Path integrals and non-Gaussianity in cosmology

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2020-10-23

Contents

Abstract

To write

1 Introduction and motivation

The Large Scale Structure of the cosmos formed as a result of gravitational clustering, starting from "seed" inhomogeneities in the density field in the early universe. The generation of these perturbations is commonly thought to have been the result of quantum fluctuations in the early universe, which have reached galactic scales in the inflationary stage of the expansion.

These perturbations may be considered as pertaining to the gravitational field Φ or to the density field ρ: the former approach has shown itself to be more fruitful in recent years.

A model which so far has not been contradicted by observation is to consider perturbations as being a realization of a Gaussian random field. Even if the field were not truly Gaussian, we would expect the Gaussian approximation to be a rather good one as it happens generally for processes which arise from the combined effects of many independent random variables [CM18, pag. 2].

However, the study of non-Gaussianities in the primordial universe is crucial. Inflationary models commonly predict a nearly-Gaussian perturbation field, and the deviations from Gaussianity can aid us in the effort to determine which one best fits the data.

It is common to model an almost-Gaussian field with something similar to a Taylor expansion [MVJ00, eq. 1], [CM18, eq. 1] in the Gaussian field φ_L:

$$\Phi = \varphi_L + f_{NL}\left(\varphi_L^2 + \left\langle \varphi_L^2 \right\rangle\right) + g_{NL}\left(\varphi_L^3 - \left\langle \varphi_L^2 \right\rangle \varphi_L\right) + \mathcal{O}\left(\varphi_L^4\right). \tag{1.1}$$

From the most recent Planck mission data, which measured the non-Gaussianity by looking at the Cosmic Microwave Background temperature and polarization anisotropies' correlations, the parameters f_{NL} and g_{NL} are both compatible with zero [Col+19].[1]

Non-Gaussianities may have a large impact on structure formation, since the over-densities which cluster into cosmic structures are heavily dependent on the *tails* of the density perturbation distribution: for example, if φ is Gaussian then its tails decay like $\exp\left(-\varphi^2\right)$, while φ^2 is chi-square distributed (with one degree of freedom), and its tails decay like $\exp\left(-\varphi\right)$.

Therefore, even a small amount of non-Gaussianity may "boost" structure formation significantly.

Modelling the initial conditions for structure formation for a general non-Gaussian field is tricky, and the formalism which is best suited for the task is that of functional integration (the Path Integral). It allows us to generate *constrained* realizations of the field, which are especially useful if we wish to study the tails of the distribution, which are naturally connected to rare occurrences.

In section 2 we will describe this formalism in detail from first principles, discussing its connection to the path integral used in Quantum Field Theory, the way to compute correlation functions, the use of spatial filters, and the saddle-point approximation.

In section 3 we will discuss the applications of these techniques to structure formation.

2 Path integral

Following [Zai83].

We start from the space of square-integrable functions $q(x)$, endowed with a product and an orthonormal basis ϕ_n. We consider (multi-)linear *functionals*, which are maps from the space of square-integrable functions (or from tuples of them) to $\mathbb{R}$ or $\mathbb{C}$. These can be represented as functions of infinitely many variables, countably so if we use the basis ϕ_n, uncountably so if we use the continuous basis x.

A functional $F[q]$ can be represented as a power series

$$F[q] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int \mathrm{d}x_i \, q(x_i) f(x_1, \ldots, x_n) \,. \tag{2.1}$$

Examples of this are the exponential series corresponding to the function $f(x)$, mapping $q(x)$ to $e^{(f,q)}$ where the brackets denote the scalar product in the space, and the Gaussian series corresponding to the kernel $K(x,y)$, mapping $q(x)$ to $e^{(q,K,q)}$, where

$$(q,K,q) = \int \mathrm{d}x \, \mathrm{d}y \, q(x) q(y) K(x,y) \,. \tag{2.2}$$

Functional derivatives describes how the output of the functional changes as the argument goes from $q(x)$ to $q(x) + \eta(x)$, where $\eta(x)$ is small. This will be a linear functional of

[1] The parameter f_{NL} is divided into different contributions, corresponding to different geometric configurations in Fourier space. We shall not go into details here, but they are independently compatible with zero.

η to first order, so we define the functional derivative with the expression

$$F[q+\eta] - F[q]\Big|_{\text{linear order}} = \int \eta(y) \frac{\delta F}{\delta q(y)} \, \mathrm{d}y \ . \tag{2.3}$$

The analogy to finite-dimensional spaces is as follows: the functional derivative $\delta F/\delta q(y)$ corresponds to the *gradient* $\nabla^i F$, while the integral in the previous expression corresponds to the *directional derivative* $(\nabla^i F)\eta^j g_{ij}$. The metric is present since the gradient is conventionally defined with a vector-like upper index; in our infinite-dimensional space the scalar product is given by the integral.

Practically speaking, the most convenient way to calculate a functional derivative is by taking $\eta(x)$ to be such that it only differs from zero in a small region near y, and let us define

$$\delta\omega = \int \eta(x) \, \mathrm{d}x \ . \tag{2.4}$$

Then, we define

$$\frac{\delta F}{\delta q(y)} = \lim_{\delta\omega \to 0} \frac{F[q+\eta] - F[q]}{\delta\omega} \, . \tag{2.5}$$

In order for the limit to be computed easily, it is convenient for $\eta(x)$ to be in the form $\delta\omega \times$ fixed function, so that we are only changing the normalization as we shrink $\delta\omega$. A common choice is then

$$\eta(x) = \delta\omega \delta(x-y) \, . \tag{2.6}$$

If we apply this procedure to the identity functional $q \to q$, we find

$$\frac{\delta q(x)}{\delta q(y)} = \lim_{\delta\omega \to 0} \frac{q(x) + \delta\omega\delta(x-y) - q(x)}{\delta\omega} = \delta(x-y) \, . \tag{2.7}$$

The variable q is one-dimensional, if instead we wanted to consider a multi-dimensional coordinate system q_α by the same reasoning we would find

$$\frac{\delta q_\alpha(x)}{\delta q_\beta(y)} = \delta_{\alpha\beta}\delta(x-y) \, . \tag{2.8}$$

An example: the functional derivative of a functional F_n defined by

$$F_n[q] = \int f(x_1, \ldots, x_n) q(x_1) \ldots q(x_n) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \ , \tag{2.9}$$

where f is a symmetric function of its arguments, is given by

$$\frac{\delta F_n}{\delta q(y)} = n \int f(x_1, \ldots, x_{n-1}, y) q(x_1) \ldots q(x_{n-1}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \ , \tag{2.10}$$

a function of y.

