Early Universe Cosmology

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Contents

Wednesday 2020-9-30, compiled 2020-10-07

Introduction

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Live lectures will be in room P1A, usually at the blackboard, sometimes with slides. There will be notes uploaded to Moodle.

Course content An up-to-date overview of the physics of the Early Universe. The goal is to be able to understand and analyze the problems from both a theoretical and observational point of view.

There are three main parts:

1. inflationary models: the issue of the inital conditions;
2. cosmological perturbations, GW of cosmological origin;
3. baryogenesis, production of DM particles.

All of these will be connected with observations: nowadays cosmology is data-driven.

There are connections with: cosmology, astroparticle physics, astrophysics, GR, theoretical physics, field theory, multimessenger astrophysics, GW.

We will do "blended learning", with shifts in the classroom. We will do 48 hours of lectures.

Textbooks: Liddle and Lyth, "The primordial Density Perturbation", "Cosmological inflation and "

Complete

There will be 2 exam dates for each session. The exam is an oral one. Office hours can be arranged anytime.

We should limit questions in the break, prefer asking them during the lecture itself.

Chapter 1

Inflationary models

1.1 A general introduction

The basic issue is to find what initial conditions would produce the universe as we currently observe it.

Observational probes of the Hot Big Bang model: the Hubble diagram, Big Bang nucleosynthesis, the CMB.

On large scales we observe a smooth universe. However, that is a "zeroth-order" approximation: there are structures and anisotropies. All the structures need initial conditions to start from and then grow through gravitational instability.

We have several observables to probe the anisotropies: CMB, LSS, clusters of galaxies, weak gravitational lensing. There are initial fluctuations on the order of

$$\frac{\delta\rho}{\rho} \sim \frac{\delta T}{T} \sim 10^{-5} \,. \tag{1.1.1}$$

What is the initial time and temperature at which these perturbations start? Is there a dynamical mechanism which produces the perturbations? How do the perturbations evolve exactly? How do they relate to baryogenesis?

Under a Newtonian treatment, relative density perturbations grow like $\delta_m \propto a(t)$. The problem we will address here is how the initial value of δ_m comes about.

The CMB is a very good blackbody, without spectral distortions except for the Sunyaev-Zel'dovich effect (inverse Compton scattering from high-energy electrons in galaxies up-scattering the CMB photons).

We recall some basic concepts about the smooth model of the universe: critical density, Hubble parameter and so on.

The standard ΛCDM model does predict a small deviation, $\mu/T \sim 1.9 \times 10^{-8}$, from the Planckian, whose phase space distribution is:

$$f = \left[\exp\left(\frac{h\nu - \mu}{k_B T}\right) - 1\right]^{-1} . \tag{1.1.2}$$

Currently we have upper bounds: $\mu/T < 9 \times 10^{-5}$ at 95 % CL.

The CMB radiation is also highly, but not perfectly, isotropic. the scale of the temperature angular anisotropies are of the order $\Delta T/T \sim 10^{-5}$ (the quoted value for $\Delta T/T$ is a root-mean-square, since the average of ΔT is zero). This is to say: in each direction we observe a very good blackbody, whose characteristic temperature changes slightly depending on the direction.

Planck 2018 had an angular resolution of 5 arcminutes, and it also measured the polarization of the CMB.

We also have redshift galaxy surveys like the Sloan Digital Sky Survey. We map galaxies in redshift space. There is a statistical pattern of the galaxies, which is connected to the origin of the inhomogeneities.

The idea is that the seeds of the perturbations are quantum mechanical, coming from the inflaton scalar field, which are made into galaxies and galaxy clusters from gravitational instabilities.

At $z \sim 20$ the DM distribution was quite smooth, it then clustered.[1]

The components of the ΛCDM model are:

1. dark energy 68 %;

2. dark matter 26 %;

3. hydrogen and helium gas 4 %;

4. stars 0.5 %;

5. neutrinos 0.26 %;

6. metals 0.025 %;

7. radiation 0.005 %.

We also need seed perturbations and baryo-leptogenesis. We will see phases in which the universe is not in thermal equilibrium.

We want to find information about energies up to 10^{16} GeV: we will see that the inflationary phase corresponds to this epoch.

GW from inflation travel basically unimpeded from inflation to us.

Today, we have radiation with $w = 1/3$, $\rho \propto a^{-4} \propto T^4$, so, Tolman's law $Ta = \text{const}$.

Baryonic matter has $\Omega_b h^2 = 0.0224 \pm 0.0001$. Its equation of state is $P = nT \ll nm$. So, $\rho \propto a^{-3}$. Dark matter is also nonrelativistic, with $P \approx 0$, and $\Omega_{DM} h^2 0.120 \pm 0.001$.

