Gravitational physics notes

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Contents

Monday 2020-3-9

0.1 Introduction

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Office hour: check by email.

Reading material: slides (to be used as an index of what is treated in the course), Hobson, Michele Maggiore.

This is a general class on gravitational physics and GW, it does not really follow any textbook: the field is young so there is not really a textbook.

The slides will be provided before lectures. There will be no home assignments.

The idea is that formulas are important, detailed calculations and derivations are not.

The target is to be able to read a research paper on GW and understand it. We will not go into very much detail on any topic: the program of the class is very large.

For the exam: it is a discussion of a GW paper (about 25 min), plus theoretical questions — focusing on the physical meaning, not on tedious derivations. It usually takes a bit less than an hour. The paper is optional.

Off session exams are OK, best if with several people (2-5 people).

Please fill out the questionnaire on the course before taking the exam.

0.1.1 Topics

Understanding what GW are: how they are described, how they are generated, what is their physical effect.

Some astrophysical and cosmological GW courses. The professor's background is more in experimental physics than in astrophysics and cosmology.

Interactions of GW with light and matter: ideas, techniques, experiments to detect GWs, especially GW interferometers.

Analysis of GW signals.

What we can learn from GW, overview of the most significant recent papers.

What follows is a long, somewhat divulgative introduction.

Einstein thought their detection impossible. Now we can not only *detect* them, we can actually *observe* them.

They are a test of GR in *extreme* conditions, where the weak-field approximation does not apply.

We can derive properties of matter in these extreme conditions, such as the equation of state for a neutron star.

GWs are "ripples" in the metric of spacetime, described by a quadrupole formula: the quadrupole is

$$Q_{jk} = \int \rho x_j x_k \, \mathrm{d}^3 x \,, \tag{1}$$

and then the perturbation propagates like

$$h_{jk} = \frac{2}{r} \frac{\mathrm{d}^2 Q_{jk}}{\mathrm{d}t^2} \,. \tag{2}$$

What generates GWs are non-spherically symmetric perturbations: by Jebsen-Birkhoff, if we have spherical symmetry there is no perturbation in the vacuum metric.

They "stretch" space by squeezing one direction and stretching a perpendicular one.

The typical relative scale of these perturbations is

$$\frac{\Delta L}{L} \sim 10^{-21} \,, \tag{3}$$

which is *really small*: if we multiply it by the radius of the Earth's orbit we get a length on the order of the size of an atom.

An interesting thing which could emit in the $\sim 1\,\mathrm{Hz}$ range are extreme Mass Ratio inspirals: we have what is effectively a test particle in a strong gravitational field.

We have different kinds of interferometers: for now we have used ground interferometers, there are also space detectors like LISA, Pulsar Timing Arrays at higher frequencies, and inflation probes (?).

In binary systems, we have different stages in the pulsation: an almost stationary one, the inspiral, the coalescence, and finally the ringdown.

In the early years, it was thought that GWs might be a coordinate artifact which could be "gauged away".

In 1959, Joseph Weber proposed a "resonant bar" detector. These are based upon a sound principle: one of the last ones was AURIGA, the issue was that the sensitivity was insufficient and they are only sensitive in a specific frequency range.

GWs were detected indirectly using Hulse-Taylor pulsars: they measured the energy loss of a binary pulsar-NS system, which implied the loss of energy through gravitational wave emission.

The famous graph is not a fit line, it is the prediction based upon the measured orbital parameters.

Now we use laser interferometers: they are broad-band (a couple orders of magnitude, from 10 Hz to 1 kHz), they are inherently differential (as opposed to the single-mode excitation of a resonant bar).

We can use Fabry-Perrot cavities in order to amplify effective length. There is also a power recycling mirror in order for the light not to go back to the laser: modern lasers are on the order of 100 kW, so there is a huge amount of power circulating in the cavities.

We can plot the sensitivity of the interferometers. On the x axis we put the frequency of the incoming wave. On the y axis we put the amplitude spectral density $h(f)$, which is measured in $\mathrm{Hz}^{-1/2}$.

Why?

The curve describes where the noise dominates. We can plot both the theoretical sensitivity and the measured one.

The signal comes out buried in noise, we must extract it in some way, like by correlating to a standard test signal.

Friday 2020-3-13, compiled 2020-04-10

What we will do in the first lessons:

1. A quick review of GR;
2. linearization and GW in free space;
3. the physical effect of GW: free falling reference frames, detector frame;
4. GW sources : binary systems, multipole expansion and quadrupole approximation, GW back reaction: energy & momentum loss, Hulse-Taylor pulsar;
5. GW sources: a rotating rigid body.

0.2 A quick review of GR

We start from special relativity. The "old" way to do transformations are galilean transformations: in 2D they are

$$t' = t \tag{4a}$$
$$x' = x - vt\,. \tag{4b}$$

There are issues with these, such as the invariance of the speed of light. So, we use Lorentz transformations:

$$ct' = \gamma(ct - \beta x) \tag{5a}$$
$$x' = \gamma(x - \beta ct)\,, \tag{5b}$$

where $\beta = v/c \leq 1$, c being the speed of light, and $\gamma = 1/\sqrt{1-\beta^2} \geq 1$.

These preserve the spacetime interval

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 . \tag{6}$$

The intervals between two events can be spacelike ($\Delta s^2 > 0$), null ($\Delta s^2 = 0$) or timelike ($\Delta s^2 < 0$).

We can express this using an infinitesimal time interval

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \, \eta_{\mu\nu} \, \mathrm{d}x^\nu \ , \tag{7}$$

where we use Einstein summation convention. We are going to use the mostly plus metric convention.

We can define the differential *proper time* along a curve, by

$$c^2 \, \mathrm{d}\tau^2 = -\, \mathrm{d}s^2 = c^2 \, \mathrm{d}t^2 \left(1 - \beta^2\right) = \frac{c^2}{\gamma^2} \, \mathrm{d}t^2 \ , \tag{8}$$

which means that $\mathrm{d}\tau = \mathrm{d}t \, / \gamma$. We can use this as a *covariant* parametrization of a spacetime curve.

For curved spacetime, we model it as a 4D semi-Riemannian manifold with signature $(1, 3)$. Since it is a manifold, the parametrization of points in spacetime must be a homeomorphism, and we ask for the *transition maps* between two regions of spacetime to be infinitely differentiable. The set of local charts is called an atlas. The charts are maps from $\mathbb{R}^4$ to the manifold.

The metric is a function of the point at which we are, and (the way it changes) describes the local geometry of the manifold. Only the symmetric part of the metric appears in the spacetime interval, therefore we say that the metric is always symmetric without losing any generality.

The metric is a bilinear form at each point of the manifold, and it transforms as a $(0, 2)$ tensor. The components of this tensor in our chosen reference frame are $g_{\mu\nu}$. The choice of coordinates is arbitrary and tricky.

In a neighborhood of a point we can always choose a reference frame (Riemann normal coordinates) such that $g_{\mu\nu} = \eta_{\mu\nu}$, and $g_{\mu\nu,\alpha} = 0$ (partial derivatives calculated *at that point*), but the second derivatives $g_{\mu\nu,\alpha\beta}$ cannot all be set to zero.

Vectors in a manifold are defined in the tangent space *at a point*. Formally, we define curves parametrically as $X^\mu(\lambda)$.

Then, we define the tangent vector to the curve as the *directional derivative* operator along the curve:

$$\vec{v}(f) = \left. \frac{\mathrm{d}f}{\mathrm{d}\lambda} \right|_C = \frac{\partial f}{\partial x^\mu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda} \, , \tag{9}$$

for any scalar field f. The motivation for this definition, as opposed to just taking the tangent vector to the curve, is the fact that there is no *intrinsic* way to to that.

If we define a curve using a coordinate as a parameter, with the other coordinates staying constant along the curve, this is called a *coordinate curve*.

Vectors defined at different points are in different spaces, we cannot compare them directly.

Tangent vectors to coordinate lines are called coordinate basis vectors $e_{(\mu)}$, where μ is not a vector index but instead it spans the basis vectors. So, any vector can be written as a linear combination as $\vec{v} = v^\mu e_\mu$. We also have $e_\mu \cdot e_\nu = g_{\mu\nu}$, so, in order to find the components of the scalar product $v \cdot w$ we need to do $v^\mu w^\nu g_{\mu\nu}$.

This is because $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}s \cdot \mathrm{d}s = \left(\mathrm{d}x^\mu\, e_\mu\right) \cdot (\mathrm{d}x^\nu\, e_\nu)$.

An orthonormal basis in one for which $e_\mu \cdot e_\nu = \eta_{\mu\nu}$.

Dual basis vectors e^μ are defined by $e^\mu e_\nu = \delta^\mu_\nu$.