A **linear transformation** is in the form

$$q(x) = \int K(x,y) q'(y)\, \mathrm{d}y \ . \tag{2.11}$$

If this transformation has an inverse, which is characterized by the kernel K^{-1}, then we must have the orthonormality relation

$$\int K(x,y) K^{-1}(y,z)\, \mathrm{d}y = \int K^{-1}(x,y) K(y,z)\, \mathrm{d}y = \delta(x-z)\,. \tag{2.12}$$

We can do **Legendre transforms**: if we have a functional F we can differentiate with respect to the coordinate q to find

$$\frac{\delta F[q]}{\delta q(x)} = p(x)\,, \tag{2.13}$$

in analogy to the momentum in Lagrangian mechanics. Then, we can map $F[q]$ to a new functional $G[p]$ which will only depend on the momentum:

$$G[p] = F[p] - \int q(x) p(x)\, \mathrm{d}x \ . \tag{2.14}$$

We can also define functional integration, by

$$\int F[q] \big[\mathrm{d}q\big] = \int \hat{F}(\{q_i\}) \prod_i \mathrm{d}q_i \ . \tag{2.15}$$

On the right-hand side we are using the expression of the functional as a function of infinitely many variables which we discussed above; we are then integrating over each of the coordinates in this infinite dimensional function space. The infinite-dimensional measure is also often denoted as $\mathcal{D}q$.

This integral will not always exist, however in the cases in which it does we can change variables. Let us consider a linear change of variable, whose kernel is $K(x,y)$, such that (compactly written) $q = Kq'$.

Then, we want to compute the integral

$$\int F[Kq'] \big[\mathrm{d}Kq'\big] \tag{2.16}$$

as an integral in $\big[\mathrm{d}q\big]$: in order to do so, we need to relate the two functional measures. We start by expressing both q and q' in terms of an orthonormal basis ϕ_i: inserting this into the linear transformation law we get

$$q(x) = \int K(x,y) q'(y)\, \mathrm{d}y \tag{2.17}$$

$$\sum_i q_i \phi_i(x) = \int K(x,y) \sum_j q'_j \phi_j(y)\, \mathrm{d}y \tag{2.18}$$

$$\sum_i q_i \underbrace{\int \phi_i(x) \phi_k(x)\, \mathrm{d}x}_{\delta_{ik}} = \sum_j q'_j \underbrace{\int K(x,y) \phi_j(y) \phi_k(x)\, \mathrm{d}y\, \mathrm{d}x}_{k_{jk}} \tag{2.19}$$

$$q_k = \sum_j q'_j k_{jk} \,. \tag{2.20}$$

Then, the measure will transform with the determinant $\det K = \det k$, which we can now express as an infinite product of the eigenvalues of k:

$$\big[\mathrm{d}q\big] = \left|\frac{\partial q}{\partial q'}\right| \big[\mathrm{d}q'\big] = \det K \big[\mathrm{d}q'\big] \,. \tag{2.21}$$

Usually functional integrals cannot be computed analytically; the exception is given by Gaussian integrals, which generalize the finite-dimensional result

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}A_{ij}x_i x_j + ib_j x_j\right) \mathrm{d}x_1 \ldots \mathrm{d}x_j = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right) . \tag{2.22}$$

Here A_{ij} is an n-dimensional real matrix (which WLOG can be taken to be symmetric) while b_i is an n-dimensional vector. The result comes from a transformation of the coordinates according to the finite-dimensional

This can be interpreted as a "functional" (still finite-dimensional, so just a function, but we will generalize soon) of b_i; we write it with an additional normalization N for convenience:

$$Z[b] = N \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}A_{ij}x_i x_j + b_j x_j\right) \mathrm{d}x_1 \ldots \mathrm{d}x_j = N\sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right) , \tag{2.23}$$

and if we rescale the normalization N so that $Z[\vec{0}] = 1$ we get

$$Z[b] = \exp\left(-\frac{1}{2}(A^{-1})_{ij} b_i b_j\right) . \tag{2.24}$$

The infinite-dimensional generalization of this result amounts to replacing all the sums (expressed implicitly with Einstein notation here) with integrals; also conventionally we change the names of the variables to $x \to q$, $A \to K$, $b \to J$:

$$Z[J] = N \int \mathcal{D}q \exp\left(-\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, K(x,y) q(x) q(y) + i \int \mathrm{d}x\, q(x) J(x)\right) \tag{2.25}$$

$$= \exp\left(-\frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, J(x) J(y) K^{-1}(x,y)\right) . \tag{2.26}$$

Let us now give some examples of applications of this result: $K(x,y) = \sigma^{-2}\delta(x-y)$ means $K^{-1}(x,y) = \sigma^2\delta(x-y)$, so

$$Z[J] = \exp\left(-\frac{\sigma^2}{2}\int \mathrm{d}x\, J^2(x)\right) . \tag{2.27}$$

This, as we shall see, can be used to give us a description of white noise, which is uncorrelated in momentum space.

Let us consider another example, whose physical application is to describe the motion of a massive scalar boson with Lagrangian

$$\mathscr{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2}_{\mathscr{L}_0} + \mathscr{L}_I(\phi)\,, \tag{2.28}$$

where the self-interaction term is some non-quadratic function of ϕ, often taken to be proportional to ϕ^3 or ϕ^4.

The Feynman path integral corresponding to this Lagrangian is given by the functional

$$Z[J] = N\int \mathcal{D}\phi \exp\left(i\int \mathscr{L}(\phi) + J\phi\,\mathrm{d}x\right). \tag{2.29}$$

Let us start with the non-interacting case, that is, we compute Z_0 with only the quadratic term in the Lagrangian. This can be expressed, in the formalism from before, using the kernel

$$K(x,y) = (-\Box_x - \mu^2)\delta(x-y)\,. \tag{2.30}$$

Now, the expression the functional is given in terms of K^{-1}: what is the inverse of this kernel? The definition reduces to

$$\int K(x,y)K^{-1}(y,z)\,\mathrm{d}y = \delta(x-z) \tag{2.31}$$

$$-\left(\Box_x + \mu^2\right)K^{-1}(x,z) = \delta(x-z)\,, \tag{2.32}$$

which is readily solved in momentum space, with a $+i\epsilon$ prescription in order to avoid the pole in the integration: what we find is called the *Green's function*,

$$K^{-1}(x,z) = G(x-z) = \frac{1}{(2\pi)^4}\int \frac{e^{-ik\cdot(x-z)}}{k^2+\mu^2-i\epsilon}\,\mathrm{d}k\,, \tag{2.33}$$

so the unperturbed functional reads

$$Z_0[J] = \exp\left(-\frac{i}{2}\int \mathrm{d}x\,\mathrm{d}y\, G(x-y)J(x)J(y)\right). \tag{2.34}$$

