Gravitational physics notes

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Contents

Monday 2020-3-9

0.1 Technical details

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Reading material: slides (to be used as an index of what is treated in the course), Hobson [HEL06], Michele Maggiore [Mag07; Mag18].

This is a general class on gravitational physics and GW, it does not really follow any textbook: the field is young so there is no textbook covering all the necessary topics, really.

The slides will be provided before lectures. There will be no home assignments.

The idea for the exam is that formulas are important, detailed calculations and derivations are not.

The target is to be able to read a research paper on GW and understand it. We will not go into very much detail on any topic: the program of the class is very large.

For the exam: it is a discussion of a GW paper (about 25 min), plus theoretical questions — focusing on the physical meaning, not on tedious derivations. It usually takes a bit less than an hour. The paper is optional.

Off session exams are OK, if the exam is live then its is best if it is organized with several people (2-5 people).

Please fill out the questionnaire on the course before taking the exam.

0.1.1 Topics

Understanding **what gravitational waves are**: how they are described, how they are generated, what is their physical effect.

Interactions of GW with light and matter: ideas, techniques, experiments to detect GW, especially GW interferometers.

Analysis of GW signals.

What we can learn from GW, overview of the most significant recent papers.

Chapter 1

Gravitational waves

1.1 Introduction

Einstein thought the detection of GW impossible; at the time it was thought that they might be a coordinate artifact which could be “gauged away”.

Now we can not only *detect* them, we can actually *observe* them, determining their position in the sky and their parameters.

They are a test of GR in *extreme* conditions, where the weak-field approximation does not apply. We can test the properties of matter in these extreme conditions, such as the equation of state for a neutron star.

GW are “ripples” in the metric of spacetime; their production is described by a quadrupole formula: the quadrupole is

$$Q_{jk} = \int \rho x_j x_k \, \mathrm{d}^3 x \,, \tag{1.1}$$

and then the perturbation propagates like

$$h_{jk} = \frac{2}{r} \frac{\mathrm{d}^2 Q_{jk}}{\mathrm{d}t^2} \,. \tag{1.2}$$

What generates GW are non-spherically symmetric perturbations: by Jebsen-Birkhoff, if we have spherical symmetry there is no perturbation in the vacuum metric. The simplest kind of object which can generate them is a binary system.

The effect of a GW is to “stretch” space by squeezing one direction and stretching a perpendicular one, in an area-preserving way. The typical relative scale of these perturbations is

$$\frac{\Delta L}{L} \sim 10^{-21} \,, \tag{1.3}$$

which is *really small*: if we multiply it by the radius of the Earth’s orbit we get a length on the order of the size of an atom.

We have different kinds of interferometers for different GW frequency ranges: for now we have only used ground interferometers, but in the works there are also space detectors like LISA, Pulsar Timing Arrays at higher frequencies, and inflation probes.

In **binary systems**, we have different stages in the pulsation: an almost stationary one, the inspiral, the coalescence, and finally the ringdown. The frequency and amplitude both increase up to the coalescence, after it the frequency is almost constant while the amplitude decreases.

In 1959, Joseph Weber proposed a "**resonant bar**" detector. These are based upon a sound principle, and this path was explored for several decades with, for example, AURIGA; the issue was that the sensitivity was insufficient, and these detectors would only be sensitive in a specific high frequency range.

GW were first detected indirectly using **Hulse-Taylor pulsars**: they measured the energy loss of a binary pulsar-NS system, which implied the loss of energy through gravitational wave emission. The famous graph is not a fit line, it is the prediction based upon the measured orbital parameters.

Now we use ground-based laser **interferometers**: they are broad-band (a couple orders of magnitude, from 10 Hz to 1 kHz), they are inherently differential (as opposed to the single-mode excitation of a resonant bar).

We can use Fabry-Pérot cavities in order to amplify effective length, by "storing photons" for several bounces in the interferometer. There is also a power recycling mirror in order for the light not to go back to the laser: with modern lasers and these systems we can get 10 kW of power circulating in the cavities.

We can plot the sensitivity of the interferometers: on the x axis we put the frequency of the incoming wave, and on the y axis we put the amplitude spectral density $h(f)$, which is measured in $\text{Hz}^{-1/2}$.

The curve describes where the noise dominates. We can plot both the theoretical sensitivity, with its various sources, and the measured one.

The signal comes out buried in noise, we must extract it in some way, like by correlating to a standard test signal.

A planned detector is **LISA**, which is a space interferometer but it works differently from the ground-based one, since it cannot reflect the beam back, and since the travel time for the beam is several seconds.

Another way to detect low-frequency GW is by using **Pulsar Timing Arrays**: the idea is to monitor several millisecond pulsars, and compute the difference in the arrival times of their signals; distortions of spacetime will modify the relative distances between us and these.

Also, we have **atomic interferometry** detectors, by having the wavefunctions of atoms do what light does in an interferometer. This is not a mature technology as of now.

We have seen several **BH-BH mergers** and some NS-NS ones, with masses between a few and about $80 M_\odot$. With these, people are starting to do studies on the populations of these objects and their formation mechanisms. Unfortunately, with our frequency range we can only see a very short part of the signal, up to a few seconds at most.

With **NS-NS mergers** we can investigate short gamma ray bursts, the formation of heavy elements, the mass of gravitational waves (which in GR is zero), the rate of expansion of the universe.

Friday 2020-3-13, compiled 2020-07-10

1.2 A quick review of GR

We start from special relativity. The "old" way to do transformations are Galilean transformations: in 2D they are

$$t' = t \tag{1.4a}$$
$$x' = x - vt\,. \tag{1.4b}$$

There are issues with these: they do not respect the equivalence principle and the invariance of the speed of light. So, we move to Lorentz transformations:

$$ct' = \gamma(ct - \beta x) \tag{1.5a}$$
$$x' = \gamma(x - \beta ct)\,, \tag{1.5b}$$

where $\beta = v/c \leq 1$, c being the speed of light, and $\gamma = 1/\sqrt{1-\beta^2} \geq 1$.

These preserve the spacetime interval, which in our mostly plus metric convention reads:

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2\,. \tag{1.6}$$

The interval between two events can be spacelike ($\Delta s^2 > 0$), null ($\Delta s^2 = 0$) or timelike ($\Delta s^2 < 0$).

We can express this using an infinitesimal time interval

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \, \eta_{\mu\nu} \, \mathrm{d}x^\nu \,, \tag{1.7}$$

where we use Einstein summation convention.

We can define the differential *proper time* along a curve, by

$$c^2 \, \mathrm{d}\tau^2 = -\,\mathrm{d}s^2 = c^2 \, \mathrm{d}t^2 \left(1 - \beta^2\right) = \frac{c^2}{\gamma^2} \, \mathrm{d}t^2 \,, \tag{1.8}$$

which means that $\mathrm{d}\tau = \mathrm{d}t\,/\gamma$. The parameter τ can then be used as a natural *covariant* parametrization of a spacetime curve.

We model spacetime it as a 4D semi-Riemannian manifold with metric signature $(1, 3)$. Since it is a manifold, the parametrization of points in spacetime must be a homeomorphism, and we ask for the *transition maps* between two regions of spacetime to be infinitely differentiable. The set of local charts is called an atlas. The charts are maps from $\mathbb{R}^4$ to the manifold.

The coordinates we use are arbitrary: this is very powerful, but it is tricky to find the right ones.

The **metric** is a function of the point at which we are, and (the way it changes) describes the local geometry of the manifold. Only the symmetric part of the metric appears in the spacetime interval, therefore we say that the metric is always symmetric without losing any generality.

The metric is a bilinear form at each point of the manifold, and it transforms as a $(0, 2)$ tensor. The components of this tensor in our chosen reference frame are $g_{\mu\nu}$.

In a neighborhood of a point we can always choose a reference frame (Riemann normal coordinates) such that $g_{\mu\nu} = \eta_{\mu\nu}$, and $g_{\mu\nu,\alpha} = 0$ (partial derivatives calculated *at that point*), but the second derivatives $g_{\mu\nu,\alpha\beta}$ cannot all be set to zero.

Vectors in a manifold are defined in the tangent space *at a point*. Intuitively, at each point we can define locally Cartesian coordinates, and the tangent is the space they span.

Formally, we define curves parametrically as functions from the real numbers to the manifold: $X^\mu(\lambda)$. Then, we define the tangent vector to the curve as the *directional derivative* operator along the curve:

$$\vec{v}(f) = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_C = \frac{\partial f}{\partial x^\mu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\,, \tag{1.9}$$

which associates to any scalar field f its directional derivative. The motivation for this definition, as opposed to just taking the tangent vector to the curve, is the fact that there is no *intrinsic* way to do that.

If we define a curve using a coordinate as a parameter, with the other coordinates staying constant along the curve, this is called a *coordinate curve*.

Vectors defined at different points are in different spaces, we cannot compare them directly.

Tangent vectors to coordinate lines are called coordinate basis vectors $e_{(\mu)}$, where μ is not a vector index but instead it spans the basis vectors. Any vector can be written as a linear combination of these as $\vec{v} = v^\mu e_\mu$. We always have $e_\mu \cdot e_\nu = g_{\mu\nu}$, so, in order to find the components of the scalar product $v \cdot w$ we need to do $v^\mu w^\nu g_{\mu\nu}$. This is because $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}s \cdot \mathrm{d}s = \left(\mathrm{d}x^\mu\, e_\mu\right) \cdot (\mathrm{d}x^\nu\, e_\nu)$.

An *orthonormal basis* in one for which $e_\mu \cdot e_\nu = \eta_{\mu\nu}$. Dual basis vectors e^μ are defined by $e^\mu e_\nu = \delta^\mu_\nu$. We write a co-vector (or dual vector) as a linear combination of these: $v = v_\mu e^{(\mu)}$.

Then, we can raise and lower indices like

$$g_{\mu\nu}v^\mu w^\nu = v \cdot w = v_\mu e^\mu \cdot w^\nu e_\nu = v_\mu w^\nu \delta^\mu_\nu = v_\mu w^\nu\,. \tag{1.10}$$

The inverse metric $g^{\mu\nu}$ is defined by the relation $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$.

Tensors are geometrical objects which belong to the dual space of the Cartesian product of n copies of the tangent space and m copies of the dual tangent space. The type of a tensor in this space is then said to be (n, m), and its rank is $n + m$. This definition means that the tensor is a *multilinear* transformation, associating a scalar to n vectors and m covectors in a multilinear way.

Once we have a coordinate system, we can move to another via a coordinate transformation $x'^\mu = x'^\mu(x^\mu)$, and then the differential of the coordinates will transform like

$$x'^\mu = x'^\mu(x^\mu) \implies \mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu\,. \tag{1.11}$$

A scalar is something which does not transform: $\phi(x) = \phi'(x')$. A vector's and a covector's components do transform: we find the transformation law by imposing $v = v'$ in components, so that we get

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}V^\nu \qquad \text{and} \qquad V_\mu = \frac{\partial x^\nu}{\partial x'^\mu}V_\nu\,. \tag{1.12}$$

This can be generalized to the transformation law of a tensor of arbitrary rank, which will transform with the product of a Jacobian for each upper index and an inverse Jacobian for each lower index.

In order to compute derivatives we need to compare vectors in different tangent spaces: we need to "connect" infinitesimally close tangent spaces, and the tool to do so is indeed called a connection, or covariant derivative. The covariant derivative of a tensor is required to still be a tensor, with a rank which is higher by one:

The covariant derivative of a vector V^μ is defined by introducing the *Christoffel symbols* Γ, which are objects with three indices which do *not* transform like tensors and which account for the "shifting of the basis vectors":

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \,. \tag{1.13}$$

For a scalar S the covariant derivative is $\nabla_\alpha S = \partial_\alpha S$.

We require the manifold we work in to be torsionless. A torsionless manifold is one in which

$$[\nabla_\mu, \nabla_\nu] S = 0 \,, \tag{1.14}$$

for a scalar field S. This means that

$$\nabla_{[\mu} \nabla_{\nu]} S = \nabla_{[\mu} \partial_{\nu]} S = \partial_{[\mu} \partial_{\nu]} S - \Gamma^\alpha_{[\mu\nu]} \partial_\alpha = 0 \implies \Gamma^\alpha_{[\mu\nu]} = 0 \,, \tag{1.15}$$

so the Christoffel symbols are symmetric in their lower indices.

Parallel transport: intuitively, we move along a curve and keep the angle with respect to the tangent vector constant. Formally, if u^μ is the tangent vector to the curve and V^μ is the vector we want to transport, we set $u^\mu \nabla_\mu V^\nu = 0$.

This parallel-transport is path-dependent: in general a vector which is transported along a curve does not come back to itself.

The Riemann tensor $R^\alpha_{\beta\mu\nu}$ is defined from the commutator of the covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta \,, \tag{1.16}$$

and it can be expressed in terms of the Christoffel symbols as

$$R^\mu_{\nu\rho\sigma} = -2 \left(\Gamma^\mu_{\nu[\rho,\sigma]} + \Gamma^\beta_{\nu[\rho} \Gamma^\mu_{\sigma]\beta} \right) . \tag{1.17}$$

This tensor measures the curvature of the manifold: if it is zero, then the manifold is flat ($g_{\mu\nu} = \eta_{\mu\nu}$). We define its trace $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ as the Ricci tensor, whose trace $R = R_{\mu\nu} g^{\mu\nu}$ is called the Ricci scalar, or curvature scalar.

The Riemann tensor has several symmetries, both differential (related to its derivatives) and not (antisymmetries and symmetries of its indices) [HEL06, eqs. 7.14-7.18], making it so that its free components in N spatial dimensions are not N^4 but instead $N^2(N^2-1)/12$. In 4D, this means 20 free components.

Geodesics: they are "the straightest possible path between two points"; they stationarize the proper length. Formally, they are curves whose tangent vector is parallel-transported along the curve (it "always points in the same direction").

The path that a massive particle follows in the absence of external forces is a geodesic. We can describe the evolution of the separation between two nearby particles which follow geodesics: this is described by the equation of geodesic deviation. We take a geodesic x^μ and another $y^\mu = x^\mu + \xi^\mu$, with ξ^μ being (at least initially) small. Let us call the starting point of x^μ P and the starting point of y^μ Q, also let us take a coordinate system in which $\Gamma^\mu_{\nu\rho}\Big|_P = 0$. This can always be done, but note that the *derivatives* of the Christoffel symbols cannot all be set to zero.

So, we can write the geodesic equation for the two curves at their starting points as

$$\left.\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}u^2}\right|_P = 0 \qquad \text{and} \qquad \left.\left(\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}u^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}y^\nu}{\mathrm{d}u}\frac{\mathrm{d}y^\rho}{\mathrm{d}u}\right)\right|_Q = 0\,, \tag{1.18}$$

where u is the tangent vector to the geodesics. We approximate the Christoffel symbols to first order as

$$\left.\Gamma^\mu_{\nu\rho}\right|_Q = \xi^\alpha \partial_\alpha \left.\Gamma^\mu_{\nu\rho}\right|_P\,. \tag{1.19}$$

If we subtract the two and only keep the first order in ξ, we get

$$0 = \left(\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}u^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}y^\nu}{\mathrm{d}u}\frac{\mathrm{d}y^\rho}{\mathrm{d}u}\right) - \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}u^2} = \frac{\mathrm{d}^2 \xi^\mu}{\mathrm{d}u^2} + \xi^\alpha\partial_\alpha\Gamma^\mu_{\nu\rho}\frac{\mathrm{d}(x^\nu+\xi^\nu)}{\mathrm{d}u}\frac{\mathrm{d}(x^\rho+\xi^\rho)}{\mathrm{d}u} \tag{1.20}$$

$$= \frac{\mathrm{d}^2 \xi^\mu}{\mathrm{d}u^2} + \xi^\alpha\partial_\alpha\Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho + \mathcal{O}(\xi^2) \tag{1.21}$$

A dot denotes u differentiation.

$$\approx \ddot{\xi}^\mu + \left(\partial_\alpha\Gamma^\mu_{\nu\rho}\right)\dot{x}^\nu\dot{x}^\rho\xi^\alpha = 0\,, \tag{1.22}$$

but the first term is not an intrinsic derivative: that would be given by

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\dot{\xi}^\mu + \Gamma^\mu_{\nu\rho}\xi^\nu\dot{x}^\rho\right) + \mathcal{O}(\xi^2) = \ddot{\xi}^\mu + \dot{x}^\alpha\partial_\alpha\Gamma^\mu_{\nu\rho}\xi^\nu\dot{x}^\rho + \mathcal{O}(\xi^2) \tag{1.23}$$

$$= \ddot{\xi}^\mu + \dot{x}^\nu\partial_\nu\Gamma^\mu_{\alpha\rho}\xi^\alpha\dot{x}^\rho + \mathcal{O}(\xi^2)\,, \tag{1.24}$$

Relabeled the contracted indices.

so we can write the differential equation we have found by inserting this expression for $\ddot{\xi}$:

$$0 = \frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} + \left(\partial_\alpha\Gamma^\mu_{\nu\rho} - \partial_\nu\Gamma^\mu_{\alpha\rho}\right)\xi^\alpha\dot{x}^\nu\dot{x}^\rho = \frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} + R^\mu_{\nu\alpha\rho}\xi^\alpha\dot{x}^\nu\dot{x}^\rho\,. \tag{1.25}$$

In the last step we used the fact that in the frame we chose, a Local Inertial Frame, the Christoffel symbols are zero so $R = \partial\Gamma + \Gamma\Gamma$ simplifies to $R = \partial\Gamma$.

