Theoretical cosmology notes

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Contents of the course

We start with a derivation of the Friedmann eqs. from the Einstein equations.

We will then discuss the properties of the CMB, deriving the spectrum, and then CMB anisotropies.

Then we will discuss star and structure formation, about the nonlinear evolution of perturbations. We will use the path-integral approach to classical field theory. We will also discuss weak gravitational lensing in the universe.

We will use some smart nonlinear approximations: the Zel'dovich approximation and the adhesion approximation.

We will use an "effective Planck constant" instead of $\hbar$: it will be a parameter which can be fit in our model.

As for references: there are handwritten notes by the professor in the Dropbox folder (for access to the folder, write to the professor). Also, there notes by a student from the previous years, in Italian [Nat17], which are to be used with caution as they contain some errors.

0.1 Friedmann equations: a brief overview

Thursday 2020-3-12, compiled 2020-03-21

In the previous course we used the approximate symmetries of the universe to write the FLRW line element:

$$ds^2 = -\,dt^2 + a^2(t)\,d\sigma^2 \,, \tag{1}$$

do note that we switch signature from the previous course: now we use the mostly plus one. The spatial part is defined by

$$d\sigma^2 = \widetilde{g}_{ij}\,dx^i\,dx^j \,, \tag{2}$$

where $\widetilde{g}_{ij}$ is the maximally symmetric metric tensor in a 3D space. There are only 3 maximally symmetric 4D spacetimes: Minkowski, dS and AdS.

Since we have maximal symmetry, the Riemann tensor is

$$R_{ijkl} = k\left(\widetilde{g}_{ik}\widetilde{g}_{jl} - \widetilde{g}_{il}\widetilde{g}_{jk}\right). \tag{3}$$

We can use spherical coordinates:

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\,, \tag{4}$$

and we can define the coordinate χ by

$$\mathrm{d}\chi = \frac{\mathrm{d}r^2}{\sqrt{1-kr^2}}\,. \tag{5}$$

The Einstein equations read

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \tag{6}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is its trace, the scalar curvature, while $T_{\mu\nu}$ is the stress energy momentum tensor.

In cosmology we assume to have the SEMT of a perfect fluid. Really, we have particles, between which there is vacuum.

We need to use the Weyl tensor, which describes the parts of the Riemann tensor which are not in the traces. "The real world" is only described by the Weyl tensor, but in cosmology we make a great approximation in ignoring it.

What we do is to insert an ansatz for the metric tensor, which we use to derive the Christoffel symbols, and from these we write the Riemann tensor. Doing it the other way around, starting from the source SEMT, is very difficult.

Claim 0.1.1. *The Christoffel symbols for the FLRW metric are:*

$$\Gamma^t_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\dot{a}a}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 a\dot{a} & 0 \\ 0 & 0 & 0 & r^2 a\dot{a}\sin^2\theta \end{bmatrix} \tag{7a}$$

$$\Gamma^r_{\mu\nu} = \begin{bmatrix} 0 & \dot{a}/a & 0 & 0 \\ \dot{a}/a & \frac{kr}{(1-kr^2)} & 0 & 0 \\ 0 & 0 & (kr^2-1)r & 0 \\ 0 & 0 & 0 & (kr^2-1)r\sin^2\theta \end{bmatrix} \tag{7b}$$

$$\Gamma^\theta_{\mu\nu} = \begin{bmatrix} 0 & 0 & \dot{a}/a & 0 \\ 0 & 0 & 1/r & 0 \\ \dot{a}/a & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{7c}$$

$$\Gamma^{\varphi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & \dot{a}/a \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & \cos\theta/\sin\theta \\ \dot{a}/a & 1/r & \cos\theta/\sin\theta & 0 \end{bmatrix} . \tag{7d}$$

In order to calculate these, we can make use of certain simplifications: the FLRW metric is diagonal, and it does not depend on φ.

Notice that the spatial Christoffel symbols are nonzero even in Minkowski ($k = 0$, $\dot{a} = \ddot{a} = 0$): why is this? This is because we are using curvilinear coordinates, the Christoffel symbols express the *extrinsic* curvature, not the *intrinsic* curvature; they are not tensors, so they can be zero in a reference and nonzero in another.

In general, the Riemann tensor is given by

$$R^{\mu}_{\nu\rho\sigma} = -2\left(\Gamma^{\mu}_{\nu[\rho,\sigma]} + \Gamma^{\alpha}_{\nu[\rho}\Gamma^{\mu}_{\sigma]\alpha}\right), \tag{8}$$

where commas denote coordinate derivation, and square square brackets denote antisymmetrization (for clarification on this notation Wikipedia does a good job [19]).

