Gravitational physics notes

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Monday 2020-3-9

0.1 Technical details

Giacomo Ciani. Room 114 DFA 0498277036 or 0498068456 giacomo.ciani@unipd.it Office hours: check by email.

Reading material: slides (to be used as an index of what is treated in the course), Hobson [HEL06], Michele Maggiore [Mag07; Mag18].

This is a general class on gravitational physics and GW, it does not really follow any textbook: the field is young so there is no textbook covering all the necessary topics, really.

The slides will be provided before lectures. There will be no home assignments.

The idea for the exam is that formulas are important, detailed calculations and derivations are not.

The target is to be able to read a research paper on GW and understand it. We will not go into very much detail on any topic: the program of the class is very large.

For the exam: it is a discussion of a GW paper (about 25 min), plus theoretical questions — focusing on the physical meaning, not on tedious derivations. It usually takes a bit less than an hour. The paper is optional.

Off session exams are OK, if the exam is live then its is best if it is organized with several people (2-5 people).

Please fill out the questionnaire on the course before taking the exam.

0.1.1 Topics

Understanding **what gravitational waves are**: how they are described, how they are generated, what is their physical effect.

Interactions of GW with light and matter: ideas, techniques, experiments to detect GW, especially GW interferometers.

Analysis of GW signals.

What we can learn from GW, overview of the most significant recent papers.

Chapter 1

Gravitational waves

1.1 Introduction

Einstein thought the detection of GW impossible; at the time it was thought that they might be a coordinate artifact which could be "gauged away".

Now we can not only *detect* them, we can actually *observe* them, determining their position in the sky and their parameters.

They are a test of GR in *extreme* conditions, where the weak-field approximation does not apply. We can test the properties of matter in these extreme conditions, such as the equation of state for a neutron star.

GW are "ripples" in the metric of spacetime; their production is described by a quadrupole formula: the quadrupole is

$$Q_{jk} = \int \rho x_j x_k \, \mathrm{d}^3 x \,, \tag{1.1.1}$$

and then the perturbation propagates like

$$h_{jk} = \frac{2}{r} \frac{\mathrm{d}^2 Q_{jk}}{\mathrm{d}t^2} \,. \tag{1.1.2}$$

What generates GW are non-spherically symmetric perturbations: by Jebsen-Birkhoff, if we have spherical symmetry there is no perturbation in the vacuum metric. The simplest kind of object which can generate them is a binary system.

The effect of a GW is to "stretch" space by squeezing one direction and stretching a perpendicular one, in an area-preserving way. The typical relative scale of these perturbations is

$$\frac{\Delta L}{L} \sim 10^{-21} \,, \tag{1.1.3}$$

which is *really small*: if we multiply it by the radius of the Earth's orbit we get a length on the order of the size of an atom.

We have different kinds of interferometers for different GW frequency ranges: for now we have only used ground interferometers, but in the works there are also space detectors like LISA, Pulsar Timing Arrays at higher frequencies, and inflation probes.

In **binary systems**, we have different stages in the pulsation: an almost stationary one, the inspiral, the coalescence, and finally the ringdown. The frequency and amplitude both increase up to the coalescence, after it the frequency is almost constant while the amplitude decreases.

In 1959, Joseph Weber proposed a "**resonant bar**" detector. These are based upon a sound principle, and this path was explored for several decades with, for example, AURIGA; the issue was that the sensitivity was insufficient, and these detectors would only be sensitive in a specific high frequency range.

GW were first detected indirectly using **Hulse-Taylor pulsars**: they measured the energy loss of a binary pulsar-NS system, which implied the loss of energy through gravitational wave emission. The famous graph is not a fit line, it is the prediction based upon the measured orbital parameters.

Now we use ground-based laser **interferometers**: they are broad-band (a couple orders of magnitude, from 10 Hz to 1 kHz), they are inherently differential (as opposed to the single-mode excitation of a resonant bar).

We can use Fabry-Pérot cavities in order to amplify effective length, by "storing photons" for several bounces in the interferometer. There is also a power recycling mirror in order for the light not to go back to the laser: with modern lasers and these systems we can get 10 kW of power circulating in the cavities.

We can plot the sensitivity of the interferometers: on the x axis we put the frequency of the incoming wave, and on the y axis we put the amplitude spectral density $h(f)$, which is measured in $\mathrm{Hz}^{-1/2}$.

The curve describes where the noise dominates. We can plot both the theoretical sensitivity, with its various sources, and the measured one.

The signal comes out buried in noise, we must extract it in some way, like by correlating to a standard test signal.

A planned detector is **LISA**, which is a space interferometer but it works differently from the ground-based one, since it cannot reflect the beam back, and since the travel time for the beam is several seconds.

Another way to detect low-frequency GW is by using **Pulsar Timing Arrays**: the idea is to monitor several millisecond pulsars, and compute the difference in the arrival times of their signals; distortions of spacetime will modify the relative distances between us and these.

Also, we have **atomic interferometry** detectors, by having the wavefunctions of atoms do what light does in an interferometer. This is not a mature technology as of now.

We have seen several **BH-BH mergers** and some NS-NS ones, with masses between a few and about $80 M_{\odot}$. With these, people are starting to do studies on the populations of these objects and their formation mechanisms. Unfortunately, with our frequency range we can only see a very short part of the signal, up to a few seconds at most.

With **NS-NS mergers** we can investigate short gamma ray bursts, the formation of heavy elements, the mass of gravitational waves (which in GR is zero), the rate of expansion of the universe.

Friday 2020-3-13, compiled 2020-07-28

1.2 A quick review of GR

We start from special relativity. The "old" way to do transformations are Galilean transformations: in 2D they are

$$
\begin{aligned}
t' &= t && (1.2.1a)\\
x' &= x - vt\,. && (1.2.1b)
\end{aligned}
$$

There are issues with these: they do not respect the equivalence principle and the invariance of the speed of light. So, we move to Lorentz transformations:

$$
\begin{aligned}
ct' &= \gamma(ct - \beta x) && (1.2.2a)\\
x' &= \gamma(x - \beta ct)\,, && (1.2.2b)
\end{aligned}
$$

where $\beta = v/c \leq 1$, c being the speed of light, and $\gamma = 1/\sqrt{1-\beta^2} \geq 1$.

These preserve the spacetime interval, which in our mostly plus metric convention reads:

$$
\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2\,. \tag{1.2.3}
$$

The interval between two events can be spacelike ($\Delta s^2 > 0$), null ($\Delta s^2 = 0$) or timelike ($\Delta s^2 < 0$).

We can express this using an infinitesimal time interval

$$
\mathrm{d}s^2 = \mathrm{d}x^\mu \, \eta_{\mu\nu} \, \mathrm{d}x^\nu \ , \tag{1.2.4}
$$

where we use Einstein summation convention.

We can define the differential *proper time* along a curve, by

$$
c^2 \, \mathrm{d}\tau^2 = -\,\mathrm{d}s^2 = c^2 \, \mathrm{d}t^2 \left(1 - \beta^2\right) = \frac{c^2}{\gamma^2} \, \mathrm{d}t^2 \ , \tag{1.2.5}
$$

which means that $\mathrm{d}\tau = \mathrm{d}t \,/\gamma$. The parameter τ can then be used as a natural *covariant* parametrization of a spacetime curve.

We model spacetime it as a 4D semi-Riemannian manifold with metric signature $(1,3)$. Since it is a manifold, the parametrization of points in spacetime must be a homeomorphism, and we ask for the *transition maps* between two regions of spacetime to be infinitely differentiable. The set of local charts is called an atlas. The charts are maps from $\mathbb{R}^4$ to the manifold.

The coordinates we use are arbitrary: this is very powerful, but it is tricky to find the right ones.

The **metric** is a function of the point at which we are, and (the way it changes) describes the local geometry of the manifold. Only the symmetric part of the metric appears in the spacetime interval, therefore we say that the metric is always symmetric without losing any generality.

The metric is a bilinear form at each point of the manifold, and it transforms as a $(0,2)$ tensor. The components of this tensor in our chosen reference frame are $g_{\mu\nu}$.

In a neighborhood of a point we can always choose a reference frame (Riemann normal coordinates) such that $g_{\mu\nu} = \eta_{\mu\nu}$, and $g_{\mu\nu,\alpha} = 0$ (partial derivatives calculated *at that point*), but the second derivatives $g_{\mu\nu,\alpha\beta}$ cannot all be set to zero.

Vectors in a manifold are defined in the tangent space *at a point*. Intuitively, at each point we can define locally Cartesian coordinates, and the tangent is the space they span.

Formally, we define curves parametrically as functions from the real numbers to the manifold: $X^\mu(\lambda)$. Then, we define the tangent vector to the curve as the *directional derivative* operator along the curve:

$$\vec{v}(f) = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_C = \frac{\partial f}{\partial x^\mu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}, \tag{1.2.6}$$

which associates to any scalar field f its directional derivative. The motivation for this definition, as opposed to just taking the tangent vector to the curve, is the fact that there is no *intrinsic* way to do that.

If we define a curve using a coordinate as a parameter, with the other coordinates staying constant along the curve, this is called a *coordinate curve*.

Vectors defined at different points are in different spaces, we cannot compare them directly.

Tangent vectors to coordinate lines are called coordinate basis vectors $e_{(\mu)}$, where μ is not a vector index but instead it spans the basis vectors. Any vector can be written as a linear combination of these as $\vec{v} = v^\mu e_\mu$. We always have $e_\mu \cdot e_\nu = g_{\mu\nu}$, so, in order to find the components of the scalar product $v \cdot w$ we need to do $v^\mu w^\nu g_{\mu\nu}$. This is because $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}s \cdot \mathrm{d}s = \left(\mathrm{d}x^\mu\, e_\mu\right) \cdot (\mathrm{d}x^\nu\, e_\nu)$.

An *orthonormal basis* in one for which $e_\mu \cdot e_\nu = \eta_{\mu\nu}$. Dual basis vectors e^μ are defined by $e^\mu e_\nu = \delta^\mu_\nu$. We write a co-vector (or dual vector) as a linear combination of these: $v = v_\mu e^{(\mu)}$.

Then, we can raise and lower indices like

$$g_{\mu\nu}v^\mu w^\nu = v \cdot w = v_\mu e^\mu \cdot w^\nu e_\nu = v_\mu w^\nu \delta^\mu_\nu = v_\mu w^\nu\,. \tag{1.2.7}$$

The inverse metric $g^{\mu\nu}$ is defined by the relation $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$.

Tensors are geometrical objects which belong to the dual space of the Cartesian product of n copies of the tangent space and m copies of the dual tangent space. The type of a tensor in this space is then said to be (n, m), and its rank is $n + m$. This definition means that the tensor is a *multilinear* transformation, associating a scalar to n vectors and m covectors in a multilinear way.

Once we have a coordinate system, we can move to another via a coordinate transformation $x'^\mu = x'^\mu(x^\mu)$, and then the differential of the coordinates will transform like

$$x'^\mu = x'^\mu(x^\mu) \implies \mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu\,. \tag{1.2.8}$$

A scalar is something which does not transform: $\phi(x) = \phi'(x')$. A vector's and a covector's components do transform: we find the transformation law by imposing $v = v'$ in components, so that we get

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}V^\nu \qquad \text{and} \qquad V_\mu = \frac{\partial x^\nu}{\partial x'^\mu}V_\nu\,. \tag{1.2.9}$$

This can be generalized to the transformation law of a tensor of arbitrary rank, which will transform with the product of a Jacobian for each upper index and an inverse Jacobian for each lower index.

In order to compute derivatives we need to compare vectors in different tangent spaces: we need to "connect" infinitesimally close tangent spaces, and the tool to do so is indeed called a connection, or covariant derivative. The covariant derivative of a tensor is required to still be a tensor, with a rank which is higher by one:

The covariant derivative of a vector V^μ is defined by introducing the *Christoffel symbols* Γ, which are objects with three indices which do *not* transform like tensors and which account for the "shifting of the basis vectors":

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \,. \tag{1.2.10}$$

For a scalar S the covariant derivative is $\nabla_\alpha S = \partial_\alpha S$.

We require the manifold we work in to be torsionless. A torsionless manifold is one in which

$$[\nabla_\mu, \nabla_\nu] S = 0 \,, \tag{1.2.11}$$

for a scalar field S. This means that

$$\nabla_{[\mu} \nabla_{\nu]} S = \nabla_{[\mu} \partial_{\nu]} S = \partial_{[\mu} \partial_{\nu]} S - \Gamma^\alpha_{[\mu\nu]} \partial_\alpha S = 0 \implies \Gamma^\alpha_{[\mu\nu]} = 0 \,, \tag{1.2.12}$$

so the Christoffel symbols are symmetric in their lower indices.

Parallel transport: intuitively, we move along a curve and keep the angle with respect to the tangent vector constant. Formally, if u^μ is the tangent vector to the curve and V^μ is the vector we want to transport, we set $u^\mu \nabla_\mu V^\nu = 0$.

This parallel-transport is path-dependent: in general a vector which is transported along a curve does not come back to itself.

The Riemann tensor $R^\alpha_{\beta\mu\nu}$ is defined from the commutator of the covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\beta\mu\nu} V^\beta \,, \tag{1.2.13}$$

and it can be expressed in terms of the Christoffel symbols as

$$R^\mu_{\nu\rho\sigma} = -2 \left(\Gamma^\mu_{\nu[\rho,\sigma]} + \Gamma^\beta_{\nu[\rho} \Gamma^\mu_{\sigma]\beta} \right) . \tag{1.2.14}$$

This tensor measures the curvature of the manifold: if it is zero, then the manifold is flat ($g_{\mu\nu} = \eta_{\mu\nu}$). We define its trace $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ as the Ricci tensor, whose trace $R = R_{\mu\nu} g^{\mu\nu}$ is called the Ricci scalar, or curvature scalar.

The Riemann tensor has several symmetries, both differential (related to its derivatives) and not (antisymmetries and symmetries of its indices) [HEL06, eqs. 7.14-7.18], making it so that its free components in N spatial dimensions are not N^4 but instead $N^2(N^2-1)/12$. In 4D, this means 20 free components.

Geodesics: they are "the straightest possible path between two points"; they stationarize the proper length. Formally, they are curves whose tangent vector is parallel-transported along the curve (it "always points in the same direction").

The path that a massive particle follows in the absence of external forces is a geodesic. We can describe the evolution of the separation between two nearby particles which follow geodesics: this is described by the equation of geodesic deviation. We take a geodesic x^μ and another $y^\mu = x^\mu + \xi^\mu$, with ξ^μ being (at least initially) small. Let us call the starting point of x^μ P and the starting point of y^μ Q, also let us take a coordinate system in which $\Gamma^\mu_{\nu\rho}\Big|_P = 0$. This can always be done, but note that the *derivatives* of the Christoffel symbols cannot all be set to zero.

So, we can write the geodesic equation for the two curves at their starting points as

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}u^2}\bigg|_P = 0 \qquad \text{and} \qquad \left(\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}u^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}y^\nu}{\mathrm{d}u}\frac{\mathrm{d}y^\rho}{\mathrm{d}u}\right)\bigg|_Q = 0\,, \tag{1.2.15}$$

where u is the tangent vector to the geodesics. We approximate the Christoffel symbols to first order as

$$\Gamma^\mu_{\nu\rho}\Big|_Q = \xi^\alpha \partial_\alpha\, \Gamma^\mu_{\nu\rho}\Big|_P\,. \tag{1.2.16}$$

If we subtract the two and only keep the first order in ξ, we get

$$0 = \left(\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}u^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}y^\nu}{\mathrm{d}u}\frac{\mathrm{d}y^\rho}{\mathrm{d}u}\right) - \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}u^2} = \frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}u^2} + \xi^\alpha\partial_\alpha\Gamma^\mu_{\nu\rho}\frac{\mathrm{d}(x^\nu+\xi^\nu)}{\mathrm{d}u}\frac{\mathrm{d}(x^\rho+\xi^\rho)}{\mathrm{d}u} \tag{1.2.17}$$

$$= \frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}u^2} + \xi^\alpha\partial_\alpha\Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho + \mathcal{O}(\xi^2) \tag{1.2.18}$$

A dot denotes u differentiation.

$$\approx \ddot{\xi}^\mu + \left(\partial_\alpha\Gamma^\mu_{\nu\rho}\right)\dot{x}^\nu\dot{x}^\rho\xi^\alpha = 0\,, \tag{1.2.19}$$

so the first term is not an intrinsic derivative: that would be given by

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} = \frac{\mathrm{D}}{\mathrm{D}u}\left(\dot{\xi}^\mu + \Gamma^\mu_{\nu\rho}\xi^\nu\dot{x}^\rho\right) + \mathcal{O}(\xi^2) = \ddot{\xi}^\mu + \dot{x}^\alpha\partial_\alpha\Gamma^\mu_{\nu\rho}\xi^\nu\dot{x}^\rho + \mathcal{O}(\xi^2) \tag{1.2.20}$$

$$= \ddot{\xi}^\mu + \dot{x}^\nu\partial_\nu\Gamma^\mu_{\alpha\rho}\xi^\alpha\dot{x}^\rho + \mathcal{O}(\xi^2)\,, \tag{1.2.21}$$

Relabeled the contracted indices.

where the two derivatives are defined by

$$\frac{\mathrm{D}}{\mathrm{D}u} = \dot{x}^\mu\nabla_\mu \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}u} = \dot{x}^\mu\partial_\mu\,. \tag{1.2.22}$$

We can then write the differential equation we have found by inserting this expression for $\ddot{\xi}$:

$$0 = \frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} + \left(\partial_\alpha\Gamma^\mu_{\nu\rho} - \partial_\nu\Gamma^\mu_{\alpha\rho}\right)\xi^\alpha\dot{x}^\nu\dot{x}^\rho = \frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}u^2} + R^\mu_{\nu\alpha\rho}\xi^\alpha\dot{x}^\nu\dot{x}^\rho\,. \tag{1.2.23}$$

In the last step we used the fact that in the frame we chose, a Local Inertial Frame, the Christoffel symbols are zero so $R = \partial\Gamma + \Gamma\Gamma$ simplifies to $R = \partial\Gamma$.

The gravitational field's dependence on the matter content of the universe is described by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}\,, \tag{1.2.24}$$

which follow from the assumptions that

1. they should be a local tensorial equation;
2. they should relate $T_{\mu\nu}$ with $g_{\mu\nu}$ and its derivatives;
3. they should reduce to Newton's equation for the gravitational potential ϕ, $\nabla^2\phi = 4\pi G\rho$, in the weak field limit.

They look rather simple, but there is a lot of hidden complexity: the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/2$ is calculated from the Riemann tensor by taking a trace; the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$ is calculated from the Christoffel symbols by differentiating and multiplying them ($R \sim \partial\Gamma + \Gamma\Gamma$), the Christoffel symbols are calculated from the metric by differentiating it and multiplying it by the inverse $\Gamma \sim g^{-1}\partial g$.

In this course we will be using the sign conventions adopted by Misner, Thorne and Wheeler [MTW73, page 3]; some authors adopt different signs for

1. the metric $\eta_{\mu\nu} = \pm\mathrm{diag}(-1,1,1,1)$;
2. the Riemann tensor $R^{\mu}_{\nu\rho\sigma} = \mp 2\left(\Gamma^{\mu}_{\nu[\rho,\sigma]} + \Gamma^{\alpha}_{\nu[\rho}\Gamma^{\mu}_{\sigma]\alpha}\right)$;
3. the Einstein Equations: $G_{\mu\nu} = \pm T_{\mu\nu}/M_P^2$.

All of these conventions are equivalent, but one must be careful when comparing different sources.

1.3 Linearized GR

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Let us assume that there exists a reference frame in which our metric tensor is almost flat: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\left|h_{\mu\nu}\right| \ll 1$.

This is a coordinate dependent statement: however the physical situation is clear — almost flat spacetime — and the way we will proceed is to work to linear order in $h_{\mu\nu}$.

Do note that the inverse metric is given by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, and that we can raise and lower the indices of $h_{\mu\nu}$ using $\eta_{\mu\nu}$, since the corrections would be second order.

Choosing a reference in which these components are small limits our gauge freedom: we will only be able to do transformations which preserve the condition. Let us now discuss explicitly **which transformations we will still be able to use**.

We will be able to apply global Lorentz transformations Λ, which act on the metric as:

$$g' = \Lambda^{-1}\Lambda^{-1}g = \Lambda^{-1}\Lambda^{-1}(\eta + h) = \eta + \Lambda^{-1}\Lambda^{-1}h\,, \tag{1.3.1}$$

so the flat metric part does not change, while h changes to $h' = \Lambda^{-1}\Lambda^{-1}h$. We are omitting indices for clarity, they are in the usual positions.

A more general class of transformation is given by *infinitesimal translations*, which can be expressed as

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu\,, \tag{1.3.2}$$

and whose Jacobian looks like

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu}\,. \tag{1.3.3}$$

We ask that $\partial\xi$ is small — formally, the first order in $\partial_{\mu}\xi_{\nu}$ is the same as the first order in $h_{\mu\nu}$.

This yields, always to first order:

$$\begin{aligned} g'_{\mu\nu} &= \left(\delta^{\rho}_{\mu} - \partial_{\mu}\xi^{\rho}\right)\left(\delta^{\sigma}_{\nu} - \partial_{\nu}\xi^{\sigma}\right)\left(\eta_{\rho\sigma} + h_{\rho\sigma}\right) && (1.3.4) \\ &= \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\xi^{\rho}\eta_{\rho\sigma} - \partial_{\nu}\xi^{\sigma}\eta_{\rho\sigma} && (1.3.5) \\ &= \eta_{\mu\nu} + h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}\,. && (1.3.6) \end{aligned}$$

For an alternative reference on this derivation, see the notes on General Relativity in the course by Marco Peloso [TM20, section 10]. The crucial thing here is that our transformation left the metric in the form "flat + first-order infinitesimal", so we still are satisfying the assumptions we set at the start.

Now, we wish to linearize the Riemann tensor: we must start from the Christoffel symbols. We can discard the derivatives of the flat metric and substitute the inverse metric at the start of the expression with the flat one, since the parenthesis is already first order:

$$\begin{aligned} \Gamma^{\sigma}_{\mu\nu} &= \frac{1}{2}g^{\sigma\rho}\left(2g_{\rho(\mu,\nu)} - g_{\mu\nu,\rho}\right) && (1.3.7a) \\ &= \frac{1}{2}\left(\partial_{\mu}h^{\sigma}_{\nu} + \partial_{\nu}h^{\sigma}_{\mu} + \partial^{\sigma}h_{\mu\nu}\right), && (1.3.7b) \end{aligned}$$

and now in the Riemann tensor $R = \partial\Gamma + \Gamma\Gamma$ the $\Gamma\Gamma$ terms are second order in h, so we ignore them. Then, we get the simplified expression

$$R^{\sigma}_{\mu\nu\rho} = \frac{1}{2}\left(\partial_{\nu}\partial_{\mu}h^{\sigma}_{\rho} + \partial_{\rho}\partial^{\sigma}h_{\mu\nu} - \partial_{\nu}\partial^{\sigma}h_{\mu\rho} - \partial_{\rho}\partial_{\mu}h^{\sigma}_{\nu}\right), \tag{1.3.8}$$

so the Ricci tensor — which we will set to zero in the vacuum — will be

$$R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma} = \frac{1}{2}\left(\partial_{\nu}\partial_{\mu}h + \Box h_{\mu\nu} - \partial_{\nu}\partial_{\sigma}h^{\sigma}_{\mu} - \partial_{\sigma}\partial_{\mu}h^{\sigma}_{\nu}\right), \tag{1.3.9}$$

and the Ricci scalar is

$$R = \eta^{\mu\nu}R_{\mu\nu} = \Box h - \partial_{\nu}\partial_{\sigma}h^{\sigma\nu}\,, \tag{1.3.10}$$

where h is the trace of the perturbation, $h = h^{\sigma}_{\sigma}$, while $\Box = \partial_{\mu}\partial^{\mu}$ is the Dalambertian operator. The field equations read

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &\approx \frac{1}{2}\left[\partial_{\nu}\partial_{\mu}h + \Box h_{\mu\nu} - \partial_{\nu}\partial_{\sigma}h^{\sigma}_{\mu} - \partial_{\sigma}\partial_{\mu}h^{\sigma}_{\nu} - \eta_{\mu\nu}\left(\Box h - \partial_{\nu}\partial_{\sigma}h^{\sigma\nu}\right)\right] && (1.3.11) \\ &= -\frac{1}{M_P^2}T_{\mu\nu}\,, && (1.3.12) \end{aligned}$$

The metric multiplying h can be written as the flat one, since h is first order already.

where $M_P = 1/\sqrt{8\pi G}$ in natural units is the reduced Planck mass.

We define the trace-reversed perturbation as

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{1.3.13}$$

so the doubly-trace reverse is the perturbation itself, $\overline{\overline{h}}_{\mu\nu} = h_{\mu\nu}$, and the trace-reversed trace is the negative of the trace: $\overline{h} = -h$. In terms of this, the linearized equations read

$$\Box\overline{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\overline{h}^{\rho\sigma} - \partial_\nu\partial_\rho\overline{h}^\rho_\mu - \partial_\mu\partial_\rho\overline{h}^\rho_\nu = -2\frac{T_{\mu\nu}}{M_P^2}\,, \tag{1.3.14}$$

which we can simplify greatly using our gauge freedom: we shall use the so-called Lorenz gauge[1].

How does the trace-reverse perturbation transform?

$$\begin{aligned} \overline{h}'^{\mu\rho} &= h^{\mu\rho} - 2\partial^{(\mu}\xi^{\rho)} - \frac{1}{2}\eta^{\mu\rho}\left(h - 2\partial_\sigma\xi^\sigma\right) && \text{(1.3.15a)}\\ &= \overline{h}^{\mu\rho} - 2\partial^{(\mu}\xi^{\rho)} + \eta^{\mu\rho}\partial_\sigma\xi^\sigma\,. && \text{(1.3.15b)} \end{aligned}$$

The derivatives of the new and old perturbations differ by

$$\partial_\rho\overline{h}'^{\mu\rho} - \partial_\rho\overline{h}^{\mu\rho} = \partial_\rho\left(-\partial^\mu\xi^\rho - \partial^\rho\xi^\mu + \eta^{\mu\rho}\partial_\sigma\xi^\sigma\right) = -\Box\xi^\mu\,, \tag{1.3.16}$$

so we can choose our gauge with a transformation defined by ξ^μ such that $\partial_\rho\overline{h}'^{\mu\rho}$, since the field $\Box\xi^\mu$ can be chosen arbitrarily with a suitable choice of ξ^μ.

This means we can remove all the $\partial\overline{h}$ terms and find:

$$\boxed{\Box\overline{h}_{\mu\nu} = -\frac{2T_{\mu\nu}}{M_P^2}\,,} \tag{1.3.17}$$

Comments on the linearized equations Since we expanded in $\eta_{\mu\nu}$, the quantities $h_{\mu\nu}$ have a geometric meaning, but we are treating them as 16 scalar fields on a flat background.

When we look at the geodesic equations, we get a prediction of the gravity having no effect on matter: there is no *backreaction*, and if we want to model the effect of GW energy and angular momentum loss we will have to insert them by hand. We are treating gravity as a linear theory, so we have the superposition principle, according to which the fields due to different particles can be added.

We are ignoring the physical principle that "gravity gravitates": curvature of spacetime is associated to a SEMT in a nonlinear way.

[1] This is not the same as Loren*t*z, after whom Lorentz covariance is called.

GW in empty space We set $T_{\mu\nu}$ to zero, so we get

$$\Box \overline{h}_{\mu\nu} = 0\,, \tag{1.3.18}$$

which is the usual wave equation: its solutions are superpositions of plane waves $\overline{h}_{\mu\nu} = A_{\mu\nu} e^{i k_\lambda x^\lambda}$.

In general $A_{\mu\nu}$ is symmetric, constant, complex. k_λ is constant and real, and by taking the derivative we find that we must have

$$\eta^{\rho\sigma} k_\rho k_\sigma A_{\mu\nu} e^{i k\cdot x} \implies k^2 = 0\,, \tag{1.3.19}$$

which means that the wave travels at light speed, since $k^\lambda = (\omega/c, \vec{k})$.

In order to have these conditions, we must still impose the Lorenz gauge condition we chose in the derivation:

$$\partial_\mu \overline{h}^{\mu\nu} = A^{\mu\nu} k_\mu e^{i k\cdot x} = 0 \implies A^{\mu\nu} k_\nu = 0\,. \tag{1.3.20}$$

The conjugate of the wave equation also holds, so after our manipulations we will always be able to take the real part.

We still have gauge freedom: we can perform transformations if they satisfy $\Box \xi^\mu = 0$, so that we do not alter the value of $\partial_\rho \overline{h}^{\mu\rho}$. We define

$$\xi^{\mu\rho} = \partial^\mu \xi^\rho + \partial^\rho \xi^\mu - \eta^{\mu\rho} \partial_\sigma \xi^\sigma\,, \tag{1.3.21}$$

which satisfies the wave equation $\Box \xi^{\mu\rho} = 0$ if ξ^μ does, since the Dalambertian commutes with the other derivatives.

So, if $\overline{h}^{\mu\rho}$ satisfies the vacuum field equations, then $\overline{h}'^{\mu\nu} = \overline{h}^{\mu\nu} - \xi^{\mu\nu}$ also does.

Then, we can use the 4 functions ξ^μ to set 4 constraints on $\overline{h}^{\mu\rho}$: we choose to set

$$\overline{h}^{0i}_{TT} = 0 \tag{1.3.22a}$$

$$\overline{h}_{TT} = 0\,, \tag{1.3.22b}$$

which conveniently means that $\overline{h}_{\mu\nu} = h_{\mu\nu}$. This is called Transverse-Traceless gauge. We want to write out **all the gauge constraints** on our perturbation. The Lorenz gauge $\partial_\rho \overline{h}^{\mu\rho} = 0$ consists of four equations: the $\mu = 0$ one reads

$$0 = \partial_\rho \overline{h}^{0\rho}_{TT} = \partial_0 \overline{h}^{00}_{TT}\,, \tag{1.3.23}$$

$h^{i0} = 0$.

which means that the metric element $\overline{h}^{00}$ is constant, so we can set it to zero, since a constant in the metric is not relevant for our study of oscillations.[2] Right now, the only nonzero components are the h_{ij}, which must be traceless and symmetric.

[2] A constant can be reabsorbed into the background metric: it amounts to measuring all time with a slightly slower or faster clock.

Strictly speaking, we have only shown that h^{00} is a constant with respect to time, but could it have nontrivial space dependence? Well, we know that $\Box h^{00} = 0$, so if the time derivative vanishes this reduces to $\nabla^2 h^{00} = 0$. The Laplacian is an invertible operator, and this means that $h^{00} = \text{const}$.

The other $\mu = j$ Lorenz gauge conditions are the three constraints

$$0 = \partial_\rho \overline{h}^{j\rho}_{TT} = \partial_i \overline{h}^{ji}_{TT}\,, \tag{1.3.24}$$

which means that, of the 5 potentially free components of the traceless symmetric h^{ij}, we actually have only 2 true degrees of freedom.

Now, if we align our reference frame so that $\vec{k} = k\hat{z}$ we will have $k^\mu = (k, 0, 0, k)$; also, the matrix $A_{\mu\nu}$ will need to be symmetric and satisfy $A^{ij}k_j = 0$, so $A^{i3} = 0$.

This means that the only nonzero components are A_{ij}, with i, j between 1 and 2; also $A_{11} = -A_{22} \overset{\text{def}}{=} h_+$ and $A_{12} = A_{21} \overset{\text{def}}{=} h_\times$. Then, in full generality under our gauge choices we shall have

$$\overline{h}^{\mu\nu}_{TT} = h^{\mu\nu}_{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{ik(t-z)}\,. \tag{1.3.25a}$$

Claim 1.3.1. *For a generic direction of propagation, we can define a projector onto the directions orthogonal to the direction of propagation k_i: $P_{ij} = \delta_{ij} - n_i n_j$, where $n_i = k_i / \left|\vec{k}\right|$.*

Proof. The conditions we need to check for this to be a projector onto the directions we want are:

1. idempotency: $P_{ij}P_{jk} = P_{ik}$;
2. orthogonality: $P_{ij} = P_{ji}$;
3. that it projects on the right space: $P_{ij}k_j = 0$ and $P_{ij}y_j = y_j$ if $y_j k_j = 0$.

The computations are:

$$P_{ij}P_{jk} = \left(\delta_{ij} - n_i n_j\right)\left(\delta_{jk} - n_j n_k\right) \tag{1.3.26}$$

$$= \delta_{ik} - 2n_i n_k + n_i \underbrace{n_j n_j}_{+1} n_k\,, \tag{1.3.27}$$

where we used the fact that n_j is chosen to be normalized, and we are using the $- + ++$ metric convention. Note that k_i is not the full four-vector k^μ, which is null, but instead it is only its spatial part. As for projecting on the right space, we have:

$$P_{ij}k_j = k_i - k_i \underbrace{k_j k_j}_{+1} = 0 \qquad \text{and} \qquad P_{ij}y_j = y_i - k_i \underbrace{k_j y_j}_{0} = y_i\,. \tag{1.3.28}$$

□

Using this P_{ij} we might try to calculate the transverse traceless A^{TT}_{ij} like

$$A^{TT}_{ij} \overset{?}{=} P^k_i P^l_j A_{kl}\,, \tag{1.3.29}$$

but the resulting tensor is not in general traceless: we need to subtract the trace, so the correct expression reads

$$A^{ij}_{TT} = \underbrace{\left(P^i_k P^j_l - \frac{1}{2} P^{ij} P_{kl} \right)}_{\Lambda_{ij,kl}} A^{kl} \,. \tag{1.3.30}$$

The tensor we defined,

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \,, \tag{1.3.31}$$

is a projector, it is transverse with respect to all of its indices, and its traces where we set $i = j$ or $k = l$ are zero [Mag07, eqs. 1.36 to 1.39].

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1.4 The physical effects of gravitational waves

We want to discuss how we can build instruments which can detect gravitational waves.

An open question for decades (from 1916 to 1957) was to theoretically determine whether the effects of gravitational waves could be removed using a proper gauge choice. At a conference in Chapel Hill a thought experiment was presented describing a non-removable gravitational wave effect: two beads on a stick which is positioned orthogonal to the GW propagation. As the GW passes by they move since their proper distance changes (while the stick is held in place by atomic forces), so they dissipate energy.

What happens to free particles in the TT gauge? The geodesic equation for the spatial indices reads

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \,, \tag{1.4.1}$$

where the parameter τ parametrize our curve. We assume that the particle starts out at rest: then its four-velocity is $\mathrm{d}x^\mu / \mathrm{d}\tau = (\mathrm{d}x^0 \big/ \mathrm{d}\tau \,, \vec{0})$. So, we get the simplification

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\Gamma^i_{00} \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 \,, \tag{1.4.2}$$

and in linearized gravity

$$\Gamma^i_{00} \approx \frac{1}{2} \left(2\partial_0 h^i_0 - \partial^i h_{00} \right) = 0 \tag{1.4.3}$$

if we use the TT gauge. This means that the derivative of the velocity is zero: so, the velocity of a stationary particle remains zero indefinitely. Let us consider geodesic deviation between two particles instead: say that the first particle has the geodesic $x(\tau)$ and the second is $x(\tau) + \xi(\tau)$. Their geodesic equations will read

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{1.4.4a}$$

$$\frac{\mathrm{d}^2(x^\sigma + \xi^\sigma)}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\mu\nu}(x+\xi)\frac{\mathrm{d}(x^\mu + \xi^\mu)}{\mathrm{d}\tau}\frac{\mathrm{d}(x^\nu + x^\nu)}{\mathrm{d}\tau} = 0\,, \tag{1.4.4b}$$

which we can expand to first order using the perturbative expression: $\Gamma^\sigma_{\mu\nu}(x+\xi) = \Gamma^\sigma_{\mu\nu}(x) + \partial_\gamma \Gamma^\sigma_{\mu\nu}\xi^\gamma$. We use this, keep only the first order terms, subtract equation (1.4.4a) and finally get

$$\frac{\mathrm{d}^2\xi^\sigma}{\mathrm{d}\tau^2} + 2\Gamma^\sigma_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma\partial_\gamma\Gamma^\sigma_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0\,, \tag{1.4.5}$$

so if we restrict ourselves to only spatial components, and assume that the particles start out stationary we get, at $\tau = 0$:

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}\tau^2} = -2\Gamma^i_{0\nu}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma\partial_\gamma\Gamma^i_{00}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \tag{1.4.6}$$

$$= -2c\Gamma^i_{0\nu}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau} + \xi^\gamma\partial_\gamma\Gamma^i_{00}c^2\,, \tag{1.4.7}$$

We are keeping $c \neq 1$ here.

so, using the expressions for the Christoffel symbols in the TT gauge, where $\Gamma^i_{00} = 0$ and $\Gamma^i_{0\nu}$ is nonzero only for $\nu = j$, we get[3]

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}\tau^2} = -2c\Gamma^i_{0j}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau} = -c\partial_0 h^{ij}\frac{\mathrm{d}\xi^j}{\mathrm{d}\tau}\,, \tag{1.4.9}$$

but $\mathrm{d}\xi^j \big/ \mathrm{d}\tau$ is zero if evaluated at $\tau = 0$ for parallel geodesics! So, parallel geodesics remain parallel: if the separation initially is stationary, it will remain so.

The issue is that in the TT gauge we are using a special set of coordinates which "follow" the gravitational wave. We see **no change in coordinate distance** since the coordinates are moving around with the gravitational wave: we did a coordinate change using ξ^μ satisfying $\Box\xi^\mu = 0$, so the coordinates are harmonically moving, together with the GW.

It is like we defined wave-like coordinates, "gauging away" the wave-like motion.

