General Relativity notes

Jacopo Tissino, Giorgio Mentasti

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Marco Peloso, marco.peloso@pd.infn.it

1 Special relativity

Definition 1.1. *An inertial frame is one in which Newton's laws hold: a free body moves with acceleration $a^i = 0$.*

Newton's first law establishes the *existence* of inertial frames.

Proposition 1.1. *The frames O and O' are both inertial frames iff O' moves with constant velocity wrt O.*

Proposition 1.2. *Coordinate transformations between inertial frames are Lorentz boosts, which in some coordinate frame can be written as*

$$t' = \gamma_v \left(t - \frac{vx}{c^2} \right) \tag{1a}$$

$$x' = \gamma_v (x - vt) \tag{1b}$$

$$y' = y \tag{1c}$$

$$z' = z\,, \tag{1d}$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$.

If $v \ll c$, so $v/c \sim 0$, they simplify to the identity for t, y, z and $x' = x - vt$: these are Galilean transformations.

If we have two events, x^μ and y^μ, they occur with some time and space separation $\Delta x^\mu = x^\mu - y^\mu$. We can compute $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$, where

$$\eta_{\mu\nu} = \operatorname{diag}\left(-c^2, 1, 1, 1 \right). \tag{2}$$

Proposition 1.3. *Under Lorentz transformations Δs^2 is invariant.*

We can classify separations between events as

- time-like when $\Delta s^2 < 0$;
- null-like when $\Delta s^2 = 0$;
- space-like when $\Delta s^2 > 0$.

We can draw spacetime diagrams. A light cone is the set of points which are null-like separated from a select point. Things can be only causally related to events inside the light-cone (with $\Delta s^2 \geq 0$).

1.1 Time dilation

Take two events which occur at the same location for O'. In the primed frame they will have coordinates $x^\mu = (t_0, x_0)$ and $y^\mu = (t_1, x_0)$.

Definition 1.2. *The* proper time *between these two events is* $t_1 - t_0 \stackrel{def}{=} \Delta\tau$.

We now see that $\Delta s'^2 = -c^2 \Delta\tau^2$. Then, any other observer will see the same $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = \Delta s'^2$.

This directly implies that $\Delta\tau \leq \Delta t$ for any observer, since $\Delta\tau^2 = \Delta t^2 - \Delta x^2/c^2$. This effect is called *time dilation*.

By how much exacly is time dilated? Of course $\Delta x = v\Delta t$, therefore $\Delta t = \gamma_v \Delta\tau$.

-> Muon problem.

Inverse Lorentz transformation have the same expression, but with $v \to -v$. This can be proved both mathematically by solving the equations and phisically by reasoning about their meaning. There is no preferential inertial frame.

A Lorentz transformation can be written in matrix form in the (ct, x) plane as:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \tag{3}$$

since there is an angle θ such that $\gamma = \cosh\theta$ and $\gamma\beta = \sinh\theta$: the angle θ will be $\theta = \tanh^{-1}(v/c)$. This is true because $\gamma^2 - \beta^2\gamma^2 = 1$.

After a boost the ct' and x' axes are respectively the lines $ct = x/\beta$ and $ct = \beta x$.

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Last lecture we saw the fact that the ct' and x' axes are rotated by equal angles from the ct and x axes towards the $ct = x$ axis.

1.2 Relativity of simultaneity

Consider two events which are simultaneous in the O' frame. Their times in this frame are $t'_A = t'_B$.

In the O frame, instead, we have

$$ct_{A,B} = \frac{v}{c}x_{A,B} + \underbrace{\sqrt{1 - \frac{v^2}{c^2}}ct'_{A,B}}_{\text{a constant}}, \tag{4}$$

so the events are not simultaneous in the O frame.

1.3 Length contraction

If in the O frame, A occurs at $t, x = 0$ while B occurs at $t = 0$, $x = L$, then L is the measured length of their spatial interval by O. We assume that this is the frame in which the object is moving, and we transform into a frame in which it is stationary: O'.

In the primed frame their coordinates will be:

$$x'_A = \gamma_v \left(x_A - \frac{v}{c}ct_A \right) \tag{5a}$$

$$x'_B = \gamma_v \left(x_B - \frac{v}{c}ct_B \right), \tag{5b}$$

therefore $x'_B - x'_A = \gamma_v(x_B - x_A)$: the length is contracted in the O frame, since $\gamma \geq 1$.

1.4 Addition of velocities

Two observers see an object moving with $v' = \mathrm{d}x'/\mathrm{d}t'$ and $v = \mathrm{d}x/\mathrm{d}t$ respectively. Their relative velocity is u. Differentiating we get:

$$v' = \frac{\gamma(\mathrm{d}x - v\,\mathrm{d}t)}{\gamma\left(\mathrm{d}t - \frac{u\mathrm{d}x}{c^2}\right)} = \frac{v - u}{1 - \frac{uv}{c^2}}. \tag{6}$$

Two interesting limits of this formula are: $v' = v - u$ if $u \ll c$ or $v \ll c$; and $v' = c$ if $v = c$ for whatever u.