The Ricci tensor is given by the contraction of the Riemann tensor along its first and third component:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = -2\left(\Gamma^{\alpha}_{\mu[\alpha,\nu]} + \Gamma^{\beta}_{\mu[\alpha}\Gamma^{\alpha}_{\nu]\beta}\right) \tag{9a}$$

$$= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} . \tag{9b}$$

A great simplification comes from the fact that, for the FLRW metric, the Ricci tensor is diagonal.[1]

Claim 0.1.2. *The components of the Ricci tensor are:*

$$R_{tt} = -3\partial_t\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \tag{10a}$$

$$= -3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2\right) \tag{10b}$$

$$= -3\frac{\ddot{a}}{a}, \tag{10c}$$

$$\begin{aligned} R_{rr} = {} & \partial_t\left(\frac{\dot{a}a}{1-kr^2}\right) + \partial_r\left(\frac{kr}{1-kr^2}\right) - \partial_r\left(\frac{kr}{1-kr^2}\right) - 2\partial_r\left(\frac{1}{r}\right) \\ & + \frac{\dot{a}a}{1-kr^2}3\frac{\dot{a}}{a} + \frac{kr}{1-kr^2}\left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) \\ & - 2\frac{\dot{a}}{a}\frac{\dot{a}a}{1-kr^2} - \left(\frac{kr}{1-kr^2}\right)^2 - 2\left(\frac{1}{r}\right)^2 \end{aligned} \tag{11a}$$

[1] If there are a certain number of coordinates the metric is independent of, the Ricci tensor has very few nonzero components [Win96]. This is not enough to prove that the Ricci tensor must be diagonal for this metric, however in the specific case of FLRW this is the case anyways.

$$
\begin{aligned}
&= \frac{\ddot{a}a + \dot{a}^2}{1-kr^2} + 3\frac{\dot{a}^2}{1-kr^2} + 2\frac{k}{1-kr^2} - 2\frac{\dot{a}^2}{1-kr^2} && \text{(11b)}\\
&= \frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2}\,, && \text{(11c)}
\end{aligned}
$$

$$
\begin{aligned}
R_{\theta\theta} &= r^2\partial_t(a\dot{a}) + \partial_r\Big((kr^2-1)r\Big) - \partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) \\
&\quad + 3\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Big(\Gamma^r_{rr} + 2\Gamma^\theta_{r\theta}\Big) - 2\Big(\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}\Big) - \frac{\cos^2\theta}{\sin^2\theta} && \text{(12a)}\\
&= r^2\Big(\ddot{a}a + \dot{a}^2\Big) + 3kr^2 - 1 + \frac{1}{\sin^2\theta} + r^2\dot{a}^2 - kr^2 - \frac{\cos^2\theta}{\sin^2\theta} && \text{(12b)}\\
&= r^2\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big)\,, && \text{(12c)}
\end{aligned}
$$

$$
\begin{aligned}
R_{\varphi\varphi} &= \partial_\alpha\Gamma^\alpha_{\varphi\varphi} - \partial_\varphi\Gamma^\alpha_{\alpha\varphi} + \Gamma^\alpha_{\varphi\varphi}\Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\varphi\alpha}\Gamma^\alpha_{\varphi\beta} && \text{(13a)}\\
&= r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big)\,. && \text{(13b)}
\end{aligned}
$$

The Ricci scalar then comes out to be

$$
\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= 3\frac{\ddot{a}}{a} + \frac{1-kr^2}{a^2}\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2} \\
&\quad + \frac{1}{a^2r^2}r^2\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) + \frac{1}{a^2r^2\sin^2\theta}r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) && \text{(14a)}\\
&= 3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{a^2} && \text{(14b)}\\
&= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right]. && \text{(14c)}
\end{aligned}
$$

The dimensions of the Ricci scalar are those of a length to the -2.

The stress energy tensor is the functional derivative of everything but the curvature in the action with respect to the metric: if our Lagrangian is

$$
L = L_g + L_{\text{fluid}}\,, \tag{15}
$$

where the gravitational Lagrangian is $L_g = M_P^2 R/2$ (and $M_P = 1/\sqrt{8\pi G}$ in natural units is the reduced Planck mass) then

$$
T_{\mu\nu} \overset{\text{def}}{=} -\frac{2}{\sqrt{-g}}\frac{\delta L_{\text{fluid}}}{\delta g^{\mu\nu}}\,. \tag{16}
$$

Discuss why this is equivalent to "flux of momentum component μ across a surface of constant x^ν".

We use perfect fluids: they have a stress-energy tensor like

$$T^{\mu\nu} = \left(\rho + P\right) u^\mu u^\nu + p g^{\mu\nu}\,, \tag{17}$$

where u^μ is the 4-velocity of the fluid element. It is diagonal *in the comoving frame*, in which $u^\mu = \left(1, \vec{0}\right)$.

If we are not comoving, we have additional heat transfer off diagonal terms (this is discussed in my thesis [Tis19, section 4.2]).

If we take the covariant divergence of the Einstein tensor $G_{\mu\nu}$ we get zero; so the stress energy tensor must also have $\nabla_\mu T^{\mu\nu} = 0$. This is *not* a conservation equation.

In SR we had an equation like $\partial_\mu T^{\mu\nu}$: this *was* a conservation equation, a local one. In GR we also need Killing vectors in order to actually have conserved quantities. In cosmology we do not have symmetry with respect to time translation, so there is no timelike Killing vector ξ_μ such that $\xi_\nu \nabla_\mu T^{\mu\nu}$ represents the conservation of energy.

This equation, $\nabla_\mu T^{\mu\nu}$ follows from the fact that our fluid follows its equations of motion.

Let us explore the meaning of these equations: if, in the equation $0 = \nabla_\mu T^\mu_0$, we find

$$0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \tag{18a}$$

$$= -\dot{\rho} - 3H\left(\rho + P\right). \tag{18b}$$

For example consider radiation: $P = \rho/3$. This means that $\dot{\rho} = -4H\rho$: so, as the Hubble parameter increases, the radiation density decreases.

The other two Friedmann equations can be derived from the time-time and space-space components on the Einstein equations: we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) \tag{19a}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,. \tag{19b}$$

The space-space equation is not a dynamical equation, since it contains no second time derivatives: it is a *constraint* on the evolution of the system.

