General Relativity exercises

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We set $c = 1$.

1 Sheet 1

1.1 Lorentz transformations

1.1.1 Inverses

We can consider a Lorentz boost with velocity v in the x direction, and we look at its representation in the (t, x) plane (since the y and z directions are unchanged). Its matrix expression looks like:

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix}, \tag{1}$$

where $\gamma = 1/\sqrt{1-v^2}$. The inverse of this matrix can be computed using the general formula for a 2x2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{2}$$

The determinant of Λ is equal to $\gamma^2(1-v^2) = 1$, therefore the inverse matrix is:

$$\Lambda = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix}. \tag{3}$$

1.1.2 Invariance of the spacetime interval

Our Lorentz transformation is

$$\mathrm{d}t' = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{4a}$$
$$\mathrm{d}x' = \gamma(-v\,\mathrm{d}t + \mathrm{d}x) \tag{4b}$$
$$\mathrm{d}y' = \mathrm{d}y \tag{4c}$$
$$\mathrm{d}z' = \mathrm{d}z \tag{4d}$$

and we wish to prove that the spacetime interval, defined by $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is preserved: $\mathrm{d}s'^2 = \mathrm{d}s^2$. Let us write the claimed equality explicitly:

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{5a}$$

1.1.3 Tensor notation pseudo-orthogonality

The invariance of the spacetime interval $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$ can be also written as $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x'^{\,\mu}\,\mathrm{d}x'^{\,\nu}$. By making the primed differentials explicit we have:

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho\,\mathrm{d}x^\rho\,\Lambda^\nu{}_\sigma\,\mathrm{d}x^\sigma\ , \tag{6}$$

but the dummy indices on the LHS can be changed to ρ and σ, so that both sides are proportional to $\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$. Doing this we get:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = (\Lambda^\top)_\rho{}^\mu\eta_{\mu\nu}\Lambda^\nu{}_\sigma\,, \tag{7}$$

or, in matrix form, $\eta = \Lambda^\top\eta\Lambda$.

1.1.4 Pseudo orthogonality

Defining $dx^\mu = (cdt, dx, dy, dz)^T$, and $dx_\mu = (cdt, dx, dy, dz) = \eta_{\mu\nu}dx^\nu$, we will have

$$\begin{aligned} ds^2 &= dx^\mu dx_\mu = dx^\mu\eta_{\mu\nu}dx^\nu \\ \mathrm{d}s,^{2\prime} &= dx^{\mu\prime}dx'_\mu = dx^{\mu\prime}\eta_{\mu\nu}dx^{\nu\prime} = dx^\rho\Lambda^\mu_\rho\eta_{\mu\nu}dx^\sigma\Lambda^\nu_\sigma \end{aligned} \tag{8}$$

and $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$ is equivalent to our thesis.

1.1.5 Pseudo orthogonality by matrix product

Let Λ be the Lorentz transformation in (1).bIn this case we have

$$\begin{aligned}
\Lambda^T\eta\Lambda &= \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} -\gamma & -v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} -\gamma^2 + v^2\gamma^2 & 0 & 0 & 0 \\ 0 & -v^2\gamma^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \eta
\end{aligned} \tag{9}$$

Where we used the fact that $\gamma^2(1 - v^2) = \frac{\gamma^2}{\gamma^2} = 1$.

1.1.6 Explicit pseudo-orthogonality

For simplicity but WLOG we consider a boost in the x direction with velocity v and Lorentz factor γ. The matrix expression to verify is:

$$\begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10a}$$

$$\begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} -\gamma & v\gamma \\ -v\gamma & \gamma \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10b}$$

$$\begin{bmatrix} -\gamma^2 + \gamma^2 v^2 & v\gamma^2 - v\gamma^2 \\ v\gamma^2 - v\gamma^2 & -v\gamma^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{10c}$$

which by $\gamma^2 = 1/(1 - v^2)$ confirms the validity of the expression.

1.2 Muons

1.2.1 Nonrelativistic approximation

The survival probability is given by $\mathbb{P}(t) = \exp\left(-t/2.2 \times 10^{-6}\,\mathrm{s}\right)$. If the ground is $h = 15\,\mathrm{km}$ away, then the muon will reach it in $t = h/v = 15\,\mathrm{km}/(0.995c) \approx 5.03 \times 10^{-5}\,\mathrm{s}$, therefore $\mathbb{P}(t) \approx 1.2 \times 10^{-10}$.

1.2.2 Relativistic effects: ground perspective

The observer on the ground will see the muon having to traverse the whole $h = 15\,\mathrm{km}$, but the muon's time will be dilated for them by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$.

1.2.3 Relativistic effects: muons perspective

The muons in their system will observe lenght contraction, with respect to Lorentz boost, by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$. This result is the same of the one predicted by ground observer, with respect to relativity principle.

1.3 Radiation

1.3.1 New angle

In the source frame the radiation velocity components are $u'_x = \cos\theta', u'_y = \sin\theta'$. From the composition of velocities we obtain:

$$\sin\theta = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\sin\theta'}{\gamma_v(1 + v\cos\theta')}$$

$$\cos\theta = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma_v(\mathrm{d}x' + v\,\mathrm{d}t')}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\cos\theta' + v}{1 + v\cos\theta'}$$

Hence

$$\frac{1}{\tan\theta} = \frac{1}{\tan\theta'} + \frac{\gamma_v v}{\sin\theta'}$$

1.3.2 Angle plot and relevant limits

See the jupyter notebook in the `python` folder for plots. For $v = 0$ we have $\theta = \theta'$ as we expected, while for $v = 1$, $\theta = 0$.

1.3.3 Radiation speed invariance

Are the components of the velocity, which we called $\sin\theta$ and $\cos\theta$, actually normalized? Let us check:

$$\sin^2\theta + \cos^2\theta = \frac{(\frac{\sin\theta'}{\gamma_v})^2 + (\cos\theta' + v)^2}{(1 + v\cos\theta')^2} = \frac{(1 - v^2)\sin^2\theta' + \cos^2\theta' + v^2 + 2v\cos\theta'}{(1 + v\cos\theta')^2} \tag{11a}$$

$$= \frac{1 + v^2(1 - \sin\theta') + 2v\cos\theta'}{(1 + v\cos\theta')^2} = 1\,, \tag{11b}$$

therefore the square modulus of the speed of the radiation is still c, as we could have assumed earlier.

1.3.4 Isotropic emission

Since we had that angle emission vary when varying inertial system we obtain that every system in relative motion respect to O with $v \neq 0$ observs a nonisotropic emission. For $v \simeq 1$ we have that in the observer system there is almost only $\theta = 0$ emission.

See the jupyter notebook in the `python` folder for interactive plots :)