This by itself might not seem very useful, the motion of a free massive boson can be calculated with easier methods. However, the real power of this path integral is the possibility to write the interacting term perturbatively: the interaction Lagrangian is a function of ϕ, which is what we find if we perform a functional integration of the argument of the exponential in $Z_0[J]$ with respect to J; so we can express the full functinal as

$$Z[J] = \exp\left(i\int \mathrm{d}x\,\mathscr{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right)\underbrace{\int \mathcal{D}\phi \exp\left(i\int \mathrm{d}x\,\left(\mathscr{L}_0 + J\phi\right)\right)}_{=Z_0[J]} \tag{2.35}$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\int \mathrm{d}x \, \mathscr{L}_I \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right]^n Z_0[J] \,. \tag{2.36}$$

We can use this to compute the Green's functions:

$$G(x_1, \ldots, x_n) = \frac{1}{i^n} \frac{\delta Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \bigg|_{J=0} \,. \tag{2.37}$$

2.1 The probability density functional

We can interpret the quantity

$$\exp\left(-\frac{1}{2}(q, K, q) \right) \mathcal{D}q \tag{2.38}$$

as a *probability density functional* $\mathrm{d}P\,[q]$, since

1. it is positive definite;
2. it is normalized, as long as we set its integral, $Z[0]$, to 1;
3. it goes to zero as $q \to \pm\infty$.

If this is the case, then we ought to be able to compute the average value of a functional $F[q]$ as

$$\big\langle F[q] \big\rangle = \int F[q] \, \mathrm{d}P\,[q] = \int \mathcal{D}q \exp\left(-\frac{1}{2}(q, K, q) \right) F[q] \,, \tag{2.39}$$

which we can generalize to any non-gaussian probability density functional by replacing the exponential $\exp\left(-\frac{1}{2}(q, K, q)\right)$ with a generic $\mathcal{P}[q]$.

A useful kind of average we can compute is given by the N-point correlation function,

$$C^{(N)}(x_1, \ldots, x_n) = \big\langle q(x_1) \ldots q(x_n) \big\rangle \,. \tag{2.40}$$

With the formula we gave earlier, this can be computed as

$$C^{(N)}(x_1, \ldots, x_n) = \int \mathcal{D}q \mathcal{P}[q] \prod_i q(x_i) \,. \tag{2.41}$$

Here we can make use of a trick: going back to the Gaussian probability case, consider the functional derivative

$$\frac{1}{i} \frac{\delta Z[J]}{\delta J(x_1)} \bigg|_{J=0} = \frac{1}{i} \, \frac{\delta}{\delta J(x)} \bigg|_{J=0} \int \mathcal{D}q \exp\left(-\frac{1}{2}(q, K, q) + i(J, q) \right) \tag{2.42}$$

$$= \int \mathcal{D}q \exp\left(-\frac{1}{2}(q, K, q) \right) q(x_1) = \big\langle q(x_1) \big\rangle = C^{(1)}(x_1) \,, \tag{2.43}$$

which actually holds for any probability density functional, we did not make use of the gaussianity. So, in general we will be able to write

$$C^{(N)}(x_1 \dots x_n) = \frac{1}{i^N} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \right|_{J=0} . \tag{2.44}$$

The correlation functions, which as we discussed in an earlier section are crucial when discussing structure formation, can be "simply" calculated by functional differentiation as long as we have the generating functional $Z[J]$. This generating functional is very similar mathematically to a partition function in statistical mechanics, and it serves an analogous role: its derivatives allow us to characterize the dynamics of the system.

Now, any functional $\mathscr{F}[q]$ can be expressed through a functional Taylor series:

$$\mathscr{F}[q] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left. \frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)} \right|_{q=0} q(x_1) \dots q(x_n) , \tag{2.45}$$

so if we compute the average value $\langle \mathscr{F}[q] \rangle$ we find

$$\langle \mathscr{F}[q] \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left. \frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)} \right|_{q=0} \underbrace{\langle q(x_1) \dots q(x_n) \rangle}_{=C^{(N)}(x_1 \dots x_n)} \tag{2.46}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \left. \frac{\delta^n \mathscr{F}[q]}{\delta q(x_1) \dots \delta q(x_n)} \right|_{q=0} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \right|_{J=0} \tag{2.47}$$

$$= \mathscr{F}\left[-i \frac{\delta}{\delta J} \right] Z[J] \Big|_{J=0} . \tag{2.48}$$

Formally this makes sense, but what does it mean to calculate the field at a derivation operator? Should this just be interpreted as a shorthand for the Taylor expansion or is there more to it?

An example: consider a Gaussian field whose partition function $Z[J]$ is given by

$$Z[J] = \exp\left(-\frac{1}{2} (J, K^{-1}, J) \right) . \tag{2.49}$$

Then, as before we can calculate the correlation functions through functional derivatives: the first ones are

$$\langle q(x) \rangle = \frac{1}{i} \left. \frac{\delta Z[J]}{\delta J(x)} \right|_{J=0} \tag{2.50}$$

$$= -i \int \mathrm{d}y \, K^{-1}(x,y) J(y) \exp\left(-\frac{1}{2} (J, K^{-1}, J) \right) \Big|_{J=0} = 0 \tag{2.51}$$

$$\langle q(y) q(x) \rangle = - \left. \frac{\delta^2 Z[J]}{\delta J(y) \delta J(x)} \right|_{J=0} \tag{2.52}$$

$$= - \left(-K^{-1}(x,y) + \int \mathrm{d}z\, \mathrm{d}w\, K^{-1}(x,z) K^{-1}(y,w) J(w) \right) \exp\left(-\frac{1}{2}(J, K^{-1}, J) \right) \Big|_{J=0} \tag{2.53}$$

$$= K^{-1}(x,y)\,. \tag{2.54}$$

So, we have our result: *for a Gaussian variable, the two-point correlation function is the inverse of the kernel*. A similar, albeit quite long, calculation allows us to compute the N-point correlation function for the same Gaussian variable: we expand the exponential in $Z[J]$ in a power series, and when we differentiate it an even number of times we find

$$C^{2N}(x_1 \dots x_{2N}) = \left[K^{-1}(x_1,x_2) K^{-1}(x_3,x_4) \dots K^{-1}(x_{2N-1},x_{2N}) \right]_{\text{symmetrized}}, \tag{2.55}$$

where "symmetrized" means that we must sum over all the permutations of the variables x_i in the argument of the inverse kernels; on the other hand, the odd correlation functions C^{2N+1} all vanish since they correspond to the integrals of odd functions over all space.

We can also apply this process in reverse: starting from the two-point correlation function we can reconstruct the kernel, and with it the probability density functional $\mathrm{d}P\,[q]$.

