

# AstroStatistics and Cosmology Homework

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Exercises 1–3 are in a Jupyter notebook in this same folder.

## 1 Exercise 4

After being given a probability distribution  $\mathbb{P}(x)$ , we define the *characteristic function*  $\phi$  as its Fourier transform, which can also be expressed as the expectation value of  $\exp(-i\vec{k} \cdot \vec{x})$ :

$$\phi(\vec{k}) = \int d^n x \exp(-i\vec{k} \cdot \vec{x}) \mathbb{P}(x) = \mathbb{E} \left[ \exp(-i\vec{k} \cdot \vec{x}) \right]. \quad (1.1)$$

**Claim 1.1.** *A multivariate normal distribution*

$$\mathcal{N}(\vec{x}|\vec{\mu}, C) = \frac{1}{(2\pi)^{n/2} \sqrt{\det C}} \exp\left(-\frac{1}{2} \vec{y}^\top C^{-1} \vec{y}\right) \Big|_{\vec{y}=\vec{x}-\vec{\mu}}, \quad (1.2)$$

has a characteristic function equal to

$$\phi(\vec{k}) = \exp\left(-i\vec{\mu} \cdot \vec{k} - \frac{1}{2} \vec{k}^\top C \vec{k}\right). \quad (1.3)$$

*Proof: completing the square.* The integral we need to compute is given, absorbing the normalization into a factor  $N$ , by

$$\phi(\vec{k}) = N \int d^n x \exp\left(-i\vec{k} \cdot \vec{x} - \frac{1}{2} \vec{y}^\top C^{-1} \vec{y}\right) \Big|_{\vec{y}=\vec{x}-\vec{\mu}}. \quad (1.4)$$

The only integrals we really know how to do are Gaussian ones, so we want to rewrite the argument of the exponential so that it is a quadratic form. The manipulation goes as

follows, considering the opposite of the argument the exponential in order to have less minus signs and defining the symmetric matrix  $V = C^{-1}$ :

$$i\vec{k} \cdot \vec{x} + \frac{1}{2}\vec{y}^\top V\vec{y} = i\vec{k} \cdot \vec{x} + \frac{1}{2}\vec{x}^\top V\vec{x} - \vec{x}^\top V\vec{\mu} + \frac{1}{2}\vec{\mu}^\top V\vec{\mu} \quad (1.5)$$

$$= \frac{1}{2}\vec{x}^\top V\vec{x} + \vec{x}^\top (i\vec{k} - V\vec{\mu}) + \frac{1}{2}\vec{\mu}^\top V\vec{\mu} \quad (1.6)$$

$$= \underbrace{\frac{1}{2}(\vec{x} + V^{-1}(i\vec{k} - V\vec{\mu}))^\top V(\vec{x} + V^{-1}(i\vec{k} - V\vec{\mu}))}_{\textcircled{1}} + \underbrace{-\frac{1}{2}(i\vec{k} - V\vec{\mu})^\top V^{-1}(i\vec{k} - V\vec{\mu}) + \frac{1}{2}\vec{\mu}^\top V\vec{\mu}}_{\textcircled{2}}, \quad (1.7)$$

which we can now integrate, since it is now a quadratic form in terms of a shifted variable,  $\vec{x} + \vec{p}$ , where the constant (with respect to  $\vec{x}$ ) vector  $\vec{p}$  is given by  $V^{-1}(i\vec{k} - V\vec{\mu})$ .<sup>1</sup>

Now, shifting the integral from one in  $d^n x$  to one in  $d^n(x + p)$  does not change the measure, since the Jacobian of a shift is the identity. Then, we have

$$\phi(\vec{k}) = N \int d^n(x + p) \exp(-\textcircled{1} - \textcircled{2}) \quad (1.12)$$

$$= N \sqrt{\frac{(2\pi)^n}{\det V}} \exp(-\textcircled{2}) \quad (1.13)$$

$$= \underbrace{\frac{1}{\sqrt{\det V \det C}}}_{=1} \exp(-\textcircled{2}), \quad (1.14)$$

since the determinant of the inverse is the inverse of the determinant.

