Zanardi's lectures

Notes taken by Jacopo Tissino

June 2019

Zanardi from the University of Southern California.

1 Distances over quantum state spaces

What we will talk about What is a metric? "Trace norm distance"

Information theore tical protocol

Battacharya distance; its infinitesimal version: the Fischer metric, the Fubini-Study metric: many body physics, Quantum Phase Transitions.

At zero temperature, we only have the ground state, there is no entropy, so how do transitions work?

The statements preceded by the word "claim" are left as exercises.

1.1 Quantum theory recap

We have *states* and *observables*. We call the system S, and its associated Hilbert space $\mathcal{H}$. (No real quantum system is going to be truly isolated).

A state ρ is a density matrix, an observable is a Hermitian operator over $\mathcal{H}$.

$$S(\mathcal{H}) = \left\{\rho \middle| \rho \geq 0, \operatorname{Tr}\rho = 1\right\} \tag{1}$$

Then $\forall \phi \in \mathcal{H} : \left\langle \phi \middle| \rho \middle| \phi \right\rangle \geq 0$, $\rho = \rho^\dagger$.

$\sigma(\rho) =$ spectrum of $\rho = \left\{p_i\right\}_i$. $d = \dim(\mathcal{H})$.

$p_i \geq 0$, $\sum_i p_i = 1$.

So quantum theory is just noncommutative probability theory.

Single qubit .

$\mathcal{H} \sim \mathbb{C}^2$, $\dim\mathcal{H} = 2$, $\rho = \frac{1}{2}\left(\mathbb{1} + \vec{\lambda} \cdot \vec{\sigma}\right)$.

$\vec{\lambda}$ is the Bloch vector, $\vec{\sigma}$ are the Pauli matrices. If $\left|\vec{\lambda}\right| = 1$, we have a pure state.

Exercise Prove that ρ being a density matrix is equivalent to $|\lambda| \leq 1$, and that $|\lambda| = 1 \iff \rho^2 = \rho \iff \rho = |\psi\rangle\langle\psi|$.

Creating convex combinations of states should always be allowed, and we see this in the fact that the space of states is convex.

States are usually decomposable as probabilistic mixtures, which are convex combinations in "Bloch space" (or its generalizations).

1.2 Telling states apart

We are given two states $\rho_{1,2}$: what is the measurement which maximizes the probability we will be able to tell one from the other?

Probability vectors: $\vec{p} = (p_i)$, $\vec{q} = (q_i)$. We can use the "ℓ_1" metric:

$$d(p,q) = \sum_i |p_i - q_i| \tag{2}$$

Claim: this is a distance.

Can we do the same for quantum states? In general, $[\rho_1, \rho_2] \neq 0$, we do not have a common eigenbasis. Could we use the Frobenius metric? Eeh, not really.

We have a map from observables to numbers: $A \to \operatorname{Tr}(\rho A) = \langle A \rangle_\rho$.

We can do $\sup \left| \langle A \rangle_{\rho_1} - \langle A \rangle_{\rho_2} \right|$ (Over $\|A\| = 1$).

$$\|A\| = \sup_{\psi \neq 0} \frac{\|A\psi\|}{\psi} \tag{3}$$

So our distance is

$$d = \sup_{\|A\|=1} \left| \operatorname{Tr}\left[A(\rho_1 - \rho_2)\right] \right| \tag{4}$$

We know that

$$\begin{aligned}
\left|\operatorname{Tr}(AB)\right| &= \sum_i b_i \langle i|A|i\rangle && (5)\\
&\leq \sum_i |b_i| \langle i|A|i\rangle && (6)\\
&\leq \|A\| \sum_i |b_i| && (7)\\
&\leq \|A\| \operatorname{Tr}|B| && (8)
\end{aligned}$$

Where $B = \sum_i b_i |i\rangle\langle i|$), and we call $\|B\|_1 = \operatorname{Tr}|B|$, where the modulus of the operator can be thought of eigenvalue-wise (diagonalizing the operator, and then flipping the sign of all the negative eigenvalues).

So

$$d \leq \|A\| \left\|\rho_1 - \rho_2\right\|_1 = \left\|\rho_1 - \rho_2\right\|_1 \tag{9}$$

Claim: this in an equality (there *always* exists an A to do the job).