In the Gaussian case, as long as we know the two-point function, which corresponds to the inverse kernel, we can reconstruct the N-point function. This can also be stated by saying that the "irreducible" N-point functions are all zero except for $N = 2$, since all the higher ones can be reduced to that one.

We shall see that all the reduced N-point functions can be recovered starting from the *generating functional* of connected correlation functions:

$$\mathscr{W}[J] = \log Z[J]\,, \tag{2.56}$$

through the expansion

$$\mathscr{W}[J] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n\, C_C^N(x_1 \dots x_N) J(x_1) \dots J(x_n)\,. \tag{2.57}$$

These connected correlation functions C_C^N are not (in principle) related to the C^N from before.

What is the relation between them, though? What is the physical interpretation of these connected correlation functions?

We could also have defined $\mathscr{W}[J] = i \log Z[J]$, this is a matter of convention.

Now, we define the *classical field*

$$q_{\text{cl}}(x) = \frac{\delta \mathscr{W}[J]}{\delta J(x)}\,, \tag{2.58}$$

and the effective action $\Gamma[q_{\text{cl}}]$ as the Legendre transform of $\mathscr{W}[J]$:

$$\Gamma[q_{\text{cl}}] = \mathscr{W}[J] - \int \mathrm{d}x\, q_{\text{cl}}(x) J(x)\,, \tag{2.59}$$

from which we can then recover $J(x)$ as

$$J(x) = -\frac{\delta\Gamma[q_{\rm cl}]}{\delta q_{\rm cl}(x)}\,. \tag{2.60}$$

In the Gaussian case we have

$$\mathscr{W}[J] = \log Z[J] = -\frac{1}{2}(J, K^{-1}, J)\,, \tag{2.61}$$

which, by direct comparison with the Taylor expansion, means that

$$C^2_C(x_1, x_2) = K^{-1}(x_1, x_2)\,, \tag{2.62}$$

while $C^N \equiv 0$ for any $N \neq 2$. Also, our expression for $q_{\rm cl}$ yields

$$q_{\rm cl}(x) = -\int \mathrm{d}y\, K^{-1}(x, y) J(y)\,, \tag{2.63}$$

from which we can express $J(x)$ by using the direct kernel $K(x, y)$:

$$\int \mathrm{d}x\, q_{\rm cl}(x) K(x, w) = -\int \mathrm{d}y\, \mathrm{d}w\, K^{-1}(x, y) K(x, w) J(y) = -\int \mathrm{d}w\, \delta(y - w) J(y) = -J(w)\,. \tag{2.64}$$

With an analogous procedure we can show that

$$(J, K^{-1}, J) = (q_{\rm cl}, K, q_{\rm cl})\,. \tag{2.65}$$

So in some sense J is a covariant vector while q is a contravariant one, right? can this be said in a better way?

The effective action then reads

$$\Gamma[q_{\rm cl}] = -\frac{1}{2}(J, K^{-1}, J) + (J, K^{-1}, J)\Big|_{J=J(q_{\rm cl})} \tag{2.66}$$

$$= \frac{1}{2}(q_{\rm cl}, K, q_{\rm cl})\,. \tag{2.67}$$

Now we have the tools to consider actual probabilities: starting from our classical field q, we want to compute the probability that it takes on a certain value $q \in (\alpha, \alpha + \mathrm{d}\alpha)$ at a point $\overline{x}$: this is expressed with a probability density function in the form

$$\frac{\mathrm{d}P_q}{\mathrm{d}\alpha} = P_{q(x)}(\alpha; \overline{x})\,. \tag{2.68}$$

We want to write this "$P(\alpha)\,\mathrm{d}\alpha$" in terms of the functional integral; in order to do so, we start from the Fourier transform

$$\int \mathrm{d}\beta \exp(i\beta\varphi) P_q(\beta; \overline{x}) = \left\langle e^{i\beta\varphi} \right\rangle_\beta\,. \tag{2.69}$$

The integral is a definite one, with bounds corresponding to the region in which P_q is nonzero, typically $\mathbb{R}$. The right-hand side is an average over the possible values taken on by the field q at the point $\overline{x}$, but it can also be computed by averaging over the possible *overall* field configurations, computed at a point $\overline{x}$:

$$\int \mathrm{d}\beta \exp\bigl(i\beta\varphi\bigr) P_q(\beta;\overline{x}) = \Bigl\langle e^{iq(\overline{x})\varphi} \Bigr\rangle_q \,. \tag{2.70}$$

This will be much harder to compute: it is a proper functional integral, to be computed according to (2.39); however it must give the same result.

If we take the Fourier antitrasform, we find

$$\begin{aligned} P_q(\alpha,\overline{x}) &= \frac{1}{2\pi}\int \mathrm{d}\varphi\, e^{-i\varphi\alpha} \Bigl\langle e^{i\varphi q(\overline{x})} \Bigr\rangle_q & (2.71)\\ &= \frac{1}{2\pi}\int \mathrm{d}\varphi \Bigl\langle e^{i\varphi(q(\overline{x})-\alpha)} \Bigr\rangle_q & (2.72)\\ &= \bigl\langle \delta(q(\overline{x})-\alpha) \bigr\rangle_q \,. & (2.73) \end{aligned}$$

This equation can be interpreted to mean the following: "the probability that the field q is equal to α at $\overline{x}$ is given by the integral of the probabilities of all the field configurations which satisfy $q(\overline{x}) = \alpha$".

This formalism can be generalized to N-point functions, the notation for the probability that for $j = 1 \dots N$ the field q takes on the value α_j at position x_j is as follows:

$$\begin{aligned} \mathrm{d}P_q^N &= P_q(\alpha_1 \dots \alpha_N; x_1 \dots x_N)\, \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_N & (2.74)\\ &= P_q\bigl([\alpha_N];[x_N]\bigr)\, \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_N & (2.75)\\ P_q\bigl([\alpha_N];[x_N]\bigr) &= \left\langle \prod_{j=1}^{N} \delta(q(x_j)-\alpha_j) \right\rangle_q \,. & (2.76) \end{aligned}$$

Statistical independence corresponds to the statement that the product can be brought out of the average, and we are interested in the general case, in which this does not happen.