However, the three Friedmann equations are not independent: the time-time one can be found from the other two.

A useful theorem is the fact that for a maximally symmetric space the Ricci tensor must be given by

$$\widetilde{R}_{\alpha\beta} = 2k\widetilde{g}_{\alpha\beta}\,. \tag{20}$$

We can write the stress energy tensor as

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}\,, \tag{21}$$

where $h_{\mu\nu}$ is the projection tensor onto the spacelike subspace $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$.

This is more physically meaningful.

Tomorrow we will start the discussion on the CMB.

Chapter 1

The CMB

Today we discuss the CMB. This is discussed in the book Modern Cosmology [Dod03], we follow the professor's notes. Friday 2020-3-13, compiled 2020-03-21

A note: in these lectures a dot will refer to conformal time derivatives only, if we differentiate with respect to cosmic time we shall write the derivative explicitly. Let us suppose we have some particle species interacting, such as $1+2 \leftrightarrow 3+4$.

The variation in time of the abundance of particle type 1, (which is given by the density times a volume: $n_1 a^3$) is given by the difference of the particles which are created and destroyed. We write the formula first, and then explain it: this is given by

$$
\begin{aligned}
a^{-3}\frac{\mathrm{d}(n_1 a^3)}{\mathrm{d}t} &= \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1}\left[\prod_{i=2}^{4}\int \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}\right]\times \\
&\times (2\pi)^4 \delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\delta(E_1+E_2-E_3-E_4)\times' \\
&\times |\mathcal{M}|^2\Big[f_3 f_4(1\pm f_1)(1\pm f_2) - f_1 f_2 (1\pm f_3)(1\pm f_4)\Big]
\end{aligned}
\tag{1.1a}
$$

where:

1. the delta functions account for momentum and energy conservation: energy is *not conserved* in general in cosmology, *but* we can use the equivalence principle to go to a reference frame which is locally Minkowski: in our description of an instantaneous process such as this, the deviations from this frame are negligible.

2. $\mathcal{M}$ is the invariant scattering amplitude between the initial and final states.

3. The f_i are the phase space distributions of the different species: the terms including these account for the quantum statistics, we use $-$ for fermions and $+$ for bosons. Bose statistics enhance the process, Pauli statistics block it.

4. The 2π-s account for the normalization of the deltas: if we were to discretize phase space and use Kronecker deltas we would not need them.

5. The energy of each particle species is given by $E = \sqrt{p^2+m^2}$. Why are there $2E$ factors in the denominators? In principle, we should integrate in $\mathrm{d}^4 p$, however we

work *on shell*. A priori, the particle does whatever it wants, however solutions to the equations of motion are preferred in the path integral. So, we impose this condition: we do

$$\int \mathrm{d}^3 p \int_0^\infty \delta(E^2 - p^2 - m^2) = \int \mathrm{d}^3 p \int_0^\infty \frac{\delta(E - \sqrt{p^2 + m^2})}{2E}\,, \tag{1.2}$$

so we include the term in the denominator.

Clarify definition of $\mathcal{M}$.

If there is no interaction, $n_1 \propto a^{-3}$.

We set $\hbar = c = k_B = 1$.

The term for particle 1, E_1, has a different origin: the time is related to the proper time by p^0, which is E_1. The factor 2 is included for symmetry, it is indifferent if we include it or not since we can normalize the helicities g_i.

Typically we have kinetic equilibrium, if the scattering time is very short with respect to the Hubble time. So, we use

$$f_{\mathrm{BE/FD}} = \left(\exp\left(\frac{E - \mu}{T}\right) \pm 1 \right)^{-1}, \tag{1.3}$$

where the sign is a $-$ for Bose Einstein statistics, while a $+$ for Fermi-Dirac statistics.

For the nonrelativistic particles (all of them, except the photons) we have $E - \mu \gg T$. If f becomes very small, then we can drop the terms $(1 \pm f_i)$. This is the Boltzmann limit.

In theory we could not do this for photons, in practice we do it and the magnitude of the error is the same as the ratio $\zeta(3) \approx 1.2$ to 1.

Then, our distributions will be given by

$$f(E) = e^{\mu/T} e^{-E/T}\,. \tag{1.4}$$

So the phase space distribution term is

$$\exp\left(-\frac{E_1 + E_2}{T}\right) \left(e^{(\mu_3 + \mu_4)/T} - e^{(\mu_1 + \mu_2)/T}\right), \tag{1.5}$$

where we used the fact that $E_1 + E_2 = E_3 + E_4$ by energy conservation, as we said. If we enforced the Saha condition, chemical equilibrium $\mu_1 + \mu_2 = \mu_3 + \mu_4$, then we get precisely zero: the number densities of the species are constant.

The mean number density of species i is given by

$$n_i = g_i e^{\mu_i/T} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T}\,, \tag{1.6}$$

where g_i is the number of helicity states.

So, we find for the whole expression inside the brackets:

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\,, \tag{1.7}$$

so we can define the time-averaged cross section

$$\langle \sigma v \rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_i \int \frac{d^3 p}{2E_i} \dots , \tag{1.8}$$

so the final equation is

$$a^{-3} \frac{d}{dt}(n_1 a^3) = \langle \sigma v \rangle \, n_1^{(0)} n_2^{(0)} \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} , \tag{1.9}$$

and the left hand side is typically $\sim n_1/t \sim n_1 H$. So, the combination on the RHS must be "squeezed to zero" eventually, which is equivalent to the Saha equation.

