Theoretical cosmology notes

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Contents

Contents of the course

We start with a derivation of the Friedmann eqs. from the Einstein equations.

We will then discuss the properties of the CMB, deriving the spectrum, and then CMB anisotropies.

Then we will discuss star and structure formation, about the nonlinear evolution of perturbations. We will use the path-integral approach to classical field theory. We will also discuss weak gravitational lensing in the universe.

We will use some smart nonlinear approximations: the Zel'dovich approximation and the adhesion approximation.

We will use an "effective Planck constant" instead of $\hbar$: it will be a parameter which can be fit in our model.

As for references: there are handwritten notes by the professor in the Dropbox folder (for access to the folder, write to the professor). Also, there notes by a student from the previous years, in Italian [Nat17], which are to be used with caution as they contain some errors.

0.1 Friedmann equations: a brief overview

Thursday 2020-3-12, compiled 2020-04-09

Throughout the course, we set $\hbar = c = k_B = 1$.

In the previous course we used the approximate symmetries of the universe to write the FLRW line element:

$$ds^2 = -\,dt^2 + a^2(t)\,d\sigma^2\ , \tag{1}$$

do note that we switch signature from the previous course: now we use the mostly plus one. The spatial part is defined by

$$d\sigma^2 = \widetilde{g}_{ij}\, dx^i\, dx^j \ , \tag{2}$$

where $\widetilde{g}_{ij}$ is the maximally symmetric metric tensor in a 3D space. There are only 3 maximally symmetric 4D spacetimes: Minkowski, dS and AdS.

Since we have maximal symmetry, the Riemann tensor is

$$R_{ijkl} = k\Big(\widetilde{g}_{ik}\widetilde{g}_{jl} - \widetilde{g}_{il}\widetilde{g}_{jk}\Big) . \tag{3}$$

We can use spherical coordinates:

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2\, d\Omega^2 \ , \tag{4}$$

and we can define the coordinate χ by

$$d\chi = \frac{dr^2}{\sqrt{1-kr^2}} . \tag{5}$$

The Einstein equations read

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \, , \tag{6}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is its trace, the scalar curvature, while $T_{\mu\nu}$ is the stress energy momentum tensor.

In cosmology we assume to have the SEMT of a perfect fluid. Really, we have particles, between which there is vacuum.

We need to use the Weyl tensor, which describes the parts of the Riemann tensor which are not in the traces. “The real world” is only described by the Weyl tensor, but in cosmology we make a great approximation in ignoring it.

What we do is to insert an ansatz for the metric tensor, which we use to derive the Christoffel symbols, and from these we write the Riemann tensor. Doing it the other way around, starting from the source SEMT, is very difficult.

Claim 0.1.1. *The Christoffel symbols for the FLRW metric are:*

$$\Gamma^t_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\dot{a}a}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 a\dot{a} & 0 \\ 0 & 0 & 0 & r^2 a\dot{a}\sin^2\theta \end{bmatrix} \tag{7a}$$

$$\Gamma^r_{\mu\nu} = \begin{bmatrix} 0 & \dot{a}/a & 0 & 0 \\ \dot{a}/a & \frac{kr}{(1-kr^2)} & 0 & 0 \\ 0 & 0 & (kr^2-1)r & 0 \\ 0 & 0 & 0 & (kr^2-1)r\sin^2\theta \end{bmatrix} \tag{7b}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & \dot{a}/a & 0 \\ 0 & 0 & 1/r & 0 \\ \dot{a}/a & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{7c}$$

$$\Gamma^{\varphi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & \dot{a}/a \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & \cos\theta/\sin\theta \\ \dot{a}/a & 1/r & \cos\theta/\sin\theta & 0 \end{bmatrix}. \tag{7d}$$

In order to calculate these, we can make use of certain simplifications: the FLRW metric is diagonal, and it does not depend on φ.

Notice that the spatial Christoffel symbols are nonzero even in Minkowski ($k = 0$, $\dot{a} = \ddot{a} = 0$): why is this? This is because we are using curvilinear coordinates, the Christoffel symbols express the *extrinsic* curvature, not the *intrinsic* curvature; they are not tensors, so they can be zero in a reference and nonzero in another.

In general, the Riemann tensor is given by

$$R^{\mu}_{\nu\rho\sigma} = -2\left(\Gamma^{\mu}_{\nu[\rho,\sigma]} + \Gamma^{\alpha}_{\nu[\rho}\Gamma^{\mu}_{\sigma]\alpha}\right), \tag{8}$$

where commas denote coordinate derivation, and square square brackets denote antisymmetrization (for clarification on this notation Wikipedia does a good job [19]).

The Ricci tensor is given by the contraction of the Riemann tensor along its first and third component:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = -2\left(\Gamma^{\alpha}_{\mu[\alpha,\nu]} + \Gamma^{\beta}_{\mu[\alpha}\Gamma^{\alpha}_{\nu]\beta}\right) \tag{9a}$$

$$= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}. \tag{9b}$$

A great simplification comes from the fact that, for the FLRW metric, the Ricci tensor is diagonal.[1]

Claim 0.1.2. *The components of the Ricci tensor are:*

$$R_{tt} = -3\partial_t\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \tag{10a}$$

$$= -3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2\right) \tag{10b}$$

$$= -3\frac{\ddot{a}}{a}, \tag{10c}$$

[1] If there are a certain number of coordinates the metric is independent of, the Ricci tensor has very few nonzero components [Win96]. This is not enough to prove that the Ricci tensor must be diagonal for this metric, however in the specific case of FLRW this is the case anyways.

$$
\begin{aligned}
R_{rr} &= \partial_t\left(\frac{\dot{a}a}{1-kr^2}\right) + \partial_r\left(\frac{kr}{1-kr^2}\right) - \partial_r\left(\frac{kr}{1-kr^2}\right) - 2\partial_r\left(\frac{1}{r}\right) \\
&\quad + \frac{\dot{a}a}{1-kr^2}3\frac{\dot{a}}{a} + \frac{kr}{1-kr^2}\left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) \\
&\quad - 2\frac{\dot{a}}{a}\frac{\dot{a}a}{1-kr^2} - \left(\frac{kr}{1-kr^2}\right)^2 - 2\left(\frac{1}{r}\right)^2 &\text{(11a)} \\
&= \frac{\ddot{a}a + \dot{a}^2}{1-kr^2} + 3\frac{\dot{a}^2}{1-kr^2} + 2\frac{k}{1-kr^2} - 2\frac{\dot{a}^2}{1-kr^2} &\text{(11b)} \\
&= \frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2}\,, &\text{(11c)}
\end{aligned}
$$

$$
\begin{aligned}
R_{\theta\theta} &= r^2\partial_t(a\dot{a}) + \partial_r\Big((kr^2-1)r\Big) - \partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) \\
&\quad + 3\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Big(\Gamma^r_{rr} + 2\Gamma^\theta_{r\theta}\Big) - 2\Big(\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}\Big) - \frac{\cos^2\theta}{\sin^2\theta} &\text{(12a)} \\
&= r^2\Big(\ddot{a}a + \dot{a}^2\Big) + 3kr^2 - 1 + \frac{1}{\sin^2\theta} + r^2\dot{a}^2 - kr^2 - \frac{\cos^2\theta}{\sin^2\theta} &\text{(12b)} \\
&= r^2\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big)\,, &\text{(12c)}
\end{aligned}
$$

$$
\begin{aligned}
R_{\varphi\varphi} &= \partial_\alpha\Gamma^\alpha_{\varphi\varphi} - \partial_\varphi\Gamma^\alpha_{\alpha\varphi} + \Gamma^\alpha_{\varphi\varphi}\Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\varphi\alpha}\Gamma^\alpha_{\varphi\beta} &\text{(13a)} \\
&= r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big)\,. &\text{(13b)}
\end{aligned}
$$

The Ricci scalar then comes out to be

$$
\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= 3\frac{\ddot{a}}{a} + \frac{1-kr^2}{a^2}\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2} \\
&\quad + \frac{1}{a^2r^2}r^2\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) + \frac{1}{a^2r^2\sin^2\theta}r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) &\text{(14a)} \\
&= 3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{a^2} &\text{(14b)} \\
&= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right]. &\text{(14c)}
\end{aligned}
$$

The dimensions of the Ricci scalar are those of a length to the -2.

The stress energy tensor is the functional derivative of everything but the curvature in the action with respect to the metric: if our Lagrangian is

$$
L = L_g + L_{\text{fluid}}\,, \tag{15}
$$

where the gravitational Lagrangian is $L_g = M_P^2 R/2$ (and $M_P = 1/\sqrt{8\pi G}$ in natural units is the reduced Planck mass) then

$$T_{\mu\nu} \stackrel{\text{def}}{=} -\frac{2}{\sqrt{-g}} \frac{\delta L_{\text{fluid}}}{\delta g^{\mu\nu}} . \tag{16}$$

Discuss why this is equivalent to "flux of momentum component μ across a surface of constant x^ν".

We use perfect fluids: they have a stress-energy tensor like

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + p g^{\mu\nu} , \tag{17}$$

where u^μ is the 4-velocity of the fluid element. It is diagonal *in the comoving frame*, in which $u^\mu = \left(1, \vec{0}\right)$.

If we are not comoving, we have additional heat transfer off diagonal terms (this is discussed in my thesis [Tis19, section 4.2]).

If we take the covariant divergence of the Einstein tensor $G_{\mu\nu}$ we get zero; so the stress energy tensor must also have $\nabla_\mu T^{\mu\nu} = 0$. This is *not* a conservation equation.

In SR we had an equation like $\partial_\mu T^{\mu\nu}$: this *was* a conservation equation, a local one. In GR we also need Killing vectors in order to actually have conserved quantities. In cosmology we do not have symmetry with respect to time translation, so there is no timelike Killing vector ξ_μ such that $\xi_\nu \nabla_\mu T^{\mu\nu}$ represents the conservation of energy.

This equation, $\nabla_\mu T^{\mu\nu}$ follows from the fact that our fluid follows its equations of motion.

Let us explore the meaning of these equations: if, in the equation $0 = \nabla_\mu T^\mu_0$, we find

$$0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \tag{18a}$$

$$= -\dot{\rho} - 3H\left(\rho + P\right) . \tag{18b}$$

For example consider radiation: $P = \rho/3$. This means that $\dot{\rho} = -4H\rho$: so, as the Hubble parameter increases, the radiation density decreases.