The cosmological constant has $P = -\rho$, and $\Omega_\Lambda = 0.6847 \pm 0.0073$.

Neutrinos have $\sum m_\nu < 0.12\,\text{eV}$, and $\Omega_\nu h^2 < 0.0012$. Both values are at 95 % CL.

Spatial curvature has $\Omega_k = 1 - \Omega_0 = 0.001 \pm 0.002$, from Planck, Baryon Acoustic Oscillations, local measurements.

The presence of discordance can surely signal systematics, but also new physics. There are certain discordances.

[1] https://www.youtube.com/watch?v=FBkYIqtYb0I.

Beyond galactic rotation curves, we also have evidence for DM from the power spectrum of inhomogeneities. We Fourier-transform the density perturbation field δ to get $\delta_{\vec{k}}$; then we can calculate the power spectrum

$$\Delta^2(k) = \frac{\partial \sigma^2}{\partial \log k} \propto k^3 \left| \delta_{\vec{k}} \right|^2 \propto k^{3+n} T^2(k) \,. \tag{1.1.3}$$

The Poisson equation reads

$$4\pi G \bar{\rho} \delta = \nabla^2 \Phi \implies \delta_{\vec{k}} \propto k^2 \Phi_{\vec{k}} \,. \tag{1.1.4}$$

If the gravitational perturbation is written as

$$\Phi_{\vec{k}} = \Phi_{\vec{k}}^{\text{primordial}} T(k) \times \text{growth function} \,, \tag{1.1.5}$$

and the primordial field perturbation squared is $\left| \Phi_k^{\text{primordial}} \right|^2 \propto k^{n-4}$, where $n = 0.9600 \pm 0.0042$ is a spectral index. This explains the last proportionality sign we wrote earlier, the density power spectrum includes information about the power-spectral index of the initial conditions. This index measures the amplitude of the inhomogeneities in DM density.

Also, if we only had baryons without dark matter the power spectrum of the matter density perturbations would look very different from what they do.

Baryon Acoustic Oscillations are an oscillatory imprint in the power spectrum, they have been measured today.

Inflation is an early epoch in the history of the universe during which expansion is accelerated. The basic predictions of inflation are so far confirmed, however we have not detected the SGWB from it, which would be a "smoking gun". Monday 2020-10-5, compiled 2020-10-07

These next few lectures, we will consider the motivations for inflationary models. The problems they solved were the **shortcomings of the Hot Big Bang** model.

1. The horizon problem;
2. the flatness problem;
3. unwanted relics / magnetic monopole problem.

We start by recalling some basic elements in cosmology. In order to describe a homogeneous and isotropic universe we use the FLRW metric:

$$ds^2 = -c^2 \, dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \, d\Omega^2 \right] , \tag{1.1.6}$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2$. The quantity $a(t)$ is called the *scale factor*. The coordinates r, θ and φ are called *comoving coordinates*.

Physical distances and comoving distances are related by

$$\lambda_{\text{phys}} = a(t) \lambda_{\text{comoving}} \,. \tag{1.1.7}$$

The constant k is the spatial curvature of the universe, which can always be rescaled so that it is equal to

1. $+1$ for a spatially closed universe;
2. 0 for a spatially flat universe;
3. -1 for a spatially open universe.

In terms of the scale factor we define the **Hubble parameter**

$$H = \frac{\dot{a}}{a}, \tag{1.1.8}$$

which describes the rate at which the universe expands.

The dynamics of gravity are described by the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1.9}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the particle species filling the universe, while $G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu}/2$ is the Einstein tensor, describing curvature.

These can be derived from an action principle through the action

$$S = \underbrace{\frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x}_{S_{EH}} + S_{\text{matter}}. \tag{1.1.10}$$

Often we use an ideal fluid energy-momentum tensor:

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}, \tag{1.1.11}$$

where $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$ is a projector onto the space orthogonal to the four-velocity. This does not account for any anisotropy, it is the most symmetric energy-momentum tensor. This is a diagonal

In order to solve the Einstein equations we can proceed with some assumptions, without needing to know the action for all the fundamental fields. The perfect fluid S-E tensor has all the FLRW symmetries, as long as ρ and P are only functions of time.

We are *not* saying that this S-E tensor is only allowed if we are in a FLRW universe.

Clarify…

Requiring the FLRW symmetries means that the S-E tensor must be diagonal, however we can have viscosity as long as it is not *shear* but *bulk* viscosity, which adds onto the diagonal terms.

Inserting the FLRW metric into the Einstein equation yields the Friedmann equations:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.1.12}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) \tag{1.1.13}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P\right). \tag{1.1.14}$$

The first two can be derived from the Einstein equations directly, the third comes from the "conservation law" ${T_{\mu\nu}}^{;\nu} = 0$.