We write a co-vector (or dual vector) as a linear combination of these: $v = v_\mu e^\mu$.

Then, we can raise and lower indices like

$$g_{\mu\nu} v^\mu w^\nu = v \cdot w = v_\mu e^\mu \cdot w^\nu e_\nu = v_\mu w^\nu \delta^\mu_\nu = v_\mu w^\nu\,. \tag{10}$$

The inverse metric is defined as $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$.

Tensors are geometrical objects which belong to the dual space to the cartesian product of n copies of the tangent space and m copies of the dual tangent space. The type of such a tensor is then said to be (n, m), and its rank is $n + m$. This definition means that the tensor is a *multilinear* transformation.

Once we have a coordinate system, we can move to another via a coordinate transformation

$$x'^\mu = x'^\mu(x^\mu) \implies \mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu\,. \tag{11}$$

A scalar does not transform: $\phi(x) = \phi'(x')$. A vector's component do transform: we find the transformation law by imposing $v = v'$ in components. This works both for covariant and contravariant vectors, and we find that these transform using either the Jacobian of the transformation or its inverse.

In order to compute derivatives we need to compare vectors in different tangent spaces: we need to "connect" infinitesimally close tangent spaces, and the tool to do so is indeed called a connection, or covariant derivative.

For a scalar field S the covariant derivative is $\nabla_\alpha S = \partial_\alpha S$.

A torsionless manifold is one in which

$$[\nabla_\mu, \nabla_\nu] S = 0\,, \tag{12}$$

for a scalar field S. This means that

$$\nabla_{[\mu}\nabla_{\nu]} S = \nabla_{[\mu}\partial_{\nu]} S = \partial_{[\mu}\partial_{\nu]} S - \Gamma^\alpha_{[\mu\nu]}\partial_\alpha = 0 \implies \Gamma^\alpha_{[\mu\nu]} = 0\,. \tag{13}$$

Parallel transport: intuitively, we move along a curve and keep the angle with respect to the tangent vector constant. Formally, if u^μ is the tangent vector to the curve and V^μ is the vector we want to transport, we set $u^\mu \nabla_\mu V^\nu = 0$.

The Riemann tensor is defined as the commutator of the covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta\,, \tag{14}$$

and it can be expressed in terms of the Christoffel symbols as

$$R^{\mu}_{\nu\rho\sigma} = -2\left(\Gamma^{\mu}_{\nu[\rho,\sigma]} + \Gamma^{\beta}_{\nu[\rho}\Gamma^{\mu}_{\sigma]\beta}\right). \tag{15}$$

Geodesics: they are "the straightest possible path between two points". They stationarize the proper length. Formally, they are curves whose tangent vector is parallel-transported along the curve.

We actually do not need to say that the derivative of the tangent vector with respect to the parameter is zero: it can be nonzero, as long as it is parallel to the tangent vector.

So, we could say that

$$h_{\nu\rho}\left(u^{\mu}\nabla_{\mu}u^{\nu}\right) = 0\,, \tag{16}$$

?

The path that a massive particle follows in the absence of external forces is a geodesic. We can describe the separation between two particles which follow geodesics: this is described by the equation of geodesic deviation. We take a geodesic x^{μ} and another $y^{\mu} = x^{\mu} + \xi^{\mu}$, with ξ^{μ} being (at least initially) small.

We can choose a coordinate system in which $\Gamma^{\mu}_{\nu\rho} = 0$. So,

$$\left.\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}u^2}\right|_{P} = 0\,, \tag{17}$$

$$\left.\left(\frac{\mathrm{d}^2 y^{\mu}}{\mathrm{d}u^2} + \Gamma^{\mu}_{\nu\rho}\frac{\mathrm{d}y^{\nu}}{\mathrm{d}u}\frac{\mathrm{d}y^{\rho}}{\mathrm{d}u}\right)\right|_{P} = 0\,, \tag{18}$$

where u is the tangent vector to the geodesics. We approximate the Christoffel symbols to first order as

$$\left.\Gamma^{\mu}_{\nu\rho}\right|_{Q} = \xi^{\alpha}\partial_{\alpha}\Gamma^{\mu}_{\nu\rho}\,. \tag{19}$$

If we subtract the two, we get

$$\ddot{\xi}^{\mu} + \left(\partial_{\alpha}\Gamma^{\mu}_{\nu\rho}\right)\dot{x}^{\nu}\dot{x}^{\rho}\xi^{\alpha} = 0\,, \tag{20}$$

but the first term is not an intrinsic derivative: that would be given by

$$\frac{\mathrm{D}^2\xi^{\mu}}{\mathrm{D}u^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{\xi}^{\mu} + \Gamma^{\mu}_{\nu\rho}\xi^{\nu}\dot{x}^{\rho}\right) = \ddot{\xi}^{\mu} + \left(\partial_{\alpha}\Gamma^{\mu}_{\nu\rho}\right)\xi^{\alpha}\dot{x}^{\nu}\dot{x}^{\rho}\,, \tag{21}$$

which means that

$$0 = \frac{\mathrm{D}^2\xi^{\mu}}{\mathrm{D}u^2} + \left(\partial_{\alpha}\Gamma^{\mu}_{\nu\rho} - \partial_{\rho}\Gamma^{\mu}_{\nu\alpha}\right)\xi^{\alpha}\dot{x}^{\mu}\dot{x}^{\nu} = \frac{\mathrm{D}^2\xi^{\mu}}{\mathrm{D}u^2} + R^{\mu}_{\nu\rho\sigma}u^{\nu}u^{\rho}\xi^{\sigma}\,. \tag{22}$$

The gravitational field is described by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}\,. \tag{23}$$

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Remember that the version of the slides which appears during the lesson might be updated.

There might be some disruption in late March because the professor is having a child.

0.3 Linearized GR

Let us assume that our metric tensor is almost flat: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\left|h_{\mu\nu}\right| \ll 1$.

This is a coordinate dependent statement: however the physical situation is clear — almost flat spacetime — and the way we will proceed is to expand in h.

Do note that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$.

Choosing a reference in which these components are small limits our gauge freedom: we will be able to do only transformations which preserve the condition.

We will be able to do global Lorentz transformations:

$$g' = \Lambda\Lambda g\,, \tag{24}$$

therefore

$$g' = \Lambda\Lambda(\eta + h) = \eta + \Lambda\Lambda h\,, \tag{25}$$

so the flat metric does not change, while h changes to $h' = \Lambda\Lambda h$. We are omitting indices, they are in the usual positions.

A more general class of transformation is given by those which can be expressed as

$$x^{\mu} \to x'^{\mu} + \xi^{\mu}\,, \tag{26}$$

so the Jacobian will look like

$$\delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu}\,, \tag{27}$$

and we ask that $\partial\xi$ is small — formally, first order in $h_{\mu\nu}$. This yields

$$g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}\,. \tag{28}$$

For an alternative reference on this derivation, see the notes on General Relativity in the course by Marco Peloso [TM20, section 10].

Now, we wish to linearize the Riemann tensor: we start from the Christoffel symbols. We get

$$\begin{aligned}
\Gamma^{\sigma}_{\mu\nu} &= \frac{1}{2} g^{\sigma\rho}\left(2g_{\rho(\mu,\nu)} - g_{\mu\nu,\rho}\right) &\text{(29a)}\\
&= \frac{1}{2}\left(\partial_{\mu} h^{\sigma}_{\nu} + \partial_{\nu} h^{\sigma}_{\mu} + \partial^{\sigma} h_{\mu\nu}\right)\,, &\text{(29b)}
\end{aligned}$$

and now in the Riemann tensor $R = \partial\Gamma + \Gamma\Gamma$ the $\Gamma\Gamma$ terms are second order in h, so we ignore them. Then, we get the simplified expression

$$R^{\sigma}_{\mu\nu\rho} = \frac{1}{2}\left(\partial_{\nu}\partial_{\mu} h^{\sigma}_{\rho} + \partial_{\rho}\partial^{\sigma} h_{\mu\nu} - \partial_{\nu}\partial^{\sigma} h_{\mu\rho} - \partial_{\rho}\partial_{\mu} h^{\sigma}_{\nu}\right)\,, \tag{30}$$

so the Ricci tensor — which we will set to zero in the vacuum — will be

$$R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma} = \frac{1}{2}\left(\partial_{\nu}\partial_{\mu} h + \Box h_{\mu\nu} - \partial_{\nu}\partial_{\sigma} h^{\sigma}_{\mu} - \partial_{\sigma}\partial_{\mu} h^{\sigma}_{\nu}\right)\,, \tag{31}$$

and the Ricci scalar is

$$R = \eta^{\mu\nu} R_{\mu\nu} = \Box h - \partial_\nu \partial_\sigma h^{\sigma\nu} \,. \tag{32}$$

Now, the field equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx \frac{1}{2} \Big[\partial_\nu \partial_\mu h + \Box h_{\mu\nu} - \partial_\nu \partial_\sigma h^\sigma_\mu - \partial_\sigma \partial_\mu h^\sigma_\nu - \eta_{\mu\nu} \big(\Box h - \partial_\nu \partial_\sigma h^{\sigma\nu} \big) \Big] = -\frac{1}{M_P^2} T_{\mu\nu} \,. \tag{33}$$

Now, we define the trace reverse perturbation as

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \,, \tag{34}$$

so the doubly-trace reverse is the perturbation itself, and $\overline{h} = -h$.