The gravitational field's dependence on the matter content of the universe is described by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}\,, \tag{1.26}$$

which follow from the assumptions that

1. they should be a local tensorial equation;

2. they should relate $T_{\mu\nu}$ with $g_{\mu\nu}$ and its derivatives;

3. they should reduce to Newton's equation for the gravitational potential ϕ, $\nabla^2\phi = 4\pi G\rho$, in the weak field limit.

They look rather simple, but there is a lot of hidden complexity: the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/2$ is calculated from the Riemann tensor by taking a trace; the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$ is calculated from the Christoffel symbols by differentiating and multiplying them ($R \sim \partial\Gamma + \Gamma\Gamma$), the Christoffel symbols are calculated from the metric by differentiating it and multiplying it by the inverse $\Gamma \sim g^{-1}\partial g$.

In this course we will be using the sign conventions adopted by Misner, Thorne and Wheeler [MTW73, page 3]; some authors adopt different signs for

1. the metric $\eta_{\mu\nu} = \pm\mathrm{diag}(-1, 1, 1, 1)$;

2. the Riemann tensor $R^{\mu}_{\nu\rho\sigma} = \mp 2\left(\Gamma^{\mu}_{\nu[\rho,\sigma]} + \Gamma^{\alpha}_{\nu[\rho}\Gamma^{\mu}_{\sigma]\alpha}\right)$;

3. the Einstein Equations: $G_{\mu\nu} = \pm T_{\mu\nu}/M_P^2$.

All of these conventions are equivalent, but one must be careful when comparing different sources.

1.3 Linearized GR

Monday 2020-3-16, compiled 2020-07-10

Let us assume that there exists a reference frame in which our metric tensor is almost flat: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\left|h_{\mu\nu}\right| \ll 1$.

This is a coordinate dependent statement: however the physical situation is clear — almost flat spacetime — and the way we will proceed is to work to linear order in $h_{\mu\nu}$.

Do note that the inverse metric is given by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, and that we can raise and lower the indices of $h_{\mu\nu}$ using $\eta_{\mu\nu}$, since the corrections would be second order.

Choosing a reference in which these components are small limits our gauge freedom: we will only be able to do transformations which preserve the condition. Let us now discuss explicitly **which transformations we will still be able to use**.

We will be able to apply global Lorentz transformations Λ, which act on the metric as:

$$g' = \Lambda^{-1}\Lambda^{-1}g = \Lambda^{-1}\Lambda^{-1}(\eta + h) = \eta + \Lambda^{-1}\Lambda^{-1}h\,, \tag{1.27}$$

so the flat metric part does not change, while h changes to $h' = \Lambda^{-1}\Lambda^{-1}h$. We are omitting indices for clarity, they are in the usual positions.

A more general class of transformation is given by *infinitesimal translations*, which can be expressed as

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu}\,, \tag{1.28}$$

and whose Jacobian looks like

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu} \,. \tag{1.29}$$

We ask that $\partial\xi$ is small — formally, the first order in $\partial_{\mu}\xi_{\nu}$ is the same as the first order in $h_{\mu\nu}$.

This yields, always to first order:

$$\begin{aligned} g'_{\mu\nu} &= \left(\delta^{\rho}_{\mu} - \partial_{\mu}\xi^{\rho}\right)\left(\delta^{\sigma}_{\nu} - \partial_{\nu}\xi^{\sigma}\right)\left(\eta_{\rho\sigma} + h_{\rho\sigma}\right) & (1.30)\\ &= \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\xi^{\rho}\eta_{\rho\sigma} - \partial_{\nu}\xi^{\sigma}\eta_{\rho\sigma} & (1.31)\\ &= \eta_{\mu\nu} + h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)} \,. & (1.32) \end{aligned}$$

For an alternative reference on this derivation, see the notes on General Relativity in the course by Marco Peloso [TM20, section 10]. The crucial thing here is that our transformation left the metric in the form "flat + first-order infinitesimal", so we still are satisfying the assumptions we set at the start.

Now, we wish to linearize the Riemann tensor: we must start from the Christoffel symbols. We can discard the derivatives of the flat metric and substitute the inverse metric at the start of the expression with the flat one, since the parenthesis is already first order:

$$\begin{aligned} \Gamma^{\sigma}_{\mu\nu} &= \frac{1}{2} g^{\sigma\rho}\left(2g_{\rho(\mu,\nu)} - g_{\mu\nu,\rho}\right) & (1.33a)\\ &= \frac{1}{2}\left(\partial_{\mu}h^{\sigma}_{\nu} + \partial_{\nu}h^{\sigma}_{\mu} + \partial^{\sigma}h_{\mu\nu}\right), & (1.33b) \end{aligned}$$

and now in the Riemann tensor $R = \partial\Gamma + \Gamma\Gamma$ the $\Gamma\Gamma$ terms are second order in h, so we ignore them. Then, we get the simplified expression

$$R^{\sigma}_{\mu\nu\rho} = \frac{1}{2}\left(\partial_{\nu}\partial_{\mu}h^{\sigma}_{\rho} + \partial_{\rho}\partial^{\sigma}h_{\mu\nu} - \partial_{\nu}\partial^{\sigma}h_{\mu\rho} - \partial_{\rho}\partial_{\mu}h^{\sigma}_{\nu}\right), \tag{1.34}$$

so the Ricci tensor — which we will set to zero in the vacuum — will be

$$R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma} = \frac{1}{2}\left(\partial_{\nu}\partial_{\mu}h + \Box h_{\mu\nu} - \partial_{\nu}\partial_{\sigma}h^{\sigma}_{\mu} - \partial_{\sigma}\partial_{\mu}h^{\sigma}_{\nu}\right), \tag{1.35}$$

and the Ricci scalar is

$$R = \eta^{\mu\nu}R_{\mu\nu} = \Box h - \partial_{\nu}\partial_{\sigma}h^{\sigma\nu} \,, \tag{1.36}$$

where h is the trace of the perturbation, $h = h^{\sigma}_{\sigma}$, while $\Box = \partial_{\mu}\partial^{\mu}$ is the Dalambertian operator. The field equations read

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &\approx \frac{1}{2}\left[\partial_{\nu}\partial_{\mu}h + \Box h_{\mu\nu} - \partial_{\nu}\partial_{\sigma}h^{\sigma}_{\mu} - \partial_{\sigma}\partial_{\mu}h^{\sigma}_{\nu} - \eta_{\mu\nu}\left(\Box h - \partial_{\nu}\partial_{\sigma}h^{\sigma\nu}\right)\right] & (1.37)\\ &= -\frac{1}{M_P^2}T_{\mu\nu} \,, & (1.38) \end{aligned}$$

The metric multiplying h can be written as the flat one, since h is first order already.

where $M_P = 1/\sqrt{8\pi G}$ in natural units is the reduced Planck mass.

We define the trace-reversed perturbation as

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{1.39}$$

so the doubly-trace reverse is the perturbation itself, $\overline{\overline{h}}_{\mu\nu} = h_{\mu\nu}$, and the trace-reversed trace is the negative of the trace: $\overline{h} = -h$. In terms of this, the linearized equations read

$$\Box\overline{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\overline{h}^{\rho\sigma} - \partial_\nu\partial_\rho\overline{h}^\rho_\mu - \partial_\mu\partial_\rho\overline{h}^\rho_\nu = -2\frac{T_{\mu\nu}}{M_P^2}\,, \tag{1.40}$$

which we can simplify greatly using our gauge freedom: we shall use the so-called Lorenz gauge[1].

How does the trace-reverse perturbation transform?

$$\overline{h}'^{\mu\rho} = h^{\mu\rho} - 2\partial^{(\mu}\xi^{\rho)} - \frac{1}{2}\eta^{\mu\rho}\left(h - 2\partial_\sigma\xi^\sigma\right) \tag{1.41a}$$

$$= \overline{h}^{\mu\rho} - 2\partial^{(\mu}\xi^{\rho)} + \eta^{\mu\rho}\partial_\sigma\xi^\sigma\,. \tag{1.41b}$$

The derivatives of the new and old perturbations differ by

$$\partial_\rho\overline{h}'^{\mu\rho} - \partial_\rho\overline{h}^{\mu\rho} = \partial_\rho\left(-\partial^\mu\xi^\rho - \partial^\rho\xi^\mu + \eta^{\mu\rho}\partial_\sigma\xi^\sigma\right) = -\Box\xi^\mu\,, \tag{1.42}$$

so we can choose our gauge with a transformation defined by ξ^μ such that $\partial_\rho\overline{h}'^{\mu\rho}$, since the field $\Box\xi^\mu$ can be chosen arbitrarily with a suitable choice of ξ^μ.

This means we can remove all the $\partial\overline{h}$ terms and find:

$$\boxed{\Box\overline{h}_{\mu\nu} = -\frac{2T_{\mu\nu}}{M_P^2}\,,} \tag{1.43}$$

Comments on the linearized equations Since we expanded in $\eta_{\mu\nu}$, the quantities $h_{\mu\nu}$ have a geometric meaning, but we are treating them as 16 scalar fields on a flat background.

When we look at the geodesic equations, we get a prediction of the gravity having no effect on matter: there is no *backreaction*, and if we want to model the effect of GW energy and angular momentum loss we will have to insert them by hand. We are treating gravity as a linear theory, so we have the superposition principle, according to which the fields due to different particles can be added.

We are ignoring the physical principle that “gravity gravitates”: curvature of spacetime is associated to a SEMT in a nonlinear way.

[1] This is not the same as Loren*t*z, after whom Lorentz covariance is called.

GW in empty space We set $T_{\mu\nu}$ to zero, so we get

$$\Box \bar{h}_{\mu\nu} = 0\,, \tag{1.44}$$

which is the usual wave equation: its solutions are superpositions of plane waves $\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\lambda x^\lambda}$.

In general $A_{\mu\nu}$ is symmetric, constant, complex. k_λ is constant and real, and by taking the derivative we find that we must have

$$\eta^{\rho\sigma} k_\rho k_\sigma A_{\mu\nu} e^{ik\cdot x} \implies k^2 = 0\,, \tag{1.45}$$

which means that the wave travels at light speed, since $k^\lambda = (\omega/c, \vec{k})$.

In order to have these conditions, we must still impose the Lorenz gauge condition we chose in the derivation:

$$\partial_\mu \bar{h}^{\mu\nu} = A^{\mu\nu} k_\mu e^{ik\cdot x} = 0 \implies A^{\mu\nu} k_\nu = 0\,. \tag{1.46}$$

The conjugate of the wave equation also holds, so after our manipulations we will always be able to take the real part.

We still have gauge freedom: we can perform transformations if they satisfy $\Box \xi^\mu = 0$, so that we do not alter the value of $\partial_\rho \bar{h}^{\mu\rho}$. We define

$$\xi^{\mu\nu} = \partial^\mu \xi^\rho + \partial^\rho \xi^\mu - \eta^{\mu\rho} \partial_\sigma \xi^\sigma\,, \tag{1.47}$$

which satisfies the wave equation $\Box \xi^{\mu\nu} = 0$ if ξ^μ does, since the Dalambertian commutes with the other derivatives.

So, if $\bar{h}^{\mu\rho}$ satisfies the vacuum field equations, then $\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \xi^{\mu\nu}$ also does.

Then, we can use the 4 functions ξ^μ to set 4 constraints on $\bar{h}^{\mu\rho}$: we choose to set

$$\bar{h}^{0i}_{TT} = 0 \tag{1.48a}$$

$$\bar{h}_{TT} = 0\,, \tag{1.48b}$$

which conveniently means that $\bar{h}_{\mu\nu} = h_{\mu\nu}$. This is called Transverse-Traceless gauge. We want to write out **all the gauge constraints** on our perturbation. The Lorenz gauge $\partial_\rho \bar{h}^{\mu\rho} = 0$ consists of four equations: the $\mu = 0$ one reads

$$0 = \partial_\rho \bar{h}^{0\rho}_{TT} = \partial_0 \bar{h}^{00}_{TT}\,, \tag{1.49}$$

$h^{i0} = 0$.

which means that the metric element $\bar{h}^{00}$ is constant, so we can set it to zero, since a constant in the metric is not relevant for our study of oscillations.[2] Right now, the only nonzero components are the h_{ij}, which must be traceless and symmetric.

The other $\mu = j$ Lorenz gauge conditions are the three constraints

$$0 = \partial_\rho \bar{h}^{j\rho}_{TT} = \partial_i \bar{h}^{ji}_{TT}\,, \tag{1.50}$$

[2] Also, a constant can be reabsorbed into the background metric: it amounts to measuring all of time with a slightly slower or faster clock.

which means that, of the 5 potentially free components of the traceless symmetric h^{ij}, we actually have only 2 true degrees of freedom.

Now, if we align our reference frame so that $\vec{k} = k\hat{z}$ we will have $k^\mu = (k, 0, 0, k)$; also, the matrix $A_{\mu\nu}$ will need to be symmetric and satisfy $A^{ij}k_j = 0$, so $A^{i3} = 0$.

This means that the only nonzero components are A_{ij}, with i, j between 1 and 2; also $A_{11} = -A_{22} \stackrel{\text{def}}{=} h_+$ and $A_{12} = A_{21} \stackrel{\text{def}}{=} h_\times$. Then, in full generality under our gauge choices we shall have

$$\overline{h}^{\mu\nu}_{TT} = h^{\mu\nu}_{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{ik(t-z)} . \tag{1.51a}$$

Claim 1.3.1. *For a generic direction of propagation, we can define a projector onto the directions orthogonal to the direction of propagation k_i: $P_{ij} = \delta_{ij} - n_i n_j$, where $n_i = k_i / \left|\vec{k}\right|$.*

Proof. The conditions we need to check for this to be a projector onto the directions we want are:

1. idempotency: $P_{ij}P_{jk} = P_{ik}$;
2. orthogonality: $P_{ij} = P_{ji}$;
3. that it projects on the right space: $P_{ij}k_j = 0$ and $P_{ij}y_j = y_j$ if $y_j k_j = 0$.

The computations are:

$$P_{ij}P_{jk} = \left(\delta_{ij} - n_i n_j\right)\left(\delta_{jk} - n_j n_k\right) \tag{1.52}$$

$$= \delta_{ik} - 2n_i n_k + n_i \underbrace{n_j n_j}_{+1} n_k , \tag{1.53}$$

where we used the fact that n_j is chosen to be normalized, and we are using the $- + ++$ metric convention. Note that k_i is not the full four-vector k^μ, which is null, but instead it is only its spatial part. As for projecting on the right space, we have:

$$P_{ij}k_j = k_i - k_i \underbrace{k_j k_j}_{+1} = 0 \qquad \text{and} \qquad P_{ij}y_j = y_i - k_i \underbrace{k_j y_j}_{0} = y_i . \tag{1.54}$$

□

Using this P_{ij} we might try to calculate the transverse traceless A^{TT}_{ij} like

$$A^{TT}_{ij} \stackrel{?}{=} P^k_i P^l_j A_{kl} , \tag{1.55}$$

but the resulting tensor is not in general traceless: we need to subtract the trace, so the correct expression reads

$$A^{ij}_{TT} = \underbrace{\left(P^i_k P^j_l - \frac{1}{2} P^{ij} P_{kl}\right)}_{\Lambda_{ij,kl}} A^{kl} . \tag{1.56}$$

The tensor we defined,

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\,, \tag{1.57}$$

is a projector, it is transverse with respect to all of its indices, and its traces where we set $i = j$ or $k = l$ are zero [Mag07, eqs. 1.36 to 1.39].

Friday 2020-3-20, compiled 2020-07-10

1.4 The physical effects of gravitational waves

We want to discuss how we can build instruments which can detect gravitational waves.

An open question for decades (from 1916 to 1957) was to theoretically determine whether the effects of gravitational waves could be removed using a proper gauge choice. At a conference in Chapel Hill a thought experiment was presented describing a non-removable gravitational wave effect: two beads on a stick which is positioned orthogonal to the GW propagation. As the GW passes by they move since their proper distance changes (while the stick is held in place by atomic forces), so they dissipate energy.