The other two Friedmann equations can be derived from the time-time and space-space components on the Einstein equations: we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) \tag{19a}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} . \tag{19b}$$

The space-space equation is not a dynamical equation, since it contains no second time derivatives: it is a *constraint* on the evolution of the system.

However, the three Friedmann equations are not independent: the time-time one can be found from the other two.

A useful theorem is the fact that for a maximally symmetric space the Ricci tensor must be given by

$$\widetilde{R}_{\alpha\beta} = 2k\widetilde{g}_{\alpha\beta} . \tag{20}$$

We can write the stress energy tensor as

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}\,, \tag{21}$$

where $h_{\mu\nu}$ is the projection tensor onto the spacelike subspace $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$.

This is more physically meaningful.

Tomorrow we will start the discussion on the CMB.

Chapter 1

The CMB

Today we discuss the CMB. This is discussed in the book Modern Cosmology [Dod03, chapter 3], we will follow the professor's notes, which are available in the Dropbox.

Friday 2020-3-13, compiled 2020-04-09

A note: in these lectures a dot will refer to conformal time derivatives only, if we differentiate with respect to cosmic time we shall write the derivative explicitly. Let us suppose we have some particle species interacting, such as $1 + 2 \leftrightarrow 3 + 4$.

The variation in time of the abundance of particle type 1, (which is given by the density times a volume: $n_1 a^3$) is given by the difference of the particles which are created and destroyed. We write the formula first, and then explain it: this is given by

$$\begin{aligned} a^{-3}\frac{\mathrm{d}(n_1 a^3)}{\mathrm{d}t} &= \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1}\left[\prod_{i=2}^{4}\int \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}\right]\times \\ &\times (2\pi)^4 \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\delta(E_1 + E_2 - E_3 - E_4)\times \\ &\times |\mathcal{M}|^2 \Big[f_3 f_4 \big(1 \pm f_1\big)\big(1 \pm f_2\big) - f_1 f_2 \big(1 \pm f_3\big)\big(1 \pm f_4\big)\Big] \end{aligned} \tag{1.1a}$$

where:

1. the delta functions account for momentum and energy conservation: energy is *not conserved* in general in cosmology, *but* we can use the equivalence principle to go to a reference frame which is locally Minkowski: in our description of an instantaneous process such as this, the deviations from this frame are negligible.

2. $\mathcal{M}$ is the *invariant scattering amplitude* between the initial and final states. In Quantum Field Theory, in order to describe a process one assumes that the particles involved usually evolve according to their free Hamiltonians, while their interaction is described by an interaction Hamiltonian $\hat{H}_{\text{int}}$ — we put ourselves in the interaction picture, so that the free time evolution is factored out. The matrix element of the time evolution between the initial state $|12\rangle$ and the final state $|34\rangle$ is called the S-matrix element:

$$S_{12,34} = \lim_{t\to\infty} \langle 12| U(t) |34\rangle \,, \tag{1.2}$$

where t is assumed to go to infinity since we are considering asymptotic initial and final states. This is in general hard to express, but it can be formally written as a

Dyson series in terms of the interaction Hamiltonian. If we only consider this series to first order we get (roughly speaking):

$$S_{12,34} \sim \underbrace{\langle 12|34\rangle}_{=0} + \langle 12| \hat{H}_{\text{int}} |34\rangle \,. \tag{1.3}$$

The invariant scattering amplitude is then defined so that

$$\left|S_{12,34}\right|^2 = (2\pi)^4 \delta^{(4)}(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu) |\mathcal{M}|^2 \,. \tag{1.4}$$

Do note that $\mathcal{M}$ is an amplitude, while $|\mathcal{M}|^2$ is a probability. If we integrate across all of momentum space, as we are doing, we get the total probability of the process happening. For more details, see Peskin [Pes19, sections 7.2, 7.3].

Also, note that in our model the processes are always time-reversible, so $\mathcal{M}_{12\to 34} = \mathcal{M}_{34\to 12}$.

3. The f_i are the phase space distributions of the different species: the terms including these account for the quantum statistics, we use $-$ for fermions and $+$ for bosons. The factors $1 \pm f_i$ account for the quantum phenomena of Bose enhancing and Pauli blocking [Nor28]. We include them here, but we will usually neglect them.

4. The 2π-s account for the normalization of the deltas: if we were to discretize phase space we would not need them, but when we move from discrete sums (and Kronecker deltas) to integrals (and Dirac deltas) we need to divide by h^3 for each integral over momentum; in our units however $\hbar = 1$, therefore $h = 2\pi$.

5. The energy of each particle species is given by $E^2 = p^2 + m^2$. Why are there $2E$ factors in the denominators? In principle, we should integrate in $\mathrm{d}^4 p$, however we work *on shell*: a priori, the particle does whatever it wants, however solutions to the equations of motion are (very much) preferred in the path integral. Because of this, it is an excellent approximation to just impose this condition: we do

$$\int \mathrm{d}^3 p \int_0^\infty \delta(E^2 - p^2 - m^2) = \int \mathrm{d}^3 p \int_0^\infty \frac{\delta(E - \sqrt{p^2 + m^2})}{2E} \,, \tag{1.5}$$

so we include the $2E$ term in the denominator: the integral over the energies has already been performed, and by E we always mean $E(\vec{p})$.

If there is no interaction, $n_1 \propto a^{-3}$, as we expect for nonrelativistic matter.

We are integrating over the momenta of all the particles except for 1 in order to account for all the possible ways the reaction could happen.

Something like:
The term for particle 1, E_1, has a different origin: the time is related to the proper time by p^0, which is E_1. The factor 2 is included for symmetry, it is indifferent if we include it or not since we can normalize the helicities g_i.
was mentioned during lecture, but why should we not integrate over $\mathrm{d}^4 p_1$ and apply the same reasoning as the other particles?

Typically we have kinetic equilibrium, as long as the scattering time is very short with respect to the Hubble time. So, we use the conventional distribution functions for Bose-Einstein or Fermi-Dirac gases,

$$f_{\mathrm{BE/FD}} = \left(\exp\left(\frac{E-\mu}{T} \right) \pm 1 \right)^{-1}, \tag{1.6}$$

where the sign is a $-$ for Bose Einstein statistics, while a $+$ for Fermi-Dirac statistics. Here, E is the energy of the particle, μ is its chemical potential, which quantifies how much the energy changes if we add particles, while T is the temperature.

For the nonrelativistic particles (all of them, except the photons[1], much less than the mass of the lightest particles involved, electrons.) we have $E - \mu \gg T$. If f becomes very small, then we can drop the terms $(1 \mp f_i)$.[2] This is the Boltzmann limit.

In theory we could not do this for photons, in practice we do it and the magnitude of the error is the same as the ratio $\zeta(3) \approx 1.2$ to 1, where ζ is the Riemann zeta function.

Then, our distributions will be given by

$$f(E) = e^{\mu/T} e^{-E/T}. \tag{1.8}$$

So the phase space distribution term is

$$\begin{aligned} f_1 f_2 - f_3 f_4 &= \exp\left(\frac{-(E_1+E_2) + (\mu_1+\mu_2)}{T} \right) - \exp\left(\frac{-(E_3+E_4) + (\mu_3+\mu_4)}{T} \right) && \text{(1.9a)} \\ &= \exp\left(-\frac{E_1+E_2}{T} \right) \left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right), && \text{(1.9b)} \end{aligned}$$

where we used the fact that $E_1 + E_2 = E_3 + E_4$ by energy conservation.

What do the chemical potentials look like? If we were to enforce the Saha condition $\mu_1 + \mu_2 = \mu_3 + \mu_4$, which is equivalent to chemical equilibrium , then we would get precisely zero for our variation of species number. This makes sense: if there is equilibrium, the number densities of the species do not change. So, we cannot restrict ourselves to chemical equilibrium if we want to describe recombination.

We can get relations for the chemical potentials starting from the number densities of the species which are already present: the mean number density of species i is given by

$$n_i = \frac{g_i}{(2\pi)^3} \int \mathrm{d}^3 p \, f(p) = g_i e^{\mu_i/T} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T}, \tag{1.10}$$

where g_i is the number of helicity states of the particle (see the notes for the course in fundamentals of astrophysics and cosmology for more details [TM20b]).

[1] As we will discuss in more detail later, the recombination temperature is around 0.3 eV

[2] What do these terms look like? We have

$$1 \mp f_i = 1 \mp \frac{1}{e^{(E-\mu)/T} \pm 1} = \frac{e^{(E-\mu)/T} \pm 1 \mp 1}{e^{(E-\mu)/T} \pm 1} = \frac{1}{1 \pm e^{-(E-\mu)/T}}. \tag{1.7}$$

So, we can define

$$n_i^{(0)} = n_i\Big|_{\mu_i=0} \implies e^{\mu_i/T} = \frac{n_i}{n_i^{(0)}}\,. \tag{1.11}$$

These quantities can be estimated if the particles are either very relativistic ($E_i \approx p_i$) or very non-relativistic ($E_i \approx m_i + p^2/2m_i$): in the second case we have[3]

$$n_i^{(0)} = g_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T} \approx \frac{g_i}{(2\pi)^3} e^{-m_i/T} \int \mathrm{d}^3 p\, e^{-\frac{p^2}{2m_i T}} \tag{1.13a}$$

$$= \frac{g_i}{(2\pi)^3} e^{-m_i/T} \sqrt{(2\pi)^3 (m_i T)^3} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}\,, \tag{1.13b}$$

while in the very relativistic case we find

$$n_i^{(0)} = g_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T} \approx \frac{g_i}{(2\pi)^3} \int \mathrm{d}^3 p\, e^{-p_i/T} \tag{1.14a}$$

$$= \frac{g_i}{(2\pi)^3} 4\pi T^3 \int e^{-x} x^2 \,\mathrm{d}x = \frac{g_i}{\pi^2} T^3\,. \tag{1.14b}$$

For photons, the correct expression accounting for their quantum mechanical statistics would be this one, multiplied by the Riemann zeta function calculated at 3:

$$\zeta(3) = \sum_{n\in\mathbb{N}} \frac{1}{n^3} \approx 1.2\,, \tag{1.15}$$

so we are wrong by about 20 % in neglecting it.