Now we shall use Lorenz gauge[1]

We shall use our gauge freedom. How does the trace-reverse perturbation transform?

$$\overline{h}'^{\mu\rho} = h^{\mu\rho} - 2\partial^{(\mu} \xi^{\rho)} - \frac{1}{2} \eta^{\mu\rho} \big(h - 2\partial_\sigma \xi^\sigma \big) \tag{35a}$$

$$= \overline{h}^{\mu\rho} - 2\partial^{(\mu} \xi^{\rho)} + \eta^{\mu\rho} \partial_\sigma \xi^\sigma \,. \tag{35b}$$

The derivatives of the new and old perturbations differ by

$$\partial_\rho \overline{h}'^{\mu\rho} - \partial_\rho \overline{h}^{\mu\rho} = -\Box \xi^\mu \,, \tag{36}$$

so we can choose a variable change ξ^μ such that $\partial_\rho \overline{h}'^{\mu\rho}$.

Then, the linearized field equations will read

$$\Box \overline{h}_{\mu\nu} + \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} - \partial_\nu \partial_\rho \overline{h}^\rho_\mu - \partial_\mu \partial_\rho \overline{h}^\rho_\nu = -2 \frac{T_{\mu\nu}}{M_P^2} \,, \tag{37}$$

which finally yields

$$\Box \overline{h}_{\mu\nu} = -\frac{2 T_{\mu\nu}}{M_P^2} \,. \tag{38}$$

Comments on the linearized equations Since we expanded in $\eta_{\mu\nu}$, the quantities $h_{\mu\nu}$ have a geometric meaning but we are treating them as 16 scalar fields.

When we look at the geodesic equations, we get a prediction of the gravity having no effect on matter. We are treating gravity as a linear theory, so we have the superposition principle.

We are ignoring the physical principle that "gravity gravitates": curvature of spacetime is associated to a SEMT in a nonlinear way.

[1] This is not the same as Loren*tz*, after whom Lorentz covariance is called.

GWs in empty space We set $T_{\mu\nu}$ to zero, so we get

$$\Box \bar{h}_{\mu\nu} = 0\,, \tag{39}$$

which is the usual wave equation, with solutions $\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\lambda x^\lambda}$.

In general $A_{\mu\nu}$ is symmetric, constant, complex. k_λ is constant and real, and we must have

$$\eta^{\rho\sigma} k_\rho k_\sigma A_{\mu\nu} e^{ik\cdot x} \implies k^2 = 0\,. \tag{40}$$

In order to have these conditions, we must still have

$$\partial_\mu \bar{h}^{\mu\nu} = A^{\mu\nu} k_\mu e^{ik\cdot x} = 0 \implies A^{\mu\nu} k_\nu = 0\,. \tag{41}$$

The conjugate of the wave equation also holds, so after our manipulations we will always be able to take the real part.

We still have gauge freedom: we can perform transformations if they satisfy $\Box \xi^\mu = 0$.

We define

$$\xi^{\mu\nu} = \partial^\mu \xi^\rho + \partial^\rho \xi^\mu - \eta^{\mu\rho} \partial_\sigma \xi^\sigma\,, \tag{42}$$

and we will have $\Box \xi^{\mu\nu} = 0$ since the Dalambertian commutes with the other derivatives.

So, if $\bar{h}^{\mu\rho}$ satisties the Vacuum field equations, then $\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \xi^{\mu\nu}$ also does.

So, we can use the 4 functions ξ^μ to set 4 constraints on $\bar{h}^{\mu\rho}$:

$$\bar{h}^{0i}_{TT} = 0 \tag{43a}$$

$$\bar{h}_{TT} = 0\,, \tag{43b}$$

which conveniently means that $\bar{h}_{\mu\nu} = h_{\mu\nu}$. This is called TT-gauge.

Combined with the Lorenz gauge, which says

$$0 = \partial_\rho \bar{h}^{0\rho}_{TT} = \partial_0 \bar{h}^{00}_{TT} = 0\,, \tag{44}$$

which means that the metric element is constant, so we can rescale time in order to set it to zero.

Also, the other Lorenz gauge conditions are

$$0 = \partial_\rho \bar{h}^{j\rho}_{TT} = \partial_i \bar{h}^{ji}_{TT}\,, \tag{45}$$

which means that, of the 6 potentially free components of the h^{ij}, we actually have only 2 degrees of freedom.

Now, if we assume $\vec{k} = k\hat{z}$ then we will have $k^\mu = (k, 0, 0, k)$; also, in the matrix $A_{\mu\nu}$ we will have $A^{ij} k_j = 0$, so $A^{i3} = 0$.

Then, in full generality under our gauge choices we shall have

$$\bar{h}^{\mu\nu}_{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{ik(t-z)}\,. \tag{46a}$$

For a generic direction of propagation, we can define a projector onto the direction orthogonal to the direction of propagation k_i: $P_{ij} = \delta_{ij} - k_i k_j$, so we will have

$$A^{ij}_{TT} = \left(P^i_k P^j_l - \frac{1}{2} P^{ij} P_{kl} \right) A^{kl} \,. \tag{47}$$

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0.4 The physical effects of gravitational waves

We want to discuss how we can build instruments which can detect gravitational waves.

An open question for decades was to see whether the effects of gravitational waves could be removed using a proper gauge choice. There was a conference in Chapel Hill (?) which showed examples of non-removable gravitational wave effects, such as the "beads on a stick", which move and dissipate energy if a GW passes through them: a non removable effects.

What happens to free particles in the TT gauge? The geodesic equation for the spatial indices reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \,, \tag{48}$$

where the parameter τ parametrize our curve. We assume that the particle starts out at rest: then its four-velocity is $\mathrm{d}x^\mu / \mathrm{d}\tau = (\mathrm{d}x^0 \big/ \mathrm{d}\tau \,, \vec{0})$. So, we get the simplification

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{00} \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 \,, \tag{49}$$

and in linearized gravity

$$\Gamma^i_{00} \approx \frac{1}{2} \Big(2 \partial_0 h^i_0 - \partial^i h_{00} \Big) = 0 \tag{50}$$

if we use the TT gauge. This means that the derivative of the velocity is zero: so, the velocity of a stationary particle remains zero indefinitely. Let us consider geodesic deviation between two particles instead: say that the first particle has the geodesic $x(\tau)$ and the second is $x(\tau) + \xi(\tau)$. Their geodesic equations will read

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{51a}$$

$$\frac{\mathrm{d}^2 (x^\sigma + \xi(\sigma))}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\mu\nu}(x + \xi) \frac{\mathrm{d}x^\mu + \xi^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu + x^\nu}{\mathrm{d}\tau} = 0 \,, \tag{51b}$$

which we can expand to first order using the perturbation: $\Gamma^\sigma_{\mu\nu}(x + \xi) = \Gamma^\sigma_{\mu\nu}(x) + \partial_\gamma \Gamma^\sigma_{\mu\nu} \xi^\gamma$, using which and expanding we finally get

$$\frac{\mathrm{d}^2 \xi^\sigma}{\mathrm{d}\tau^2} + 2\Gamma^\sigma_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma \partial_\gamma \Gamma^\sigma_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \,, \tag{52}$$

so if we restrict ourselves to only spatial components, and assume that the particles start out stationary we get

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}\tau^2} = -2\Gamma^i_{0\nu}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma\partial_\gamma\Gamma^i_{00}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\,, \tag{53}$$

so, using the expressions for the Christoffel symbols in the TT gauge we get

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}\tau^2} = -2c\Gamma^i_{0j}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau} = -c\partial_0 h^{ij}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau}\,, \tag{54}$$

so, parallel geodesics remain parallel: if the separation initially is stationary, it will remain so.

The issue is that in the TT gauge we are using a special set of coordinates which "follow" the gravitational wave. We see no change in coordinate distance since the coordinates are moving around with the gravitational wave: we did a coordinate change using ξ^μ satisfying $\Box\xi^\mu = 0$, so the coordinates are harmonically moving, together with the GW.

It is like we defined wave-like coordinates, "gauging away" the wave-like motion.

