General Relativity notes

Jacopo Tissino, Giorgio Mentasti

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Marco Peloso, marco.peloso@pd.infn.it

1 Special relativity

Definition 1.1. *An inertial frame is one in which Newton's laws hold: a free body moves with acceleration* $a^i = 0$*.*

Newton's first law establishes the *existence* of inertial frames.

Proposition 1.1. *The frames* O *and* O' *are both inertial frames iff* O' *moves with constant velocity wrt* O*.*

Proposition 1.2. *Coordinate transformations between inertial frames are Lorentz boosts, which in some coordinate frame can be written as*

$$t' = \gamma_v \left(t - \frac{vx}{c^2} \right) \tag{1a}$$

$$x' = \gamma_v (x - vt) \tag{1b}$$

$$y' = y \tag{1c}$$

$$z' = z\,, \tag{1d}$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$*.*

If $v \ll c$, so $v/c \sim 0$, they simplify to the identity for t, y, z and $x' = x - vt$: these are Galilean transformations.

If we have two events, x^μ and y^μ, they occur with some time and space separation $\Delta x^\mu = x^\mu - y^\mu$. We can compute $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$, where

$$\eta_{\mu\nu} = \operatorname{diag}\left(-c^2, 1, 1, 1 \right). \tag{2}$$

Proposition 1.3. *Under Lorentz transformations Δs^2 is invariant.*

We can classify separations between events as

- time-like when $\Delta s^2 < 0$;
- null-like when $\Delta s^2 = 0$;
- space-like when $\Delta s^2 > 0$.

We can draw spacetime diagrams. A light cone is the set of points which are null-like separated from a select point. Things can be only causally related to events inside the light-cone (with $\Delta s^2 \geq 0$).

1.1 Time dilation

Take two events which occur at the same location for O'. In the primed frame they will have coordinates $x^\mu = (t_0, x_0)$ and $y^\mu = (t_1, x_0)$.

Definition 1.2. *The* proper time *between these two events is* $t_1 - t_0 \stackrel{def}{=} \Delta\tau$.

We now see that $\Delta s'^2 = -c^2 \Delta\tau^2$. Then, any other observer will see the same $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = \Delta s'^2$.

This directly implies that $\Delta\tau \leq \Delta t$ for any observer, since $\Delta\tau^2 = \Delta t^2 - \Delta x^2/c^2$. This effect is called *time dilation*.

By how much exacly is time dilated? Of course $\Delta x = v\Delta t$, therefore $\Delta t = \gamma_v \Delta\tau$.
-> Muon problem.

Inverse Lorentz transformation have the same expression, but with $v \to -v$. This can be proved both mathematically by solving the equations and phisically by reasoning about their meaning. There is no preferential inertial frame.

A Lorentz transformation can be written in matrix form in the (ct, x) plane as:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \tag{3}$$

since there is an angle θ such that $\gamma = \cosh\theta$ and $\gamma\beta = \sinh\theta$: the angle θ will be $\theta = \tanh^{-1}(v/c)$. This is true because $\gamma^2 - \beta^2\gamma^2 = 1$.

After a boost the ct' and x' axes are respectively the lines $ct = x/\beta$ and $ct = \beta x$.

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Last lecture we saw the fact that the ct' and x' axes are rotated by equal angles from the ct and x axes towards the $ct = x$ axis.

1.2 Relativity of simultaneity

Consider two events which are simultaneous in the O' frame. Their times in this frame are $t'_A = t'_B$.

In the O frame, instead, we have

$$ct_{A,B} = \frac{v}{c}x_{A,B} + \underbrace{\sqrt{1-\frac{v^2}{c^2}}ct'_{A,B}}_{\text{a constant}}, \tag{4}$$

so the events are not simultaneous in the O frame.

1.3 Length contraction

If in the O frame, A occurs at $t, x = 0$ while B occurs at $t = 0$, $x = L$, then L is the measured length of their spatial interval by O. We assume that this is the frame in which the object is moving, and we transform into a frame in which it is stationary: O'.