Do we get back the classical case if the matrices commute? Claim: yes.

We call $D(\rho_1, \rho_2) = \frac{1}{2}\left\|\rho_1 - \rho_2\right\|_1$. (since the d we used before is upper-bounded by 2, by the triangular inequality).

"the duals of self-adjoint operators are traceles"?

Measurements Von Neumann orthogonal measurement we know about.

Generalized measurement: we have an ancillary system, mearure this system and then trace over it. This is not described by an orthogonal projection:

Positive Operator-Valued Measurement We have finitely many $\{E_i\}_i$, $E_i \geq 0$, $\sum_i E_i = \mathbb{1}$.

$\rho \to p_i = \operatorname{Tr}(\rho E_i)$.

2-element POVM: $E_{1,2} \geq 0$, we have our states $\rho_{1,2}$.

Say we get the states with 50% probability each, and we wish to distinguish them:

$$P(\text{success}) = \frac{1}{2}\left[\operatorname{Tr}(E_1\rho_1) + \operatorname{Tr}(E_2\rho_2)\right] \tag{10}$$

$$P(\text{error}) = \frac{1}{2}\left[\operatorname{Tr}(E_1\rho_2) + \operatorname{Tr}(E_2\rho_1)\right] \tag{11}$$

We want to maximize $P(\text{success})$. We can rewrite it as:

$$\begin{aligned} P(\text{success}) &= \frac{1}{2}\left[\operatorname{Tr}(E_1\rho_1) + \operatorname{Tr}\left((\mathbb{1} - E_1)\rho_2\right)\right] &(12)\\ &= \frac{1}{2}\left[1 + \operatorname{Tr}\left(E_1(\rho_1 - \rho_2)\right)\right] &(13)\end{aligned}$$

to maximize over E_1. The optimum (Claim) is

$$P(\text{success}) = \frac{1}{2}\left[1 + \frac{1}{2}\left\|\rho_1 + \rho_2\right\|_1\right] \tag{14}$$

Hellstrom optimal measurement? This is 1 if they are maximally different, 1/2 if they are indistinguishable.

E_1 should be the projection over the positive eigenvalues of the difference between the matrices.

Bhattacharyya distance We have two probability vectors $\vec{p} = (p_i)_i$, $\vec{q} = (q_i)_i$. Normalized in the euclidean metric, if we take the square root component by component: $V_p = \left(\sqrt{p_i}\right)_i$.

$$d_B(\vec{p}, \vec{q}) = \cos^{-1}(\vec{V}_p, \vec{V}_q) \tag{15}$$

Claim: this is a distance.

$$d_B(\vec{p}, \vec{q}) = \cos^{-1}\left(\sum_i \sqrt{p_i q_i}\right) \tag{16}$$

Quantize it! Let us focus on the pure state case: we have a POVM $\mathbb{E} = \{E_i\}$, and two probability distributions $\rho = |\phi\rangle\langle\phi|$, $\sigma = |\psi\rangle\langle\psi|$

$$P_\phi(i) \stackrel{\text{def}}{=} \operatorname{Tr}\big(E_i \rho\big) \;=\; \langle E_i|\phi|E_i\rangle \tag{17}$$

and similarly for ψ.
The Bhattacharyya distance is

$$d_B(\phi, \psi) = \sup_{\mathbb{E}} d_B\left(\vec{P}_\phi, \vec{P}_\psi\right) \tag{18}$$

Theorem:

$$d_B(\phi, \psi) = \cos^{-1}\left|\langle\phi|\psi\rangle\right| \tag{19}$$

this is the Fubini-Study metric over a projective Hilbert space.
We can generalize this to differential geometry.

In this particular case this can be expressed as $\left|\langle\phi|\psi\rangle\right| = \sqrt{\operatorname{Tr}\left(\rho\sigma\right)}$ but the result does not generalize.

Questions How do we implement the POVM we described last time? That is left to the experimentalists.