This is basically saying that we eventually reach chemical equilibrium.

1.1 Hydrogen recombination

The process is

$$e^- + p \leftrightarrow H + \gamma , \tag{1.10}$$

so the Saha equation yields

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} , \tag{1.11}$$

and charge neutrality implies $n_e = n_p$, *not* $n_e^{(0)} = n_p^{(0)}$.

At this stage in evolution, there are already some Helium nuclei, but we ignore them.

We define the ionization fraction

$$X = \frac{n_e}{n_e + H} . \tag{1.12}$$

This then yields

$$\frac{1 - X_e^n}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_e} \right)^{3/2} \exp(\epsilon_0 / T) , \tag{1.13}$$

where $\epsilon_0 = m_p + m_e - m_H = 13.6\,\mathrm{eV}$ is the ionization energy of Hydrogen.

Then, we get that the temperature of recombination is $T_{\mathrm{rec}} \approx 0.3\,\mathrm{eV}$.

The evolution of the ionization fraction is

$$\frac{dX_e}{dt} = (1 - X_e)\beta(T) - X_e^2 n_b \alpha^{(2)}(T) , \tag{1.14}$$

where we defined the ionization rate

$$\beta(T) = \langle \sigma v \rangle \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T} , \tag{1.15}$$

and the recombination rate $\alpha^{(2)} = \langle \sigma v \rangle$.

The value of this can be solved numerically: the difference between this and the Saha equation is not great in the prediction in the recombination redshift; however the prediction of the residual ionized hydrogen is different: there is much more than Saha would predict.

The universe gets reionized at $z \gtrsim 6$; this is still under discussion.

There are many ingredients in the interaction of the universe. We are interested in the photons: we want to predict the anisotropies in the CMB. There is a dipole due to the movement of the solar system through the CMB. Now, we want to see what our predictions are if we subtract this.

[Scheme of the interactions.]

The metric interacts with everything, photons interact with electrons through Compton scattering, electrons interact with protons through Coulomb scattering, dark energy, dark matter and neutrinos interact only with the metric.

Instead of Compton scattering, we use its nonrelativistic limit which applies here.

Scattering between electrons and protons is suppressed since protons are very massive. The other terms in the universe affect the geometry and we could see them through this.

There are models which include DM-DE coupling, and quintessence models, and models in which dark energy clusters.

We are not going to consider these.

We go back to first principles:

$$\hat{\mathbb{L}}[f] = \hat{\mathbb{C}}[f] \,, \tag{1.16}$$

where $f = f(x^\alpha, p^\alpha)$, however actually we do not have that much freedom in the phase space distribution. If there are no collisions: $\hat{\mathbb{L}}[f] = 0$, which is equivalent to

$$\frac{\mathrm{D}f}{\mathrm{D}\lambda} = 0 \,, \tag{1.17}$$

where λ is the affine parameter.

In the nonrelativistic case,

$$\hat{\mathbb{L}} = \frac{\partial}{\partial t} + \dot{x} \cdot \nabla_x + \dot{v} \cdot \nabla_v = \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_x + \frac{F}{m} \cdot \nabla_v \,, \tag{1.18}$$

while in the GR case we need to account for the geodesic equation: and we write

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \,, \tag{1.19}$$

where the affine parameter λ has the dimensions of a mass, in order to have dimensional consistency.

Then, the Liouville operator is

$$\hat{\mathbb{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \stackrel{\text{def}}{=} \frac{\mathrm{D}}{\mathrm{D}\lambda} \,. \tag{1.20}$$

This is a total derivative in phase space with respect to the affine parameter.

In the FLRW background, $f = f(|p|, t)$ and

$$\hat{\mathbb{L}} = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}|p|^2\frac{\partial f}{\partial E}\,, \tag{1.21}$$

so if we define the number density

$$n(t) = \frac{g}{(2\pi)^3}\int \mathrm{d}^3 p\, f(E, t)\,, \tag{1.22}$$

so if we integrate over momenta we get

$$\int \frac{\mathrm{d}^3 p}{E}\hat{\mathbb{L}}[f]\,, \tag{1.23}$$

we find the equation from before:

$$\dot{n} + 3\frac{\dot{a}}{a}n\,, \tag{1.24}$$

??? to check

1.1.1 Metric perturbations

We want to write the most general perturbed metric solution to the Einstein Equations: it will look like $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{FLRW}} + h_{\mu\nu}$, for a small perturbation $h_{\mu\nu}$. Recall that the FLRW metric, in the zero-curvature case, is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\ . \tag{1.25}$$

We will neglect spatial curvature because it does not affect the CMB anisotropies in the universe much.

In general this perturbation will be gauge-dependent, and we want to write only a number of independent components corresponding to the number of physical degrees of freedom. This is done via the scalar-vector-tensor decomposition [Ber00, section 2.1].

We decompose the components of $h_{\mu\nu}$ as:

$$h_{00} = -2\Phi\,, \qquad h_{0i} = 2a\omega_i \qquad h_{ij} = a^2\Big(-2\Psi\delta_{ij} + \chi_{ij}\Big)\,. \tag{1.26}$$

So, we have distinguished two scalar degrees of freedom Φ and Ψ, plus a vector ω_i and a tensor χ_{ij}. Since we can absorb any variations of the trace of the spatial part of the metric into Ψ, we can assume χ_{ij} to be traceless, which takes away one degree of freedom from the 6 of the tensor. This would mean we have $2 + 3 + 6 - 1 = 10$ degrees of freedom, the right amount before accounting for gauge.