The question we generally ask is: what is this probability? We can try to find an expression for it in terms of the partition function $Z[J]$: we use once again the fact that

$$\delta(x) = \frac{1}{2\pi}\int \mathrm{d}\varphi\, e^{i\varphi x} \tag{2.77}$$

to see that

$$P_q([\alpha_N];[X_N]) = \frac{1}{(2\pi)^N}\int \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_N \exp\left(-i\sum_{j=1}^{N}\varphi_j\alpha_j\right) \left\langle \exp\left(i\sum_{j=1}^{N}\varphi_j q(x_j)\right)\right\rangle_q \,. \tag{2.78}$$

This can be brought back to the partition function by making use of the fact that

$$Z[J] = \Bigl\langle \exp\bigl(i(J,q)\bigr) \Bigr\rangle_q \tag{2.79}$$

$$= \int \mathcal{D}qP[q] \exp\left(i \int \mathrm{d}x\, J(x)q(x) \right), \tag{2.80}$$

therefore

$$\left\langle \exp\left(i \sum_{j=1}^{N} \varphi_j q(x_j) \right) \right\rangle_q = Z\left[\sum_{j=1}^{N} \varphi_j \delta(x - x_j) \right] = Z[\widetilde{J}_\varphi], \tag{2.81}$$

where we have used the fact that

$$i \int \mathrm{d}x\, q(x) \left(\sum_{j=1}^{N} \varphi_j \delta(x - x_j) \right) = i \sum_{j=1}^{N} \varphi_j q(x_j). \tag{2.82}$$

So, we can compute the probability that the field reaches the values α_j at the points x_j as long as we can compute the partition function at $Z[\widetilde{J}_\varphi]$, and then integrate N times:

$$P_q([\alpha_N], [x_N]) = \frac{1}{(2\pi)^N} \int \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_N \exp\left(-i \sum_{j=1}^{N} \varphi_j \alpha_j \right) Z[\widetilde{J}_\varphi]. \tag{2.83}$$

Let us compute this for the case of a Gaussian random field $q(x)$. The partition function evaluated at $\widetilde{J}_\varphi = \sum_j \varphi_j \delta(x - x_j)$ is equal to

$$Z[\widetilde{J}_\varphi] = \exp\left(-\frac{1}{2} (\widetilde{J}_\varphi, K^{-1}, \widetilde{J}_\varphi) \right) \tag{2.84}$$

$$= \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} \varphi_i \varphi_j K^{-1}(x_i, x_j) \right). \tag{2.85}$$

Since it appears in the expression for the partition function, let us define the *covariance matrix*

$$M_{ij} = K^{-1}(x_i, x_j) = C^{(2)}(x_i, x_j), \tag{2.86}$$

so that the probability reads, using the usual result about Gaussian integrals,

$$P_q([\alpha_N], [x_N]) = \frac{1}{(2\pi)^N} \int \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_N \exp\left(-i \sum_{j=1}^{N} \varphi_j \alpha_j \right) \exp\left(-\frac{1}{2} \varphi_i M_{ij} \varphi_j \right) \tag{2.87}$$

$$= \frac{1}{\sqrt{(2\pi)^N \det M}} \exp\left(-\frac{1}{2} \alpha_i (M^{-1})_{ij} \alpha_j \right). \tag{2.88}$$

This is the standard expression for an N-variate Gaussian distribution whose covariance matrix is M_{ij}.

Application to Brownian motion.

Avoiding divergences: high- and low-pass filters. If $W_R(x)$ is our filter function, then the general correlation function reads

$$C_R^{(N)}(x_1 \dots x_N) = \int \mathcal{D}qP[q] \prod_{r=1}^{N} \int \mathrm{d}y_r \, q(y_r) W_R(|y_r - x_r|) \tag{2.89}$$

$$= \int \prod_{r=1}^{N} \mathrm{d}y_r \, W_R(|y_r - x_r|) C^{(N)}(y_1 \dots y_N) \,. \tag{2.90}$$

The same holds for the *connected* correlation functions, so we can calculate the $C_{R,C}^{(N)}$: *smooth, connected* N-point correlation functions.

We can calculate these "smoothed" correlation functions through $Z[J]$ by choosing $J(x)$ in the form

$$J(x) = \int \mathrm{d}y \, \varphi(x+y) W_R(y) \,, \tag{2.91}$$

where φ is a generic function.

Since these regularized correlation functions do not explode at vanishing distances, we can define the N-th order **moment**:

$$\left\langle q_R^N \right\rangle = C_R^{(N)}(x \dots x) \tag{2.92}$$

and the N-th order **cumulant**:

$$\left\langle q_R^{(N)} \right\rangle_C = C_{R,C}^{(N)}(x \dots x) \,. \tag{2.93}$$

Due to homogeneity and isotropy, neither of these depends on x. We expect these to diverge if we perform no filtering, which is equivalent to taking the filtering scale $R \to 0$. On the opposite limit, taking $R \to \infty$ amounts to averaging over all space, so we expect $\left\langle q_R^N \right\rangle \sim \left\langle q \right\rangle^N$, while $\left\langle q_R^N \right\rangle_C \sim 0$.

The moments of q_R can be obtained through the moment generating function

$$Z(\varphi) = Z[J_\varphi(x)] = Z[\varphi W_R(|x|)] \,, \tag{2.94}$$

and we can define the *cumulant* generating function $W(\varphi)$ analogously. We can then define an effective action through a Legendre transform:

$$\Gamma(q_{R,\mathrm{cl}}) = W(\varphi) - q_{R,\mathrm{cl}} \varphi \,, \tag{2.95}$$

where the classical field $q_{R,\mathrm{cl}}$ is given by

$$q_{R,\mathrm{cl}} = \frac{\mathrm{d}W(\varphi)}{\mathrm{d}\varphi} \,. \tag{2.96}$$

The function $Z(\varphi)$ is just the Fourier transform of $P_R(\alpha, \overline{x})$:

$$Z(\varphi) = \left\langle e^{i\varphi q_R} \right\rangle_q = \int \mathrm{d}\alpha \, e^{i\varphi\alpha} P_{q_R}(\alpha; \overline{x}) \,, \tag{2.97}$$

so we can insert the explicit expression for the probability inside the inverse of this transform:

$$P_R(\alpha;\overline{x}) = \frac{1}{2\pi}\int \mathrm{d}\varphi\, e^{-i\alpha\varphi} Z(\varphi) \tag{2.98}$$

$$= \frac{1}{2\pi}\int \mathrm{d}\varphi\, e^{-i\alpha\varphi} \int \mathcal{D}qP[q] \exp\left(i\varphi \int \mathrm{d}y\, W_R(|y-\overline{x}|)q(y)\right). \tag{2.99}$$

We can expand $Z(\varphi)$ and $W(\varphi)$ as

$$Z(\varphi) = 1 + \sum_{N=1}^{\infty} \frac{i^N}{N!}\varphi^N \left\langle q_R^N \right\rangle \tag{2.100}$$

$$W(\varphi) = \sum_{N=1}^{\infty} \frac{i^N}{N!}\varphi^N \left\langle q_R^N \right\rangle_C , \tag{2.101}$$

why?

therefore

$$\left\langle q_R^N \right\rangle = \int \mathrm{d}\alpha\, \alpha^N P_{q_R}(\alpha;\overline{x}) , \tag{2.102}$$

which clarifies what was meant by saying that these are *moments*.