Proof of the theorem This is useful because it shows us technicques, tools.
We have these E_i from the POVM.
We wish to prove

$$\cos^{-1}\left|\langle\phi|\psi\rangle\right| = \cos^{-1}\left(\sqrt{p_i^{\mathbb{E}}(\phi)}\sqrt{p_i(\psi)^{\mathbb{E}}}\right) \tag{20}$$

$$\left|\langle\phi|\psi\rangle\right| = \left|\sum_i \langle\phi| E_i |\psi\rangle\right| \tag{21}$$

$$\leq \sum_i \left|\langle\phi| \sqrt{E_i}\sqrt{E_i}\,|\psi\rangle\right| \tag{22}$$

$$= \sum_i \left|\langle\hat{\phi}_i|\hat{\psi}_i\rangle\right| \tag{23}$$

$$\leq \sum_i \left\|\sqrt{E_i}\,|\phi\rangle\right\|\left\|\sqrt{E_i}\,|\psi\rangle\right\| \tag{24}$$

$$= \sum_i \sqrt{\langle\phi|E_i|\phi\rangle}\sqrt{\langle\psi|E_i|\psi\rangle} \tag{25}$$

$$= \sum_i \sqrt{p_i^{\mathbb{E}}(\phi)}\sqrt{p_i(\psi)^{\mathbb{E}}} \tag{26}$$

so we just apply the $\cos^{-1}$ to both sides.

$$\mathbb{E} = \left\{ |\phi\rangle\langle\phi|\,|, \left\{ |\phi_i\rangle\langle\phi_i| \right\} \text{span the } |\phi\rangle\langle\phi|^{\perp} \right\} \tag{27}$$

so the first term just gives us the upper bound, the other terms in the sum are 0: the inequality is saturated. We can just measure the projector associated with the state we want to know about.

Infinitesimal distance What is $\mathrm{d}_B\left(\vec{p}, \vec{p} + \mathrm{d}\vec{p}\right)$?

$$\cos^{-1}\left\{\sum_i \sqrt{p_i(p_i + \mathrm{d}p_i)}\right\} = \cos^{-1}\left\{\sum_i p_i \sqrt{1 + \frac{\mathrm{d}p_i}{p_i}}\right\} \tag{28}$$

$$\sim \cos^{-1}\left(\sum_i p_i \left(1 + \underbrace{\cancelto{0}{\frac{1}{2}\frac{\mathrm{d}p_i}{p_i}}}_{\text{The probabilities are normalized}} - \frac{1}{8}\left(\frac{\mathrm{d}p_i}{p_i}\right)^2\right)\right) \tag{29}$$

but the term $\sum_i \mathrm{d}p_i$ vanishes, so we have

$$\cos \mathrm{d}s = 1 - \frac{1}{8}\frac{\mathrm{d}p_i^2}{p_i} \tag{30}$$

but $\cos \mathrm{d}s \sim 1 - 1/2\, \mathrm{d}s_B^2$, so

$$\mathrm{d}s_B^2 = \frac{1}{4}\sum_i \frac{\left(\mathrm{d}p_i\right)^2}{p_i} \tag{31}$$

This is the *Fisher metric*. More differential-geometry-like: take some $\lambda \in \mathcal{M} = \left\{\text{manifold of control parameters of dimension N}\right\}$, such that $p_i = p_i(\lambda)$ and $\mathrm{d}p_i = \sum_\mu (\partial_\mu p_i)\,\mathrm{d}\lambda_\mu$.

Then

$$g_{\mu\nu} = \frac{1}{4}\sum_i \frac{(\partial_\mu p_i)(\partial_\nu p_i)}{p_i} \tag{32}$$

$$\mathrm{d}s_B^2 = g_{\mu\nu}\,\mathrm{d}\lambda_\mu\,\mathrm{d}\lambda_\nu \tag{33}$$

2 Parameter estimation

Take a $P_\theta(x)$, where x is a random variable, $\langle\Theta\rangle = \theta = \int p_\theta(x)\Theta(x)\,\mathrm{d}x$. θ is our parameter.

We prove a super-famous bound. Differentiate the previous equation wrt θ.

$$1 = \int p'_\theta(x)\Theta(x)\,\mathrm{d}x \tag{34}$$

$$= \left\langle \frac{p'_\theta}{p_\theta} \middle| \Theta \right\rangle_p \tag{35}$$

Claim: the notation $\langle f|g\rangle = \int p_\theta f g\,\mathrm{d}x$ defines a scalar product.