They are not independent, only two are. We have too many parameters: a, ρ and P, but only two independent equations, so we "close" the system of equations with an equation of state, commonly $P = P(\rho) = w\rho$.

These equations of state describe many kinds of fluids (approximately): dust with $w = 0$, which means $\rho \propto a^{-3}$; radiation with $w = 1/3$, which means $\rho \propto a^{-4}$; a cosmological constant with $w = -1$, which means $\rho = \text{const}$.

In general as long as $w \neq 1$ we have $\rho \propto a^{-3(1+w)}$ and $a \propto t^{2/3(1+w)}$.

1.1.1 The horizon problem

The **particle horizon**, denoted as $d_H(t)$, is given by

$$d_H(t) = a(t) \int_0^t \frac{c\,\mathrm{d}\tau}{a(\tau)}\,, \tag{1.1.15}$$

and it sets the radius of a sphere centered at an observer O. The points inside this sphere have been able to have causal interactions with observer O in the time from the Big Bang to t.

It is the proper distance (as measured today) which could have been travelled by light starting at the beginning and moving in a geodesic. It can be derived from the FLRW metric by assuming radial light-like motion

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a(t)\frac{\mathrm{d}r^2}{1-kr^2} = 0\,, \tag{1.1.16}$$

and setting $k = 0$:[2]

$$c\,\mathrm{d}t = \pm a(t)\,\mathrm{d}r\,, \tag{1.1.17}$$

which we can use to calculate the *comoving distance* from the point of emission to today, which we then multiply by the scale factor calculated at a chosen point.

We know that the scale factor goes to 0 as t goes to 0, so the integral giving us r_H could diverge. We can show that d_H is finite as long as $\alpha = 2/3(1+w)$ is smaller than one, meaning that $w > -1/3$, which is equivalent to $\ddot{a} < 0$. In a decelerating universe, the particle horizon is finite.

In general, the calculation yields

$$d_H(t) = \frac{3(1+w)}{1+3w}ct\,. \tag{1.1.18}$$

With $w = 0$, a spatially flat matter-dominated universe, $d_H = 3ct$. This is called an Einstein-De Sitter universe. With $w = 1/3$, a spatially flat radiation-dominated universe, we have $d_H = 2ct$.

[2] This is a good approximation for early times, even if the universe is not flat.

Another way to characterize causality is the Hubble radius:

$$r_C(t) = \frac{c}{H(t)}\,. \tag{1.1.19}$$

The characteristic time of expansion is $\tau(H) = H^{-1}$. We can show that in a FLRW universe, typically after a Hubble time the scale factor doubles.

Since

$$H(t) = \frac{2}{3(1+w)}\frac{1}{t}\,, \tag{1.1.20}$$

we can write

$$R_H \approx \frac{1+3w}{2} d_H(t) \approx d_H(t)\,. \tag{1.1.21}$$

check! exercise

This will *not* happen in an inflationary universe. The two are similar in a regular FLRW universe, while they differ a lot if there is inflation. The particle horizon takes into account all the past history of an observer, the Hubble radius does not care about it: it only described causal connections taking place in a time interval taking place in a Hubble time.

Let us introduce the *comoving Hubble radius*: $r_H(t)$, given by

$$r_H(t) = \frac{r_C(t)}{a(t)}\,. \tag{1.1.22}$$

Let us plot this for a matter or radiation-dominated FLRW universe.

In radiation domination, $r_H \propto \sqrt{t}$, while in matter domination $a \sim t^{2/3}$ so $r_H \propto t^{1/3} \sim a^{1/2}$.

This comoving radius is then always increasing, initially faster and then slower.

Instead, consider the comoving particle horizon: $d_H(t)/a(t)$, so just the integral in the definition of $d_H(t)$:

$$\frac{d_H(t)}{a(t)} = \int \frac{c\,\mathrm{d}t}{a} = \int \frac{\mathrm{d}a}{a}\underbrace{\frac{c}{aH}}_{r_H}\,, \tag{1.1.23}$$

so we can see that the comoving *particle horizon* is the logarithmic integral over the scale factor of the comoving *Hubble radius*: as we mentioned before, this takes into account the whole past history.

In a matter dominated universe, $d_H = 2r_C \approx 5h^{-1}$Gpc.

Now we discuss the horizon problem, which is best understood in a comoving plot. We neglect dark energy for simplicity.

If we choose a fixed comoving size λ, we get in our model that in early times λ is super-horizon, then at a certain point it crosses the horizon, becoming smaller than r_H. The time at which $r_H = \lambda$ is called the *horizon crossing* time, $t_H(\lambda)$.