What happens to free particles in the TT gauge? The geodesic equation for the spatial indices reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,, \tag{1.58}$$

where the parameter τ parametrize our curve. We assume that the particle starts out at rest: then its four-velocity is $\mathrm{d}x^\mu/\mathrm{d}\tau = (\mathrm{d}x^0 \big/ \mathrm{d}\tau\,, \vec{0})$. So, we get the simplification

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2\,, \tag{1.59}$$

and in linearized gravity

$$\Gamma^i_{00} \approx \frac{1}{2}\left(2\partial_0 h^i_0 - \partial^i h_{00}\right) = 0 \tag{1.60}$$

if we use the TT gauge. This means that the derivative of the velocity is zero: so, the velocity of a stationary particle remains zero indefinitely. Let us consider geodesic deviation between two particles instead: say that the first particle has the geodesic $x(\tau)$ and the second is $x(\tau) + \xi(\tau)$. Their geodesic equations will read

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{1.61a}$$

$$\frac{\mathrm{d}^2(x^\sigma + \xi^\sigma)}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\mu\nu}(x+\xi)\frac{\mathrm{d}(x^\mu + \xi^\mu)}{\mathrm{d}\tau}\frac{\mathrm{d}(x^\nu + x^\nu)}{\mathrm{d}\tau} = 0\,, \tag{1.61b}$$

which we can expand to first order using the perturbative expression: $\Gamma^\sigma_{\mu\nu}(x+\xi) = \Gamma^\sigma_{\mu\nu}(x) + \partial_\gamma \Gamma^\sigma_{\mu\nu}\xi^\gamma$. We use this, keep only the first order terms, subtract equation (1.61a) and finally get

$$\frac{\mathrm{d}^2 \xi^\sigma}{\mathrm{d}\tau^2} + 2\Gamma^\sigma_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma\partial_\gamma\Gamma^\sigma_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0\,, \tag{1.62}$$

so if we restrict ourselves to only spatial components, and assume that the particles start out stationary we get, at $\tau = 0$:

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}\tau^2} = -2\Gamma^i_{0\nu}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma\partial_\gamma\Gamma^i_{00}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \tag{1.63}$$

$$= -2c\Gamma^i_{0\nu}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma\partial_\gamma\Gamma^i_{00}c^2\,, \tag{1.64}$$

We are keeping $c \neq 1$ here.

so, using the expressions for the Christoffel symbols in the TT gauge, where $\Gamma^i_{00} = 0$ and $\Gamma^i_{0\nu}$ is nonzero only for $\nu = j$, we get[3]

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}\tau^2} = -2c\Gamma^i_{0j}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau} = -c\partial_0 h^{ij}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau}\,, \tag{1.66}$$

but $\mathrm{d}\xi^j \big/ \mathrm{d}\tau$ is zero if evaluated at $\tau = 0$ for parallel geodesics! So, parallel geodesics remain parallel: if the separation initially is stationary, it will remain so.

The issue is that in the TT gauge we are using a special set of coordinates which "follow" the gravitational wave. We see **no change in coordinate distance** since the coordinates are moving around with the gravitational wave: we did a coordinate change using ξ^μ satisfying $\Box\xi^\mu = 0$, so the coordinates are harmonically moving, together with the GW.

It is like we defined wave-like coordinates, "gauging away" the wave-like motion.

This is only an issue with our coordinates: the physically measurable quantities are *proper distances*, not coordinate distances, which in general are computed as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}x^2 + h^{TT}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\ . \tag{1.67}$$

Let us apply this to the case of a GW propagating along the z axis, for two particles initially separated along the x axis, whose coordinates are x_1 and x_2 (initially and also later, since as we saw the coordinate distance does not change in the TT gauge). The full metric perturbation looks like

$$h^{\mu\nu}_{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{i\omega(t-z/c)}\,, \tag{1.68a}$$

then the distance, in the case of an h_+ polarized wave, becomes

$$s = (x_1 - x_2)\sqrt{1 + h_+\cos(\omega t)} \approx (x_1 - x_2)\left(1 + \frac{1}{2}h_+\cos(\omega t)\right). \tag{1.69}$$

[3] We use the fact that, since in the TT gauge $h^0_\mu = 0$,

$$\Gamma^i_{0j} = \frac{1}{2}\left(\partial_0 h^i_j + \partial_j h^i_0 - \partial^i h_{0j}\right) = \frac{1}{2}\partial_0 h^i_j\,. \tag{1.65}$$

So, the amplitude of the oscillation in the distance is given by $h_+/2$. For two general events separated by the spacelike vector L^μ, whose norm is $L > 0$:

$$s^2 = \Big(\eta_{\mu\nu} + h_{\mu\nu}\Big) L^\mu L^\nu \approx L\left(1 + \frac{1}{2L^2} h_{ij} L^i L^j\right). \tag{1.70}$$

We would, however, like to work in coordinates which do not oscillate with the GW.

Free-falling frames

The useful frame to define is the *free-falling frame*, whose coordinates are rigid and not perturbed by the GW.

In order to build such a frame **in theory**, we will need to define 4 orthogonal vectors on the point P:

$$\eta_{\mu\nu} e^\mu_\alpha e^\nu_\beta = \eta_{\alpha\beta}\,. \tag{1.71}$$

Consider a geodesic through point P whose tangent vector at P is a unit vector $\hat{n}$. If this unit vector is spacelike, we parametrize the geodesic by s (defined with $\mathrm{d}s^2$, from the metric), if it is timelike we parametrize it with τ (defined by $\mathrm{d}\tau^2 = -\,\mathrm{d}s^2$). We denote as λ either of s or τ.

Now, the coordinates of point Q are generically $\lambda\hat{n}$, if the geodesic starting with unit vector $\hat{n}$ reaches Q when its parameter is λ.

We can reach almost every point this way, the points which are only connected through null geodesics to P can be reached by continuity, and in a small enough region the coordinates of a point Q are unique — that is, the geodesics do not cross.

In this frame, then, $g_{\mu\nu}(P) = \eta_{\mu\nu}(P)$; also, in the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\,, \tag{1.72}$$

we have that the second derivatives are zero since $x^\mu(\lambda)$ is linear in λ, so we must have $\Gamma^\mu_{\nu\rho} n^\nu n^\rho = 0$. This must be true for any unit vector n^μ, therefore we have $\Gamma^\mu_{\nu\rho} = 0$. The linear system giving $g_{\mu\nu,\rho}(P)$ from $\Gamma^\mu_{\nu\rho}$ is nondegenerate, so the first derivatives of the metric also vanish: $g_{\mu\nu,\rho}(P) = 0$.[4] These are called **Riemann normal coordinates**.

The conditions on the metric and its derivatives only hold at the point. We can do slightly better with *Fermi normal coordinates*, where we require a gyroscope's angular momentum to be parallel-transported along the geodesics, so that an observer moving along a geodesic is indeed free-falling.

How do we make such a frame **experimentally**? We might think to use free-falling particles, and put them in orbit. This is not actually that simple. A satellite which accomplishes this task is called a *drag-free satellite*.

[4] Do note that this reasoning works only at the point, since if we moved along a geodesic we do not have access to the other unit vectors anymore (in these coordinates).

This better be so: otherwise we would have proven that the first derivatives of the metric are zero in a neighborhood of a generic point P, so the metric is constant in the whole neighborhood, which is nonphysical.

Consider a particle orbiting the Sun. The Sun's radiation pressure pushes the particle away from a geodesic. The way to solve this issue is to put a thrusted spacecraft around our test mass to balance the Sun's radiation pressure, constantly measuring the distance to the test mass without touching it, and then balancing the thrusters by keeping at a constant distance from it.

LISA's drag-free navigation

This is the idea behind the satellites making up the space-based GW interferometer LISA (which we will discuss in greater detail later in the course). The distances between the spacecrafts should be about 5 Gm apart. We do not measure the distances between the spacecrafts, but instead the distances between the test masses inside them, which are 2 kg, 4.6 cm side, gold-platinum shielded cubes.

The interferometric measurements have pico-meter (10^{-12} m) sensitivity. It takes about 30 s for light to move between the mirrors: this time-delayed interferometry needs special consideration.

We have a *gravitational reference sensor*, a cubic shell around the cube: we keep measuring the distances between the two. We also need to precisely discharge the masses with a Charge Management System, otherwise electrostatic forces are too strong. Also, the thrusters need to be very weak, on the order of the μN.

The LISA Pathfinder mission successfully tested all of these technologies, except for time-delayed interferometry. It only used one spacecraft, and measured how well the drag-free navigation worked.

In the final mission there will be two masses inside each satellite, so we will need to account for the gravitational pull between them. Even accounting for this by relaxing the acceleration precision requirement 10-fold, the results of LISA Pathfinder were exceptional.

Let us discuss the sources of noise: at high frequencies, inertia prevents a force from creating significant displacement. This applies to external forces, not to gravitational forces, since the latter are proportional to the mass. So, there is a limit at high frequencies because of our inability to measure that fast. At low frequencies, it is easy to measure, but it is hard to verify whether the mass is indeed in free fall. We can also have issues with the parasitic coupling of the test mass to the spacecraft.

In the end, it was verified that we can do

$$S_a^{1/2} \leq 3 \times 10^{-14}\,\mathrm{m/s^2/\sqrt{Hz}} \tag{1.73}$$

at 1 mHz. Solar radiation pressure is two orders of magnitude higher. The LISA Pathfinder mission greatly outperformed its original requirements — see figure 6 in the paper published by the LISA Pathfinder collaboration [LIS17].

1.4.1 Proper detector frame

Let us now come back to theory by discussing the *proper detector frame*: coordinates defined by a rigid ruler. Rigid rulers do not really exist, but we can approximate it well enough. If the gravitational pull is small compared to the restoring forces in the ruler, then its length will approximately not change.

Let us put ourselves in a free-falling frame in Fermi local coordinates, so that in the origin the metric is flat. Then, we can expand it to second order in the spatial coordinates

$$g_{\mu\nu}(x) \approx g_{\mu\nu}(0) + x^i \partial_i \, g_{\mu\nu}\Big|_{x=0} + \frac{1}{2} x^i x^j \, \partial_i \partial_j g_{\mu\nu}\Big|_{x=0} + \dots \tag{1.74a}$$

$$= \eta_{\mu\nu} + \frac{1}{2} x^i x^j \, \partial_i \partial_j g_{\mu\nu}\Big|_{x=0} , \tag{1.74b}$$

which we can rewrite in terms of the Riemann tensor by making use of the expression of the Riemann tensor in the LIF, which is

$$R_{iklm} = \frac{1}{2} \left(\partial_k \partial_l g_{im} + \partial_i \partial_m g_{kl} - \partial_k \partial_m g_{il} - \partial_i \partial_l g_{km} \right) , \tag{1.75}$$

we get

$$\mathrm{d}s^2 \approx -c^2 \, \mathrm{d}t^2 \left(1 + R_{0i0j} x^i x^j \right) - 2c \, \mathrm{d}t \, \mathrm{d}x^i \left(\frac{2}{3} R_{0ijk} x^j x^k \right) + \mathrm{d}x^i \, \mathrm{d}x^j \left(\delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l \right) . \tag{1.76}$$

The corrections to the flat metric are of the order $\mathcal{O}\left(r^2 / L_B^2\right)$, where r^2 is the square distance from the origin, while L_B is the typical spatial scale of the variation of the metric, such that $R_{0ijk} = \mathcal{O}\left(L_B^{-2}\right)$. This L_B is the wavelength of the GW, if we are describing a GW.

So, the flat coordinate description works as long as the scale of the region we are describing is very small compared to the characteristic scale of the variations in the metric.

Geodesic deviation in the proper detector frame

Monday 2020-3-23, compiled 2020-07-10

What happens in this frame? The equation of geodesic deviation can be calculated from (1.63):

$$0 = \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + 2\Gamma^i_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} + \xi^\sigma \left(\partial_\sigma \Gamma^i_{\nu\rho} \right) \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \tag{1.77a}$$

$$= \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + \xi^j \left(\partial_j \Gamma^j_{00} \right) \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 , \tag{1.77b}$$

where we made the nonrelativistic approximation where the spacelike components of the four-velocity are negligible, and accounted for the fact that in this frame we have

$$\Gamma^\mu_{\nu\rho} = 0 \qquad \text{and} \qquad \partial_0 \Gamma^i_{0j} = 0 \implies R^i_{0j0} = \partial_j \Gamma^j_{00} , \tag{1.78}$$

so we can write the geodesic equation in terms of the Riemann tensor as:

$$0 = \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + R^i_{0j0} \xi^j \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 , \tag{1.79}$$

and since in linearized gravity the Riemann tensor is *invariant* (rather than covariant) under coordinate transformations[5] such as those we used to move between the TT gauge and the detector frame, we can compute it in the TT gauge starting from equation (1.75):

$$R^i_{0j0} = \frac{1}{2}\left(\partial_j\partial_0 h^i_0 + \partial_0\partial^i h_{0j} - \partial_j\partial^i h_{00} - \partial_0\partial_0 h^i_j\right) = R_{i0j0} \tag{1.81}$$

$$= -\frac{1}{2}\partial_0\partial_0 h_{ij} = -\frac{1}{2}\ddot{h}^{TT}_{ij}\,, \tag{1.82}$$

so our final result for the geodesic deviation equation in the detector frame is:

$$\boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{TT}_{ij}\xi^j\,,} \tag{1.83}$$

which is physically significant since we can interpret the effect of the GW as that of a Newtonian force, given by

$$F^i = \frac{m}{2}\ddot{h}^{TT}_{ij}\xi^j\,. \tag{1.84}$$

This seems great! We can have particles move under the influence of the GW, work with h_{ij} in the simple TT gauge, while still being in almost flat spacetime.

However, to get here we made some approximations, and we need to check whether they are justified. We imposed $r^2/L_B^2 \ll 1$, where L_B is the scale of the variations of the metric while r is the scale of our detector. For ground-based detectors which are $\sim 3\,\text{km}$ long and sensitive in the $\sim 100\,\text{Hz}$ range (corresponding to $\lambda \sim 3000\,\text{km}$) this is perfectly fine. For space-based detectors like LISA it is not!

What we have calculated applies to proper distances as well, which are the same as coordinate distances up to first order in our coordinates. So, since proper distances are invariant, we have an approximate expression which we can use in general, by substituting s for ξ in the differential equation.

Effects of GW

This result allows us to see that the GWs are **transverse**: for a wave along the z direction the equation reads

$$\ddot{\xi}^3 = \frac{1}{2}\ddot{h}^{TT}_{3j}\xi^j = 0\,, \tag{1.85}$$

so the particle does not move along the direction of propagation; on the other hand we can do the calculations for particles starting out with small separations along x or y.

[5] This can be seen by plugging the transformation law which is allowed in linearized gravity, $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}h_{\nu)}$, into the LIF expression for the Riemann tensor,

$$R_{\mu\nu\rho\sigma} = -2g_{[\mu|[\rho|,|\nu]\sigma]}\,, \tag{1.80}$$

which yields no change (the expression is a compact way to write that the indices to antisymmetrize are both $\mu\nu$ and $\rho\sigma$). This is discussed by Maggiore [Mag07, below eq. 1.13].

We consider an initial displacement vector $\xi_i(t=0) = (x_0, y_0, 0)$, allow it to vary denoting it as $\xi_i(t) = (x_0 + \delta x, y_0 + \delta y, 0) = \xi_i(t=0) + \delta\xi_i(t)$ and compute:

$$\ddot{x}_i = \delta\ddot{\xi}_i = \frac{1}{2}\ddot{h}_{ij}^{TT}(\xi_j + \delta\xi_j) \approx \frac{1}{2}\ddot{h}_{ij}^{TT}\xi_j\,, \tag{1.86}$$

since the variation $\delta\xi$ is of the same perturbative order as h_{ij}, making the term containing it second order.

So, after we take the real part of the exponential in h_{ij} and ignore the z dependence (which just gives a constant phase, since z is fixed) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{bmatrix}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\cos(\omega t))\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{1.87}$$

$$= \frac{1}{2}\begin{bmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\left(-\omega^2\cos(\omega t)\right). \tag{1.88}$$

When we integrate to get δx and δy the factor $-\omega^2$ gets reabsorbed.

For the plus polarization h_+ (setting $h_\times = 0$) we find

$$\delta x = \frac{1}{2}h_+ x_0\cos(\omega t) \tag{1.89a}$$

$$\delta y = -\frac{1}{2}h_+ y_0\cos(\omega t)\,, \tag{1.89b}$$

while for the cross polarization $h_\times$ (setting $h_+ = 0$) we get

$$\delta x = \frac{1}{2}h_\times y_0\cos(\omega t) \tag{1.90a}$$

$$\delta y = \frac{1}{2}h_\times x_0\cos(\omega t)\,. \tag{1.90b}$$

There is a wrong sign in the slides here.

Are these considerations valid for **Earth-based detectors**? They are definitely not free-falling, in fact:

1. at zeroth order the metric is flat;
2. at first order we have the Newtonian forces, such as the Earth's gravity, the Coriolis force, the centrifugal force and such;
3. at second order we get the curvature contributions from GW and the background metric.