We can overcome this issue by calculating *proper distances* instead of coordinate distances: for two space-like separated objects along the x axis we get

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}x^2 + h^{TT}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\ . \tag{55}$$

Let us apply this to the case of a GW propagating along the z axis, for particles separated along the x axis. The full metric perturbation looks like

$$h^{\mu\nu}_{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{i\omega(t-z/c)}\,, \tag{56a}$$

then the distance, in the case of an h_+ polarized wave, becomes

$$s = (x_1 - x_2)\sqrt{1 + h_+\cos(\omega t)} \approx (x_1 - x_2)\left(1 + \frac{1}{2}h_+\cos(\omega t)\right)\,. \tag{57}$$

So, the fractional change is given by $h_+/2$.

For two general events separated by the 4-vector L:

$$s^2 = \left(\eta_{\mu\nu} + h_{\mu\nu}\right)L^\mu L^\nu \approx L\left(1 + \frac{1}{2}h_{ij}L^i L\right)\,. \tag{58}$$

We would, however, like to work in coordinates. The useful frame to define is the *free falling frame*, whose coordinates are rigid and not perturbed by the GW.

In order to build such a frame, a local inertial frame which will be inertial, we will need to define 4 orthogonal vectors on the point P:

$$\eta_{\mu\nu}e^\mu_\alpha e^\nu_\beta = \eta_{\alpha\beta}\,. \tag{59}$$

Consider a geodesic through point P whose tangent vector at P is a unit vector $\hat{n}$. If it is spacelike, we parametrize it by s (defined with $\mathrm{d}s^2$ from the metric), if it is timelike we parametrize it with τ (defined by $\mathrm{d}\tau^2 = -\mathrm{d}s^2$). We call λ a generic one of those.

Now, the coordinates of point Q are generically $\lambda\hat{n}$, if the geodesic starting with unit vector $\hat{n}$ reaches Q when its parameter is λ.

We can reach almost every point this way, the points which are only connected through null geodesics can be reached by continuity, and in a small enough region the coordinates of a point Q are unique — that is, the geodesics do not cross.

In this frame, then, $g_{\mu\nu}(P) = \eta_{\mu\nu}(P)$; also, in the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\,, \tag{60}$$

we have that the second derivatives are zero since $x^\mu(\lambda)$ is linear, so we must have $\Gamma^\mu_{\nu\rho}n^\nu n^\rho = 0$. This must be true for any unit vector, therefore we have $\Gamma^\mu_{\nu\rho} = 0$. The system giving $g_{\mu\nu,\rho}(P)$ from $\Gamma^\mu_{\nu\rho}$ is nondegenerate, so we also have $g_{\mu\nu,\rho}(P) = 0$.[2] These are called **Riemann normal coordinates**.

The conditions on the metric and its derivatives only hold at the point. We can do slightly better with *Fermi normal coordinates*, where we require a gyroscope's angular momentum to be parallel-transported along the geodesics.

How do we make such a frame? we use free falling particles: we could put them in orbit. This is not actually that simple: consider *drag-free satellites*.

Consider a particle orbiting the Sun. The Sun's radiation pressure pushes the particle away from a geodesic.

Maybe we could put a thrusted spacecraft around our test mass: it balances the sun's radiation pressure; we measure the distance to the test mass without touching it, and then balance the thrusters by keeping at a constant distance from it.

This is the idea behind LISA. The distances between the spacecrafts should be about 5 Gm apart. We do not measure the distances between the spacecrafts, but instead the distances between the test masses inside them, which are 2 kg, 4.6 cm side, gold-platinum shielded cubes: this is known as *drag-free* navigation.

The interferometric measurements have pico-meter (10^{-12} m) sensitivity. It takes about 30 s for light to move between the mirrors: this time-delayed interferometry needs special consideration.

We have a *gravitational reference sensor*, a cubic shell around the cube: we keep measuring the distances between the two. We also need to discharge the masses, otherwise electrostatic forces are too strong. Also, the thrusters need to be very weak, on the order of the µN.

The LISA Pathfinder mission checked all of these boxes, except for time-delayed interferometry. It only used one spacecraft, and measured how well the drag-free navigation worked.

Using two masses, we need to account for the gravitational pull between them.

Even accounting for this by relaxing the acceleration precision requirement 10-fold, the results were exceptional.

[2] Do note that this reasoning works only at the point, since if we moved along a geodesic we do not have access to the other unit vectors anymore (in these coordinates).

Let us discuss the sources of noise: at high frequencies, inertia prevents a force from creating significant displacement. This applies to external forces, not to gravitational forces, since the latter are proportional to the mass. So, there is a limit at high frequencies because of our inability to measure that fast.

At low frequencies, it is easy to measure, but it is hard to verify whether the mass is indeed in free fall.

In the end, it was verified that we can do

$$s_a^{1/2} \leq 3 \times 10^{-14}\,\mathrm{m/s^2/\sqrt{Hz}} \tag{61}$$

at 1 mHz. Solar radiation pressure is two orders of magnitude higher.

We can have issues with the parasitic coupling of the test mass to the spacecraft.

Let us now come back to theory by discussing the *proper detector frame*: coordinates defined by a rigid ruler. Rigid rulers do not really exist, but we can approximate it well enough. If the gravitational pull is small compared to the restoring forces in the ruler than its length will approximately not change.

Let us put ourselves in a free falling frame in Fermi local coordinates; in the origin, the metric is always flat. Let us expand to second order in the spatial coordinates

$$\begin{aligned} g_{\mu\nu}(x) &\approx g_{\mu\nu}(0) + x^i \partial_i \, g_{\mu\nu}\Big|_{x=0} + \frac{1}{2} x^i x^j \, \partial_i \partial_j g_{\mu\nu}\Big|_{x=0} && \text{(62a)} \\ &= \eta_{\mu\nu} + \frac{1}{2} x^i x^j \, \partial_i \partial_j g_{\mu\nu}\Big|_{x=0} \,, && \text{(62b)} \end{aligned}$$

which we can rewrite in terms of the Riemann tensor by making use of the expression of the Riemann tensor in the LIF: we get

$$\mathrm{d}s^2 \approx -c^2\,\mathrm{d}t^2 \left(1 + R_{0i0j} x^i x^j\right) - 2c\,\mathrm{d}t\,\mathrm{d}x^i \left(\frac{2}{3} R_{0ijk} x^j x^k\right) + \mathrm{d}x^i\,\mathrm{d}x^j \left(\delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l\right), \tag{63}$$

so the corrections are of the order $\mathcal{O}\left(r^2/L_B^2\right)$, where r^2 is the square distance from the origin, while L_B is the typical spatial scale of the variation of the metric, such that $R_{0ijk} = \mathcal{O}\left(L_B^{-2}\right)$. This L_B is the wavelength of the GW if we are describing a GW.

So, this coordinate description works as long as the scale of the region we are describing is small compared to the wavelength of the GW.

Monday 2020-3-23, compiled 2020-04-10

We were discussing the proper detector frame: it is defined by rigid rulers around one point in free fall.

What happens in this frame? the equation of geodesic deviation looks like

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + 2\Gamma^i_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} + \xi^\sigma \left(\partial_\sigma \Gamma^i_{\nu\rho}\right) \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} && \text{(64a)} \\ &= \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + \xi^j \left(\partial_j \Gamma^j_{00}\right) \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2 , && \text{(64b)} \end{aligned}$$

where we made the nonrelativistic approximation, and accounted for the fact that in this frame we have

$$\Gamma^{\mu}_{\nu\rho} = 0 \qquad \text{and} \qquad \partial_0 \Gamma^i_{0j} = 0 \implies R^i_{0j0} = \partial_j \Gamma^j_{00}\,, \tag{65}$$

so we have

$$0 = \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + R^i_{0j0} \xi^j \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 , \tag{66}$$

and since the Riemann tensor is *invariant* (rather than covariant) under coordinate transformations, we can compute it in the TT gauge:

$$R^i_{0j0} = -\frac{1}{2} \partial_0 \partial_0 h_{ij} = -\frac{1}{2} \ddot{h}^{TT}_{ij}\,, \tag{67}$$

so our final result reads

$$\ddot{\xi}^i = \frac{1}{2} \ddot{h}^{TT}_{ij} \xi^j\,, \tag{68}$$

which is physically significant since we can interpret the effect of the GW as that of a Newtonian force:

$$F^i = \frac{m}{2} \ddot{h}^{TT}_{ij} \xi^j\,. \tag{69}$$

How do we decide whether these approximations are justified? We need to impose $r^2/L_B^2 \ll 1$, where L_B is the scale of the variations of the metric while r is the scale of our detector.