This is only an issue with our coordinates: the physically measurable quantities are *proper distances*, not coordinate distances, which in general are computed as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}x^2 + h^{TT}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,. \tag{1.4.10}$$

Let us apply this to the case of a GW propagating along the z axis, for two particles initially separated along the x axis, whose coordinates are x_1 and x_2 (initially and also later, since as we saw the coordinate distance does not change in the TT gauge). The full metric perturbation looks like

$$h^{\mu\nu}_{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{i\omega(t-z/c)}\,, \tag{1.4.11a}$$

[3] We use the fact that, since in the TT gauge $h^0_\mu = 0$,

$$\Gamma^i_{0j} = \frac{1}{2}\left(\partial_0 h^i_j + \partial_j h^i_0 - \partial^i h_{0j}\right) = \frac{1}{2}\partial_0 h^i_j\,. \tag{1.4.8}$$

then the distance, in the case of an h_+ polarized wave, becomes

$$s = (x_1 - x_2)\sqrt{1 + h_+ \cos(\omega t)} \approx (x_1 - x_2)\left(1 + \frac{1}{2}h_+ \cos(\omega t)\right). \tag{1.4.12}$$

So, the amplitude of the oscillation in the distance is given by $h_+/2$. For two general events separated by the spacelike vector L^μ, whose norm is $L > 0$:

$$s^2 = \left(\eta_{\mu\nu} + h_{\mu\nu}\right)L^\mu L^\nu \approx L\left(1 + \frac{1}{2L^2}h_{ij}L^i L^j\right). \tag{1.4.13}$$

We would, however, like to work in coordinates which do not oscillate with the GW.

Free-falling frames

The useful frame to define is the *free-falling frame*, whose coordinates are rigid and not perturbed by the GW.

In order to build such a frame **in theory**, we will need to define 4 orthogonal vectors on the point P:

$$\eta_{\mu\nu}e^\mu_\alpha e^\nu_\beta = \eta_{\alpha\beta}\,. \tag{1.4.14}$$

Consider a geodesic through point P whose tangent vector at P is a unit vector $\hat{n}$. If this unit vector is spacelike, we parametrize the geodesic by s (defined with $\mathrm{d}s^2$, from the metric), if it is timelike we parametrize it with τ (defined by $\mathrm{d}\tau^2 = -\mathrm{d}s^2$). We denote as λ either of s or τ.

Now, the coordinates of point Q are generically $\lambda\hat{n}$, if the geodesic starting with unit vector $\hat{n}$ reaches Q when its parameter is λ.

We can reach almost every point this way, the points which are only connected through null geodesics to P can be reached by continuity, and in a small enough region the coordinates of a point Q are unique — that is, the geodesics do not cross.

In this frame, then, $g_{\mu\nu}(P) = \eta_{\mu\nu}(P)$; also, in the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\,, \tag{1.4.15}$$

we have that the second derivatives are zero since $x^\mu(\lambda)$ is linear in λ, so we must have $\Gamma^\mu_{\nu\rho}n^\nu n^\rho = 0$. This must be true for any unit vector n^μ, therefore we have $\Gamma^\mu_{\nu\rho} = 0$. The linear system giving $g_{\mu\nu,\rho}(P)$ from $\Gamma^\mu_{\nu\rho}$ is nondegenerate, so the first derivatives of the metric also vanish: $g_{\mu\nu,\rho}(P) = 0$.[4] These are called **Riemann normal coordinates**.

The conditions on the metric and its derivatives only hold at the point. We can do slightly better with *Fermi normal coordinates*, where we require a gyroscope's angular momentum to be parallel-transported along the geodesics, so that an observer moving along a geodesic is indeed free-falling.

[4] Do note that this reasoning works only at the point, since if we moved along a geodesic we do not have access to the other unit vectors anymore (in these coordinates).

This better be so: otherwise we would have proven that the first derivatives of the metric are zero in a neighborhood of a generic point P, so the metric is constant in the whole neighborhood, which is nonphysical.

How do we make such a frame **experimentally**? We might think to use free-falling particles, and put them in orbit. This is not actually that simple. A satellite which accomplishes this task is called a *drag-free satellite*.

Consider a particle orbiting the Sun. The Sun's radiation pressure pushes the particle away from a geodesic. The way to solve this issue is to put a thrusted spacecraft around our test mass to balance the Sun's radiation pressure, constantly measuring the distance to the test mass without touching it, and then balancing the thrusters by keeping at a constant distance from it.

LISA's drag-free navigation

This is the idea behind the satellites making up the space-based GW interferometer LISA (which we will discuss in greater detail later in the course). The distances between the spacecrafts should be about 5 Gm apart. We do not measure the distances between the spacecrafts, but instead the distances between the test masses inside them, which are 2 kg, 4.6 cm side, gold-platinum shielded cubes.

The interferometric measurements have pico-meter (10^{-12} m) sensitivity. It takes about 30 s for light to move between the mirrors: this time-delayed interferometry needs special consideration.

We have a *gravitational reference sensor*, a cubic shell around the cube: we keep measuring the distances between the two. We also need to precisely discharge the masses with a Charge Management System, otherwise electrostatic forces are too strong. Also, the thrusters need to be very weak, on the order of the μN.

The LISA Pathfinder mission successfully tested all of these technologies, except for time-delayed interferometry. It only used one spacecraft, and measured how well the drag-free navigation worked.

In the final mission there will be two masses inside each satellite, so we will need to account for the gravitational pull between them. Even accounting for this by relaxing the acceleration precision requirement 10-fold, the results of LISA Pathfinder were exceptional.

Let us discuss the sources of noise: at high frequencies, inertia prevents a force from creating significant displacement. This applies to external forces, not to gravitational forces, since the latter are proportional to the mass. So, there is a limit at high frequencies because of our inability to measure that fast. At low frequencies, it is easy to measure, but it is hard to verify whether the mass is indeed in free fall. We can also have issues with the parasitic coupling of the test mass to the spacecraft.

In the end, it was verified that we can do

$$S_a^{1/2} \leq 3 \times 10^{-14}\,\mathrm{m/s^2/\sqrt{Hz}} \tag{1.4.16}$$

at 1 mHz. Solar radiation pressure is two orders of magnitude higher. The LISA Pathfinder mission greatly outperformed its original requirements — see figure 6 in the paper published by the LISA Pathfinder collaboration [LIS17].

1.4.1 Proper detector frame

Let us now come back to theory by discussing the *proper detector frame*: coordinates defined by a rigid ruler. Rigid rulers do not really exist, but we can approximate it well enough. If the gravitational pull is small compared to the restoring forces in the ruler, then its length will approximately not change.

Let us put ourselves in a free-falling frame in Fermi local coordinates, so that in the origin the metric is flat. Then, we can expand it to second order in the spatial coordinates

$$\begin{aligned} g_{\mu\nu}(x) &\approx g_{\mu\nu}(0) + x^i \partial_i \, g_{\mu\nu}\Big|_{x=0} + \frac{1}{2} x^i x^j \, \partial_i \partial_j g_{\mu\nu}\Big|_{x=0} + \dots & (1.4.17a) \\ &= \eta_{\mu\nu} + \frac{1}{2} x^i x^j \, \partial_i \partial_j g_{\mu\nu}\Big|_{x=0} \,, & (1.4.17b) \end{aligned}$$

The derivatives of the metric vanish in the free-falling frame.

which we can rewrite in terms of the Riemann tensor by making use of the expression of the Riemann tensor in the LIF, which is

$$R_{iklm} = \frac{1}{2} \left(\partial_k \partial_l g_{im} + \partial_i \partial_m g_{kl} - \partial_k \partial_m g_{il} - \partial_i \partial_l g_{km} \right) , \tag{1.4.18}$$

we get

$$\mathrm{d}s^2 \approx -c^2 \,\mathrm{d}t^2 \left(1 + R_{0i0j} x^i x^j \right) - 2c \,\mathrm{d}t \,\mathrm{d}x^i \left(\frac{2}{3} R_{0ijk} x^j x^k \right) + \mathrm{d}x^i \,\mathrm{d}x^j \left(\delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l \right) . \tag{1.4.19}$$

The corrections to the flat metric are of the order $\mathcal{O}\left(r^2 / L_B^2\right)$, where r^2 is the square distance from the origin, while L_B is the typical spatial scale of the variation of the metric, such that $R_{0ijk} = \mathcal{O}\left(L_B^{-2}\right)$. This L_B is the wavelength of the GW, if we are describing a GW.

So, the flat coordinate description works as long as the scale of the region we are describing is very small compared to the characteristic scale of the variations in the metric.

Geodesic deviation in the proper detector frame

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What happens in this frame? The equation of geodesic deviation can be calculated from (1.4.6):

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + 2\Gamma^i_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} + \xi^\sigma \left(\partial_\sigma \Gamma^i_{\nu\rho} \right) \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} & (1.4.20a) \\ &= \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + \xi^j \left(\partial_j \Gamma^j_{00} \right) \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 , & (1.4.20b) \end{aligned}$$

where we made the nonrelativistic approximation where the spacelike components of the four-velocity are negligible, and accounted for the fact that in this frame we have

$$\Gamma^\mu_{\nu\rho} = 0 \qquad \text{and} \qquad \partial_0 \Gamma^i_{0j} = 0 \implies R^i_{0j0} = \partial_j \Gamma^i_{00} \,, \tag{1.4.21}$$

so we can write the geodesic equation in terms of the Riemann tensor as:

$$0 = \frac{\mathrm{d}^2 \xi^i}{\mathrm{d}\tau^2} + R^i_{0j0} \xi^j \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \right)^2 , \tag{1.4.22}$$

and since in linearized gravity the Riemann tensor is *invariant* (rather than covariant) under coordinate transformations[5] such as those we used to move between the TT gauge and the detector frame, we can compute it in the TT gauge starting from equation (1.4.18):

$$R^i_{0j0} = \frac{1}{2} \left(\partial_j \partial_0 h^i_0 + \partial_0 \partial^i h_{0j} - \partial_j \partial^i h_{00} - \partial_0 \partial_0 h^i_j \right) = R_{i0j0} \tag{1.4.24}$$

$$= -\frac{1}{2} \partial_0 \partial_0 h_{ij} = -\frac{1}{2} \ddot{h}^{TT}_{ij} , \tag{1.4.25}$$

so our final result for the geodesic deviation equation in the detector frame is:

$$\boxed{\ddot{\xi}^i = \frac{1}{2} \ddot{h}^{TT}_{ij} \xi^j ,} \tag{1.4.26}$$

which is physically significant since we can interpret the effect of the GW as that of a Newtonian force, given by

$$F^i = \frac{m}{2} \ddot{h}^{TT}_{ij} \xi^j . \tag{1.4.27}$$

This seems great! We can have particles move under the influence of the GW, work with h_{ij} in the simple TT gauge, while still being in almost flat spacetime.

However, to get here we made some approximations, and we need to check whether they are justified. We imposed $r^2 / L_B^2 \ll 1$, where L_B is the scale of the variations of the metric while r is the scale of our detector. For ground-based detectors which are $\sim 3\,\mathrm{km}$ long and sensitive in the $\sim 100\,\mathrm{Hz}$ range (corresponding to $\lambda \sim 3000\,\mathrm{km}$) this is perfectly fine. For space-based detectors like LISA it is not!

What we have calculated applies to proper distances as well, which are the same as coordinate distances up to first order in our coordinates. So, since proper distances are invariant, we have an approximate expression which we can use in general, by substituting s for ξ in the differential equation.

[5] This can be seen by plugging the transformation law which is allowed in linearized gravity, $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu} h_{\nu)}$, into the LIF expression for the Riemann tensor,

$$R_{\mu\nu\rho\sigma} = -2 g_{[\mu|[\rho|,|\nu]\sigma]} , \tag{1.4.23}$$

which yields no change (the expression is a compact way to write that the indices to antisymmetrize are both $\mu\nu$ and $\rho\sigma$). This is discussed by Maggiore [Mag07, below eq. 1.13].

Effects of GW

This result allows us to see that the GWs are **transverse**: for a wave along the z direction the equation reads

$$\ddot{\xi}^3 = \frac{1}{2}\ddot{h}_{3j}^{TT}\xi^j = 0\,, \tag{1.4.28}$$

so the particle does not move along the direction of propagation; on the other hand we can do the calculations for particles starting out with small separations along x or y.

We consider an initial displacement vector $\xi_i(t=0) = (x_0, y_0, 0)$, allow it to vary denoting it as $\xi_i(t) = (x_0 + \delta x, y_0 + \delta y, 0) = \xi_i(t=0) + \delta\xi_i(t)$ and compute:

$$\ddot{x}_i = \delta\ddot{\xi}_i = \frac{1}{2}\ddot{h}_{ij}^{TT}(\xi_j + \delta\xi_j) \approx \frac{1}{2}\ddot{h}_{ij}^{TT}\xi_j\,, \tag{1.4.29}$$

since the variation $\delta\xi$ is of the same perturbative order as h_{ij}, making the term containing it second order.

So, after we take the real part of the exponential in h_{ij} and ignore the z dependence (which just gives a constant phase, since z is fixed) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \frac{1}{2}\begin{bmatrix}h_+ & h_\times\\ h_\times & -h_+\end{bmatrix}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\cos(\omega t))\begin{bmatrix}x_0\\ y_0\end{bmatrix} \tag{1.4.30}$$

$$= \frac{1}{2}\begin{bmatrix}h_+ & h_\times\\ h_\times & -h_+\end{bmatrix}\begin{bmatrix}x_0\\ y_0\end{bmatrix}\Big(-\omega^2\cos(\omega t)\Big)\,. \tag{1.4.31}$$

When we integrate to get δx and δy the factor $-\omega^2$ gets reabsorbed.

For the plus polarization h_+ (setting $h_\times = 0$) we find

$$\delta x = \frac{1}{2}h_+ x_0\cos(\omega t) \tag{1.4.32a}$$

$$\delta y = -\frac{1}{2}h_+ y_0\cos(\omega t)\,, \tag{1.4.32b}$$

while for the cross polarization $h_\times$ (setting $h_+ = 0$) we get

$$\delta x = \frac{1}{2}h_\times y_0\cos(\omega t) \tag{1.4.33a}$$

$$\delta y = \frac{1}{2}h_\times x_0\cos(\omega t)\,. \tag{1.4.33b}$$

There is a wrong sign in the slides here.

Are these considerations valid for **Earth-based detectors**? They are definitely not free-falling, in fact:

1. at zeroth order the metric is flat;

2. at first order we have the Newtonian forces, such as the Earth's gravity, the Coriolis force, the centrifugal force and such;

3. at second order we get the curvature contributions from GW and the background metric.

So, how do we distinguish these second-order GW effects from other second-order effects? We can isolate them by Fourier analysis: we only look at the Fourier window in which they are dominant.

1.5 GW generation

The assumptions we will make are

1. expanding around flat spacetime;
2. consider nonrelativistic systems;
3. assume the stress energy tensor is conserved: to first order, this reads

$$\partial^\mu T_{\mu\nu} = 0\,. \tag{1.5.1}$$

If the system we consider is self-gravitating (like a binary), then being nonrelativistic means that it is also large with respect to its Schwarzschild radius:

$$E_{\text{kin}} = -\frac{1}{2}U \implies \frac{1}{2}\mu v^2 = \frac{1}{2}G\frac{\mu M}{r} \implies \frac{v^2}{c^2} = \frac{GM}{c^2 r} = \frac{R_S}{r} \ll 1\,. \tag{1.5.2}$$

The expression of the gravitational force is that since, by the definition of the reduced mass, we have $m_1 m_2 = \mu M$.

Recall the $\Lambda_{ij,kl}$ tensor (1.3.31), which can be used to project a rank-two spacelike tensor to the TT gauge.

We will proceed as section 17.6 in Hobson [HEL06]. In order to solve the linearized equations (1.3.17), we use Green's functions, which are defined by:

$$\Box_x G(x-y) = \delta^{(4)}(x-y)\,, \tag{1.5.3}$$

where x is our variable, while y is a coordinate which will span the positions in the source. The operator $\Box_x$ is the Dalambertian with respect to the x coordinates.

The idea of this method is to calculate the wave response to a single impulsive source, and then superimpose the effects of many of these. We introduce $\kappa = 1/M_P^2$ for simplicity, multiply the previous equality by $T_{\mu\nu}(y)$ and integrate in $\mathrm{d}^4 y$ so we get

$$-2\kappa \int \mathrm{d}^4 y\, \Box_x G(x-y) T_{\mu\nu}(y) = -2\kappa \int \mathrm{d}^4 y\, \delta^{(4)}(x-y) T_{\mu\nu}(x) \tag{1.5.4}$$

$$-2\kappa \Box_x \left(\int \mathrm{d}^4 y\, G(x-y) T_{\mu\nu}(y) \right) = -2\kappa T_{\mu\nu}(x) = \Box_x \bar{h}_{\mu\nu}(x)\,, \tag{1.5.5}$$

The argument of the right $T_{\mu\nu}$ can be switched to x because of the delta.

so we will have as a solution a superposition of the homogeneous solution and the source term:

$$\bar{h}_{\mu\nu}(x) = \underbrace{\bar{h}^{(0)}_{\mu\nu}(x)}_{\Box \bar{h}^{(0)}_{\mu\nu} = 0} - 2\kappa \int \mathrm{d}^4 y\, G(x-y) T_{\mu\nu}(y)\,. \tag{1.5.6}$$

In order to make the Green's function explicit, we write it as centered around the origin:

$$\partial_\mu \partial^\mu G(x^\sigma) = \delta^{(4)}(x^\sigma)\,, \tag{1.5.7}$$

and we integrate this equality over a hypercylinder V (the product of a 3-sphere of radius $r = |\vec{x}|$ and an interval $[-ct, ct] \subset \mathbb{R}$, where $ct > r$) we have

$$\int_V \mathrm{d}^4x\, \delta^{(4)}(x^\sigma) = 1 = \int_V \mathrm{d}^4x\, \partial_\mu \partial^\mu G(x^\sigma) \tag{1.5.8a}$$

$$= \int \mathrm{d}S \left(\partial_\mu G(x^\sigma)\right) n^\mu\,, \tag{1.5.8b}$$

but the only points which can contribute are in the future light-cone because of causality, so the dependence of G upon x^σ must be in the form $G(x^\sigma) = f(r)\delta(ct - r)[ct \geq 0]$.[6]

We write $\mathrm{d}S = c\,\mathrm{d}t\, r^2\,\mathrm{d}\Omega$,[7] and we call $n^\mu \partial_\mu = \partial_r$:

$$1 = \int \mathrm{d}S \left(\partial_\mu G(x^\sigma)\right) n^\mu \tag{1.5.9a}$$

$$= 4\pi r^2 \int_0^\infty \mathrm{d}t\, \partial_r \big(f(r)\delta(ct - r)\big) c \tag{1.5.9b}$$

$$= 4\pi r^2 \partial_r f(r) c + 4\pi r^2 f(r) \underbrace{\int_0^\infty \partial_r \delta(ct - r) c\, \mathrm{d}t}_{=0}\,, \tag{1.5.9c}$$

The derivative of the delta evaluates the derivative of the thing it multiplies, which is a constant.

so we can get an explicit expression for the function $f(r)$:

$$4\pi r^2 \partial_r f(r) = 1 \implies f(r) = -\frac{1}{4\pi r} \implies G(x^\sigma) = -\frac{\delta(x^0 - |\vec{x}|)}{4\pi|\vec{x}|}\theta_H(x^0)\,. \tag{1.5.10}$$

The integration constant is set to zero so that the Green function vanishes at infinity.

We can then plug this into the general solution (1.5.6) to find

$$\bar{h}_{\mu\nu}(t, \vec{x}) = (-)^2 2\kappa \int \mathrm{d}^4y\, \frac{\delta(x^0 - y^0 - |\vec{x} - \vec{y}|)}{4\pi|\vec{x} - \vec{y}|}\theta_H(x^0 - y^0) T_{\mu\nu}(y) \tag{1.5.11}$$

$$= \frac{4G}{c^4} \int \mathrm{d}^4y\, \frac{\delta(y^0 - (ct - |\vec{x} - \vec{y}|))}{|\vec{x} - \vec{y}|} T_{\mu\nu}(y^0, \vec{y}) \tag{1.5.12}$$

$\kappa = 8\pi G/c^4$.

$$= \frac{4G}{c^4} \int \mathrm{d}^3y\, \frac{T_{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|}\,. \tag{1.5.13}$$

As long as we are outside the source we can move to the TT gauge, since there the equation $\Box \bar{h}_{\mu\nu}$ satisfies the EFE, so we can do the required gauge change of variables with $\Box \xi^{\mu\nu} = 0$. Now, in order to move to the TT gauge we can use the Λ tensor. It is equivalent to apply it to the trace-reversed $\bar{h}_{ij}$ or to h_{ij}, since the tensor is projected into the space of traceless tensors anyways.[8]

[6] The bracket is an Iverson bracket [Knu92], it evaluates to 1 or 0 depending on whether the expression inside it is true or false.

[7] The r^2 is missing in Hobson [HEL06, pag. 477] as well, but it should be there.

[8] Formally, this is shown as

$$\Lambda_{ij,kl}\bar{h}_{kl} = \Lambda_{ij,kl}\left(h_{kl} - \frac{1}{2}\eta_{kl} h\right) = \Lambda_{ij,kl} h_{kl} - \frac{1}{2}\underbrace{\Lambda_{ij,kk}}_{=0} h = \Lambda_{ij,kl} h_{kl}\,. \tag{1.5.14}$$

So, the general expression for our TT-gauge tensor measured at a position $\vec{x}$ outside the source, with $\hat{n} = \vec{x}/|x|$:

$$\boxed{h_{ij}^{TT}(ct, \vec{x}) = \Lambda_{ij,kl}(\hat{n})\bar{h}_{kl} = \frac{4G}{c^4}\Lambda_{ij,kl}(\hat{n}) \int \mathrm{d}^3y \, \frac{T_{kl}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \,. \qquad (1.5.15)}$$

Far from the source, $|\vec{x}| \gg |\vec{y}|$ for any $\vec{y}$ inside the source. So, we can expand:[9]

$$|\vec{x} - \vec{y}| = r\left(1 - \frac{\vec{y}\cdot\hat{n}}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right)\right) . \tag{1.5.18}$$

Keeping only the terms at $\mathcal{O}(1/r)$ we get[10]

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl} \int \mathrm{d}^3y \, T_{kl}\left(t - \frac{r}{c} + \frac{\vec{y}\cdot\hat{n}}{c}, \vec{y}\right) . \tag{1.5.19}$$

If the object is moving periodically with frequency ω_s, then we will have

$$\frac{1}{\omega_s} \sim \frac{d}{v} , \tag{1.5.20}$$

and we assume $d/c \ll d/v$, which is equivalent to $v \ll c$: the characteristic velocities of the source should be nonrelativistic. Under these assumptions we can expand the stress-energy tensor in powers of $\xi = \vec{y}\cdot\hat{n}/c \sim d/c \ll d/v$:

$$T_{kl}\left(t - \frac{r}{c} + \xi, \vec{y}\right) = T_{kl}\left(t - \frac{r}{c}, \vec{y}\right) + \left.\frac{\partial T_{kl}}{\partial \xi}\right|_{\xi=0} \xi + \frac{1}{2}\left.\frac{\partial^2 T_{kl}}{\partial \xi^2}\right|_{\xi=0} \xi^2 + \mathcal{O}\left(\xi^3\right) \tag{1.5.21}$$

$$\approx T_{kl}\left(t - \frac{r}{c}, \vec{y}\right) + \xi \partial_0 T_k l + \frac{\xi^2}{2}\partial_0\partial_0 T_{kl} \tag{1.5.22}$$

$$= T_{kl}\left(t - \frac{r}{c}, \vec{y}\right) + \frac{y^i n^i}{c}\partial_0 T_k l + \frac{y^i n^i y^j n^j}{2c^2}\partial_0\partial_0 T_{kl} \,. \tag{1.5.23}$$

Inserting this into the expression we get

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl} \int \mathrm{d}^3y \left[T_{kl} + \frac{y^m n^m}{c}\partial_0 T_{kl} + \frac{y^m y^p n^m n^p}{2c^2}\partial_0^2 T_{kl} + \mathcal{O}\left(\xi^3\right)\right]_{\text{ret}} , \tag{1.5.24}$$

[9] The full calculation goes as follows:

$$|x - y| = \sqrt{(x-y)^2} = \sqrt{x^2 + y^2 - 2x\cdot y} = r\sqrt{1 - 2\frac{\hat{n}\cdot y}{r} + \frac{y^2}{r^2}} \tag{1.5.16}$$

$$\approx r\left(1 - \frac{\hat{n}\cdot y}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right)\right) , \tag{1.5.17}$$

where d is the length scale of the source, such that $|y| \leq d$.

[10] At this point we change the first argument of the stress-energy tensor's dimensionality from a space ct to a time t; this is just a matter of convention, it makes it easier to write the Taylor expansion later.

where "ret" means that the stress-energy tensor should be computed at a retarded time: $t - r/c$ instead of t.

We define the multipole moments: they are tensors with an arbitrary amount of indices,

$$S^{ij,m_1 m_2 \cdots}(t) = \int \mathrm{d}^3 y \, T^{ij}(t,y) \prod_\alpha y^{m_\alpha} \,. \tag{1.5.25}$$

In terms of these, the expression for h_{ij}^{TT} can be written as

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\left(S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m} + \frac{1}{2c^2} n_m n_p \ddot{S}^{kl,mp} + \mathcal{O}\left(\xi^3\right)\right)_{\text{ret}} . \tag{1.5.26}$$

As we go up a perturbative order, we insert a factor $1/c$, we add an index to the multipole, which corresponds to a multiplication by $y \sim d$, and we differentiate, corresponding to a division by a timescale t of the evolution of the system: so, the n-th order perturbative term is of order $(d/ct)^n \sim (v/c)^n$. Since the system is nonrelativistic, v/c is small compared to 1, so we can stop at the first order and still get a good result:

$$h_{ij}^{TT}(t,\vec{x}) \approx \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\left(S^{kl} + \frac{1}{c} n_m \dot{S}^{kl,m}\right)_{\text{ret}} . \tag{1.5.27}$$

Let us define the moments of the energy and momentum densities:

$$M = \frac{1}{c^2}\int \mathrm{d}^3 y \, T^{00}(t,\vec{y}) \sim \frac{1}{c^2} S^{00} \tag{1.5.28}$$

$$M^i = \frac{1}{c^2}\int \mathrm{d}^3 y \, T^{00}(t,\vec{y}) y^i \sim \frac{1}{c^2} S^{00,i} \tag{1.5.29}$$

$$M^{ij} = \frac{1}{c^2}\int \mathrm{d}^3 y \, T^{00}(t,\vec{y}) y^i y^j \sim \frac{1}{c^2} S^{00,ij} \tag{1.5.30}$$

$$P^i = \frac{1}{c^2}\int \mathrm{d}^3 y \, T^{0i}(t,\vec{y}) \sim \frac{1}{c^2} S^{0i} \tag{1.5.31}$$

$$P^{i,j} = \frac{1}{c^2}\int \mathrm{d}^3 y \, T^{0i}(t,\vec{y}) y^j \sim \frac{1}{c^2} S^{0i,j} \tag{1.5.32}$$

$$P^{i,jk} = \frac{1}{c^2}\int \mathrm{d}^3 y \, T^{0i}(t,\vec{y}) y^j y^k \sim \frac{1}{c^2} S^{0i,jk} \,, \tag{1.5.33}$$

where we have an analogy ($\sim$) instead of an equality because the $S^{ij,m_1 m_2 \cdots}$ are defined only with spatial indices.

We might want to compute the **backreaction** of the GW emission onto the system, the energy lost per unit time: since M corresponds to the total energy, we could try to compute $\dot{M}$.[11] Let us try to do this, recalling that by the stress-energy tensor's conservation we have $\partial_\mu T^{\mu\nu} = 0$, which means $\partial_0 T^{00} = -\partial_i T^{0i}$:

$$c\dot{M} = \int_V \mathrm{d}^3 y \, \partial_0 T^{00} = -\int_V \mathrm{d}^3 y \, \partial_i T^{0i} = -\int_{\partial V} \mathrm{d}S_i \, T^{0i} \to 0 \,, \tag{1.5.34}$$

[11] A small technical note: since we are bothering to keep the cs, we should recall that the time derivative denoted by a dot is actually a derivative with respect to ct.

since the flux is computed in a region outside the source, where its stress-energy tensor vanishes. What this means is that the leading order is too low to see the energy loss. If we move up an order we find

$$c\dot{M}^i = \int_V \mathrm{d}^3y\, y^i \underbrace{\partial_0 T^{00}}_{=-\partial_j T^{0j}} = +\int_V \mathrm{d}^3y \left(\partial_j y^i\right) T^{0j} = cP^i\,, \tag{1.5.35}$$

which means that the time derivative of the center of mass M^i gives the total linear momentum. Like the total energy, this is conserved: it can be shown like

$$\dot{P}^i = \int_V \mathrm{d}^3y\, \partial_0 T^{0i} = -\int_V \mathrm{d}^3y\, \partial_j T^{ji} = -\int_{\partial V} \mathrm{d}S_j\, T^{ji} \to 0\,. \tag{1.5.36}$$

We also can show that $\dot{M}^{ij} = 2P^{(i,j)}$, $\dot{M}^{ijk} = P^{i,jk} + P^{j,ik} + P^{k,ij}$ and $\dot{P}^{i,j} = 2S^{ij}$, which means $\ddot{M}^{ij} = 2S^{ij}$. The computation is done by repeatedly applying integration by parts and the continuity equation:

$$\ddot{M}^{kl} = \int \partial_0\partial_0 T^{00}(t,\vec{y}) y^k y^l \tag{1.5.37}$$

$$= -\int \partial_0 \left(\partial_i T^{i0}\right) y^k y^l = \int \partial_0 T^{i0} \partial_i \left(y^k y^l\right) \tag{1.5.38}$$

$$= \int \partial_0 T^{i0} \left(\delta_i^k y^l + \delta_i^l y^k\right) = -\int \partial_j T^{ij} \left(\delta_i^k y^l + \delta_i^l y^k\right) \tag{1.5.39}$$

$$= \int T^{ij} \left(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k\right) = \int T^{kl} + T^{lk} = 2S^{kl}\,. \tag{1.5.40}$$

To give a physical intuition: the spatial components S^{kl} describe the *pressure* and *shears* in the source — they make up the stress tensor. On the other hand, M^{kl} is the second moment of the mass density; its second derivative is somewhat analogous to a mass times an acceleration. So, this equation is a way to relate the forces and the accelerations inside the source.

Quadrupole radiation

If we only keep the first order in the expression for h_{ij}^{TT} we get:

$$h_{ij}^{TT}(t,\vec{x}) \approx \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl} S^{kl} = \frac{1}{r}\frac{2G}{c^4} \Lambda_{ij,kl} \ddot{M}^{kl}\,. \tag{1.5.41}$$

The quadrupole moment is defined as the traceless part of the moment M^{ij}:

$$Q^{kl} = M^{kl} - \frac{1}{3}\delta^{kl} M_{ii} = \int \mathrm{d}^3x\, \rho(t,\vec{x}) \left(x^i x^j - \frac{1}{3} r^2 \delta_{ij}\right). \tag{1.5.42}$$

Since $\Lambda_{ij,kl}\delta^{kl} = 0$, we can substitute Q^{kl} for M^{kl}:

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4} \Lambda_{ij,kl} \ddot{Q}^{kl}\left(t - \frac{r}{c}\right) = \frac{1}{r}\frac{2G}{c^4} \ddot{Q}_{ij}^{TT}\left(t - \frac{r}{c}\right). \tag{1.5.43}$$

If a wave is propagating along the $\hat{n} = \hat{z}$ direction, we get

$$\Lambda_{ij,kl} \ddot{M}_{kl} = \frac{1}{2} \begin{bmatrix} \left(\ddot{M}_{11} - \ddot{M}_{22}\right)/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & \left(\ddot{M}_{22} - \ddot{M}_{11}\right)/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.5.44a}$$

since the projection tensor P_{ij} is $P_{ij} = \operatorname{diag}(1, 1, 0)$, while

$$\Lambda_{ij,kl} \ddot{M}_{kl} = \left(P \ddot{M} P\right)_{ij} - \frac{1}{2} P_{ij} \operatorname{Tr} \ddot{M}. \tag{1.5.45}$$

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This means that the two polarizations' amplitudes are:

$$h_+ = \frac{1}{r} \frac{G}{c^4} \left(\ddot{M}_{11} - \ddot{M}_{22}\right) \qquad \text{and} \qquad h_\times = \frac{2}{r} \frac{G}{c^4} \ddot{M}_{12}. \tag{1.5.46}$$

For the full details on this, see Maggiore [Mag07, eqs. 3.60 to 3.72].

But how do we compute the full angular distribution? We can brute-force it using the full Λ projection tensor, but a more conceptual way is to put ourselves in a frame in which the generic vector $\hat{n}$ is $\hat{z}$. Then we use the rotation matrices: we need two of them, one for each unit vector in a rank-2 tensor M_{ij}, which will then transform like

$$\ddot{M}'_{ij} = R_{ik} R_{jl} \ddot{M}_{kl}, \tag{1.5.47}$$

Then we use the simple expression for h_+ and $h_\times$, substituting in the $\ddot{M}_{11}$, $\ddot{M}_{12}$ and so on in the primed system.

The main results to take away are:

1. There is no monopole radiation: $\dot{M} = 0$, since mass is conserved.
2. We can move the origin so that the dipole is zero: $M^i = 0$. This corresponds to linear momentum being conserved: $\dot{P}^i = 0$. In Electromagnetism, on the other hand, we cannot eliminate the dipole radiation. This is due to there being positive and negative electric charges, while there are no negative masses.
3. We did not account for back-action: our GWs do not carry "away" energy or momentum. This is unphysical, we will account for it!

Application to a self-gravitating system

The stress-energy tensor for a system of point masses looks like

$$T^{\mu\nu}(t, \vec{x}) = \sum_A \frac{p^\mu_A p^\nu_A}{\gamma_A m_A} \delta^{(3)}(x - x_A(t)), \tag{1.5.48}$$

where γ is the Lorentz factor, while p^μ is the four-momentum.

In order to apply the calculations from before to our system, it must be closed; we cannot consider any particle trajectory, since they must move on geodesics in order to conserve the stress-energy tensor. This means that there cannot be any external forces.

However, we can use the relative coordinates in a self-gravitating system: if two particles have coordinates x_1 and x_2 we can define their relative distance $x_0 = x_1 - x_2$, the total mass $m = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$, and the center of mass position $x_{\mathrm{CM}} = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$.

If we set the position of the COM to zero identically, all the terms containing it in the second mass moment vanish, and we are left with;

$$M^{ij} = \int \mathrm{d}^3 x\, T^{00} x^i x^j = m_1 x_1^i x_1^i + m_2 x_2^i x_2^j = \mu x_0^i x_0^j \,. \tag{1.5.49}$$

The quadrupole is then given by

$$Q^{ij}(t) = \mu \left(x_0^i(t) x_0^j(t) - \frac{r_0^2}{3} \delta^{ij} \right) , \tag{1.5.50}$$

where $r_0^2 = |x_0|^2$.

We consider two particles in the xy plane moving on a circular trajectory; for now we do not consider the Newtonian equations of motion and let all the parameters of this trajectory be independent of each other, so we write

$$x_0(t) = R \cos\left(\omega_s t + \frac{\pi}{2} \right) \tag{1.5.51a}$$

$$y_0(t) = R \sin\left(\omega_s t + \frac{\pi}{2} \right) \tag{1.5.51b}$$

$$z_0(t) = 0 \,, \tag{1.5.51c}$$

so we can compute the mass moments: we get products of sines and cosines which can be written as:

$$M_{11} = \frac{\mu R^2}{2} \big(1 - \cos(2\omega_s t) \big) \tag{1.5.52}$$

$$M_{22} = \frac{\mu R^2}{2} \big(1 + \cos(2\omega_s t) \big) \tag{1.5.53}$$

$$M_{12} = \frac{\mu R^2}{2} \sin(2\omega_s t) \,, \tag{1.5.54}$$

so the second derivatives also **oscillate at twice the rotational frequency** ω_s:

$$\ddot{M}_{11} = -\ddot{M}_{22} = 2\mu R^2 \omega_s^2 \cos(2\omega_s t) \tag{1.5.55a}$$

$$\ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin(2\omega_s t) \,. \tag{1.5.55b}$$

If we look at this emission as an observer placed in a generic direction $\hat{n}$, described by the angles θ and φ, we will receive

$$h_+(t, \theta, \varphi) = \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \frac{1 + \cos^2\theta}{2} \cos\big(2\omega_s t_{\mathrm{ret}} + 2\varphi \big) \tag{1.5.56a}$$

$$h_\times(t,\theta,\varphi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos(\theta)\sin\left(2\omega_s t_{\text{ret}} + 2\varphi\right), \tag{1.5.56b}$$

so the two scale differently as θ varies. The variation of φ does not change the amplitude of the wave, as we would expect, but only the phase.

Do note that h_+, and $+h_\times$ looks like a circular polarization moving counter-clockwise, while h_+, $-h_\times$ moves clockwise. This is what we see for $\theta = 0, \pi$; on the other hand for $\theta = \pi/2$ we only have h_+. So, if we can measure the magnitudes and relative phases of the $+$ and $\times$ polarizations then we can determine in which direction the binary is spinning relative to us.

Plot effects!

Let us put some numbers into these equations: consider the Earth-Sun system, as seen from $r \sim 10^5\,\text{lyr} \sim 10^{21}\,\text{m}$ away. The reduced mass is $\mu \sim m_\oplus \sim 6 \times 10^{24}\,\text{kg}$, the frequency is $\omega_s \sim 2 \times 10^{-7}\,\text{rad/s}$, the radius of the orbit is $R \sim 150 \times 10^9\,\text{m}$.