So, how do we distinguish these second-order GW effects from other second-order effects? We can isolate them by Fourier analysis: we only look at the Fourier window in which they are dominant.

1.5 GW generation

The assumptions we will make are

1. expanding around flat spacetime;
2. consider nonrelativistic systems;
3. assume the stress energy tensor is conserved: to first order, this reads

$$\partial^\mu T_{\mu\nu} = 0\,. \tag{1.91}$$

If the system we consider is self-gravitating (like a binary), then being nonrelativistic means that it is also large with respect to its Schwarzschild radius:

$$E_{\text{kin}} = -\frac{1}{2}U \implies \frac{1}{2}\mu v^2 = \frac{1}{2}G\frac{\mu M}{r} \implies \frac{v^2}{c^2} = \frac{GM}{c^2 r} = \frac{R_S}{r} \ll 1\,. \tag{1.92}$$

The expression of the gravitational force is that since, by the definition of the reduced mass, we have $m_1 m_2 = \mu M$.

Recall the $\Lambda_{ij,kl}$ tensor (1.57), which can be used to project a rank-two spacelike tensor to the TT gauge.

We will proceed as section 17.6 in Hobson [HEL06]. In order to solve the linearized equations (1.43), we use Green's functions, which are defined by:

$$\Box_x G(x-y) = \delta^{(4)}(x-y)\,, \tag{1.93}$$

where x is our variable, while y is a coordinate which will span the positions in the source. The operator $\Box_x$ is the Dalambertian with respect to the x coordinates.

The idea of this method is to calculate the wave response to a single impulsive source, and then superimpose the effects of many of these. We introduce $\kappa = 1/M_P^2$ for simplicity, multiply the previous equality by $T_{\mu\nu}(y)$ and integrate in $\mathrm{d}^4 y$ so we get

$$-2\kappa \int \mathrm{d}^4 y\, \Box_x G(x-y) T_{\mu\nu}(y) = -2\kappa \int \mathrm{d}^4 y\, \delta^{(4)}(x-y) T_{\mu\nu}(x) \tag{1.94}$$

$$-2\kappa \Box_x \left(\int \mathrm{d}^4 y\, G(x-y) T_{\mu\nu}(y) \right) = -2\kappa T_{\mu\nu}(x) = \Box_x \bar{h}_{\mu\nu}(x)\,, \tag{1.95}$$

The argument of the right $T_{\mu\nu}$ can be switched to x because of the delta.

so we will have as a solution a superposition of the homogeneous solution and the source term:

$$\bar{h}_{\mu\nu}(x) = \underbrace{\bar{h}^{(0)}_{\mu\nu}(x)}_{\Box \bar{h}^{(0)}_{\mu\nu} = 0} - 2\kappa \int \mathrm{d}^4 y\, G(x-y) T_{\mu\nu}(y)\,. \tag{1.96}$$

In order to make the Green's function explicit, we write it as centered around the origin:

$$\partial_\mu \partial^\mu G(x^\sigma) = \delta^{(4)}(x^\sigma)\,, \tag{1.97}$$

and we integrate this equality over a hypercylinder V (the product of a 3-sphere of radius $r = |\vec{x}|$ and an interval $[-ct, ct] \subset \mathbb{R}$, where $ct > r$) we have

$$\int_V \mathrm{d}^4x \, \delta^{(4)}(x^\sigma) = 1 = \int_V \mathrm{d}^4x \, \partial_\mu \partial^\mu G(x^\sigma) \tag{1.98a}$$

$$= \int \mathrm{d}S \left(\partial_\mu G(x^\sigma) \right) n^\mu \,, \tag{1.98b}$$

but the only points which can contribute are in the future light-cone because of causality, so the dependence of G upon x^σ must be in the form $G(x^\sigma) = f(r)\delta(ct - r)[ct \geq 0]$.[6]

We write $\mathrm{d}S = c \, \mathrm{d}t \, r^2 \, \mathrm{d}\Omega$,[7] and we call $n^\mu \partial_\mu = \partial_r$:

$$1 = \int \mathrm{d}S \left(\partial_\mu G(x^\sigma) \right) n^\mu \tag{1.99a}$$

$$= 4\pi r^2 \int_0^\infty \mathrm{d}t \, \partial_r \big(f(r) \delta(ct - r) \big) c \tag{1.99b}$$

$$= 4\pi r^2 \partial_r f(r) c + 4\pi r^2 f(r) \underbrace{\int_0^\infty \partial_r \delta(ct - r) c \, \mathrm{d}t}_{=0} \,, \tag{1.99c}$$

The derivative of the delta evaluates the derivative of the thing it multiplies, which is a constant.

so we can get an explicit expression for the function $f(r)$:

$$4\pi r^2 \partial_r f(r) = 1 \implies f(r) = -\frac{1}{4\pi r} \implies G(x^\sigma) = -\frac{\delta(x^0 - |\vec{x}|)}{4\pi |\vec{x}|} \theta_H(x^0) \,. \tag{1.100}$$

The integration constant is set to zero so that the Green function vanishes at infinity.

We can then plug this into the general solution (1.96) to find

$$\bar{h}_{\mu\nu}(t, \vec{x}) = (-)^2 2\kappa \int \mathrm{d}^4y \, \frac{\delta(x^0 - y^0 - |\vec{x} - \vec{y}|)}{4\pi |\vec{x} - \vec{y}|} \theta_H(x^0 - y^0) T_{\mu\nu}(y) \tag{1.101}$$

$$= \frac{4G}{c^4} \int \mathrm{d}^4y \, \frac{\delta(y^0 - (ct - |\vec{x} - \vec{y}|))}{|\vec{x} - \vec{y}|} T_{\mu\nu}(y^0, \vec{y}) \tag{1.102}$$

$\kappa = 8\pi G / c^4$.

$$= \frac{4G}{c^4} \int \mathrm{d}^3y \, \frac{T_{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \,. \tag{1.103}$$

As long as we are outside the source we can move to the TT gauge, since there the equation $\Box \bar{h}_{\mu\nu}$ satisfies the EFE, so we can do the required gauge change of variables with $\Box \xi^{\mu\nu} = 0$. Now, in order to move to the TT gauge we can use the Λ tensor. It is equivalent to apply it to the trace-reversed $\bar{h}_{ij}$ or to h_{ij}, since the tensor is projected into the space of traceless tensors anyways.[8]

So, the general expression for our TT-gauge tensor measured at a position $\vec{x}$ outside the source, with $\hat{n} = \vec{x}/|x|$:

[6] The bracket is an Iverson bracket [Knu92], it evaluates to 1 or 0 depending on whether the expression inside it is true or false.

[7] The r^2 is missing in Hobson [HEL06, pag. 477] as well, but it should be there.

[8] Formally, this is shown as

$$\Lambda_{ij,kl} \bar{h}_{kl} = \Lambda_{ij,kl} \left(h_{kl} - \frac{1}{2} \eta_{kl} h \right) = \Lambda_{ij,kl} h_{kl} - \underbrace{\frac{1}{2} \Lambda_{ij,kk}}_{=0} h = \Lambda_{ij,kl} h_{kl} \,. \tag{1.104}$$

$$
\boxed{h_{ij}^{TT}(ct, \vec{x}) = \Lambda_{ij,kl}(\hat{n})\overline{h}_{kl} = \frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n}) \int \mathrm{d}^3y \, \frac{T_{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \,.} \tag{1.105}
$$

Far from the source, $|\vec{x}| \gg |\vec{y}|$ for any $\vec{y}$ inside the source. So, we can expand:[9]

$$
|\vec{x} - \vec{y}| = r \left(1 - \frac{\vec{y} \cdot \hat{n}}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right) \right) . \tag{1.108}
$$

Keeping only the terms at $\mathcal{O}(1/r)$ we get[10]

$$
h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl} \int \mathrm{d}^3y \, T_{kl}\left(t - \frac{r}{c} + \frac{\vec{y} \cdot \hat{n}}{c}, \vec{y}\right) . \tag{1.109}
$$

If the object is moving periodically with frequency ω_s, then we will have

$$
\frac{1}{\omega_s} \sim \frac{d}{v} \,, \tag{1.110}
$$

and we assume $d/c \ll d/v$, which is equivalent to $v \ll c$: the characteristic velocities of the source should be nonrelativistic. Under these assumptions we can expand the stress-energy tensor in powers of $\xi = \vec{y} \cdot \hat{n}/c$

Expanding the stress energy tensor, we find

$$
h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl} \int \mathrm{d}^3y \left[T_{kl} + \frac{y^m n^p}{c}\partial_0 T_{kl} + \frac{y^m y^p n^m n^p}{2c^2}\partial_0^2 T_{kl} + \dots \right] . \tag{1.111}
$$

If we define the multipole moments:

$$
S^{ijm_1 m_2 \dots} = \int \mathrm{d}^3x \, T^{ij} \prod_\alpha x^{m_\alpha} \,. \tag{1.112}
$$

[definitions and stuff, too fast to write]

We get that the quadrupole term dominates:

$$
h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl} S^{kl} \,. \tag{1.113}
$$

---

[9] The full calculation goes as follows:

$$
\begin{aligned}
|x - y| = \sqrt{(x-y)^2} = \sqrt{x^2 + y^2 - 2x \cdot y} &= r\sqrt{1 - 2\frac{\hat{n} \cdot y}{r} + \frac{y^2}{r^2}} && (1.106) \\
&\approx r\left(1 - \frac{\hat{n} \cdot y}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right)\right) , && (1.107)
\end{aligned}
$$

where d is the length scale of the source, such that $|y| \leq d$.

[10] At this point we change the first argument of the stress-energy tensor's dimensionality from a space ct to a time t; this is just a matter of convention, it makes it easier to write the Taylor expansion later.

The quadrupole moment is defined as

$$Q^{kl} = M^{kl} - \frac{1}{3}\delta^{kl} M_{ii} = \int \mathrm{d}^3 x \, \rho(t, \vec{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta_{ij} \right) . \tag{1.114}$$

Since $\Lambda_{ij,kl}\delta^{kl} = 0$, we can substitute Q for M: we get

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{2G}{c^4} \Lambda_{ij,kl} \ddot{Q}^{kl} \left(t - \frac{r}{c} \right) = \frac{1}{r}\frac{2G}{c^4} \ddot{Q}_{ij}^{TT} \left(t - \frac{r}{c} \right) . \tag{1.115}$$

If a wave is propagating along the $\hat{n} = \hat{z}$ direction, we get

$$\Lambda_{ij,kl} \ddot{M}_{kl} = \begin{bmatrix} \left(\ddot{M}_{11} - \ddot{M}_{22} \right) / 2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & \left(\ddot{M}_{22} - \ddot{M}_{11} \right) / 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{1.116a}$$

Last time we saw how to expand the solution into multipole moments, and focused on the quadrupole. Friday 2020-3-27, compiled 2020-07-10

We finally got, for a GW propagating along $\hat{z}$:

$$h_+ = \frac{1}{r}\frac{G}{c^4} \left(\ddot{M}_{11} - \ddot{M}_{22} \right) \tag{1.117a}$$

$$h_\times = \frac{2}{r}\frac{G}{c^4} \ddot{M}_{12} . \tag{1.117b}$$

But how do we compute the full angular distribution? We can brute-force it using the full Λ projection tensor, but a more conceptual way is to put ourselves in a frame in which the generic vector $\hat{n}$ is $\hat{z}$. Then we use the rotation matrices: we need two of them, one for each unit vector in a rank-2 tensor M_{ij}. Then we use the simple expression for h_+ and $h_\times$, substituting in the $\ddot{M}_{11}$, $\ddot{M}_{12}$ and so on in the primed system.

1. There is no monopole radiation: $\dot{M} = 0$, since mass is conserved.
2. We can move the origin so that the dipole is zero: $M^i = 0$. This corresponds to linear momentum being conserved: $\dot{P}^i = 0$. This is in stark contrast to Electromagnetism: there, we cannot eliminate the dipole radiation. This is due to there being positive and negative electric charges, while there are no negative masses.
3. We did not account for back-action: our GWs do not carry "away" energy or momentum. This is unphysical, we will account for it!

A stress-energy tensor for a system of point masses looks like

$$T^{\mu\nu}(t, \vec{x}) = \sum_A \frac{p_A^\mu p_A^\nu}{\gamma_A m_A} \delta^{(3)}(x - x_A(t)) . \tag{1.118}$$

In order to apply our system, our system must be closed: we cannot consider any particle trajectory, since they must move on geodesics in order to conserve the stress-energy tensor.

However, we can use the relative coordinate in a self-gravitating system: then we have the relative $x_0 = x_1 - x_2$, the total mass $m = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$, an the center of mass position $x_{CM} = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$.

If we set the position of the COM to zero identically, we find

$$M^{ij} = \mu x_0^i x_0^j \,. \tag{1.119}$$

We consider two particles in the XY plane moving on a trajectory, which for now we assign. so that we get

$$x_0(t) = R\cos\left(\omega_s t + \frac{\pi}{2}\right) \tag{1.120a}$$

$$y_0(t) = R\sin\left(\omega_s t + \frac{\pi}{2}\right) \tag{1.120b}$$

$$z_0(t) = 0\,, \tag{1.120c}$$

so we find

$$\ddot{M}_{11} = -\ddot{M}_{22} = 2\mu R^2 \omega_s^2 \cos(2\omega_s t) \tag{1.121a}$$

$$\ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin(2\omega_s t)\,, \tag{1.121b}$$

which means the GW emission has twice the rotational frequency.

If we look at emission in a generic direction $\hat{n}$, described by the angles θ and φ, we will receive

$$h_+(t, \theta, \varphi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\frac{1+\cos^2\theta}{2}\cos\left(2\omega_s t_{\text{ret}} + 2\varphi\right) \tag{1.122a}$$

$$h_\times(t, \theta, \varphi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos(\theta)\sin\left(2\omega_s t_{\text{ret}} + 2\varphi\right)\,, \tag{1.122b}$$

so the two scale differently as θ varies.

Do note that $h_+ + h_\times$ looks like a circular polarization moving counter-clockwise, while $h_+ - h_\times$ moves clockwise. This is what we see for $\theta = 0, \pi$; on the other hand for $\theta = \pi/2$ we only have h_+.

Plot effects!

Some numbers: the Earth-Sun system, seen 1×10^5 lyr away, has a strain on the order 5×10^{-32}. Let us try a BNS in our galaxy: we use Newtonian orbits to model the binary.

Note that the strain is inversely proportional to r: this is because we are not measuring the intensity, but directly the amplitude.

If we put in the expression for the radius from Kepler's law: we find

$$h_+ \sim h_\times \sim \frac{4G^{5/3}\omega_s^{2/3}\mu M^{2/3}}{rc^4}\,. \tag{1.123}$$

Let us consider a BNS with masses $m_1 = m_2 = 1.5M_\odot$ in our galaxy: then, the strain will be of the order 10^{-20}.

We have seen a BNS outside of our galaxy, at 40 Mpc away.

Do this with GW150914: $m_1 \sim 36 M_\odot$, $m_2 \sim 31 M_\odot$, $f_s \sim 250\,\text{Hz}$, $r \sim 440\,\text{Mpc}$.

When their distance becomes too small, we cannot model them using Kepler's laws anymore.

Now, back-action: we saw no energy loss, which is "by design": GW energy is quadratic in h, while our theory is linear.

Another approach: if our theory is local, it cannot describe the energy of GWs. GWs do carry energy, however we cannot describe it in a local way, since for any single particle we can gauge them away. However, we can look at the tidal effects between two particles.

We need to perform an average in spacetime.

If we introduce a stress-energy tensor for our GWs, this will curve the background spacetime: then we have

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{1.124}$$

but how do we decide which deviation from flat is part of the background and what is part of h?

There is no formal way to define this, but we can use a heuristic, based on what they describe. We say that the background $\overline{g}_{\mu\nu}$ varies spatially slowly, across a distance L_B, such that all the GWs we are considering have reduced wavelengths ($k = \lambda / 2\pi$) which are smaller than a fixed maximum wavelengths, so that $\lambda \leq \lambda_{GW} \ll L_B$.

This is like distinguishing waves and tides in the ocean: intuitively it is easy to see how they differ, based on the scale of their effects.

We also impose that all the GWs vary much faster than the background, temporally: $f > f_{GW} \gg f_B$. Do not that these two are independent, since while GWs travel at the speed of light[11] the background does not. We refer to both as the short-wave approximation.

How do GW detectors then distinguish GW from background? If we impose $f_{GW} = c/\lambda_{GW}$ with λ_{GW} around a kilometer we get $f \sim 300\,\text{kHz}$. This is not interesting, and technically difficult. Also, ground-based detectors do not measure the metric along the length, but only an integrated effect.