Now, we make use of the Helmholtz decomposition: any vector field x^i can be written as $x^i = x^i_\perp + x^i_\parallel$, where $x^i_\perp$ is solenoidal ($\nabla_i x^i_\perp = 0$), while $x^i_\parallel$ is irrotational ($\epsilon^{ijk}\nabla_i x_j = 0$).

This means that we can split the three degrees of freedom of the vector field into one in $x^i_\parallel$ — since it is irrotational, it can be written as the gradient of a scalar function —, and two in $x^i_\perp$, since it is a 3D vector field for which one component is determined.

We can do a similar thing for the tensor perturbation, but it is slightly more complicated: it it written in the form

$$\chi_{ij} = \underbrace{\chi^{\parallel}_{ij}}_{\text{1 dof}} + \underbrace{\chi^{\perp}_{ij}}_{\text{3 dof}} + \underbrace{\chi^{T}_{ij}}_{\text{2 dof}}\,, \tag{1.27}$$

where $\chi^\parallel$ contains only one degree of freedom, since it can be written as

$$\chi^{\parallel}_{ij} = \left(\nabla_i \nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\phi_S\,, \tag{1.28}$$

while $\chi^\perp_{ij}$ contains three degrees of freedom, since it can be written as the symmetrized gradient of a vector field S^T:

$$\chi^{\perp}_{ij} = 2\nabla_{(i}S^T_{j)}\,, \tag{1.29}$$

where the vector field is divergence-free: $\nabla_j S^j_T = 0$.

The last term is fully transverse: its divergence vanishes, $\nabla_i \chi^{ij}_T = 0$. This term then contains two physical degrees of freedom, since it starts out from five — it is a symmetric spatial tensor with zero trace — but we impose 3 equations.

Now, the term $\chi^\parallel$ can be absorbed into Φ so we remove it. Let us enumerate the degrees of freedom we have, distinguishing them into "modes" based on how they transform:

1. The **tensor mode** is given by χ^T_{ij}, the part of the spatial metric perturbation which cannot be obtained as a gradient. It has two degrees of freedom, and it transforms as a spin-2 field (which means it is unchanged by rotations of angle π about its axis). This corresponds to gravitational radiation.

2. The **vector mode** is given by the vectors ω_i and S^T_i. It has

Now we use a perturbed FLRW metric, in the Poisson gauge.

$$ds^2 = -e^{2\Phi}\,dt^2 + 2a\omega_i\,dx^i\,dt + a^2\left(e^{-2\Psi}\delta_{ij} + \chi_{ij}\,dx^i\,dx^j\right)\,, \tag{1.30}$$

where we neglect spatial curvature (which we will do from now on). This is because we would never be able to see the effect of spatial curvature in the geometry (although we could see it in the dynamics).

Let us describe the quantities we introduced. We have 10 degrees of freedom in the metric, 6 of which are physical, while 4 of which are gauge. We account for them like this: Φ and Ψ are scalar, ω is a vector, χ_{ij} is a tensor. This is explained in more detail in the class by Nicola Bartolo ("early Universe"). These are GR perturbations.

"Perturbation" means that we compare the physical spacetime and the idealized FLRW metric. We need to do this since we cannot solve the EFE if there is no symmetry. So, we say that spacetime is *close* to the idealized spacetime.

We need a map between the physical and idealized spacetimes: this is called a *gauge*. Perturbations are classified with respect to their effect on FLRW. In euclidean space we know scalars, vectors, tensors. The perturbations will behave as such, under a change of coordinates in the Cartesian space which is the 3D space-like slice of FLRW.

ω_i carries a 3D vector index. χ_{ij} contains the off-diagonal perturbations. Let us start with ω_i. In principle: Helmholtz's theorem says that we can decompose $\omega_i = \omega_{i,\text{transverse}} + \partial_i \omega$. We say that the part we are interested in is the transverse one, but we still have a gradient.

We choose our χ such that $\chi^i_j = \chi^i_{j,i} = 0$. χ could also contain vectors (objects with a vector index), as long as they are divergenceless.

It can also contain tensors: these are GWs.

So, we have 3 scalars (a, Ψ, Φ), 3 components of a transverse vector (ω_i), plus the divergence ω, while in the tensor we have $6 - 3 - 1 = 2$ degrees of freedom.

So we have 10 total degrees of freedom. The reason we do this is that the degrees of freedom obey independent eqs. of motion.

There is gauge ambiguity in our problem: we could change the mapping between physical space and FLRW. We could do a change such as $x^\mu \to x'^\mu = x^\mu +$ a perturbation.

This is explained better in the notes called "GR perturbations".

Gauge freedom allows us to drop 2 of the 4 scalars we have.

We use Poisson or longitudinal gauge, which is sometimes incorrectly called "Newtonian gauge" even though it is not Newtonian.

Our next goal will be to solve the geodesic eqs. for the motion of the particles in this gauge.

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We come back to the discussion from last time, about the Boltzmann equation in a perturbed universe.

When can we drop some terms using a gauge transformation? We can do it for scalar and vector perturbations. We shall use the longitudinal, or Poisson gauge, in which the scalar perturbations are reduced to Φ and Ψ, so we can take the tensor terms to be traceless and covariantly constant.