We can always subtract θ in the integrand (?).

$$1 \leq \left\| \frac{p'_\theta}{p_\theta} \right\|_p^2 \|\Theta\|_p^2 \tag{36}$$

$$= \left(\int p_\theta \frac{(p'_\theta)^2}{p_\theta^2}\,\mathrm{d}x\right)\left(\int p_\theta(\Theta(x)-\theta)^2\,\mathrm{d}x\right) \tag{37}$$

$$= F\mathrm{var}(\Theta) \tag{38}$$

Where F is just the Fischer metric. (To check: $\bar{\Theta} = \Theta - \theta$)

But this means $\mathrm{var}(\Theta) \geq F^{-1}$: this is the *Cramer-Rao* inequality.

2.1 Quantize it!

Unbiased estimator $\mathrm{Tr}\left(\rho_\theta' \hat{\Theta}\right) = 1$.

We can do $\mathrm{Tr}\left(\rho_\theta \rho_\theta^{-1} \rho_\theta' \hat{\Theta}\right) = 1$

But this is noncommutative! We can do

$$L_\rho(x) = \frac{1}{2}(\rho X + X\rho) = \rho' \tag{39}$$

This is the Symmetric Logarithmic Derivative, SLD. (recall the logarithmic derivative $\rho'/\rho = \mathrm{d}\log\rho/\mathrm{d}x$).

Now

$$\frac{1}{2}\mathrm{Tr}\left[(\rho x + x\rho)\bar{\Theta}\right] = \mathrm{Re}\,\mathrm{Tr}(\rho X \Theta) \tag{40}$$

and we can take this equation in absolute value. Then,

$$1 \leq \left|\mathrm{Tr}(\rho X \bar{\Theta})\right|^2 = \left|\left\langle X \middle| \bar{\Theta}\right\rangle\right|_\rho^2 \tag{41}$$

And like before

$$1 \leq \|X\|_\rho^2 \|\bar{\Theta}\|_\rho^2 = \mathrm{Tr}\left(\rho X^2\right) Tr(\rho(\Theta - \theta)^2) \tag{42}$$

so then $\mathrm{var}(\Theta) \geq 1/F_Q$.

Claim: take $\rho = \sum_i |i\rangle\langle i|$, then $X_{ij} = 2\langle i|\rho'|j\rangle / \left(p_i + p_j\right)$.

So we can compute

$$F_Q = \mathrm{Tr}\left(\rho X^2\right) = \sum_i p_i \langle i|X^2|i\rangle \tag{43}$$

$$= \sum_{ij} p_i \langle i| X |j\rangle\langle j| X |i\rangle \tag{44}$$

$$= \sum_{ij} p_i \frac{2}{p_i + p_j} \langle i|\rho'|j\rangle\langle j|\rho'|i\rangle \frac{2}{p_i + p_j} \tag{45}$$

$$= 2\sum_{ij} \frac{\left|\langle i|\rho'|j\rangle\right|^2}{p_i + p_j} \tag{46}$$

This is Quantum Fischer. It really is the result of *classical* optimization.

The denominator diverges! but the numerator goes to zero quadratically (?).

Fischer metric for pure states Take $\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ gound-eigenstate of a many-body system.

$$F_Q^{\text{pure}} \sim \left\langle \psi'_\theta \middle| \psi'_\theta \right\rangle - \left| \left\langle \psi'_\theta \middle| \psi_\theta \right\rangle \right|^2 \tag{47}$$

where $\psi' = \partial_\theta \psi$. This is a way to solve the Lyapunov equation $L_\theta(\rho) = \rho'$.

$\Pi = |\psi\rangle\langle\psi|$. Then $\Pi^2 = \Pi$, we differentiate it. So $\Pi'\Pi + \Pi\Pi' = \Pi'$: in the pure state case, the solution is then just $\Pi' = X$.

Claim: if we do $\operatorname{Tr}\left(\rho F_Q^2\right)$...

First try The derivative of ρ is $\rho' = |\psi\rangle\langle\psi'| - |\psi'\rangle\langle\psi|$.

Then computing $\operatorname{Tr}\left(\rho\rho'^{\dagger}\rho'\right) = -\operatorname{Tr}\left(\rho\rho'\rho'^{\dagger}\right)$ we get

$$\operatorname{Tr}\left(\rho\rho'^{\dagger}\rho'\right) = \left\langle \psi' \middle| \psi' \right\rangle - \left| \left\langle \psi' \middle| \psi \right\rangle \right|^2 \tag{48}$$