Instead, the detectors monitor local temporal variations of $g_{\mu\nu}(t, x_0)$. So, our detectors work best between 100 Hz to 1000 Hz; the Earth's gravitational field is not smooth along the corresponding length scale. However, it's close to static: its variations are slower than a few Hz. So, we can apply our distinction: GW and background can be distinguished in frequency.

Monday 2020-3-30, compiled 2020-07-10

Last time we discussed: how can we distinguish what is a GW and what is not?

We do this by separating them by frequency. We cannot really measure the nature of tensor perturbation of GWs, since we are only measuring integrated effects.

We can precisely map the effect of the GW in time, by sampling with a frequency which is much higher than the one of the GW.

So, let us fo this formally: if we expand the Ricci tensor to second order in $h_{\mu\nu}$ we get

$$\frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) = \overline{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} \,, \tag{1.125}$$

[11] Which we know to within a part in 10^{-14}.

where the second order in $h_{\mu\nu}$ term, $R^{(2)}_{\mu\nu}$, has both high and low frequency components.

This is because, when we have a term like

$$\left(\sin(\omega_1 t)\sin(\omega_2 t)\right)^2, \tag{1.126}$$

by the prostapheresis formulas we will gets terms like $\omega_1 + \omega 2$ and $\omega_1 - \omega_2$, one of which is high frequency and the other is low frequency, since we are considering a high-frequency wavepacket.

Recall, we expand in two parameters: h and λ / L_B. Then, we have

$$\overline{R}_{\mu\nu} \sim \partial^2 \overline{g}_{\mu\nu} \sim \frac{1}{L_B^2}, \tag{1.127}$$

while

$$\left[R^{(2)}_{\mu\nu}\right]^{\text{low frequency}} \sim (\partial h)^2 \sim \left(\frac{h}{\lambda}\right)^2. \tag{1.128}$$

See Maggiore, page 31 for more details. So, the EFE low-frequency components are

$$\overline{R}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{low freq}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)^{\text{low freq}}, \tag{1.129}$$

so we either have $T_{\mu\nu} = 0$, in which case

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda}\right)^2, \tag{1.130}$$

or

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda}\right)^2 + T_{\mu\nu} \gg \left(\frac{h}{\lambda}\right)^2, \tag{1.131}$$

so we will have $h \ll \lambda / L_B$: so we can take averages on a scale ℓ such that $\lambda_{GW} \ll \ell \ll L_B$: then we wil find

$$\overline{R}_{\mu\nu} = -\left\langle R^{(2)}_{\mu\nu}\right\rangle + \frac{8\pi G}{c^4}\left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right\rangle. \tag{1.132}$$

After some math (see Maggiore): we can define a stress tensor of the GW, which looks like

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu} R^{(2)}\right\rangle, \tag{1.133}$$

and a "smoothed out" SEMT of matter:

$$\left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right\rangle \overset{\text{def}}{=} \overline{T}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{T}, \tag{1.134}$$

soo the equation which will hold is

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \frac{8\pi G}{c^4}\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right). \tag{1.135}$$

Do note that if we work with these, the tensor which is conserved is $\overline{T}_{\mu\nu} + t_{\mu\nu}$:

$$\nabla^\mu\left(\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R}\right) = 0 = \nabla^\mu\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right). \tag{1.136}$$

How does this look like far from the source? there, we can approximate $\overline{g} \approx \eta_{\mu\nu}$ and $\nabla_\mu \approx \partial_\mu$.

This $t_{\mu\nu}$ only has 2 physical degrees of freedom: how do we gauge the others away? The Lorentz gauge plus $h = 0$ eliminates 5 degrees of freedom.

When we have terms like $h\partial\partial h$, we can integrate by parts on a sufficiently large volume to turn them into $\partial(h\partial) - \partial h\partial h$. Using the facts $\partial^\mu h_{\mu\nu} = h = \Box h_{\mu\nu} = 0$ we can simplify several terms: in the end we get

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle .} \tag{1.137}$$

This is actually coordinate-independent, it can be computed in any frame we like. In the TT-gauge, we will have

$$t^{00} = \frac{c^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle . \tag{1.138}$$

Also, we will have $t^{01} = t^{02} = 0$ by symmetry, and also $t^{03} = t^{00}$.

Then, if we are far enough away from the source we can compute

$$\mathrm{d}E = \mathrm{d}A\, c\, \mathrm{d}t\, \frac{c^2}{32\pi G}\left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right\rangle \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle . \tag{1.139}$$

so, in order to get the total power $\mathrm{d}E/\mathrm{d}t$ we can integrate this expression in $R^2\,\mathrm{d}\Omega$. In order to compute the momentum carried away, we can do a similar thing, and get the momentum flux.

In order to do this calculation, we can use the explicit expression for the $\Delta_{ij,kl}$ projection tensor. We find

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{r^2c^3}{32\pi G}\int \mathrm{d}\Omega\left\langle \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{Q}^{kl}\frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{Q}^{kl}\right\rangle \tag{1.140a}$$

$$= \frac{G}{8\pi c^5}\int \mathrm{d}\Omega\,\Lambda_{ij,kl}\left\langle \dddot{Q}^{ij}\dddot{Q}^{kl}\right\rangle , \tag{1.140b}$$

where the only expression depending on the angle is $\Lambda_{ij,kl}$: then we integrate and find

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dots . \tag{1.141}$$

When we do the same from the momentum density, we get an integral in the form

$$\frac{\mathrm{d}P^k}{\mathrm{d}t} \propto \int \mathrm{d}\Omega\, \ddot{Q}\partial^k \dddot{Q}\,, \tag{1.142}$$

but the integrand is odd under spatial inversion, so there is no contribution! This is not true if we go beyond the quadrupole, instead full GR calculations/simulations show that there are kicks at the merger.

We can calculate the angular distribution in a relatively simple way, since it is easy to go to TT gauge at a point.

The energy lost by the source at $t_{\mathrm{ret}} = t - r/c$ is the same as the energy measured in GW.

Clarify: why are the two expressions calculated at the same time?

If we model the back-reaction as a force, we have

$$\frac{\mathrm{d}E_{\mathrm{source}}}{\mathrm{d}t} = \left\langle \int \mathrm{d}^3 x\, \frac{\mathrm{d}F_i}{\mathrm{d}V} \dot{x}_i \right\rangle == -\frac{G}{5c^5} \left\langle \frac{\mathrm{d}Q_{ij}}{\mathrm{d}t} \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} \right\rangle . \tag{1.143}$$

Equating terms and making some considerations, we get

$$\frac{\mathrm{d}F_i}{\mathrm{d}V} = -\frac{2G}{5c^5} \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} \rho(t, \vec{x}) x_j\,, \tag{1.144}$$

so finally

$$F_i = -\frac{2G}{5c^5} \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} m \overline{x}_j\,, \tag{1.145}$$

where $\overline{x}_j$ is the center-of-mass coordinate.

Then, we can calculate the torque explicitly: we get

$$T_i = -\frac{2G}{5c^5} \epsilon_{ijk} Q_{il} \frac{\mathrm{d}^5 Q_{kl}}{\mathrm{d}t^5}\,, \tag{1.146}$$

so if we take the average we get

$$\left\langle \frac{\mathrm{d}L_i}{\mathrm{d}t} \right\rangle = -\frac{2G}{5c^5} \epsilon_{ijk} \left\langle \ddot{Q}_{jl} \dddot{Q}_{kl} \right\rangle . \tag{1.147}$$

1.6 Back-reaction and the evolution of binary systems

We use the reduced mass formalism, the amplitude is

$$A = \frac{4G^{5/3} \omega_s^{2/3} \mu M^{2/3}}{r c^4}\,, \tag{1.148}$$

so we have the polarizations

$$h_+ = A \frac{1 - \cos^2\theta}{2} \cos\left(2\omega_s t_{\mathrm{ret}} + 2\varphi\right) \tag{1.149a}$$

$$h_\times = A\cos\theta\sin\left(2\omega_s t_{\text{ret}} + 2\varphi\right), \tag{1.149b}$$

we define the chirp mass:

$$M_c = \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}, \tag{1.150}$$

also the frequency of the GW is twice the frequency of the orbit.

Coming back to the radiated power: it is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}\left(\dddot{M}_{kk}\right)^2 \right\rangle, \tag{1.151}$$

so we can calculate this explicitly for our binary: we need to average over a period, so we get a factor $\left\langle \sin^2\varphi \right\rangle = 1/2$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{GW}}{2c^3}\right)^{10/3}, \tag{1.152}$$

where we used the fact $R^3 = GM/\omega_s^2$. We can do a similar thing for the angular momentum:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dots. \tag{1.153}$$

Now, let us consider masses in quasi-circular orbit: by the virial theorem,

$$\dot{R} = -\frac{2R^2}{Gm_1m_2}\dot{E}_{GW}. \tag{1.154}$$

Therefore, as the GWs carry away energy the radius shrinks. But we assumed the orbits to be circular! This is fine: they are almost-circular usually. This is fine for most of the inspiral, until the merger phase.

Friday 2020-4-3, compiled 2020-07-10

As we were discussing yesterday, we can approximate every orbit as a circular one.

We finally get the relation

$$\dot{\omega}_{GW} = \frac{12}{5}2^{1/3}\left(\frac{M_cG}{c^3}\right)^{5/3}\omega_{GW}^{11/3}, \tag{1.155}$$

where M_c is the chirp mass. So, we can integrate this: since

$$\frac{\mathrm{d}f_{GW}}{\mathrm{d}t} = kf_{GW}^{11/3} \implies -\frac{3}{8k}f^{-8/3} = t - t_{\text{coalescence}} \overset{\text{def}}{=} -\tau, \tag{1.156}$$

so that we can get the frequency as a function of the time until coalescence:

$$\tau = \frac{5}{256}\left(\frac{1}{\pi f_{GW}}\right)^{8/3}\dots. \tag{1.157}$$

We can also get an expression for the radius at a given time from coalescence:

$$R(\tau) = R_0\left(\frac{\tau}{\tau_0}\right)^{1/4}, \tag{1.158}$$

where τ_0 is the time to coalescence at t_0. If we plot this, it has a "plunge" phase, and up to it we can trust our plot.

1.6.1 Chirping waveform

The *phase* is the argument of the cosine, so we write $\cos\big(\phi(t)\big)$. The angular frequency is given by $\omega_{GW} = \phi'$.

The chirping waveform cannot be trusted near the end. We know that if

$$R < R_{\text{ISCO}} = \frac{6GM}{c^2} \tag{1.159}$$

then orbits are not stable. This formula only holds for extreme mass ratios (we actually could have these SMBHs merging with solar mass ones!). Anyway, we use it as a guideline to see when our approximations break down.

The shape of the chirping waveform is basically correct; it goes out of phase with the numerical relativity calculation, but it works somewhat.

Numerical relativity has a *lower* frequency than the quadrupole approx!

What about eccentric binaries? We can also analyze them. The formula is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{5}\frac{G\mu^2}{c^5} a^4 \omega_0^6 f(e)\,, \tag{1.160}$$

where

$$f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \geq 1\,. \tag{1.161}$$

We have emission at frequencies other than the orbital one; also, the GW emission has the effect of circularizing the orbit. So, we usually observe circular systems.

1.7 Hulse-Taylor binaries

It is debatable whether the observation of this was the first observation of gravitational waves.

This is a binary system in which one star is a pulsar.

What is a pulsar? It's a kind of neutron star. Not a moral judgement, but you are completely empty.

A pulsar has a large magnetic field; at a distance $r_c = c/\omega$ the field lines cannot close so a radio beam escapes. This provides a clock!

"Taking the pulse" of a pulsar: they usually have a certain well-defined shape, if we average over a few periods.

The procedure is: take signal, FFT to get the fundamental, average over periods.

The period can then be measured precisely, and we can observe its variations.

Some relevant frequencies: the radio waves are on the order of $10^8\,$Hz, the pulsar's frequency is of the order $10\,$Hz, the frequency of the binary period is $10^{-5}\,$Hz, the motion of the Earth around the Sun at $10^{-8}\,$Hz is also relevant.

Since the pulsar frequency is very small, we can still average many pulses and still be measuring at what is basically "a single point". Monday 2020-4-6, compiled 2020-07-10

We are discussing the Hulse-Taylor pulsar, which is a very precise clock. There are many effects by which the time-of-arrival is shifted, if we take them all out we can get the effect from the source.

We will have the Shapiro delay from the Sun, the Einstein delay at the receiver from the curvature of the spacetime around the Earth, which is given by

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} \approx 1 + \phi(x_{\text{obs}}) - \frac{v_{\text{obs}}^2}{2c^2} , \tag{1.162}$$

so

$$\tau \approx t + \int^t \mathrm{d}\widetilde{t} \left(\phi(x_{\text{obs}}) - \frac{v_{\text{obs}}^2}{2c^2} \right) = t - \Delta_{\oplus,\odot} . \tag{1.163}$$

If most of the velocity is due to the motion of the Earth in its elliptic orbit, we have from conservation of energy

$$\frac{v_{\text{obs}}^2}{2} = \frac{GM}{r} - \frac{GM}{2a} , \tag{1.164}$$

so that we find

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} \approx \frac{v^2}{2c^2} - \phi = \frac{GM_\odot}{c^2} \left(\frac{1}{r} - \frac{1}{2a} - \frac{1}{r} \right) = R_{\text{earth-Sun}} \left(\frac{1}{r} - \frac{1}{4a} \right) . \tag{1.165}$$

However, we must also consider the group velocity of the signal which travels through the ISM, which is ionized gas. Then, we get a delay which depends on the frequency:

$$t_L = \frac{L}{c} + \frac{e^2}{2\pi m_e c} \frac{1}{\nu^2} DM , \tag{1.166}$$

where DM is the Dispersion Measurement. For the HT pulsar, this spreads the time over a 4 MHz bandwidth: however, we can measure precisely in the spectral domain, and we can "connect the dots" to find what corresponds to a single pulse.

This allows us to reconstruct the original pulse.

Taking these effects out, we get

$$t_{\text{ssb}} = \tau - \frac{D}{\nu^2} + \Delta_{E,\odot} - \Delta_{S,\odot} + \Delta_{R,\odot} . \tag{1.167}$$

This is "time in solar-system barycenter". where

$$D = \frac{e^2}{2\pi m_e c} DM . \tag{1.168}$$

We need to look at the gravitational time delay at the source: there is a contribution from the gravitational field of the NS itself, which is hard to calculate but constant, so we do not worry about it. The combined gravitational field, instead, is time-varying: so, we find

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 - \frac{Gm_c}{c^2 \left| x_p - x_c \right|} - \frac{v_p^2}{2c^2} \tag{1.169a}$$

$$\frac{\mathrm{d}T}{\mathrm{d}u} \approx \frac{P_b}{2\pi}\left(1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p+m_c)}\right)\left(1 - e\cos u\left(1+\frac{G}{c^2}\right)\right)\dots, \tag{1.169b}$$

[to finish] where u is the angular parameter describing the orbit, while e is the eccentricity.

Also, we have the Romer delay: $\Delta_R = \hat{z}\cdot x_{pb}/c$.

The coordinates, for a Keplerian orbit, are

$$r_{pb} = r_1 = a_1(1 - e\cos u) \qquad \text{and} \qquad \cos\psi = \frac{\cos u - e}{1 - e\cos u}. \tag{1.170}$$

The Romer delay then is

$$\Delta_R = r_1 \sin\iota \sin(\omega+\psi) = r_1 \sin\iota(\cos\psi\sin\omega + \cos\omega\sin\psi), \tag{1.171}$$

where ι is the observation angle, while ψ is the angle from the line of notes (see drawing).

The relativistic effect, however, is large. We will not do the calculation, we find

$$\Delta_R = a_1 \sin\iota\left((\cos u - e_r)\sin\omega + \sqrt{1-e_\theta^2}\sin u\cos\omega\right), \tag{1.172}$$

where

$$e_{r,\theta} = e(1+\delta_{r,\theta}) \tag{1.173a}$$

$$\delta_r = \frac{G}{c^2}\frac{3m_p^2 + 6m_p m_c + 2m_c^2}{a(m_p+m_c)} \tag{1.173b}$$

$$\delta_\theta = , \tag{1.173c}$$

also here the advance of the periastron is much more significant than it is for Mercury.

The Shapiro delay at the source must also be accounted for.

If we get all the Keplerian parameters and two of the post-Newtonian ones then we should know everything.

We measure $P_b, T_0, x = a\sin\iota/c, e, \omega$ and the post-Newtonian $\dot{\omega}, \gamma$ and finally we make a prediction for $\dot{P}$. This matches the data very well.

