General Relativity exercises

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October 4, 2019

We set $c = 1$.

1 Sheet 1

1.1 Lorentz transformations

1.1.1 Inverses

We can consider a Lorentz boost with velocity v in the x direction, and we look at its representation in the (t, x) plane (since the y and z directions are unchanged). Its matrix expression looks like:

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} , \tag{1}$$

where $\gamma = 1/\sqrt{1-v^2}$. The inverse of this matrix can be computed using the general formula for a 2x2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} . \tag{2}$$

The determinant of Λ is equal to $\gamma^2(1-v^2) = 1$, therefore the inverse matrix is:

$$\Lambda = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix} . \tag{3}$$

1.1.2 Invariance of the spacetime interval

Our Lorentz transformation is

$$\mathrm{d}t' = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{4a}$$
$$\mathrm{d}x' = \gamma(-v\,\mathrm{d}t + \mathrm{d}x) \tag{4b}$$
$$\mathrm{d}y' = \mathrm{d}y \tag{4c}$$
$$\mathrm{d}z' = \mathrm{d}z \tag{4d}$$

and we wish to prove that the spacetime interval, defined by $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is preserved: $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$. Let us write the claimed equality explicitly:

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \stackrel{?}{=} -\gamma^2(\mathrm{d}t - v\,\mathrm{d}x)^2 + \gamma^2(-v\,\mathrm{d}t + \mathrm{d}x)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{5a}$$

$$(1 - v^2)(-\mathrm{d}t^2 + \mathrm{d}x^2) \stackrel{?}{=} -(\mathrm{d}t - v\,\mathrm{d}x)^2 + (-v\,\mathrm{d}t + \mathrm{d}x)^2 \tag{5b}$$

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + v^2\,\mathrm{d}t^2 - v^2\,\mathrm{d}x^2 \stackrel{?}{=} -\mathrm{d}t^2 - v^2\,\mathrm{d}x^2 + 2v\,\mathrm{d}t\,\mathrm{d}x + v^2\,\mathrm{d}t^2 + \mathrm{d}x^2 - 2v\,\mathrm{d}x\,\mathrm{d}t \tag{5c}$$

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + v^2\,\mathrm{d}t^2 - v^2\,\mathrm{d}x^2 = -\mathrm{d}t^2 - v^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}t^2 + \mathrm{d}x^2 \tag{5d}$$

where we simplified the y and z differentials, multiplied by $1/\gamma^2 = 1 - v^2$, expanded the squares of the binomials and simplified the mixed terms.

1.1.3 Tensor notation pseudo-orthogonality

The invariance of the spacetime interval $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$ can be also written as $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x'^{\,\mu}\,\mathrm{d}x'^{\,\nu}$. By making the primed differentials explicit we have:

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho\,\mathrm{d}x^\rho\,\Lambda^\nu{}_\sigma\,\mathrm{d}x^\sigma\ , \tag{6}$$

but the dummy indices on the LHS can be changed to ρ and σ, so that both sides are proportional to $\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$. Doing this we get:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = (\Lambda^\top)_\rho{}^\mu\eta_{\mu\nu}\Lambda^\nu{}_\sigma\,, \tag{7}$$

or, in matrix form, $\eta = \Lambda^\top\eta\Lambda$.

1.1.4 Explicit pseudo-orthogonality

For simplicity but WLOG we consider a boost in the x direction with velocity v and Lorentz factor γ. The matrix expression to verify is:

$$\begin{bmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \tag{8a}$$

$$\begin{bmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-\gamma & v\gamma\\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \tag{8b}$$

$$\begin{bmatrix}-\gamma^2 + \gamma^2 v^2 & v\gamma^2 - v\gamma^2\\ v\gamma^2 - v\gamma^2 & -v\gamma^2 + \gamma^2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \tag{8c}$$

which by $\gamma^2 = 1/(1 - v^2)$ confirms the validity of the expression.

1.2 Muons

1.2.1 Nonrelativistic approximation

The survival probability is given by $\mathbb{P}(t) = \exp\Big(-t/2.2\times10^{-6}\,\mathrm{s}\Big)$. If the ground is $h = 15\,\mathrm{km}$ away, then the muon will reach it in $t = h/v = 15\,\mathrm{km}/(0.995c) \approx 5.03\times10^{-5}\,\mathrm{s}$, therefore $\mathbb{P}(t) \approx 1.2\times10^{-10}$.

1.2.2 Relativistic effects: ground perspective

The observer on the ground will see the muon having to traverse the whole $h = 15\,\mathrm{km}$, but the muon's time will be dilated for them by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\Big(-t/(\gamma_v \times 2.2\times10^{-6}\,\mathrm{s})\Big) \approx 0.1$.

1.2.3 Relativistic effects: muon perspective

1.3 Radiation