Do we have a similar relation for the connected moments?

P_{q_R} can be reconstructed starting from these moments.

We can take a different path, by transforming the probability measure: we start from the probability density functional evaluated at q, and calculate the expectation value of a generic functional $F[q]$ as

$$\int \mathcal{D}qP[q]F[q] \propto \int \mathcal{D}q_R P[(W^{-1}, q_R)]F[(W^{-1}, q_R)] . \tag{2.103}$$

The proportionality is because we need to include a Jacobian determinant, however probability densities must always be normalized to 1 so any constant is inessential.

Applying this to a Gaussian field amounts to mapping the kernel K to a "smoothed" kernel K_R, which corresponds to the inverse of the smoothed two-point correlation function: $\langle q_R(x)q_R(y)\rangle$.

Now, we move to a more physical example. We consider a Gaussian field q in three dimensions, whose mean is zero and whose two-point function in Fourier space is

$$\langle q(k)q(k')\rangle = (2\pi)^3 P(|k|)\delta^{(3)}(k-k') , \tag{2.104}$$

where $P(|k|)$ is called the **power spectral density** of the field q: it quantifies how much of the power of the field is transmitted at each frequency.

The **Wiener–Khinchin theorem** tells us that the smoothed two-point function is given by

$$\langle q_R(x)q_R(y)\rangle = G_R(|x-y|) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, P(k)\widetilde{W}_R^2(k)e^{ik\cdot(x-y)} \tag{2.105}$$

$$= \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k\, k^2 P(k) \widetilde{W}_R^2(k) j_0(k|x-y|)\,, \tag{2.106}$$

where j_0 is a Bessel function of the first kind. From the first expression we can read off the Fourier transform of K_R^{-1}, whose inverse will then be

$$\widetilde{K}_R(k) = \frac{1}{P(k)\widetilde{W}_R^2(k)}\,. \tag{2.107}$$

Now, let us consider a typical power-spectrum: $P(k) \propto k^n$. The smoothed two-point function will be asymptotically (in the $r \ll R$ or $r \gg R$ limits) be proportional to

$$G_R(r) \propto \left[\max(R,r)\right]^{-(n+3)}\,. \tag{2.108}$$

The $n = 0$ case is **white noise**, for which $G_R(0) \sim R^{-3}$.

Connection with a Poisson process...not clear

The $n = -3$ case is **flicker noise** corresponds to $1/f$ noise in 1D.

In the $n > 0$ case the smoothed two-point function $G_R(r)$ must be equal to zero somewhere, since we have the constraint[2]

$$\int_0^\infty \mathrm{d}r\, r^2 G_R(r) = 0\,. \tag{2.110}$$

What does this have to do with $n > 0$? Isn't this constraint always present?

The first zero-crossing of the correlation function, r_0 such that $G_R(r_0) = 0$, is called the **coherence length**.

If $n < 0$, we have fractal behaviour: as long as $G_R(r) \gg 1$, the fractal dimension is $D_F = -n$.

Clarify this...how would we show it?

A common technique in QFT is the **Wick rotation** $t \to -it_E$, needed in order to move from a generally ill-defined oscillatory path integral of an exponential $\exp\big(iS[q]\big)$ in 4D space with an indefinite signature, to a Euclidean path integral of an exponential $\exp\big(-S_E[q]\big)$ in 4D space with a Cartesian signature, so that the D'Alambert operator is mapped to the Laplace-Beltrami operator.

In general, our probability measure can be written as

$$P[q] = \exp\left(-\frac{1}{2}(q,K,q) - V[q]\right)\,, \tag{2.111}$$

[2] This comes from the following line of reasoning: since the field has zero mean,

$$\int \mathrm{d}^3y\, \langle q_R(x) q_R(y)\rangle = \left\langle q_R(x) \int \mathrm{d}^3y\, q_R(y)\right\rangle = 0\,. \tag{2.109}$$

where the potential term encompasses all the non-quadratic parts of the probability. Linear terms can be removed with a change of variable, so we can expand it starting from third order:

$$V[q] = \sum_{n=3}^{\infty} \mathrm{d}x_1 \dots \mathrm{d}x_n \, K^{(n)}(x_1 \dots x_n) q(x_1) \dots q(x_n) \,. \tag{2.112}$$

Now, if we define

$$Z[J] = \int \mathcal{D}q \exp\left(-\frac{1}{2}(q, K, q) - V[q] + i(q, J) \right), \tag{2.113}$$

then we can expand the potential term perturbatively:

$$Z[J] = \exp\left(-V\left(-i\frac{\delta}{\delta J} \right) \right) \int \mathcal{D}q \exp\left(-\frac{1}{2}(q, K, q) + i(q, J) \right) \tag{2.114}$$

$$= \exp\left(-V\left(-i\frac{\delta}{\delta J} \right) \right) \underbrace{\exp\left(-\frac{1}{2}(J, K^{-1}, J) \right)}_{Z_0[J]} \tag{2.115}$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \left[V\left(-i\frac{\delta}{\delta J} \right) \right]^n Z_0[J] \,. \tag{2.116}$$

With this approach, we can **recover the probability** of the field attaining a certain value as a function of the cumulants.

The **saddle point expansion** is the following: we want to compute an integral in the form

$$J = \int \mathrm{d}\tau \exp\big(-f(\tau) \big) \,, \tag{2.117}$$

so we choose a τ_0 such that $f'(\tau_0) = 0$, and then approximate f up to second order:

$$J \approx \int \mathrm{d}\tau \exp\left(-f(\tau_0) - \frac{1}{2} f''(\tau_0)(\tau - \tau_0)^2 \right) \tag{2.118}$$

$$\approx \sqrt{\frac{2\pi}{f''(\tau_0)}} \exp\big(-f(\tau_0) \big) \,. \tag{2.119}$$

We are effectively approximating the integrand as a Gaussian.