Now, we only need to simplify  $\textcircled{2}$ :

$$\textcircled{2} = -\frac{1}{2} \left[ -\vec{k}^\top V^{-1}\vec{k} - 2i\vec{\mu}^\top V V^{-1}\vec{k} + \vec{\mu}^\top V V^{-1} V\vec{\mu} \right] + \frac{1}{2}\vec{\mu}^\top V\vec{\mu} \quad (1.15)$$

$$= \frac{1}{2}\vec{k}^\top C\vec{k} + i\vec{\mu}^\top \vec{k}, \quad (1.16)$$

inserting which into the exponent yields the desired result.  $\square$

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<sup>1</sup> In the last step we applied the matrix square completion formula: for a symmetric matrix  $A$  and vectors  $\vec{x}$ ,  $\vec{b}$  we have

$$\frac{1}{2}(\vec{x} + A^{-1}\vec{b})^\top A(\vec{x} + A^{-1}\vec{b}) - \frac{1}{2}\vec{b}^\top A^{-1}\vec{b} = \quad (1.8)$$

$$= \frac{1}{2} \left[ \vec{x}^\top A\vec{x} + \vec{x}^\top A A^{-1}\vec{b} + (A^{-1}\vec{b})^\top A\vec{x} + (A^{-1}\vec{b})^\top A A^{-1}\vec{b} - \vec{b}^\top A^{-1}\vec{b} \right] \quad (1.9)$$

$$= \frac{1}{2} \left[ \vec{x}^\top A\vec{x} + \vec{x}^\top \vec{b} + \vec{b}^\top (A^{-1})^\top A\vec{x} + \vec{b}^\top (A^{-1})^\top \vec{b} - \vec{b}^\top A^{-1}\vec{b} \right] \quad (1.10)$$

$$= \frac{1}{2}\vec{x}^\top A\vec{x} + \vec{b}^\top \vec{x}, \quad (1.11)$$

which we used with  $\vec{b} = i\vec{k} - V\vec{\mu}$ .

*Proof: by diagonalization.* We now follow a different approach: the covariance matrix  $C$  is symmetric, so we will always be able to find an orthogonal matrix  $O$  (satisfying  $O^\top = O^{-1}$ ) such that  $C = O^\top D O$ , where  $D$  is diagonal. We will then also have  $V = C^{-1} = O^\top D^{-1} O$ . Let us denote the eigenvalues of  $D$  as  $\lambda_i$ , and the eigenvalues of  $D^{-1}$  as  $d_i = \lambda_i^{-1}$ .

Defining  $\vec{z} = O\vec{x}$ ,  $\vec{m} = O\vec{\mu}$ ,  $\vec{u} = O\vec{k}$  the negative of the argument of the integral becomes:

$$i\vec{k} \cdot \vec{x} + \frac{1}{2}(\vec{x} - \vec{\mu})^\top C^{-1}(\vec{x} - \vec{\mu}) = i\vec{u} \cdot \vec{z} + \frac{1}{2}(\vec{z} - \vec{m})^\top D^{-1}(\vec{z} - \vec{m}) \quad (1.17)$$

$$= i\vec{u} \cdot \vec{z} + \frac{1}{2} \sum_i d_i (z_i - m_i)^2 \quad (1.18)$$

$$= \sum_i \left[ iu_i z_i + \frac{d_i}{2} (z_i^2 + m_i^2 - 2m_i z_i) \right] \quad (1.19)$$

$$= \sum_i \left[ z_i^2 \frac{d_i}{2} + z_i(iu_i - m_i d_i) + \frac{d_i}{2} m_i^2 \right]. \quad (1.20)$$

With this, and since by  $\det O = 1$  we have  $d^n z = d^n x$ , we can decompose our Gaussian integral into a product of Gaussian integrals:

$$\phi(\vec{k}) = N \int d^n x \exp \left( -i\vec{k} \cdot \vec{x} - \frac{1}{2}(\vec{x} - \vec{\mu})^\top C^{-1}(\vec{x} - \vec{\mu}) \right) \quad (1.21)$$

$$= N \int d^n z \exp \left( - \sum_i \left[ z_i^2 \frac{d_i}{2} + z_i(iu_i - m_i d_i) + \frac{d_i}{2} m_i^2 \right] \right) \quad (1.22)$$

$$= N \prod_i \int dz_i \exp \left( -z_i^2 \frac{d_i}{2} - z_i(iu_i - m_i d_i) - \frac{d_i}{2} m_i^2 \right) \quad (1.23)$$

$$= N \prod_i \sqrt{\frac{2\pi}{d_i}} \exp \left( \frac{(iu_i - m_i d_i)^2}{2d_i} - \frac{d_i m_i^2}{2} \right) \quad (1.24)$$

$$= \frac{1}{\sqrt{\det C \det V}} \prod_i \exp \left( \frac{-u_i^2 + m_i^2 d_i^2 - 2iu_i m_i d_i}{2d_i} - \frac{d_i m_i^2}{2} \right) \quad (1.25)$$

$$= \exp \left( \sum_i \left[ -\frac{u_i^2}{2d_i} - iu_i m_i \right] \right) \quad (1.26)$$

$$= \exp \left( -\frac{1}{2} \vec{u}^\top C \vec{u} - i\vec{u} \cdot \vec{m} \right) \quad (1.27)$$

$$= \exp \left( -\frac{1}{2} \vec{k}^\top C \vec{k} - i\vec{k} \cdot \vec{\mu} \right), \quad (1.28)$$

where we have used the expression for the single-variable Gaussian integral:

$$\int dz \exp(-az^2 + bz + c) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right), \quad (1.29)$$

which comes from the one-variable completion of the square:

$$-az^2 + bz + c = -a \left( z - \frac{b}{2a} \right)^2 + \frac{b^2}{4a} + c. \quad (1.30)$$