Then, we find

$$h_+ \sim h_\times \sim \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4} \sim 5 \times 10^{-32}, \tag{1.5.57}$$

which is definitely undetectable with current technologies. Note that the strain is inversely proportional to r: this is because we are not measuring the intensity, but directly the amplitude.

Let us try a BNS in our galaxy, using Newtonian orbits to model the binary. Now we will have $r \sim 10^{21}\,\text{m}$, $m_1 = m_2 \sim M_\odot \sim 10^{30}\,\text{kg}$, $\omega_s \sim 2\pi\text{Hz}$, and we need to calculate R. Kepler's third law for circular orbits gives simply

$$\omega_s^2 R^3 = GM, \tag{1.5.58}$$

which we can put into the expression from before as $R^2 = G^{2/3}M^{2/3}\omega_s^{-4/3}$ to find

$$h_+ \sim h_\times \sim \frac{4G^{5/3}\omega_s^{2/3}\mu M^{2/3}}{rc^4} \sim 10^{-20}. \tag{1.5.59}$$

This was for a BNS at 10^5 lyr; we have seen a BNS outside our galaxy, at 40 Mpc $\sim 10^8$ lyr away, whose signal had an amplitude as low as 10^{-23}.

We can try to do the computation for the actual event GW150914, for which $m_1 \sim 36M_\odot$, $m_2 \sim 29M_\odot$, $f_{GW} = 2f_s \sim 35\,\text{Hz}$ in the inspiral phase, $r \sim 410\,\text{Mpc}$. The amplitude we find is approximately 8×10^{-22}. The amplitude which was measured was $\sim 5 \times 10^{-22}$ [LIG+16, fig. 2] for the inspiral phase; the difference of a factor $5/8$ is due to the angular distribution — we are not looking at the source head-on, but instead at a certain angle.

When the distance between the bodies in the binary becomes too small, we cannot model them using Kepler's laws anymore.

1.6 Energy and momentum of GW

Now we will treat the back-action of gravitational waves on the source. As we saw, our approximations predicted no energy loss in the source ($\dot{M} = 0$), which is "by design": GW energy is quadratic in h, while our theory is linear. Also, at each point we can move to the LIF and completely remove the effects of GW.

GWs do carry energy, as shown by the "sticky beads" thought experiment; however we cannot describe this in a local way, since for any single particle we can gauge them away. The effect is related to the *tidal effects* of the GW, which we must calculate by comparing different points in spacetime.

We cannot define a *local* stress-energy tensor for the waves, an energy *density*: we need to perform an average in spacetime.

In our simplified treatment we perturbed around flat spacetime, but in the real world we will also have other sources of curvature, and we will need to perturb from a non-flat background metric $\overline{g}_{\mu\nu}$:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}\,, \tag{1.6.1}$$

but how do we decide which deviation from flat is part of the background and what is part of the gravitational wave? There is no formal way to define this,[12] but we can use a heuristic, based on what they describe. What we will do is like distinguishing waves and tides in the ocean: intuitively it is easy to see how they differ, based on the scale of their effects, although it is harder to see how much each contributes to the height of a specific point.

We say that the background $\overline{g}_{\mu\nu}$ varies **spatially slowly**, across distances with scales L_B, such that all the GW we are considering have reduced wavelengths ($k = \lambda/2\pi$) satisfy $\lambda \leq \lambda_{GW} \ll L_B$, where λ_{GW} is a fixed maximum wavelength.

We can also impose that all the GWs vary much faster than the background, temporally: $f > f_{GW} \gg f_B$.

Do note that these two conditions are independent, since while GW travel at the speed of light[13] the background's temporal and spatial variations are in general not constrained. We refer to the combination of the two conditions as the **short-wave approximation**.

How do ground-based **GW detectors** practically distinguish GW from background?

Their length is at most on the order of single kilometers: so, their characteristic frequency will be $f_{GW} = c/\lambda_{GW} \sim 300\,\text{kHz}$: the idea of measuring a GW's variation in space and checking that its wavelength is small with respect to the background is doomed to fail. We cannot measure at these frequencies or higher, and do not expect to find interesting astrophysical sources in this range. Also, ground-based detectors do not measure the metric along the whole length, but only an integrated effect.

Instead, the detectors monitor local temporal variations of $g_{\mu\nu}(t, x_0)$ (at what is basically a fixed point). They work best between 100 Hz to 1000 Hz; the Earth's gravitational field is

[12] There is a theoretical way to do so, which however is not really practical: if we consider the general form of a perturbation to the metric, we can decompose it into degrees of freedom which transform differently under the Poincaré group: scalar, vector and tensor perturbations (gravitational waves).

Tensor perturbations are spin-2, so they come back to themselves after a 180° rotation. This distinguishes them from other background effects, but it is not an easy characteristic to measure, especially since our detectors cannot measure the shape of the wave point-by-point but instead only as an integrated effect.

[13] We know this to be true to within a part in 10^{-14} based on the event GW170817, for which the GW and the EM counterpart arrived within a couple of seconds from several Mlyr.

not smooth along the corresponding length scale, since the contributions from mountains, oceans, tides and such have amplitudes which are larger than those of the GW. However, the Earth's gravitational field is close to static at these frequencies: its variations are slower than a few Hz.

So, we can apply our distinction: GW and background can be **distinguished in frequency**: they will be the greatest gravitational contribution at the 100 Hz to 1 kHz range.

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So, let us do this formally: we consider $g = \overline{g} + h$ where $\overline{g}$ only contains low frequency components while h contains high frequency ones, we expand the Ricci tensor in powers of $h_{\mu\nu}$ up to second order and rewrite the Einstein equations as

$$\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) = \overline{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}\,, \tag{1.6.2}$$

where the second order in $h_{\mu\nu}$ term, $R^{(2)}_{\mu\nu}$, has both high and low frequency components, while the term $\overline{R}_{\mu\nu}$ is exclusively low frequency and $R^{(1)}_{\mu\nu}$ is exclusively high frequency.

Let us give a heuristic argument for this statement: the gravitational waves contained in $h_{\mu\nu}$ will have several high frequencies; let us call two of them ω_1 and ω_2, so as we compute a term which is quadratic in h we will get

$$\begin{aligned}
\big(\sin(\omega_1 t) + \sin(\omega_2 t)\big)^2 &= \qquad (1.6.3)\\
= \frac{1}{2}\left(-\cos\left(2t\omega_1\right) - \cos\left(2t\omega_2\right) - 2\cos\left(t\omega_1 + t\omega_2\right) + 2\cos\left(t\omega_1 - t\omega_2\right) + 2\right)\,, &\qquad (1.6.4)
\end{aligned}$$

where we used the prostapheresis formulas: we get terms oscillating with $\omega_1 + \omega_2$ as well as $\omega_1 - \omega_2$, one of which is high frequency while the other is low frequency, since we are considering a high-frequency wavepacket.

So, what we want to do is to separate the field equation into its low and high frequency parts:

$$\overline{R}_{\mu\nu} = -\left(R^{(2)}_{\mu\nu}\right)^{\text{low}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{low}} \tag{1.6.5}$$

$$R^{(1)}_{\mu\nu} = -\left(R^{(2)}_{\mu\nu}\right)^{\text{high}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{high}}\,. \tag{1.6.6}$$

How might we define these formally? Recall, we expand in two parameters: h and $\lambda/L_B \sim f_B/f$. Then, in terms of orders of magnitude we have[14]

$$\overline{R}_{\mu\nu} \sim \partial^2 \overline{g}_{\mu\nu} \sim \frac{1}{L_B^2}\,, \tag{1.6.7}$$

while

$$\left[R^{(2)}_{\mu\nu}\right]^{\text{low}} \sim (\partial h)^2 \sim \left(\frac{h}{\lambda}\right)^2. \tag{1.6.8}$$

[14] The fact that the derivatives correspond to divisions by lengths is justified since we are looking at harmonic expansions, the metric will be a sum of sinusoids around a characteristic frequency.

See Maggiore [Mag07, pag. 31] for more details. So, inserting these order-of-magnitude estimates inside equation (1.6.5) we get

$$\frac{1}{L_B^2} \sim \frac{h^2}{\lambda^2} + \Big(T_{\mu\nu} \text{ contribution} \Big) . \tag{1.6.9}$$

If we are in a vacuum ($T_{\mu\nu} = 0$) then

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda} \right)^2 \implies h \sim \frac{\lambda}{L_B} , \tag{1.6.10}$$

while if the stress-energy tensor is nonvanishing we get

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda} \right)^2 + (T_{\mu\nu} \text{ contribution}) \gg \left(\frac{h}{\lambda} \right)^2 , \tag{1.6.11}$$

which means $h \ll \lambda / L_B$. This shows that **the linearized expansion cannot be extended beyond linear order** starting from a flat background: if we start from a flat metric then $1/L_B = 0$, so any $h > 0$ violates the condition $h \lesssim \lambda / L_B$. We can also see that the notion of a gravitational wave only makes sense as long as h is small: if h were close to 1, then we would have $\lambda \sim L_B$, but our only way to distinguish GW from background is through their wavelength!

Unclear what the problem is: we cannot use the flat metric, but the real world's metric is not flat anyways! Why can we not go to second order with a curved background?

The solution to this problem is to take averages on a scale ℓ such that $\lambda_{GW} \ll \ell \ll L_B$, so that the gravitational wave is fully averaged out, while the background metric is approximately constant across the integration volume. This low-frequency projection reads:

$$\overline{R}_{\mu\nu} = - \left\langle R^{(2)}_{\mu\nu} \right\rangle + \frac{8\pi G}{c^4} \left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle . \tag{1.6.12}$$

After some math (see Maggiore [Mag07, eqs. 1.122-1.123]) we can define a stress tensor of the GW, which looks like

$$t_{\mu\nu} = - \frac{c^4}{8\pi G} \left\langle R^{(2)}_{\mu\nu} - \frac{1}{2} \overline{g}_{\mu\nu} R^{(2)} \right\rangle , \tag{1.6.13}$$

where the Ricci scalar is computed using the smoothed metric: $R^{(2)} = \overline{g}^{\mu\nu} R^{(2)}_{\mu\nu}$. Also, we define a "smoothed out" stress-energy tensor of matter:

$$\left\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right\rangle \stackrel{\text{def}}{=} \overline{T}_{\mu\nu} - \frac{1}{2} \overline{g}_{\mu\nu} \overline{T} , \tag{1.6.14}$$

so the equation which will hold is the equivalence of the smoothed Einstein tensor with the sum of the smoothed and GW stress-energy tensors:

$$\boxed{\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R} = \frac{8\pi G}{c^4}\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right) .} \tag{1.6.15}$$

Do note that if we work with these, the stress-energy tensor which is conserved is $\overline{T}_{\mu\nu} + t_{\mu\nu}$: if we take the divergence of the equation we get, by the Bianchi identities,

$$\nabla^\mu\left(\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{R}\right) = 0 = \nabla^\mu\left(\overline{T}_{\mu\nu} + t_{\mu\nu}\right) . \tag{1.6.16}$$

This means that gravitational waves and matter can exchange energy and momentum. If the stress-energy tensor is slowly-varying enough, as often happens practically, we can just use $\overline{T}^{\mu\nu} \approx T^{\mu\nu}$.

How does this look like far from the source? There, we can approximate $\overline{g} \approx \eta_{\mu\nu}$ and $\nabla_\mu \approx \partial_\mu$. This $t_{\mu\nu}$ only has 2 physical degrees of freedom (those of the gravitational wave): we need to gauge the others away.

The Lorentz gauge plus $h = 0$ eliminates 5 degrees of freedom.

When we have terms like $h\partial\partial h$, we can integrate by parts on a sufficiently large volume to turn them into $\partial(h\partial) - \partial h\partial h$. Using the facts $\partial^\mu h_{\mu\nu} = h = \Box h_{\mu\nu} = 0$ we can also simplify several terms: in the end we get

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle .} \tag{1.6.17}$$

This is invariant under the residual gauge transformations, and coordinate independent: we can compute it in any frame we like.

Explicit TT-gauge expression

Let us compute this in the simplest case: a gravitational wave travelling along the z axis, described in the TT-gauge. The 00 component will read:

$$t^{00} = \frac{c^4}{32\pi G}\left\langle \partial^0 h_{\alpha\beta}\partial^0 h^{\alpha\beta}\right\rangle = \frac{c^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle , \tag{1.6.18}$$

since:

$$\partial^0 h_{\alpha\beta}\partial^0 h^{\alpha\beta} = \frac{\dot{h}_{ij}^{TT}}{c}\frac{\dot{h}_{ij}^{TT}}{c} = \frac{1}{c^2}\sum_{i,j=1}^{2}\left(\dot{h}_{ij}^{TT}\right)^2 = \frac{2}{c^2}\left(\dot{h}_+^2 + \dot{h}_\times^2\right) . \tag{1.6.19}$$

As for the other components, we will have $t^{01} = t^{02} = 0$ by symmetry (they would represent momentum transfer in a direction orthogonal to the propagation, also formally $\partial_{1,2} h_{ij}^{TT} = 0$ since it only depends on t and z), and also $t^{03} = t^{00}$, since the perturbation is a function of $(t - z/c)$: so, we have

$$\partial_3 h_{ij}^{TT} = -\partial_0 h_{ij}^{TT} = +\partial^0 h_{ij}^{TT} . \tag{1.6.20}$$

1.6.1 Energy and momentum flux far from the source

If we are far enough away from the source, we can compute the energy crossing a surface $\mathrm{d}A$ in a time $\mathrm{d}t$ as the spacetime density contained in a volume $\mathrm{d}A\, c\, \mathrm{d}t$: if we are considering a specific direction (say, the flux coming the way of the Earth) then we can say that the wave's propagation is aligned with the z axis and so we can write

$$\mathrm{d}E = \mathrm{d}A\, c\, \mathrm{d}t\, \frac{c^2}{32\pi G} \left\langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \right\rangle \tag{1.6.21}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t\, \mathrm{d}A} = \frac{c^3}{16\pi G} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle . \tag{1.6.22}$$

In order to get the total power $\mathrm{d}E/\mathrm{d}t$ which is emitted by the source we can integrate this expression in $R^2\, \mathrm{d}\Omega$; however we will need to use the general expression $\left\langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \right\rangle$ since we cannot have alignment with the z axis across the whole sphere.

To get the momentum density the reasoning is similar: we start from $\mathrm{d}P^k = \mathrm{d}A\, c\, \mathrm{d}t\, t^{0k}/c$ to get

$$\frac{\mathrm{d}P^k}{\mathrm{d}A\, \mathrm{d}t} = \frac{c^4}{32\pi G} \left\langle \partial^0 h_{\alpha\beta} \partial^k h^{\alpha\beta} \right\rangle = -\frac{c^3}{32\pi G} \left\langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \right\rangle , \tag{1.6.23}$$

where we used the fact that $\partial^0 = -\partial_0 = -\, \partial/\partial(ct)$.

So, the general expressions for the energy and momentum density emitted at a distance will be

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{c^3}{32\pi G} r^2 \int \left\langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \right\rangle \mathrm{d}\Omega \qquad \text{and} \qquad \frac{\mathrm{d}P^k}{\mathrm{d}t} = -\frac{c^3}{32\pi G} r^2 \int \left\langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \right\rangle \mathrm{d}\Omega . \tag{1.6.24}$$

We want to express this explicitly in terms of the quadrupole moment. We start from the expression of the TT-gauge amplitude in terms of the quadrupole (1.5.43): its derivative reads

$$\dot{h}_{ij}^{TT}(t, \vec{x}) = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl} \dddot{Q}^{kl}(t - r/c) . \tag{1.6.25}$$

Inserting this into the expression we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{r^2 c^3}{32\pi G} \int \mathrm{d}\Omega \left\langle \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl} \dddot{Q}^{kl} \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,mn} \dddot{Q}^{mn} \right\rangle \tag{1.6.26a}$$

$$= \frac{G}{8\pi c^5} \int \mathrm{d}\Omega\, \Lambda_{ij,kl} \left\langle \dddot{Q}^{ij} \dddot{Q}^{kl} \right\rangle , \tag{1.6.26b}$$

Λ is idempotent and symmetric under swaps of index pairs: so $\Lambda_{ij,kl}\Lambda_{ij,mn} = \Lambda_{kl,mn}$.

where the only expression depending on the angle is $\Lambda_{ij,kl}$, since the quadrupole only depends on the source, not on an observer's position.

Claim 1.6.1. *The explicit expression for the projection tensor $\Lambda_{ij,kl}(\hat{n})$ in terms of the unit vector $\hat{n}$ is:*

$$\Lambda_{ij,kl}(\hat{n}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l , \tag{1.6.27}$$

and its integral in $\mathrm{d}\Omega$ is given by

$$\int \mathrm{d}\Omega\, \Lambda_{ij,kl} = \frac{4\pi}{30} \Big(11 \delta_{ik}\delta_{jl} - 4 \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} \Big) . \tag{1.6.28}$$

With this result, we can write

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{8\pi c^5}\frac{2\pi}{15}\left(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right)\left\langle \dddot{Q}^{ij}\dddot{Q}^{kl}\right\rangle , \tag{1.6.29}$$

but since the quadrupole moment derivatives $\dddot{Q}^{ij}$ are both traceless and symmetric the first and third delta combinations are equal (so the factor multiplying them will be 12), while the second combination will vanish. So, we will get

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{8\pi c^5}\frac{2\pi}{15}12\left\langle \dddot{Q}^{ij}\dddot{Q}^{ij}\right\rangle = \frac{G}{5c^5}\left\langle \dddot{Q}^{ij}\dddot{Q}^{ij}\right\rangle = \frac{G}{5c^5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}\dddot{M}_{kk}^2\right\rangle .} \tag{1.6.30}$$

When we do the same from the **momentum loss**, we get an integral in the form

$$\frac{\mathrm{d}P^k}{\mathrm{d}t} \propto \int \mathrm{d}\Omega\, \dddot{Q}_{ij}^{TT}\partial^k \ddot{Q}_{ij}^{TT} , \tag{1.6.31}$$

which is odd under spatial inversion because of the spatial derivative, so there is no contribution!

This is not true if we go beyond the quadrupole approximation: full GR calculations/simulations show that there can be kicks at the merger.

In linearized gravity, we can calculate the power loss by energy conservation:

$$\frac{\mathrm{d}E_{\text{source}}}{\mathrm{d}t} = -\frac{\mathrm{d}E_{\text{far-field}}}{\mathrm{d}t} = -\frac{G}{5c^5}\left\langle \dddot{Q}^{ij}\dddot{Q}^{ij}\right\rangle , \tag{1.6.32}$$

although this is not exact moment-by-moment in full GR, since there some of the emission can be delayed. In our approximation, the two sides of the equation should be calculated at the same retarded time: the power loss by the source at $t - r/c$ will be the power detected a distance r away at a time t.

1.6.2 Angular momentum loss from GW

Effective back-action force

We can model the back-action from GW as a force applied on the source [Mag07, sec. 3.3.4], whose power is the average of $\vec{F}\cdot\vec{v}$ as in classical mechanics; since we consider a continuous medium we have:

$$-\frac{G}{5c^5}\left\langle \frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3}\right\rangle = \frac{\mathrm{d}E_{\text{source}}}{\mathrm{d}t} = \left\langle \int \mathrm{d}^3x\, \frac{\mathrm{d}F_i}{\mathrm{d}V}\dot{x}_i \right\rangle . \tag{1.6.33}$$

We can integrate by parts twice to write the two third derivatives as

$$\left\langle \frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3}\right\rangle = \left\langle \frac{\mathrm{d}Q_{ij}}{\mathrm{d}t}\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}\right\rangle . \tag{1.6.34}$$

Also, the 0th component of the conservation of the stress-energy tensor $\partial_\mu T^{0\mu}$ for a classical source reads $\partial_t \rho + \partial_i(\rho v_i) = 0$. Using this fact, we can rewrite the first derivative of Q_{ij} as

$$\frac{\mathrm{d}Q_{ij}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\,\rho\left(x_i x_j - \underbrace{\frac{1}{3}r^2\delta_{ij}}_{\text{contracted with traceless } \dddot{Q}_{ij}}\right) = -\int \mathrm{d}^3x\,\partial_k(\rho v_k)x_i x_j \tag{1.6.35}$$

$$= +\int \mathrm{d}^3x\,\rho v_k \partial_k\left(x_i x_j\right) = 2\int \mathrm{d}^3x\,\rho v_{(i}x_{j)}\,. \tag{1.6.36}$$

Missing symmetrization in the slides! Although it is implicit when contracting with Q...

Then our equation reads:

$$\left\langle \int \mathrm{d}^3x\,\frac{\mathrm{d}F_i}{\mathrm{d}V}\dot{x}_i \right\rangle = -\frac{2G}{5c^5}\left\langle \int \mathrm{d}^3x\,\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}\rho\dot{x}_i x_j \right\rangle, \tag{1.6.37}$$

so we can identify the terms to get

$$\frac{\mathrm{d}F_i}{\mathrm{d}V} = -\frac{2G}{5c^5}\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}\rho(t,\vec{x})x_j\,, \tag{1.6.38}$$

so finally, since the only position-dependent terms on the right-hand side are the density and the position we get that the effective force is:

$$\boxed{F_i = -\frac{2G}{5c^5}\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}m\overline{x}_j\,,} \tag{1.6.39}$$

where $\overline{x}_j$ is the center-of-mass coordinate.

Angular momentum

With this effective force we can calculate the torque explicitly: in general it is given by $T_i = \epsilon_{ijk}x_j F_k$, which we can make into a local relation by substituting T_i and F_k with their densities. With this we can write

$$\frac{\mathrm{d}T_i}{\mathrm{d}V} = \epsilon_{ijk}x_j\frac{\mathrm{d}F_k}{\mathrm{d}V} = -\epsilon_{ijk}x_j\frac{2G}{5c^5}\frac{\mathrm{d}^5Q_{kl}}{\mathrm{d}t^5}\rho(t,\vec{x})x_l\,, \tag{1.6.40}$$

so the total torque is given by

$$T_i = -\frac{2G}{5c^5}\epsilon_{ijk}\frac{\mathrm{d}^5Q_{kl}}{\mathrm{d}t^5}\int \mathrm{d}^3x\,\rho(t,\vec{x})x_l x_j \underbrace{-\frac{1}{3}r^2\delta_{lj}}_{\text{does not contribute, since } Q \text{ is traceless}} \tag{1.6.41}$$

$$= -\frac{2G}{5c^5}\epsilon_{ijk}\frac{\mathrm{d}^5Q_{kl}}{\mathrm{d}t^5}Q_{lj}\,, \tag{1.6.42}$$

so if we take the average, substituting the derivative of the angular momentum for the torque and integrating by parts twice to get a second and a third derivative, we find:

$$\left\langle \frac{\mathrm{d}L_i}{\mathrm{d}t} \right\rangle = -\frac{2G}{5c^5}\epsilon_{ijk} \left\langle \ddot{Q}_{jl}\dddot{Q}_{kl} \right\rangle . \tag{1.6.43}$$

1.7 Back-reaction and the evolution of binary systems

1.7.1 Compact circular inspiral

We use the reduced mass formalism for a two-body problem: we define the total mass $M = m_1 + m_2$, the reduced mass $\mu = m_1 m_2 / M$, the relative coordinate $\vec{x} = \vec{x}_1 - \vec{x}_2$ whose modulus is $R = |\vec{x}|$. The angular velocity of the circular motion is ω_s, satisfying Kepler's law $\omega_s^2 R^3 = GM$.

As we saw, the amplitude of the emitted gravitational waves is given by

$$A = \frac{4G^{5/3}\omega_s^{2/3}\mu M^{2/3}}{rc^4} , \tag{1.7.1}$$

and the amplitudes of the two polarizations in time are given in terms of A by

$$h_+ = A\frac{1-\cos^2\theta}{2}\cos\big(2\omega_s t_{\mathrm{ret}} + 2\varphi\big) \tag{1.7.2a}$$

$$h_\times = A\cos\theta\sin\big(2\omega_s t_{\mathrm{ret}} + 2\varphi\big) . \tag{1.7.2b}$$

In order to make these expressions easier to interpret, we define the chirp mass:

$$M_c = \mu^{3/5}M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} . \tag{1.7.3}$$

Also, since the frequency of the emitted GW is double that of the binary we define $f_{GW} = 2f_s = 2\omega_s/(2\pi)$, and $\omega_{GW} = 2\pi f_{GW}$. The reduced wavelength corresponding to this frequency is $\lambdabar = c/\omega_{GW}$; also we can define a Schwarzschild radius corresponding to the chirp mass: $R_C = 2GM_c/c^2$.

With these definitions we can rewrite A as:

$$A = \frac{4}{r}\left(\frac{GM_C}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}}{c}\right)^{2/3} = \frac{1}{\sqrt[3]{2}}\frac{R_C}{r}\left(\frac{R_C}{\lambdabar}\right)^{2/3} , \tag{1.7.4}$$

As we have shown, in the quadrupole approximation the radiated power is:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}\left(\dddot{M}_{kk}\right)^2 \right\rangle , \tag{1.7.5}$$

which we can calculate explicitly for our binary. We have already derived expressions for the second time derivatives of the second mass moment, $\ddot{M}_{ij}$ for motion in the xy plane (1.5.55); taking their derivative we get

$$\dddot{M}_{11} = -\dddot{M}_{22} = -4\mu R^2\omega_s^3\sin(2\omega_s t) \qquad \text{and} \qquad \dddot{M}_{12} = 4\mu R^2\omega_s^3\cos(\omega_s t) , \tag{1.7.6}$$

so the term $\left(\dot{\dot{M}}_{ij}\right)^2$ vanishes ($\dot{\dot{M}}_{ij}$ is traceless) and $\dot{\dot{M}}_{11}^2 = \dot{\dot{M}}_{22}^2$, so the contribution we get is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}\left\langle 2\dot{\dot{M}}_{11}^2 + 2\dot{\dot{M}}_{12}^2 \right\rangle \tag{1.7.7}$$

$$= \frac{G}{5c^5} 2\left(4\mu R^2 \omega_s^3\right)^2 \underbrace{\left(\left\langle \sin^2(2\omega_s t)\right\rangle + \left\langle \cos^2(2\omega_s t)\right\rangle\right)}_{=1/2+1/2=1}, \tag{1.7.8}$$

since the average of the squared sine or cosine is 1/2. Now, we can simplify this expression by using $R^3 = GM/\omega_s^2$ and substituting in the GW angular velocity:

$$\frac{\mathrm{d}E_{GW}}{\mathrm{d}t} = \frac{32G}{5c^5}\mu^2 R^4 \omega_s^6 \tag{1.7.9}$$

$$= \frac{32G}{5c^5}\mu^2 \left(\frac{GM}{\omega_s}\right)^{4/3} \omega_s^6 \tag{1.7.10}$$

$$= \frac{32}{5c^5}\mu^2 G^{7/3} M^{4/3} \omega_s^{10/3} \tag{1.7.11}$$

$$= \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c \omega_{GW}}{2c^3}\right)^{10/3}. \tag{1.7.12}$$

We can apply a similar reasoning for the angular momentum loss (here $L = \left|\vec{L}\right|$):

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{GW}}{2c^3}\right)^{10/3}\frac{2}{\omega_{GW}} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c}{c^3}\right)^{10/3}\left(\frac{\omega_{GW}}{2}\right)^{7/3}. \tag{1.7.13}$$

If we have masses in circular orbit, then by the virial theorem their total energy is given by half of the potential energy:

$$E = -\frac{1}{2}\frac{Gm_1 m_2}{R}, \tag{1.7.14}$$

which we can differentiate with respect to time to get $\dot{E} = Gm_1m_2/(2R^2)\dot{R}$; this can be also written as

$$\dot{R} = -\frac{2R^2}{Gm_1 m_2}\dot{E}_{GW}, \tag{1.7.15}$$

as long as the orbital energy is only lost through gravitational wave emission.

This means that, as the GW carry away energy, the radius shrinks; this corresponds to an increase in frequency, and a corresponding increase in gravitational wave emission.

In all our calculations we assumed the orbits to be circular! This is fine, as long as they are almost-circular: the condition to require is that the variation of the radius across a single orbit is very small:

$$\left|\frac{\dot{R}T}{R}\right| \ll 1, \tag{1.7.16}$$

where T is the period. In order to express this conditions in terms of the orbital angular velocity ω_s we use Kepler's law: if we differentiate $R = \left(GM/\omega_s^2\right)^{1/3}$ we get

$$\dot{R} = -\frac{2}{3}\omega_s^{-5/3}(GM)^{1/3}\dot{\omega}_s = -\frac{2}{3}\frac{R}{\omega_s}\dot{\omega}_s \,. \tag{1.7.17}$$

This equality can also be written in terms of logarithmic derivatives as

$$\frac{\dot{R}}{R} + \frac{2}{3}\frac{\dot{\omega}_s}{\omega_s} = 0 \,. \tag{1.7.18}$$

Starting from this, and using $T = 2\pi/\omega_s$ we can write

$$\left|\frac{\dot{R}T}{R}\right| = \frac{2}{3}\frac{R}{\omega_s}\dot{\omega}_s\frac{2\pi}{\omega_s}\frac{1}{R} = \frac{4\pi}{3}\frac{\dot{\omega}_s}{\omega_s^2} \,, \tag{1.7.19}$$

which means that we need to require $\dot{\omega}_s \ll \omega_s^2$.

In practice, this condition is quite well satisfied for most of the inspiral, right up until the merger phase. Motion satisfying this condition is called **quasi-circular**.

1.7.2 Frequency evolution and time to coalesce

Friday 2020-4-3, compiled 2020-07-28

From the equation for the evolution of the binary's radius (1.7.15), its explicit expression in terms of ω_s (1.7.17) and the expression for the gravitational wave emitted power (1.7.12) we can write

$$-\frac{2}{3}\frac{\dot{\omega}_s}{\omega_s}R = -\frac{R^2}{GM\mu}\dot{E}_{GW} \tag{1.7.20}$$

$$\dot{\omega}_s = \frac{3}{\mu}(GM)^{-2/3}\omega_s^{1/3}\frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{GW}}{2c^3}\right)^{10/3} \tag{1.7.21}$$

Substituted in $\dot{E}$, used $R = (GM)^{1/3}\omega_s^{-2/3}$.

$$= \frac{6}{5}\sqrt[3]{2}\left(\frac{M_cG}{c^3}\right)^{5/3}\omega_{GW}^{11/3} \,. \tag{1.7.22}$$

If we substitute in ω_{GW} from ω_s we finally get the relation

$$\dot{\omega}_{GW} = \frac{12}{5}\sqrt[3]{2}\left(\frac{M_cG}{c^3}\right)^{5/3}\omega_{GW}^{11/3} \qquad \text{or} \qquad \dot{f}_{GW} = \underbrace{\frac{96}{5}\pi^{8/3}\left(\frac{M_cG}{c^3}\right)^{5/3}}_{k} f_{GW}^{11/3} \,. \tag{1.7.23}$$

This can be integrated directly, by separating the variables as $f^{-11/3}\,\mathrm{d}f = \mathrm{d}t$: the result is

$$-\frac{3}{8k}f^{-8/3} = t - t_{\text{coal}} \overset{\text{def}}{=} -\tau \,, \tag{1.7.24}$$

where t_{coal} is an integration constant, chosen so that $t = t_{\text{coal}}$ when the frequency diverges. We can get the frequency as a function of the time until coalescence τ:

$$f_{GW} = \left(\frac{3}{8k\tau}\right)^{3/8} = \frac{1}{\pi}\left(\frac{5}{256\tau}\right)^{3/8}\left(\frac{c^3}{GM_c}\right)^{5/8} \,. \tag{1.7.25}$$

Wrong sign in the slides at this point.

Inverting this expression we can get the time until coalescence from the frequency:

$$\tau = \frac{5}{256} \left(\frac{1}{\pi f_{GW}} \right)^{8/3} \left(\frac{c^3}{G M_c} \right)^{5/3} . \tag{1.7.26}$$

Now, we want to get an expression for the **evolution of the radius**. We know that $\omega_{GW} \propto \tau^{-3/8}$, so we can calculate

$$\frac{1}{\omega_{GW}} \frac{\mathrm{d}\omega_{GW}}{\mathrm{d}t} = -\frac{1}{\omega_{GW}} \frac{\mathrm{d}\omega_{GW}}{\mathrm{d}\tau} = +\frac{3}{8} \frac{\tau^{-11/8}}{\tau^{-3/8}} = \frac{3}{8} \frac{1}{\tau} , \tag{1.7.27}$$

which is also equal to $\dot{\omega}_s / \omega_s$, since it is a constant multiple of ω_{GW}. Then, by using the relation between the logarithmic derivatives of R and ω_s (1.7.18) we can write

$$\frac{\dot{\omega}_s}{\omega_s} = \frac{3}{8} \frac{1}{\tau} = -\frac{3}{2} \frac{\dot{R}}{R} \implies \frac{\dot{R}}{R} = -\frac{1}{4\tau} , \tag{1.7.28}$$

which is solved by

$$R(\tau) = R_0 \left(\frac{\tau}{\tau_0} \right)^{1/4} = R_0 \left(\frac{t_{\text{coal}} - t}{t_{\text{coal}} - t_0} \right)^{1/4} , \tag{1.7.29}$$

where τ_0 is the time to coalescence at t_0, and R_0 is the orbital radius at t_0.

If we plot this, it has a "plunge" phase near $\tau = 0$, near which our assumptions of a nonrelativistic system break down.

Add plot maybe

The radius and frequency at $R = R_0$ are related by Kepler's third law:

$$R_0 = \sqrt[3]{\frac{GM}{\omega_s^2(\tau_0)}} = \sqrt[3]{\frac{GM}{\pi^2 f_{GW}^2(\tau_0)}} , \tag{1.7.30}$$

which can be combined with the equation for the evolution of the GW frequency (1.7.25) to yield the following expression for the time until coalescence given the orbital parameters:

$$\tau_0 = \frac{5}{256} \frac{c^5 R_0^4}{G^3 M^2 \mu} . \tag{1.7.31}$$

1.7.3 Chirping waveform

The argument of the cosine is in general denoted as $\phi(t)$ and called the **phase**, it is defined by the relation

$$\phi(t) = \int_{t_0}^{t} \omega_{GW}(t') \, \mathrm{d}t' = -\int_{\tau_0}^{\tau} \mathrm{d}\tau' \, \frac{2\pi}{\pi} \left(\frac{5}{256} \frac{1}{\tau'} \right)^{3/8} \left(\frac{c^3}{G M_c} \right)^{5/8} \tag{1.7.32}$$

$$= -2\left(\frac{c^3}{5GM_c}\right)^{5/8}\tau^{5/8} + \phi_0\,, \tag{1.7.33}$$

which for a circular orbit is instead just a linear function of t; the angular frequency is given by its constant derivative $\omega_{GW} = \phi'$.

For a quasi-circular orbit we need to generalize, and the mass moments will look like

$$M_{11} = \frac{\mu R^2(t)\Big(1+\cos\big(\phi(t)\big)\Big)}{2} \qquad \text{and} \qquad M_{12} = \frac{\mu R^2(t)\sin\big(\phi(t)\big)}{2}\,, \tag{1.7.34}$$

however since $\dot{\omega}_{GW} \ll \omega_{GW}^2$ and $\dot{R} \ll R\omega_{GW}$ we will be able to neglect the derivatives of R and ω_{GW} when computing the second time derivatives of the mass moments. So, our two polarizations' amplitudes will read

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}(t_{\mathrm{ret}})}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos\big(\phi(t_{\mathrm{ret}})\big) \tag{1.7.35}$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}(t_{\mathrm{ret}})}{c}\right)^{2/3}\cos\theta\sin\big(\phi(t_{\mathrm{ret}})\big)\,, \tag{1.7.36}$$

which can be expressed in terms of the time to coalescence τ as

$$h_+(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\frac{1+\cos^2\theta}{2}\cos\big(\phi(\tau)\big) \tag{1.7.37}$$

$$h_\times(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\cos\theta\sin\big(\phi(\tau)\big)\,, \tag{1.7.38}$$

since as long as we are measuring from a fixed position the time to coalescence is the same for any observer seeing the same part of the signal: we can compute $\tau = t_{\mathrm{coal}} - t = (t_{\mathrm{coal}} + \Delta t) - (t + \Delta t)$ for any fixed shift Δt.

This model predicts a signal with a **chirping waveform**, whose amplitude increases and frequency both increase. The chirping waveform cannot be trusted near the end, when our quasi-circular orbit approximation breaks down.

A result from the study of the orbit of a test particle around a Schwarzschild black hole is that when the radius of the orbit goes below the Innermost Stable Circular Orbit:

$$R < R_{\mathrm{ISCO}} = \frac{6GM}{c^2} \tag{1.7.39}$$

the orbit becomes unstable. This formula holds for extreme mass ratios which can approximate the scenario of a test mass in a large masses' gravitational field. We have not seen any mergers like this yet, although it is definitely possible that they might exist.

Anyway, we can use the Schwarzschild radius as a rough guideline to see when the system becomes very relativistic and our approximations break down.

The frequency corresponding to the ISCO can be also calculated:

$$f_{GW,ISCO} = \frac{1}{6\sqrt{6}\pi}\frac{c^3}{GM}\,. \tag{1.7.40}$$

From numerical full-GR simulations and the measured GW signals we can see that the shape of the chirping waveform is basically correct even after the approximations break down; it only slightly goes out of phase with the numerical relativity calculation.

Numerical relativity has a *lower* frequency than the quadrupole approx!

1.7.4 Eccentric binaries

We only present the results of the rather long computations; radiated power is higher than the corresponding circular binary, it is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{5}\frac{G\mu^2}{c^5}a^4\omega_0^6 f(e)\,, \tag{1.7.41}$$

where

$$f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \geq 1\,. \tag{1.7.42}$$

Here the parameter e is the eccentricity of the elliptical orbit: the major semiaxis is given in terms of the minor one by $a = b/\sqrt{1-e^2}$. This enhancement is due to the fact that the emission is large at the point of closest approach.