We can apply this to the calculation of path integrals: if the probability $P(q)$ can be written in terms of $Z(\varphi) = \exp\big(W(\varphi)\big)$, then we have

$$P(q) = \frac{1}{2\pi} \int \mathrm{d}\varphi \exp\big(-i\varphi q + W(\varphi) \big) \,. \tag{2.120}$$

The classical variable $q_{R,\mathrm{cl}} = \overline{q}$ is defined as $\overline{q} = W'(\varphi)$, and the effective action is written in terms of it: $\Gamma(\overline{q}) = W(\varphi) - \overline{q}\varphi$. Therefore,

$$\frac{\mathrm{d}\overline{q}}{\mathrm{d}\varphi} = W''(\varphi) \,, \tag{2.121}$$

while the derivative of the classical action reads

$$\Gamma''(\bar{q}) = \frac{\mathrm{d}}{\mathrm{d}\bar{q}}(-\varphi) = -\left[W''(\bar{q})\right]^{-1}. \tag{2.122}$$

This means that $\mathrm{d}\varphi = -\Gamma''(\bar{q})\,\mathrm{d}\bar{q}$, which allows us to change variable:

$$P(q) = -\frac{1}{2\pi}\int \Gamma''(\bar{q})\,\mathrm{d}\bar{q}\exp\left(iq\Gamma'(\bar{q}) + \Gamma(\bar{q}) - \bar{q}\Gamma'(\bar{q})\right), \tag{2.123}$$

where we used the fact that

$$-i\varphi q + W(\varphi) = -i\varphi q + \Gamma(\bar{q}) + \bar{q}\varphi \tag{2.124}$$

$$= \Gamma(\bar{q}) + (\bar{q} - iq)\frac{\mathrm{d}\Gamma(\bar{q})}{\mathrm{d}\bar{q}}. \tag{2.125}$$

For large q, this oscillating integral is dominated by the points at which the phase does not change much: these are the stationary points of the argument of the exponential,

$$iq\Gamma''(\bar{q}) + \Gamma'(\bar{q}) - \Gamma'(\bar{q}) - \bar{q}\Gamma''(\bar{q}) = \Gamma''(\bar{q})(iq - \bar{q}) = 0, \tag{2.126}$$

meaning that $\bar{q} = iq$.

We can then apply the saddle-point approximation:

$$P(q) \approx \frac{e^{i\Gamma(iq)}}{2\pi}\Gamma''(iq)\int \mathrm{d}\bar{q}\exp\left(-\Gamma''(iq)\frac{\left(\bar{q} - iq\right)^2}{2}\right) \tag{2.127}$$

$$\approx \sqrt{\frac{\Gamma''(iq)}{2\pi}}\exp\left(\Gamma(iq)\right). \tag{2.128}$$

In the Gaussian case the probability reads

$$P(q) = \frac{1}{\sqrt{2\pi\sigma_R^2}}\exp\left(-\frac{q^2}{2\sigma_R^2}\right), \tag{2.129}$$

so we can recognize $\Gamma(\bar{q}) = \bar{q}^2/(2\sigma_R^2)$, while $W(\varphi) = \varphi^2/(2\sigma_R^2)$.

We can apply this approximation to the evaluation of the general case:

$$Z[J] = \int \mathcal{D}q \underbrace{\exp\left(-\frac{1}{2}(q,K,q) - V[q] + i(q,J)\right)}_{\exp\left(\mathscr{F}[q,J]\right)}. \tag{2.130}$$

The functional integral will be dominated by the stationary points of $\mathscr{F}[q,J]$: we define q_0 as the field configuration satisfying

$$\left.\frac{\delta\mathscr{F}[q,J]}{\delta q}\right|_{q_0} = (K,q_0) + \left.\frac{\delta V[q]}{\delta q}\right|_{q_0} - iJ = 0. \tag{2.131}$$

This will depend on J in general. We then expand up to second order, as usual:

$$\mathscr{F}[q,J] \approx \mathscr{F}[q_0,J] + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\frac{\delta^2\mathscr{F}[q,J]}{\delta q(x)\delta q(y)}\bigg|_{q_0}(q(x)-q_0(x))(q(y)-q_0(y))\bigg|_{\text{symm}}\,. \tag{2.132}$$

Thanks to the functional equation for q_0 (2.131), we can write $\mathscr{F}[q_0,J]$ as

$$\mathscr{F}[q_0,J] = \frac{1}{2}(q_0,K,q_0) + V[q_0] - i(J,q_0) \tag{2.133}$$

$$= (K,q_0) + \frac{\delta V[q]}{\delta q}\bigg|_{q_0} - \frac{i}{2}(q_0,J)\,. \tag{2.134}$$

This is usually the "bottleneck" in the calculation, practically speaking. It is really hard to solve this functional equation. Also, we define

$$\frac{\delta^2\mathscr{F}}{q(x_1)q(x_2)}\bigg|_{q_0} = K(x_1,x_2) + \frac{\delta^2 V[q]}{\delta q(x_1)\delta q(x_2)}\bigg|_{q_0} = \mathscr{Q}(x_1,x_2)\,. \tag{2.135}$$

This confuses me: earlier we said that V starts from third order, since the quadratic term is brought out. What are we doing here? If we only consider the Gaussian part of V what is even the point of including it?

We then perform the integration with respect to the field $q - q_0$ (this amounts to adding a constant, so the measure does not change) and find

$$Z[q_0,J] \approx \mathscr{N}\frac{\exp\left(-\mathscr{F}[q_0,J]\right)}{\sqrt{\mathscr{Q}[q_0,J]}}\,. \tag{2.136}$$

Using the identity $\log\det M = \operatorname{Tr}\log M$ we can compute

$$W[q_0,J] \approx \mathscr{F}[q_0,J] - \frac{1}{2}\operatorname{Tr}\log\mathscr{Q}[q_0,J] + \text{const}\,. \tag{2.137}$$

3 Applications

Definitions from [MLB86].

The second connected correlation function, computed with a smoothing at a scale R, can be used to define a variance: $\xi^{(2)}(x,x) \sim \sigma_R^2$. With this we can then quantify "peaks" in the density perturbation $\delta(x) = (\rho - \langle\rho\rangle)/\langle\rho\rangle$.

Specifically, we define a function $\rho_{\nu,R}$ so that it is equal to 1 when the perturbation is ν sigmas above the average (which is $\delta = 0$ by definition), and 0 otherwise:

$$\rho_{\nu,R} = \Theta(\delta(x) - \nu\sigma_R)\,. \tag{3.1}$$

We can define an N-point correlation for this "boolean density":

$$\Pi^{(N)}_{\nu,R}(x_1\dots x_N) = \left\langle\prod_{r=1}^{N}\rho_{\nu,R}(x_r)\right\rangle \tag{3.2}$$

quantifies the probability that the perturbation is above the threshold at all the positions x_r. We then define the *peak* correlation function

$$\xi^{(N)}_{\mathrm{dis},\nu,R} = \left\langle \frac{\prod_{r=1}^{N} \rho_{\nu,R}(x_r)}{\langle \rho_{\nu,R} \rangle^N} \right\rangle - 1 \,. \tag{3.3}$$

According to equations 16 and 22 in [MLB86] the two-point correlation function with two different smoothing scales and thresholds is allowed to have zero-crossings.