We will not discuss vectors, since if they are zero at the beginning they cannot be generated, they stay at zero. In our natural units, the perturbations will be small: $\Phi \ll 1$ and $\Psi \ll 1$. Recall that we are neglecting spatial curvature.

Photons have $P^2 = 0$: so we can express this as

$$-(1+2\Phi)(p^0)^2 + p^2 = 0\,, \tag{1.31}$$

where p^2 is defined as $p^2 = g_{ij}P^iP^j$, since in our gauge choice the spatial part of the metric has a Kronecker delta this will only include the spatial parts. So, we get

$$P^0 = \frac{p}{\sqrt{1+2\Phi}} \approx p(1-\Phi)\,. \tag{1.32}$$

Now, we will write the Liouville operator by dividing through by P^0:

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\cdot\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}\,, \tag{1.33}$$

where we split the three-momentum into absolute value p and the unit vector $\hat{p}$, which has $\hat{p}^i = \hat{p}_i$ and $\delta_{ij}\hat{p}^i\hat{p}^j$. We have $\mathrm{d}x^i \big/ \mathrm{d}t = P^i / P^0$.

We are going to expand only to first order. Higher order are more important for small angular scales, and for secondary CMB anisotropies, these are interesting but we are not going to treat them.

To first order, the last term of the RHS is zero, since it is a product of two terms which are both zero to zeroth order (in an unperturbed universe the phase space distribution is perfectly isotropic and a particle keeps travelling in the same direction).

Now we define the amplitude A by $P^i = A\hat{p}^i$: now we will have

$$p^2 = g_{ij}P^iP^j = a^2\delta_{ij}(1-2\Psi)\hat{p}^i\hat{p}^jA^2 \tag{1.34a}$$
$$= a^2(1-2\Psi)A^2\,, \tag{1.34b}$$

therefore, taking the square root and staying to first order we get

$$A \approx p\frac{1+\Psi}{a}\,, \tag{1.35}$$

so

$$P^i = p\hat{p}^i\frac{1+\Psi}{a}\,, \tag{1.36}$$

and the division by a can be interpreted as a redshift effect. Inserting this term we get

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{P^i}{P^0} = \hat{p}^i\frac{1+\Psi+\Phi}{a}\,, \tag{1.37}$$

and we can notice that $\mathrm{d}x^i \big/ \mathrm{d}t$ multiplies the term $\partial f \big/ \partial x^i$, which is only nonzero to first order: so we must consider this term to zeroth order. So, we get

$$\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}\,, \tag{1.38}$$

and now we shall show that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right), \tag{1.39}$$

which will imply that

$$\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right). \tag{1.40}$$

We will use the geodesic equation for photons: it is enough to consider its zeroth component, which is

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta\,, \tag{1.41}$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(p(1+\Phi)\big) = -\Gamma^0_{\alpha\beta} P^\alpha P^\beta \frac{1+\Phi}{p}\,, \tag{1.42}$$

where we brought a P^0 from the left to the right side. This means that we have

$$(1-\Phi)\frac{\mathrm{d}p}{\mathrm{d}t} = p\frac{\mathrm{d}\Phi}{\mathrm{d}t} - \Gamma^0_{\alpha\beta} P^\alpha P^\beta \frac{1+\Phi}{p}\,, \tag{1.43}$$

and now we multiply both sides by $1+\Phi$, the inverse of $1-\Phi$ to linear order:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right) - \Gamma^0_{\alpha\beta} P^\alpha P^\beta \frac{1+2\Phi}{p}\,, \tag{1.44}$$

and now we have to start calculating the Christoffel symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)\,, \tag{1.45}$$

so we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\left(2g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\right)\frac{P^\alpha P^\beta}{p}\,, \tag{1.46}$$

but g^{0i} are zero, since we are ignoring vector perturbations, and $g^{00} = -1+2\Phi$ (since it is the contravariant metric). Then we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Phi}{2}\left(2g_{0\alpha,\beta} - g_{\alpha\beta,0}\right)\frac{P^\alpha P^\beta}{p}\,, \tag{1.47}$$

and we also have

$$\frac{\partial g_{0\alpha}}{\partial x^\beta} = -2\frac{\partial\Phi}{\partial x^\beta}\delta_{\alpha 0}\,, \tag{1.48}$$

so we distinguish the components and find:

$$\begin{aligned} -\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} &= -\frac{\partial g_{00}}{\partial t}\frac{(P^0)^2}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^i P^j}{p} &\text{(1.49a)}\\ &= 2\frac{\partial\Phi}{\partial t}p - a^2\delta_{ij}\left(-2\frac{\partial\Psi}{\partial t} + 2H(1-2\Psi)\right)\frac{P^i P^j}{p}\,, &\text{(1.49b)}\end{aligned}$$

and we already have shown that $\delta_{ij}P^iP^j = p^2(1+2\Psi)/a^2$.

So on the whole we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = (-1+2\Phi)\left[-\frac{\partial\Phi}{\partial t}p - 2\frac{\partial\Phi}{\partial x^i}\frac{p\hat{p}^i}{a} + p\left(\frac{\partial\Psi}{\partial t} - H\right)\right]\,, \tag{1.50}$$

so putting everything together [extra passage] we get

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right). \tag{1.51}$$

Now we need to choose how to perturb the photon distribution function. At zeroth order it is the Planckian:

$$f \approx \frac{1}{e^{p/T} - 1}. \tag{1.52}$$

In general we will have dependence on the position $\vec{x}$, the momentum $(p, \hat{p})$ and time t.