This means that the time to coalescence is *shorter* for elliptical binaries: for an initial eccentricity $e_0 \sim 1$ we have $\tau \sim 2\left(1 - e_0^2\right)^{7/2}\tau_{\text{circ}}$.

We have emission at frequencies which are integer multiples of the circular-orbit one; the peak of this spectrum shifts upward as e increases.

GW emission has the effect of circularizing the orbit rapidly — by the time the merger arrives it is usually quite close to circular, so we usually detect circular systems.

1.8 Hulse-Taylor binaries

The detection of gravitational waves in 2015 [LIG+16] was the first *direct* one, but the existence of gravitational waves was established before that through indirect observations. By what we have shown in the last section, GW emission causes the decay of the orbit of a binary.

Weisberg and Taylor measured this effect [TW82, fig. 6 especially] on a binary pulsar discovered by Hulse and Taylor in the seventies [HT75]. It is debatable whether this was the first observation of gravitational waves, but anyway it is an interesting case study for us.

In this section we will explore the nature of pulsar radioastronomy, the sources of delays in the signal and the compatibility of the period decay result with theoretical GR predictions.

1.8.1 Pulsars

A **neutron star** is a kind of stellar remnant which is very dense, although not quite dense enough to form a black hole. It has the density of nuclear matter, $\rho \sim 10^{17}\,\mathrm{kg/m^3}$, since in it electronic repulsion between atoms has been overcome. Its mass is generally of the order of the mass of the Sun, so its radius is of the order of 10 km — on the scale of the Schwarzschild radius, but with a bit of a margin.

Neutron stars generally have very high angular velocities, which follows from conservation of angular momentum at the moment of collapse. Their moments of inertia, $I = (2/5)MR^2 = (8\pi/15)\rho R^5 \sim 10^{38}\,\mathrm{kgm^2}$, are very high, so their angular velocities are very stable.

They also generally have very high magnetic field, on the order of $B \sim 10^8\,\mathrm{T}$; the magnetic poles are often misaligned with the rotation axis of the NS, causing the magnetic field lines to spin with the star.

At a distance $r_c = c/\omega$ the field lines cannot close since they would need to move faster than c, so a radio beam escapes. This beam sweeps around with the same frequency as the pulsar, and since the period is very stable it provides a stable clock.

Pulsar signal measurement

Generally the radio signals coming from pulsars arrive buried in noise. However, we can still measure the signal, since it is characterized by a precise periodicity. We take a *Fourier transform* of the radio signal and identify a peak corresponding to the fundamental frequency of the pulsar's emission. Then, we average the signal across periods corresponding to this frequency.

The external noise will then average out, while the signal builds with each additional period.

The period can then be measured precisely, and we can observe its variations.

The pulsar which was discovered by Hulse and Taylor is called PSR 1913+16 [HT75]: it was discovered in 1974; its period (corrected for the effects which we will now mention) is $T = (59.030 \pm 0.001)\,\mathrm{ms}$.

The slides say that the 20 µs measurement error is random and does not accumulate: we can identify the pulse number even after years of not observing the pulsar. However, the paper reports a 1 µs error: where is the 20 µs figure coming from? Is it referring to the daily variations?

We can identify fluctuations in the radio signal period of around 80 µs daily: this is very large compared to what is typically observed for pulsars, variations around 10 µs *yearly*. This atypical behavior was understood by H&T to correspond to a **binary** pulsar, where the mass of the pulsar is $M_p = 1.4414 M_\odot$ (around the Chandrasekhar mass!), while the mass of the companion is $M_c = 1.3867 M_\odot$. As for the orbital parameters, the period is $T \approx 0.323\,\mathrm{d}$, the eccentricity is $e \approx 0.617\,133\,8$.

These are modern measurements, right? the paper has a much larger uncertainty on e.

These parameters are gathered by assuming that in the pulsar's frame the emission is stable, and that the delays come from effects pertaining, for example, to the orbit.

1.8.2 All the delays

Frequencies' timescales and effect summary

Let us discuss the magnitudes of the frequencies at hand: the radio waves are on the order of $f_{\text{radio}} \sim 10^8\,\text{Hz}$, the pulsar's frequency is of the order of $f_{\text{pulsar}} \sim 10\,\text{Hz}$, the frequency of the binary period is $f_{\text{binary}} \sim 10^{-5}\,\text{Hz}$, the motion of the Earth around the Sun at $f_{\text{Earth}} \sim 10^{-8}\,\text{Hz}$ is also relevant.

It is advantageous for us that there are many orders of magnitude between these: since the pulsar frequency is very small, we can still average many pulses and still be measuring at what is basically "a single point" in the orbit of the binary.

The effects which we will account for are:

1. Rømer delay (observer): due to the position of the Earth with respect to the solar system barycenter;
2. Einstein delay (observer): gravitational and Doppler shift at the observer due to the Earth's motion and gravitational field;
3. Shapiro delay (observer): delay due to propagation in the Sun's gravitational field;
4. propagation in the interstellar medium;
5. Rømer delay and aberration (emitter): position of the source with respect to the barycenter of the binary, must account for a relativistic orbit;
6. Einstein delay (emitter): gravitational and Doppler shift due to the pulsar's motion and gravitational field;
7. Shapiro delay (emitter): delay due to the propagation in the companion's gravitational field;
8. **secular changes**: reduction of the period due to GW emission!

Rømer delay (observer)

Monday 2020-4-6, compiled 2020-07-28

This effect is denoted as $\Delta_{R,\odot}$, it is due to the fact that if Earth is on the far side of the Sun the radio signal takes longer to get to it than it would if it were on the near side to the pulsar. The simplest way to model it is

$$\Delta_{R,\odot} = t_0 \cos(\Omega t - \lambda) \cos\big(\beta\big)\,, \tag{1.8.1}$$

where $t_0 = R/c$, R being the radius of the Earth's orbit (here taken to be circular), while λ and β define the position of the source in the orbital plane.

Proper modelling would require us to also take into account the ellipticity of the orbit and the Earth's rotation. In general, the effect can be written as

$$\Delta_{R,\odot} = -\frac{\vec{r}_{\text{ob}} \cdot \hat{n}}{c} , \tag{1.8.2}$$

where $\hat{n}$ denotes the direction of the source in barycentric coordinates for the solar system, while $\vec{r}_{\text{ob}}$ is the vector connecting the barycenter of the Solar System to the observer, which can be decomposed as a sum of vectors to the Sun's center, Earth's center and finally to the observer:

$$\vec{r}_{\text{ob}} = \vec{r}_{\text{ob},\oplus} + \vec{r}_{\oplus,\odot} + \vec{r}_{\odot,\text{barycenter}} . \tag{1.8.3}$$

Modeling this allows us get information about λ and β, which define $\hat{n}$. The **effect size** is dominated by the Earth's motion around the Sun, and it has a magnitude of about $\pm 10^3\,\text{s}$ in the span of six months.

Shapiro delay (observer)

This effect is due to the curvature introduced by the gravitational potential of the Sun. The Schwarzschild metric in Cartesian coordinates reads:

$$\begin{aligned} \mathrm{d}s^2 &= -\big(1+2\phi(x)\big)c^2\,\mathrm{d}t^2 + \frac{1}{\big(1+2\phi(x)\big)}\,\mathrm{d}\vec{x}^2 && (1.8.4) \\ &\approx -\big(1+2\phi(x)\big)c^2\,\mathrm{d}t^2 + \big(1-2\phi(x)\big)\,\mathrm{d}\vec{x}^2 , && (1.8.5) \end{aligned}$$

where $\phi(x) = -GM/|x|c^2 \ll 1$ is the gravitational potential in the weak-field approximation. Photons travel along null geodesics, so they have $\mathrm{d}s^2 = 0$: this means that

$$c\,\mathrm{d}t = \pm\sqrt{\frac{1-2\phi}{1+2\phi}}|\mathrm{d}x| \approx \pm\big(1-2\phi\big)|\mathrm{d}x| , \tag{1.8.6}$$

so we can compute the Shapiro delay as the correction to the total travel time:

$$c(t_{\text{obs}} - t_{\text{emit}}) = \int_{r_{\text{obs}}}^{t_{\text{emit}}} |\mathrm{d}x|\big|1-2\phi(r)\big| , \tag{1.8.7}$$

which automatically incorporates the Rømer delay, since we find

$$t_{\text{obs}} - t_{\text{emit}} = \underbrace{\frac{\big|\vec{r}_{\text{emit}} - \vec{r}_{\text{bary}}\big|}{c}}_{\text{simple propagation}} + \underbrace{\frac{\vec{r}_{\text{obs}} \cdot \hat{n}}{c}}_{\text{Rømer delay}} \underbrace{-\frac{2}{c}\int_{r_{\text{obs}}}^{r_{\text{emit}}} |\mathrm{d}x|\phi(x)}_{\text{Shapiro delay: } \Delta_{S,\odot}} . \tag{1.8.8}$$

The sign of the Rømer effect is different here! Intuitively we should expect the delay Δ_R to be > 0 when $\hat{n} \cdot \vec{r} < 0$, since then the Earth is *behind the barycenter*.

Let us explicitly compute $\Delta_{S,\odot}$: we consider a reference frame in which the Sun is at the center, the Earth is on the x axis at a distance $r_{\oplus\odot}$, while the pulsar is at a distance ρ from the Earth and at an angle θ with respect to the Sun-Earth axis: so, the Sun-pulsar distance r is given by

$$r^2 = \left(r_{\oplus\odot} + \rho\cos\theta\right)^2 + \left(\rho\sin\theta\right)^2 = r_{\oplus\odot}^2\left(u^2 + 2u\cos\theta + 1\right), \tag{1.8.9}$$

Used $\cos^2\theta + \sin^2\theta = 1$.

where $u = \rho / r_{\oplus\odot}$. The delay is given by

$$\begin{aligned}
\Delta_{S,\odot} &= -\frac{2}{c}\int_{r_{\text{obs}}}^{r_{\text{emit}}} |\mathrm{d}x|\phi(x) = \frac{2}{c}\int_0^d \mathrm{d}\rho\,\frac{GM_\odot}{c^2 r} && (1.8.10)\\
&= \frac{2GM_\odot}{c^3}\int_0^{d/r_{\oplus\odot}} \mathrm{d}u\,\frac{r_{\oplus\odot}}{r} && (1.8.11)\\
&= \frac{R_{S,\odot}}{c}\int_0^{d/r_{\oplus\odot}} \mathrm{d}u\,\frac{1}{\sqrt{u^2 + 2u\cos\theta + 1}}, && (1.8.12)
\end{aligned}$$

where $R_{S,\odot}$ is the Sun's Schwarzschild radius, while d is the distance from the Earth to the pulsar. We used the fact that $|\mathrm{d}x| = \mathrm{d}\rho$: the differential $|\mathrm{d}x|$ represents the *coordinate distance* along the path from the Earth to the pulsar, which is linearly parametrized by ρ; r on the other hand can be used to parametrize the curve but that would be a *nonlinear* parametrization, not useful for our purposes.

We need to estimate this integral; it is a divergent one but the divergence is only logarithmic. It is reasonable to estimate it assuming $d/r_{\oplus\odot} \gg 1$ — realistically this is of the order of 10^9, since the distance to the pulsar is of the order 5 kpc [HT75, pag. L53].

So, we add and subtract $1/\sqrt{u^2+1}$ in the integrand: then we find

$$\Delta_{S,\odot} = \frac{R_{S,\odot}}{c}\left[\int_0^{d/r_{\oplus\odot}} \mathrm{d}u\,\frac{1}{\sqrt{u^2+1}} + \int_0^{d/r_{\oplus\odot}} \mathrm{d}u\,\frac{1}{\sqrt{u^2 + 2u\cos\theta + 1}} - \frac{1}{\sqrt{u^2+1}}\right], \tag{1.8.13}$$

where the first integral is a hyperbolic arcsine, which can be estimated by $\log(2d/r_{\oplus\odot})$; the second term can be estimated with the following

Claim 1.8.1.

$$\int_0^\infty \mathrm{d}u\left[\frac{1}{\sqrt{u^2 + 2u\cos\theta + 1}} - \frac{1}{\sqrt{u^2+1}}\right] = -\log\left(1 + \cos\theta\right). \tag{1.8.14}$$

Proof. Mathematica says so. □

Then finally our estimate is

$$\boxed{\Delta_{S,\odot} = \frac{R_{S,\odot}}{c}\left(\log\left(\frac{2d}{r_{\oplus\odot}}\right) - \log(1 + \cos\theta)\right),} \tag{1.8.15}$$

where θ varies seasonally, as the Earth moves around the Sun, while d is basically constant.

It might seem that this can diverge as $\theta \to \pi$, but this is not actually the case: that would correspond to the radiation coming through the center of the Sun, which it cannot.

Actually, there is a range of values of θ around π for which the radiation cannot reach us since the Sun is in the way. The maximum value of θ that can be reached, as the radiation is tangent to the surface of the Sun, is called the grazing angle θ_g. This angle can be estimated by $\theta_g \approx \pi - R_\odot / r_{\oplus\odot}$.

The delay is maximal when the radiation grazes the Sun, and minimal when it is coming from the opposite direction as the Sun: $\theta = 0$. So, the **maximum size of the effect** is given by the difference

$$\Delta_{S,\odot}(\theta = \theta_g) - \Delta_{S,\odot}(\theta = 0) = \frac{R_{S,\odot}}{c}\left[\log\left(\frac{2d}{r_{\oplus\odot}}\right) - \log\left(1 + \theta_g\right) - \log\left(\frac{2d - 2r_{\oplus\odot}}{r_{\oplus\odot}}\right)\right] \tag{1.8.16}$$

$\log 1 = 0$.

$$\approx -\frac{R_{S,\odot}}{c}\log\left(1 + \cos\left(\pi - \frac{R_\odot}{r_{\oplus\odot}}\right)\right) \tag{1.8.17}$$

$$\approx -\frac{R_{S,\odot}}{c}\log\left(1 - 1 + \frac{1}{2}\left(\frac{R_\odot}{r_{\oplus\odot}}\right)^2\right) \tag{1.8.18}$$

Used $\cos(\pi - x) = -\cos(x)$ and expanded.

$$\approx -\frac{R_{S,\odot}}{c}\log\left(\frac{1}{2}\left(\frac{R_\odot}{r_{\oplus\odot}}\right)^2\right) \approx +56\,\mu\text{s}\,, \tag{1.8.19}$$

where we discarded the difference of the two logarithms with d since the relative difference between their arguments is on the order 10^{-9}.

In the slides this is reported as being on the scale of $100\,\mu$s with the logarithm's argument being $2r_{\oplus\odot}/d$: I do not see how that could come about, and a dependence on d seems not to make sense physically.

Einstein delay (observer)

This effect is about the gravitational shift and *transverse* Doppler shift due to gravitational field of the Earth. We start from the proper time at the observer:

$$c^2\,\mathrm{d}\tau^2 \approx \left(1 + 2\phi(x_{\text{obs}})\right)\mathrm{d}t^2 - \left(1 - 2\phi(x_{\text{obs}})\right)\mathrm{d}\vec{x}^2_{\text{obs}} \tag{1.8.20}$$

$$\frac{\mathrm{d}\tau^2}{\mathrm{d}t^2} \approx 1 + 2\phi(x) - \frac{1 - 2\phi(x)}{c^2}\frac{\mathrm{d}x^2}{\mathrm{d}t^2} \tag{1.8.21}$$

$$= 1 + 2\phi(x) - \frac{v^2}{c^2} + \mathcal{O}(\phi v^2/c^2)\,, \tag{1.8.22}$$

and if we discard the higher order terms (both ϕ and v/c are small for an Earth-bound observer, so a term containing their product is negligible) we can expand the square root as

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} \approx 1 + \phi(x_{\text{obs}}) - \frac{v^2_{\text{obs}}}{2c^2}\,, \tag{1.8.23}$$

so

$$\tau \approx t + \int^t \mathrm{d}\widetilde{t}\left(\phi\left(x_{\text{obs}}(\widetilde{t})\right) - \frac{v^2_{\text{obs}}(\widetilde{t})}{2c^2}\right) = t - \Delta_{\oplus,\odot}\,, \tag{1.8.24}$$

where the lower bound of the integration is arbitrary, amounting only to a shift in the origin from which we measure times.

If most of the velocity is due to the motion of the Earth in its elliptic orbit, from Keplerian dynamics we know that the energy is given by the *vis viva* equation:

$$E = -\frac{GM\mu}{2a} = \frac{1}{2}\mu v_{\text{obs}}^2 - \frac{GM\mu}{r}\,, \tag{1.8.25}$$

which means that the square velocity can be written as

$$\frac{v_{\text{obs}}^2}{2} = \frac{GM}{r} - \frac{GM}{2a}\,, \tag{1.8.26}$$

so that we find

$$\frac{\mathrm{d}\Delta_{E,\odot}}{\mathrm{d}t} \approx \frac{v^2}{2c^2} - \phi = \frac{GM_\odot}{c^2}\left(\frac{1}{r} - \frac{1}{2a} + \frac{1}{r}\right) = R_{S,\odot}\left(\frac{1}{r} - \frac{1}{4a}\right), \tag{1.8.27}$$

where $R_{S,\odot} = 2GM_\odot/c^2$ is the Schwarzschild radius of the Sun.

Wrong sign in the slides!

The term depending on r is relevant to the modulation of our signal; the other one is a correction to apply to our estimates of the parameters of the binary and pulsar but it is not relevant to the modulation as it is a constant.

In order to estimate the magnitude of this modulation we need to ask by how much the radius of the Earth's orbit changes: the average orbital distance is around 150×10^9 m, with annual variations of the order of 2×10^9 m. So, the magnitude of the first term is around 10^{-8} seconds per second, with annual variations on the order of 10^{-10} seconds per second.

Not sure about the ± 4 ns/s: that would mean 4×10^{-9} seconds per second... but the number we get is not the *amplitude* of the oscillation, it is the value of the correction: the oscillation is much smaller, corresponding to the variation of the orbital radius of the Earth!

Interstellar medium propagation

The ISM is made of ionized gas mostly, and it causes the group velocity of the radio signal to not be precisely c: there is a correction depending on the frequency ν of the signal, given by

$$v_g \approx c\left(1 - \frac{n_e e^2}{2\pi m_e}\frac{1}{\nu^2}\right), \tag{1.8.28}$$

which causes a total time lag in the arrival of a pulse of fixed frequency:

$$t_d = \int_0^d \frac{\mathrm{d}l}{v_g} \approx \frac{d}{c} + \frac{e^2}{2\pi m_e}\frac{1}{\nu^2}\int_0^d n_e \,\mathrm{d}l \tag{1.8.29}$$

First order approximation of $1/(1-x) \sim 1 + x$.

$$= \frac{d}{c} + \frac{e^2}{2\pi m_e c}\frac{1}{\nu^2} DM\,, \tag{1.8.30}$$

where DM is the Dispersion Measurement: it corresponds to the column number density of electrons between us and the pulsar, and it has the units of an inverse area.

For the HT pulsar, the bandwidth of the interesting signal is around 4 MHz, and from the top to the bottom of the band the time delay is of the order of 70 ms!

Fortunately we can measure precisely in the spectral domain: the arrival times for spectral bands which are near to each other will be very close, and they will form a continuous line: so, we can "connect the dots" to find what corresponds to a single pulse. The process of doing this and correcting for the DM is called **de-dispersion**; after doing it we can sum all the components in order to get the original pulse.

Combination of local effects

Taking all the effects pertaining to the source into account, we get

$$t_{\mathrm{ssb}} = \tau - \frac{D}{\nu^2} + \Delta_{E,\odot} - \Delta_{S,\odot} + \Delta_{R,\odot}\,, \tag{1.8.31}$$

where t_{ssb} means "time in solar-system barycenter", and

$$D = \frac{e^2}{2\pi m_e c} DM\,. \tag{1.8.32}$$

Einstein delay at the source

We need to look at the gravitational time delay at the source: the gravitational field will be able to be written as

$$\phi(x) = -\frac{Gm_p}{c^2\left|\vec{x} - \vec{x}_p\right|} - \frac{Gm_c}{c^2\left|\vec{x} - \vec{x}_c\right|}\,, \tag{1.8.33}$$

where p and c denote "pulsar" and "companion" respectively. When we compute the delay due to the source we cannot apply the weak-field approximation, which makes the calculation hard; however, this effect gives a constant frequency shift, which we cannot observe.

The potential from the companion, on the other hand, modulates the signal and is therefore observable. So, like we did for the Earth-Sun system we can write the proper-to-coordinate time ratio as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 - \frac{Gm_c}{c^2\left|x_p - x_c\right|} - \frac{v_p^2}{2c^2} \tag{1.8.34a}$$

$$\frac{\mathrm{d}T}{\mathrm{d}u} \approx \frac{P_b}{2\pi}\left(1 - \frac{G}{c^2}\frac{2m_c m_p + 3m_c^2}{2a(m_p + m_c)}\right)\left(1 - e\cos u\left(1 + \frac{G}{c^2}\frac{m_c(m_p + 2m_c)}{a(m_p + m_c)}\right)\right) \tag{1.8.34b}$$

$$= \frac{P_b}{2\pi}(1 - e\cos u) - \gamma\cos u\,, \tag{1.8.34c}$$

where we plugged in the equation for relativistic eccentric motion; u is the adimensional angular parameter describing the orbit, which is related to the coordinate time by $u - e\sin u = (2\pi/P_b)(t - t_0)$, while e is the eccentricity. This expression is post-Newtonian: it is valid up to first order in G. Its simpler formulation is given in terms of the **Einstein parameter**

$$\gamma = e\frac{P_b}{2\pi}\frac{G}{c^2}\frac{m_c(m_p + 2m_c)}{a(m_p + m_c)} = e\left(\frac{P_b}{2\pi}\right)^{1/3}\frac{G^{2/3}}{c^2}\frac{m_c(m_p + 2m_c)}{(m_p + m_c)^{4/3}}\,. \tag{1.8.35}$$

What is the magnitude of this effect? Well, we define the Einstein time difference Δ_E by $T = t - \Delta_E$; we can differentiate this equality with respect to u to find

$$\frac{\mathrm{d}\Delta_E}{\mathrm{d}u} = \frac{\mathrm{d}t}{\mathrm{d}u} - \frac{\mathrm{d}T}{\mathrm{d}u} = \frac{P_b}{2\pi}(1 - e\cos u) - \frac{P_b}{2\pi}(1 - e\cos u) + \gamma\cos u = \gamma\cos u\,, \tag{1.8.36}$$

where we computed $\mathrm{d}t/\mathrm{d}u$ starting from $t = (P_b/2\pi)(u - e\sin u) + \text{const}$, which comes from the definition. So, the amplitude of the modulation comes from the parameter γ, which has the units of a time, roughly on the order of magnitude of the period of the binary multiplied by the ratio between the Schwarzschild radius of the companion and the semimajor axis of the binary: for the Hulse-Taylor pulsar it comes out to be of the order 4.292 ms.

Rømer delay (source)

Like in the Solar System, we have the Rømer delay, which is given in general by

$$\Delta_R = \frac{\hat{z}\cdot x_{pb}}{c}\,, \tag{1.8.37}$$

where $\hat{z}$ is the direction of the Earth, while x_{pb} is the vector connecting the binary barycenter to the pulsar.

The polar coordinates r_1 and ψ for a Keplerian orbit are given by

$$r_{pb} = r_1 = a_1(1 - e\cos u) \qquad \text{and} \qquad \cos\psi = \frac{\cos u - e}{1 - e\cos u}\,, \tag{1.8.38}$$

where u is the angular parameter of the orbit, while a_1 is its semimajor axis.

The Rømer delay can then be written in terms of the *observation angle* ι (the angle between the observation direction and the normal to the orbital plane) and the *argument of periapsis* ω (the angle of the orbit at the point in which the body is closest to us — this is simply $\pi/2$ for circular orbits, it can differ for elliptic ones):

$$\begin{aligned}\Delta_R = r_1\sin\iota\sin(\omega + \psi) &= r_1\sin\iota(\cos\psi\sin\omega + \cos\omega\sin\psi) &(1.8.39)\\ &= r_1\sin\iota\Big((\cos u - e)\sin\omega + \sqrt{1 - e^2}\sin u\cos\omega\Big)\,. &(1.8.40)\end{aligned}$$

Note that if ψ is measured from the *line of nodes*, the line where the orbital plane crosses the plane drawn from the barycenter and normal to the observation direction, $\omega + \psi$ gives the angle of the pulsar from the line of nodes.

This is all classical — the relativistic corrections, however, are large. We will not do the calculation, instead we show the result: it is written similarly,

$$\Delta_R = a_1 \sin\iota \left((\cos u - e_r) \sin\omega + \sqrt{1 - e_\theta^2} \sin u \cos\omega \right), \tag{1.8.41}$$

where

$$e_{r,\theta} = e(1 + \delta_{r,\theta}) \tag{1.8.42a}$$

$$\delta_r = \frac{G}{c^2} \frac{3m_p^2 + 6m_p m_c + 2m_c^2}{a(m_p + m_c)} \tag{1.8.42b}$$

$$\delta_\theta = \frac{G}{c^2} \frac{(7/2)m_p^2 + 6m_p m_c + 2m_c^2}{a(m_p + m_c)}. \tag{1.8.42c}$$

To get a feeling for the numbers: if the masses of the objects are comparable and similar to m, the order of magnitude of the corrections is given by the ratio Gm/ac^2, the ratio of the Schwarzschild radius corresponding to the mass to the binary's radius.

Also, note that here the advance of the periastron is much more significant than it is for Mercury, it is given by

$$\langle\dot{\omega}\rangle = \frac{3G}{c^2} \left(m_p + m_c \right)^{2/3} \left(\frac{2\pi}{P_b} \right)^{5/3} \frac{1}{1 - e^2} \sim 4.2\,^\circ/\mathrm{yr}. \tag{1.8.43}$$

Shapiro delay (source)

The computation to get the Shapiro delay at the source is similar to that of the Solar System. We only need to account for the companion's gravitational field, the pulsar's is included in the Einstein delay.

The expression comes out to be

$$\Delta_S = -2\frac{Gm_c}{c^3} \log\left((1 - e\cos u) - \sin\iota \left(\sin\omega(\cos u - e) + \sqrt{1 - e^2} \cos\omega \sin u \right) \right). \tag{1.8.44}$$

To get all the corrections we need to account for the *relativistic aberration* due to the motion of the source, as well as the *Doppler shift* due to the relative motion of the barycenters of the Solar System and the binary system.

1.8.3 Putting all the parameters together

The parameters of the system can be categorized into:

1. the pulsar's parameters: position, frequency, phase, spin-down and others;

2. the Keplerian parameters of the orbit: $P_b, T_0, x = (a/c)\sin\iota, e, \omega$;

3. the relativistic corrections to the orbit: $\dot{\omega}, \gamma, P_b, r, s, \delta_\theta, \dot{e}, \dot{x}$.

The system is fully characterized by all the classical parameters, plus two of the relativistic ones. If we can measure three relativistic parameters, we can use the third as a GR test.

We extract from the data the classical parameters, the precession of the periastron $\langle \dot{\omega} \rangle$ and the Einstein parameter γ. Then, we use the derivative of the orbital period $\dot{P}$ to test GR.

The prediction from the calculations is

$$\dot{P}_{b,GR} = -\frac{192\pi G^{5/3}}{5c^5} \frac{m_p m_c}{(m_p + m_c)^{1/3}} \left(\frac{P_b}{2\pi} \right)^{-5/3} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right) \tag{1.8.45}$$

$$\approx -(2.402\,42 \pm 0.000\,02) \times 10^{-12} \,, \tag{1.8.46}$$

and the measured value is $-(2.4056 \pm 0.0051) \times 10^{-12}$. The GR prediction (not a fit to the data, the theoretical prediction starting from the measured parameters!) fits the data extremely well.

1.9 GW from a rotating rigid body

Classical mechanics recap

The moment of inertia tensor can be defined as

$$I^{ij} = \int \mathrm{d}^3 x \, \rho(x) \left(r^2 \delta^{ij} - x^i x^j \right) . \tag{1.9.1}$$

There exists an *orthogonal* frame in which this tensor is diagonal, its eigenvalues are the moments of inertia, its eigenvectors are the axes of inertia: $I'^{,ij} = \mathrm{diag}(I_1, I_2, I_3)$. This is called the **body frame**. The moments in this frame can be written as:

$$I_1' = \int \mathrm{d}^3 x' \, \rho \left(\left(x_1'^{,2} + x_2'^{,2} + x_3'^{,2} \right) \delta^{11} - x'^{,1} x'^{,1} \right) = \int \mathrm{d}^3 x \, \rho \left(x_2'^{,2} + x_3'^{,2} \right) . \tag{1.9.2}$$

If the body is shaped like an ellipsoid with axes a, b, c and mass m then we have

$$I_1 = \frac{m}{5} \left(b^2 + c^2 \right) . \tag{1.9.3}$$

In general the angular momentum is given in terms of the angular velocity by the tensor equation $J_i = I_{ij} \omega_j$; in the body frame this simplifies to three scalar equations: $J_1' = I_1' \omega_1'$ and so on.

The rotational kinetic energy is given by

$$E_{\mathrm{rot}} = \frac{1}{2} I_{ij} \omega_i \omega_j \tag{1.9.4a}$$

$$= \frac{1}{2} I_i' \omega_i'^{,2} \,, \tag{1.9.4b}$$

where the last equality holds in the body frame.

1.9.1 GW emission from a rigid body

Suppose we have a body spinning its principal axis x'_3 with angular velocity ω_r; then the position of any point can be recovered from the body frame using a time-dependent rotation matrix R_{ij}:

$$x'_i = R_{ij}x_j = \begin{bmatrix} \cos(\omega_r t) & \sin(\omega_r t) & 0 \\ -\sin(\omega_r t) & \cos(\omega_r t) & 0 \\ 0 & 0 & 1 \end{bmatrix}_{ij} x_j \,. \tag{1.9.5}$$

The inertia tensor transforms as $I'_{ij} = R_{ik}R_{jl}I_{kl} = (RIR^\top)_{ij}$, and we can reverse this relation to express the time-dependent inertia tensor as a time-dependent rotation of the fixed body frame tensor:

$$I = R^\top I' R \,. \tag{1.9.6}$$

Let us write this transformation law explicitly, using $c = \cos(\omega_r t)$ and $s = \sin(\omega_r t)$ for compactness and writing only the xy components, since the I_{3j} components for any j are unchanged:

$$\begin{aligned}
R^\top I' R &= \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} && (1.9.7)\\
&= \begin{bmatrix} cI_1 & -sI_2 \\ cI_1 & cI_2 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} && (1.9.8)\\
&= \begin{bmatrix} c^2 I_1 - s^2 I_2 & cs(I_1 - I_2) \\ cs(I_1 - I_2) & s^2 I_1 - c^2 I_2 \end{bmatrix} && (1.9.9)\\
&= \begin{bmatrix} \frac{I_1+I_2}{2} + \frac{I_1-I_2}{2}\cos(2\omega_r t) & \frac{I_1-I_2}{2}\sin(2\omega_r t) \\ \frac{I_1-I_2}{2}\sin(2\omega_r t) & \frac{I_1+I_2}{2} - \frac{I_1-I_2}{2}\cos(2\omega_r t) \end{bmatrix}, && (1.9.10)
\end{aligned}$$

where we used the identities $2\cos^2 x = \cos(0) + \cos x$, $2\sin^2(x) = \cos(0) - \cos x$ and $2\sin x \cos x = \sin(2x)$.

The second mass moment M^{ij} can be written in the nonrelativistic approximation as

$$M^{ij} = \frac{1}{c^2}\int \mathrm{d}^3x\, T^{00}(x)x^i x^j \approx \int \mathrm{d}^3x\, \rho(x)x^i x^j = -I^{ij} + \int \mathrm{d}^3x\, \rho(x) r^2 \delta^{ij} \,. \tag{1.9.11}$$

Claim 1.9.1. *When we compute the time derivative of M^{ij} the integral of $\rho r^2 \delta^{ij}$ does not contribute.*

Proof. We can write the term (without the delta, which is a constant anyway) as

$$\int \mathrm{d}^3x\, \rho(x) r^2 = \mathrm{Tr}\left[\int \mathrm{d}^3x\, \rho(x) x^i x^j\right] = \mathrm{Tr}\left[K^{ij}\right], \tag{1.9.12}$$

but this K^{ij} is a tensor, so it can be written as $K = R^\top K' R$ where K' is in the body frame. Then, we can use the cyclic property of the trace to show that $\mathrm{Tr}\left[R^\top K' R\right] = \mathrm{Tr}\left[K' R R^\top\right] = \mathrm{Tr}\left[K'\right]$, which is a constant. □

So, we can write $\ddot{M}^{ij} = -\ddot{I}^{ij}$: explicitly,

$$\ddot{M}^{11} = -\ddot{M}^{22} = -\ddot{I}^{11} = +\ddot{I}^{22} = 2\omega_r^2 (I_1 - I_2)\cos(2\omega_r t) \tag{1.9.13}$$

$$\ddot{M}^{12} = -\ddot{I}^{12} = 2\omega_r^2 (I_1 - I_2)\sin(2\omega_r t)\,, \tag{1.9.14}$$

while the other components of $\ddot{M}^{ij}$ vanish.

We have general expressions for the GW amplitudes in the two polarizations if we observe from a generic direction (θ, ϕ). Let us select a direction $(\theta, \phi) = (\iota, 0)$ — we do not lose generality by fixing ϕ, since it only amounts to a phase shift. Then, the expressions give us:

$$h_+ = \frac{G}{rc^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\cos^2\iota\right) = \frac{4}{r}\frac{G\omega_r^2}{c^4}(I_1 - I_2)\frac{1+\cos^2\iota}{2}\cos(2\omega_r t) \tag{1.9.15}$$

$$h_\times = \frac{G}{rc^4} 2\ddot{M}_{12}\cos\iota = \frac{4}{r}\frac{G\omega_r^2}{c^4}(I_1 - I_2)\cos\iota\sin(2\omega_r t)\,. \tag{1.9.16}$$

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We define the **ellipticity** $\epsilon = (I_1 - I_2)/I_3$. Typical values of this parameter for astrophysical objects are at most of the order of 10^{-6}, which can be calculated as $\epsilon \sim (\delta R/R_0)^2$, where δR is the scale of the radial anomaly while R_0 is the scale of the radius of the object. For a neutron star this corresponds to "mountains" of about $\delta R \sim 10\,\text{m}$.

Then, we can define a typical amplitude h_0 as:

$$h_0 = \frac{4\pi^2 G}{c^4}\frac{f_{GW}^2}{r} I_3 \epsilon\,, \tag{1.9.17}$$

where, as usual, $f_{GW} = \omega_r/\pi = 2 f_{\text{rotation}}$.

In terms of typical orders of magnitude, this variable looks like

$$h_0 \sim 10^{-25}\left(\frac{\epsilon}{10^{-6}}\right)\left(\frac{I_3}{10^{38}\,\text{kgm}^2}\right)\left(\frac{10\,\text{kpc}}{r}\right)\left(\frac{f_{GW}}{1\,\text{kHz}}\right)^2. \tag{1.9.18}$$

With this, we can rewrite the amplitudes in the two polarizations as

$$h_+ = h_0 \frac{1+\cos^2\iota}{2}\cos\left(2\pi f_{GW} t\right) \tag{1.9.19}$$

$$h_\times = h_0 \cos\iota \sin\left(2\pi f_{GW} t\right)\,. \tag{1.9.20}$$

To find the radiated power by this mechanism we can use the quadrupole formula (1.6.30):

$$\frac{\mathrm{d}E_{GW}}{\mathrm{d}t} = \frac{G}{5c^5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}\underbrace{\left(\dddot{M}_{kk}\right)^2}_{=0}\right\rangle \tag{1.9.21}$$

$$= \frac{G}{5c^5} 2\left\langle \dddot{M}_{11}^2 + \dddot{M}_{12}^2\right\rangle \tag{1.9.22}$$

$$= \frac{2G}{5c^5} \left(4\omega_r^3 (I_1 - I_2) \right)^2 \underbrace{\left\langle \cos^2(2\omega_r t) + \sin^2(2\omega_r t) \right\rangle}_{=1/2+1/2} \tag{1.9.23}$$

$$= \frac{32G}{5c^5} \omega_r^6 \epsilon^2 I_3^2 \,, \tag{1.9.24}$$

so by conservation of energy the neutron star will lose just as much energy. The rotational energy is given by $E_{\text{rot}} = I_3 \omega_r^2 / 2$, so we have

$$\frac{\mathrm{d}E_{\text{rot}}}{\mathrm{d}t} = -\frac{\mathrm{d}E_{GW}}{\mathrm{d}t} = I_3 \omega_r \dot{\omega}_r \tag{1.9.25}$$

$$-\frac{32G}{5c^5} \omega_r^6 \epsilon^2 I_3^2 = I_3 \omega_r \dot{\omega}_r \tag{1.9.26}$$

$$\dot{\omega}_r = -\frac{32G}{5c^5} \omega_r^5 \epsilon^2 I_3 < 0 \,, \tag{1.9.27}$$

so, *as opposed to binaries*, the orbit **slows down** because of GW emission. Observations of binaries show $\dot{\omega} \sim -\omega^n$ with $n < 5$, meaning that there probably is another breaking mechanism contributing.

1.9.2 Precession

Now, let us consider a body whose angular momentum $\vec{J}$ is *not aligned* with its axes of inertia [Mag07, sec. 4.2.2].

We want to proceed like we did before, so we will need two reference frames. The first reference, S, is a frame in which $\vec{J} = J\hat{z}$; this will be at least approximately an inertial reference frame, so $\vec{J}$ will be conserved. The second reference, S', is the *body frame* of the object, in which it is stationary, and whose axes coincide with the principal axes of rotation.