Inserting these $n_i^{(0)}$ we get the simpler expression

$$e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} = \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\,. \tag{1.16}$$

We can define the thermally-averaged cross section to encompass all the terms which do not depend on the number densities at a specific time:

$$\langle \sigma v \rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_{i=1}^{4} \int \frac{\mathrm{d}^3 p}{2E_i} e^{-(E_1+E_2)/T} (2\pi)^4 \delta^{(4)}\left(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu\right) |\mathcal{M}|^2\,, \tag{1.17}$$

so the final equation is

$$a^{-3} \frac{\mathrm{d}}{\mathrm{d}t}(n_1 a^3) = \langle \sigma v \rangle\, n_1^{(0)} n_2^{(0)} \left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right]. \tag{1.18}$$

[3] We need to use a gaussian integral: in general, we have the theorem

$$\int \mathrm{d}^n x \exp\left(-\frac{1}{2} A_{ij} x_i x_j\right) = \sqrt{\frac{(2\pi)^n}{\det A}}\,, \tag{1.12}$$

and in our case $A_{ij} = (m_i T)^{-1} \delta_{ij}$, so $\det A = (m_i T)^{-3}$.

The left hand side is typically of the order $\sim n_1/t \sim n_1 H$, since the order of magnitude of the age of the universe is the Hubble time H^{-1}.

So, as the universe ages, the combination on the RHS must be "squeezed to zero": this is equivalent to the Saha equation, since it means that

$$e^{(\mu_1+\mu_2)/T} = e^{(\mu_3+\mu_4)/T} \implies \mu_1 + \mu_2 = \mu_3 + \mu_4 \,. \tag{1.19}$$

This is also called "chemical equilibrium" by particle physicists, or nuclear statistical equilibrium by people studying Big Bang Nucleosynthesis.

1.1 Hydrogen recombination

The process is

$$e^- + p \leftrightarrow H + \gamma \,, \tag{1.20}$$

so the Saha equation yields

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \,, \tag{1.21}$$

and charge neutrality implies $n_e = n_p$, *not* $n_e^{(0)} = n_p^{(0)}$.

At this stage in evolution, there are already some Helium nuclei, but we ignore them.

We define the ionization fraction

$$X = \frac{n_e}{n_e + H} \,. \tag{1.22}$$

This then yields

$$\frac{1 - X_e^n}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_e}\right)^{3/2} \exp(\epsilon_0/T) \,, \tag{1.23}$$

where $\epsilon_0 = m_p + m_e - m_H = 13.6\,\text{eV}$ is the ionization energy of Hydrogen.

Then, we get that the temperature of recombination is $T_{\text{rec}} \approx 0.3\,\text{eV}$.

The evolution of the ionization fraction is

$$\frac{\mathrm{d}X_e}{\mathrm{d}t} = (1 - X_e)\beta(T) - X_e^2 n_b \alpha^{(2)}(T) \,, \tag{1.24}$$

where we defined the ionization rate

$$\beta(T) = \langle \sigma v \rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} \,, \tag{1.25}$$

and the recombination rate $\alpha^{(2)} = \langle \sigma v \rangle$.

The value of this can be solved numerically: the difference between this and the Saha equation is not great in the prediction in the recombination redshift; however the prediction of the residual ionized hydrogen is different: there is much more than Saha would predict.

The universe gets reionized at $z \gtrsim 6$; this is still under discussion.

There are many ingredients in the interaction of the universe. We are interested in the photons: we want to predict the anisotropies in the CMB. There is a dipole due to the movement of the solar system through the CMB. Now, we want to see what our predictions are if we subtract this.

[Scheme of the interactions.]

The metric interacts with everything, photons interact with electrons through Compton scattering, electrons interact with protons through Coulomb scattering, dark energy, dark matter and neutrinos interact only with the metric.

Instead of Compton scattering, we use its nonrelativistic limit which applies here.

Scattering between electrons and protons is suppressed since protons are very massive. The other terms in the universe affect the geometry and we could see them through this.

There are models which include DM-DE coupling, and quintessence models, and models in which dark energy clusters.

We are not going to consider these.

We go back to first principles:

$$\hat{\mathbb{L}}[f] = \hat{\mathbb{C}}[f] \,, \tag{1.26}$$

where $f = f(x^\alpha, p^\alpha)$, however actually we do not have that much freedom in the phase space distribution. If there are no collisions: $\hat{\mathbb{L}}[f] = 0$, which is equivalent to

$$\frac{\mathrm{D}f}{\mathrm{D}\lambda} = 0 \,, \tag{1.27}$$

where λ is the affine parameter.

In the nonrelativistic case,

$$\hat{\mathbb{L}} = \frac{\partial}{\partial t} + \dot{x} \cdot \nabla_x + \dot{v} \cdot \nabla_v = \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_x + \frac{F}{m} \cdot \nabla_v \,, \tag{1.28}$$

while in the GR case we need to account for the geodesic equation: and we write

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \,, \tag{1.29}$$

where the affine parameter λ has the dimensions of a mass, in order to have dimensional consistency.

Then, the Liouville operator is

$$\hat{\mathbb{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \stackrel{\text{def}}{=} \frac{\mathrm{D}}{\mathrm{D}\lambda} \,. \tag{1.30}$$

This is a total derivative in phase space with respect to the affine parameter.

In the FLRW background, $f = f(|p|, t)$ and

$$\hat{\mathbb{L}} = E \frac{\partial f}{\partial t} - \frac{\dot{a}}{a} |p|^2 \frac{\partial f}{\partial E} \,, \tag{1.31}$$

so if we define the number density

$$n(t) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, f(E,t)\,, \tag{1.32}$$

so if we integrate over momenta we get

$$\int \frac{\mathrm{d}^3 p}{E} \hat{\mathbb{L}}[f]\,, \tag{1.33}$$

we find the equation from before:

$$\dot{n} + 3\frac{\dot{a}}{a} n\,, \tag{1.34}$$

??? to check

1.1.1 Metric perturbations

We want to write the most general perturbed cosmological metric solution to the Einstein Equations, starting from the FLRW metric: it will look like $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{FLRW}} + h_{\mu\nu}$, for a small perturbation $h_{\mu\nu}$. Recall that the FLRW metric, in the zero-curvature case, is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2 \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\ . \tag{1.35}$$

We will neglect spatial curvature because its effect on the *geometry* of the universe is negligible; however we can see its effect in the dynamics.

In general this perturbation will be gauge-dependent, and we want to write only a number of independent components corresponding to the number of physical degrees of freedom. This is done via the scalar-vector-tensor decomposition [Ber00, section 2.1].

We decompose the components of $h_{\mu\nu}$ as:

$$h_{00} = -2\Phi\,, \qquad h_{0i} = a\omega_i \qquad h_{ij} = a^2\Big(-2\Psi\delta_{ij} + \chi_{ij}\Big)\,. \tag{1.36}$$

So, we have distinguished two scalar degrees of freedom Φ and Ψ, plus a vector ω_i and a traceless tensor χ_{ij}, which satisfies $\chi_i^i = 0$. We can assume χ_{ij} to be traceless since we can absorb any variations of the trace of the spatial part of the metric into Ψ; this takes away one degree of freedom from the 6 of the tensor.

This means we have $2 + 3 + 6 - 1 = 10$ degrees of freedom, the right amount before accounting for gauge.

Now, we make use of the Helmholtz decomposition: any vector field x^i can be written as $x^i = x^i_\perp + x^i_\parallel$, where $x^i_\perp$ is solenoidal ($\nabla_i x^i_\perp = 0$), while $x^i_\parallel$ is irrotational ($\epsilon^{ijk}\nabla_i x_j = 0$).

This means that we can split the three degrees of freedom of the vector field into one in $x^i_\parallel$ — since it is irrotational, it can be written as the gradient of a scalar function —, and two in $x^i_\perp$, since it is a 3D vector field for which one component is determined. Explicitly, this is

$$\omega_i = \underbrace{\partial_i \omega}_{\text{1 scalar dof}} + \underbrace{\omega^i_\parallel}_{\text{2 vector dof}}\,, \tag{1.37}$$

We have several names for a vector field x^i satisfying $\nabla_i x^i$: it can be called solenoidal (since the magnetic field, in a solenoid but also anywhere else, satisfies this property), divergence-free, or transverse. The last one can be figuratively interpreted to say that the vector is "orthogonal to the gradient operator"; more formally, in Fourier space the condition reads $k_i \widetilde{x}^i = 0$.

We can do a similar thing for the tensor perturbation, but it is slightly more complicated: it it written in the form

$$\chi_{ij} = \underbrace{\chi_{ij}^{\parallel}}_{\text{1 scalar dof}} + \underbrace{\chi_{ij}^{\perp}}_{\text{2 vector dof}} + \underbrace{\chi_{ij}^{T}}_{\text{2 tensor dof}} , \tag{1.38}$$

where the derivative combination is dictated by the condition of χ_{ij} being traceless. The tensor $\chi^{\parallel}$ contains only one degree of freedom, since it can be written as

$$\chi_{ij}^{\parallel} = \left(\nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \phi_S , \tag{1.39}$$

while $\chi_{ij}^{\perp}$ contains two degrees of freedom, since it can be written as the symmetrized gradient of a vector field S^T:

$$\chi_{ij}^{\perp} = 2 \nabla_{(i} S^T_{j)} , \tag{1.40}$$

where the vector field is divergence-free: $\nabla_j S_T^j = 0$.

The last term is fully transverse: its divergence vanishes, $\nabla_i \chi_T^{ij} = 0$. This term then contains two physical degrees of freedom, since it starts out from five — it is a symmetric spatial tensor with zero trace — but we impose 3 equations.

Let us enumerate the degrees of freedom we have, distinguishing them into "modes" based on how they transform:

1. The **tensor mode** is given by χ_{ij}^T, the part of the spatial metric perturbation which cannot be obtained as a gradient. It has two degrees of freedom, and it transforms as a spin-2 field (which means it is unchanged by rotations of angle π about its axis). This corresponds to gravitational radiation. It is gauge-invariant.

2. The **vector mode** is given by the transverse (divergence-free) vectors $\omega_i^{\parallel}$ and S_i^T. It has four degrees of freedom, two for each transverse vector. Two of these degrees of freedom are gauge, two are physical and correspond to the phenomenon of gravitomagnetism, which causes Lense-Thirring precession.[4] They transform like a spin-1 field: they must be rotated by 2π to come back to the starting point.

3. The **scalar mode** is given by Φ, Ψ, ω and ϕ_S. It has four degrees of freedom, two of which are physical and two of which are gauge. They transform as a spin-0 field (so they are invariant under rotations).