We do not observe spectral distortions in the CMB: it is always described by a Planckian, with anisotropies in the *temperature*. So, we parametrize it as

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left(\frac{p}{T(t)\big(1 + \Theta(\vec{x}, \hat{p}, t)\big)}\right) - 1\right]^{-1}, \tag{1.53}$$

where we assumed that $\Theta = \delta T/T$ does *not* depend on the momentum of the photon p: otherwise, we would have a spectral distortion. This is certainly true, at least to linear order.

So, we expand in Θ:

$$f \approx \frac{1}{e^{p/T} - 1} + \left(\frac{\partial}{\partial T}\Big(\exp\big(p/T\big) - 1\Big)^{-1}\right)T\Theta \tag{1.54a}$$

$$= f^{(0)} - p\frac{\partial f^{(1)}}{\partial p}\Theta, \tag{1.54b}$$

since

$$T\frac{\partial f^{(0)}}{\partial T} = -p\frac{\partial f^{(0)}}{\partial p}. \tag{1.55}$$

At zeroth order we have

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0, \tag{1.56}$$

since we do not have collision terms. We can write this differently using

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}t}\frac{\partial f^{(0)}}{\partial p}, \tag{1.57}$$

where we used the change in derivative variable from before. So we get

$$\left[-\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right]\frac{\partial f^{(0)}}{\partial p} = 0, \tag{1.58}$$

which means $\dot{T}/T + \dot{a}/a = 0$, or $T \propto 1/a$, which is Tolman's law. Now, let us go to first order.

$$\frac{\mathrm{D}f}{\mathrm{D}t} = -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - p\frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} + Hp\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) + p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Psi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right]. \tag{1.59}$$

The final expression we get is

$$\frac{\mathrm{D}f}{\mathrm{D}t} = -p\frac{\partial f^{(0)}}{\partial p}\left[\underbrace{\frac{\partial \theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \theta}{\partial x^i}}_{\text{free-streaming}} + \underbrace{\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}}_{\text{gravitational}}\right], \tag{1.60}$$

in which we can distinguish two terms which have to do with the propagation of the anisotropies from emission to now — this is not precise, but it is the reason we call them "free-streaming". The other two terms arise from the self-gravity of the matter appearing on the RHS of the Einstein equations.

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Now, we will discuss the collision terms. The interaction we wish to consider is Compton scattering, which has the form

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}') + \gamma(\vec{p}'), \tag{1.61}$$

and we are interested to see how this affects the momentum distribution of the photons.

The collision term in the equation

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \hat{\mathbb{C}}[f(\vec{p})] \tag{1.62}$$

reads

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 q'}{(2\pi)^3 2E_e(q')}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2(2\pi)^4 \\ \times \delta^{(3)}(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\delta(E(p)+E_e(q)-E(p')-E_e(q')) \\ \times \left[f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right]\end{aligned}. \tag{1.63}$$

The factor $1/p$ at the start comes from the LHS, since we differentiated with respect to t and not λ on the LHS.

For the photons the energy is $E(p') = p'$, for the electrons instead $E_e(q) \approx m_e + q^2/2m_e$, since the electrons are nonrelativistic — the temperatures are of the order 0.3 eV at recombination, a vanishingly small fraction of the electrons' mass of 511 keV.

So, we should perform all the integrations on the RHS: we start with the integration over q'. We get rid of a δ function and get

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{\pi}{2m_e^2}\frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2 \\ \times \delta\left[p + \frac{q^2}{2m_e} - p' - \frac{(\vec{p}+\vec{q}-\vec{p}')^2}{2m_e}\right] \\ \times \left[f_e(\vec{q}+\vec{p}-\vec{p}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right]\end{aligned}. \tag{1.64}$$

For nonrelativistic Compton scattering we have

$$E_e(\vec{q}) - E_e(\vec{q}+\vec{p}-\vec{p}\,') = \frac{q^2}{2m_e} - \frac{\left(\vec{q}+\vec{p}-\vec{p}\,'\right)^2}{2m_q} \approx \frac{(\vec{p}-\vec{p}\,')\cdot\vec{q}}{m_e}\,, \tag{1.65}$$

which is true as long as $q \gg p, p'$. Also, the scattering is close to being elastic:

$$\frac{(\vec{p}-\vec{p}\,')\cdot\vec{q}}{m_e} \sim \frac{Tq}{m_e} \sim Tv_b \ll T\,, \tag{1.66}$$

since the velocity of the electrons is nonrelativistic.

So,

$$\frac{\Delta E_e}{E} \sim \frac{Tv_b}{Tc} \sim \frac{v_b}{c} \ll 1\,. \tag{1.67}$$

Let us motivate $q \ll p, p'$: we know that, since

$$\frac{q^2}{2m_e} \sim T\,, \tag{1.68}$$

we have $q \sim (m_e T)^{1/2}$, which is equal to

$$q \sim \left(\frac{m_e}{T}\right)^{1/2} T \gg T\,, \tag{1.69}$$

so $q \gg T \sim p$.

