Quantum optics

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Course given for the SGSS by professors Paolo Villoresi and Giuseppe (Pino) Vallone.

The work of the team on quantum communication started in 2003, now there is a lot of interest on it.

The aim of this course is to discuss the *implementation* of the concepts in quantum information. The field is relatively young: anyone working on it needs to work with both theory and experiment.

Quantum information comes from merging information theory and quantum theory.

References:

1. "Introductory quantum optics", Gerry & Knight;
2. "Quantum metrology, imaging, and communication", Simon, Jaeger, . . .
3. Lebellac, "Quantum Physics"

add references from slides

Bell inequalities: 1964, no physical theory of local hidden variables can ever reproduce all the predictions of quantum mecnanics.

There are quantum experiments with relativistic distances and speeds.

In 2016 there was a Quantum Manifesto.

1 Meet the photon

A complete explanation of the photoelectric effect was given by Einstein. He pointed out the difference in approaches at his time between the atomic theory of matter and the continuous functions representing light in Maxwell's theory.

We follow Gerry & Knight for the quantization of the EM field.

We start from the vacuum Maxwell equations:

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{1}$$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \tag{2}$$

$$\nabla \cdot B = 0 \tag{3}$$

$$\nabla \cdot E = 0\,, \tag{4}$$

and seek trigonometric solutions in a box-shaped cavity: they look like

$$E_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0} \right)^{1/2} \sin(kz) q(t)\,, \tag{5}$$

where $k = \omega/c$. If we fix the boundary conditions of $E_x(0,t) = E_x(L,t) = 0$ we find $k = m\pi/L$.

Here V is the volume of our cavity. The magnetic field corresponding to this is

$$B_y(z,t) = \left(\frac{\mu_0 \epsilon_0}{k} \right) \left(\frac{2\omega^2}{V\epsilon_0} \right)^{1/2} \dot{q}(t) \cos(kz)\,, \tag{6}$$

where $\dot{q}$ corresponds precisely to the conjugate momentum to q: $\dot{q} = p$.

Then, the Hamiltonian can be shown to be

$$H = \frac{1}{2} \int \mathrm{d}V \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) = \frac{1}{2} \left(p^2 + \omega^2 q^2 \right). \tag{7}$$

In order to quantize the field, we use the correspondence principle to replace $p \to \hat{p}$ and $q \to \hat{q}$. These are Hermitian operators acting on the space $L^2(V)$ and thus correspond to observables, their commutator is $[\hat{q}, \hat{p}] = i\hbar$.

Now, we can introduce the creation and annnihilation operators:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q} + i\hat{p}) \tag{8}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q} - i\hat{p})\,, \tag{9}$$

and their product will give the number operator: $\hat{N} = \hat{a}^\dagger \hat{a}$. These are not Hermitian and thus not observable.

Then, we have

$$\hat{E}_x = \mathcal{E}_0 \left(\hat{a} + \hat{a}^\dagger \right) \sin(kz) \tag{10}$$

$$\hat{B}_y = -i\mathcal{B}_0 \left(\hat{a} - \hat{a}^\dagger \right) \cos(kz)\,, \tag{11}$$

for some normalization.

The Hamiltonian is given by $\hat{H} = \hat{N} + 1/2$.

The time-evolution in the Heisenberg picture of the creation and annihilation operators can be shown to be given by

$$\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = -i\omega\hat{a}\,, \tag{12}$$

so the evolution is given by circular motion.

If $|n\rangle$ is an eigenvector of $\hat{H}$ with energy E_n, then $\hat{a}^\dagger |k\rangle$ is an eigenvector with energy $E_n + \hbar\omega$. Then it is clear why this operator is called a creation operator: it *creates* a quantum of energy.

Similarly, $\hat{a}$ decreases the energy by $\hbar\omega$. The ground state is the one for which $\hat{a}\,|\psi\rangle = 0$, it is called $|0\rangle$ and has energy $\hbar\omega/2$. This is *zero-point energy*.

This ground state must exist since the eigenvalues of $\hat{N}$ must be positive.

The interpretation for this is then the fact that the excitation number gives us the number of photons in the cavity. We can do some calculations to show that the normalization we need in order to retain a normalized vector when applying the creation operator to the state $|N\rangle$ is $1/\sqrt{N+1}$, since

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n\rangle\ , \tag{13}$$

so we get a formula for a generic state starting from the ground state:

$$|n\rangle = \frac{(\hat{a}^\dagger)}{\sqrt{n!}}\,|0\rangle\ . \tag{14}$$

We can find eigenbases $|i\rangle$ from these operators, and write completeness relations:

$$\sum_i |i\rangle\langle i| = \mathbb{1}\,. \tag{15}$$

The only nonzero matrix elements of the creation and annihilation operators are the ones which are just off-diagonal by one in the basis of the Hamiltonian.

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The energy of the states $|n\rangle$ is well defined, but there are still issues to sort out: for instance, the expectation value of the electric field is zero *at each point*,

$$\langle n|\, E_x(z,t)\,|n\rangle \propto \langle n|\,\hat{a}\,|n\rangle + \langle n|\,\hat{a}^\dagger\,|n\rangle = 0\,. \tag{16}$$

However, the expectation value of the *square* of the electric field is nonzero:

$$\left\langle E_x^2(z,t) \right\rangle = 2\mathcal{E}_0^2 \sin^2(kz) \left(n + \frac{1}{2} \right). \tag{17}$$

This is in accordance with the uncertainty principle, since the operator $\hat{n}$ does not commute with the electric field operator $\hat{E}_x$. We can write the undetermination relation

$$\Delta n \Delta E \geq \frac{1}{2} \mathcal{E}_0 \left| \sin(kz) \right| \left| \left\langle \hat{a}^\dagger - \hat{a} \right\rangle \right|. \tag{18}$$

We expect to be able to find a notion of phase such that there is a number-phase uncertainty relation, similarly to the time-energy uncertainty relation.