[4] See the General Relativity notes [TM20a, equation 341]: in order to derive the Lense-Thirring precession effect we must have a perturbation in the g_{03} component of the metric, corresponding to the vector ω_i in our notation.

We have separated the perturbations based on how they transform under the group of rotations. If we stick to linear perturbation theory, they are decoupled: they evolve independently.

As we discussed, we can set 2 scalar fields and 2 vector fields to zero by our gauge choice: we choose the Poisson gauge, in which we set ϕ_S, ω to zero for the scalars, and S_i^T to zero for the vectors.

So, the full metric we get looks like:

$$\boxed{\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + 2a\omega_i\,\mathrm{d}x^i\,\mathrm{d}t + a^2\Big(e^{-2\Psi}\delta_{ij} + \chi_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\Big)\,,} \tag{1.41}$$

where ω_i is a transverse vector: $\nabla^i\omega_i = 0$; while in χ_{ij} we only have the tensor mode, so the tensor is traceless and transverse: we have $\chi_i^i = 0 = \nabla^i\chi_{ij}$.

This is explained in more detail in the class by Nicola Bartolo ("early Universe"). These are GR perturbations.

We can compare the physical spacetime and the idealized FLRW metric. We need to do this since we cannot solve the EFE if there is no symmetry. So, we say that spacetime is *close* to the idealized spacetime.

Our next goal will be to solve the geodesic eqs. for the motion of the particles in this gauge.

Thursday 2020-3-19, compiled 2020-04-09

We come back to the discussion from last time, about the Boltzmann equation in a perturbed universe.

When can we drop some terms using a gauge transformation? We can do it for scalar and vector perturbations. We shall use the longitudinal, or Poisson gauge, in which the scalar perturbations are reduced to Φ and Ψ, so we can take the tensor terms to be traceless and covariantly constant.

We will not discuss vectors, since if they are zero at the beginning they cannot be generated, they stay at zero. In our natural units, the perturbations will be small: $\Phi \ll 1$ and $\Psi \ll 1$. Recall that we are neglecting spatial curvature.

Photons have $P^2 = 0$: so we can express this as

$$-(1+2\Phi)(p^0)^2 + p^2 = 0\,, \tag{1.42}$$

where p^2 is defined as $p^2 = g_{ij}P^iP^j$, since in our gauge choice the spatial part of the metric has a Kronecker delta this will only include the spatial parts. So, we get

$$P^0 = \frac{p}{\sqrt{1+2\Phi}} \approx p(1-\Phi)\,. \tag{1.43}$$

Now, we will write the Liouville operator by dividing through by P^0:

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\cdot\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}\,, \tag{1.44}$$

where we split the three-momentum into absolute value p and the unit vector $\hat{p}$, which has $\hat{p}^i = \hat{p}_i$ and $\delta_{ij}\hat{p}^i\hat{p}^j$. We have $\mathrm{d}x^i\Big/\mathrm{d}t = P^i/P^0$.

We are going to expand only to first order. Higher order are more important for small angular scales, and for secondary CMB anisotropies, these are interesting but we are not going to treat them.

To first order, the last term of the RHS is zero, since it is a product of two terms which are both zero to zeroth order (in an unperturbed universe the phase space distribution is perfectly isotropic and a particle keeps travelling in the same direction).

Now we define the amplitude A by $P^i = A\hat{p}^i$: now we will have

$$p^2 = g_{ij}P^iP^j = a^2\delta_{ij}(1-2\Psi)\hat{p}^i\hat{p}^jA^2 \tag{1.45a}$$
$$= a^2(1-2\Psi)A^2\,, \tag{1.45b}$$

therefore, taking the square root and staying to first order we get

$$A \approx p\frac{1+\Psi}{a}\,, \tag{1.46}$$

so

$$P^i = p\hat{p}^i\frac{1+\Psi}{a}\,, \tag{1.47}$$

and the division by a can be interpreted as a redshift effect. Inserting this term we get

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{P^i}{P^0} = \hat{p}^i\frac{1+\Psi+\Phi}{a}\,, \tag{1.48}$$

and we can notice that $\mathrm{d}x^i \big/ \mathrm{d}t$ multiplies the term $\partial f \big/ \partial x^i$, which is only nonzero to first order: so we must consider this term to zeroth order. So, we get

$$\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}\,, \tag{1.49}$$

and now we shall show that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right)\,, \tag{1.50}$$

which will imply that

$$\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right)\,. \tag{1.51}$$

We will use the geodesic equation for photons: it is enough to consider its zeroth component, which is

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta\,, \tag{1.52}$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(p(1+\Phi)\big) = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta\frac{1+\Phi}{p}\,, \tag{1.53}$$

where we brought a P^0 from the left to the right side. This means that we have

$$(1-\Phi)\frac{\mathrm{d}p}{\mathrm{d}t} = p\frac{\mathrm{d}\Phi}{\mathrm{d}t} - \Gamma^0_{\alpha\beta}P^\alpha P^\beta \frac{1+\Phi}{p}\,, \tag{1.54}$$

and now we multiply both sides by $1+\Phi$, the inverse of $1-\Phi$ to linear order:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right) - \Gamma^0_{\alpha\beta}P^\alpha P^\beta \frac{1+2\Phi}{p}\,, \tag{1.55}$$

and now we have to start calculating the Christoffel symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)\,, \tag{1.56}$$

so we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\left(2g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\right)\frac{P^\alpha P^\beta}{p}\,, \tag{1.57}$$

but g^{0i} are zero, since we are ignoring vector perturbations, and $g^{00} = -1+2\Phi$ (since it is the contravariant metric). Then we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Phi}{2}\left(2g_{0\alpha,\beta} - g_{\alpha\beta,0}\right)\frac{P^\alpha P^\beta}{p}\,, \tag{1.58}$$

and we also have

$$\frac{\partial g_{0\alpha}}{\partial x^\beta} = -2\frac{\partial\Phi}{\partial x^\beta}\delta_{\alpha 0}\,, \tag{1.59}$$

so we distinguish the components and find:

$$-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} = -\frac{\partial g_{00}}{\partial t}\frac{(P^0)^2}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^iP^j}{p} \tag{1.60a}$$

$$= 2\frac{\partial\Phi}{\partial t}p - a^2\delta_{ij}\left(-2\frac{\partial\Psi}{\partial t} + 2H(1-2\Psi)\right)\frac{P^iP^j}{p}\,, \tag{1.60b}$$

and we already have shown that $\delta_{ij}P^iP^j = p^2(1+2\Psi)/a^2$.

So on the whole we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = (-1+2\Phi)\left[-\frac{\partial\Phi}{\partial t}p - 2\frac{\partial\Phi}{\partial x^i}\frac{p\hat{p}^i}{a} + p\left(\frac{\partial\Psi}{\partial t} - H\right)\right]\,, \tag{1.61}$$

so putting everything together [extra passage] we get

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right)\,. \tag{1.62}$$

Now we need to choose how to perturb the photon distribution function. At zeroth order it is the Planckian:

$$f \approx \frac{1}{e^{p/T} - 1} \,. \tag{1.63}$$

In general we will have dependence on the position $\vec{x}$, the momentum $(p, \hat{p})$ and time t.

We do not observe spectral distortions in the CMB: it is always described by a Planckian, with anisotropies in the *temperature*. So, we parametrize it as

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left(\frac{p}{T(t)\big(1 + \Theta(\vec{x}, \hat{p}, t)\big)} \right) - 1 \right]^{-1} , \tag{1.64}$$

where we assumed that $\Theta = \delta T / T$ does *not* depend on the momentum of the photon p: otherwise, we would have a spectral distortion. This is certainly true, at least to linear order.

So, we expand in Θ:

$$\begin{aligned} f &\approx \frac{1}{e^{p/T} - 1} + \left(\frac{\partial}{\partial T} \Big(\exp\big(p/T\big) - 1 \Big)^{-1} \right) T\Theta &\text{(1.65a)} \\ &= f^{(0)} - p \frac{\partial f^{(1)}}{\partial p} \Theta \,, &\text{(1.65b)} \end{aligned}$$

since

$$T \frac{\partial f^{(0)}}{\partial T} = -p \frac{\partial f^{(0)}}{\partial p} \,. \tag{1.66}$$

At zeroth order we have

$$\frac{\mathrm{D} f}{\mathrm{D} t} = \frac{\partial f^{(0)}}{\partial t} - Hp \frac{\partial f^{(0)}}{\partial p} = 0 \,, \tag{1.67}$$

since we do not have collision terms. We can write this differently using

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T} \frac{\mathrm{d} T}{\mathrm{d} t} = -\frac{P}{T} \frac{\mathrm{d} T}{\mathrm{d} t} \frac{\partial f^{(0)}}{\partial p} \,, \tag{1.68}$$

where we used the change in derivative variable from before. So we get

$$\left[-\frac{1}{T} \frac{\mathrm{d} T}{\mathrm{d} t} - \frac{1}{a} \frac{\mathrm{d} a}{\mathrm{d} t} \right] \frac{\partial f^{(0)}}{\partial p} = 0 \,, \tag{1.69}$$

which means $\dot{T}/T + \dot{a}/a = 0$, or $T \propto 1/a$, which is Tolman's law. Now, let us go to first order.

$$\frac{\mathrm{D} f}{\mathrm{D} t} = -p \frac{\partial}{\partial t} \left[\frac{\partial f^{(0)}}{\partial p} \Theta \right] - p \frac{\hat{p}^i}{a} \frac{\partial \Theta}{\partial x^i} \frac{\partial f^{(0)}}{\partial p} + Hp\Theta \frac{\partial}{\partial p} \left(p \frac{\partial f^{(0)}}{\partial p} \right) + p \frac{\partial f^{(0)}}{\partial p} \left[\frac{\partial \Psi}{\partial t} - \frac{\hat{p}^i}{a} \frac{\partial \Phi}{\partial x^i} \right] . \tag{1.70}$$

The final expression we get is

$$\frac{\mathrm{D}f}{\mathrm{D}t} = -p\frac{\partial f^{(0)}}{\partial p}\left[\underbrace{\frac{\partial \theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \theta}{\partial x^i}}_{\text{free-streaming}} + \underbrace{\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}}_{\text{gravitational}}\right], \tag{1.71}$$

in which we can distinguish two terms which have to do with the propagation of the anisotropies from emission to now — this is not precise, but it is the reason we call them "free-streaming". The other two terms arise from the self-gravity of the matter appearing on the RHS of the Einstein equations.