This allows us to see that the GWs are transverse: for a wave along the z direction we get

$$\ddot{\xi}^3 = \frac{1}{2} \ddot{h}^{TT}_{3j} \xi^j = 0\,, \tag{70}$$

while we can do the calculations for small displacements along x or y: for the plus polarization we get

$$\delta x = \frac{1}{2} h_+ x_0 \cos(\omega t) \tag{71a}$$

$$\delta y = -\frac{1}{2} h_+ y_0 \cos(\omega t)\,, \tag{71b}$$

while for the cross polarization we get

$$\delta x = \frac{1}{2} h_\times y_0 \cos(\omega t) \tag{72a}$$

$$\delta y = \frac{1}{2} h_\times x_0 \cos(\omega t)\,. \tag{72b}$$

(possibly minus) We can also have circular polarizations, like $h_+ \pm i h_\times$.

What about Earth-based detectors? They are definitely not free-falling, in fact:

1. at zeroth order the metric is flat;
2. at first order we have the Newtonian forces;
3. at second order we get the GW and the background metric.

So, how do we distinguish these effects? We can isolate them by Fourier analysis, there will be a dominant frequency.

How do we know that our “free-falling” mass is moving because of the GW and not because of other noises? We can bound the mass and distance of a source of GW we detect in a certain frequency region.

At the low frequencies, we have objects on the Earth making noise.

Now, let us move towards GW *generation*.

Chapter 1

GW generation

Some assumptions: we will

1. expand around flat spacetime;
2. consider nonrelativistic systems;
3. assume the stress energy tensor is conserved: to first order, this reads

$$\partial^\mu T_{\mu\nu} = 0\,. \tag{1.1}$$

If the system is nonrelativistic, we have that it is also large with respect to its Schwarzschild radius:

$$E_{\text{kin}} = -\frac{1}{2}U \implies \frac{1}{2}\mu v^2 = \frac{1}{2}G\frac{\mu M}{r} \implies \frac{v^2}{c^2} = \frac{GM}{c^2 r} = \frac{R_S}{r} \ll 1\,. \tag{1.2}$$

The expression of the gravitational force is that since we have $m_1 m_2 = \mu M = m_1 m_2/(m_1 + m_2) * (m_1 + m_2)$.

The lambda tensor is a projector onto the subspace orthogonal to $\hat{n} = \vec{k}/\left|\vec{k}\right|$. It obeys some identities [see slides].

In order to solve the linearized equations, we use Green's functions:

$$\Box_x G(x-y) = \delta^{(4)}(x-y)\,, \tag{1.3}$$

where x is our variable, while y is just a parameter.

So, we get

$$-2k \int \mathrm{d}^4 y\, \Box_x G(x-y) T_{\mu\nu}(y) = -2k \int \mathrm{d}^4 y\, \delta^{(4)}(x-y) T_{\mu\nu}(x)\,, \tag{1.4}$$

so we will have as a solution a superposition of the homogeneous solution, and the source term:

$$\overline{h}_{\mu\nu}(x) = \overline{h}^{(0)}_{\mu\nu}(x) - 2k \int \mathrm{d}^4 y\, G(x-y) T_{\mu\nu}(y)\,. \tag{1.5}$$

To get the solution, we do

$$\partial_\mu \partial^\mu G(x^\sigma) = \delta^{(4)}(x^\sigma)\,, \tag{1.6}$$

so if we integrate over a hypersphere of radius $r = |\vec{x}|$ we have

$$\int_V \mathrm{d}^4x\, \delta^{(4)}(x^\sigma) = 1 \tag{1.7a}$$

$$= \int_V \mathrm{d}^4x\, \partial_\mu \partial^\mu G(x^\sigma) \tag{1.7b}$$

$$= \int \mathrm{d}S\, \Big(\partial_\mu G(x^\sigma)\Big) n^\mu\,, \tag{1.7c}$$

but the only points which matter are in the future light-cone. We get $\mathrm{d}S = c\,\mathrm{d}t\,\mathrm{d}\Omega$, and we call $n^\mu \partial_\mu = \partial_r$:

$$1 = \int \mathrm{d}S\, \Big(\partial_\mu G(x^\sigma)\Big) n^\mu \tag{1.8a}$$

$$= 4\pi r^2 \int_0^\infty \mathrm{d}t\, \partial_r \big(f(r)\delta(ct - r)\big) c\,, \tag{1.8b}$$

but if we integrate by parts the first integral vanishes, and the second is equal to 1:

$$4\pi r^2 \partial_r f(r) = 1 \implies f(r) = -\frac{1}{4\pi r} \implies G(x^\sigma) = -\frac{\delta(x^0 - |\vec{x}|)}{4\pi |\vec{x}|} \theta_H(x^0)\,, \tag{1.9}$$

and after some more passages we finally get

$$\bar{h}_{\mu\nu}(t, \vec{x}) = \frac{4G}{c^4} \int \mathrm{d}^3y\, \frac{T_{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|}\,. \tag{1.10}$$

We can move to the TT gauge if we are outside of the source, since there the equation $\Box \bar{h}_{\mu\nu}$ satisfies the FE.

Far from the source, $|\vec{x}| \gg |\vec{y}|$ for any $\vec{y}$ inside the source. So, we can expand:

$$|\vec{x} - \vec{y}| = r(1 - \frac{\vec{y} \cdot \hat{n}}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right))\,. \tag{1.11}$$

If we want to keep only the terms at $\mathcal{O}(1/r)$ we get

$$h^{TT}_{ij}(t, \vec{x}) = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl} \int \mathrm{d}^3y\, \frac{1}{|\vec{x} - \vec{y}|} T_{kl}\left(t - \frac{r}{c} + \frac{\vec{y} \cdot \hat{n}}{c}, \vec{y}\right)\,. \tag{1.12}$$

If the object is moving periodically with frequency ω, then we will have

$$\frac{1}{\omega} \sim \frac{d}{v}\,, \tag{1.13}$$

so we assume $d/c \ll d/v$. Expanding the stress energy tensor, we find

$$h^{TT}_{ij}(t, \vec{x}) = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl} \int \mathrm{d}^3y \left[T_{kl} + \frac{y^m n^p}{c} \partial_0 T_{kl} + \frac{y^m y^p n^m n^p}{2c^2} \partial_0^2 T_{kl} + \dots\right]\,. \tag{1.14}$$

If we define the multiplole moments:

$$S^{ijm_1m_2\cdots} = \int \mathrm{d}^3x\, T^{ij} \prod_\alpha x^{m_\alpha}\,. \tag{1.15}$$

[definitions and stuff, too fast to write]
We get that the quadrupole term dominates:

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}S^{kl}\,. \tag{1.16}$$

The quadrupole moment is defined as

$$Q^{kl} = M^{kl} - \frac{1}{3}\delta^{kl}M_{ii} = \int \mathrm{d}^3x\,\rho(t,\vec{x})\left(x^ix^j - \frac{1}{3}r^2\delta_{ij}\right). \tag{1.17}$$

Since $\Lambda_{ij,kl}\delta^{kl} = 0$, we can substitute Q for M: we get

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{Q}^{kl}\left(t - \frac{r}{c}\right) = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{TT}\left(t - \frac{r}{c}\right). \tag{1.18}$$

If a wave is propagating along the $\hat{n} = \hat{z}$ direction, we get

$$\Lambda_{ij,kl}\ddot{M}_{kl} = \begin{bmatrix} \left(\ddot{M}_{11} - \ddot{M}_{22}\right)/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & \left(\ddot{M}_{22} - \ddot{M}_{11}\right)/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.19a}$$

Last time we saw how to expand the solution into multipole moments, and focused on the quadrupole. Friday 2020-3-27, compiled 2020-04-10

We finally got, for a GW propagating along $\hat{z}$:

$$h_+ = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right) \tag{1.20a}$$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}\,. \tag{1.20b}$$

But how do we compute the full angular distribution? We can brute-force it using the full Λ projection tensor, but a more conceptual way is to put ourselves in a frame in which the generic vector $\hat{n}$ is $\hat{z}$. Then we use the rotation matrices: we need two of them, one for each unit vector in a rank-2 tensor M_{ij}. Then we use the simple expression for h_+ and $h_\times$, substituting in the $\ddot{M}_{11}$, $\ddot{M}_{12}$ and so on in the primed system.

1. There is no monopole radiation: $\dot{M} = 0$, since mass is conserved.

2. We can move the origin so that the dipole is zero: $M^i = 0$. This corresponds to linear momentum being conserved: $\dot{P}^i = 0$. This is in stark contrast to Electromagnetism: there, we cannot eliminate the dipole radiation. This is due to there being positive and negative electric charges, while there are no negative masses.

3. We did not account for back-action: our GWs do not carry "away" energy or momentum. This is unphysical, we will account for it!