The time evolution of the electric field operator is given by

$$E_x = \mathcal{E}_0 \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right) \sin(kz)\,, \tag{19}$$

and we define the quadrature operators:

$$X_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger \right) \qquad \text{and} \qquad X_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger \right). \tag{20}$$

We have effectively decomposed the electric field into two oscillating parts, out of phase with each other by 90°. We have the commutator $[\hat{X}_1, \hat{X}_2] = i/2$. Even for the vacuum state the fluctuations of these operators are nonzero.

We will now distinguish two different kinds of radiation: blackbody radiation and coherent (laser-like) radiation.

For the first case, we know that the distribution of the energy levels is given by the Boltzmann distribution,

$$P(n) = \frac{1}{Z} \exp\left(-\frac{E_n}{k_B T} \right), \tag{21}$$

where Z is a normalization factor. We now give an intuitive justification.

Let us suppose that we have a system of four particles, with three quanta of energy, which we write as ΔE. How can this energy be distributed?

0	ΔE	$2\Delta E$	$3\Delta E$	$4\Delta E$	Possibilities
3	0	0	1	0	4
2	1	1	0	0	12
1	3	0	0	0	4

So, for $0\Delta E$ we have $12 + 24 + 4 = 40$ total possibilities, for $1\Delta E$ we have $12 + 12 = 24$ total possibilities, for $2\Delta E$ we have 12 possibilities, for $3\Delta E$ we have 4 possibilities. The total is then 80.

This kiind of looks like an exponential decrease, I guess if we were to do a more precise calculation we would get an exponential exactly.

In a quantum setting, we will have a density matrix looking like

$$\rho = \frac{1}{\operatorname{tr} \exp\left(-\hat{H}/k_B T\right)} \exp\left(-\hat{H}/k_B T\right); \tag{22}$$

which gives a familiar result:

$$\rho = \sum_n \frac{\exp(-E_n/k_B T)}{Z} |n\rangle\langle n| \,. \tag{23}$$

The average number of photons can be found to be given by

$$\langle n \rangle = \frac{1}{\exp(\hbar\omega/k_B T) - 1} \,. \tag{24}$$

In the limits $\hbar\omega/k_B T$ going to either infinity or zero we get $\langle n \rangle \to \hbar\omega/k_B T$ or its inverse.

We can write the relation

$$\exp(-\hbar\omega/k_B T) = \frac{\overline{n}}{1+\overline{n}}, \tag{25}$$

where $\overline{n} = \langle n \rangle$. Then we will have

$$\rho = \frac{1}{1+\overline{n}} \sum_n \left(\frac{\overline{n}}{1+\overline{n}}\right)^n |n\rangle\langle n| \,. \tag{26}$$

It can be shown that

$$\left\langle \hat{n}^2 \right\rangle = \overline{n} + 2\overline{n}^2 , \tag{27}$$

which implies

$$\Delta n = \left(\overline{n} + \overline{n}^2\right)^{1/2}, \tag{28}$$

so we have $\Delta n \sim \overline{n} + \frac{1}{2}$ asymptotically. Therefore, there never is a well-defined number of photons in the box.

We can write an expression for the average energy density $U(\omega)$:

$$U(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_B T) - 1} \,. \tag{29}$$

The average energy of these photons is given by $\hbar\omega\overline{n}$.

From these expressions we can recover Wien's law and Stefan-Boltzmann's law.

How do we represent a plane wave in a QFT? If we want a nonzero electric field we need a superposition of number states differing by ± 1.

Another way to put it is: are there eigenstates $|\alpha\rangle$ of the annihilation operator $\hat{a}$? They will look like

$$|\alpha\rangle = C_0 \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ . \tag{30}$$

By normalization, $C_0 = \exp\left(-|\alpha|^2/2\right)$. This gives us a coherent state, which like we wanted has a nonzero expected electric field. It looks precisely like a plane wave:

$$\left\langle \hat{E}_x \right\rangle_\alpha = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left(\alpha \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right) - \alpha^* \exp\left(-i\vec{k}\cdot\vec{x} + i\omega t\right)\right) . \tag{31}$$

We find also the expectation value of the *square* of the electric field

$$\left\langle E_x^2 \right\rangle_\alpha = \frac{\hbar\omega}{2\epsilon_0 V}\left(1 + 4|\alpha|^2 \sin^2(\omega t - \vec{k}\cdot\vec{r} - \theta)\right) , \tag{32}$$

where $\alpha = |\alpha| \exp(i\theta)$.

This means that not even in the vacuum state we can have a zero expected electric field. The vectors $|\alpha\rangle$ are *over-complete*, since they are bidimensional while a one-dimensional continuous basis would be enough to span the Hilbert space.

The average of the number of photons $\hat{n}$ for an eigestate $|\alpha\rangle$ is quickly calculated to be $|\alpha|^2$: then we can see that $|\alpha|^2 = \overline{n}$.

So, with these states we have $\left\langle \hat{n}^2 \right\rangle_\alpha = \overline{n}^2 + \overline{n}$. Therefore $\Delta n = \sqrt{\overline{n}}$. The probability of detecting n photons is given by $\left|\langle n|\alpha\rangle\right|^2$:

$$P_\alpha(n) = \exp(-\overline{n})\frac{\overline{n}^n}{n!} , \tag{33}$$

a Poissonian distribution. If the number of photons gets large then the Poissonian approaches a Gaussian.