1.8 GW from a rotating rigid body

The moment of inertia tensor can be defined as

$$I^{ij} = \int \mathrm{d}^3x\,\rho(x)\left(r^2\delta^{ij} - x^i x^j\right). \tag{1.174}$$

There exists a frame in which this tensor is diagonal, its eigenvalues are the moments of inertia, its eigenvectors are the axes of inertia. They are then defined by equations like

$$I_1 = \int \mathrm{d}^3x\,\rho\left(x_2^2 + x_3^2\right). \tag{1.175}$$

For an ellipsoid with axes a, b, c and mass m we have

$$I_1 = \frac{m}{5}\left(b^2 + c^2\right). \tag{1.176}$$

The rotational kinetic energy is given by

$$E_{\text{rot}} = \frac{1}{2} I_{ij}\omega_i\omega_j \tag{1.177a}$$

$$= \frac{1}{2} I_i \omega_i^2 \,, \tag{1.177b}$$

where the last equality holds in the body frame.

Suppose we have a body spinning around an axis, such that the position of any point shifts by a rotation matrix R_{ij}.

The inertia tensor shifts by $I \to R^\top I R$.

The tensor we defined before,

$$M^{ij} = \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(x) x^i x^j \approx -I^{ij} + \int \mathrm{d}^3x\, \rho(x) r^2 \delta^{ij} \,, \tag{1.178}$$

which we can substitute in. This is the trace of the inertia tensor.

Suppose we had a body whose angular momentum is not aligned with the moment of inertia: we can use Euler angles to express the rotation matrix.

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Now we are considering a body which is axisymmetric, and rotating along an axis which is not aligned with its axes of inertia.

An astrophysical example of this will usually look like an ellipsoid. We should "clean our minds" from the idea of a spinning spintop precessing.

Here, the axis going around is the faster motion, the rotation of the body around its axis is slower.

This wobbling motion is similar to the one of a coin thrown on a table.

If we compute the evolution of the inertial tensor, we get terms both at ω and at 2ω.

We are interested in the projection of this variation onto the plane orthogonal to the direction of a propagation.

If we have a distribution which looks like a coin ($I_1 \sim I_2 \ll I_3$) then it looks to us like a binary if we look at it from the top (in terms of periodicity at least), so we expect 2ω emission, since the system looks the same to us, since it repeats after a rotation of π.

If, instead, we look at it from the side, the periodicity is the full period: after half a rotation the coin is edge-on, but it appears at two different angles with respect to the vertical direction.

Therefore, we both have ω and 2ω emission.

If we were able to determine the amplitude at different inclinations, we would be able to determine the inclination ι.

[formula for back reaction is wrong!]

In order to calculate the backreaction we assume that the motion is approximately constant during a single period.

We find differential equations telling us that $\dot{\beta}$ and α both decrease: the first means that the motion is slowing down; the second means that the wobbling is decreasing, as the rotation is aligning with the angular momentum.

We can define the parameter $u(t) = \dot{\beta}/\dot{\beta}_0$, and a characteristic time τ_0:

$$\tau_0 = \left(\frac{2G}{5c^5} \frac{(I_1 - I_3)^2}{I_1} \dot{\beta}_0^4 \right)^{-1}, \tag{1.179}$$

which has a typical value of

$$\tau_0 = 1.8 \times 10^6 \,\mathrm{yr} \left(10^{-7} \frac{I_3}{I_1 - I_3} \right)^2 \left(\frac{1\,\mathrm{kHz}}{f_0} \right)^4 \left(10^{38} \frac{\mathrm{kgm}^2}{I_1} \right), \tag{1.180}$$

and we can write differential equations for $\dot{u}$ and $\dot{\alpha}$: but we must have $\dot{\beta}\cos(\alpha) = \text{const}$. This implies that the boundary conditions must be $\alpha_\infty = 0$ and $u_\infty = \cos\alpha_0$. Also, asymptotically,

$$\dot{\alpha}_{t\to\infty} = \dots . \tag{1.181}$$

These conditions do not apply in general, neutron stars are not truly rigid bodies since they have an internal structure. In a generic case we will have emission at different frequencies.

We have not seen pulsars yet in GW, but we can put upper bounds to the amplitude of their emission.

"Beating the spin-down limit" means that we know that we would be able to see the GW emission in a certain case if the spin-down was only due to GW.

Could we differentiate a pulsar rotating and seen head-on and a binary system? Surely they are phenomena which happen in different frequency ranges, and last very much different times. If the binary is spinning at those frequencies it's evolving very rapidly, instead a pulsar can give out a stable signal.

Also, in full numerical relativity the waveform looks different.

Chapter 2

Detectors

2.1 Noise theory

2.1.1 A simple experiment

Friday 2020-4-17, compiled 2020-07-10

We want to use a pendulum to measure g. We want to use small oscillations, which should have $g = \omega^2 L$, and we want to fit $a(t)$.

However, our signal can be very noisy compared to the theoretical waveform.

It *is* a good idea to use the whole timeseries instead of just counting oscillations, since we are using more data then.

The physical system gives us a signal $s(t)$, we have some measurement apparatus which gives us an output $s'(t)$. The output $s'(t)$ can be also affected by noise: we can have physical noise $n_1(t)$ in the form of a random concurrent phenomenon, noise $n_2(t)$ in the transducer, and $n_3(t)$ in the readout.

In our example: n_1 can be due to air currents, vibrations of the suspension points. The transducer in our case is the accelerometer, so we can have input voltage instability, imperfect mechanical coupling, thermal vibrations contributing to n_2. In the readout, we have have reference voltage instability, quantization if we digitize the signal, and electronic pick-up.

We want to distinguish the output signal s' from the output noise n'.

We define **precision experiments** as experiments where the signal amplitude is generally comparable or smaller than the noise amplitude.

We call **noise** any unwanted signal.

In the classical realm, $n(t)$ is the sum of deterministic processes, but in practice it is random since there are so many of them. We usually assume that they are zero-mean.

2.1.2 Random processes

A random variable is a number x associated to a possible experimental outcome. Any outcome has an associated probability. In the continuous realm, we use probability density functions:

$$\mathbb{P}(x_0 \leq x \leq x_1) \int_{x_0}^{x_1} f(x)\, \mathrm{d}x \,, \tag{2.1}$$

and we can use it to compute mean values:

$$\int g(x)f(x)\,\mathrm{d}x = \langle g(x)\rangle_f \,, \tag{2.2}$$

while the variance is

$$\sigma^2 = \left\langle (x - \langle x\rangle)^2 \right\rangle = \left\langle x^2 \right\rangle - \langle x\rangle^2 \,. \tag{2.3}$$

Do note that these are ensemble averages, what we expect to be the average of a sample. They are not time averages.

A random signal is a time series $x(t)$. Every time we repeat the experiment we get a new $x(t)$.

Knowing $x(\widetilde{t})$ at a specific time $\widetilde{t}$ we have partial predictive power for $x(\widetilde{t} + T)$.

At a fixed time $\widetilde{t}$, the possible values of $x(\widetilde{t})$ have a certain probability distribution $f(x; \widetilde{t})$. Then we have the following functions of time: $\mu(t) = \langle x(t)\rangle$ and $\sigma^2(t)$.

Ideally these are averages over an infinite number of realizations. We can, however, assume *ergodicity*: this means that in stead of an ensemble mean value we can compute a time average, and similarly for other statistical properties.

Also, we assume stationarity: the statistical properties are time-independent.

Also, we can assume gaussianity: $x(t)$ is normally distributed at fixed t. This is qualitatively justified by the central limit theorem.

These assumptions are almost always made, but it is important to keep them in mind.

We assume that the noise is **stochastic**, so it results from many different uncorrelated processes, and the correlation between the noise at a time t and $t + \Delta t$ decreases quickly with Δt.

So, we meand that the noise is high-frequency?

Stationarity: the apparatus is stable in time, and so are the processes that generate the noise. This is valid only for somewhat small periods.

2.1.3 Fourier transforms

For square-integrable functions,

$$\int \left|s(t)\right|^2 \mathrm{d}t < \infty \,, \tag{2.4}$$

we define

$$s(\omega) = \int s(t)e^{-i\omega t}\,\mathrm{d}t \qquad \text{and} \qquad s(t) = \frac{1}{2\pi}\int s(\omega)e^{i\omega t}\,\mathrm{d}\omega \,. \tag{2.5}$$

Important properties are linearity, the fact that in Fourier space derivatives become multiplication by $i\omega$. Also, the convolution theorem: multiplication in time domain is the same as convolution in frequency domain:

$$\int s(t)q(t)e^{-i\omega t}\,\mathrm{d}t = \frac{1}{2\pi}\int s(\omega')q(\omega - \omega')\,\mathrm{d}\omega' \,. \tag{2.6}$$

The transform of the Dirac $\delta(t)$ function is 1. Almost any signal can be described as an infinite superposition of oscillatory terms.

The transform is a complex-valued function, telling us the amplitude and phase of any component of the oscillation.

In practice, the frequency domain is very useful. Many interesting signals are well-defined in frequency.

Even multiple signals can be easily separated in frequency. Also, random noise is easier to characterize in frequency domain: the shape of the noise is interesting.

Some other properties: the Fourier transform preserves the energy (in the sense of the integral of the square modulus). It also preserves the information. It returns a Hermitian function.

We have the uncertainty principle:

$$\Delta\omega\Delta t \geq \frac{1}{2} \,. \tag{2.7}$$

Depending on the physical characteristics of the signal, we should select a large observation time or a large frequency band.

2.1.4 Power spectral density

The phase of the noise will be completely different from one realization to the other.

We define the autocorrelation function:

$$R(t, t') = \left\langle x(t)x(t') \right\rangle = R(\tau) \,, \tag{2.8}$$

where $\tau = t - t'$. So, we define the power spectral density (PSD) as:

$$S_x(\omega) = \int \mathrm{d}\tau \, R(\tau) e^{i\omega\tau} \,. \tag{2.9}$$

White noise has no correlation between a point and another: its auto-correlation function is a delta. It has all the frequency components. The autocorrelation function measures how fast the signal loses memory.

A more loose and intuitive definition is the ensemble average:

$$S_x(\omega)\delta(\omega - \omega') = \left\langle x(\omega) x^*(\omega') \right\rangle \,. \tag{2.10}$$

It is the average of the amplitude square, which is a power.

This is not the formal definition since $x(\omega)$ may not exist.

If we build a window filter which only allows $[\omega_1 \leq \omega \leq \omega_2]$, then the residual power will be

$$P = \int_{\omega_1}^{\omega_2} S(\omega) \, \mathrm{d}\omega \,. \tag{2.11}$$

For real signals, S is symmetric: $S_x(-\omega) = S_x(\omega)$. If we do not care about negative frequencies, we keep only the $[\omega \geq_0]$ region and multiply by 2.

The root mean square of the signal in time is the integral of the PSD:

$$\sigma_x^2 = \int_0^\infty S_x(\omega)\,. \tag{2.12}$$

Here we are assuming that the signal has zero mean. If two signals are uncorrelated, the power spectral density of their sum is the sum of their PSDs.

The amplitude, or linear, spectral density, is $\sqrt{S_x(\omega)}$.

If our signal has units of m, then $[S_x(\omega)] = \mathrm{m}^2/\mathrm{Hz}$, so the linear PSD has units of $[\sqrt{S_x(\omega)}]\mathrm{m}/\sqrt{\mathrm{Hz}}$.

In practice, we measure for a limited time T. So, x will only have Fourier components at frequencies $f_n = n/T$. The frequency resolution is $\Delta f = 1/T$.

Having a measurement which is limited in time with a window $w(t)$, we are actually measuring $x(t)w(t)$: so, in frequency space we have

$$x_{\text{measured}}(\omega) = \int \mathrm{d}\widetilde{\omega}\, x(\widetilde{\omega})w(\omega - \widetilde{\omega})\,, \tag{2.13}$$

so the Fourier transform of the window spreads out our signal.

We can deconvolve if we know what the window looks like. However, we do not usually do it. In fact, if we were to take out this windowing effect it would mean we are assuming that if we were to observe our signal for a longer time than we did we would see the same thing. This might be justified sometimes, some other times it is not.

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The Fourier transform is stochastic since the noise is stochastic: however, the PSD encompasses the statistical properties of the signal in a way that is stationary and well-defined.

A physical system is a functional F which transforms input time series $i_j(t)$ into output time series $o_j(t) = F(i_j(t))$.

Real systems are causal: there cannot be causality going backward in time.

Also, often we can approximate systems as linear ones, as long as we work near a single point. Also, we can sometimes approximate them as stationary.

Under all of these assumptions, we can express everything using an impulse response function:

$$o(t) = \int \mathrm{d}\widetilde{t}\, i(\widetilde{t})F(\delta(t - \widetilde{t})) = \int \mathrm{d}\widetilde{t}\, h(t - \widetilde{t})\,, \tag{2.14}$$

and this is a convolution: so we can express it as

$$o(\omega) = i(\omega)h(\omega)\,. \tag{2.15}$$

The power spectral density then transforms as $S_o(\omega) = \left|h(\omega)\right|^2 S_i(\omega)$, and if we have systems in series we can just multiply the impulse responses together.

2.1.5 Sampling

Often we sample signals digitally. Analogic systems are faster, but electronics are getting very fast as well and they are easier to use.

The signal is quantized in two ways: we quantize both in time by sampling at an interval t_s and in amplitude, by encoding it with a finite number of bits.

This introduces noise, which is however well-known and easy to calculate.

If we have a signal at a frequency f and we want to reconstruct it, we need to sample at a frequency $2f$. If we sample at $100\,\text{Hz}$, we can only accurately describe signals up to $50\,\text{Hz}$. This is the Nyquist theorem.

This is true if we want to fit the data with the slowest sinusoid, if we know in which frequency range we should look. If we work below the Nyquist frequency we can be sure of each frequency.

2.2 Gravitational wave detection

We can use an **elastic body** which resonates: we have the pro that this might enhance the effect of a GW through resonance and extend the duration of burst signals. However, it is only sensitive around its resonant frequency. Also, since it extends the signal it is hard to precisely reconstruct the time profile of the signal.

These need to be isolated solid objects: they will fit in a lab, but the GW displacements are small.

On the other side, we have **interferometers** which measure the distance between free falling masses.

If we have a perfect harmonic oscillator with rest position x_0: then we will have

$$x(\omega) = \frac{kx_0 + F_{\text{ext}}(\omega)}{k - m\omega^2}\,, \tag{2.16}$$

while if there is damping we will have

$$m\ddot{x} = -k\big(x(t) - x_0(t)\big) - \beta\dot{x}(t) + F_{\text{ext}}(t)\,, \tag{2.17}$$

which means we have

$$x(\omega) = \frac{kx_0 + F_{\text{ext}}(t)}{k\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + \frac{i\omega\beta}{k}\right)}\,, \tag{2.18}$$

otherwise, we can have structural internal damping, which looks like

$$m\ddot{x} = -k(1 + i\delta)\big(x(t) - x_0(t)\big) + F_{\text{ext}}(t)\,. \tag{2.19}$$

The transfer function is more peaked for less damping.

How do we see the effect of GW on an elastic body? Consider two masses, which are free falling, and connect them by a spring: they now will not move along geodesics.

We will have the equation

$$F_{\text{GW}} - k(L - \Delta x) = m\Delta\ddot{x}\,. \tag{2.20}$$

This is given by

$$F = \frac{m}{2} \ddot{h}_{xx}^{TT} \Delta x \approx \frac{m}{2} L \ddot{h}_{xx}^{TT} \,. \tag{2.21}$$

This, however, is only valid as long as $L \ll \lambda_{\text{GW}}$.
For a continuous bas, we will have

$$\mathrm{d}m \left(\frac{\partial^2 u}{\partial t^2} - v_s^2 \frac{\partial^2 u}{\partial x^2} \right) = \mathrm{d}F_x = \mathrm{d}m \, \frac{1}{2} x \ddot{h}_{xx}^{TT} \,. \tag{2.22}$$

We further assume that

$$\left. \frac{\partial u}{\partial x} \right|_{x = \pm L/2} = 0 \,. \tag{2.23}$$

The general solution will be given by a sum of sines and cosines, but the cosines will move the center of the bar.

We will have

$$u(t, x) = \sum_{n=0}^{\infty} \xi_n \sin\left(\frac{\pi x}{L} (2n+1) \right) , \tag{2.24}$$

and we can take the scalar product in the L^2 space with the basis sinusoids: so we find

$$\ddot{\xi}_n + \omega_n^2 \xi_n = \frac{(-)^n}{(2n+1)^2} \frac{2L}{\pi^2} \ddot{h}_{xx}^{TT} \,. \tag{2.25}$$

We have basically eliminated the spatial part. We can analyze the time evolution by itself.

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Recover ten minutes

Resonant bars are resonant: their transfer function is very much peaked.

The transfer function is the response to the strain as a function of signal frequency.