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Now, we will discuss the collision terms. The interaction we wish to consider is Compton scattering, which has the form

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}') + \gamma(\vec{p}'), \tag{1.72}$$

and we are interested to see how this affects the momentum distribution of the photons.

The collision term in the equation

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \hat{\mathbb{C}}[f(\vec{p})] \tag{1.73}$$

reads

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 q'}{(2\pi)^3 2E_e(q')}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2 (2\pi)^4 \\ \times \delta^{(3)}(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\delta(E(p)+E_e(q)-E(p')-E_e(q')) \\ \times \left[f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right]\end{aligned}. \tag{1.74a}$$

The factor $1/p$ at the start comes from the LHS, since we differentiated with respect to t and not λ on the LHS.

For the photons the energy is $E(p') = p'$, for the electrons instead $E_e(q) \approx m_e + q^2/2m_e$, since the electrons are nonrelativistic — the temperatures are of the order $0.3\,\mathrm{eV}$ at recombination, a vanishingly small fraction of the electrons' mass of $511\,\mathrm{keV}$.

So, we should perform all the integrations on the RHS: we start with the integration over q'. We get rid of a δ function and get

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{\pi}{2m_e^2}\frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2 \\ \times \delta\left[p + \frac{q^2}{2m_e} - p' - \frac{\left(\vec{p}+\vec{q}-\vec{p}'\right)^2}{2m_e}\right] \\ \times \left[f_e(\vec{q}+\vec{p}-\vec{p}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right]\end{aligned}. \tag{1.75a}$$

For nonrelativistic Compton scattering we have

$$E_e(\vec{q}) - E_e(\vec{q}+\vec{p}-\vec{p}') = \frac{q^2}{2m_e} - \frac{\left(\vec{q}+\vec{p}-\vec{p}'\right)^2}{2m_q} \approx \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}, \tag{1.76}$$

which is true as long as $q \gg p, p'$. Also, the scattering is close to being elastic:

$$\frac{(\vec{p} - \vec{p}') \cdot \vec{q}}{m_e} \sim \frac{Tq}{m_e} \sim Tv_b \ll T\,, \tag{1.77}$$

since the velocity of the electrons is nonrelativistic.

So,

$$\frac{\Delta E_e}{E} \sim \frac{Tv_b}{Tc} \sim \frac{v_b}{c} \ll 1\,. \tag{1.78}$$

Let us motivate $q \ll p, p'$: we know that, since

$$\frac{q^2}{2m_e} \sim T\,, \tag{1.79}$$

we have $q \sim (m_e T)^{1/2}$, which is equal to

$$q \sim \left(\frac{m_e}{T}\right)^{1/2} T \gg T\,, \tag{1.80}$$

so $q \gg T \sim p$.

Now, the change to the electron kinetic energy is small so we can expand:

$$\delta\left[p + \frac{q^2}{2m_e} - p' - \frac{\left(\vec{q} + \vec{p} - \vec{p}'\right)^2}{2m_e}\right] \tag{1.81a}$$

$$\approx \delta(p - p') + \left[E_e(q') - E_e(q)\right] \times \frac{\partial}{\partial E_e(q')}\delta(p + E_e(q) - p' - E_e(q')) \tag{1.81b}$$

$$\approx \delta(p - p') + \frac{(\vec{p} - \vec{p}) \cdot \vec{q}}{m_e}\frac{\partial \delta(\vec{p} - \vec{p}')}{\partial p'}\,, \tag{1.81c}$$

where we used the fact that

$$\frac{\partial(x - y)}{\partial x} = -\frac{\partial(x - y)}{\partial y}\,. \tag{1.82}$$

The derivative of the delta-function is defined as a functional yielding the derivative of the function it is integrated with.

This then gives us

$$\begin{aligned} \hat{\mathbb{C}}[f(\vec{p})] = \frac{\pi}{2m_e^2}\frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2 \\ \times \left[\delta(p - p') + \frac{(\vec{p} - \vec{p}) \cdot \vec{q}}{m_e}\frac{\partial \delta(\vec{p} - \vec{p}')}{\partial p'}\right]\left(f(\vec{p}') - f(\vec{p})\right) \end{aligned}\,. \tag{1.83a}$$

Now, it is a fact from QFT that we can compute

$$|\mathcal{M}|^2 = 6\pi\sigma_T m_e^2\left(1 + \cos^2\theta\right), \tag{1.84}$$

where $\cos\theta = \hat{p}\cdot\hat{p}'$.

For simplicity we replace this angle-dependent quantity with its angular average: the integral of the cosine gives us $1/3$, so we get a multiplier $6(1+1/3) = 8$:

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\pi\sigma_T m_e^2 \,. \tag{1.85}$$

Now, we insert this in the expression and integrate over q: this yields a mean velocity, and also we expand the phase space distributions to first order:

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{2\pi n_e \sigma_T}{p} \int \frac{\mathrm{d}^3 p'}{(2\pi)^3 p'} &\left[\delta(p-p')\left(\vec{p}-\vec{p}'\right)\cdot\vec{v}_b \frac{\partial\delta(p-p')}{\partial p'}\right] \\ \times &\left[f^{(0)}(\vec{p}') - f^{(0)}(\vec{p}) - p'\frac{\partial f^{(0)}}{\partial p'}\theta(\vec{p}') + p\frac{\partial f^{(0)}}{\partial p}\theta(\vec{p})\right].\end{aligned} \tag{1.86a}$$

[passages]

We do the angular integral, and simplify it by defining the *monopole* contribution:

$$\theta_0(\vec{x},t) = \frac{1}{4\pi}\int \mathrm{d}\Omega'\,\theta(\vec{x},\hat{p}',t)\,. \tag{1.87}$$

Then, finally, we integrate over p', which gives us the result

$$\hat{\mathbb{C}}[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e\sigma_T\left[\theta_0 - \theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]. \tag{1.88}$$

The factor due to the electron spin, g_e, is accounted for in the electron momentum distribution f_e.

So, for the photons we have

$$\frac{\partial\theta}{\partial t}\frac{\hat{p}^i}{a}\frac{\partial\theta}{\partial x^i} - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T\left(\theta_0 - \theta + \hat{p}\cdot\vec{v}_b\right). \tag{1.89}$$

Our last step is to move to conformal time η, defined by $\mathrm{d}\eta = \mathrm{d}t\,/a$; denoting derivatives with respect to conformal time with a dot (and multiplying everything by a) we get:

$$\dot{\theta} + \hat{p}^i\frac{\partial\theta}{\partial x^i} - \dot{\Psi} + \hat{p}^i\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T a\left(\theta_0 - \theta + \hat{p}\cdot\vec{v}_b\right). \tag{1.90}$$

Now, in order to solve this equation we perform a Fourier transform:

$$\theta(\vec{x}) = \int\frac{\mathrm{d}^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}\widetilde{\theta}(\vec{k})\,, \tag{1.91}$$

and we define the *cosine* of the angle between the photon momentum $\vec{p}$ and the momentum $\vec{k}$: $\mu = \hat{k}\cdot\hat{p}$.

The optical depth is defined by

$$\tau(\eta) = \int_{\eta}^{\eta_0}\mathrm{d}\overline{\eta}\,a(\overline{\eta})n_e\sigma_T\,, \tag{1.92}$$

and it is large at early times, small at late times since the density decreases. Its derivative is

$$\dot{\tau} = \frac{\mathrm{d}\tau}{\mathrm{d}\eta} = -n_e \sigma_T a\,, \tag{1.93}$$

which is a term in the Boltzmann equation! Substituting it, we get

$$\dot{\widetilde{\theta}} + ik\mu\widetilde{\theta} - \dot{\widetilde{\Psi}} + ik\mu\widetilde{\Psi} = -\dot{\tau}\left(\widetilde{\theta}_0 - \widetilde{\theta} + \mu\widetilde{v}_b\right). \tag{1.94}$$

It is an assumption we are making that the velocities are irrotational, so they can be expressed as a gradient: so, in Fourier space we get $\widetilde{\vec{v}}_b = \hat{k}\widetilde{v}_b$.

We still need to solve the Einstein equations in order to determine Φ and Ψ. In order to do this we need to describe all the component of the universe.

Also, we need to determine the velocity of the baryons v_b; it is an assumption we are making that in order to preserve local as well as global neutrality the local velocities of baryons and leptons are equal.

1.2 Boltzmann equation for CDM

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[From yesterday: the distribution function for Compton scattering is general, as long as electrons and photons are in thermal equilibrium.]

Last time we derived the first order Boltzmann equation for the thermal anisotropy.

Now, however, we need to discuss CDM, and then we will discuss baryons..

For Dark Matter the collision term is negligible (since by the observations of galactic dynamics we know that DM is collisionless): so in the Boltzmann equation we have

$$\frac{\mathrm{D}f_{\mathrm{dm}}}{\mathrm{D}\lambda} = 0\,. \tag{1.95}$$

The key fact is that DM is nonrelativistic. The calculation, except for this, is quite analogous. The momentum P^μ has to be normalized: $P^\mu P_\mu = g_{\mu\nu}P^\mu P^\nu = -m^2$, as opposed to 0, which we had with the photon. We define the norm of the three-momentum

$$p^2 = g_{ij}P^i P^j\,, \tag{1.96}$$

and with it the energy: $E = \sqrt{p^2 + m^2}$. Then, using our perturbed metric we get

$$P^0 = E(1-\Phi) \qquad \text{and} \qquad P^i = p\hat{p}^i \frac{1+\Psi}{a}\,, \tag{1.97}$$

where $\hat{p}^i$ is a unit vector in the direction of p^i. This makes sense: the velocity, without perturbation, decays like a^{-1}.