The transformation between these two frames will be a rotation matrix R (such that $x' = Rx$), which we decompose as

$$R = R_\gamma^{(z)} R_\alpha^{(x)} R_\beta^{(z)} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{1.9.28}$$

We call the *line of nodes* the intersection between the plane orthogonal to x_3 and that orthogonal to x_3'. The β rotation brings x_1 on the line of nodes, the α rotation brings x_3 onto x_3', the γ rotation aligns x_1 with x_1'. In order to understand this, it is customary to look at the figure [Mag07, fig. 4.15] and fiddle around with your fingers in the "right-hand-rule" position.

All three of these angles will in general be time-dependent, and their time evolution will completely determine the motion of the body.

We can recover the angular velocity vector $\vec{\omega}$ by looking at the components of the three angular velocity vectors in the body frame:

$$\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}t} = \dot{\alpha} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \end{bmatrix}^\top \tag{1.9.29}$$

$$
\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}t} = \dot{\beta} \begin{bmatrix} \sin\alpha\sin\gamma & \sin\alpha\cos\gamma & \cos\alpha \end{bmatrix}^\top \tag{1.9.30}
$$

$$
\frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}t} = \dot{\gamma} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top , \tag{1.9.31}
$$

so that then $\vec{\omega} = \vec{\alpha} + \vec{\beta} + \vec{\gamma}$. These expressions can be derived geometrically by looking at the figure. So, in the body frame the components of the angular velocity are

$$
\vec{\omega} = \begin{bmatrix} \dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma \\ -\dot{\alpha}\sin\gamma + \dot{\beta}\sin\alpha\cos\gamma \\ \dot{\gamma} + \dot{\beta}\cos\alpha \end{bmatrix} . \tag{1.9.32}
$$

In the body frame the angular momentum $\vec{J}$ is *not* conserved: we can recover its time-dependent expression in the body frame J' by applying a rotation, and then we can use $J'_i = I_i\omega'_i$: this gives us

$$
J'_1 = I_1\omega'_1 \qquad \Longrightarrow \qquad J\sin\alpha\sin\gamma = I_1\Big(\dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma\Big) \tag{1.9.33}
$$

$$
J'_2 = I_2\omega'_2 \qquad \Longrightarrow \qquad J\sin\alpha\cos\gamma = I_2\Big(-\dot{\alpha}\sin\gamma + \dot{\beta}\sin\alpha\cos\gamma\Big) \tag{1.9.34}
$$

$$
J'_3 = I_3\omega'_3 \qquad \Longrightarrow \qquad J\cos\alpha = I_3\Big(\dot{\gamma} + \dot{\beta}\cos\alpha\Big) . \tag{1.9.35}
$$

Now we make the assumption that $I_1 = I_2$: we consider an **axisymmetric body**. An astrophysical example of this will usually look like an ellipsoid.

Then, we perform the following manipulation (written in a formally peculiar way, which should make it easier to remember — we are "applying a rotation matrix to the system of equations"):

$$
\begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} J\sin\alpha\sin\gamma = I_1\Big(\dot{\alpha}\cos\gamma + \dot{\beta}\sin\alpha\sin\gamma\Big) \\ J\sin\alpha\cos\gamma = I_2\Big(-\dot{\alpha}\sin\gamma + \dot{\beta}\sin\alpha\cos\gamma\Big) \end{bmatrix} = \tag{1.9.36}
$$

$$
= \begin{bmatrix} I_1\dot{\alpha}\Big(\cos^2\gamma + \sin^2\gamma\Big) = 0 \\ J\sin\alpha\Big(\cos^2\gamma + \sin^2\gamma\Big) = \dot{\beta}I_1\sin\alpha\Big(\sin^2\gamma + \cos^2\gamma\Big) \end{bmatrix} = \begin{bmatrix} \dot{\alpha} = 0 \\ \dot{\beta} = J/I_1 \overset{\text{def}}{=} \Omega \end{bmatrix} . \tag{1.9.37}
$$

So, α is constant while β changes linearly. We can substitute these relations into the third equation to get

$$
J\cos\alpha = I_3\left(\dot{\gamma} + \frac{J\cos\alpha}{I_1}\right) \tag{1.9.38}
$$

$$
\dot{\gamma} = \frac{J\cos\alpha}{I_3} - \frac{J\cos\alpha}{I_1} = J\cos\alpha\frac{I_1 - I_3}{I_1 I_3} = \Omega\cos\alpha\frac{I_1 - I_3}{I_3} \overset{\text{def}}{=} -\omega_p , \tag{1.9.39}
$$

where the sign is a convention, such that when $I_3 > I_1$ (an oblate object, like a grapefruit or a coin) we have $\omega_p > 0$.

Now, what do these represent? The fact that $\dot{\alpha} = 0$ means that the angle between x_3 and x_3' stays the same. The rotation around β is the "main" one, as $\vec{\beta}$ is aligned with $\vec{J}$, and $\dot{\beta} = \Omega \gg \left|\omega_p\right| = |\dot{\gamma}|$ typically.

The rotation around $\vec{\gamma}$ corresponds to a *precession* of the angular velocity vector around the x_3' axis: the body's rotation axis precesses around its third principal axis.

Not that this is not the same as the precession of the body axis around the angular momentum. We should "clean our minds" from the idea of a spinning spintop precessing, this is not what is happening here. This wobbling motion is similar to the one of a coin thrown on a table, although this is a *free* wobble, happening without any external torque.

The time evolution of the inertia tensor reads

$$I(t) = R^\top I' R = R_\beta^\top R_\alpha^\top R_\gamma^\top I' R_\gamma R_\alpha R_\beta\,, \tag{1.9.40}$$

but the matrices R_γ and $R_\gamma^\top$ rotate the xy components of a matrix between each other: if $I_1 = I_2$ the components $I'_{11} = I'_{22}$, so we have $R_\gamma^\top I' = I' = I' R_\gamma$. So, we can write the expression as

$$I(t) = R_\beta^\top R_\alpha^\top I' R_\alpha R_\beta\,, \tag{1.9.41}$$

so the only time dependence which is left is inside $\beta(t) = \Omega t$. We can expand the calculation, the result is given by Maggiore [Mag07, eq. 4.245]. We are only interested in the projection of this variation onto the plane orthogonal to the direction of a propagation. The amplitudes in the two GW polarizations are also given by Maggiore [Mag07, eq. 4.246 – 252].

What we find is both emission at Ω and at 2Ω, while the frequency corresponding to the precession ω_p does not appear:

$$h_+ = h_0' \left[\sin(2\alpha)\sin\iota\cos\iota\cos(\Omega t) + 2\sin^2\alpha\left(1+\cos^2\iota\right)\cos(2\Omega t)\right] \tag{1.9.42}$$

$$h_\times = h_0' \left[\sin(2\alpha)\sin\iota\sin(\Omega t) + 4\sin^2\alpha\cos\iota\sin(2\Omega t)\right] \tag{1.9.43}$$

$$h_0' = -\frac{G}{c^4}\frac{I_3 - I_1}{r}\Omega^2\,, \tag{1.9.44}$$

where this h_0' should be compared with (1.9.17).

Not sure about what comparison should be drawn: the formulas are the same with $\omega_s \to \Omega$ and $I_2 \to I_3$…

We have four measurable amplitudes (corresponding to two polarizations and two frequencies), and we need to reconstruct the unknowns α, ι, r and $I_3 - I_1$. This would in general be possible, however because of correlations we need to measure one more parameter externally (like the distance r).

To get an **intuition** for the biperiodicity: if we have a distribution which looks like a coin ($I_1 \sim I_2 \ll I_3$) then it looks to us like a binary if we look at it from the top (in terms of periodicity at least), so we expect 2ω emission, since the system looks the same to us after a rotation of π.

If, instead, we look at it from the side, the periodicity is the full period: after half a rotation the coin is edge-on (and this happens every π), but it will appear at two different angles with respect to the vertical direction, so the real periodicity is 2π. Therefore, we both have ω and 2ω emission.

Backreaction

The radiated power is given by [Mag07, eq. 4.254]:

$$\frac{\mathrm{d}E_{\text{rot}}}{\mathrm{d}t} = -\frac{G}{5c^5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij}\right\rangle \tag{1.9.45}$$

$$= -\frac{2G}{5c^5}(I_1 - I_3)^2\Omega^6 \sin^2\alpha\left(\underbrace{\cos^2\alpha}_{\text{at }\Omega} + \underbrace{16\sin^2\alpha}_{\text{at }2\Omega}\right). \tag{1.9.46}$$

Wrong sign in the slides!

So, we can see that the emission at Ω is dominant for $\alpha \sim 0$ (systems for which x_3 and x_3' are almost aligned — the "coin seen head-on"), while the emission at 2Ω is dominant for larger α (the "coin seen edge-on").

The radiated angular momentum is instead given by

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{2G}{5c^5}\epsilon_{3jk}\left\langle \ddot{Q}_{jl}\dddot{Q}_{kl}\right\rangle \tag{1.9.47}$$

$$= -\frac{4G}{5c^5}\left\langle \ddot{M}_{1a}\dddot{M}_{2a}\right\rangle \tag{1.9.48}$$

$$= -\frac{2G}{5c^5}(I_1 - I_3)^2\Omega^5 \sin^2\alpha\left(\cos^2\alpha + 16\sin^2\alpha\right) = \frac{1}{\Omega}\frac{\mathrm{d}E_{\text{rot}}}{\mathrm{d}t}, \tag{1.9.49}$$

where we swapped Q for M since the terms $\epsilon^{3kl}\delta_{ka}Q_{la}$ and $\epsilon^{3kl}Q_{kl}$ do not contribute (by symmetry and tracelessness of Q respectively); also we integrated by parts.[15]

In order to understand how the rotation decays we need to express this in terms of the angles: recall the definition of $\Omega = \dot{\beta} = J/I_1$. This means that

$$\ddot{\beta} = \frac{1}{I_1}\frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{2G}{5c^5}\frac{(I_1 - I_3)^2}{I_1}\dot{\beta}^5 \sin^2\alpha\left(\cos^2\alpha + 16\sin^2\alpha\right), \tag{1.9.50}$$

which tells us that $\dot{\beta} = \Omega$ is decreasing.

To find the evolution of α we need to write the rotational energy as

$$E_{\text{rot}} = \frac{1}{2}I_i'\omega_i'^2 = \frac{1}{2}\frac{J_i'^2}{I_i} = \frac{J^2}{2}\left(\frac{\sin^2\alpha\sin^2\gamma}{I_1} + \frac{\sin^2\alpha\sin^2\gamma}{I_2} + \frac{\cos^2\alpha}{I_3}\right) \tag{1.9.51}$$

[15] To move from $\left\langle \dddot{M}_{1a}\ddot{M}_{2a} - \dddot{M}_{2a}\ddot{M}_{1a}\right\rangle$ to $2\left\langle \ddot{M}_{1a}\dddot{M}_{2a}\right\rangle$.

$$= \frac{J^2}{2}\left(\frac{\sin^2\alpha}{I_1} + \frac{\cos^2\alpha}{I_3}\right), \tag{1.9.52}$$

$I_1 = I_2$.

which can be differentiated to yield

$$\dot\alpha = -\frac{2G}{5c^5}\frac{(I_1 - I_3)^2}{I_1}\dot\beta^4 \sin\alpha\cos\alpha\left(\cos^2\alpha + 16\sin^2\alpha\right), \tag{1.9.53}$$

so α also decreases due to GW emission. This means that the wobble is decreasing, the rotation is aligning with the angular momentum.

However, the combination $J\cos\alpha$ is constant:

$$\frac{\mathrm{d}}{\mathrm{d}t}(J\cos\alpha) = \frac{\mathrm{d}J}{\mathrm{d}t}\cos\alpha - J\dot\alpha\sin\alpha \tag{1.9.54}$$

$$\begin{aligned} &= -\frac{2G}{5c^5}(I_1 - I_3)^2\dot\beta^5\sin^2\alpha\left(\cos^2\alpha + 16\sin^2\alpha\right)\cos\alpha + \\ &\quad - \dot\beta I_1\sin\alpha\left(-\frac{2G}{5c^5}\frac{(I_1 - I_3)^2}{I_1}\dot\beta^4\sin\alpha\cos\alpha\left(\cos^2\alpha + 16\sin^2\alpha\right)\right) \end{aligned} \tag{1.9.55}$$

$$= 0, \tag{1.9.56}$$

which means that $\omega_3' = J\cos\alpha / I_3$ is a constant: this is the rotation speed of the body around its axis; $J\cos\alpha$ is the projection of the angular momentum of the body around its axis x_3'.

Backreaction

In order to study the differential equations for $\ddot\beta$ and $\dot\alpha$ we can define the parameter $u(t) = \dot\beta/\dot\beta_0$, and a characteristic time τ_0:

$$\tau_0 = \left(\frac{2G}{5c^5}\frac{(I_1 - I_3)^2}{I_1}\dot\beta_0^4\right)^{-1}, \tag{1.9.57}$$

which has a typical value of

$$\tau_0 = 1.8\times 10^6\,\mathrm{yr}\left(10^{-7}\frac{I_3}{I_1 - I_3}\right)^2\left(\frac{1\,\mathrm{kHz}}{f_0}\right)^4\left(\frac{10^{38}\,\mathrm{kgm}^2}{I_1}\right), \tag{1.9.58}$$

and we can write differential equations for $\dot u$ and $\dot\alpha$ as

$$\dot u = -\frac{u^5}{\tau_0}\sin^2\alpha\left(\cos^2\alpha + 16\sin^2\alpha\right) \tag{1.9.59}$$

$$\dot\alpha = -\frac{u^4}{\tau_0}\sin\alpha\cos\alpha\left(\cos^2\alpha + 16\sin^2\alpha\right), \tag{1.9.60}$$

with initial conditions at the origin $u(0) = 1$ (meaning $\beta = \beta_0$) and $\alpha(0) = \alpha_0$.

We have shown that $J\cos\alpha$ is a constant: this means that we must have $\dot\beta\cos(\alpha) = \text{const}$. This can aid us in the search of a steady state, by providing a constraint. The constant can be calculated at any time, so we compute it at $t = 0$: then we get $\dot\beta_0\cos(\alpha_0) = \cos(\alpha_0) = \text{const}$.

This implies that the boundary condition at infinity must satisfy $u_\infty \cos(\alpha_\infty) = \cos\alpha_0$, which in general is different from 0 (unless $\alpha_0 = \pi/2$, but it can be shown that this is an unstable equilibrium).

Having a steady state means that we require $\dot{\alpha} = \dot{u} = 0$. Since the factor $\cos^2 + 16\sin^2$ is always positive for $\alpha \in [0, \pi]$ this can be either satisfied by $u = 0$ or $\sin\alpha = 0$, meaning $\alpha = 0$ or π.

The condition $u = 0$ cannot in general obey the $J\cos\alpha$ constraint, so we are left with $\alpha = 0, \pi$.

Can we discard $\alpha > \pi/2$ because otherwise we could just flip the axes until it became $\alpha < \pi/2$?

So, we get the asymptotic state

$$\alpha_\infty = 0 \qquad \text{and} \qquad u_\infty = \cos\alpha_0\,, \tag{1.9.61}$$

and the way α approaches this value asymptotically is

$$\dot{\alpha} \sim \alpha \frac{u_\infty^4}{\tau_0} \qquad \text{as} \qquad t \to \infty\,, \tag{1.9.62}$$

so asymptotically the decay is exponential, with a timescale $\sim \tau_0/u_\infty^4 \gtrsim \tau_0$.

1.9.3 Observations

The conditions we discussed do not apply in general: first of all, neutron stars are not truly rigid bodies since they have an internal structure. Even if they were, a generic rigid body's principal axes are all different. Maggiore discusses the triaxial case briefly [Mag07, pagg. 211–214].

In the triaxial case we will have emission at different frequencies, for example $2\omega_r$, $2\omega_r + \omega_p$, $2(\omega_r + \omega_p)$. There are even more: for each base frequency ω, radiation is emitted with decreasing amplitude for $\omega + n\omega_p$, $n \in \mathbb{N}$.

We have not seen pulsars yet in GW, but we can put upper bounds to the amplitude of their emission. What we do is to search for **quasi-stationary** GW signals close to known pulsars, accounting for the modulations due to the proper motion of the Earth, of the source etc. There are about 400 pulsars in the LIGO-Virgo bandwidth which would be eligible for this kind of observation.

"Beating the spin-down limit" means that we know that we would be able to see the GW emission in a certain case if the spin-down was only due to GW. We have beaten it by a factor 10 for the Crab and Vela pulsars: a very tiny fraction $\lesssim 1\,\%$ of the rotational energy is lost to GW.

Scorpius X-1 is low-mass X-ray binary: we see X-ray emission caused by accretion of the NS from the companion. This is a plausible mechanism for the deformation of the NS. We know its position and orbital period, not its spin frequency! The amplitude of its GW emission is expected to be of the order of $h_0 \sim 5 \times 10^{-25}$.

What is this bit about? It does not seem really relevant.

Could we differentiate a pulsar rotating and seen head-on and a binary system? Surely they are phenomena which happen in different frequency ranges, and last for different times. If the binary is spinning at those frequencies it's evolving very rapidly, instead a pulsar can give out a stable signal.

Also, in full numerical relativity the waveform looks different.

Chapter 2

Detectors

2.1 Noise theory

A simple experiment

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In order to guide our description of noise theory, we start from a simple example: suppose we want to use a pendulum to measure the gravitational acceleration g. We want to use small oscillations, which should satisfy $g \approx \omega^2 L$, and we want to fit $a(t)$.

Our signal can be very noisy compared to the theoretical waveform. If we perform a **precision experiment** then the magnitude of the noise is comparable to (or smaller than) that of the signal: this is not a bad thing necessarily, it means we are operating at the very limit of our experimental capabilities.

It *is* a good idea to fit the whole timeseries instead of just counting oscillations, since we are using more data then.

2.1.1 Noise in experiments: a general formulation

The physical system gives us a signal $s(t)$, we have some measurement apparatus which gives us an output $s'(t)$.

The output $s'(t)$ can be also affected by noise: we can have physical noise $n_1(t)$ in the form of a *random concurrent phenomenon*, noise $n_2(t)$ in the *transducer*, and $n_3(t)$ in the *readout*. The term n_1 comes from the phenomenon itself, the terms n_2 and n_3 come from our measurement apparatus. This is a description of a very simple system, real-world experiments often are even more complicated than this!

In our example: n_1 could be due to air currents or vibrations of the suspension points. The transducer in our case is the accelerometer, so we can have input voltage instability, imperfect mechanical coupling, thermal vibrations contributing to n_2. In the readout noise n_3 we can have reference voltage instability, quantization if we digitize the signal, and electronic pick-up.

Also, in the signal $s(t)$ there could be phenomena we do not want to measure and would regard as noise!

We call **noise** any unwanted signal. In the classical realm, $n(t)$ is the sum of deterministic processes, but in practice it is random since there are so many of them. We usually assume that they are zero-mean, while $s(t)$ is macroscopically deterministic.

The question the experimenter must ask are: how do we **distinguish** signal from noise, and how can we **recover** the original signal? How do we **characterize** the noise in a given experiment?

2.1.2 Random processes

A random variable is a number x associated to a possible experimental outcome. Any outcome has an associated probability. In the continuous realm, this is described by a probability density function $f(x)$, defined so that:

$$\mathbb{P}(x_0 \leq x \leq x_1) = \int_{x_0}^{x_1} f(x)\,\mathrm{d}x\,, \tag{2.1.1}$$

which we can use to compute mean values:

$$\int g(x) f(x)\,\mathrm{d}x = \left\langle g(x) \right\rangle_f\,, \tag{2.1.2}$$

and variances:

$$\sigma^2 = \int \mathrm{d}x\, f(x)(x - \langle x \rangle)^2 = \left\langle (x - \langle x \rangle)^2 \right\rangle = \left\langle x^2 \right\rangle - \langle x \rangle^2\,. \tag{2.1.3}$$

Do note that at this point these are *ensemble averages*: what we expect to be the average of a sample, *not* time averages. The way to ideally compute the ensemble average of a signal would be to repeat the experiment an "infinite number of times".

A random signal is a time series $x(t)$; every time we repeat the experiment we get a new realization of $x(t)$.

Knowing $x(\widetilde{t})$ at a specific time $\widetilde{t}$ we only have *partial* predictive power for $x(\widetilde{t} + T)$.

At a fixed time t, the possible values of $x(t)$ have a certain probability distribution $f(x; t)$. Then we can define the following functions of time: $\mu(t) = \left\langle x(t) \right\rangle$ and $\sigma^2(t)$.

Some properties our signal can have are:

1. **stochasticity**: the noise results from many uncorrelated processes, and correlation between the signal at a time t and at a time $t + \Delta t$ decreases quickly with Δt;

2. **ergodicity**: this means that instead of an ensemble mean value we can compute a time average, and similarly for other statistical properties;

3. **stationarity**: statistical properties are time-independent;

4. **gaussianity**: $x(t)$ is normally distributed for a fixed t.

The assumption of gaussianity for a specific signal is qualitatively justified by the central limit theorem, as long as the noise signal is stochastic.

The assumption of stationarity is not justified in general: there are several seasonal and daily phenomena which can alter the experimental apparatus; it is important to tread carefully here and only apply this assumption for small enough times.

Apart from this last consideration, these assumptions are almost always made in order to analyze the signal, but it is important to keep them in mind.

2.1.3 Fourier transforms

In order to move to Fourier space we must make sure that our time signal $s(t)$ is square-integrable:

$$\int \left|s(t)\right|^2 \mathrm{d}t < \infty \,, \tag{2.1.4}$$

where the integration bounds are implied to be $-\infty$ to ∞, unless otherwise specified.

As long as this is the case, we can define the Fourier transform and antitransform:

$$s(\omega) = \int s(t) e^{-i\omega t} \,\mathrm{d}t \qquad \text{and} \qquad s(t) = \frac{1}{2\pi} \int s(\omega) e^{i\omega t} \,\mathrm{d}\omega \;. \tag{2.1.5}$$

The normalization factors before the integrals are conventional — their product however needs to be $(2\pi)^{-1}$ in order for the composition of the transform and antitransform to give back the original function. The signs of the arguments of the exponentials could be swapped as well, they just need to be opposite of each other.

Important properties are: linearity; the fact that in Fourier space derivatives become multiplication by $i\omega$;[1] the convolution theorem: multiplication in time domain is the same as convolution in frequency domain,

$$\int s(t) q(t) e^{-i\omega t} \,\mathrm{d}t = \frac{1}{2\pi} \int s(\omega') q(\omega - \omega') \,\mathrm{d}\omega' \;. \tag{2.1.6}$$

The transform of the Dirac $\delta(t)$ function is 1: a signal which is very well localized in time is very delocalized in frequency (and vice versa). The Fourier transform corresponds to a *unitary* transformation of the space of square-integrable functions: no information is lost by transforming, so under these assumptions any signal can be described as an infinite superposition of oscillatory terms in an equivalent.

The transform is a complex-valued function, telling us the amplitude and phase of any component of the oscillation.

In practice, the frequency domain is very useful. Many interesting signals are well-defined in frequency, less so in time.

Also, random noise is easier to characterize in frequency domain, especially if it is *stationary*: the specific value of the noise at a specific time is random, but the statistics of the Fourier transform are fixed.

Some other properties: the Fourier transform preserves the energy (in the sense of the integral of the square modulus): $\int \left|s(t)\right|^2 \mathrm{d}t = \int \left|s(\omega)\right|^2 \mathrm{d}\omega$.

[1] This can be shown by integrating by parts, the boundary terms must vanish since our function is in $L^2(\mathbb{R})$.

If the signal $s(t)$ is real-valued, then we have $s(\omega) = s^*(-\omega)$, so all the information is contained in the $\omega > 0$ part of the transform.

We can associate a *probability density* to the signal, by

$$f(t) = \frac{\left|s(t)\right|^2}{\int \mathrm{d}t \left|s(t)\right|^2} \qquad \text{and} \qquad f(\omega) = \frac{\left|s(\omega)\right|^2}{\int \mathrm{d}\omega \left|s(\omega)\right|^2}, \tag{2.1.7}$$

and we can compute the moments of this probability density like we would for any other PDF. The **average values** of ω and t (can be made to) **vanish**: for the frequency we have

$$\langle \omega \rangle = \int \mathrm{d}\omega\, \omega f(\omega) \propto \int \mathrm{d}\omega\, \omega \left|s(\omega)\right|^2, \tag{2.1.8}$$

which is odd under $\omega \to -\omega$, since $s(-\omega) = s(\omega)^*$, whose square modulus is the same. For the time, on the other hand, we will in general have some nonzero result:

$$\langle t \rangle = \int \mathrm{d}t\, t f(t) = t_0, \tag{2.1.9}$$

which can be made to vanish by shifting t by t_0.

On the other hand, the second moments are nonvanishing: they are defined by

$$\left\langle \omega^2 \right\rangle = \Delta\omega^2 = \int \mathrm{d}\omega\, \omega^2 f(\omega) = \frac{\int \mathrm{d}\omega \left|\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right|^2}{\int \mathrm{d}t \left|s(t)\right|^2} \qquad \text{and} \qquad \left\langle t^2 \right\rangle = \Delta t^2 = \int \mathrm{d}t\, t^2 f(t). \tag{2.1.10}$$

We have the uncertainty principle, giving an intrinsic trade off between the localization of a signal in momentum space and in position space:[2]

$$\Delta\omega \Delta t \geq \frac{1}{2}. \tag{2.1.12}$$

Depending on the physical characteristics of the signal, we should select a large observation time or a large frequency band, knowing that with a short observation time we will not be able to determine a frequency precisely.

2.1.4 Power spectral density

The phase of the noise will be completely different from one realization to the other: if the noise is stochastic then its Fourier transform is stochastic as well.

[2] In order to prove this result, one writes $\Delta\omega^2\Delta t^2$ explicitly and applies the Cauchy-Schwartz inequality: with $E = \int \left|s(\omega)\right|^2 \mathrm{d}\omega$ we can write

$$\Delta\omega^2\Delta t^2 = \frac{1}{E^2}\int \mathrm{d}t \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|^2 \int \mathrm{d}t'\, t'^2 \left|s(t')\right|^2 \geq \frac{1}{E^2}\left|\int t \frac{\mathrm{d}s}{\mathrm{d}t} s(t)\,\mathrm{d}t\right|^2 = \frac{1}{4E^2}\left|\int t \frac{\mathrm{d}}{\mathrm{d}t}\left(s^2\right)\mathrm{d}t\right|^2 = \frac{1}{4}, \tag{2.1.11}$$

since $\int t \frac{\mathrm{d}(s^2)}{\mathrm{d}t}\,\mathrm{d}t = -\int s^2\,\mathrm{d}t = -E$.

We define the autocorrelation function: for a *stationary process* it is given by

$$R(t,t') = \langle x(t)x(t') \rangle = R(\tau = t - t')\,. \tag{2.1.13}$$

This function measures how quickly the signal loses memory. For a random process we expect R to go to zero relatively quickly as τ increases. So, we define the power spectral density (PSD) as the **Fourier transform of the autocorrelation function**:

$$S_x(\omega) = \int \mathrm{d}\tau\, R(\tau) e^{i\omega\tau}\,. \tag{2.1.14}$$

White noise has no correlation between a point and another: its auto-correlation function is a delta. It has all the frequency components. So, its PSD will be flat.

A more loose and intuitive definition is the *ensemble average* of the frequency components of the signal:

$$S_x(\omega)\delta(\omega - \omega') = \langle x(\omega)x^*(\omega') \rangle\,, \tag{2.1.15}$$

so that we can recover the power of the signal by integrating it [Mag07, eq. 7.12]:

$$\left\langle s^2 \right\rangle = \int S_x(\omega)\,\mathrm{d}\omega\,. \tag{2.1.16}$$

This is not the formal definition since $x(\omega)$ may not exist, but it clarifies the meaning: the phase of the signal is randomly distributed, but its *amplitude* has a well-defined average.

If we build a window filter which only allows $[\omega_1 \leq \omega \leq \omega_2]$, then the residual power will be

$$P_{\text{window}} = \int_{\omega_1}^{\omega_2} S(\omega)\,\mathrm{d}\omega\,. \tag{2.1.17}$$

For real signals, S is symmetric: $S_x(-\omega) = S_x(\omega)$. If we do not care about negative frequencies, we keep only the $[\omega \geq 0]$ region and multiply by 2.

The mean square of the signal in time is the integral of the PSD:

$$\sigma_x^2 = \int_0^\infty S_x(\omega)\,. \tag{2.1.18}$$

If two signals are uncorrelated, the power spectral density of their sum is the sum of their PSDs:

$$S_{x+y}(\omega) = S_x(\omega) + S_y(\omega)\,. \tag{2.1.19}$$

Often instead of the *power* spectral density we use the amplitude, or linear, spectral density, defined by $\sqrt{S_x(\omega)}$.

If our signal has units of m, then the dimensions of the power spectral density are $[S_x(\omega)] = \mathrm{m}^2/\mathrm{Hz}$, while the linear PSD has units of $[\sqrt{S_x(\omega)}] = \mathrm{m}/\sqrt{\mathrm{Hz}}$.

PSD, bandwidth and windowing

In practice, we measure for a limited time T, either because the funding runs out or because the signal has an intrinsically short duration. So, the measured signal $x(t)$ will only have Fourier components at frequencies $f_n = n/T$, where $n \in \mathbb{N}$. We express this by saying that the frequency resolution is $\Delta f = 1/T$.

In general, we express a time-constrained measurement as a product of the signal $x(t)$ with a window $w(t)$, so that the measured signal is $x(t)w(t)$, where the window $w(t)$ is only nonzero in a certain region of length $\leq T$. In frequency space products become convolutions, so we get

$$x_{\text{measured}}(\omega) = \int d\widetilde{\omega}\, x(\widetilde{\omega}) w(\omega - \widetilde{\omega}) \ : \tag{2.1.20}$$

the Fourier transform of the window *spreads out* our signal. We must also be careful: windows with sharp boundaries (like a box window) introduce high-frequency components, dirtying the Fourier representation of our signal. In order to avoid this effect we must "ease into" the window somehow, for example we can consider a window shaped like a Gaussian. Counterintuitively, this means that we must *ignore* or consider less some of our data points.

We could deconvolve to get the signal back if we knew what the window looks like. We often know all about the window, but we do not usually apply this procedure. In fact, if we were to take out this windowing effect it would mean we are assuming that if we were to observe our signal *for a longer time than we actually did* we would see the same thing over and over.

This is quite a strong assumption, which might be justified sometimes, but usually it is not.

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A physical system is a **functional** F which transforms one or many input time series $i_j(t)$ into one or many output time series $o_j(t) = F(i_j(t))$.

In principle, any output time series at any time could be a function of any input time series at any time. However, real systems are causal: there cannot be causality going backward in time, so $o(t_0) = F(i(t))\big|_{t \leq t_0}$.

We can also make two assumptions: that the functional F is **linear** — this is justified as long as we are always working near a fixed point, so that the higher order terms are negligible; and that the functional F is **stationary**: this means that it is invariant under translations $t \to t + a$.

Under all of these assumptions, we can express the effect of the system through an **impulse response function** h, defined so that:

$$o(t) = F\left[\int d\widetilde{t}\, i(\widetilde{t})\delta(t - \widetilde{t})\right] = \int d\widetilde{t}\, i(\widetilde{t}) F(\delta(t - \widetilde{t})) = \int d\widetilde{t}\, i(\widetilde{t}) h(t - \widetilde{t})\,. \tag{2.1.21}$$

We can say that $F[i(\widetilde{t})\delta] = i(\widetilde{t})F[\delta]$ because t' is fixed inside the integral, so $i(\widetilde{t})$ is just a constant.

We used stationarity to write $h(\widetilde{t}, t)$ as $h(\widetilde{t} - t)$; also, causality tell us that the IRF $h(\tau)$ must satisfy $h(\tau) = 0$ for $\tau < 0$.

The expression for the output as a function of the input is a convolution, so in Fourier space it is a product;

$$o(\omega) = i(\omega)h(\omega) . \tag{2.1.22}$$

The power spectral density then transforms as $S_o(\omega) = \left|h(\omega)\right|^2 S_i(\omega)$, and if we have systems in series we can just multiply the impulse responses together, like

$$o(\omega) = i(\omega) \prod_{j=1}^{N} h_j(\omega) . \tag{2.1.23}$$

2.1.5 Sampling

Often we sample signals digitally. Analogic systems can be faster, but electronics are getting very fast as well, and they are easier to use.

The signal is quantized in two ways: we quantize both in time by sampling at an interval t_s and in amplitude, by encoding it with a finite number of bits. This introduces noise, which is well-known and easy to calculate. We still need to do Fourier analysis, but we will use a discrete Fourier transform.

Aliasing

If we have a signal at a frequency f, and we want to reconstruct it, we need to sample at a frequency $\geq 2f$.

If we sample at 100 Hz, we can only accurately describe signals up to 50 Hz. This is the **Nyquist-Shannon sampling theorem**.

This is true if we want to fit the data with the slowest sinusoid possible; if we know in which frequency range we should look we can try to fit higher-frequency sinusoids but this is risky business. If we work below the Nyquist frequency we can be sure of each frequency we see.

2.2 Resonant bar detectors

2.2.1 Two paths to GW detection

Most modern and planned GW detectors operate by constructing **free-falling masses**: ground-based interferometers bounce signals off of suspended mirrors, space based ones have masses in actual geodesic motion. These detectors are *broad-band*, which is useful, but each frequency has to be detected at its native amplitude with no amplification. They can be made on *large scales*, on the order of 10^3 m on Earth, 10^9 m in space. This is very useful scientifically, since then the GW-induced displacement is larger; however it requires a lot of infrastructure and investment. These must then be built and operated by large collaborations, with hundreds of people at least.

Another option, which was quite popular a few years ago, is to use an **elastic body** which resonates at a specific frequency. This might *enhance* the effect of a GW through resonance and *extend* the duration of burst signals.

However, this kind of detector is only sensitive *around its resonant frequency*. Also, since it extends the signal it is hard to precisely reconstruct the *temporal profile* of the signal.

These need to be isolated solid objects: they will fit in a lab (at scales of 10^1 m at most), but the GW-induced displacements will be small.

2.2.2 Harmonic oscillators and GW

Harmonic oscillators

Suppose we have a perfect harmonic oscillator with a time-dependent rest position $x_0(t)$ and a time-dependent external force $F_{\text{ext}}(t)$: its evolution will be determined by the differential equation

$$m\ddot{x} = -k(x(t) - x_0(t)) + F_{\text{ext}}(t)\,, \tag{2.2.1}$$

which in Fourier space can be written as

$$\begin{aligned} -m\omega^2 x(\omega) &= -k(x(\omega) - x_0(\omega)) + F_{\text{ext}}(\omega) && (2.2.2)\\ x(\omega) &= \frac{kx_0(\omega) + F_{\text{ext}}(\omega)}{k - m\omega^2} = \frac{kx_0(\omega) + F_{\text{ext}}(\omega)}{k\big(1 - \omega^2/\omega_0^2\big)} && (2.2.3)\\ &= \underbrace{\frac{\omega_0^2}{1 - \omega^2/\omega_0^2}}_{H_{x_0}(\omega)} x_0(\omega) + \underbrace{\frac{F_{\text{ext}}(\omega)}{k(1 - \omega^2/\omega_0^2)}}_{H_{F_{\text{ext}}}(\omega)} F_{\text{ext}}(\omega)\,, && (2.2.4) \end{aligned}$$

where we defined $\omega_0 = \sqrt{k/m}$ and the two transfer functions H_{x_0} and $H_{F_{\text{ext}}}$.

This diverges for $\omega = \omega_0$; but let us consider the effect of **velocity damping**: we add a term $-\beta\dot{x}$ to the RHS of the differential equation,

$$m\ddot{x} = -k\big(x(t) - x_0(t)\big) - \beta\dot{x}(t) + F_{\text{ext}}(t)\,, \tag{2.2.5}$$

which in Fourier space becomes:

$$x(\omega) = \frac{kx_0(\omega) + F_{\text{ext}}(\omega)}{k\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 - \frac{i\omega\beta}{k}\right)}\,, \tag{2.2.6}$$

since every derivative becomes $-i\omega$.

Another kind of damping we can have is called **structural internal damping**, which means modifying the differential equation as:

$$m\ddot{x} = -k(1 + i\delta)\big(x(t) - x_0(t)\big) + F_{\text{ext}}(t)\,; \tag{2.2.7}$$

concretely speaking this means that there is some *delay* between the action of the force and the response of the system. In Fourier space, this means

$$x(\omega) = \frac{k x_0(\omega) + F_{\text{ext}}(\omega)}{k\left(1 - \left(\frac{\omega}{\omega_0}\right)^2 + i\delta\right)}, \tag{2.2.8}$$

so, since both terms add a purely imaginary constant to the denominator, we encapsulate them into a term i/Q, for an arbitrary Q.

This Q quantifies damping (inversely: large Q means small damping). For non-infinite values of Q, the transfer function does not diverge.

GW interactions

How do we see the effect of GW on an elastic body? Consider two masses, which start out free-falling, and connect them by a spring: they now will not move along geodesics. If we move to the **proper detector frame**, the effect of a GW can be described as a Newtonian force on the test masses, which together with the reaction of the spring determines the motion of the system:

$$F_{\text{GW}} - k(L - \Delta x) = m \Delta \ddot{x}, \tag{2.2.9}$$

where the force is given by (equation (1.4.26) multiplied by m):

$$F_{\text{GW}} = \frac{m}{2} \ddot{h}_{xx}^{TT} \Delta x \approx \frac{m}{2} L \ddot{h}_{xx}^{TT}. \tag{2.2.10}$$

Since we can only see h_{xx}, we are only sensitive to the h_+ polarization: this is not surprising, since our detector is one-dimensional.