A stress-energy tensor for a system of point masses looks like

$$T^{\mu\nu}(t,\vec{x}) = \sum_A \frac{p_A^\mu p_A^\nu}{\gamma_A m_A} \delta^{(3)}(x - x_A(t)) \,. \tag{1.21}$$

In order to apply our system, our system must be closed: we cannot consider any particle trajectory, since they must move on geodesics in order to conserve the stress-energy tensor.

However, we can use the relative coordinate in a self-gravitating system: then we have the relative $x_0 = x_1 - x_2$, the total mass $m = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$, an the center of mass position $x_{\text{CM}} = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$.

If we set the position of the COM to zero identically, we find

$$M^{ij} = \mu x_0^i x_0^j \,. \tag{1.22}$$

We consider two particles in the XY plane moving on a trajectory, which for now we assign. so that we get

$$x_0(t) = R\cos\left(\omega_s t + \frac{\pi}{2}\right) \tag{1.23a}$$

$$y_0(t) = R\sin\left(\omega_s t + \frac{\pi}{2}\right) \tag{1.23b}$$

$$z_0(t) = 0 \,, \tag{1.23c}$$

so we find

$$\ddot{M}_{11} = -\ddot{M}_{22} = 2\mu R^2 \omega_s^2 \cos(2\omega_s t) \tag{1.24a}$$

$$\ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin(2\omega_s t) \,, \tag{1.24b}$$

which means the GW emission has twice the rotational frequency.

If we look at emission in a generic direction $\hat{n}$, described by the angles θ and φ, we will receive

$$h_+(t,\theta,\varphi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\frac{1+\cos^2\theta}{2}\cos\left(2\omega_s t_{\text{ret}} + 2\varphi\right) \tag{1.25a}$$

$$h_\times(t,\theta,\varphi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos(\theta)\sin\left(2\omega_s t_{\text{ret}} + 2\varphi\right) \,, \tag{1.25b}$$

so the two scale differently as θ varies.

Do note that $h_+ + h_\times$ looks like a circular polarization moving counter-clockwise, while $h_+ - h_\times$ moves clockwise. This is what we see for $\theta = 0, \pi$; on the other hand for $\theta = \pi/2$ we only have h_+.

Plot effects!

Some numbers: the Earth-Sun system, seen 1×10^5 lyr away, has a strain on the order 5×10^{-32}. Let us try a BNS in our galaxy: we use Newtonian orbits to model the binary.

Note that the strain is inversely proportional to r: this is because we are not measuring the intensity, but directly the amplitude.

If we put in the expression for the radius from Kepler's law: we find

$$h_+ \sim h_\times \sim \frac{4G^{5/3}\omega_s^{2/3}\mu M^{2/3}}{rc^4} \,. \tag{1.26}$$

Let us consider a BNS with masses $m_1 = m_2 = 1.5M_\odot$ in our galaxy: then, the strain will be of the order 10^{-20}.

We have seen a BNS outside of our galaxy, at 40 Mpc away.

Do this with GW150914: $m_1 \sim 36M_\odot$, $m_2 \sim 31M_\odot$, $f_s \sim 250\,\text{Hz}$, $r \sim 440\,\text{Mpc}$.

When their distance becomes too small, we cannot model them using Kepler's laws anymore.

Now, back-action: we saw no energy loss, which is "by design": GW energy is quadratic in h, while our theory is linear.

Another approach: if our theory is local, it cannot describe the energy of GWs. GWs do carry energy, however we cannot describe it in a local way, since for any single particle we can gauge them away. However, we can look at the tidal effects between two particles.

We need to perform an average in spacetime.

If we introduce a stress-energy tensor for our GWs, this will curve the background spacetime: then we have

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{1.27}$$

but how do we decide which deviation from flat is part of the background and what is part of h?

There is no formal way to define this, but we can use a heuristic, based on what they describe. We say that the background $\overline{g}_{\mu\nu}$ varies spatially slowly, across a distance L_B, such that all the GWs we are considering have reduced wavelengths ($k = \lambda/2\pi$) which are smaller than a fixed maximum wavelengths, so that $\lambda \leq \lambda_{GW} \ll L_B$.

This is like distinguishing waves and tides in the ocean: intuitively it is easy to see how they differ, based on the scale of their effects.

We also impose that all the GWs vary much faster than the background, temporally: $f > f_{GW} \gg f_B$. Do not that these two are independent, since while GWs travel at the speed of light[1] the background does not. We refer to both as the short-wave approximation.

How do GW detectors then distinguish GW from background? If we impose $f_{GW} = c/\lambda_{GW}$ with λ_{GW} around a kilometer we get $f \sim 300\,\text{kHz}$. This is not interesting, and technically difficult. Also, ground-based detectors do not measure the metric along the length, but only an integrated effect.

Instead, the detectors monitor local temporal variations of $g_{\mu\nu}(t, x_0)$. So, our detectors work best between 100 Hz to 1000 Hz; the Earth's gravitational field is not smooth along the corresponding length scale. However, it's close to static: its variations are slower than a few Hz. So, we can apply our distinction: GW and background can be distinguished in frequency.

Monday 2020-3-30, compiled 2020-04-10

[1] Which we know to within a part in 10^{-14}.

Last time we discussed: how can we distinguish what is a GW and what is not?

We do this by separating them by frequency. We cannot really measure the nature of tensor perturbation of GWs, since we are only measuring integrated effects.

We can precisely map the effect of the GW in time, by sampling with a frequency which is much higher than the one of the GW.

So, let us fo this formally: if we expand the Ricci tensor to second order in $h_{\mu\nu}$ we get

$$\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) = \overline{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}\,, \tag{1.28}$$

where the second order in $h_{\mu\nu}$ term, $R^{(2)}_{\mu\nu}$, has both high and low frequency components.

This is because, when we have a term like

$$\left(\sin(\omega_1 t)\sin(\omega_2 t)\right)^2\,, \tag{1.29}$$

by the prostapheresis formulas we will gets terms like $\omega_1 + \omega 2$ and $\omega_1 - \omega_2$, one of which is high frequency and the other is low frequency, since we are considering a high-frequency wavepacket.

Recall, we expand in two parameters: h and λ / L_B. Then, we have

$$\overline{R}_{\mu\nu} \sim \partial^2 \overline{g}_{\mu\nu} \sim \frac{1}{L_B^2}\,, \tag{1.30}$$

while

$$\left[R^{(2)}_{\mu\nu}\right]^{\text{low frequency}} \sim (\partial h)^2 \sim \left(\frac{h}{\lambda}\right)^2. \tag{1.31}$$

See Maggiore, page 31 for more details. So, the EFE low-frequency components are

$$\overline{R}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{low freq}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{low freq}}\,, \tag{1.32}$$

so we either have $T_{\mu\nu} = 0$, in which case

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda}\right)^2\,, \tag{1.33}$$

or

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda}\right)^2 + T_{\mu\nu} \gg \left(\frac{h}{\lambda}\right)^2\,, \tag{1.34}$$

so we will have $h \ll \lambda / L_B$: so we can take averages on a scale ℓ such that $\lambda_{GW} \ll \ell \ll L_B$: then we wil find

$$\overline{R}_{\mu\nu} = -\left\langle R^{(2)}_{\mu\nu}\right\rangle + \frac{8\pi G}{c^4}\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle\,. \tag{1.35}$$

After some math (see Maggiore): we can define a stress tensor of the GW, which looks like

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}R^{(2)}\right\rangle , \tag{1.36}$$

and a "smoothed out" SEMT of matter:

$$\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle \stackrel{\text{def}}{=} \overline{T}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{T} , \tag{1.37}$$

soo the equation which will hold is

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \frac{8\pi G}{c^4}\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right) . \tag{1.38}$$

Do note that if we work with these, the tensor which is conserved is $\overline{T}_{\mu\nu} + t_{\mu\nu}$:

$$\nabla^\mu\left(\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R}\right) = 0 = \nabla^\mu\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right) . \tag{1.39}$$

How does this look like far from the source? there, we can approximate $\overline{g} \approx \eta_{\mu\nu}$ and $\nabla_\mu \approx \partial_\mu$.

This $t_{\mu\nu}$ only has 2 physical degrees of freedom: how do we gauge the others away? The Lorentz gauge plus $h = 0$ eliminates 5 degrees of freedom.

When we have terms like $h\partial\partial h$, we can integrate by parts on a sufficiently large volume to turn them into $\partial(h\partial) - \partial h\partial h$. Using the facts $\partial^\mu h_{\mu\nu} = h = \Box h_{\mu\nu} = 0$ we can simplify several terms: in the end we get

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle .} \tag{1.40}$$

This is actually coordinate-independent, it can be computed in any frame we like. In the TT-gauge, we will have

$$t^{00} = \frac{c^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle . \tag{1.41}$$

Also, we will have $t^{01} = t^{02} = 0$ by symmetry, and also $t^{03} = t^{00}$.