1.5 Tensor notation

The position four-vector is $x^\mu = (ct, x, y, z)$. The Euclidean scalar product is given by $x \cdot y = \delta_{\mu\nu} x^\mu x^\nu$. If we substitute the identity $\delta_{\mu\nu}$ with another metric we can find a more general metric space.

The Minkowski metric is $\eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, 1)$. The separation 4-vector is $\mathrm{d}x^\mu = (c\,\mathrm{d}t\,, \mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z)$.

Using Einstein summation notation, we can write the spacetime interval as $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$.

Specifically for the Minkowski metric we have the relation $\eta_{\mu\nu} = \eta^{\mu\nu}$: it is its own inverse. For a general metric $g_{\mu\nu}$ this will not hold.

How do we express the Lorentz boosts? They preserve $\mathrm{d}s^2$, therefore they look like $x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu$, with the $(1,1)$ tensors $\Lambda_\mu{}^\nu$ satisfying $\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$. This is called the *pseudo-orthogonality* relation.

The metric allows us to raise and lower indices. Raising an index in the pseudo-orthogonality relation gives us: $\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta \eta^{\beta\sigma} = \delta_\alpha{}^\sigma$, therefore $\eta_{\mu\nu}\Lambda^\nu{}_\beta \eta^{\beta\sigma}$ is the inverse of a Lorentz transformation.

Four-vectors can also have their indices down, and they will transform according to the inverse of Lorentz transformations:

$$
\begin{aligned}
(\eta_{\alpha\mu} x^\mu)' &= \eta_{\alpha\mu} \Lambda^\mu{}_\nu x^\nu && \text{(7a)}\\
&= \Lambda_{\alpha\sigma} \delta^\sigma{}_\nu x^\nu && \text{(7b)}\\
&= \Lambda_{\alpha\sigma} \eta^{\sigma\beta} \eta_{\beta\nu} x^\nu && \text{(7c)}\\
&= \Lambda_\alpha{}^\beta x_\beta \,. && \text{(7d)}
\end{aligned}
$$

We will write our laws as tensorial equations, which are covariant.

By pseudo-orthogonality, the scalar product $A_\mu B^\mu$ is a covariant (that is, invariant) scalar. Of course it is equal to $A^\mu B_\mu$.

Definition 1.3 (Tensor). *A* (p, q) tensor *is an object* $M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p}$ *with many components indexed by* $p + q$ *indices, which transforms as:*

$$
M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p} \to \Lambda_{\mu_1}{}^{\mu_1'} \dots \Lambda_{\mu_p}{}^{\mu_p'} \Lambda^{\nu_1}{}_{\nu_1'} \dots \Lambda^{\nu_q}{}_{\nu_q'} M^{\nu_1' \dots \nu_q'}_{\mu_1' \dots \mu_p'} \tag{8}
$$

under Lorentz transformations $\Lambda_\mu{}^\nu$.

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Last lecture we introduced tensors.

An example of those is the EM tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_x & B_y \\ -E_y/c & B_x & 0 & -B_z \\ -E_z/c & -B_y & B_z & 0 \end{bmatrix}, \tag{9}$$

which, it can be checked, transforms as a $(0,2)$ tensor. Also, we can define the current vector $j^\mu = (c\rho, j^i)$. Then, the Maxwell equations read:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \qquad \text{and} \qquad \partial_{[\mu} F_{\nu\rho]} \,. \tag{10}$$

They are covariant!

1.6 Particles in motion

In Newtonian mechanics, the motion of a particle is described by a function of time $x^i = x^i(t)$.

In special relativity, we introduce the concept of *worldline*. It must be parametrized with respect to some parameter λ, such that $x^\mu = x^\mu(\lambda)$. A preferred choice for λ is the proper time of the particle, $\lambda = \tau$.

We then define the 4-velocity:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \,. \tag{11}$$

It is a tensor since it is the product of a scalar and a tensor.

Multiplying $u^\mu u_\mu$ we always get $-c^2$, since:

$$u^\mu u_\mu = \frac{\mathrm{d}x^\mu \,\mathrm{d}x_\mu}{\mathrm{d}\tau^2} = -c^2 \frac{\mathrm{d}s^2}{\mathrm{d}s^2} \tag{12}$$

We can make the expression explicit using $\mathrm{d}\tau = \gamma \,\mathrm{d}t$, which gives us $u^\mu = (\gamma c, \gamma v^i)$. In the frame of the particle, $u^\mu = (c, 0)$.