Note that this expression is only valid as long as we are in the short arm approximation: $L \ll \lambda_{\text{GW}}$, which means $f_{\text{GW}} \ll c/L \approx 3 \times 10^8\,\text{Hz}$, if $L \approx 1\,\text{m}$.

Intuitively, what the equation is describing is the force of the GW competing with the intrinsic one of the oscillator to move the mass.

The oscillator was a convenient approximation to give an idea of the system, but really for our detector we will use a continuous **resonant bar**; we can describe its movement by introducing the variable $u(x,t)$, which denotes the displacement from equilibrium at a certain point (still in only *one dimension*). The dynamics of the bar can be shown to obey the law

$$\mathrm{d}m \left(\frac{\partial^2 u}{\partial t^2} - v_s^2 \frac{\partial^2 u}{\partial x^2} \right) = \mathrm{d}F_x = \mathrm{d}m \frac{1}{2} x \ddot{h}_{xx}^{TT}, \tag{2.2.11}$$

where v_s is the speed of sound in the medium.

Lagrangian or Eulerian?

We assume that the ends of the bar are kept stationary:

$$\left. \frac{\partial u}{\partial x} \right|_{x = \pm L/2} = 0. \tag{2.2.12}$$

The general solution will be given by a sum of sines and cosines, but the cosines will move the center of the bar.

They will, but we imposed the ends being stationary, not the center (and why should the center be stationary)! Why should we not use that condition instead? It works just as well, since *cosines* satisfy the condition of being zero at $\pm L/2$, and we get cosines from the first derivative of sines.

Keeping only the physical sines we will then have the harmonic decomposition

$$u(t,x) = \sum_{n=0}^{\infty} \xi_n \sin\left(\frac{\pi x}{L}(2n+1)\right), \tag{2.2.13}$$

which we can plug into the differential equation: computing the derivatives explicitly we find

$$\sum_{n=0}^{\infty} \left(\ddot{\xi}_n + \underbrace{\left(\frac{v_s \pi (2n+1)}{L}\right)^2}_{\omega_n^2}\right) \sin\left(\frac{\pi x}{L}(2n+1)\right) = \frac{1}{2} x \ddot{h}_{xx}^{TT}, \tag{2.2.14}$$

which, in L^2 space, is in the form $\sum_n c_n \hat{e}_n = \vec{v}$, where $\hat{e}_n$ are orthogonal basis vectors while $\vec{v}$ is a vector (recall that $\ddot{h}_{xx}^{TT}$ is approximately constant with respect to x, but it is multiplied by x). In order to solve it, we take its scalar L^2 product with an arbitrary basis vector, which amounts to multiplying by another sinusoid and integrating.

The sinusoids $\hat{e}_n = \sin\big((2n+1)\pi x/L\big)$ are not orthonormal, they instead satisfy $\hat{e}_n \cdot \hat{e}_m = (L/2)\delta_{nm}$ as can be checked by direct computation. On the other side of the equation we find

$$\begin{aligned}
\hat{e}_m \cdot \vec{v} &= \frac{1}{2} \ddot{h}_{xx}^{TT} \int_{-L/2}^{L/2} \mathrm{d}x \, x \sin\left(\frac{\pi x}{L}(2m+1)\right) && (2.2.15) \\
&= \frac{1}{2} \ddot{h}_{xx}^{TT} \frac{L^2}{\pi^2 (2m+1)^2} \underbrace{\sin\left(\frac{\pi x}{L}(2m+1)\right)\Bigg|_{-L/2}^{L/2}}_{=2(-)^m}, && (2.2.16)
\end{aligned}$$

The indefinite integral also has a term like $x\cos(x)$, which is odd and vanishes.

so the final equation reads:

$$\ddot{\xi}_n + \omega_n^2 \xi_n = \frac{(-)^n}{(2n+1)^2} \frac{2L}{\pi^2} \ddot{h}_{xx}^{TT}. \tag{2.2.17}$$

We have eliminated ("integrated out") the spatial part: we can analyze the time evolution by itself.

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It is interesting to consider the fundamental mode for the oscillator, which is given by setting $n = 0$: this yields

$$\ddot{\xi}_0 + \omega_0^2 \xi_n = \frac{2L}{\pi^2} \ddot{h}_{xx}^{TT}, \tag{2.2.18}$$

where $\omega_0 = \pi v_s / L$. The total energy in the bar can be recovered from this mode, by

$$E = \frac{M}{4}\left(\dot{\xi}_0^2 + \omega_0^2 \xi_0^2\right) = \frac{1}{2}\int \mathrm{d}m \left(\dot{u}^2 + v_s^2 \left(\frac{\partial u}{\partial x}\right)^2\right), \tag{2.2.19}$$

where M is the total mass of the bar. This basically amounts to integrating $1/2mv^2$, where v is calculated including both temporal and spatial variations.

The equation for this fundamental mode has the same form as the one of a simple harmonic oscillator, whose mass is $m_0 = M/2$, whose frequency is ω_0 and which is subject to an effective force

$$F(t) = \frac{2m_0 L}{\pi^2}\ddot{h}_{xx}^{TT} = \frac{LM}{\pi^2}\ddot{h}_{xx}^{TT}. \tag{2.2.20}$$

Like we did in the harmonic oscillator case we introduce a damping term to model the physical imperfections in the resonance: then the equation will read

$$\ddot{\xi}_0 + \gamma_0 \dot{\xi}_0 + \omega_0^2 \xi_0 = \frac{2L}{\pi^2}\ddot{h}_{xx}^{TT}, \tag{2.2.21}$$

where $\gamma_0 = \omega_0 / Q_0$. After Fourier-transforming we find

$$\xi_0(\omega) = \frac{2L}{\pi^2}\frac{\ddot{h}_{xx}^{TT}(\omega)}{\omega^2 - \omega_0^2 - i\omega\gamma_0} \tag{2.2.22}$$

$$= \underbrace{-\frac{2L}{\pi^2}\frac{\omega^2}{\omega^2 - \omega_0^2 - i\omega\gamma_0}}_{H_{h\to\xi_0}(\omega)} h_{xx}^{TT}(\omega), \tag{2.2.23}$$

so we have the expression from the transfer function, allowing us to quantify the response of the system to the oscillatory stimulus of the GW.

Typical values for the parameters at play are a length of $L \sim 3\,\mathrm{m}$, a mass of $M \sim 2 \times 10^3\,\mathrm{kg}$ and a speed of sound of the order $v_s \sim 5 \times 10^3\,\mathrm{m/s}$, which means we have a characteristic frequency of the order of $\omega_0 \sim 5 \times 10^3\,\mathrm{rad/s}$, so the long-wavelength approximation is valid.

The value of the decay parameter Q_0 is of the order 10^6 typically, which means that the characteristic time for an oscillation to die out is of the order $\tau_0 \sim 1/\gamma_0 = Q_0/\omega_0 \sim 10^2 \div 10^3\,\mathrm{s}$: oscillations need **several minutes to die out**.

This only applies when ω from the GW is **very close** to ω_0, as the transfer function is very peaked.

Response to periodic signals

If we have an incoming GW propagating along the z axis, while the bar is positioned along the x axis, then the GW amplitude will be described by

$$h_{xx}^{TT}(t, z) = h_0 \,\mathrm{Re}\left[\exp\left(-i\omega(t - z/c)\right)\right], \tag{2.2.24}$$

so, fixing $z = 0$ for simplicity (and without losing generality, it is only a phase shift) its second derivative is

$$\ddot{h}_{xx}^{TT} = -\omega^2 h_0 \,\mathrm{Re}[\exp(-i\omega t)]\,. \tag{2.2.25}$$

A steady-state solution to the coupled differential equation (2.2.21) is:

$$\xi_0(t) = \frac{2L\omega^2 h_0}{\pi^2}\,\mathrm{Re}\left[\frac{\exp(-i\omega t)}{\omega^2 - \omega_0^2 + i\omega\gamma_0}\right] \tag{2.2.26}$$

$$= \frac{2L\omega^2 h_0}{\pi^2}\frac{(\omega^2 - \omega_0^2)\cos(\omega t) - \gamma_0 \sin(\omega t)}{(\omega^2-\omega_0^2)^2 + \omega^2\gamma_0^2}\,, \tag{2.2.27}$$

which has a simpler expression in Fourier space:

$$\xi_0(\omega) = -\frac{2L}{\pi^2}\frac{\omega^2}{\omega_0^2 - \omega^2 - i\omega\gamma_0} h_0 \delta(\omega)\,. \tag{2.2.28}$$

Response to bursts

It is a fact from classical mechanics that the absorbed energy E of an oscillator with mass m_0 subject to an external impulsive force $F(t)$ is:[3]

$$E = \frac{1}{2m_0}\left|\int_{-\infty}^{\infty} F(t) e^{-i\omega_0 t}\,\mathrm{d}t\right|^2\,, \tag{2.2.29}$$

so if we suppose our force has the form $F(t) = (2/\pi^2) m_0 L \ddot{h}_{xx}$, where $m_0 = M/2$ (both of these come from what we found for the equivalence to a simple harmonic oscillator of the fundamental mode of the resonant bar, (2.2.20)), we find

$$E = \frac{ML^2}{\pi^4}\left|\int_{-\infty}^{\infty} \ddot{h}_{xx}^{TT}(t) e^{-i\omega_0 t}\,\mathrm{d}t\right|^2\,, \tag{2.2.30}$$

and since the force is impulsive $h(t)$ goes to zero quickly for $t \to \pm\infty$, so we can integrate by parts twice:

$$E = \frac{ML^2(2\pi f_0)^4}{\pi^4}\left|\int_{-\infty}^{\infty} \mathrm{d}t\, h_{xx}^{TT}(t) e^{-i\omega_0 t}\right|^2 \tag{2.2.31}$$

$$= 16ML^2 f_0^4 \left|h_{xx}^{TT}(f_0)\right|^2\,, \tag{2.2.32}$$

since the integral is precisely the Fourier transform of $h(t)$ evaluated at f_0. This is the energy of the GW component at frequency ω_0: we can invert this expression to recover the energy of the component of the GW at a frequency f_0 from the energy we measure at the detector, E:

$$\boxed{\left|h_{xx}^{TT}(f_0)\right|^2 = \frac{1}{16L^2 f_0^4}\frac{E}{M}\,.} \tag{2.2.33}$$

[3] The reference from Maggiore is to Landau and Liftsis, volume 1, equation 22.12.

Ringdown

The transfer function in (2.2.22) can be written as

$$\tilde{\xi}_0(\omega) = -\frac{2L}{\pi^2}\frac{\omega^2}{(\omega-\omega_+)(\omega-\omega_-)}\tilde{h}^{TT}_{xx}(\omega)\,, \tag{2.2.34}$$

where the frequencies $\omega_\pm$ are defined by:

$$\omega_\pm = \pm\sqrt{\omega_0^2 - \left(\frac{\gamma_0}{2}\right)^2} - i\frac{\gamma_0}{2} \approx \pm\omega_0 - i\frac{\gamma_0}{2}\,, \tag{2.2.35}$$

where the approximation holds when the quality factor $Q = \omega_0/\gamma_0$ is large.

The minus sign in this equation is not there in Maggiore! Should check all signs here. . .

We suppose that the incoming burst is a delta in time: then, its spectral decomposition will be uniform. Specifically, let us rescale it so that it has the same integral (so, the same energy) as a rectangular impulse:

$$h^{TT}_{xx}(t) = h_0\tau_{GW}\delta(t) \implies \tilde{h}^{TT}_{xx}(\omega) = h_0\tau_{GW}\,. \tag{2.2.36}$$

Then, we get

$$\begin{aligned}
\xi_0(t) &= \int \frac{d\omega}{2\pi}\tilde{\xi}_0(\omega)e^{-i\omega t} && (2.2.37)\\
&= \frac{2L}{\pi^2}h_0\tau_{GW}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\omega^2}{(\omega-\omega_+)(\omega-\omega_-)}e^{-i\omega t} && (2.2.38)\\
&\approx \frac{2L}{\pi^2}h_0\omega_0\tau_{GW}e^{-\gamma_0 t/2}\sin(\omega_0 t)\,. && (2.2.39)
\end{aligned}$$

Probably the integral is solved by looking at the complex residuals. . .

The GW "burst" excites the fundamental mode, which then decays with a characteristic time $1/\gamma_0$.

By saying that the signal is a delta, do we mean that $\tau_{GW} \ll 1/\omega_0$? Yes, where ω_0 is the frequency of the oscillator, which is much faster than the characteristic frequency of the dissipation.

Antenna pattern

The sensitivity of the detector depends both on the direction from which the GW comes and the polarization.

If $\vec{J}$ is the orientation of the bar, then we have

$$h_{\text{out}} = \hat{J}^i\hat{J}^j h_{ij}(t)\,. \tag{2.2.40}$$

Let us consider a "detector-centric" reference frame in which the bar is along the x axis, while the source is positioned in the xz plane (this is fully general, we rotate around the x

axis until the condition is satisfied). Let us also consider a "source-centric" reference frame in which the z' axis points towards the source of the GW, while the $x'z'$ plane coincides with the xz one.

Since these two planes coincide, the $\hat{y}$ vector also will: we will be able to move between the two references with a rotation around it,

$$R = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}, \tag{2.2.41}$$

where $\alpha = \pi/2 - \theta$, and θ is the angle between the bar and the observation direction.

If we consider the polarizations of the GW to be expressed with respect to the primed frame:

$$h'_{ij} = \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.2.42}$$

then by applying the rotation matrix (twice, once for each index) we will see:

$$h_{\text{out}} = h_{xx} = R_{1k} R_{1l} h'_{kl} = h_+ \sin^2\theta \,. \tag{2.2.43}$$

This reference for the source is rather restrictive; we might want to align our axes so that a binary system is more easily described.

To move to a generic source reference frame, we must a rotation of angle φ around the axis z' of the source system, so that the $x'z'$ plane is free to differ from the xz one.

With this generalization, we find:

$$h_{\text{out}}(t) = h'_+ \sin^2\theta \cos 2\varphi + h'_\times \sin^2\theta \sin 2\varphi \,. \tag{2.2.44}$$

Resonant amplification

The response of the bar is quite small: we cannot really measure it; however we can transfer the energy to a lighter oscillator called the **transducer**, which will move more. The coupled equations of motion read:

$$\ddot{\xi}_0 + \omega_0^2 \xi_0 + \mu\omega_t^2(\xi_0 - \xi_t) = \frac{F_0}{m_0} \tag{2.2.45}$$

$$\ddot{\xi}_t + \mu\omega_t^2(\xi_0 - \xi_t) = \frac{F_t}{m_t}, \tag{2.2.46}$$

where the label t refers to the transducer while 0 is the fundamental mode of the resonant bar, and μ is defined as the mass ratio m_t/m_0.

The force on the transducer F_t can be approximated to be zero if the experimental apparatus is well-built, since the influence of GW onto such a small mass will be negligible.

If we apply an impulsive force in the form $F_0 = a_0\delta(t)$ we will see a response which can be calculated directly from the transfer function: for the two oscillators we find

$$\xi_0(\omega) = a_0 \frac{\omega_t^2 - \omega^2}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)} \tag{2.2.47}$$

$$\xi_t(\omega) = a_0 \frac{\omega_t^2}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)}\,, \tag{2.2.48}$$

where $\omega_\pm$ are the solutions (in terms of ω) of

$$\omega^4 - (\omega_0^2 + (1+\mu)\omega_t^2)\omega^2 + \omega_0^2\omega_t^2 = 0\,, \tag{2.2.49}$$

so the response of this coupled system has **two resonant frequencies**, and with the aid of the Cauchy residue theorem we can calculate the temporal form of the solution, a linear combination of sinusouds at $\omega_\pm$ [Mag07, eqs. 8.79–80], for both oscillators.

We need to tune the bar so that its resonant frequency is the same as the one of the other bar: then we will get a **resonant transducer**, and the resonant frequencies will be

$$\omega_\pm = \omega_0 \left(1 \pm \frac{\sqrt{\mu}}{2} + \mathcal{O}(\mu)\right) = \omega_0 \pm \omega_b\,, \tag{2.2.50}$$

and the temporal form of the two oscillations will be

$$\xi_0(t) \approx \frac{a_0}{\omega_0} \sin(\omega_0 t)\cos(\omega_b t) \tag{2.2.51}$$

$$\xi_t(t) \approx -\frac{a_0}{\omega_0\sqrt{\mu}} \cos(\omega_0 t)\sin(\omega_b t)\,, \tag{2.2.52}$$

which means (since they are out of phase) that **energy is transferred** back and forth between the two oscillators; but the amplitudes differ by a factor $\sqrt{\mu}$:

$$A_t = \frac{A_0}{\sqrt{\mu}}\,, \tag{2.2.53}$$

and we typically go as low as $\mu \lesssim 10^{-4}$, if we go lower other noise sources dominate. This amplitude ratio is the best we could get by **energy conservation**: the energy is proportional to $m\omega^2 A^2$, so if we impose the energy being *completely* transferred back and forth between the two oscillators we $A_0^2/A_t^2 = m_0/m_t$, since ω is the same for the two oscillators.

Still, this amplifies the signal at our desired frequency by two orders of magnitude!

The transfer function with a transducer, and introducing dissipation for both oscillators, looks like:

$$H_{h\to\xi_t}(\omega) = -\frac{2L}{\pi^2} \frac{\omega_0^2\omega^2}{(\omega^2 - \omega_+^2 + i\omega\overline{\gamma})(\omega^2 - \omega_-^2 + i\omega\overline{\gamma})}\,, \tag{2.2.54}$$

where $\overline{\gamma} = (\gamma_0 + \gamma_t)/2$. This should be compared with (2.2.22): since now we have *two* terms going to zero quadratically with $\omega \to \omega_0$ (at least near zero, because of the dissipation) the graph now has two peaks which are both higher than the single one we had before.

2.2.3 Thermal noise

We saw that we can represent the stages of our processing by their transfer functions $H_i(\omega)$. These are multiplied in the amplitude modulation, while we have to take the square modulus of their product if we want to compute the PSD:

$$y(\omega) = x(\omega) \prod_{i=1}^{n} H_i(\omega) \qquad \text{and} \qquad S_y(\omega) = S_x(\omega) \left| \prod_{i=1}^{n} H_i(\omega) \right|^2 . \tag{2.2.55}$$

At each stage, though, we can get noise, which is then processed by the later transfer functions: if we have three stages we will get

$$y = H_3\Big(H_2\big(H_1(x) + n_2\big) + n_3\Big) + n_4 \tag{2.2.56}$$

$$= H_1 H_2 H_3 x + H_2 H_3 n_2 + H_2 n_3 + n_4 . \tag{2.2.57}$$

Now, from the output y we want to reverse-engineer the signal: this will give us a signal x', calculated as

$$x' = \frac{y}{H_1 H_2 H_3} = x + \underbrace{\frac{n_4}{H_1 H_2 H_3}}_{x_{n_4}} + \underbrace{\frac{n_3}{H_1 H_2}}_{x_{n_3}} + \underbrace{\frac{n_2}{H_1}}_{x_{n_2}} . \tag{2.2.58}$$

A similar reasoning can be applied to the power spectral density: then we will find

$$S_{x_{n_2}} = \frac{S_{n_2}}{|H_1|^2} \tag{2.2.59}$$

$$S_{x_{n_3}} = \frac{S_{n_3}}{|H_1 H_2|^2} \tag{2.2.60}$$

$$S_{x_{n_4}} = \frac{S_{n_4}}{|H_1 H_2 H_3|^2} . \tag{2.2.61}$$

Within our assumptions the signal $z(t)$ will also consist of the sum of a deterministic part $z_d(t)$ plus a stochastic, noisy part $z_n(t)$.

We can model the thermal contribution as a stochastic force upon the oscillator, which is responsible both for the dissipation and fluctuations around the mean: we can then decompose it into a mean part and a zero-mean part,

$$m_0 \ddot{x} + k_0 x = F_{\text{th}} = \underbrace{-m_0 \gamma_0 \dot{x}}_{=\langle F_{\text{th}} \rangle} + F_{\text{n, th}} , \tag{2.2.62}$$

where $\langle F_{\text{n, th}} \rangle = 0$. This is called the **Nyquist force**, and we will just denote it as $F_{\text{n, th}} = F(t)$.

We can assume that thermal fluctuations are completely uncorrelated: they are due to billions of atomic interactions, so the process loses memory quickly — instantaneously, for the time scales we are interested in:

$$\big\langle F(t_1) F(t_2) \big\rangle = A_0 \delta(t_1 - t_2) , \tag{2.2.63}$$

for some constant A_0. This means that the noise is white noise: its PSD is constant in frequency.

Our harmonic oscillator can be written as

$$\ddot{\xi}_0 + \gamma_0 \dot{\xi}_0 + \omega_0^2 \xi_0 = \frac{F(t)}{m_0}\,, \tag{2.2.64}$$

which we can solve with a Green's function:

$$x(t) = \underbrace{x_0(t)}_{\text{hom. sol.}} + \frac{1}{m_0} \int_{-\infty}^{\infty} \mathrm{d}t'\, G(t-t') F(t')\,, \tag{2.2.65}$$

where

$$G(t) = \frac{1}{\omega_0} \theta(t) \exp\left(-\frac{\gamma_0 t}{2}\right) \sin(\omega_0 t) \tag{2.2.66}$$

is the Green's function, the impulse response of our system. The theta function is included in order to respect causality. Plugging this into the expression for the particular solution $x(t)$ we find:

$$x(t) = \frac{1}{m_0 \omega_0} \int_0^t \mathrm{d}t' \exp\left(-\frac{\gamma_0}{2}(t-t')\right) \sin\big(\omega_0(t-t')\big) F(t')\,. \tag{2.2.67}$$

Since $F(t')$ corresponds to the completely-uncorrelated thermal noise, we can show by an explicit calculation that, as long as $\gamma_0 \ll \omega_0$ (which means that the characteristic time of dissipation is much longer than the resonant frequency's period):

$$\left\langle x^2(t) \right\rangle = \frac{A}{m_0 \omega_0} \int_0^t \mathrm{d}u\, e^{-\gamma_0 u} \sin^2(\omega_0 u) \tag{2.2.68}$$

$$\approx \frac{A}{m_0^2 \omega_0^2} \left(1 - e^{-\gamma_0 t}\right), \tag{2.2.69}$$

since when we write out the average of x^2 we find the average of the product of two copies of F, yielding a delta-function. The average energy of the mode is in general given by

$$\left\langle E(t) \right\rangle = \frac{1}{2} m_0 \omega_0^2 \left\langle x^2(t) \right\rangle + \frac{1}{2} m_0 \left\langle \dot{x}^2(t) \right\rangle \approx \frac{A}{2 m_0 \gamma_0} \left(1 - e^{-\gamma_0 t}\right). \tag{2.2.70}$$

If the mode starts out with zero energy and then is immersed in a thermal bath with temperature T, then by the equipartition theorem each of its quadratic degrees of freedom will gain an energy $k_B T/2$: so, the average energy will asymptotically approach

$$\left\langle E(t) \right\rangle \to \frac{1}{2} k_B T + \frac{1}{2} k_B T = k_B T\,, \tag{2.2.71}$$

which means that the normalization constant A must satisfy

$$k_B T = \frac{A}{2 m_0 \gamma_0} \implies A = 2 k_B T m_0 \gamma_0\,. \tag{2.2.72}$$

2.2.4 The Fluctuation-Dissipation theorem

The Power Spectral Density is defined as the Fourier transform of the autocorrelation (times two, since we are computing it single-sided):

$$S_F(\omega) = 2\int_{-\infty}^{\infty} \mathrm{d}t' \left\langle F(t)F(t') \right\rangle e^{i\omega(t-t')} = 2A\int_{-\infty}^{\infty} \mathrm{d}\tau\, \delta(\tau) e^{i\omega\tau} = 2A\,, \tag{2.2.73}$$

where $\tau = t - t'$. Since we know the expression for A, this tells us that the PSD of the noise is flat, and equal to:

$$S_F(\omega) = 4k_B T m_0 \gamma_0\,. \tag{2.2.74}$$

For a generic linear system, we can express the spectral shape of the force as a multiple of the velocity's spectral shape:

$$F(\omega) = Z(\omega)\dot{x}(\omega)\,, \tag{2.2.75}$$

where Z is called the **impedance** of the system. The electrical analog of this is the law relating the voltage and current: $V(\omega) = Z(\omega)I(\omega)$.

The **Fluctuation-Dissipation Theorem** tells us that the single-sided power spectral density of the force which is due to fluctuations is given in terms of the impedance of the system as:

$$\boxed{S_{\text{F, th}}(\omega) = 4k_B T\,\mathrm{Re}\big\{Z(\omega)\big\}\,. \tag{2.2.76}}$$

Note that taking the real part of Z means considering the *imaginary* term in the equation of motion: the real part of Z quantifies the dissipation.

The velocity in the linear system law can be written as $\dot{x}(\omega) = i\omega x(\omega)$, so by the transformation law of the PSD under the application of a "transfer function" (which is $i\omega$ in this case) we find

$$S_{x,\text{th}}(\omega) = \frac{S_{\dot{x},\text{th}}}{|i\omega|^2} = \frac{S_{\text{F, th}}(\omega)}{\omega^2 \big|Z(\omega)\big|^2} = \frac{4k_B T}{\omega^2}\frac{\mathrm{Re}\big\{Z(\omega)\big\}}{\big|Z(\omega)\big|^2}\,. \tag{2.2.77}$$

This result quantifies the thermal noise we get when we reverse-engineer the original signal, as opposed to the noise in the output.

The energy of the DoF must be asymptotically constant and equal to $k_B T/2$, but we have dissipation! Since the system is in thermal equilibrium, there is a pumping mechanism providing the energy which is lost to dissipation — this is a force, and it creates noise! The higher the dissipation, the harder this force needs to push, so the noise increases accordingly.

Another way to see it is that the loss term quantifies the "conductance" of energy between the thermal bath and the oscillator, so if it is low the time needed to transfer energy from the noisy thermal bath to the oscillator is very long.

In real resonant-bar detectors, the temperatures are of the order of milliKelvin: we cool them to close to zero by isolating them and making liquid helium go through everything. In this case the approximation of constant temperature is quite good.

Thermal noise examples

For an electrical circuit, we have $Z(\omega) \equiv R$, therefore $S_{V,\text{th}}(\omega) = 4k_B TR$.

The equivalent of x in this context is the charge Q, so that $I(\omega) = \dot{Q}(\omega)$. Its PSD contribution due to thermal noise will be

$$S_{\text{Q, th}}(\omega) = \frac{4k_B T}{\omega^2} \frac{\operatorname{Re}\{Z(\omega)\}}{\left|Z(\omega)\right|^2} = \frac{4k_B T}{\omega^2 R}\,, \tag{2.2.78}$$

therefore the noise on the current $I(\omega) = \dot{Q}(\omega)$ will be

$$S_{\text{I, th}}(\omega) = \omega^2 S_{\text{Q, th}}(\omega) = \frac{4k_B T}{R}\,. \tag{2.2.79}$$

How to interpret the inverse dependence on the dissipation?

Monday 2020-4-27, compiled 2020-07-28

Let us consider a lossy harmonic oscillator, whose equation of motion is

$$m_0 \ddot{x} + m_0 \gamma_0 \dot{x} + k_0 x = F\,, \tag{2.2.80}$$

so in Fourier space the force can be expressed as

$$F(\omega) = m_0 \left(-\omega^2 - i\omega\gamma_0 + \frac{k_0}{m_0}\right) x(\omega) \tag{2.2.81}$$

$$= \underbrace{\frac{im_0}{\omega}\left(-\omega^2 - i\omega\gamma_0 + \omega_0^2\right)}_{Z(\omega)} \dot{x}(\omega)\,, \tag{2.2.82}$$

which means that we can write the impedance as

$$Z(\omega) = \underbrace{m_0\gamma_0}_{\operatorname{Re}\{Z\}} + i\frac{m_0}{\omega}\left(\omega_0^2 - \omega^2\right). \tag{2.2.83}$$

So, the thermal noise PSD of the reverse-engineered signal will be

$$S_x(\omega) = \frac{4k_B T}{\omega^2} \frac{m_0\gamma_0}{m_0^2\gamma_0^2 + \left(\frac{m_0}{\omega}\left(\omega_0^2 - \omega^2\right)\right)^2} = S_F(\omega)\left|H_{F\to x}(\omega)\right|^2. \tag{2.2.84}$$

For **structural damping** δ, we have $\operatorname{Re}\{Z(\omega)\} = m_0\delta\omega_0^2/\omega$, so the force PSD is

$$S_F(\omega) = 4k_B T m_0 \delta \frac{\omega_0^2}{\omega}\,. \tag{2.2.85}$$

Thermal noise for the resonant bar

What does the noise look like in the bar-transducer system? We need to compare the generic $Z(\omega)$ we wrote for the harmonic oscillator to the transfer function we found for the fundamental mode of the resonant bar, $H_{h\to\xi_0}(\omega)$: we find that they are related by

$$H_{h\to\xi_0}(\omega) = -\frac{2L}{\pi^2} m_0 \omega i \frac{1}{Z(\omega)}\,, \tag{2.2.86}$$

which, together with the fact that $\mathrm{Re}\{Z\} = m_0\gamma_0$, means that the noise PSD for the fundamental mode will read

$$S_{\xi_0} = \frac{4k_B T}{\omega^2} \frac{\mathrm{Re}\{Z(\omega)\}}{\left|Z(\omega)\right|^2} = \frac{4k_B T}{\omega^2} \frac{m_0\gamma_0}{\left|\frac{2L}{\pi^2} m_0 \omega_0 \frac{1}{H}\right|^2} \tag{2.2.87}$$

$$= \frac{4k_B T\gamma_0}{m_0\omega^4} \left(\frac{\pi^2}{2L}\right)^2 \left|H_{h\to\xi_0}(\omega)\right|^2. \tag{2.2.88}$$

If we try to reverse-engineer the original GW input, we must divide by the absolute value of the transfer function, which cancels out: so, we find

$$S_{h,\mathrm{th}} = \frac{S_{\xi_0}}{\left|H_{h\to\xi_0}(\omega)\right|^2} = \frac{4k_B T\gamma_0}{m_0\omega^4} \left(\frac{\pi^2}{2L}\right)^2 = \frac{\pi}{Q_0} \frac{k_B T}{M v_s^2} \frac{f_0^3}{f^4}, \tag{2.2.89}$$

where we used the relations $\gamma_0 = \omega_0/Q_0$, $m_0 = M/2$ and $L = \pi v_s/\omega_0$. So, a large quality factor reduces thermal noise.

Thermal noise for the transducer

To treat the thermal noise for the bar-transducer system, we assume that both of them are at equilibrium at a certain temperature T, but their damping factors are independent:

$$S_{F_0} = 4k_B T m_0 \gamma_0 \qquad \text{and} \qquad S_{F_t} = 4k_B T m_t \gamma_t\,. \tag{2.2.90}$$

The displacement of the transducer due to the two stochastic forces is given by

$$\xi_t(\omega) = \frac{\pi^2}{2L} \frac{H_{h\to\xi_t}(\omega)}{\omega^2} \left(\frac{F_0}{m_0} - \frac{F_t(\omega)}{m_t} \frac{\omega^2 - \omega_0^2 + i\omega\gamma_0}{\omega_0^2}\right). \tag{2.2.91}$$

Using these two relations we can compute the transducer's power spectral density:

$$S_{\xi_t}(\omega) = \frac{\pi^4}{4L^2} \frac{\left|H^2_{h\to\xi_t}\right|(\omega)}{\omega^4} 4k_B T \left(\frac{\gamma_0}{m_0} + \frac{\gamma_t}{m_t} \frac{(\omega^2-\omega_0^2)^2 + \omega\gamma_0^2}{\omega_0^2}\right). \tag{2.2.92}$$

Missing square modulus of the transfer function in the notes! See [Mag07, eq. 8.134].

In terms of the input, this will look like:

$$S_{h,\mathrm{th}} = \pi \frac{k_B T}{M v_s^2} \frac{f_0^3}{f^4} \left(\frac{1}{Q_0} + \frac{1}{\mu Q_t} \frac{\left(f^2 - f_0^2\right)^2 + (f f_0/Q_0)^2}{f_0^4}\right). \tag{2.2.93}$$

While Q_0 is quite large, both Q_t and μ are rather small: this might seem like a big issue, since it means that the thermal noise contribution from the transducer is quite large.

Fortunately, near resonance ($f \approx f_0$) the first term in the numerator goes to zero; while the second term always stays small since it is suppressed by a factor Q_0^{-2}. This means that, while far from resonance the thermal noise from the transducer dominates, near resonance it is quite small and the thermal noise from the bar dominates.

A qualitative explanation for this behavior is that since the response of the system is resonant even an oscillation with a large amplitude corresponds to a small GW amplitude when we reconstruct the signal.

2.2.5 Readout noise

Without getting into the specifics of the electronic setup, we try to construct a system so that small displacements of the bar are mapped linearly to voltages which we can measure:

$$V_{\text{out}} = \alpha \xi_t \,, \tag{2.2.94}$$

and, since the PSD is 2-homogeneous, if we have some readout noise whose PSD is $S_{V_{\text{out}}}$, then the effective PSD of the noise for the reconstructed displacement signal will be

$$S_{\xi_t} = \frac{1}{\alpha^2} S_{V_{\text{out}}} \,. \tag{2.2.95}$$

This means that we can interpret the readout noise as an effective displacement. If we go further and consider the effective GW signal modification, we will see

$$S_{h,\text{ro}} = \frac{1}{\left| H_{h \to \xi_t} \right|^2} S_{\xi_t} = \frac{1}{\left| H_{h \to \xi_t} \right|^2 \alpha^2} S_{V_{\text{out}}} \,. \tag{2.2.96}$$

Usually, the spectral shape of the readout noise is quite uniform, but since the transfer function is featured the effective GW PSD due to readout will be featured as well.

2.2.6 Effective temperature

We want to find out how much energy the GW must carry in order to be visible above the noise.

For **thermal noise**, we have seen that the characteristic time for the decay of the decay of a fundamental mode oscillation is $\tau_0 = 1/\gamma_0 = Q_0/\omega_0$, of the order of several minutes. This characterizes *all* the mode's interactions with the thermal bath, so the time it takes for the bath to spoil the oscillation is τ_0 as well: this means that, while we might naïvely expect the noise threshold for the detection of a GW to be $E_{GW} \geq k_B T$, it is in fact much lower. Specifically, if Δt is our sampling time, then the noise threshold is actually $E_{GW} \geq k_B T \Delta t / \tau_0$.

As for the **readout noise**, if our bandwidth (the inverse of the sampling time) is Δf, then the variance of the readout noise in terms of displacement of the transducer can be recovered from the PSD by

$$\left\langle \xi_t^2 \right\rangle = \int_{f_0 - \Delta f/2}^{f_0 + \Delta f/2} S_{\xi_t,\text{ro}} \, \mathrm{d}f \sim S_{\xi_t,\text{ro}} \Delta f \,, \tag{2.2.97}$$

which corresponds to an energy of $E_{\text{ro}} \sim m_t \omega_0 \left\langle \xi_t^2 \right\rangle \sim m_t \omega_0 S_{\xi_t, \text{ro}} \Delta f$.

The interesting thing to note is that the dependence of the thermal noise threshold energy on the sampling time is **direct** (if we sample for a long time the bath has a long time to interact with the oscillator), while the corresponding quantity for the readout noise has an **inverse dependence** on Δt (sampling very fast is error-prone since we are exposed to more high-frequency noise):

$$\Delta E_{\text{min}} \sim k_B T \frac{\Delta t}{\tau_0} + \frac{m_t \omega_0 S_{\xi_t, \text{ro}}}{\Delta t} . \tag{2.2.98}$$

We need to trade off these two contributions; the minimum can be found by differentiating and it comes out to be

$$\Delta f_{\text{opt}} = \frac{1}{\Delta t_{\text{opt}}} \approx \pi \frac{f_0}{Q\sqrt{\Gamma}} \qquad \text{where} \qquad \Gamma = \frac{m_t \omega_0^3 S_{\xi_t, \text{ro}}}{4 Q k_B T} . \tag{2.2.99}$$

An important distinction to make is the one between the useful bandwidth Δf_{opt}, which is quite large (tens of Hertz), and the width of the resonance peak *of the transfer function*, which is $\Delta f_{\text{res}} \sim f_0 / Q \sim 1\,\text{mHz}$. The detector's actual resonance peaks are **broad**, with a bandwidth comparable to Δf_{opt}.

At the optimal sampling rate, the two contributions to the ΔE are equal, therefore we have

$$\Delta E_{\text{opt}} \sim 2 k_B T \frac{\Delta t_{\text{opt}}}{\tau_0} \sim 2 k_B T \frac{\omega_0}{Q \Delta f} = k_B \underbrace{T \frac{4\pi f_0}{Q \Delta f}}_{T_{\text{eff}}} , \tag{2.2.100}$$

which can be interpreted as a new effective temperature at which the oscillator is immersed: substituting in we can find that it is given by

$$T_{\text{eff}} \sim 4\sqrt{\Gamma} T , \tag{2.2.101}$$

which can be much lower than the real temperature of the object, since Γ can be made to be of the order 10^{-8} to 10^{-9}: this means that the effective temperature can be three orders of magnitude lower than the thermodynamic one.

The definitive formula for the sum of these contributions can be found in Maggiore [Mag07, eq. 8.150]; it is interesting to compare the shape of the PSD at varying values of Γ: if it is rather high ($\sim 10^{-7}$) then the PSD is quite high as well with two conspicuous drops at $\omega_0 \pm \omega_p$. Here, the transducer thermal noise dominates almost everywhere.

If Γ becomes lower, of the order of 10^{-9}, then the PSD becomes uniformly low in the range $[\omega_0 - \omega_p, \omega_0 + \omega_p]$.