Then, if we are far enough away from the source we can compute

$$\mathrm{d}E = \mathrm{d}A\, c\, \mathrm{d}t\, \frac{c^2}{32\pi G}\left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right\rangle \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle . \tag{1.42}$$

so, in order to get the total power $\mathrm{d}E/\mathrm{d}t$ we can integrate this expression in $R^2\,\mathrm{d}\Omega$. In order to compute the momentum carried away, we can do a similar thing, and get the momentum flux.

In order to do this calculation, we can use the explicit expression for the $\Delta_{ij,kl}$ projection tensor. We find

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{r^2c^3}{32\pi G}\int \mathrm{d}\Omega \left\langle \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\dddot{Q}^{kl}\frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\dddot{Q}^{kl}\right\rangle && (1.43a)\\
&= \frac{G}{8\pi c^5}\int \mathrm{d}\Omega\,\Lambda_{ij,kl}\left\langle \dddot{Q}^{ij}\dddot{Q}^{kl}\right\rangle\,, && (1.43b)
\end{aligned}$$

where the only expression depending on the angle is $\Lambda_{ij,kl}$: then we integrate and find

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dots. \tag{1.44}$$

When we do the same from the momentum density, we get an integral in the form

$$\frac{\mathrm{d}P^k}{\mathrm{d}t} \propto \int \mathrm{d}\Omega\,\ddot{Q}\partial^k\dddot{Q}\,, \tag{1.45}$$

but the integrand is odd under spatial inversion, so there is no contribution! This is not true if we go beyond the quadrupole, instead full GR calculations/simulations show that there are kicks at the merger.

We can calculate the angular distribution in a relatively simple way, since it is easy to go to TT gauge at a point.

The energy lost by the source at $t_{\mathrm{ret}} = t - r/c$ is the same as the energy measured in GW.

Clarify: why are the two expressions calculated at the same time?

If we model the back-reaction as a force, we have

$$\frac{\mathrm{d}E_{\mathrm{source}}}{\mathrm{d}t} = \left\langle \int \mathrm{d}^3x\,\frac{\mathrm{d}F_i}{\mathrm{d}V}\dot{x}_i\right\rangle == -\frac{G}{5c^5}\left\langle \frac{\mathrm{d}Q_{ij}}{\mathrm{d}t}\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}\right\rangle\,. \tag{1.46}$$

Equating terms and making some considerations, we get

$$\frac{\mathrm{d}F_i}{\mathrm{d}V} = -\frac{2G}{5c^5}\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}\rho(t,\vec{x})x_j\,, \tag{1.47}$$

so finally

$$F_i = -\frac{2G}{5c^5}\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}m\overline{x}_j\,, \tag{1.48}$$

where $\overline{x}_j$ is the center-of-mass coordinate.

Then, we can calculate the torque explicitly: we get

$$T_i = -\frac{2G}{5c^5}\epsilon_{ijk}Q_{jl}\frac{\mathrm{d}^5Q_{kl}}{\mathrm{d}t^5}\,, \tag{1.49}$$

so if we take the average we get

$$\left\langle\frac{\mathrm{d}L_i}{\mathrm{d}t}\right\rangle = -\frac{2G}{5c^5}\epsilon_{ijk}\left\langle \ddot{Q}_{jl}\dddot{Q}_{kl}\right\rangle\,. \tag{1.50}$$

1.1 Back-reaction and the evolution of binary systems

We use the reduced mass formalism, the amplitude is

$$A = \frac{4G^{5/3}\omega_s^{2/3}\mu M^{2/3}}{rc^4}\,, \tag{1.51}$$

so we have the polarizations

$$h_+ = A\frac{1-\cos^2\theta}{2}\cos\left(2\omega_s t_{\text{ret}} + 2\varphi\right) \tag{1.52a}$$

$$h_\times = A\cos\theta\sin\left(2\omega_s t_{\text{ret}} + 2\varphi\right)\,, \tag{1.52b}$$

we define the chirp mass:

$$M_c = \mu^{3/5}M^{2/5} = \frac{\left(m_1 m_2\right)^{3/5}}{(m_1+m_2)^{1/5}}\,, \tag{1.53}$$

also the frequency of the GW is twice the frequency of the orbit.

Coming back to the radiated power: it is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}\left(\dddot{M}_{kk}\right)^2\right\rangle\,, \tag{1.54}$$

so we can calculate this explicitly for our binary: we need to average over a period, so we get a factor $\left\langle \sin^2\varphi\right\rangle = 1/2$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{GW}}{2c^3}\right)^{10/3}\,, \tag{1.55}$$

where we used the fact $R^3 = GM/\omega_s^2$. We can do a similar thing for the angular momentum:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dots\,. \tag{1.56}$$

Now, let us consider masses in quasi-circular orbit: by the virial theorem,

$$\dot{R} = -\frac{2R^2}{Gm_1m_2}\dot{E}_{GW}\,. \tag{1.57}$$

Therefore, as the GWs carry away energy the radius shrinks. But we assumed the orbits to be circular! This is fine: they are almost-circular usually. This is fine for most of the inspiral, until the merger phase.

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As we were discussing yesterday, we can approximate every orbit as a circular one.

We finally get the relation

$$\dot{\omega}_{GW} = \frac{12}{5}2^{1/3}\left(\frac{M_cG}{c^3}\right)^{5/3}\omega_{GW}^{11/3}\,, \tag{1.58}$$

where M_c is the chirp mass. So, we can integrate this: since

$$\frac{\mathrm{d}f_{GW}}{\mathrm{d}t} = k f_{GW}^{11/3} \implies -\frac{3}{8k} f^{-8/3} = t - t_{\text{coalescence}} \stackrel{\text{def}}{=} -\tau\,, \tag{1.59}$$

so that we can get the frequency as a function of the time until coalescence:

$$\tau = \frac{5}{256} \left(\frac{1}{\pi f_{GW}} \right)^{8/3} \dots \,. \tag{1.60}$$

We can also get an expression for the radius at a given time from coalescence:

$$R(\tau) = R_0 \left(\frac{\tau}{\tau_0} \right)^{1/4}\,, \tag{1.61}$$

where τ_0 is the time to coalescence at t_0. If we plot this, it has a "plunge" phase, and up to it we can trust our plot.

1.1.1 Chirping waveform

The *phase* is the argument of the cosine, so we write $\cos(\phi(t))$. The angular frequency is given by $\omega_{GW} = \phi'$.

The chirping waveform cannot be trusted near the end. We know that if

$$R < R_{\text{ISCO}} = \frac{6GM}{c^2} \tag{1.62}$$

then orbits are not stable. This formula only holds for extreme mass ratios (we actually could have these SMBHs merging with solar mass ones!). Anyway, we use it as a guideline to see when our approximations break down.

The shape of the chirping waveform is basically correct; it goes out of phase with the numerical relativity calculation, but it works somewhat.

Numerical relativity has a *lower* frequency than the quadrupole approx!

What about eccentric binaries? We can also analyze them. The formula is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{5} \frac{G\mu^2}{c^5} a^4 \omega_0^6 f(e)\,, \tag{1.63}$$

where

$$f(e) = \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right) \geq 1\,. \tag{1.64}$$

We have emission at frequencies other than the orbital one; also, the GW emission has the effect of circularizing the orbit. So, we usually observe circular systems.

1.2 Hulse-Taylor binaries

It is debatable whether the observation of this was the first observation of gravitational waves.

This is a binary system in which one star is a pulsar.

What is a pulsar? It's a kind of neutron star. Not a moral judgement, but you are completely empty.

A pulsar has a large magnetic field; at a distance $r_c = c/\omega$ the field lines cannot close so a radio beam escapes. This provides a clock!

"Taking the pulse" of a pulsar: they usually have a certain well-defined shape, if we average over a few periods.

The procedure is: take signal, FFT to get the fundamental, average over periods.

The period can then be measured precisely, and we can observe its variations.

Some relevant frequencies: the radio waves are on the order of 10^8 Hz, the pulsar's frequency is of the order 10 Hz, the frequency of the binary period is 10^{-5} Hz, the motion of the Earth around the Sun at 10^{-8} Hz is also relevant.

Since the pulsar frequency is very small, we can still average many pulses and still be measuring at what is basically "a single point". Monday 2020-4-6, compiled 2020-04-10

We are discussing the Hulse-Taylor pulsar, which is a very precise clock. There are many effects by which the time-of-arrival is shifted, if we take them all out we can get the effect from the source.