Notice that we can make the derivative with respect to the affine parameter λ to one with respect to t, the factor $\mathrm{d}t/\mathrm{d}\lambda$ can be moved to the RHS and does not affect our discussion. The LHS of the Boltzmann equation (1.95) then reads

$$\frac{\mathrm{D}f_{\mathrm{dm}}}{\mathrm{D}t} = \frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\partial f_{\mathrm{dm}}}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \underbrace{\frac{\partial f_{\mathrm{dm}}}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}}_{\text{higher order}}\,, \tag{1.98}$$

where the last term is second order since it is a product of factors which are both zero to zeroth order. With steps analogous to the massless case we find

$$0 = \frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{\mathrm{dm}}}{\partial x^i} - \frac{\partial f_{\mathrm{dm}}}{\partial E}\left[\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\frac{p^2}{E} - \frac{p^2}{E}\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial \Phi}{\partial x^i}\right]. \tag{1.99}$$

The factor $\mathrm{d}a/\mathrm{d}t\,/a$ is just H, we write it this way in order not to overload the dot (which represents derivatives with respect to conformal time only in our notation).

In order to work with this, we shall consider only terms up to first order in p/E, which is justified since DM is non-relativistic. Also, we will take moments: in general, a moment is an integral of a function (with angular dependence) multiplied by some power of the cosine of the angle between the direction considered and some fixed other direction. In principle, the hierarchy goes to any order: in practice, we can usually truncate the hierarchy to some order according to some physical principle. Fortunately, DM behaves like a fluid, which allows us to work to first order only.

Are the NS equations not exact?

This approach is fully relativistic, but linear. In star formation, we use Newtonian approximation, but we account for nonlinearity. Having both is impossible.

The lowest (0th) moment is just given by the integral in $\mathrm{d}^3p\,/(2\pi)^3$ of the distribution, that is, over all angles: we find

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\int \frac{\mathrm{d}^3p}{(2\pi)^3} f_{\mathrm{dm}} + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{\mathrm{d}^3p}{(2\pi)^3} f_{\mathrm{dm}}\frac{p\hat{p}^i}{E} \\ &\quad - \left[\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} - \frac{\partial \Psi}{\partial t}\right]\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{p^2}{E} - \underbrace{\frac{1}{a}\frac{\partial \Phi}{\partial x^i}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial f_{\mathrm{dm}}}{\partial E}\hat{p}^i p}_{\text{higher order}}.\end{aligned} \tag{1.100a}$$

We take out of the integral any term not depending on the angles in momentum space.

why is the last term higher order? There is no way to have a nonzero result to zeroth order in the integral by isotropy, and it is multiplied by a first-order perturbation.

Now, we introduce the dark matter density n_{dm} — for more details, see the notes for the course in Fundamentals of Astrophysics and Cosmology [TM20b, section 3.2], but do note that the number density defined there is global, while the one we are considering here is a local number density, which can be different for different points in spacetime. Also, we define its velocity v^i:

$$n_{\mathrm{dm}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} f_{\mathrm{dm}} \tag{1.101a}$$

$$v^i = \left\langle \frac{P^i}{E} \right\rangle_{\text{particle}} = \frac{1}{n_{\mathrm{dm}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3} f_{\mathrm{dm}}\frac{p\hat{p}^i}{E}. \tag{1.101b}$$

Note that f_{dm} accounts for the number of degrees of freedom g_{dm}.

These are not covariant, right? Since n is the 0th component of a 4-vector...

In the first and second terms we can insert n_{dm} and v^i directly (to first order — we always do manipulations neglecting second order terms), while for the third term we need to integrate by parts, using $\mathrm{d}E/\mathrm{d}p = p/E$:[5] we obtain:

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\partial f_{\mathrm{dm}}}{\partial E} \frac{p^2}{E} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p \frac{\partial f_{\mathrm{dm}}}{\partial p} \tag{1.103a}$$

Changed the derivative of the distribution

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}p\, p^3 \left\langle \frac{\partial f_{\mathrm{dm}}}{\partial p} \right\rangle \tag{1.103b}$$

$$= -3 \frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}p\, p^2 \left\langle f_{\mathrm{dm}} \right\rangle = -3 n_{\mathrm{dm}} \,, \tag{1.103c}$$

which is a calculation discussed in the FAC course [TM20b] [I'll insert the section when I get to it].

This yields:

$$\frac{\partial n_{\mathrm{dm}}}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x^i} \left(n_{\mathrm{dm}} v^i \right) + 3 \left[\frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} - \frac{\partial \Psi}{\partial t} \right] n_{\mathrm{dm}} = 0 \,, \tag{1.104}$$

which to 0th order yields

$$\frac{\partial n^{(0)}_{\mathrm{dm}}}{\partial t} + 3 \frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} n^{(0)}_{\mathrm{dm}} \iff a^3 n^{(0)}_{\mathrm{dm}} = \mathrm{const} \,. \tag{1.105}$$

Now, we can write the full number density as

$$n_{\mathrm{dm}} = n^{(0)}_{\mathrm{dm}} \left[1 + \delta(\vec{x}, t) \right] \,, \tag{1.106}$$

where δ is the standard notation in cosmology for these kinds of dimensionless fractional perturbations.

Then, after some simplifications the evolution of this perturbation looks like

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial v^i}{\partial x^i} - 3 \frac{\partial \Psi}{\partial t} = 0 \,, \tag{1.107}$$

which is our first order continuity equation.

This was for the first moment, for the next one we integrate after multiplying by $\hat{p}^j/E$. We neglect a term because it is too relativistic, that is, second order in p/E.

Integrating by parts, for a different term we get

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\partial f_{\mathrm{dm}}}{\partial p} \frac{p^2 \hat{p}^i}{E} = \int \frac{\mathrm{d}\Omega\, \hat{p}^j}{(2\pi)^3} \int \dots lni \qquad = -4 n_{\mathrm{dm}} v^i \,. \tag{1.108}$$

We finally obtain

$$\frac{\partial (n_{\mathrm{dm}} v^i)}{\partial t} + 4 \frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} n_{\mathrm{dm}} v^j + \frac{n_{\mathrm{dm}}}{a} \frac{\partial \Phi}{\partial x^i} = 0 \,, \tag{1.109}$$

[5] This comes from $E = \sqrt{p^2 + m^2}$:

$$\frac{\partial E}{\partial p} = \frac{2p}{2\sqrt{p^2 + m^2}} = \frac{p}{E} \,. \tag{1.102}$$

so we factor out the background and get, to first order,

$$\frac{\partial v^j}{\partial t} + Hv^j + \frac{1}{a}\frac{\partial \Phi}{\partial x^j} = 0\,. \tag{1.110}$$

This is basically a linear Euler equation. In Fourier space, we get

$$\dot{\widetilde{\delta}} + ik\widetilde{v} - 3\dot{\widetilde{\psi}} = 0 \tag{1.111a}$$

$$\dot{\widetilde{v}} + \frac{\dot{a}}{a}\widetilde{v} + ik\widetilde{\Phi} = 0\,. \tag{1.111b}$$

1.3 Boltzmann equation for baryons

Which interaction are relevant? At recombination, electrons are tightly coupled with photons, while the interaction between protons and photons is suppressed by a factor $(m_e/m_p)^2$ with respect to to electron-photon scattering. We need to deal with Coulomb scattering between electrons and proton. So, we can say that the quantity

$$\frac{\rho_e - \rho_p^{(0)}}{\rho_e^{(0)}} = \rho_b\,, \tag{1.112}$$

is a perturbation. So we write two Boltzmann equations, for electrons and protons, assuming that they move with the same velocity. The collision term looks like

$$C_{e\gamma} = (2\pi)^4\delta^{(4)}(p+q-p'-q')\frac{|\mathcal{M}|^2\left[f_e(q')f_\gamma(p') - f_e(q)f_\gamma(p)\right]}{8E(p)E(p')E_e(q)E_e(q')}\,, \tag{1.113}$$

So the zeroth moment equation for electrons is the same as the one for Dark Matter: both of the terms in the RHS vanish in

$$\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}(n_e v_b^i) + 3\left[H - \frac{\partial \Psi}{\partial t}\right]n_e = \left\langle C_{ep}\right\rangle + \left\langle C_{e\gamma}\right\rangle\,, \tag{1.114}$$

since they multiply antisymmetric terms in exchange of momenta [clear up].

For the first moment we sum the proton and electron equations multiplied by the respective charges. The symmetry of the momenta of electrons and protons cancels the term of Coulomb scattering in the first moment equation. So, we get

$$\frac{\partial v_b^j}{\partial t} + Hv_b^j + \frac{1}{a}\frac{\partial \Phi}{\partial x^j} = \frac{1}{\rho_b}\left\langle c_{e\gamma}q^j\right\rangle_{pp'q'q}\,, \tag{1.115}$$

but by total momentum conservation we can switch the momentum q^j in the equation to a p^j. We Fourier transform, and multiply by $\hat{k}^j$: then we get $\vec{p}\cdot\hat{k} = p\mu$, where $\mu = \cos\alpha$.

Two pages from Dodelson [Dod03].

In the perturbed Einstein Equations we will need the stress-energy tensor: it does not require the full phase space distribution, but only certain kinds of integrated information.

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With our approach, we did not assume that DE, DM and such are fluids: instead, we have shown it.

We can combine the equations for electrons and protons into a unique equation for the baryons:

$$m_p \frac{\partial}{\partial t}\left(n_b v_b^j\right) + 4Hm_p n_b v_b^j \dots . \tag{1.116}$$

The quantity $-\left\langle c_{e\gamma} p\mu \right\rangle / \rho_b$ is similar to what we had for photons: a monopole term, the temperature anisotropy, and the Doppler term, which is called that since it involves a velocity. Then we get

$$\begin{aligned} -\left\langle c_{e\gamma} p\mu \right\rangle / \rho_b &= \frac{n_e \sigma_T}{\rho_b} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p} \mu \left[\widetilde{\theta}_0 - \widetilde{\theta}(\mu) + \widetilde{v}_b \mu\right] && (1.117\text{a}) \\ &= \frac{n_e \sigma_T}{\rho_b} \int_0^\infty \frac{\mathrm{d}p}{2\pi^2} p^4 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^1 \frac{\mathrm{d}\mu}{2} \mu \left[\widetilde{\theta}_0 - \widetilde{\theta}(\mu) + \widetilde{v}_b \mu\right], && (1.117\text{b}) \end{aligned}$$

notice that the terms inside the bracket are independent of the modulus of p. Now we define the *dipole*:

$$\theta_1 = i \int_{-1}^1 \frac{\mathrm{d}\mu}{2} \mu \theta(\mu) , \tag{1.118}$$

so we finally get

$$\dot{\widetilde{v}}_b + \frac{\dot{a}}{a} \widetilde{v}_b + ik\widetilde{\Phi} = \dot{\tau} R \left(3i\widetilde{\theta}_1 + \widetilde{v}_b\right) , \tag{1.119}$$

where we defined

$$R = \frac{3\rho_b^{(0)}}{4\rho_\gamma^{(0)}} . \tag{1.120}$$

In general we define the multipole moments as

$$\theta_\ell = \frac{1}{(-i)^\ell} \int_{-1}^1 \frac{\mathrm{d}\mu}{2} P_\ell(\mu) \theta(\mu) , \tag{1.121}$$

where $P_e(\mu)$ are the Legendre polynomials.