2.3 Gravitational Wave Interferometry

2.3.1 Mach-Zender interferometer

The setup here is the following: the laser impacts onto a first beamsplitter, is split onto two orthogonal paths which are made to converge onto a second beamsplitter through two mirrors. After this second beamsplitter, the signal is measured. Depending on the phase, the interference is either constructive or destructive at the second beamsplitter.

The laser's electric field will be given by

$$\vec{E}_{\text{in}} = \vec{E}_0 \exp\left(-i(\omega_l t - k_l t)\right), \tag{2.3.1}$$

where the subscript l means "laser", we include it in order to distinguish these parameters from the GW ones.

In general:

1. at reflection the beam picks up a phase π;
2. at transmission the beam picks up no phase;
3. between the two paths there is a phase difference due to their different lengths: $\Delta\phi$.

So, the fields incoming to the second beamsplitter are:

$$E_T = \frac{E_{\text{in}}}{\sqrt{2}} e^{i\pi} \qquad \text{and} \qquad E_R = \frac{E_{\text{in}}}{\sqrt{2}} e^{i\Delta\phi}. \tag{2.3.2}$$

At the second beamsplitter there are two outputs: the outgoing fields are

$$E_{\text{out, 1}} = \frac{E_T}{\sqrt{2}} e^{i\pi} + \frac{E_R}{\sqrt{2}} = \frac{E_{\text{in}}}{2}\left(1 + e^{i\Delta\phi}\right) \tag{2.3.3}$$

$$E_{\text{out, 2}} = \frac{E_T}{\sqrt{2}} + \frac{E_R}{\sqrt{2}} e^{i\pi} = \frac{E_{\text{in}}}{2}\left(-1 - e^{i\Delta\phi}\right), \tag{2.3.4}$$

so the initial power is multiplied by $(1 + \cos(\Delta\phi))/2$. So, is energy not conserved? This output is the same on both ends of the beamsplitter, and these two do not sum to the initial power.

This result is due to an oversight: the phase of π is actually picked up only if the index of refraction increases along the path of the beam. Correcting for this, we find

$$E_{\text{out, 2}} = \frac{E_{\text{in}}}{2}\left(-1 + e^{i\Delta\phi}\right), \tag{2.3.5}$$

so

$$\left|E_{\text{out, 2}}\right|^2 = \frac{\left|E_{\text{in}}\right|^2}{2}\left(1 - \cos(\Delta\phi)\right), \tag{2.3.6}$$

so we recover energy conservation: $\left|E_{\text{out, 1}}\right|^2 + \left|E_{\text{out, 2}}\right|^2 = \left|E_{\text{in}}\right|^2$.

2.3.2 Michelson-Morley interferometer

Now the setup is different: the first and second beamsplitters are the same one, and the mirrors reflect the laser directly backwards.

The electric fields in the path from the beamsplitter to either mirror are denoted as E_x and E_y; they are given by

$$E_x = \frac{E_0}{\sqrt{2}\sqrt{2}} \exp\Big(i\big(kx - \omega_l t + \phi_x\big)\Big) \tag{2.3.7}$$

$$E_y = \frac{E_0}{\sqrt{2}\sqrt{2}} \exp\left(i\Big(ky - \omega_l t + \phi_y\Big)\right), \tag{2.3.8}$$

where the double $\sqrt{2}$ factor is due to the fact that each beam goes through the beamsplitter twice; while the factors $\phi_{x,y}$ are the phases picked up upon reflection on each side.

The output electric field (for either output of the beamsplitter — we will distinguish between them by varying ϕ) is given by

$$E_{\text{out}} = E_x + E_y = \frac{E_0}{2}\left(\exp\Big(i\big(kx - \omega_l t + \phi_x\big)\Big) + \exp\left(i\Big(ky - \omega_l t + \phi_y\Big)\right)\right), \tag{2.3.9}$$

so the output intensity is

$$I_{\text{out}} = |E_{\text{out}}|^2 = \frac{E_0^2}{4}\left(2 + \operatorname{Re}\left\{\exp\Big(i\big(k(x-y) - (\phi_x - \phi_y)\big)\Big)\right\}\right) \tag{2.3.10}$$

$$= \frac{E_0^2}{2}\left(1 + \cos\Big(k(x-y) + (\phi_x - \phi_y)\Big)\right). \tag{2.3.11}$$

Now, for output 1 (perpendicular to the original laser beam) we have $\phi_{x1} = \pi$ and $\phi_{y1} = 2\pi$, which can be gathered by counting the number of times the x (y) beam is reflected, and adding a π for each. Similarly, for output 2 we have $\phi_{x2} = \pi$ and $\phi_{y2} = 3\pi$. So, we have $\Delta\phi_1 = -\pi$ and $\Delta\phi_2 = -2\pi$.

Plugging this in, and using the fact that $\cos(x+\pi) = -\cos(x)$, we find

$$I_{\text{out, 1}} = \frac{E_0^2}{2}\Big(1 - \cos\big(k(x-y)\big)\Big) = E_0^2 \sin^2\left(\frac{k}{2}(x-y)\right) \tag{2.3.12}$$

$$I_{\text{out, 2}} = \frac{E_0^2}{2}\Big(1 + \cos\big(k(x-y)\big)\Big) = E_0^2 \sin^2\left(\frac{k}{2}(x-y) + \frac{\pi}{2}\right). \tag{2.3.13}$$

So, both of the outputs heavily depend on the path length difference between the two beams.

2.3.3 GW interactions in the detector frame

In the detector frame we can treat the GW as a Newtonian force acting on the mirrors, and we can compute it using the TT-gauge perturbation $h_{xx}^{TT} = h_0 \cos(\omega_{GW} t)$ since the Riemann tensor is invariant in linearized gravity:

$$F_x \approx \frac{m}{2} x_0 \ddot{h}_{xx}^{TT}, \tag{2.3.14}$$

since the perturbation is really small, we can treat x_0 as a constant. This means that the length of the arm changes according to the differential equation

$$\ddot{x} = \frac{1}{2} x_0 \ddot{h}_{xx}^{TT} \,. \tag{2.3.15}$$

This only holds in the short arm approximation: $x \ll \lambda_{GW}$, which means $f_{GW} \ll c/L \approx 100\,\mathrm{kHz}(L/3\,\mathrm{km})$ — fortunately this is not a great constraint, since modern GW interferometers are only sensitive up to a few kHz anyway.

Let us consider a plus-polarized GW travelling along the z axis, and let us assume that the beamsplitter is the origin. Then, let us analyze the motion of the two test masses which are in free-fall: the two mirrors, one for each arm. Their positions will be, respectively, $(x_{XTM}, y_{XTM}, z_{XTM})$ and $(x_{YTM}, y_{YTM}, z_{YTM})$. Because of the polarization of the wave, the only coordinates which will evolve in time will be

$$x_{XTM}(t) = L_x + \frac{h_0 L_x}{2} \cos(\omega_{GW} t) \tag{2.3.16}$$

$$y_{YTM}(t) = L_y - \frac{h_0 L_y}{2} \cos(\omega_{GW} t) \,. \tag{2.3.17}$$

If we insert the displacement for the mirrors, assuming $L_x \sim L_y \sim L$ we find

$$I_{\text{out}} = E_0^2 \sin^2(k(x-y)) \tag{2.3.18}$$

$$= E_0^2 \sin^2 \left(k \left[L_x + \frac{h_0 L_x}{2} \cos(\omega_{GW} t) - L_y + \frac{h_0 L_y}{2} \cos(\omega_{GW} t) \right] \right) \tag{2.3.19}$$

$$= E_0^2 \sin^2 \left(k \Big(L_x - L_y + h_0 L \cos(\omega_{GW} t) \Big) \right) \,. \tag{2.3.20}$$

One might think that we should not be able to see any effect, since the GW also stretches the wavelength of the light: this is not the case, since each wavefront will still be travelling at c and so its travel time will change for a change in the spatial components of the metric.

However, this detector-frame approach is not great, since it neglects all time-of-flight effects. Let us consider the problem in the TT-gauge.

2.3.4 GW interferometry in the TT gauge

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The TT gauge is a coordinate system in which the mirrors are free-falling (in the xy plane at least, and for the frequencies we are considering; all the delicate considerations we must make in order to distinguish the GW from the background still hold).

So, the mirrors are still; however the light propagating through spacetime is affected by the fact that the spacetime is “stretched”: for our photons the metric reads

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \big(1 + h_+(t)\big)\,\mathrm{d}x^2 + \big(1 - h_+(t)\big)^2\,\mathrm{d}y^2 + \mathrm{d}z^2 = 0 \,, \tag{2.3.21}$$

so in the x arm, depending on the direction of propagation we will have:

$$\mathrm{d}x = \sqrt{\frac{c^2\,\mathrm{d}t^2}{1 + h_+(t)}} \approx \pm c\,\mathrm{d}t \left(1 - \frac{1}{2} h_+(t)\right) . \tag{2.3.22}$$

We can integrate this along the photon's path to recover the effective travel length: for the first leg of the journey we get

$$L_x = \int_0^{L_x} \mathrm{d}x = \int_{t_0}^{t_1} c\,\mathrm{d}t = c(t_1 - t_0) - \frac{c}{2} \int_{t_0}^{t_1} \mathrm{d}t\, h_+(t)\,, \tag{2.3.23}$$

and for the second leg:

$$-L_x = \int_{L_x}^{0} \mathrm{d}x = -\int_{t_1}^{t_2} c\,\mathrm{d}t \left(1 - \frac{h_+(t)}{2}\right) = -c(t_2 - t_1) + \frac{c}{2} \int_{t_1}^{t_2} \mathrm{d}t\, h_+(t)\,, \tag{2.3.24}$$

where $t_{0,1,2}$ are the times at which the photon leaves the beamsplitter, bounces off the mirror, returns to the mirror. so the time difference is given by

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2} \int_{t_0}^{t_2} \mathrm{d}t\, h_+(t)\,, \tag{2.3.25}$$

where we can consider $t_2 \approx t_0 + 2L_x/c = \tau$ in the integration bound, since the correction would be second order.

Let us then compute the integral:

$$\frac{1}{2} \int_{t_0}^{\tau} h_+(t)\,\mathrm{d}t = \frac{1}{2} \int_{t_0}^{\tau} \mathrm{d}t\, h_0 \cos(\omega_{GW} t) = \frac{h_0}{2\omega_{GW}} \sin(\omega_{GW} t)\Big|_{t=t_0}^{t=\tau}\,. \tag{2.3.26}$$

We can simplify the expression making use of the trigonometric relation

$$\sin\big(\alpha + 2\beta\big) - \sin(\alpha) = 2\sin\big(\beta\big)\cos\big(\alpha + \beta\big)\,, \tag{2.3.27}$$

where our $\alpha = \omega_{GW} t_0$, and $\beta = \omega_{GW} L_x/c$. This yields:

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{L_x}{c} h_+\left(t_0 + \frac{L_x}{c}\right) \frac{\sin\left(\frac{\omega_{gw} L_x}{c}\right)}{\frac{\omega_{gw} L_x}{c}}\,, \tag{2.3.28}$$

while for the y arm the sign of the corrective term is inverted — the travel time diminishes. Note that in this last expression we are computing h_+ at the time $t_0 + L_x/c$.

We have divided and multiplied by L_x/c in order to recover a sinc function, $\operatorname{sinc} x = \sin x / x$. The argument of this function is the ratio of the length of our arm to the wavelength of the GW. If $L_x/c \ll T_{GW} = 2\pi/\omega_{GW}$, the perturbation is essentially "frozen" during the light's travel; in the opposite limit during the travel time of the light the perturbation oscillates back and forth, cancelling out most of the effect.

We have a maximum of the effect, then, when the GW perturbation is effectively static during the time in which we are detecting: this suggests we should build *short* GW interferometers. Keep in mind, though, that the effect size also scales with L_x: a very small detector wouldn't work. We must trade off between these two. If we fix the frequency, making the detector longer and longer does not help: further full oscillations of the path length will not have any effect.

The light will arrive at the beamsplitter at a time $t = t_2$, from which we can (at least approximately) recover $t_0 \approx t - 2L/c$. Similarly, we can calculate the starting times of the two beams by inverting the relation and plugging in what we have just found:

$$t_0^x = t - \frac{2L_x}{c} - \frac{L_x}{c} h_+ \left(t - \frac{L_x}{c} \right) \operatorname{sinc} \left(\frac{\omega_{GW} L_x}{c} \right) \tag{2.3.29}$$

$$t_0^y = t - \frac{2L_y}{c} + \frac{L_y}{c} h_+ \left(t - \frac{L_y}{c} \right) \operatorname{sinc} \left(\frac{\omega_{GW} L_y}{c} \right) . \tag{2.3.30}$$

If the light arrives at a certain time t to the beamsplitter from both arms, then we can compute the phase difference of the beams by starting from $t_0^x - t_0^y$: $\Delta\phi = \omega_l (t_0^x - t_0^y)$.

The interesting thing is the phase difference: if in both arms the light reaches the detector at t_2 we have

$$\Delta\phi = \omega_l \left(t_0^x - t_0^y \right) \tag{2.3.31a}$$

$$= \underbrace{\omega_l 2 \frac{L_x - L_y}{c}}_{\Delta\phi_0} + \underbrace{\omega_l \frac{2L}{c} \operatorname{sinc} \left(\frac{\omega_{gw} L}{c} \right) h_0 \cos \left(\omega_{gw} t + \alpha \right)}_{\Delta\phi_{GW}} , \tag{2.3.31b}$$

where we substituted $h_+(t - L/c)$ with its explicit expression, with the constant phase $\alpha = -\omega_{GW} L/c$.

The term $\Delta\phi_0$ is controlled by the experimenter, while the term $\Delta\phi_{GW}$ is due to the GW.

We know that the output intensity will look like:

$$I_{\text{out}} = E_0^2 \sin^2 (\Delta\phi_0 + \Delta\phi_{GW}) . \tag{2.3.32}$$

There will be two main contributions to $\Delta\phi_0$:

1. a microscopic term we can vary to change the working point of the interferometer, meaning the output with no GW;
2. a macroscopic term called the **Schnupp asymmetry** which allows the *sideband frequency* $\omega_l + \omega_{sb}$ to leak.

We want to maximize $\Delta\phi_{GW}$, so we have a tradeoff: we want to stay before the first zero of the sinc, but we also want to have a relatively large detector, since there is an L multiplying everything. Specifically, we have

$$\Delta\phi_{GW} = \omega_l \frac{2L}{c} \operatorname{sinc} \left(\frac{\omega_{GW} L}{c} \right) h_0 \cos(\omega_{GW} t + \alpha) \propto L \operatorname{sinc} \left(\frac{2\pi L}{\lambda_{GW}} \right) \propto \sin \left(\frac{2\pi L}{\lambda_{GW}} \right) , \tag{2.3.33}$$

which reaches a maximum when the argument of the sine is $\pi/2$: so, we have

$$\frac{\pi}{2} = \frac{2\pi L}{\lambda_{GW}} \implies L = \frac{\lambda_{GW}}{4} \approx 750\,\text{km} \left(\frac{100\,\text{Hz}}{f_{GW}} \right) . \tag{2.3.34}$$

So, our optimal length is of the order of a quarter of the wavelength. This is the same as saying we want to keep the photon in-flight for a quarter of the period of the GW.

If we do the computation accounting for the oscillation of the laser light, we get that the field out of the BS is

$$E = \frac{E_0}{2}\exp\left(i\omega_l t + i\Delta\phi_0 + i\Delta\phi_{GW}\right) \approx \frac{E_0}{2}\exp\left(i\omega_l t + i\Delta\phi_0\right)\left(1 + i\Delta\phi_{GW}\right) \tag{2.3.35}$$

$$= \frac{E_0}{2}e^{-i\omega_l(t-2L/c)}\left(1 + i\omega_l\frac{L}{c}\operatorname{sinc}\left(\frac{\omega_{GW}L}{c}\right)\frac{e^{i\omega_{GW}t+i\alpha} + e^{-i\omega_{GW}t-i\alpha}}{2}\right) \tag{2.3.36}$$

$$= \frac{E_0}{2}e^{-i\gamma}\left(e^{-i\omega_l t} + \beta e^{-i\alpha}e^{-i(\omega_l-\omega_{gw})} + \beta e^{i\alpha}e^{-i(\omega_l+\omega_{gw})t}\right), \tag{2.3.37}$$

with a suitable definition of γ and β — it does not really matter, the important thing is that we now have components of the oscillation at $\omega_l \pm \omega_{GW}$.

We can interpret the addition of two sidebands which are in phase with the signal as an *amplitude modulation*:

$$\cos(\omega_c t) + A_{sb}\cos\left((\omega_c + \omega_{sb})t\right) + A_{sb}\cos\left((\omega_c - \omega_{sb})t\right) = \left(1 + A_m\cos(\omega_m t)\right)\cos(\omega_c t), \tag{2.3.38}$$

while if the sidebands are out of phase with the signal we can interpret them as a phase modulation:

$$\cos(\omega_c t) + A_{sb}\sin\left((\omega_c + \omega_{sb})t\right) + A_{sb}\sin\left((\omega_c - \omega_{sb})t\right) = \cos\left(\omega_c t + A_m\cos(\omega_m t)\right). \tag{2.3.39}$$

These two scenarios are identical (amplitude modulation and phase modulation) if we look at the amplitude of the Fourier transform, but with different phases.

The phase modulation scenario is approximate, since there is a slight amplitude modulation (which however is second order).

2.3.5 Lasers and cavities

We have seen that the optimal length of our detector is of the order of several hundreds of kilometers: this is an issue! In this section we will see how to "fold" our interferometer so that we can reach this sensitivity.

Dielectric mirrors

An ideal mirror would be a sharp interface between two mediums, where the incoming electric field E_{in} is split into $rE_{\text{in}} = E_r$ and $tE_{\text{in}} = E_t$. Due to energy conservation, these must satisfy $r^2 + t^2 = 1$.

For a perfectly reflecting mirror, the transmission coefficients t are symmetric for the swap of the two materials, while r changes sign if we go from the denser to the less dense material or vice versa.

The best mirrors in the world are built for GW detectors: these are dielectric mirrors.

The idea is to stack dielectric interfaces on top of each other by alternating layers of high and low index of refraction, so that each layer has an optical depth of $\lambda/4$ at the desired wavelength λ. So, going through two of them changes the phase by π (half of a wavelength has gone by). Also, if we go through two of them then we pass *one* low-to-high index of refraction transition, yielding a phase difference of π. So, the global phase is $2\pi \equiv 0$.

So, all the reflected light keeps going back as the interference is constructive. This only holds as long as the light is of the correct frequency, and its angle of incidence is a certain one.

Cavities

A **cavity** is an arrangement of mirrors such that we have a closed path for light.

Mirrors are symmetric, if we can input some light then we are also losing the same amount. In the round-trip the beam can lose some energy to the environment: this is described as a "round-trip loss".

We consider a horizontal cavity with two mirrors: we will have an incident field E_{in}, a circulating field E_c and a transmitted field E_t. The "in" mirror is labelled 1, the "out" mirror is labelled 2. Both of these have reflection and transmission coefficients $r_{1,2}$ and $t_{1,2}$ respectively.

In general the reflection and transmission coefficients will be complex since the laser can acquire a phase while passing through the mirror, however this can be disregarded: it amounts to a global phase, which can be discarded by moving the mirrors until the desired working point is reached.

The circulating field can be obtained by adding the transmitted component of the incoming field, and the circulating field itself which has been reflected by both mirrors:

$$E_c = t_1 E_{\text{in}} + r_1 r_2 E_c e^{-ik2L}\,, \tag{2.3.40}$$

and we can simplify this to

$$E_c = E_{\text{in}} \frac{t_1}{1 - r_1 r_2 e^{-ik2L}}\,. \tag{2.3.41}$$

Although it might seem that this calculation is simplistic, neglecting the field which has done more than one round-trip, the result is in fact the same as the more complete calculation.

Here k is the wavevector of the electric field. This expression gives us the *round-trip gain*: the term in the denominator can become lower than 1.

The reflected field, which is in the same place as the incoming field but moving in the opposite direction, is given by

$$\begin{aligned}
E_r &= -r_1 E_{\text{in}} + r_2 t_1 E_c e^{-ik2L} && \text{(2.3.42a)}\\
&= E_{\text{in}} \left(-r_1 + \frac{r_2 t_1^2 e^{-ik2L}}{1 - r_1 r_2 e^{-ik2L}} \right) && \text{(2.3.42b)}\\
&= -E_{\text{in}} \frac{r_1 - r_2 e^{-ik2L}}{1 - r_1 r_2 e^{-ik2L}}\,. && \text{(2.3.42c)}
\end{aligned}$$

On the other hand, the transmitted field going out of mirror 2 is

$$E_t = t_2 E_c = E_{\text{in}} \frac{t_1 t_2}{1 - r_1 r_2 e^{-ik2L}} . \tag{2.3.43}$$

The circulating intensity can be calculated by the square modulus of the circulating field:

$$I_c = E_{\text{in}}^2 \left| \frac{t_1}{1 - r_1 r_2 e^{-ik2L}} \right|^2 , \tag{2.3.44}$$

which we want to be large, so we want to minimize the denominator: this means we should tune the length to that the exponential is equal to 1, so $ik2L = 2\pi n$, which implies $L = n\pi/k$, for some $n \in \mathbb{R}$. If we perform the optimal choice, we will find

$$I_c = \left| \frac{t_1}{1 - r_1 r_2} \right| . \tag{2.3.45}$$

We can plot the intensity as k varies (which means we are varying the wavelength of the laser: as long as $r_1 r_2$ is close to 1, we see distinct peaks, whose distance is fixed even as we vary $r_1 r_2$, since it only depends on the length of the cavity. This distance, which is equal to $c/2L$, is called the **free spectral range**.

As we increase the reflectivity (which is measured by $r_1 r_2$) the peaks get narrower and higher.

The *finesse* is defined as the free spectral range divided by the FWHM of the peaks, and it can be shown that it can be expressed as

$$\mathcal{F} = \frac{c/2L}{\text{FWHM}} = \frac{\pi \sqrt{r_1 r_2}}{1 - r_1 r_2} . \tag{2.3.46}$$

Actually, we take the last expression to be the definition of the finesse; it is only approximately the ratio of FSR and FWHM.

Using this, we can estimate the storage time of a photon inside the cavity:

$$\tau_s \approx \frac{L\mathcal{F}}{c\pi} . \tag{2.3.47}$$

For example, if $r_1^2 = 0.99$, $r_2 = 1$ and $L = 3\,\text{km}$ then $\mathcal{F} \approx 625$ and $\tau_s \approx 2\,\text{ms}$.

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We can have a lot of energy stored in the cavity, so that the power of the stored laser beam is much larger than the input power.

This does not violate conservation of energy: the energy is stored, but we cannot extract more power than what is coming in.

We can treat the losses inside the cavity by introducing an intensity loss parameter, l_I, which is the ratio of the lost power to the circulating power; then we can define

$$r_l = 1 - l_E = \sqrt{1 - l_I} , \tag{2.3.48}$$

and then each time r_2 appears we can substitute it by $r_2 r_l$ — this effectively amounts to attributing the power loss to the output coupler.

Then, the fields inside and outside will be given by

$$E_c = E_{\text{in}} \frac{t_1}{1 - r_1 r_2 r_l e^{-ik2L}} \tag{2.3.49a}$$

$$E_r = -E_{\text{in}} \frac{r_1 - r_2 r_l e^{-ik2L}}{1 - r_1 r_2 r_l e^{-ik2L}} \,. \tag{2.3.49b}$$

As we saw, the reflected electric field has a component which is due to the promptly reflected incoming field ($E_{r,p}$), and a component which is due to the loss of part of the circulating field ($E_{r,l}$): we can write this as $E_r = E_{r,p} + E_{r,l}$.

These are of the same order of magnitude: the mirrors are very reflective, but inside the cavity the power is much greater than outside. Since they are comparable, the interference of these two contributions is interesting.

We can then distinguish three different behaviors by comparing the losses in the first mirror (described by r_1) to the losses in the path and in the second mirror (described by $r_2 r_l$):

1. the cavity is **overcoupled** when $r_1 < r_2 r_l$ (mirror 1 does not reflect enough): then around resonance we will have $E_{r,p} < E_{r,l}$;

2. the cavity is **undercoupled** when $r_1 > r_2 r_l$ (mirror 1 reflects too much): then around resonance we will have $E_{r,p} > E_{r,l}$;

3. the cavity is **impedence matched** when $r_1 = r_2 r_l$, so that $E_{r,p} = E_{r,l}$ around resonance.

In this case the reflected intensity goes to zero, since by having the same intensity the two fields can destructively interfere — they *do* destructively interfere, since the phase difference is given by $-e^{-ik2L} = -1$ if the cavity is tuned. We also have a discontinuity in phase in the impedance matched case, but this is not an issue since the intensity goes to zero.

Beams

Let us define a **beam** properly: its electric field at a fixed time will be in the form

$$\vec{E}(x, y, z) = u(x, y, z) e^{-ikz} \,, \tag{2.3.50}$$

where the function $u(x, y, z)$ is a complex amplitude, and $k = \omega_l / c$ is the wavenumber. We are assuming that the beam propagates along the z axis.

This must obey the wave equation $\Box E = 0$, which can be written as

$$\left(\nabla^2 + k^2\right) E \propto \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - 2ik \frac{\partial u}{\partial z} - k^2 u + k^2 u = 0 \,. \tag{2.3.51}$$

The term $\partial^2 u \big/ \partial z^2$ has magnitude much smaller than the other terms: so, we neglect it. This is called the **paraxial approximation** — we are basically saying that as the wave propagates it diverges out in a fashion which is approximately linear in z. This allows us to rewrite the wave equation as

$$\nabla_t^2 u(\vec{s}, z) - 2ik \frac{\partial u(\vec{s}, z)}{\partial z} = 0\,, \tag{2.3.52}$$

where ∇_t^2 is the Laplacian operator only in the transverse directions, while $\vec{s}$ are two coordinates such as x, y describing these directions.

An approximate solution to this differential equation is given by the **Gaussian beam**[4]

$$u(x, y, z) = E_0 \frac{w_0}{w(z)} \exp\left(-\frac{x^2 + y^2}{w^2(z)} - i \left(kz + k \frac{x^2 + y^2}{2R(z)} - \psi(z) \right) \right)\,, \tag{2.3.53}$$

where we define the **beam waist**

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_r} \right)^2}\,, \tag{2.3.54}$$

the **radius of curvature** of the wavefronts

$$R(z) = z \left(1 + \left(\frac{z_r}{z} \right)^2 \right)\,, \tag{2.3.55}$$

and the **Gouy phase**:

$$\psi(z) = \operatorname{atan}(z/z_r)\,, \tag{2.3.56}$$

where the **Rayleigh range** z_r defines the scale of the distortions of the beam: it is the length after which the beam has doubled in area, and it is given by

$$z_r = \frac{\pi w_0^2}{\lambda}\,. \tag{2.3.57}$$

Qualitatively speaking, the beam is *squeezed*: it is not a plane wave, its width can be made to be small in a certain region but the smaller it is there the faster it becomes wide outside it.

All the parameters can be determined if we give the width of the beam at the center w_0 and the wavelength of the light λ.

The Gouy phase means that passing through the narrow beam region gives the beam a phase of π.

The divergence angle of the beam is given by $\theta = w(z)/z$, and since we can approximate $w(z) \approx w_0 z/z_r$ we can write $\theta \approx w_0/z_r = \lambda/\pi w_0$, which is a sort of uncertainty principle:

$$w_0 \theta = \frac{\lambda}{\pi}\,, \tag{2.3.58}$$

[4] Notice that there is no propagation term $\exp(ikz)$ since we factored it out earlier! The expression for E will include it.

meaning that we have uncertainty between the width of the beam at its narrowest and its angle of dispersion.

Starting from the Gaussian beam, we can define orthonormal bases which describe any beam in the paraxial approximation: the Laguerre-Gauss (cylindrical, flower-looking) and Hermite-Gauss (rectangular, boxy-looking) bases.

These are eigenfunctions of the paraxial wave equation: they propagate without changing their shape (although they are scaled). This is not the case for combinations of them, because of the Guoy phase.

For both bases, we have two "quantum numbers" l, m labelling the eigenfunctions. We generally try to work with the 00 mode (which is a Gaussian in both bases), because it is easier.

Resonance in a cavity means constructive interference of the beam with itself after a round-trip: we must have resonance both in phase and in the transverse profile.

For the **phase** resonance, the round-trip length of the cavity must equal an integer times the wavelength, plus the Gouy phase. Because of the latter effect, different eigensolutions will resonate at different frequencies.

For the **profile** resonance, the beam must come back from the round trip with the same spatial profile: this is selected through the geometry of the cavity. As the beam spreads the wavefronts curve, we need to account for this: the mirrors must be exactly parallel to the wavefronts. So, we need to make them curved.

What is the length scale of this curvature? For LIGO-VIRGO we have a *confocal* cavity, so the sum of the curvatures of the mirrors is of the order of the length of the cavity. For the mirrors we use in the lab, their radii of curvature are from centimeters to meters.

If a gaussian beam is reflected on the mirror and comes back to itself, one might think that all the higher orders modes should come back to themselves as well: they do not do so because of the Gouy phase difference. The very high order modes are also suppressed since they are more affected by the imperfections of the mirror. These are desired effects, we try to clean the beam until there is almost only the fundamental frequency.

2.3.6 Realistic GW interferometers

We can do a back-of-the-envelope calculation for the sensitivity of a Michelson interferometer.

We want to detect a GW amplitude whose amplitude spectral density is of the order

$$\frac{\Delta L}{L} \approx 10^{-21}\,\mathrm{Hz}^{-1/2}\,, \tag{2.3.59}$$

and since our interferometer is a few km long this means that we must have

$$\Delta L \approx 10^{-18}\,\mathrm{m}/\sqrt{\mathrm{Hz}}\,, \tag{2.3.60}$$

so, since the light we use is in the visible to near-infrared range $\lambda \sim 10^{-6}\,\mathrm{m}$, or equivalently we want an interferometric sensitivity of around

$$\frac{\Delta L}{\lambda} \approx 10^{-12}\,\mathrm{Hz}^{-1/2}\,. \tag{2.3.61}$$

The error in photon counting will be given by Poisson statistics, $\Delta N \sim \sqrt{N}$; if we want it to be comparable to the signal then we must have

$$\frac{\Delta L}{\lambda} \approx \frac{\Delta N}{N} = \frac{1}{\sqrt{N}} = 10^{-12}\,\mathrm{Hz}^{-1/2}\,, \tag{2.3.62}$$

which means we must get a number of photons per second the order of

$$N \approx 10^{-24}\,\mathrm{Hz}\,. \tag{2.3.63}$$

This means we need a power of approximately

$$P = N\hbar\omega \approx 200\,\mathrm{kW}\,. \tag{2.3.64}$$

This is a very large requirement for a laser. Fortunately, Fabry-Perot cavities are the solution: they increase the power inside the cavity by a factor $\mathcal{F}/2\pi$, the average number of bounces a photon makes before being lost; they also useful to increase the effective length: we want $L_{\mathrm{arm}} \approx 750\,\mathrm{km}\left(100\,\mathrm{Hz}/f_{\mathrm{gw}}\right)$, while the real arm length is of the order $3\,\mathrm{km}$.

So, we have different reasons to introduce cavities: they increase power, they increase effective length.

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With a FP cavity we increase the interaction time of a photon with the GW by a factor $\mathcal{F}/2\pi$, yielding

$$\tau_s = \frac{L}{c}\frac{\mathcal{F}}{\pi}\,. \tag{2.3.65}$$

The cavity, in a way, acts as a low-pass filter, removing any effects which happen on timescales shorter than τ_s.

Ideally, we would want to make the storage time as long as possible, but still smaller than the period of the GW:

$$\tau_s \sim \frac{1}{2f_{GW}} \implies \mathcal{F} \sim \frac{\pi c}{L2f_{GW}} \sim 10^3\left(\frac{100\,\mathrm{Hz}}{f_{GW}}\right)\,. \tag{2.3.66}$$

This is an intuitive way to write it, let us make it more precise: we can model power loss at a mirror i as $r_i^2 + t_i^2 = 1 - p_i$, where p_i is a small parameter quantifying the fraction of power lost at the mirror.

We can define an **effective power loss** p by $(1-p_1)r_2^2 = 1 - p$. Then, we will have

$$r_1^2 = 1 - p_1 - t_1^2 < 1 - p_1 \implies r_1^2 r_2^2 < 1 - p\,, \tag{2.3.67}$$

This whole calculation is kind of unclear...

so the finesse will satisfy the inequality

$$r_1 r_2 \approx 1 - \frac{\pi}{\mathcal{F}} \leq 1 - \frac{p}{2}\,. \tag{2.3.68}$$

Let us define the **coupling factor** $\sigma = p\mathcal{F}/\pi$: it will be between 0 and 2, and its value will determine whether the cavity is over- or under-coupled; for $\sigma = 1$ it will be impedance-matched.

This is interesting since it allows us to study the response of the phase of the reflected field

$$E_r = -E_{\text{in}} \frac{r_1 - r_2 e^{-ik2L}}{1 - r_1 r_2 e^{-ik2L}}, \tag{2.3.69}$$

to a small perturbation $\epsilon = 2k_l \delta L$: its phase $\phi = \arg E_r$ responds as

$$\phi = \pi + \arctan\left(\frac{\mathcal{F}\epsilon}{\pi} \frac{1}{1-\sigma}\right) + \arctan\left(\frac{\mathcal{F}\epsilon}{\pi}\right), \tag{2.3.70}$$

so for $\sigma > 1$ we have partial cancellation of the reflected field, if $\sigma = 1$ there is no reflected light at all, while if $\sigma < 1$ gets small we get ever more light but less sensitivity.

Why are we interested in the phase of the *reflected* field?

If we suppose $r_2 = 1$, $r_1 \sim 1$ and $p_1 = 0$ we get $p = 1 - r_2^2 = 0$, so $\sigma = 0$, which means that the phase sensitivity is

$$\frac{\partial \phi}{\partial \epsilon} = \frac{2\mathcal{F}}{\pi}, \tag{2.3.71}$$

since we have two equal arctangent terms, both of which are linear near the origin. This is much better than what we have with a simple Michelson interferometer, $\partial\phi/\partial\epsilon = 1$.

GW response of the cavity in the detector frame

Suppose we have a Michelson interferometer with a FP cavity, which is invested by a h_+ polarized GW which propagates perpendicular to the interferometer plane: then, the phase response will look like

$$\Delta\phi_{FP} = \Delta\phi_x - \Delta\phi_y = 2 \times 2k_l \frac{1}{2} L h_0 \cos(\omega_{GW} t) \frac{2\mathcal{F}}{\pi}, \tag{2.3.72}$$

where the first factor of 2 is because the light goes back and forth, while the second factor of 2 is because the effects on the two arms are opposite in sign; the phase difference is computed as laser wavevector times GW length difference (which in the detector frame is due to the mirrors moving around), and finally the factor $2\mathcal{F}/\pi$ is the enhancement due to the FP cavity. The amplitude of this variation is then

$$\left|\Delta\phi_{FP}\right| = \frac{4\mathcal{F}}{\pi} k_l L h_0 . \tag{2.3.73}$$

GW response of the cavity in the TT gauge

We treat the problem in the sidebands picture; the carrier angular frequency is ω_l, and we get some power in the sidebands $\omega_l \pm \omega_{gw}$.

At each round-trip of the light, some of the power is moved from the carrier to the sidebands. We express this as a vector equation: we define a vector containing the amplitudes of the fields, $\vec{B} = (\text{carrier}, \text{upper sideband}, \text{lower sideband})$; if $\vec{B}$ is calculated after a round trip while $\vec{A}$ is calculated before it then we will have

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} X_{00} & 0 & 0 \\ X_{10} & X_{11} & 0 \\ X_{20} & 0 & X_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} . \tag{2.3.74a}$$

The diagonal terms describe the in-cavity propagation of the three modes, while the X_{10} and X_{20} terms describe the GW-caused scattering of the carrier field into the sidebands.

Going through the algebra, we find the following expression:

$$\left|\Delta\phi_x\right| = h_0 k_l L \operatorname{sinc}\left(\frac{\omega_{gw} L}{c}\right) \frac{r_2\left(1 - r_1^2 - p\right)}{r_2(1-p) - r_1} \frac{1}{\left|e^{2i\omega_{gw}\frac{L}{c}} - r_1 r_2\right|} \tag{2.3.75}$$

$$= h_0 k_l L \operatorname{sinc}\left(\frac{\omega_{gw} L}{c}\right) \frac{r_2\left(1 - r_1^2 - p\right)}{r_2(1-p) - r_1} \frac{1}{\sqrt{1 + (r_1 r_2)^2 - 2 r_1 r_2 \cos(2\omega_{GW} L/c)}} . \tag{2.3.76}$$

We can make some approximations: a typical interferometer will work with $p \approx 0$, $r_2 \approx 1$, $r_1 \sim 1$. Applying these, we can write

$$\frac{r_2\left(1 - r_1^2 - p\right)}{r_2(1-p) - r_1} \approx \frac{1 - r_1^2}{1 - r_1} = 1 + r_1 \approx 2\big(1 + \epsilon(r1, r2, p)\big) , \tag{2.3.77}$$

where we defined the small parameter ϵ: what we are saying is that globally the term will be close to 2.