We will have the Shapiro delay from the Sun, the Einstein delay at the receiver from the curvature of the spacetime around the Earth, which is given by

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} \approx 1 + \phi(x_{\text{obs}}) - \frac{v_{\text{obs}}^2}{2c^2}, \tag{1.65}$$

so

$$\tau \approx t + \int^t \mathrm{d}\widetilde{t} \left(\phi(x_{\text{obs}}) - \frac{v_{\text{obs}}^2}{2c^2} \right) = t - \Delta_{\oplus,\odot}. \tag{1.66}$$

If most of the velocity is due to the motion of the Earth in its elliptic orbit, we have from conservation of energy

$$\frac{v_{\text{obs}}^2}{2} = \frac{GM}{r} - \frac{GM}{2a}, \tag{1.67}$$

so that we find

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} \approx \frac{v^2}{2c^2} - \phi = \frac{GM_\odot}{c^2} \left(\frac{1}{r} - \frac{1}{2a} - \frac{1}{r} \right) = R_{\text{earth-Sun}} \left(\frac{1}{r} - \frac{1}{4a} \right). \tag{1.68}$$

However, we must also consider the group velocity of the signal which travels through the ISM, which is ionized gas. Then, we get a delay which depends on the frequency:

$$t_L = \frac{L}{c} + \frac{e^2}{2\pi m_e c} \frac{1}{\nu^2} DM, \tag{1.69}$$

where DM is the Dispersion Measurement. For the HT pulsar, this spreads the time over a 4 MHz bandwidth: however, we can measure precisely in the spectral domain, and we can "connect the dots" to find what corresponds to a single pulse.

This allows us to reconstruct the original pulse.

Taking these effects out, we get

$$t_{\text{ssb}} = \tau - \frac{D}{\nu^2} + \Delta_{E,\odot} - \Delta_{S,\odot} + \Delta_{R,\odot} \,. \tag{1.70}$$

This is "time in solar-system barycenter". where

$$D = \frac{e^2}{2\pi m_e c} DM \,. \tag{1.71}$$

We need to look at the gravitational time delay at the source: there is a contribution from the gravitational field of the NS itself, which is hard to calculate but constant, so we do not worry about it. The combined gravitational field, instead, is time-varying: so, we find

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 - \frac{Gm_c}{c^2 \left| x_p - x_c \right|} - \frac{v_p^2}{2c^2} \tag{1.72}$$

$$\frac{\mathrm{d}T}{\mathrm{d}u} \approx \frac{P_b}{2\pi} \left(1 - \frac{G}{c^2} \frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)} \right) \left(1 - e\cos u \left(1 + \frac{G}{c^2} \right) \right) \dots \,, \tag{1.73}$$

[to finish] where u is the angular parameter describing the orbit, while e is the eccentricity.

Also, we have the Romer delay: $\Delta_R = \hat{z} \cdot x_{pb}/c$.

The coordinates, for a Keplerian orbit, are

$$r_{pb} = r_1 = a_1(1 - e\cos u) \qquad \text{and} \qquad \cos\psi = \frac{\cos u - e}{1 - e\cos u} \,. \tag{1.74}$$

The Romer delay then is

$$\Delta_R = r_1 \sin\iota \sin(\omega + \psi) = r_1 \sin\iota (\cos\psi \sin\omega + \cos\omega \sin\psi) \,, \tag{1.75}$$

where ι is the observation angle, while ψ is the angle from the line of notes (see drawing).

The relativistic effect, however, is large. We will not do the calculation, we find

$$\Delta_R = a_1 \sin\iota \left((\cos u - e_r) \sin\omega + \sqrt{1 - e_\theta^2} \sin u \cos\omega \right) , \tag{1.76}$$

where

$$e_{r,\theta} = e(1 + \delta_{r,\theta}) \tag{1.77}$$

$$\delta_r = \frac{G}{c^2} \frac{3m_p^2 + 6m_p m_c + 2m_c^2}{a(m_p + m_c)} \tag{1.78}$$

$$\delta_\theta = , \tag{1.79}$$

also here the advance of the periastron is much more significant than it is for Mercury.

The Shapiro delay at the source must also be accounted for.

If we get all the Keplerian parameters and two of the post-Newtonian ones then we should know everything.

We measure $P_b, T_0, x = a \sin \iota / c, e, \omega$ and the post-Newtonian $\dot{\omega}, \gamma$ and finally we make a prediction for $\dot{P}$. This matches the data very well.

1.3 GW from a rotating rigid body

The moment of inertia tensor can be defined as

$$I^{ij} = \int \mathrm{d}^3 x \, \rho(x) \left(r^2 \delta^{ij} - x^i x^j \right) . \tag{1.80}$$

There exists a frame in which this tensor is diagonal, its eigenvalues are the moments of inertia, its eigenvectors are the axes of inertia. They are then defined by equations like

$$I_1 = \int \mathrm{d}^3 x \, \rho \left(x_2^2 + x_3^2 \right) . \tag{1.81}$$

For an ellipsoid with axes a, b, c and mass m we have

$$I_1 = \frac{m}{5} \left(b^2 + c^2 \right) . \tag{1.82}$$

The rotational kinetic energy is given by

$$\begin{aligned} E_{\mathrm{rot}} &= \frac{1}{2} I_{ij} \omega_i \omega_j & (1.83) \\ &= \frac{1}{2} I_i \omega_i^2 \,, & (1.84) \end{aligned}$$

where the last equality holds in the body frame.

Suppose we have a body spinning around an axis, such that the position of any point shifts by a rotation matrix R_{ij}.

The inertia tensor shifts by $I \rightarrow R^\top I R$.

The tensor we defined before,

$$M^{ij} = \frac{1}{c^2} \int \mathrm{d}^3 x \, T^{00}(x) x^i x^j \approx -I^{ij} + \int \mathrm{d}^3 x \, \rho(x) r^2 \delta^{ij} \,, \tag{1.85}$$

which we can substitute in. This is the trace of the inertia tensor.

Suppose we had a body whose angular momentum is not aligned with the moment of inertia: we can use Euler angles to express the rotation matrix.

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Now we are considering a body which is axisymmetric, and rotating along an axis which is not aligned with its axes of inertia.

An astrophysical example of this will usually look like an ellipsoid. We should "clean our minds" from the idea of a spinning spintop precessing.

Here, the axis going around is the faster motion, the rotation of the body around its axis is slower.

This wobbling motion is similar to the one of a coin thrown on a table.

If we compute the evolution of the inertial tensor, we get terms both at ω and at 2ω.

We are interested in the projection of this variation onto the plane orthogonal to the direction of a propagation.

If we have a distribution which looks like a coin ($I_1 \sim I_2 \ll I_3$) then it looks to us like a binary if we look at it from the top (in terms of periodicity at least), so we expect 2ω emission, since the system looks the same to us, since it repeats after a rotation of π.

If, instead, we look at it from the side, the periodicity is the full period: after half a rotation the coin is edge-on, but it appears at two different angles with respect to the vertical direction.

Therefore, we both have ω and 2ω emission.

If we were able to determine the amplitude at different inclinations, we would be able to determine the inclination ι.

[formula for back reaction is wrong!]

In order to calculate the backreaction we assume that the motion is approximately constant during a single period.

We find differential equations telling us that $\dot{\beta}$ and α both decrease: the first means that the motion is slowing down; the second means that the wobbling is decreasing, as the rotation is aligning with the angular momentum.

We can define the parameter $u(t) = \dot{\beta}/\dot{\beta}_0$, and a characteristic time τ_0:

$$\tau_0 = \left(\frac{2G}{5c^5} \frac{(I_1 - I_3)^2}{I_1} \dot{\beta}_0^4 \right)^{-1}, \tag{1.86}$$

which has a typical value of

$$\tau_0 = 1.8 \times 10^6 \,\mathrm{yr} \left(10^{-7} \frac{I_3}{I_1 - I_3} \right)^2 \left(\frac{1\,\mathrm{kHz}}{f_0} \right)^4 \left(10^{38} \frac{\mathrm{kgm}^2}{I_1} \right), \tag{1.87}$$

and we can write differential equations for $\dot{u}$ and $\dot{\alpha}$: but we must have $\dot{\beta}\cos(\alpha) = \mathrm{const}$. This implies that the boundary conditions must be $\alpha_\infty = 0$ and $u_\infty = \cos\alpha_0$. Also, asymptotically,

$$\dot{\alpha}_{t\to\infty} = \dots . \tag{1.88}$$

These conditions do not apply in general, neutron stars are not truly rigid bodies since they have an internal structure. In a generic case we will have emission at different frequencies.

We have not seen pulsars yet in GW, but we can put upper bounds to the amplitude of their emission.

"Beating the spin-down limit" means that we know that we would be able to see the GW emission in a certain case if the spin-down was only due to GW.

Could we differentiate a pulsar rotating and seen head-on and a binary system? Surely they are phenomena which happen in different frequency ranges, and last very much different times. If the binary is spinning at those frequencies it's evolving very rapidly, instead a pulsar can give out a stable signal.

Also, in full numerical relativity the waveform looks different.

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