What does $\xi_{cc} \propto \xi_{gg}$ mean? These are the two-point correlation functions for rich cluster and galaxies. Do they correspond to $\xi^{(N)}_{\mathrm{dis},\nu,R}$ for different values of ν? What is the connection between this observable quantity and the mathematical formalism of peak correlation functions?

The paper also finds *scaling relations* between a certain order N-point peak correlation function and lower order ones, as well as the background correlation functions [MLB86, eqs. 24–25]. The result is interesting since it is general, allowing for the computation to be performed with a general non-Gaussian background; in the Gaussian case it reduces to

$$\zeta(1,2,3) = \xi(1,2)\xi(2,3) + \xi(1,3)\xi(2,3) + \xi(1,2)\xi(1,3) + \xi(1,2)\xi(2,3)\xi(1,3) \tag{3.4}$$

$$\zeta \sim \sum \xi^2 + \xi^3 \,, \tag{3.5}$$

where ζ is the three-point peak correlation function, $\xi$is the two point one, and we denote $x_i \equiv i$ for compactness. The second expression is just a compact form for the first.

This expression does not seem to fit observational data: specifically, the ξ^3 term has not been found in observations. The aforementioned general expression can be written as

$$\zeta = F\left(\sum \xi^2 + \xi^3\right) + (F-1)\left(1 + \sum \xi\right), \tag{3.6}$$

where $F = F(1,2,3) = 1$ in the Gaussian case.

I do not understand the motivation behind equation 28 in the same paper? Is it meant to make it so the ξ^3 term vanishes?

Is the ξ^3 term predicted by Gaussian statistics but not observed?

Moving on to [Ber87]. Perturbative methods work well in the initial stages, then the non-linearity kicks in and the higher order terms become very relevant: Monte Carlo methods are then more suitable to calculate the functional integrals.

They are able to put constrains on the realizations of the Gaussian random field! BBKS gives us peaks; the path integral approach allows us to more precisely estimate the surrounding distribution, by the use of constraints.

In [HR91] an improved algorithm for constrained realizations of Gaussian fields is presented.

To understand: what constrained realizations are practically useful? Do we impose the presence of a (proto-galaxy?) and generate the density distribution around it?

Bertschinger [Ber87] found a method with complexity $\mathcal{O}\left((N_c^2+1)N\right)$ working in Fourier space (so, in a plane wave basis), Binney and Quinn (1990) improved upon it by working in

spherical harmonics — for certain symmetric types of constrains this decreases N_c significantly.

Hoffman and Ribak [HR91] work by constraining the Fourier *phases* of the components of the random field.

I don't see where $\overline{f}(r) = \langle f(r)|\Gamma\rangle = \xi_i(r)\xi_{ij}^{-1}c_j$ comes from in general — I can see it in the specific case $C_i = f(r_i)$ through.

Looking at [MVJ00]. They account for non-gaussianity through field (overdensity or gravitational potential) in the form $\alpha\phi + \epsilon(\phi^2 - \langle\phi^2\rangle)$, where ϕ is Gaussian with zero mean. They use the Press–Schechter formalism, considering the regions in which the linearized density perturbation goes beyond $\Delta_c \approx 1.687$. Figure 4: the effect of adding the non-Gaussian tails is to boost by an order of magnitude the probability $\mathbb{P}(> \delta_c|z_c, R)$, where z_c is the redshift we are considering while R is the filtering scale.

If we perturb the density we need $|\epsilon| \lesssim 10^{-2}$, if we perturb the gravitational field we need $|\epsilon| \gtrsim 10^2$, in either case these are needed in order to see a significant departure from the Gaussian case. The latter statements tells us that non-Gaussianity predicted by inflationary models would be undetectable from structure formation.

Does equation 51 solve the problem of halos-inside-halos which was mentioned as an issue in early Press-Schechter theory?

Section 5: using high-z galaxies is advantageous because "they directly sample the galaxy scale, and are always in virialized halos". I get the fist point, but not the second: are more recent galaxies not in virialized halos?

At higher z the density PDF is more sensitive to non-Gaussianity: the critical overdensity is $\delta_c = \Delta_c / D(z_c|\Omega_{0,m}, \Omega_{0,\Lambda})$, where D is the linear growth factor: it seems like for most cosmological models $D^{-1} \propto 1 + z$, with an order-1 constant, from figure 3 in [MVJ00]. More details should be found in [Ham01].

In the paper the functional integrals are explicitly performed to find

$$\mathbb{P}(> \delta_c|z_c, R) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{\lambda}\exp\big(-i\lambda\delta_c(z_c) + \mathscr{W}(\lambda)\big) + \frac{1}{2}, \tag{3.7}$$

where $\mathscr{W}(\lambda)$ is the *cumulant generator*, while λ is related to the magnitude of the source term. The explicit expression for $\mathscr{W}$ is

$$\begin{aligned}\mathscr{W}(\lambda) = -i\lambda C - \frac{1}{2}\int \mathrm{d}^3y\,\mathrm{d}^3z\,\Bigg[&\lambda^2\alpha^2 F_R(|y|)\big[\mathscr{K}'_\lambda\big]^{-1}(y,z)F_R(|z|) + \\ &+ \delta_D(y-z)\log\Big[\delta_D(y-z) - 2i\lambda\epsilon F_R(|y|)\xi_\phi(y-z)\Big]\Bigg],\end{aligned} \tag{3.8}$$

where:

Add more details of the computation, and definitions of the parameters (?)

So, δ_c increases with redshift, therefore we see from equation 66 that the skewness term becomes less and less relevant. This can be seen in figs. 4 and 5 as well.

Connection with observation! We have a sample of 6 $z \sim 5 \div 6$ galaxies, whose masses, SFR and σ_v are estimated from spectroscopic data. Approximately speaking, from the amount of galaxies found per area surveyes, at this redshift we find $\sim$ 1 galaxy per $3 \times 10^{-4}\,\mathrm{deg}^2$.

Estimating the number of galaxies formed with masses within $2 \times 10^{10} M_\odot < M < 4 \times 10^{11} M_\odot$ and formation redshifts within $6 < z_c < 8$ yields a number density of galaxies about 16 times lower than the observed value; non-Gaussianity can raise the tail of the density distribution to give the required order-10 increase of galaxy formation.

Precise mass determination is crucial to figure out whether we actually need non-Gaussianity to explain the data.

Equation 4.25 in Verde et al. [Ver+13] is a calculation tool: it allows us to calculate the average of an observable A for a non-Gaussian random field in a discretized space.

Collaboration et al. [Col+19]: I guess I can interpret the PDF for the nongaussianity parameter, but not much else...

Celoria and Matarrese [CM18]: is EEE, TTT and such related to the three-point functions computed with different polarizations?

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