In the primed frame their coordinates will be:

$$x'_A = \gamma_v\left(x_A - \frac{v}{c}ct_A\right) \tag{5a}$$

$$x'_B = \gamma_v\left(x_B - \frac{v}{c}ct_B\right), \tag{5b}$$

therefore $x'_B - x'_A = \gamma_v(x_B - x_A)$: the length is contracted in the O frame, since $\gamma \geq 1$.

1.4 Addition of velocities

Two observers see an object moving with $v' = \mathrm{d}x'/\mathrm{d}t'$ and $v = \mathrm{d}x/\mathrm{d}t$ respectively. Their relative velocity is u. Differentiating we get:

$$v' = \frac{\gamma(\mathrm{d}x - v\,\mathrm{d}t)}{\gamma\left(\mathrm{d}t - \frac{u\mathrm{d}x}{c^2}\right)} = \frac{v-u}{1-\frac{uv}{c^2}}. \tag{6}$$

Two interesting limits of this formula are: $v' = v - u$ if $u \ll c$ or $v \ll c$; and $v' = c$ if $v = c$ for whatever u.

1.5 Tensor notation

The position four-vector is $x^\mu = (ct, x, y, z)$. The Euclidean scalar product is given by $x \cdot y = \delta_{\mu\nu} x^\mu x^\nu$. If we substitute the identity $\delta_{\mu\nu}$ with another metric we can find a more general metric space.

The Minkowski metric is $\eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, 1)$. The separation 4-vector is $\mathrm{d}x^\mu = (c\,\mathrm{d}t\,, \mathrm{d}x\,, \mathrm{d}y\,, \mathrm{d}z)$.

Using Einstein summation notation, we can write the spacetime interval as $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$.

Specifically for the Minkowski metric we have the relation $\eta_{\mu\nu} = \eta^{\mu\nu}$: it is its own inverse. For a general metric $g_{\mu\nu}$ this will not hold.

How do we express the Lorentz boosts? They preserve $\mathrm{d}s^2$, therefore they look like $x'^{\,\mu} = \Lambda^\mu{}_\nu x^\nu$, with the $(1, 1)$ tensors $\Lambda_\mu{}^\nu$ satisfying $\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$. This is called the *pseudo-orthogonality* relation.

The metric allows us to raise and lower indices. Raising an index in the pseudo-orthogonality relation gives us: $\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta \eta^{\beta\sigma} = \delta_\alpha{}^\sigma$, therefore $\eta_{\mu\nu}\Lambda^\nu{}_\beta \eta^{\beta\sigma}$ is the inverse of a Lorentz transformation.

Four-vectors can also have their indices down, and they will transform according to the inverse of Lorentz transformations:

$$
\begin{aligned}
(\eta_{\alpha\mu} x^\mu)' &= \eta_{\alpha\mu} \Lambda^\mu{}_\nu x^\nu && \text{(7a)} \\
&= \Lambda_{\alpha\sigma} \delta^\sigma{}_\nu x^\nu && \text{(7b)} \\
&= \Lambda_{\alpha\sigma} \eta^{\sigma\beta} \eta_{\beta\nu} x^\nu && \text{(7c)} \\
&= \Lambda_\alpha{}^\beta x_\beta\,. && \text{(7d)}
\end{aligned}
$$

We will write our laws as tensorial equations, which are covariant.

By pseudo-orthogonality, the scalar product $A_\mu B^\mu$ is a covariant (that is, invariant) scalar. Of course it is equal to $A^\mu B_\mu$.

Definition 1.3 (Tensor). *A* (p, q) tensor *is an object* $M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p}$ *with many components indexed by several indices, which transforms as:*

$$
M^{\nu_1 \dots \nu_q}_{\mu_1 \dots \mu_p} \to \Lambda_{\mu_1}{}^{\mu'_1} \dots \Lambda_{\mu_p}{}^{\mu'_p} \Lambda^{\nu_1}{}_{\nu'_1} \dots \Lambda^{\nu_q}{}_{\nu'_q} M^{\nu'_1 \dots \nu'_q}_{\mu'_1 \dots \mu'_p} \tag{8}
$$

under Lorentz transformations $\Lambda_\mu{}^\nu$.