Now, the change to the electron kinetic energy is small so we can expand:

$$\delta\left[p + \frac{q^2}{2m_e} - p' - \frac{\left(\vec{q}+\vec{p}-\vec{p}\,'\right)^2}{2m_e}\right] \tag{1.70}$$

$$\approx \delta(p-p') + \left[E_e(q') - E_e(q)\right] \times \frac{\partial}{\partial E_e(q')}\delta(p + E_e(q) - p' - E_e(q')) \tag{1.71}$$

$$\approx \delta(p-p') + \frac{(\vec{p}-\vec{p})\cdot\vec{q}}{m_e}\frac{\partial\delta(\vec{p}-\vec{p}\,')}{\partial p'}\,, \tag{1.72}$$

where we used the fact that

$$\frac{\partial(x-y)}{\partial x} = -\frac{\partial(x-y)}{\partial y}\,. \tag{1.73}$$

The derivative of the delta-function is defined as a functional yielding the derivative of the function it is integrated with.

This then gives us

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{\pi}{2m_e^2}\frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2 \\ \times\left[\delta(p-p') + \frac{(\vec{p}-\vec{p})\cdot\vec{q}}{m_e}\frac{\partial\delta(\vec{p}-\vec{p}\,')}{\partial p'}\right]\left(f(\vec{p}\,') - f(\vec{p})\right)\end{aligned}\,. \tag{1.74}$$

Now, it is a fact from QFT that we can compute

$$|\mathcal{M}|^2 = 6\pi\sigma_T m_e^2 \Big(1+\cos^2\theta\Big)\,, \tag{1.75}$$

where $\cos\theta = \hat{p}\cdot\hat{p}'$.

For simplicity we replace this angle-dependent quantity with its angular average: the integral of the cosine gives us $1/3$, so we get a multiplier $6(1+1/3) = 8$:

$$\Big\langle |\mathcal{M}|^2 \Big\rangle = 8\pi\sigma_T m_e^2\,. \tag{1.76}$$

Now, we insert this in the expression and integrate over q: this yields a mean velocity, and also we expand the phase space distributions to first order:

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{2\pi n_e\sigma_T}{p}\int\frac{\mathrm{d}^3p'}{(2\pi)^3p'}\left[\delta(p-p')\big(\vec{p}-\vec{p}'\big)\cdot\vec{v}_b\frac{\partial\delta(p-p')}{\partial p'}\right] \\ \times\left[f^{(0)}(\vec{p}') - f^{(0)}(\vec{p}) - p'\frac{\partial f^{(0)}}{\partial p'}\theta(\vec{p}') + p\frac{\partial f^{(0)}}{\partial p}\theta(\vec{p})\right].\end{aligned} \tag{1.77}$$

[passages]

We do the angular integral, and simplify it by defining the *monopole* contribution:

$$\theta_0(\vec{x},t) = \frac{1}{4\pi}\int\mathrm{d}\Omega'\,\theta(\vec{x},\hat{p}',t)\,. \tag{1.78}$$

Then, finally, we integrate over p', which gives us the result

$$\hat{\mathbb{C}}[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p}n_e\sigma_T\big[\theta_0 - \theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\big]\,. \tag{1.79}$$

The factor due to the electron spin, g_e, is accounted for in the electron momentum distribution f_e.

So, for the photons we have

$$\frac{\partial\theta}{\partial t}\frac{\hat{p}^i}{a}\frac{\partial\theta}{\partial x^i} - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T\big(\theta_0 - \theta + \hat{p}\cdot\vec{v}_b\big)\,. \tag{1.80}$$

Our last step is to move to conformal time η, defined by $\mathrm{d}\eta = \mathrm{d}t\,/a$; denoting derivatives with respect to conformal time with a dot (and multiplying everything by a) we get:

$$\dot{\theta} + \hat{p}^i\frac{\partial\theta}{\partial x^i} - \dot{\Psi} + \hat{p}^i\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T a\big(\theta_0 - \theta + \hat{p}\cdot\vec{v}_b\big)\,. \tag{1.81}$$

Now, in order to solve this equation we perform a Fourier transform:

$$\theta(\vec{x}) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\widetilde{\theta}(\vec{k})\,, \tag{1.82}$$

and we define the *cosine* of the angle between the photon momentum $\vec{p}$ and the momentum $\vec{k}$: $\mu = \hat{k}\cdot\hat{p}$.

The optical depth is defined by

$$\tau(\eta) = \int_{\eta}^{\eta_0} \mathrm{d}\overline{\eta}\, a(\overline{\eta}) n_e \sigma_T \,, \tag{1.83}$$

and it is large at early times, small at late times since the density decreases. Its derivative is

$$\dot{\tau} = \frac{\mathrm{d}\tau}{\mathrm{d}\eta} = -n_e \sigma_T a \,, \tag{1.84}$$

which is a term in the Boltzmann equation! Substituting it, we get

$$\dot{\widetilde{\theta}} + ik\mu\widetilde{\theta} - \dot{\widetilde{\Psi}} + ik\mu\widetilde{\Psi} = -\dot{\tau}\left(\widetilde{\theta}_0 - \widetilde{\theta} + \mu\widetilde{v}_b\right). \tag{1.85}$$

It is an assumption we are making that the velocities are irrotational, so they can be expressed as a gradient: so, in Fourier space we get $\widetilde{\vec{v}}_b = \hat{k}\widetilde{v}_b$.

We still need to solve the Einstein equations in order to determine Φ and Ψ. In order to do this we need to describe all the component of the universe.

Also, we need to determine the velocity of the baryons v_b; it is an assumption we are making that in order to preserve local as well as global neutrality the local velocities of baryons and leptons are equal.

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