Also, our finesse is large, of the order $\mathcal{F} L/c \sim 1/\omega_{GW}$, so $\omega_{GW} L/c$ is very small: therefore the value of $\operatorname{sinc}(\omega_{GW} L/c)$ is around 1, and

$$\cos\left(\frac{2\omega_{GW} L}{c}\right) \approx 1 - \frac{1}{2}\left(\frac{2\omega_{GW} L}{c}\right)^2 . \tag{2.3.78}$$

With all of these approximations, we can write the phase response of teh cavity as

$$\left|\Delta\phi_x\right| \approx 2 h_0 k_l L \frac{1 + \epsilon(r_1, r_2, p)}{1 - r_1 r_2} \frac{1}{\sqrt{1 + \frac{r_1 r_2}{(1 - r_1 r_2)^2}\left(\frac{2\omega_{GW} L}{c}\right)^2}} \tag{2.3.79}$$

$$\approx 2 h_0 k_l L \frac{\mathcal{F}}{\pi} \frac{1}{\sqrt{1 + \left(4\pi f_{GW} \tau_s\right)^2}} , \tag{2.3.80}$$

where we have used the definition of the finesse: $\mathcal{F} = \pi\sqrt{r_1 r_2}/(1 - r_1 r_2)$ and the storage time $\tau_s = (L/c)(\mathcal{F}/\pi)$.

The square should not include the 1! mistake in the slides.

This is only for an arm: we recover the factor 2 which appears in the detector-frame expression if we consider both.

So, we can see an additional effect which was hidden in the detector frame expression: the cavity acts as a low-pass filter, whose cutoff is called the *cavity pole*,

$$f_p = \frac{1}{4\pi\tau_s} \approx \frac{c}{4\mathcal{F}L}\,, \tag{2.3.81}$$

using which we can write the response of the cavity as

$$\boxed{\left|\Delta\phi_{FP}\right| = h_0 \frac{4\mathcal{F}}{\pi} k_l L \frac{1}{\sqrt{1+\left(\frac{f_{gw}}{f_p}\right)^2}}\,.} \tag{2.3.82}$$

This is the **transfer function** of the cavity, from the GW signal h to the phase difference! We can also express it in terms of the laser wavelength $\lambda_l = 2\pi/k_l$ as

$$\boxed{T_{FP}(f_{GW}) = \frac{8\mathcal{F}L}{\lambda_l} \frac{1}{\sqrt{1+(f_{GW}/f_p)^2}}\,.} \tag{2.3.83}$$

This is of the order of 10^{13} radians per unit strain for typical values of the finesse $\mathcal{F} \sim 500$, arm length $L \sim 3\,\mathrm{km}$ and laser wavelength $\lambda_l \sim 1000\,\mathrm{nm}$, if $f_{GW} < f_p$. The cavity pole is of the order $f_p \sim 50\,\mathrm{Hz}$.

For GW frequencies of the order of 500 Hz the response drops to about 10^{12}.

Double-recycled Fabry-Perot interferometer

This is the setup used by actual interferometers: we have

1. an input laser with a power of 100 W and a wavelength of $\lambda_l \approx 1064\,\mathrm{nm}$;
2. an **input mode cleaner**, which filters out the high-order modes and stabilizes the frequency;
3. a first **power-recycling cavity**, inside which the power is of the order 5 kW, and which increases the effective power seen by the actual FP cavities;
4. the beamsplitter;
5. for each arm, the 3 km-long **FP arm cavity**; the circulating power inside which is of the order 750 kW;
6. the output of the beamsplitter, to which is connected a **signal extraction cavity**, which resonantly enhances the GW sidebands, and a **output mode cleaner**, which rejects any high-order modes which might have been generated.

The FP arm cavities also act as mode filters and frequency stabilizers. The signal extraction cavity can be tuned to change the response of the interferometer.

Locking and alignment

So far, we have always assumed that the cavity is at resonance. Monday 2020-5-11, compiled 2020-07-28

We have an issue if noise is strong enough to move our mirrors out of lock, even if we do not care to observe GW at the frequency of that noise.

Our cavities have a finesse of $\mathcal{F} \sim 500$, and the free spectral range will be of the order of $c/2L \sim 50\,\text{kHz}$. This means that the FWHM of the peaks will be of the order of FWHM $\approx$ FSR$/\mathcal{F} \approx 100\,\text{Hz} = \Delta f$.

If the variation of the frequency is due to a variation of the length of the cavity then the relative variations will be equal (at least to the linear level): $\Delta L/L = \Delta f/f$, which means that, *at the very least*, we will need to control the length of the cavity to the order of

$$\Delta L = L\frac{\Delta f}{f} \approx 10^{-9}\,\text{m}\,. \tag{2.3.84}$$

In reality, we are able to control arm lengths to within $10^{-15}\,\text{m}$ (root-mean-square of the position variation). There are many sources of noise in this respect: the seismic motion of the ground, the moon's pull, the intrinsic laser noise. Fortunately, there is a technique we can use to **keep** the cavity locked onto the wavelength of the laser.

The first thing we might try is to measure the transmitted intensity of the laser light from the FP cavity to check whether the length is the correct one. This has two issues: if the power decreases we cannot tell whether the cavity is slightly too *long* or too *short*; also, we cannot distinguish an intensity fluctuation due to a length imperfection from an intrinsic fluctuation of the laser.

The solution is the **Pound-Drever-Hall** technique: an electro-optical modulator is used in order to insert sidebands at $\omega_l \pm \Omega$ by doing phase modulation. These can be used as oscillators which detect any departure from resonance: they are *not* at resonance in the cavity, so while a length fluctuation of the cavity affect the carrier frequency a lot, it leaves them basically unchanged. So, we see a term oscillating at Ω whose amplitude is linear in $\Delta\phi$, measuring it we can tell the sign of the length variation, and we can distinguish it from a laser power oscillation.

If we know this, we can then actuate the cavity to follow the laser.

Also, we need actuators to control the beam position: the angular control we need in order to prevent noise is of the order of $10^{-9}\,\text{rad}$.

Antenna pattern

The antenna pattern of the interferometer is described by the **detector tensor** D_{ij}, which transforms the perturbation h_{ij} into the observed time-dependent scalar as $h(t) = D_{ij}h_{ij}(t)$. It is given by

$$D_{ij} = \frac{1}{2}\left(\hat{x}_i\hat{x}_j - \hat{y}_i\hat{y}_j\right)\,, \tag{2.3.85}$$

so the output of the detector will look like

$$h(t) = \frac{1}{2}\left(\ddot{h}_{xx} - \ddot{h}_{yy}\right)\,, \tag{2.3.86}$$

as long as the arms are aligned with the $\hat{x}$ and $\hat{y}$ axes. Note that this is similar to the antenna pattern of the resonant bar, but now we have two arms, as opposed to the single "arm" we had in that case.

We must perform a rotation with two angles ϕ and θ to go from the orthogonal frame of the source and the orthogonal frame of the detector: it will look like

$$R = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.3.87}$$

Missing ones in the rotation matrices in the slides.

and applying it (twice, once for each index of the perturbation tensor) we will find

$$h_{xx} = h_+ \left(\cos^2\theta\cos^2\phi - \sin^2\phi\right) + 2h_\times \cos\theta\sin\phi\cos\phi \tag{2.3.88}$$

$$h_{yy} = h_+ \left(\cos^2\theta\cos^2\phi - \cos^2\phi\right) - 2h_\times \cos\theta\sin\phi\cos\phi\,, \tag{2.3.89}$$

so the output timeseries will look like

$$h(t) = F_+(\theta,\phi)h_+ + F_\times(\theta,\phi)h_\times \tag{2.3.90}$$

$$F_+(\theta,\phi) = \frac{1}{2}\left(1+\cos^2\theta\right)\cos 2\phi \tag{2.3.91}$$

$$F_\times(\theta,\phi) = \cos\theta\sin 2\phi\,. \tag{2.3.92}$$

We have no way to distinguish these two components if we only have one detector; we must compare the outputs of different ones.

2.3.7 The interferometer's noise budget

The main sources of noise in the interferometer are

1. **quantum noise**: it is not actually fundamental, we can decrease it by clever design;
2. **seismic noise**: ground vibrations, this can be suppressed with better suspensions;
3. **gravity gradients**: this is also a metric perturbation, so it is the hardest to work around, it is fundamental in a way;
4. **thermal noise**, especially in the mirror coatings (which are exposed to hundreds of kW of laser power!) is currently a big limiting factor.

At high frequencies, the problem is that it is hard to measure small displacements with small integration time. At low frequencies, the problem is that the mirrors move too much.

Quantum noise

The **shot noise** is the error in the count of photons — this is a Poisson process, since the photon arrivals are uncorrelated (the autocorrelation function is a $\delta(t)$, the power spectrum is flat). The power seen at the beamsplitter in an observation time T will look like

$$P_0 = \frac{N_\gamma \overline{h} \omega_L}{T} = \frac{\Delta E}{T} \,. \tag{2.3.93}$$

Its square fluctuation will be given by

$$\Delta P^2 = \frac{\Delta E^2}{T^2} = \frac{\Delta N^2 \hbar^2 \omega_l^2}{T^2} = N \frac{\hbar^2 \omega_l^2}{T^2} = \frac{P_0 \hbar \omega_l}{T} = \frac{1}{2} \int_0^{1/T} S_P(\omega) \, \mathrm{d}\omega = \frac{1}{2} \frac{S_P(\omega)}{T} \,, \tag{2.3.94}$$

so we get $S_P(\omega) = 2 P_0 \hbar \omega_l$.

Wrong factor of 2 in the slides!

Note that this is a power-PSD: it measures the average square *power*, so it has the dimensions of a power squared over a frequency ($\mathrm{W}^2/\mathrm{Hz} = \mathrm{WJ}$).

The output of the detector, $\Delta\phi$, is proportional to P_0.. It can be shown that the contribution to the noise PSD of the phase due to shot noise will be

$$\sqrt{S_{\Delta\phi, \text{ shot}}(\omega)} = \frac{C}{P_0} \sqrt{S_P(\omega)} = C \sqrt{\frac{2\hbar\omega_l}{P_0}} \,, \tag{2.3.95}$$

where C is a dimensionless constant of order 1, accounting for the working point and the photodetector efficiency. We can refer this to the input by making use of the T_{FP} transfer function: we will then have

$$\sqrt{S_{h, \text{ shot}}} = \frac{\sqrt{S_{\Delta\phi, \text{ shot}}}}{T_{FP}} = \frac{c}{8 \mathcal{F} L} \sqrt{\frac{4\pi\hbar c \lambda_l}{P_0}} \sqrt{1 + \left(\frac{f_{GW}}{f_p} \right)^2} \,. \tag{2.3.96}$$

This then diminishes as we increase the effective length of the cavity. So, one might say, why would we build a cavity which is several km long, instead of a tabletop experiment with a very high finesse? We shall answer shortly.

We also have **radiation pressure** noise, which scales differently: it is due to the fact that each photon impacting on the mirror gives it a bump of momentum $2\omega_l \hbar / c$. With a reasoning not unlike the previous one we find

$$\sqrt{S_{F, \text{ rp}}} = 2 \sqrt{\frac{2 P_0 \omega_l}{c^2}} \,, \tag{2.3.97}$$

so the spectral density of the displacement of the mirror will be

$$\sqrt{S_{x, \text{ rp}}} = \frac{2}{M\omega^2} \sqrt{\frac{2 P_0 \hbar \omega_l}{c^2}} \,, \tag{2.3.98}$$

which means that the amplitude spectral density of the input will be

$$\sqrt{S_{h,\,\mathrm{rp}}} = \frac{16\sqrt{2}\mathcal{F}}{ML(2\pi f_{GW})^2}\sqrt{\frac{P_0\hbar}{2\pi c^2\lambda_l}}\frac{1}{\sqrt{1+(f_{GW}/f_p)^2}}\,. \tag{2.3.99}$$

The interesting thing to note here is that this noise scales *directly* with the finesse. This answers the question: if we try to raise the finesse too much, the power inside the laser increases by a lot, and this creates a huge amount of radiation pressure noise on the mirrors. The fact that we found a factor $\mathcal{F}$ is due to the fact that a photon makes $\mathcal{F}/2\pi$ bounces inside the cavity, creating noise for each.

So, we must reach a compromise for the finesse (or for the circulating power P_0).

The shot noise is proportional to $P_0^{-1/2}$, the radiation pressure noise is proportional to $P_0^{1/2}$. Their sum gives the total quantum noise, whose expression is

$$\sqrt{S_{h,\mathrm{qn}}}(f) = \frac{1}{L\pi f_0}\sqrt{\frac{\hbar}{M}}\sqrt{\left(1+\frac{f^2}{f_p^2}\right)+\frac{f_0^4}{f^4}\frac{1}{1+f^2/f_p^2}}\,, \tag{2.3.100}$$

where

$$f_0 = \frac{4\mathcal{F}}{\pi}\sqrt{\frac{P_0}{\pi\lambda_l cM}}\,. \tag{2.3.101}$$

The shot noise is flat in frequency (for $f < f_p$), while the radiation pressure noise decreases when frequency increases. By changing P_0 we can raise one and lower the other; for each frequency we have a minimum for the uantum noise, which is called the **Standard Quantum Limit**. This is given by minimizing the noise. The optimal value for the frequency f_0 comes out to be the one satisfying

$$1+\frac{f^2}{f_p^2} = \frac{f_0^2}{f^2}\,, \tag{2.3.102}$$

and the result for the SQL is

$$\sqrt{S_{h,\,\mathrm{SQL}}}(f) = \frac{1}{2\pi f L}\sqrt{\frac{8\hbar}{M}}\,. \tag{2.3.103}$$

Note that this limit can be reached for a specific, fixed frequency f: we cannot achieve it for all the spectrum.

We can go below this limit using Quantum Vacuum Squeezing, in which we gain precision in the measurement of one variable (photon number) at the expense of another (phase).

Thermal Noise

We have contribution from all dissipation sources, be they mechanical or not. There is thermo-elastic noise: as a material bends, the side which compresses heats up a little, while the side which expands cools a little.

The ultimate limit is the internal dissipation: by the fluctuation-dissipation theorem,

$$S_{F,\,\mathrm{th}} = 4k_B T\,\mathrm{Re}[Z(\omega)]\,, \tag{2.3.104}$$

where $Z(\omega)$ is the characteristic impedance of the system.

For the **mirror suspensions** we can have loss comparable to the seismic noise. If we use high-loss materials we get lots of noise, so we try to use low-loss materials like fused silica, which can reach quality factors like $Q \sim 10^9$. Lowering the temperature is also a thing to do: detectors are going cryogenic.

We also have **mirror coating Browian motion**: unfortunately, the multilayer coating of the dielectric mirrors is relatively high-loss.

The expression for this kind of noise is

$$S_x(f,T) = \frac{2k_B T}{\pi^2 f}\frac{d}{w^2 Y}\phi\left(\frac{Y'}{Y} + \frac{Y}{Y'}\right), \tag{2.3.105}$$

where w is the beam radius, ϕ is the coating mirror loss and d is the coating thickness.

what are the other variables? Eh.

People are investigating techniques which could lower ϕ: new materials, heat treatment which could aid with the relaxation of the material, new layer structures, monolithic crystalline coatings.

We could also try to have larger mirrors and/or materials with higher optical contrast.

Seismic noise

The ground vibrations have a very large amplitude, of the order of 10^{-6} m, which is about ten orders of magnitude larger than the precision we need. Fortunately, most of it has a very low frequency compared to GW: human activity gives vibrations at around 1 Hz to 10 Hz, while the "oceanic peak" is at around 0.1 Hz.

Nevertheless, this pushes the mirrors out of alignment, so we need active stabilization.

The seismic noise which is in our band is more concerning, although its amplitude is lower than the peak of the seismic noise. This might **mimic a signal**: we must suppress it.

We use a passive approach: a cascade of pendula, for each of them we get a transfer function

$$H_{x_n \to x_{n+1}}(f) \sim \frac{1}{1 - f^2/f_0^2} \qquad \text{where} \qquad f_0 \sim \sqrt{\frac{g}{l}} \sim 1\,\mathrm{Hz}\,, \tag{2.3.106}$$

so each pendulum acts as a high-pass filter: if $f \gg f_0$ (which is the case for the high-frequency seismic noise we want to eliminate) we get an attenuation $\sim (f/f_0)^{-2\#\,\text{pendulums}}$. We use about 5 pendula, so very roughly this will be $100^{2\times 5}$.

Newtonian noise

This noise is about stochastic variations of the local gravitational field. Its main causes are seismic movements and variations in atmospheric pressure. We cannot shield from it: it is a metric perturbation like the GW, we cannot distinguish them *a priori*!

There are two main ways to deal with this:

1. the active approach is to have many sensors detecting ground displacements and atmospheric pressure variations, model the expected disturbances and subtract this from the GW signal;
2. the passive approach is to move the detector underground, where we do not need to worry about surface waves and atmospheric effects are reduced.

Quantum Vacuum Squeezing

A good reference for this is Miao's PhD thesis [Mia10, sec. 2.8].

We quantize the electromagnetic field:

$$\hat{E} = u(x,y,z) \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\frac{2\pi\hbar\omega}{\mathcal{A}c}} \left[\hat{a}_\omega e^{ikz-i\omega t} + \hat{a}^\dagger_\omega e^{-ikz+i\omega t} \right], \tag{2.3.107}$$

where $\mathcal{A}$ is the beam area. We will need creation and annihilation operators corresponding to the sidebands: $\hat{a}_\pm = \hat{a}_{\omega_0 \pm \Omega}$, and we define

$$\hat{a}_1 = \frac{\hat{a}_+ + \hat{a}^\dagger_-}{\sqrt{2}} \qquad \text{and} \qquad \hat{a}_2 = \frac{\hat{a}_+ - \hat{a}^\dagger_-}{i\sqrt{2}}. \tag{2.3.108}$$

The state of the laser light is a **coherent state**; it is not an eigenstate of photon number, and it can be defined as

$$|\alpha\rangle = \exp\left(\int \frac{d\Omega}{2\pi} \left(\alpha_\Omega \hat{a}^\dagger_\Omega - \alpha^*_\Omega \hat{a}_\Omega \right) \right) |0\rangle . \tag{2.3.109}$$

The $\hat{a}_{1,2}$ are the *quadratures*. The product of their standard deviations $S_{\hat{a}_{1,2}}(\Omega)$ must be larger than a certain constant by Heisenberg, but we need not have a circular distribution: we can squeeze it in one direction and stretch it in the orthogonal one.

The result for the spectral density at the input is

$$S^h(\Omega) = \left[\frac{S_{\hat{a}_2(\Omega)}}{k} + k S_{\hat{a}_1}(\Omega) \right] \frac{h^2_{SQL}}{2}, \tag{2.3.110}$$

which still does not allow us to beat the standard quantum limit: the bound is the same, although we can work on it without increasing the power which could be useful.

The trick comes from the fact that we can do **frequency-dependent** squeezing! The relation of $\Delta\hat{a}_{1,2}$ at a specific frequency is independent of that at another!

So, we can define

$$\frac{\Delta\hat{a}_1}{\Delta\hat{a}_2} = k(\omega), \tag{2.3.111}$$

which allows us to get, if we do the rotation optimally:

$$S^h(\Omega) = e^{-2r} \left[\frac{1}{k} + k \right] \frac{h^2_{SQL}}{2}. \tag{2.3.112}$$

The way to **generate squeezed light** is through the use of nonlinear crystals (whose polarization has non-negligible terms depending on higher-than-linear powers of the field).

If we input into a nonlinear crystal a combination of a seed field at a frequency ω and a pump field a 2ω, we can control the squeezing through their relative phase.

We can think of squeezing as generating correlations between the sidebands' oscillations.

We can describe a fixed-frequency EM signal with two operators in QM, for example amplitude and phase.

A squeezed vacuum is a vacuum state whose fluctuations are asymmetric. At low frequency, we want to squeeze amplitude, so that the radiation pressure is more predictable and we have less test mass motion; at high frequency we want to squeeze phase, so that the number of photons is more predictable.

Hold on though: different **GW** frequencies, not different laser frequencies! How can we change what we do for different GW frequencies before seeing the signal?

2.4 Elements of data analysis

Friday 2020-5-15, compiled 2020-07-28

The issue is that our data are noise-dominated: extracting the signal is quite hard.

The signal is something like 3 orders of magnitude below the noise. Also, we want to say something about the source, extracting parameters like the masses of the BHs, the distance, the spins.

We classify signals into: transient versus persistent (how long does the signal last?), and modeled versus unmodeled (do we know of a speficic waveform?).

1. Transient modeled signals are usually coalescing binaries;
2. persistent modeled signals can be binaries far from coalescence or rotating neutron stars;
3. transient unmodeled signals can be supernovae or some other sources;
4. persistent unmodeled signals are some form of stochastic background.

2.4.1 Matched filtering

We will discuss matched filtering, which applies to **transient modeled** signals. The assumption is that the signal is in the form

$$s(t) = \underbrace{h(t)}_{\text{known}} + \underbrace{n(t)}_{\text{noise}}, \tag{2.4.1}$$

where by saying that h is known we mean that we have an idea of the waveform it should have, we do not know its precise parameters nor where it is in the signal.

If we define the time averaging

$$\langle x \rangle = \frac{1}{T} \int_0^T x(t)\, \mathrm{d}t \,, \tag{2.4.2}$$

then we can calculate $\langle sh \rangle$, which is equal to $\left\langle h^2 \right\rangle + \langle nh \rangle$. We know that $\langle nh \rangle \to 0$, since the noise is not correlated to the signal; specifically, $\langle nh \rangle \sim T^{-1/2}$.

On the other hand, $\langle hh \rangle$ will approach a constant nonzero value, so this integral will have a positive value if the signal is there. So, asymptotically we will expect $\langle sh \rangle \to \left\langle h^2 \right\rangle$.

Suppose we know what $h(t)$ is, and we want to build a **linear filter** $K(t)$ which returns a low value if the signal seems to not be in the data, and a high value if the signal seems not to be in the data. The response of this filter will look like

$$s(t) \to \hat{s} = \int_{-\infty}^{\infty} \mathrm{d}t\, s(t) K(t)\,. \tag{2.4.3}$$

We assume that $s(t) = \alpha h(t) + n(t)$, leaving the amplitude of the signal $h(t)$ as a variable. We want to define the signal-to-noise ratio S/N, where by S we mean the expectation value of $\hat{s}$ when $\alpha \neq 0$ and by N we mean the expectation value of $\hat{s}$ when $\alpha = 0$.

The value of $K(t)$ is fixed (we built the filter after all) so we can factor it out of ensemble averages. So, we find

$$\begin{aligned}
S = \left\langle \hat{s}(t) \right\rangle &= \int \mathrm{d}t \left\langle s(t) \right\rangle K(t) &\quad (2.4.4)\\
&= \alpha \int \mathrm{d}t \left\langle h(t) \right\rangle K(t) + \int \mathrm{d}t\, \cancel{\left\langle n(t) \right\rangle} K(t) &\quad (2.4.5)\\
&= \alpha \int \mathrm{d}t \left\langle h(t) \right\rangle \left(\int \mathrm{d}f\, e^{2\pi i f t} K^*(f) \right) &\quad (2.4.6)\\
&= \alpha \int \mathrm{d}f\, K^*(f) \left(\int \mathrm{d}t\, e^{2\pi i f t} \left\langle h(t) \right\rangle \right) &\quad (2.4.7)\\
&= \alpha \int \mathrm{d}f\, K^*(f) h(f)\,, &\quad (2.4.8)
\end{aligned}$$

where we have used the fact that, since $K(t)$ is real ($K(t) = K^*(t)$)

$$K(t) = \int \mathrm{d}f\, e^{-2\pi i f t} K(f) = \int \mathrm{d}f\, e^{2\pi i f t} K^*(f)\,, \tag{2.4.9}$$

and we insert the latter expression so that we can recover the expression for the *antitrasform* of $h(f)$. For the noise we need to compute the square modulus:

$$\begin{aligned}
N^2 &= \left\langle \hat{s}^2(t) \right\rangle - \cancel{\langle \hat{s} \rangle^2} \Big|_{\alpha=0} &\quad (2.4.10)\\
&= \int \mathrm{d}t_1\, \mathrm{d}t_2\, K(t_1) K(t_2) \left\langle n(t_1) n(t_2) \right\rangle &\quad (2.4.11)\\
&= \int \mathrm{d}t_1\, \mathrm{d}t_2\, K(t_1) K(t_2) \int \mathrm{d}f_1\, \mathrm{d}f_2\, e^{2\pi i t_1 f_1} e^{2\pi i t_2 f_2} \left\langle n(f_1) n(f_2) \right\rangle &\quad (2.4.12)\\
&= \int \mathrm{d}t_1\, \mathrm{d}t_2\, K(t_1) K(t_2) \int \mathrm{d}f_1\, \mathrm{d}f_2\, e^{-2\pi i t_1 f_1} e^{2\pi i t_2 f_2} \underbrace{\left\langle n^*(f_1) n(f_2) \right\rangle}_{=\frac{1}{2}\delta(f_1 - f_2) S_N(f_1)} &\quad (2.4.13)\\
&= \frac{1}{2} \int \mathrm{d}t_1\, \mathrm{d}t_2\, K(t_1) K(t_2) \int \mathrm{d}f_1\, S_n(f_1) \underbrace{e^{2\pi i f_1 (t_2 - t_1)}}_{=\delta(t_2 - t_1)} &\quad (2.4.14)
\end{aligned}$$

$$= \frac{1}{2} \int \mathrm{d}t_1\, K(t_1) K(t_1) \int \mathrm{d}f_1\, S_n(f_1) \tag{2.4.15}$$

$$= \frac{1}{2} \int \mathrm{d}f_3\, |K(f_3)|^2 \int \mathrm{d}f_1\, S_n(f_1) \tag{2.4.16}$$

$$= \frac{1}{2} \int \mathrm{d}f\, |K(f)|^2 S_n(f)\,. \tag{2.4.17}$$

A step is not clear here either...

Then, we can write

$$\frac{S}{N} = \alpha \frac{\int \mathrm{d}f\, K^*(f) h(f)}{\sqrt{\int \mathrm{d}f\, |K(f)|^2 S_n(f)/2}}\,, \tag{2.4.18}$$

and we wish to optimize K in order to maximize this.

We can express it in a clearer way if we define a scalar product for real functions of f:

$$(A|B) = \operatorname{Re} \int \mathrm{d}f\, \frac{A^*(f) B(f)}{S_n(f)/2} = 4 \operatorname{Re} \int_0^\infty \mathrm{d}f\, \frac{A^*(f) B(f)}{S_n(f)}\,, \tag{2.4.19}$$

The integrand is even since we are taking the real part and $A(f) = A^*(-f)$.

so we can write the SNR as

$$\frac{S}{N} = \alpha \frac{(u|h)}{(u|u)}\,, \tag{2.4.20}$$

where we defined the vector $u = S_n(f) K(f)/2$.

So, we want $u(f)$ to be parallel (i. e. proportional) to h in order to maximize the product: this means that $SK/2 = \beta h$ for some β, or

$$K(f) = 2\beta \frac{h(f)}{S_n(f)}\,. \tag{2.4.21}$$

This choice is called the **Wiener filter**, and with it we find the SNR

$$\frac{S}{N} = \alpha \frac{2\beta \int \mathrm{d}f\, h^*(f) h(f)/S_n(f)}{\sqrt{2\beta^2 \int \mathrm{d}f\, |h(f)|^2 / S_n(f)}} = \sqrt{2} \alpha \sqrt{\int_{-\infty}^{\infty} \mathrm{d}f\, \frac{|h(f)|^2}{S_n(f)}}\,. \tag{2.4.22}$$

This means that our best filter is *not* simply $h(f)$: we must weigh the signal by the detector noise is, the noisier regions are weighted less.

We do not actually know $h(t)$, since we do not know the time at which the signal will arriver: we can slide our filter over our signal, varying the time of arrival. We do this by defining

$$\hat{s}(\widetilde{t}) = \int \mathrm{d}t\, s(t) K(t - \widetilde{t})\,, \tag{2.4.23}$$

which will exhibit a peak for $\widetilde{t}$ equal to the arrival time of the signal.

In practice, we do not know the parameters of $h(t)$ either. What we do is generate hundreds of thousands of filters spanning a reasonable parameter range and move them through the data.

We must decide on a coverage for our parameter space in order to run our filters, while still having a manageable number of iterations to do.

Some problems with this: the detector noise is non-gaussian, its tails of extreme events are quite high. Also, the signal is often very weak compared to detector noise.

How to reject false alarms due to local noise? Coincidences between a detector network! We allow time differences of L/c at most.

How to be very confident that two local sources of noise did not happen to coincide? We delay the output intentionally by more than L/c, in order to get an estimate of what we would see without GW signals (since we effectively eliminate any GW signals there could be in the data). This gives us the expected "no-signal" output. This works as long as the events are rare.

We can plot the probability density function we get for the detection statistic. If we leave the true signal in, this looks quite dirty since we have coincidences of noise with the GW; if we remove the GW, the PDF becomes a nice powerlaw.

The analysis must be done blind: we look at the delayed data stream first, and then we set the threshold.

Do people use KDE in the estimation of PDFs?

2.4.2 Probability

We use the Kolmogorov axioms: consider events belonging to the powerset of S, then we say that

1. probabilities are positive;
2. the probabilities of disjoint events are additive;
3. the probability of S is 1.

An implementation is the frequentist approach: we assign probabilities according to the frequency of occurrence of an event after many repetitions.

So, in this approach Bayes's theorem does not really make sense: what is the probability of a die being true, given that we have gotten 1006 times the number 1 over 6000 tries? The die either is or isn't true.

The alternate, Bayesian, approach is to define probabilities, instead, as subjective beliefs.

Then, we can update our subjective belief about a hypothesis like

$$\mathbb{P}(\text{hyp}|\text{data}) = \frac{\mathbb{P}(\text{data}|\text{hyp})\mathbb{P}(\text{hpy})}{\mathbb{P}(\text{data})} . \tag{2.4.24}$$

Parameter estimation

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Now, suppose we have found a candidate signal. How do we estimate the most likely set of parameters? Let us denote as $\vec{p}$ a vector containing all the parameters of the source.

We do it by

$$\mathbb{P}(\vec{p}|\text{data}) = \mathbb{P}(\text{data}|\vec{p})\mathbb{P}(\vec{p})\,, \tag{2.4.25}$$

where we usually assume a flat prior for the parameters.

The signal we see is in the form $s(t) = h(t, \widetilde{p}) + n_0(t)$, the GW signal depending on a true set of parameters $\widetilde{p}$ plus the noise.

If the noise is stationary and Gaussian, then the probability of getting this specific realization of the noise, n_0, is given by

$$\mathbb{P}(n_0) = N \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \frac{\left|n_0(f)\right|^2}{S_n(f)^2}\,\mathrm{d}f\right) = N \exp\left(-\frac{(n_0|n_0)}{2}\right), \tag{2.4.26}$$

by the scalar product we defined earlier, where N is a normalization factor. This comes from the relation

$$\sigma^2_{n(f)} = \left\langle n^*(f)n(f')\right\rangle = \frac{1}{2}\delta(f-f')S_n(f)\,. \tag{2.4.27}$$

We can imagine this as calculating a sort of likelihood of the noise, multiplying the probability densities of each of its frequency components.

The noise can also be expressed as

$$n = s - h(\widetilde{p})\,, \tag{2.4.28}$$

with which we can evaluate $\Lambda(s|\widetilde{p}) = \mathbb{P}(\text{data}|\widetilde{p})$, the likelihood of the data given the parameter vector $\widetilde{p}$. It is given by

$$\Lambda(s|\widetilde{p}) = N \exp\left(-\frac{(s-\widetilde{h}|s-\widetilde{h})}{2}\right) = N \exp\left(-\frac{(s|s)+(h|h)+2(s|h)}{2}\right), \tag{2.4.29}$$

which we can calculate explicitly; the term given by $(s|s)$ can be included in the normalization N, since it is a constant with respect to the signal given.

Let us quickly interpret this heuristically: templates h with a higher likelihood are those for whom $(s|h)$ and $(h|h)$ are lower; the scalar product is given by a product divided by $S_n(f)$ in the frequency domain. Saying that $(h|h)$ should be low means that templates which have high values in high-noise regions in the frequency domain are suppressed. The term $(s|h)$ describes the fact that we are more interested in whether the signal "lines up" with the template in low-noise regions than whether it does in high-noise ones.

The **posterior** is then given in terms of the likelihood Λ as

$$\mathbb{P}(\widetilde{p}|\text{data}) = \underbrace{N \exp\left(-\frac{(h|h)+(s|h)}{2}\right)}_{\Lambda(s|\widetilde{p})} \mathbb{P}(\widetilde{p})\,. \tag{2.4.30}$$

If we apply this procedure we get a likelihood and a posterior distribution. How do we extract our best estimate for the actual values of the parameters? There are several approaches for this choice of **estimator**, which all have in common the few things we require of a good estimator:

1. **consistency**, that is, as the amount of data increases the estimate should converge to the true value;
2. **efficiency**, that is, the variance of the estimate should be as low as possible;
3. **robustness**, that is, if we make slight changes to either the data or our priors we should not see abrupt variations in the estimate.

With these things in mind, a few useful estimators are:

1. using **maximum likelihood** means that we maximize $\Lambda(s|\hat{p})$. This ignores the prior, and is equivalent to maximizing S/N for matched filtering.
2. using **maximum posterior** means we maximize $\mathbb{P}(\hat{p}|s)$. This is very useful if our priors are robust, but it has the drawback that if we marginalize over a parameter the estimate of another may change.
3. usign **Bayes estimator** means we compute

$$\hat{p}_i^B = \int \mathrm{d}\vec{p}\, p_i \mathbb{P}(\vec{p}|s)\,, \tag{2.4.31}$$

   which corresponds to minimizing the error

$$\sum_{ij} \int \mathrm{d}\vec{p}\, \left(p_i - \hat{p}_i^B\right)\left(p_j - \hat{p}_j^B\right)\mathbb{P}(\vec{p}|s)\,. \tag{2.4.32}$$

   If we use this, the estimated parameters are ensured to be independent of each other.

Coherent wave bursts

This technique applies to unmodeled transient signals, like GW emission from asymmetric supernovae. It can also be applied to signals for which we only have partial models.

We Fourier-transform the signal multiplied by a narrow windowing function of width w_0, giving us a 2D power spectrum, whose resolution is limited by the uncertainty principle between frequency $\sim 1/w_0$ and time ($\sim w_0$).

We need not use the same window for all frequencies, we can make it change according to the frequency range we are looking at.

We then compute the **normalized time-frequency map** for the detector k:

$$w_k(i) = \frac{x_k(i)}{\sqrt{S_k(i)}}\,, \tag{2.4.33}$$

where i denotes the pixel in this 2D time-frequency space, while S_k is the noise PSD of detector k. The value of x_k indicates the amplitude at the pixel.

We can then compute the energies of the pixel across all detectors:

$$E(i) = \max_{\text{time-of-flight delays}} \sum_k w_k^2(i) \,, \tag{2.4.34}$$

amongst which we can look for anomalies.

We also might want to filter what we find by requiring a "sweep" (frequency increasing over time).

Unclear last point in slide.

In order to do all of this analysis we need to know what the PSD looks like: we must give an estimate of the PSD which is "mesoscopic": we average over something like ten minutes, so that we average over any GW signal we could see, but do not average on so long a timescale that we start to have normalization differences — we must recalibrate the detectors often, because of wind, day-night and so on.

Continuous signals

We can also search for monochromatic signals: the sensitivity of the detector around a frequency f_0 is approximately

$$\frac{S_n^{1/2}(f_0)}{\sqrt{T}} \,, \tag{2.4.35}$$

so the SNR roughly increases like $\sqrt{T}$.

Generally, even stable sources are not perfectly monochromatic: the source can have slow changes (i. e. due to spindown) or we can have all kinds of time shifts, for an enumeration of the possibilities one can look at the discussion of the Hulse-Taylor radio pulsar.

This analysis must be done for a long time, and for all the data stream.

Stochastic background

As for the stochastic background: we could extract it from the PSD of the signal if we knew the detector PSD precisely, since as it is unresolved we can only look at it through its PSD $S_h(f)$. For ground-based detectors, this is unpractical, however LISA might achieve it.

We could distinguish this kind of noise from local detector noise by correlating signals from detectors at a distance D such that

$$\lambda_{GW} \geq D \geq L_{\text{noise}} \,, \tag{2.4.36}$$

where L_{noise} is the correlation length of the local disturbances. In calculating this, we must account for the different antenna patterns of the detectors.

If these inequalities are respected, then all the detectors see "the same part" of the SGWB, but they are far enough apart that *everything else* they see is uncorrelated.

2.5 LISA

It's approved by ESA to launch in 2034.

2.5.1 Mission characteristics

It will be surely sensitive in the 0.1 to 100 mHz region, possibly extending down to 20 µHz and up to 1 Hz.

The length of the arms will be of the order of 2.5×10^9 m, or about 8 light-seconds. This is comparable with the wavelengths of the highest-frequency GWs which will be able to be seen by LISA.

The detector-frame picture, therefore, will definitely not hold here.

This is expected to be an extremely signal-rich region of the spectrum, since we would be able to see many stable and long-lived sources.

We might be able to identify long-lived sources by the modulation in the signal due to the variation of the orientation and velocity of the LISA satellites in their orbit.

The shape of LISA will be triangular, with three different interferometers between three satellites.

Technology

The masses making up LISA will truly be free-falling.

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2.6 Pulsar Timing Array

We seek a method to measure SMBH mergers and other extremely long wavelength GW. What we do is measure the distance from millisecond pulsars.

The light travel time depends on how the space is curved by GW.

The key to the data analysis is that the GW signal is one of the only correlated signals across the sky. The contribution is

$$r_{\alpha,GW}(t) = \int_0^t \mathrm{d}t' \, \frac{\delta \nu_\alpha}{\nu_\alpha}(t') \,. \tag{2.6.1}$$

Here α is an index representing which pulsar we are interested in. We can divide the signal r into a part at Earth, $r_\alpha^e(t)$, and a part at the pulsar, $r_\alpha^p(t)$.

Is there an averaging effect in PTA? If the GW is propagating in the same direction as the pulsar signal the signal is always in the same gravitational field…not clear really

2.7 Atom interferometry

2.8 Overview of GW detections and science

Discussion on current GW detections and prospects for the future.

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