The displacement function ξ_0 is a sinusoid, it looks like

$$\xi_0(t) = \frac{2L\omega^2 h_0}{\pi^2} \frac{(\omega^2 - \omega_0^2) \cos(\omega t) - \gamma_0 \sin(\omega t)}{(\omega^2 - \omega_0^2)^2 + \omega^2 \gamma_0^2} \,. \tag{2.26}$$

In Fourier space this is easier:

$$\xi_0(\omega) = -\frac{2L}{\pi^2} \frac{\omega^2}{\omega_0^2 - \omega^2 - i\omega\gamma_0} h_0 \delta(\omega) \,. \tag{2.27}$$

It is a fact from classical mechanics that the absorbed energy of an oscillator subject to an external impulsive force is

$$E = \frac{1}{2m_0} \left| \int_{-\infty}^{\infty} F(t) e^{-i\omega_0 t} \, \mathrm{d}t \right|^2 = \frac{ML^2}{\pi^4} \left| \int_{-\infty}^{\infty} \ddot{h}_{xx}^{TT}(t) e^{-i\omega_0 t} \, \mathrm{d}t \right|^2 , \tag{2.28}$$

and since the force is impulsive we can integrate by parts twice: so, we get

$$E = 16ML^2 f_0^4 \left| h_{xx}^{TT}(f_0) \right|^2 , \tag{2.29}$$

which is the energy of the GW component at frequency ω_0: we can invert it as

$$\left| h_{xx}^{TT}(f_0) \right|^2 = \frac{1}{16L^2 f_0^4} \frac{E}{M} . \tag{2.30}$$

The transfer function can be written as

$$\xi_0(\omega) = -\frac{2L}{\pi^2} \frac{\omega^2}{(\omega - \omega_+)(\omega - \omega_-)} h_{xx}^{TT}(\omega) \qquad \text{where} \qquad \omega_\pm = \pm\sqrt{\omega_0^2 - \left(\frac{\gamma_0}{2}\right)^2} - i\frac{\gamma_0}{2} . \tag{2.31}$$

We suppose that the incoming burst is a delta in time: then, it has components at all possible frequencies.

Then, we get

$$\xi_0(t) = \frac{2L}{\pi^2} h_0 \tau_{GW} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2}{(\omega - \omega_+)(\omega - \omega_-)} e^{-i\omega t} \approx \frac{2L}{\pi^2} h_0 \omega_0 \tau_{GW} e^{-\gamma_0 t/2} \sin(\omega_0 t) . \tag{2.32}$$

By saying that the signal is a delta, do we mean that $\tau_{GW} \ll 1/\omega_0$? YES, where ω_0 is the frequency of the oscillator, which in our case is faster than the characteristic frequency of the dissipation.

2.2.1 Antenna pattern

The sensitivity of the detector depends both on the direction from which che GW comes and the polarization.

Specifically, if $\vec{J}$ is the orientation of the bar, then we have

$$h_{\text{out}} = \hat{J}^i \hat{J}^j h_{ij}(t) . \tag{2.33}$$

If θ is the angle between the bar direction and the source-Earth vector, we will have

$$h_{\text{out}} = h_+ \sin^2\theta . \tag{2.34}$$

For a generic rotation, including a rotation of angle φ around the axis z' of the source system where the $+$ and $\times$ polarizations are defined, we get

$$h_{\text{out}}(t) = h'_+ \sin^2\theta \cos 2\varphi + h'_\times \sin^2\theta \sin 2\varphi . \tag{2.35}$$

The response of the bar is quite small: we cannot really measure it; however we can transfer the energy to a lighter oscillator, which will move more.

We need to tune the bar so that its resonant frequency is the same as the one of the other bar.

The ratio of masses is μ, then we will have

$$A_t = \frac{A_0}{\sqrt{\mu}} , \tag{2.36}$$

and we typically can choose $\mu \lesssim 10^{-4}$, if we go lower other noise sources dominate.

2.2.2 Thermal and readout noise

We saw that we can represent the stages of our processing by their transfer functions $H_i(\omega)$. At each stage, though, we can get noise.

This noise is then processed by the later transfer functions.

Say we have

$$y(\omega) = H_1(\omega) H_2(\omega) H_3(\omega) x(\omega)\,, \tag{2.37}$$

and we have noise after each stage.

We can have thermal noise: noisy signals are described by a deterministic part plus a stochastic noisy part.

We can model this thermal noise using a Green's function: we get

$$x(t) = \dots\,. \tag{2.38}$$

For thermal noise, we can assume that it is completely uncorrelated: it is due to billions of atomic interactions, so it loses memory quickly. This means that the noise is white noise: the PSD is constant in freqency. Therefore, we can show that

$$\left\langle x^2(t) \right\rangle \approx \frac{A}{m_0^2 \omega_0^2} \left(1 - e^{-\gamma_0 t} \right)\,. \tag{2.39}$$

Then, the average total energy of the mode is

$$\left\langle E(t) \right\rangle = \frac{1}{2} m_0 \omega_0^2 \left\langle x^2(t) \right\rangle + \frac{1}{2} m_0 \left\langle \dot{x}^2(t) \right\rangle \approx \frac{A}{2 m_0 \gamma_0} \left(1 - e^{-\gamma_0 t} \right)\,. \tag{2.40}$$

So, we get asymptotically

$$\left\langle E(t) \right\rangle \to 2 k_B T m_0 \gamma_0\,, \tag{2.41}$$

which is the fluctuation-dissipation theorem: the PSD of the noise is

$$S_F(\omega) = 4 k_B T m_0 \gamma_0\,. \tag{2.42}$$

We can always write

$$F(\omega) = Z(\omega) \dot{x}(\omega)\,, \tag{2.43}$$

for any linear system. So, the FDT gives us

$$S_{\text{F, th}}(\omega) = 4 k_B T \operatorname{Re}\left\{ Z(\omega) \right\}\,. \tag{2.44}$$

This is in terms of the PSD of the force, which can be connected to that of the velocity.

If the energy of the DoF must be asymptotically constant, there needs to be a lossy mechanism giving energy to the DoF.

Is there a problem with many transducers?

Could we analyze different wavelengths by modifying the frequency

The temperatures are of the order of milliKelvin, we cool them to close to zero by isolating them and making liquid helium go through everything.

In this case the approximation of constant temperature is quite good. Monday 2020-4-27, compiled 2020-07-10

We discussed the Fluctuation Dissipation Theorem last time: we have qualitative arguments, if by equipartition each DoF has energy $1/2 k_B T$ and it is dissipative there must be some forcing mechanism.

A lossy harmonic oscillator gets from its environment a PSD given by

$$S_x(\omega) = \frac{4 k_B T}{\omega^2} \frac{m_0 \gamma_0}{m_0^2 \gamma_0^2 + \left(\frac{m_0}{\omega}\left(\omega_0^2 - \omega^2\right)\right)^2} . \tag{2.45}$$

What does the noise look like in the bar-transducer system? The noise by the transducer is very small at resonance, the noise of the bar does not have features there: the PSD is

$$S_{\text{h, th}} = \pi \frac{k_B T}{M v_s^2} \frac{f_0^3}{f^4} \left(\frac{1}{Q_0} + \frac{1}{\mu Q_t} \frac{\left(f^2 - f_0^2\right)^2 + \left(f f_0 / Q_0\right)^2}{f_0^4} \right) . \tag{2.46}$$

Missing bit... still to understand

2.3 Gravitational Wave Interferometry

2.3.1 Mach-Zender interferometer

We consider a laser with the electric field

$$\vec{E}_{\text{in}} = \vec{E}_0 \exp\left(-i(\omega_l t - k_l t)\right) , \tag{2.47}$$

where the subscript l means "laser", we include it to remind ourselves with the GW parameters.

After the first beamsplitter the reflected wave picked up a phase:

$$E_T = \frac{E_{\text{in}}}{\sqrt{2}} e^{i\pi} \qquad \text{and} \qquad E_R = \frac{E_{\text{in}}}{\sqrt{2}} e^{i\Delta\phi} , \tag{2.48}$$

where we insert some prism which delays the reflected wave by $\Delta\phi$.

So the output value is

$$E_{\text{out}} = \frac{E_T}{\sqrt{2}} e^{i\pi} + \frac{E_R}{\sqrt{2}} = E_{\text{in}} \left(1 + e^{i\Delta\phi}\right) , \tag{2.49}$$

so the initial power is multiplied by $(1 + \cos(\Delta\phi))/2$.

So, is energy not conserved? This output is the same on both ends of the beamsplitter. The phase of π is picked up only if the index of refraction increases. This includes a correction: then energy is conserved.

2.3.2 Michelson-Morley interferometer

In the detector frame we can treat the GW as a Newtonian force acting on the mirrors:

$$F_x \approx \frac{m}{2} x_0 \ddot{h}_{xx}^{TT} \,, \tag{2.50}$$

so

$$\ddot{x} = \frac{1}{2} x_0 \ddot{h}_{xx}^{TT} \,. \tag{2.51}$$

This only holds if $x \ll \lambda_{GW}$, which means $f \ll c/L \approx 100\,\mathrm{kHz}(L/3\,\mathrm{km})$.

If we insert the displacement for the mirrors we find

$$I_{\mathrm{out}} = E_0^2 \sin^2 \left(k \Big(L_x - L_y + h_0 L \cos(\omega_{GW} t) \Big) \right) . \tag{2.52}$$

Friday 2020-5-1, compiled 2020-07-10

We were discussing the basics of GW interferometry.

Moving to the TT gauge is a coordinate system in which the mirrors are free-falling. However, the light propagating through spacetime is affected by the fact that the spacetime is "stretched": for our photons,

$$\mathrm{d}s^2 = -c^2 \,\mathrm{d}t^2 + \big(1 + h_+(t)\big) \,\mathrm{d}x^2 + \big(1 - h_+(t)\big)^2 \,\mathrm{d}y^2 + \mathrm{d}z^2 = 0 \,, \tag{2.53}$$

so in the x arm, depending on the direction of propagation we will have:

$$\mathrm{d}x = \sqrt{\frac{c^2 \,\mathrm{d}t^2}{1 + h_+(t)}} \approx \pm c \,\mathrm{d}t \left(1 - \frac{1}{2} h_+(t) \right) , \tag{2.54}$$

so if we integrate the whole round-trip of the photon we get, for the first leg:

$$L_x = \int_0^{L_x} \mathrm{d}x = \int_{t_0}^{t_1} c \,\mathrm{d}t = c(t_1 - t_0) - \frac{c}{2} \int_{t_0}^{t_1} \mathrm{d}t \, h_+(t) \,, \tag{2.55}$$

and for the second leg:

$$-L_x = \int_{L_x}^{0} = -\int_{t_1}^{t_2} c \,\mathrm{d}t \left(1 - \frac{h_+(t)}{2} \right) = -c(t_2 - t_1) + \frac{c}{2} \int_{t_1}^{t_2} \mathrm{d}t \, h_+(t) \,, \tag{2.56}$$

so the time difference is given by

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2} \int_{t_0}^{t_2} \mathrm{d}t \, h_+(t) \,, \tag{2.57}$$

where we can consider $t_2 \approx t_0 + 2L_x/c$ in the integration bound, since the correction would be second order.

To first order, then, if the GW is sinusoidal we have for the x arm:

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{L_x}{c} h_+ \left(t_0 + \frac{L_x}{c} \right) \frac{\sin\left(\frac{\omega_{gw} L_x}{c} \right)}{\frac{\omega_{gw} L_x}{c}} \,, \tag{2.58}$$

while for the y arm the sign is inverted.

The argument is basically the ratio of the length of our arm to the wavelength of the GW. We have a maximum of the effect, then, when the GW perturbation is effectively static during the time in which we are detecting.

Keep in mind, though, that the effect also scales with L_x: a very small detector wouldn't work.

If the length, on the other hand, is very large the perturbation cancels out during the flight of the photon. If we fix the frequency, making the detector longer and longer does not help: further full oscillations of the path length do not have any effect.

The interesting thing is the phase difference: if in both arms the light reaches the detector at t_2 we have

$$\Delta\phi = \omega_l \left(t_0^x - t_0^y \right) \tag{2.59a}$$

$$= \omega_l \left(2\frac{L_x - L_y}{c} + \frac{2L}{c} \operatorname{sinc}(\frac{\omega_{gw} L}{c}) h_0 \cos\left(\omega_{gw} t + \alpha \right) \right) = \Delta\phi_0 + \Delta\phi_{gw} \,. \tag{2.59b}$$

So, we have a tradeoff: we want to stay before the first zero of the sinc, but we also want to have a relatively large detector.

So, our optimal length is of the order of a quarter of the wavelength.

This is the same as saying we want to keep the photon in-flight for a quarter of the period of the GW.

This would mean, for a frequency of 100 Hz, that we would need a detector of around 750 km.

If we do the computation accounting for the oscillation of the laser light, we get that the field out of the BS is

$$E = \frac{E_0}{2} e^{-i\gamma} \left(e^{-i\omega_l t} + \beta e^{-i\alpha} e^{-i(\omega_l - \omega_{gw})} + \beta e^{i\alpha} e^{-i(\omega_l + \omega_{gw})t} \right) , \tag{2.60}$$

which is the same of a single field which is modulated in amplitude.

2.4 Lasers and cavities

The best mirrors in the world are built for GW detectors. These are dielectric mirrors. We stack dielectric interfaces on top of each other, so that the wave coming back has constructive interference while the wave being transmitted interferes destructively with the next layer.

A **cavity** is an arrangement of mirrors such that we have a closed path for light.

Mirrors are symmetric, if we can input some light then we are also losing the same amount.

We have an incident field E_{in}, a circulating field E_c and a transmitted field E_t. The "in" mirror is labelled 1, the "out" mirror is labelled 2.

So, we have

$$E_c = t_1 E_{\text{in}} + r_1 r_2 E_c e^{-ik2L} \,, \tag{2.61}$$

so

$$E_c = E_{\text{in}} \frac{t_1}{1 - r_1 r_2 e^{-ik2L}} \,. \tag{2.62}$$

Here k is the wavevector of the electric field. The exponential is called the *round-trip gain*.

The reflected field is given by

$$\begin{aligned} E_r &= -r_1 E_{\text{in}} + r_2 t_1 E_c e^{-ik2L} && \text{(2.63a)} \\ &= -E_{\text{in}} \frac{r_1 - r_2 e^{-ik2L}}{1 - r_1 r_2 e^{-ik2L}} \,. && \text{(2.63b)} \end{aligned}$$

On the other hand, the transmitted field is

$$E_t = t_2 E_c \,. \tag{2.64}$$

The circulating intensity can be calculated by the square modulus of the field, we then get

$$I_c = E_{\text{in}}^2 \left| \frac{t_1}{1 - r_1 r_2 e^{-ik2L}} \right|^2 , \tag{2.65}$$

so we want to minimize the denominator: this means we want $L = n\pi/k$.

As we increase reflectivity the peaks get narrower and higher.

The distance in frequency between the peaks, $c/2L$, is called the *free spectral range*.

The *finesse* is defined as the free spectral range divided by the FWHM of the peaks,

$$\mathcal{F} = \frac{\pi \sqrt{r_1 r_2}}{1 - r_1 r_2} \,. \tag{2.66}$$

This is 2π/losses.

Using this, we can estimate the storage time of a photon inside the cavity:

$$\tau_s \approx \frac{L\mathcal{F}}{c\pi} \,. \tag{2.67}$$

We are discussing resonance and cavities in order to study the LASERs used in GW interferometry. Monday 2020-5-4, compiled 2020-07-10

We can have a lot of energy stored in the cavity, so that the power of the stored laser beam is much larger than the input power. This does not violate conservation of energy: the energy is stored, but we cannot extract more power than what is coming in.

As the reflectivity increases, the peaks of intensity become sharper and sharper.

We can treat the losses inside the cavity by introducing:

$$r_l = 1 - l_E = \sqrt{1 - l_I} \,, \tag{2.68}$$

which is derived from $r_l^2 + l_I = r_l^2 + t_l^2 = 1$.

We have the fields inside and outside given by

$$E_c = E_{\text{in}} \frac{t_1}{1 - r_1 r_2 r_l e^{-ik2L}} \tag{2.69a}$$

$$E_r = -E_{\text{in}} \frac{r_1 - r_2 r_l e^{-ik2L}}{1 - r_1 r_2 r_l e^{-ik2L}} . \tag{2.69b}$$

The total reflected field inside the cavity can be overcoupled if $r_1 < r_2 r_l$, so that the reflected field is less than the one outside; also we can have undercoupling and finally the **impedence matched** case is obtained when $r_1 = r_2 r_l$.

In this case the reflected intensity goes to zero. We also have discontinuity in phase, but this is not an issue.

Let us define a **beam** properly:

$$\vec{E}(x, y, z) = u(x, y, z) e^{-ikz} , \tag{2.70}$$

where we are neglecting the temporal dependence. The function $u(x, y, z)$ is a complex amplitude.