Then we get

$$\dot{\widetilde{\theta}} + ik\mu\widetilde{\theta} - \dot{\widetilde{\Psi}} + ik\mu\widetilde{\Phi} = -\dot{\tau}\left[\widetilde{\theta}_0 - \widetilde{\theta} + \mu v_b - \frac{1}{2} P_2(\mu) \widetilde{\theta}_2\right] , \tag{1.122}$$

where the addition of the quadrupole term accounts for (???), see the notes by Natale [Nat17].

Dropping the tildes, we get

$$\dot{\theta} + ik\mu\theta = \dot{\Psi} - ik\mu\Phi - \dot{\tau}\left[\theta_0 - \theta + \mu v_b - \frac{1}{2} P_2(\mu) \Pi\right] \tag{1.123a}$$

$$\Pi = \theta_2 - \theta_{P_2} + \theta_{P_0} \tag{1.123b}$$

$$\dot{\theta}_p + ik\mu\theta_p = -\dot{\tau}\left[-\theta_p + \frac{1}{2}\big(1 - P_2(\mu)\Pi\big)\right] \dots , \tag{1.123c}$$

anisotropy creates polarization, polarization does not affect polarization much.

Why do we not have a quadrupole term? Its integral is zero since it is multiplied by μ.

What about neutrinos? We can model the anisotropies in the cosmic neutrino background using Fermi-Dirac statistics, with

$$f = \frac{1}{\exp(d)} \dots , \tag{1.124}$$

so now repeating the considerations we made for the photons, we get

$$\dot{w} \dots . \tag{1.125}$$

1.4 Perturbing the Einstein Equations

Why do we consider only the scalar perturbations? Vectors do not obey dynamical equation but only conservation equations, and they die away quickly in the expansion. The Lense-Thirring effect is about the g_{0i} terms, and it is not governed by the (???) theorem, but we neglect it as well.

Tensor perturbations are gravitational waves. They evolve separately from the scalar perturbations: we decompose the anisotropy into

$$\theta^T(k, \mu, \phi) = \theta^T_+(k, \mu)\left(1 - \mu^2\right)\cos(2\phi) + \theta^t_\times(k, \mu)(1 - \mu^2) , \tag{1.126}$$

and both of the polarizations ($\epsilon = +, \times$) satisfy

$$\dot{\theta}^T_\epsilon + ik\mu\theta^T_\epsilon + \frac{1}{2}\dot{h}_\epsilon = \dot{\tau}\left[\theta^T_\epsilon - \frac{1}{10}\theta^T_{\epsilon,0} - \frac{1}{7}\theta^T_{\epsilon,2} - \frac{3}{7}\theta^T_{\epsilon,4}\right] . \tag{1.127}$$

Let us then perturb Einstein's equations: we use conformal time, so the metric (in Euclidean flat 3-space) is

$$\mathrm{d}s^2 = a^2\left[-(1 + 2\Phi)\,\mathrm{d}\eta^2 + (1 - 2\Psi)\,\mathrm{d}\ell^2\right] . \tag{1.128}$$

We compute the Christoffel symbols, and from these we get the Ricci tensor, the Ricci scalar and the Einstein tensor. It is generally much more convenient to write the equations with mixed indices

$$G^\mu_\nu = 8\pi G T^\mu_\nu . \tag{1.129}$$

To first order we get

$$\Gamma^0_{00} = \frac{\dot{a}}{a} + \dot{\Phi} \tag{1.130a}$$

$$\Gamma^i_{0j} = \left(\frac{\dot{a}}{a} - \dot{\Psi} \right) \delta^i_j \tag{1.130b}$$

$$\Gamma^0_{ij} = \left[\frac{\dot{a}}{a} (1 - 2\Phi - 2\Psi) - \dot{\Psi} \right] \delta_{ij} \tag{1.130c}$$

$$\Gamma^0_{0i} = \partial_i \Phi \tag{1.130d}$$

$$\Gamma^i_{00} = \partial^i \Phi \tag{1.130e}$$

$$\Gamma^i_{jk} = - \partial_j \Psi \delta^i_k - \partial_k \Psi \delta^i_j + \partial^i \Psi \delta_{jk} \,, \tag{1.130f}$$

so the Einstein tensor reads:

$$G^0_0 = \frac{1}{a^2} [\dots] \,, \tag{1.131}$$

Often people make the quasi-static approximation, in which we neglect the time-derivatives of the potentials. We do not do that here. The stress-energy tensor has the following 00 component:

$$T^0_0 = - \sum_i g_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_i(p) f_i(\vec{x}, \vec{p}, t) \,. \tag{1.132}$$

For nonrelativistic matter, $E_i \sim m_i$: then we get

$$T^{0,\mathrm{dm}}_0 = - \rho_{\mathrm{dm}} (1 + \delta) \,. \tag{1.133}$$

For the photons we have

$$T^{0,(\gamma)}_0 = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \theta \right] = - \rho_\gamma [1 + 4\Theta_0] \,, \tag{1.134}$$

and for the neutrinos we get the exact same result, substituting the variables:

$$T^{0,(\nu)}_0 = - \rho_\nu [1 + 4 w_0] \,. \tag{1.135}$$

Dark Energy is usually considered to be smooth. Then we have, for the 00 EFE:

$$\nabla^2 \Psi - 3 \frac{\dot{a}}{a} \left(\dot{\Psi} + \Phi \frac{\dot{a}}{a} \right) = 4\pi G a^2 \left(\rho_{\mathrm{dm}} \delta + \rho_b \delta_b + 4 \rho_\gamma \Theta_0 + 4 \rho_\nu w_0 \right) . \tag{1.136}$$

Recall that we want to solve for Ψ and Φ: we have a lot of redundancy in the EFE. Another convenient independent equation is the traceless part of the ij components. The simplest way to do these calculations is to use projectors in Fourier space.

On the left hand side we get

$$\left(\hat{k}_i \hat{k}^j - \frac{1}{3} \delta^i_j \right) \hat{G}^i_j = \left(\hat{k}_i \hat{k}^j - \frac{1}{3} \delta^j_i \right) \frac{k^i k_j}{a^2} (\Phi - \Psi) \tag{1.137a}$$

$$= \frac{2}{3} \frac{k^2}{a^2} (\Phi - \Psi) \,. \tag{1.137b}$$

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On the Right Hand Side, instead, since the stress energy tensor is given by

$$T^i_j = \sum_{\text{all species } a} g_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{p^i p_j}{E_a(p)} f_a(\vec{x}, \vec{p}, t) \tag{1.138}$$

we obtain

$$\left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^i_j\right) \widetilde{T}^i_j = \sum_a g_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{p^2\mu^2 - p^2/3}{E_a(p)} f_a(\vec{p})\,, \tag{1.139}$$

where μ is the cosine of the angle between $\hat{k}$ and $\hat{p}$.

But we know that

$$\mu^2 - \frac{1}{3} = \frac{2}{3} P_2(\mu)\,, \tag{1.140}$$

where P_2 is the second Legendre polynomial; recall that these Legendre polynomials are precisely those used in the definition of the multipole moments θ_k [definition ref]. Using this, we get

$$\begin{aligned}
\left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^i_j\right) \widetilde{T}^i_j &= -2 \int_0^\infty \frac{\mathrm{d}p\, p^2}{2\pi^2} \frac{\partial f^{(0)}}{\partial p} p^2 \int_{-1}^1 \frac{\mathrm{d}\mu}{2} \frac{2}{3} P_2(\mu) \theta(\mu) &\text{(1.141a)}\\
&= 2\frac{2}{3}\widetilde{\theta}_2 \int_0^\infty \frac{\mathrm{d}p\, p^2}{2\pi^2} p^2 \frac{\partial f^{(0)}}{\partial p} = -\frac{8}{3} \rho_\gamma \widetilde{\theta}_2\,, &\text{(1.141b)}
\end{aligned}$$

we have integrated by parts, and the quadrupole contribution has vanished by the orthogonality property of the Legendre polynomials.

[why is it $\rho_{\gamma,0}$? we are not computing it now, right?] We are writing ρ_γ and the such we really mean $\rho^{(0)}_\gamma$, an average at monopole order.

In a universe without neutrinos, we would have no difference between Φ and Ψ.

What would be the geometric meaning of the difference between Φ and Ψ?

So, if we collect terms we find that the whole Einstein equation reads:

$$k^2(\Phi - \Psi) = -32\pi G a^2 \underbrace{\left[\rho_\gamma \widetilde{\theta}_2 + \rho_\nu \widetilde{w}_2\right]}_{\rho_r \theta_{r,2}}\,. \tag{1.142}$$

Since we have added the neutrino contribution, we are accounting for all of radiation. See Dodelson [Dod03, page 99].

The Einstein 00 equation (1.136) can be written in Fourier space as

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Psi} + \frac{\dot{a}}{a}\Phi\right) = -4\pi G a^2 \left(\rho_m \delta_m + 4\rho_r \widetilde{\theta}_{r,0}\right)\,, \tag{1.143}$$

where

$$\rho_m \delta_m = \rho_m \delta + \rho_b \delta_b \qquad \text{and} \qquad \rho_r \theta_{r,0} = \rho_\gamma \theta + \rho_\nu w\,. \tag{1.144}$$

Dark Matter and baryons do not contribute to the quadrupole (at first order).