Also, we used the fact that orthogonal transformation do not change fully-contracted objects, such as scalar products or bilinear forms.  $\square$

## 2 Exercise 5

We can calculate the moments of a distribution through its characteristic function:

$$\mathbb{E}\left[x_\alpha^{n_\alpha} \dots x_\beta^{n_\beta}\right] = \frac{\partial^{n_\alpha \dots n_\beta} \phi(\vec{k})}{\partial(-ik_\alpha)^{n_\alpha} \dots \partial(-ik_\beta)^{n_\beta}} \Big|_{\vec{k}=0}. \quad (2.1)$$

In the multivariate Gaussian case we can then calculate the mean (component by component) as

$$\mathbb{E}(x_\alpha) = \frac{\partial \phi(\vec{k})}{\partial(-ik_\alpha)} \Big|_{\vec{k}=0} \quad (2.2)$$

$$= \frac{\partial}{\partial(-ik_\alpha)} \Big|_{\vec{k}=0} \exp\left(-\frac{1}{2} \vec{k}^\top C \vec{k} - i \vec{k} \cdot \vec{\mu}\right) \quad (2.3)$$

$$= \left[-i \sum_\beta k_\beta C_{\beta\alpha} + \mu_\alpha\right] \exp\left(-\frac{1}{2} \vec{k}^\top C \vec{k} - i \vec{k} \cdot \vec{\mu}\right) \Big|_{\vec{k}=0} \quad (2.4)$$

$$= \mu_\alpha, \quad (2.5)$$

where we used the fact that the differentiation of a symmetric bilinear form is as follows:

$$\frac{\partial}{\partial k_\alpha} \left( \sum_{\beta\gamma} k_\beta k_\gamma C_{\beta\gamma} \right) = 2 \sum_{\beta\gamma} \delta_{\beta\alpha} k_\gamma C_{\beta\gamma} = 2 \sum_\gamma k_\gamma C_{\alpha\gamma}. \quad (2.6)$$

The covariance matrix can be computed by linearity as

$$\tilde{C}_{\alpha\beta} = \mathbb{E}\left[(x_\alpha - \mathbb{E}(x_\alpha))(x_\beta - \mathbb{E}(x_\beta))\right] = \mathbb{E}[x_\alpha x_\beta] - \mu_\alpha \mu_\beta, \quad (2.7)$$

the first term of which reads as follows:

$$\mathbb{E}[x_\alpha x_\beta] = \frac{\partial^2 \phi(\vec{k})}{\partial(-ik_\beta) \partial(-ik_\alpha)} \Big|_{\vec{k}=0} \quad (2.8)$$

$$= \frac{\partial}{\partial(-ik_\beta)} \Big|_{\vec{k}=0} \left[-i \sum_\beta k_\beta C_{\beta\alpha} + \mu_\alpha\right] \exp\left(-\frac{1}{2} \vec{k}^\top C \vec{k} - i \vec{k} \cdot \vec{\mu}\right) \quad (2.9)$$

$$= C_{\alpha\beta} + \mu_\alpha \mu_\beta, \quad (2.10)$$

therefore, as expected,  $\tilde{C}_{\alpha\beta}$  is indeed  $C_{\alpha\beta}$ .

### 3 Exercise 6

**Claim 3.1.** *The characteristic function of a multivariate Gaussian is, up to normalization, a multivariate Gaussian.*

*Proof.* The characteristic function is the exponential of (minus)

$$\frac{1}{2}\vec{k}^\top C\vec{k} + i\vec{k} \cdot \vec{\mu} = \frac{1}{2}(\vec{k} + iC^{-1}\vec{\mu})^\top C(\vec{k} + iC^{-1}\vec{\mu}) + \frac{1}{2}\vec{\mu}^\top C^{-1}\vec{\mu}, \quad (3.1)$$

which means that the characteristic function is in the form

$$\phi(\vec{k}) = \text{const} \times \exp\left(-\frac{1}{2}(\vec{k} - \vec{m})^\top C(\vec{k} - \vec{m})\right), \quad (3.2)$$

a multivariate normal with mean  $\vec{m} = -iC^{-1}\vec{\mu}$  and covariance matrix  $C^{-1}$ , the inverse of the covariance matrix of the corresponding MVN.  $\square$