The *four-momentum* of a particle is defined as:

$$p^\mu = m u^\mu = (m\gamma c, m\gamma v^i) \,. \tag{13}$$

The component p^0 is mc at $v = 0$. What does it mean? we can expand it for small v:

$$\frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}} \sim mc\left(1 + \frac{v^2}{2c^2}\right) = mc + \frac{1}{c}\frac{mv^2}{2} \,. \tag{14}$$

We get the mass, plus a kinetic energy term: more explicitly, $cp^0 = mc^2 + {}^1\!/\!{}_2 mv^2$.

We can rewrite Newton's first law in SR:

Proposition 1.4 (Newton I)**.** *A free particle moves with constant u^μ, or*

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \tag{15}$$

To express this in an easier way we introduce the 4-acceleration:

$$a^\mu \stackrel{\text{def}}{=} \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} \tag{16}$$

We now with to introduce the concept of a path minimizing proper time. Recall Snell's law, which allows us to relate the angles of incidence of light when it passes between one medium to another, if they have different indices of refraction:

$$\frac{\sin(\theta_2)}{\sin(\theta_1)} = \frac{n_1}{n_2} = \frac{v_2}{v_1}\,. \tag{17}$$

This can be shown to be equivalent to light minimizing the time it takes to move from a point in one medium to a point in the other.

Analogously, saying that a massive particle travels along the worldline which minimizes τ is equivalent to Newton's first principle.

We want to perturb a generic worldline x^μ with some $\mathrm{d}x^\mu$, and consider the proper time functional τ which gives the proper time of a generic trajectory: we impose

$$\frac{\tau[x^\mu + \varepsilon^\mu] - \tau[x^\mu]}{|\varepsilon^\mu|} = \frac{\delta\tau}{\delta x^\mu} \stackrel{!}{=} 0\,, \tag{18}$$

where a limit $|\varepsilon^\mu| \to 0$ is implied, and only the linear terms are considered.

The proper time functional for paths between A and B is given $\tau = \int_A^B \mathrm{d}\tau$. We can rewrite it as:

$$\tau = \int_A^B \mathrm{d}\tau\, \frac{\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = \int_A^B \mathrm{d}\tau\, \frac{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}{\mathrm{d}\tau^2}\,. \tag{19}$$

We now consider a perturbation $\varepsilon^\mu = \delta^\mu_1 \delta x$:

$$\tau_{AB}[x+\varepsilon] = \int_A^B \mathrm{d}\tau \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\mathrm{d}\delta x}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \right]. \tag{20}$$

We can discard a second order term $\left(\mathrm{d}\delta x/\mathrm{d}\tau\right)^2$, and subtract off $\tau_{AB}[x]$: we are left with

$$\delta\tau = -\frac{2}{c^2}\int_A^B \mathrm{d}\tau\, \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\delta x}{\mathrm{d}\tau} \tag{21}$$

Now, we integrate by parts, disregard the boundary terms since the endpoints of the path cannot be deformed, and get:

$$\frac{\delta \tau_{AB}}{\delta x} = +\frac{2}{c^2}\int_A^B \mathrm{d}\tau \, \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2}\,, \tag{22}$$

which proves the equivalence for this type of perturbation, the others are analogous.

The generalization of Newton's second law, which at low speeds is $F^i = ma^i$, can be similarly restated as $\delta S = 0$, for the action $S = \int \mathrm{d}\tau$.

1.7 Motion of light rays

For light we cannot compute u^μ with the previous definition, since its proper time is always zero.

Instead, we *define* u^μ to be a normalized null-like vector, such that $x^\mu = \lambda u^\mu$ for some λ.

We know from quantum mechanics that $E = \hbar\omega$, where $\hbar = h/(2\pi)$ and $\omega = 2\pi/T = 2\pi f$.

The momentum is proportional to the wavevector k^i: $p^i = \hbar k^i/c$. The relativistic generalization of this fact is

$$p^\mu = \left(\frac{\hbar\omega}{c}, \frac{\hbar k^i}{c}\right) = \frac{\hbar k^\mu}{c}\,. \tag{23}$$

Since the momentum of light must be null we have that necessarily $\omega = |k|$.

1.8 Doppler effect

We take a special case: radiation goes in the same direction as the observer. In the O frame we have $k^\mu = (\omega, \omega, 0, 0)$.

The observer, moving with velocity v, measures $k'^{,\mu}$. This can be easily computed with a Lorentz transformation: $k'^{,\mu} = \Lambda^\mu{}_\nu k^\nu$.

We are mostly interested in $k'^{,0} = \omega'$: it comes out to be $\omega' = \gamma\omega + (-\gamma\beta)\omega = (1 - v/c)\gamma\omega$.

Some notes: at slow speeds $\omega' \approx (1 - v/c)\omega$; we have $f' < f$ when source and observer are moving away from each other.