Recall that the EFE are not independent: we could also have chosen the ${}^0_i$ one: those components of the stress-energy tensor are given by

$$T^0_i = \sum_\alpha a\rho_\alpha \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p_i f_\alpha(\vec{x}, \vec{p}, t)\,, \tag{1.145}$$

so with reasoning similar to what was done before we find that the ${}^0_i$ equation reads

$$\dot{\Psi} + aH\Phi = -\frac{4\pi G a^2}{ik}\left[\rho_m v_m - 4i\rho_r\theta_{r,1}\right], \tag{1.146}$$

where

$$\rho_m v_m = \rho_{\mathrm{dm}} v + \rho_b v_b \qquad \text{and} \qquad \rho_r\theta_{r,1} = \rho_g\theta_1 + \rho_\nu w_1\,. \tag{1.147}$$

In this case then we would find

$$k^2\Psi = -4\pi G a^2\left[\rho_m\delta_m + 4\rho_r\theta_{r,0} + \frac{32H}{k}\left(i\rho_m v_m + 4\rho_r\theta_{r,1}\right)\right]. \tag{1.148}$$

Tensor models

Now that we have found the equation of motion of the scalar field, let us discuss tensor modes: we can find the equation of motion for these starting from the transverse traceless part of the ${}^i_j$ equation. We set $\chi_{ij} = 2h_{ij}$, and with this we can write the traceless Christoffel symbols:

$$\Gamma^0_{ij} = \frac{1}{2}\dot{h}_{ij} + \frac{\dot{a}}{a}h_{ij} \tag{1.149a}$$

$$\Gamma^i_{0j} = \frac{1}{2}\dot{h} \tag{1.149b}$$

$$\Gamma^i_{jk} = h^i_{(j,k)} - \frac{1}{2}{h_{jk}}^{,i} \tag{1.149c}$$

$$, \tag{1.149d}$$

which yield the traceless Einstein tensor

$$G^i_j = \frac{\dot{a}}{a}\dot{h}^i_j - \frac{1}{2}\nabla^2 h^i_j + \frac{1}{2}\ddot{h}^i_j\,. \tag{1.150}$$

In order to get the correct contribution on the RHS we need to project it: we apply the projection operator

$$\mathcal{P}^{k\,j}_{\;i\,l} = \mathcal{P}^k_i\mathcal{P}^j_l - \frac{1}{2}\mathcal{P}^k_l\mathcal{P}^j_i\,, \tag{1.151}$$

where

$$\mathcal{P}^i_j = \delta^i_j - \hat{k}^i\hat{k}_j\,. \tag{1.152}$$

The work [CM05] is mentioned, but the work only the two-index $\mathcal{P}$ is defined (in position space and not in momentum space, but it's the same) (eq. 13); but the four-index combination is not explicitly written…

What is usually done is to completely neglect the right hand side of the EFE (since it is second order in v/c?), to get the homogeneous expression

$$\ddot{h}_{ij} + 2\frac{\dot{a}}{a}h_{ij} - \nabla^2 h_{ij} = 0\,. \tag{1.153}$$

We can express the tensor perturbation in Fourier space as

$$h_{ij} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, e^{-\vec{k}\cdot\vec{x}} h_{ij}(\vec{k}, t)\,, \tag{1.154}$$

where $h_{ij}(\vec{k}, t)$ can be decomposed into the two basis polarizations $+$ and $\times$ so that our equation will read

$$\ddot{h}_\epsilon + 2\frac{\dot{a}}{a}\dot{h}_\epsilon + k^2 h_\epsilon = 16\pi G \pi_\epsilon\,, \tag{1.155}$$

where $\epsilon = +, \times$ (see Pritchard & Kamionkowski [PK05]). Here π_ϵ is the traceless tensor part of the stress-energy tensor, which contains a negligible contribution from photons and a contribution from neutrinos which has a non-negligible effect on gravity waves; the latter leads to

$$\ddot{h}_\epsilon + 2\frac{\dot{a}}{a}\dot{h}_\epsilon + k^2 h_\epsilon = -24 f_\nu(\eta)\left(\frac{\dot{a}}{a}\right)^2 \int_0^\eta \mathrm{d}\widetilde{\eta}\, K(k(\eta - \widetilde{\eta}))\dot{h}_\epsilon(\widetilde{\eta})\,, \tag{1.156}$$

where $f_\nu = p_\nu^{(0)}/p^0$ and

$$K(s) = -\frac{\sin s}{s^3} - \frac{3\cos s}{s^4} + \frac{3\sin s}{s^5}\,. \tag{1.157}$$

This is a damping, not a source term

Is it? as long as $s < 5$, sure, but is that guaranteed? around $s = 5.5$ the function changes sign…

Solutions to the tensor perturbation equations are given by

$$h_{\mathrm{rad}}(\eta) = h(0)\ldots \tag{1.158}$$
$$h_{\mathrm{mat}}(\eta) = 3h(0)\ldots\,. \tag{1.159}$$

Initial conditions

Plot: comoving scale versus log of scale factor. From Lucchin-Coles? [CL02].

We can say that the universe starts at the end of inflation.

Let us put the curvature k back into the equations we found. At early times, but after inflation, (???) are outside the horizon so we have $k\eta \ll 1$ and $\dot{\tau} \to \infty$, so the initial conditions look like

$$\dot{\theta}_0 = \dot{\Psi} \qquad \dot{w}_0 = \dot{\psi} \tag{1.160}$$

$$\dot{\delta} = 3\dot{\psi} \qquad \dot{\delta}_b = 3\dot{\psi}\,; \tag{1.161}$$

these equations assume that all the multipoles from the dipole onward are suppressed.

This is called the "separate universe" approximation. Locally, the universe looks like a separate FRLW curved universe.

Clarify

Also, the dipole must be much smaller than the monopole, and we must have the velocity of the baryons v_b be equal to $-3i\theta$ because of tight coupling.

In this limit, photons behave like a fluid, and this fluid has the same behavior as the e^- fluid.

We assume that the neutrino quadrupole is negligible. We are considering a radiation-dominated epoch, so we neglect matter.

We have then $\Psi = \Phi$, and

$$3\frac{\dot{a}}{a}\left(\dot{\Psi} + \frac{\dot{a}}{a}\Phi\right) = -16\pi G a^2 \rho_r \Theta_{r,0}\,, \tag{1.162}$$

which becomes

$$\frac{\dot{\Phi}}{\eta} + \frac{1}{\eta^2}\Phi = -\frac{16\pi G}{3}\rho_r a^2 \left(\frac{\rho_\gamma}{\rho_r\theta_0 + \frac{\rho_\nu}{\rho_r}w_0}\right). \tag{1.163}$$

Then the phase space distribution looks like

$$f_\nu = \frac{\rho_\nu}{\rho_\gamma + \rho_\nu}\,, \tag{1.164}$$

since $a \propto \eta$ and $\rho_r \propto a^{-4}$ and $\rho_r = \rho_\gamma + \rho_\nu$.

If we assume $\dot{\theta}_0 = \dot{w}_0 = \dot{\Phi}$. This finally yields

$$\ddot{\Phi}\eta + 4\dot{\Phi} = 0\,, \tag{1.165}$$

which has a vanishing solution, and a constant solution. We choose the latter.

We require the perturbation to be isentropic (see [CL02]). This yields

$$\delta_m = 3\theta_0 = \frac{3}{4}\delta_r\,. \tag{1.166}$$

Why do we assume this? If the perturbation is sourced by a single scalar field, then we can only consider a single mode: the isentropic mode.

With these, the 00 EFE becomes

$$\Phi = -2\left[(1 - f_\nu)\theta_0 + f_\nu w_0\right], \tag{1.167}$$

where we must have $f_\nu = \text{const}$.

Adiabaticity implies

$$\phi = -2\theta_0 \implies \delta = -\frac{3}{2}\Phi\,. \tag{1.168}$$

If we had kept the neutrino quadrupole, we would have gotten

$$\Phi = \Psi\left(1 + \frac{2f_\nu}{5}\right). \tag{1.169}$$

f_ν is a constant at any order: it is a number fixed by the number of neutrino species. Friday 2020-4-3, compiled 2020-04-09

We work in the tight coupling limit.

[around equation 39]

We drop the quadrupole term.

The term multiplying $\dot{\tau}$ in eq 40 goes to infinity like $1/\dot{\tau}$: we need to replace it by a finite contribution. Thursday 2020-4-9, compiled 2020-04-09

We still need information about $\ddot{\Psi}$ and Φ. This should be provided by the EFE. We approximate them as being independent of θ_0 and θ_1.

The trick to get θ_0 is to differentiate eq 39 with respect to η, and then take $\dot{\theta}_1$ from the other eq.

So we get eq 42: it describes the evolution of θ_0, on its right hand side we have a forcing term which we denote by $F(k, \eta)$. The factor $R/(1+R)$ is the ratio of the baryon energy density to the total energy density.

This is solved numerically, we want a simple analytic solution.

We basically have a damped wave equation, with a forcing term [43]. This is solved, as usual, by solving the homogeneous equation first. We solve the homogeneous equation with the use of an integrating factor.

People sometimes drop the viscous term: this is a good approximation if we are within the "sound horizon" (which we shall see shortly).

We replace the c_s, which is dependent on time in principle, by its integral in η. (?) The sound horizon is given by

$$r_\eta = . \tag{1.170}$$

Once we have solved the homogeneous equation, we use the Green function method to solve the inhomogeneous equation.

We use the cosine instead of the sine in order to account for the boundary conditions.

The wavenumber corresponding to equality is:

$$k_{\text{eq}} = 0.073 \text{Mpc}^{-1} \Omega_{0m} h^2 . \tag{1.171}$$

So we have that $\theta_0 \sim \cos$, while $\theta_1 \sim \sin$: they are in opposition of phase in the tight coupling limit.

Anisotropic stress means neutrinos. The perturbations enter the horizon in the radiation dominated epoch as long as…

Meszaros effect: perturbations which enter the horizon during radiation dominance are damped.

The gravitational perturbation is constant outside the horizon, it is damped under radiation dominance.

We now add the quadrupole, but we neglect the octupole.

In order to solve these equations, we can apply the same procedure as before, or we can use a trick: we get an approximate dispersion relation, compared to the previous solution we get an imaginary term, which correspond to damping in θ_0 and θ_1.

λ_D tells us how small a perturbation's wavelength can be if it is damped.

1.4.1 Free-streaming

We wish to describe what we expect to see. We discovered we can have anisotropies, so we can have hot and cold spots in the radiation content of the universe.

For photons, the distance in comoving time $\Delta\eta$ and the comoving distance ΔL are equal, since $\mathrm{d}s = 0$.

The quantity $\eta_0 - \eta_*$ is usually called *lookback time*. A multipole of order ℓ is maximized when ℓ is of the order $1/\theta$, where $\theta = k/(\eta_0 - \eta_*)$.

We define a new temporary function $\widetilde{S}$.

The most important equation is equation 48.

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