sin^2\alpha = \cos(2\alpha) \tag{539a}$$

$$\hat{L}^\top_{12} e_\times \hat{L}_{12} = 2\sin\alpha\cos\alpha = \sin(2\alpha) , \tag{539b}$$

and similarly for $\hat{L}_{13}$, where we can substitute $\alpha \to \alpha + \pi/2$. We get:

$$\frac{\Delta T}{T} = \frac{1}{2}\cos\left(kt - \vec{k}\cdot\vec{x}\right)h_+\left(\left(\cos(2\alpha) - \cos(2\alpha+\pi)\right)\right) + h_\times\left(\left(\sin(2\alpha) - \sin(2\alpha+\pi)\right)\right) \tag{540a}$$

$$= \cos\left(kt - \vec{k}\cdot\vec{x}\right)\left(h_+\cos(2\alpha) + h_\times\sin(2\alpha)\right), \tag{540b}$$

Simplified factors of 2

where we used the fact that $\cos(x+\pi) = -\cos(x)$ and similarly for the sine.

So, this is a $\mathcal{O}(h)$ effect, with $h \sim 10^{-21}$, which means $\Delta L \sim 4\,\mathrm{km} \times 10^{-21} \sim 4 \times 10^{-18}\,\mathrm{m}$.

The fact that we have many photons in the laser helps, and the fact that they bounce several times also does.