This obeys the wave equation, as long as

$$\left(\nabla^2 + k^2 \right) E = \nabla^2 u - 2ik \frac{\partial u}{\partial z} . \tag{2.71}$$

The term

$$\frac{\partial^2 u}{\partial z^2} \tag{2.72}$$

has magnitude much smaller than the other terms. So, we neglect it. This is the paraxial approximation. The function is then given by

$$u(x, y, z) = E_0 \frac{w_0}{w(z)} \exp \left(- \frac{x^2 + y^2}{w^2(z)} - i \left(k \frac{x^2 + y^2}{2R(z)} - \psi(z) \right) \right) , \tag{2.73}$$

where we define

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_r} \right)^2} \qquad \text{and} \qquad R(z) = z \left(1 + \left(\frac{z_r}{z} \right)^2 \right) \qquad \text{and} \qquad \psi(z) = \operatorname{atan}(z/z_r) , \tag{2.74}$$

so the beam is squeezed: it is not a plane wave.

The factor $z_r = \pi w_0^2 / \lambda$. The radius of the beam can only be small in a certain region, as we go further it widens.

This also affects the phase, by the factor $\psi(z)$.

The divergence angle is given by $\theta \approx w(z)/z \approx \lambda / \pi w_0$, which is a manifestation of the uncertainty principle:

$$w_0 \theta = \frac{\lambda}{\pi} . \tag{2.75}$$

Starting from the gaussian beam, we can define orthonormal bases which describe any beam in the paraxial approximation: the Laguerre-Gauss (cylindrical) and Hermite-Gauss (rectangular) bases.

These are eigenfunctions of the paraxial wave equation: they propagate without changing their shape (although they are scaled). This is not the case for combination of them, because of the Guoy phase.

We generally try to work with the 00 mode, because it is easier.

Resonance means constructive interference in the cavity: we must have resonance both in phase and in the transverse profile.

As the beam spreads the wavefronts curve, we need to account for this: the mirrors must be exactly parallel to the wavefronts. So, we need to make them curved.

For LIGO-VIRGO we have a *confocal* cavity, so the sum of the curvatures of the mirrors is of the order of the length of the cavity.

For the mirrors we use in the lab, their radii of curvature are from centimeters to meters.

If a gaussian beam is reflected on the mirror and comes back to itself, all the higher orders modes come back to themselves as well. However, they do not do so because of the phase difference.

The very high order modes are suppressed, since they are more affected by the imperfections of the mirror.

We can do a back-of-the-envelope calculation for the sensitivity of a Michelson interferometer. We want to get

$$\frac{\Delta}{L} \approx 10^{-21}\,\mathrm{Hz}^{-1/2}\,, \tag{2.76}$$

so we want

$$\Delta L \approx 10^{-18}\,\mathrm{mHz}^{-1/2}\,, \tag{2.77}$$

so we want an interferometric sensitivity of around

$$\frac{\Delta L}{\lambda} = \frac{\Delta L}{\lambda} \approx 10^{-12}\,\mathrm{Hz}^{-1/2}\,. \tag{2.78}$$

The relative error in photon counting is given by

$$\frac{\Delta L}{\lambda} \approx \frac{\Delta N}{N} = \frac{1}{\sqrt{N}}\,, \tag{2.79}$$

which means we must get something on the order of

$$N \approx 10^{-24}\,\mathrm{Hz}\,. \tag{2.80}$$

This means we need a power of approximately

$$P = N\hbar\omega \approx 200\,\mathrm{kW}\,. \tag{2.81}$$

Fabry-Perot cavities are also useful to increase the effective length: we want $L_{\mathrm{arm}} \approx 750\,\mathrm{km}\left(100\,\mathrm{Hz}/f_{\mathrm{gw}}\right)$, while the arm length is of the order $3\,\mathrm{km}$.

So, we have different reasons to introduce cavities: they increase power, they increase effective length.

Friday 2020-5-8, compiled 2020-07-10

About an hour of only listening

In the TT gauge, the carrier angular frequency is ω_l and we get some power in the sidebands $\omega_l \pm \omega_{gw}$. We express this as a vector equation: we define the vector $\vec{B} = (\text{carrier}, \text{sidebands})$ after a round-trip, so we have

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} X_{00} & 0 & 0 \\ X_{10} & X_{11} & 0 \\ X_{20} & 0 & X_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} , \tag{2.82a}$$

and in the end we get

$$\left|\Delta\phi_x\right| = h_0 k_l L \operatorname{sinc}\left(\frac{\omega_{gw} L}{c}\right) \frac{r_2\left(1 - r_1^2 - p\right)}{r_2(1-p) - r_1} \frac{1}{\left|e^{2i\omega_{gw}\frac{L}{c}}\right| - r_1 r_2} . \tag{2.83}$$

We can make some approximations: $p \approx 0$, $r_2 \approx 1$, $r_1 \sim 1$. We get

$$\frac{r_2\left(1 - r_1^2 - p\right)}{r_2(1-p) - r_1} \approx 1 + r_1 \approx 2\left(1 + \epsilon(r1, r2, p)\right) . \tag{2.84}$$

Also, our finesse is large so that the value of the sinc is around 1.

The cavity acts as a low-pass frequency: the cutoff is called the *cavity pole*,

$$f_p = \frac{1}{4\pi\tau_s} \approx \frac{c}{4\mathcal{F}L} , \tag{2.85}$$

so the response looks like

$$\left|\Delta\phi_{FP}\right| = h_0 \frac{4\mathcal{F}}{\pi} k_l L \frac{1}{\sqrt{1 + \left(\frac{f_{gw}}{f_p}\right)^2}} . \tag{2.86}$$

Monday 2020-5-11, compiled 2020-07-10

We have an issue if noise is strong enough to move our mirrors out of lock, even if we do not care to observe GW at the frequency of that noise.

Today we will discuss the **antenna pattern**: the detector tensor is given by

$$D_{ij} = \frac{1}{2}\left(\hat{x}_i \hat{x}_j - \hat{y}_i \hat{y}_j\right) , \tag{2.87}$$

so that

$$h(t) = \frac{1}{2}\left(\ddot{h}_{xx} - \ddot{h}_{yy}\right) , \tag{2.88}$$

as long as the arms are aligned with the $\hat{x}$ and $\hat{y}$ axes. The function D_{ij} describes the sensitivity of our detector in different directions.

These kinds of detectors return a scalar (a timeseries, yes, but a scalar with respect to 3D space). This scalar will be linear in the tensor h_{ij}, so we can express the observation as

$$h(t) = D_{ij} h_{ij}(t) \,. \tag{2.89}$$

We must perform a rotation with two angles ϕ and θ to go from the source frame and the detector frame.

We must make a choice to select what we call h_+ and $h_\times$. In the end, we find

$$h(t) = F_+(\theta, \phi) h_+ + F_\times(\theta, \phi) h_\times \tag{2.90a}$$

$$F_+ = \frac{1}{2}\Big(1 + \cos^2\theta\Big)\cos\big(2\phi\big) \tag{2.90b}$$

$$F_\times = \cos(\theta)\sin\big(2\phi\big) \,. \tag{2.90c}$$

2.5 The interferometer's noise budget

The noise is dominated by the quantum noise: quantum fluctuations of the laser light. The other source of noise giving us problems in the 100 Hz region is the coating Brownian noise.

At high frequencies, the problem is that it is hard to measure small displacements with small integration time. At low frequencies, the problem is that the mirrors move too much.

2.5.1 Quantum noise

The fluctuation in square power — the *shot noise*, error in the count of photons — is given, since it is a Poisson process, by

$$\Delta P^2 = \frac{\Delta E^2}{T^2} = \frac{\Delta N^2 \hbar^2 \omega_l^2}{T^2} = N \frac{\hbar^2 \omega_l^2}{T^2} = \frac{P_0 \hbar \omega_l}{T} = 2\frac{S_P(\omega)}{T} = \frac{1}{2}\int_0^{1/T} S_P(\omega)\,\mathrm{d}\omega \,, \tag{2.91}$$

so we get $S_P(\omega) = 2P_0\omega_l$.

We also have *radiation pressure*, which scales differently. So, we must reach a compromise with both power and finesse.

The shot noise is flat in frequency, the RP noise decreases with frequency. At each frequency, we can define a Standard Quantum Limit, which is the lowest noise we could have at that frequency.

We can go below this limit using Quantum Vacuum Squeezing.

2.5.2 Thermal Noise

We have contribution from all dissipation sources. There is thermo-elastic noise: as a material bends, the side which compresses heats up a little.

The limit is the internal dissipation:

$$S_{F,\mathrm{th}} = 4k_B T \,\mathrm{Re}[Z(\omega)] \,. \tag{2.92}$$

2.5.3 Seismic noise

2.5.4 Newtonian noise

2.6 Elements of data analysis

Friday 2020-5-15, compiled 2020-07-10

The issue is that our signal is noise-dominated. Extracting the signal is quite hard. About half of the people working on GW do this.

The signal is something like 3 orders of magnitude below the noise. Also, we want to say something about the object.

We classify signals into: transient versus persistent (how long does the signal last?), and modeled versus unmodeled (do we know of a speficic waveform?).

1. Transient modeled signals are usually coalescing binaries;
2. persistent modeled signals can be binaries far from coalescence or rotating neutron stars;
3. transient unmodeled signals can be supernovae or some other sources;
4. persistent unmodeled signals are some form of stochastic background.

We will discuss matched filtering, which applies to transient modeled signals. The assumption is that the signal is in the form

$$s(t) = \underbrace{h(t)}_{\text{known}} + \underbrace{n(t)}_{\text{noise}} . \tag{2.93}$$

The thing we can do is to calculate $\langle sh \rangle$, which is equal to $\left\langle h^2 \right\rangle + \langle nh \rangle$, but we know that $\langle nh \rangle \to 0$ since the noise is not correlated to the signal. Specifically, $\langle nh \rangle \sim T^{-1/2}$.

On the other hand, $\langle hh \rangle \geq 0$, so this integral will have a positive value if the signal is there. This grows much faster than the standard deviation of the uncorrelated signal.

Suppose we know what $h(t)$ is, and we want to build a linear filter $K(t)$ which returns a low value if the signal seems to not be in the data, and a high value if the signal is in the data.

We assume that $s(t) = \alpha h(t) + n(t)$, where it is convenient to leave the amplitude of the signal as a variable.

We wat to define the signal to noise ratio S/N.

What do we mean by ensemble average inside the time integral?

We find

$$S = \left\langle \hat{s}(t) \right\rangle = \alpha \int \mathrm{d}f \, K^*(f) h(f) , \tag{2.94}$$

while

$$N^2 = \left\langle \hat{n}^2(t) \right\rangle - \langle \hat{n} \rangle^2 = \frac{1}{2} \int \mathrm{d}f \, \left| K(f) \right|^2 S_n(f) . \tag{2.95}$$

Then, we can write

$$\frac{S}{N} = \alpha \frac{\int \mathrm{d}f\, K^*(f) h(f)}{\sqrt{\int \mathrm{d}f\, \left|K(f)\right|^2 S_n(f)/2}}\,, \tag{2.96}$$

and we wish to optimize K in order to maximize this.

We define a scalar product for real functions of f:

$$A \cdot B = \mathrm{Re} \int \mathrm{d}f\, \frac{A^*(f) B(f)}{S_n(f)/2}\,, \tag{2.97}$$

so that

$$\frac{S}{N} = \alpha \frac{u \cdot h}{u \cdot u}\,, \tag{2.98}$$

where $u = S_n(f) K(f)/2$. So, we want $u(f)$ to be parallel to h: therefore, we want

$$K(f) \propto \frac{h(f)}{S_n(f)}\,. \tag{2.99}$$

Missing a square root in the denominator?

This means that our best filter is *not* h, as we thought: we must weigh the filter by how high the detector noise is. The first thing we do not know is the *time of coalescence*: we can "slider" our filter over our signal, varying the time of arrival.

In practice we do not know the form $h(t)$ and we do not know the time of arrival. What we do is generate hundreds of thousands of filters and move them through the data.

We must decide on a coverage for our parameter space in order to run our filters, while still having a manageable thing.

The problem is that the noise is non-gaussian: its tails of extreme events are very large. How to reject false alarms due to local noise? Coincidences!

We allow time differences of L/c at most. The amount of signal which is retained after these coincidences is very little.

How to be very confident that two local sources of noise did not happen to coincide? We delay the output intentionally, in order to get an estimate of what we would see without GW signals.

This works as long as the events are rare.

The analysis must be done blind: we look at the delayed data stream first, and then we set the threshold.

Do people use KDE in the estimation of PDFs?

2.6.1 Probability

We use the Kolmogorov axioms: consider events belonging to the powerset of S, then we say that

1. probabilities are positive;
2. the probabilities of disjoint events are additive;
3. the probability of S is 1.

An implementation is the frequentist approach: we assign probabilities according to the frequency of occurrence of an event after many repetitions.

So, in this approach Bayes's theorem does not really make sense: what is the probability of a die being true, given that we have gotten 1006 times the number 1 over 6000 tries?

We can define probabilities, instead, as subjective beliefs.

2.6.2 Parameter estimation

Monday 2020-5-18, compiled 2020-07-10

Now, suppose we have found a candidate signal. How do we estimate the most likely set of parameters?

We do it by

$$\mathbb{P}(\vec{p}|\text{data}) = \mathbb{P}(\text{data}|\vec{p})\mathbb{P}(\vec{p}) \,, \tag{2.100}$$

where we usually assume a flat prior for the parameters.

Thee output signal is in the form $s(t) = h(t, \widetilde{p}) + n_0(t)$, the GW plus the noise. So, we get

$$\mathbb{P}(n_0) = N \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \frac{|n_0(f)|^2}{S_n(f)^2} \,\mathrm{d}f\right) = N \exp\left(-\frac{(n_0 \cdot n_0)}{2}\right) , \tag{2.101}$$

by the scalar product we defined earlier. This gives the probability of a specific realization of the noise. So, we get

$$n = s - h(\widetilde{p}) \,, \tag{2.102}$$

so we can evaluate $\Lambda(s|\widetilde{p})$, which is defined as the likelihood of the data given the parameter vector $\widetilde{p}$, as

$$\Lambda(s|\widetilde{p}) = N \exp\left(-\frac{(s-\widetilde{h}) \cdot (s-\widetilde{h})}{2}\right) = N \exp\left(-\frac{s \cdot s + h \cdot h + 2s \cdot h}{2}\right) , \tag{2.103}$$

which we can calculate explicitly. The term given by $s \cdot s$ can be included in the normalization, since it is a constant with respect to the signal given.

This allows us to compute the likelihood of the parameters.

$$\mathbb{P}(\widetilde{p}|\text{data}) = N \exp\left(-\frac{h \cdot h + 2s \cdot h}{2}\right)\mathbb{P}(\widetilde{p}) \,, \tag{2.104}$$

where the scalar product is always the functional one which includes the PSD.

Then, we have different approaches:

1. **maximum likelihood** means that we maximize $\Lambda(s|\hat{p})$. This ignores the prior, and is equivalent to maximizing S/N for matched filtering.
2. **maximum posterior** means we maximize $\mathbb{P}(\hat{p}|s)$.
3. **Bayes estimator**: we compute

$$\hat{p}_i^B = \int \mathrm{d}\vec{p}\, p_i \mathbb{P}(\vec{p}|s) \,. \tag{2.105}$$

   This minimizes the error for each parameter.

Some other data analysis methods: coherent wave bursts is for short-lived signals for which we do not have a model.

We must give an estimate of the PSD which is "mesoscopic": we average on something like ten minutes, so that we average over any GW signal we could see, but do not average on too long a timescale so we have things which are normalized appropriately — we must recalibrate often, because of wind, day-night and so on.

We can search for monochromatic signals: the stdev of their distribution in frequency around a specific one should go as the observation time to the $-1/2$.

This must be done during long times.

As for the stochastic background: we could extract it from the PSD of the signal if we knew the detector PSD precisely.

Ideally, we would want to correlate signals from detectors at a distance D such that

$$\lambda_{GW} \geq D \geq L_{\text{noise}} \,. \tag{2.106}$$

2.7 LISA

It's approved by ESA to launch in 2034. There's no seismic noise in space. It will be sensitive in the 0.1 to 100 mHz signals. Possibly, it will be able to see even further. Now, the wavelengths will be in the length scale of the arm of the detector.

We are discussing LISA.

Friday 2020-5-22, compiled 2020-07-10

2.8 Pulsar Timing Array

We seek a method to measure SMBH mergers and other extremely long wavelength GW. What we do is measure the distance from millisecond pulsars.

The light travel time depends on how the space is curved by GW.

The key to the data analysis is that the GW signal is one of the only correlated signals across the sky. The contribution is

$$r_{\alpha,GW}(t) = \int_0^t \mathrm{d}t' \, \frac{\delta\nu_\alpha}{\nu_\alpha}(t') \,. \tag{2.107}$$

Here α is an index representing which pulsar we are interested in. We can divide the signal r into a part at Earth, $r_\alpha^e(t)$, and a part at the pulsar, $r_\alpha^p(t)$.

Is there an averaging effect in PTA? If the GW is propagating in the same direction as the pulsar signal the signal is always in the same gravitational field... not clear really

2.9 GW detections

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Discussion on current GW detections and prospects for the future.

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