For times earlier than $t_H(\lambda)$, by definition it is impossible for points at a distance λ to be causally connected. This happens for every scale, and it means that for many regions we are interested in there cannot have been causal connection in the early universe. But, today we observe the universe to exhibit the same properties across the whole sky, even though the regions were causally disconnected earlier.

This is most directly expressed in terms of CMB photons. They would have become causally connected at the quadrupole scale (separations of 90°) almost *today*.

We can compute the size of the horizon at the last scattering epoch: this subtends an angle in the sky of around 1°; however we observe photons with the same temperature on much larger scales, this was already seen by COBE with an angular resolution of 7°.

re-do calculation!

Photons which could not have been in causal contact in the HBB model are observed to have the same temperature.

The inflationary solution to this issue is to think that, before the radiation-dominated epoch, the comoving Hubble radius decreased for a certain period of time.

This allows the parts of the sky to have been in causal contact in the early universe.

This means that $\ddot{a} > 0$ in the inflationary phase, or $w < -1/3$. These are only the *kinematics* of inflation, we are not yet discussing how it might come about.

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We come back to the horizon problem. [Plot of the comoving Hubble radius r_H as a function of time]

The problem is solved if there is an early epoch in which r_H decreases in time, due to accelerated expansion.

After the end of this inflation, the regular FLRW universe's history starts, with the radiation, then matter, then cosmological constant dominated phases. An accelerated expansion, however, is not enough to solve the horizon problem: what we need is for *every* observable scale, up to the largest ones, was causally connected in the early universe. In other words, the inflation phase must last *long enough*.

More specifically, our constraint on inflation is that it must start when the Hubble radius was at least as large as it is today. This can be expressed in terms of the *number of e-folds*:

$$N = \log\left(\frac{a_f}{a_{\text{in}}}\right) = \int_{t_{\text{in}}}^{t_f} H(t)\, \mathrm{d}t \,, \tag{1.1.24}$$

the ratio of the scale factor at the beginning and at the end of inflation. The number of elapsed e-folds is a natural measure of time in the epoch of inflation.

We can give the bound $N \gtrsim 60 \div 70$ in order to solve the horizon problem. This is a *huge* expansion! Typical atomic scales of $10^{-15}\,\text{m}$ get stretched to the typical scales of the Solar System, $10^{11}\,\text{m}$.

The condition is $r_H(t_{\text{in}}) \gtrsim r_H(t_0)$. We can express this as

$$\frac{1}{a_{\text{in}} H_{\text{in}}} =\gtrsim \frac{1}{a_0 H_0} \tag{1.1.25}$$

$$\frac{a_f}{a_{\text{in}}} = e^N \gtrsim \frac{H_i}{H_0} \frac{a_f}{a_0} \,. \tag{1.1.26}$$

See on Moodle: paper with the exact computation.

We want to bring on the left all the quantities in the inflationary epoch. Recall that $H^2 \propto \rho \propto a^{-3(1+w_i)/2}$, which we will apply to the inflationary epoch with an equation of state $w_i < -1/3$. This means that

$$\frac{H_i}{H_f} H_f = \left(\frac{a_i}{a_f} \right)^{-3(1+w_i)/2} H_f \,, \tag{1.1.27}$$

so

$$\frac{a_f}{a_i} \left(\frac{a_f}{a_i} \right)^{-3(1+w_i)/2} = \gtrsim \frac{H_f}{H_0} \frac{a_f}{a_0} \tag{1.1.28}$$

$$\left(\frac{a_f}{a_i} \right)^{\frac{-(1+3w_i)}{2}} \gtrsim \frac{T_0}{H_0} \frac{H_f}{T_f} \,, \tag{1.1.29}$$

where we applied Tolman's law, $T \sim 1/a$, neglecting the matter dominated phase — this is a reasonable approximation, we find a similar result to the complete calculation. This yields

$$N \gtrsim -\frac{2}{1+3w_i} \left[\log \frac{T_0}{H_0} + \log \frac{H_f}{T_f} \right] , \tag{1.1.30}$$

where $T_0 = 2.7\,\mathrm{K} \approx 10^{-13}\,\mathrm{GeV}$, while $H_0 \sim 10^{-42}\,\mathrm{GeV}$ in natural units. Therefore, the first logarithm is of the order ~ 67. We also need the *pre-heating* temperature and Hubble parameter: H_f and T_f. This is model-dependent: it is what gives the theoretical uncertainty. The dependence, however, is weak: only logarithmic.

With current measurements, we are starting to be able to measure this term as well. Let us give an estimate for it:

$$H_f^2 \approx \frac{8\pi G}{3} \rho_{\mathrm{rad}} \,, \tag{1.1.31}$$

where $\rho_{\mathrm{rad}} \frac{\pi^2}{30} g_* T^4$. This then yields

$$H_f^2 = \frac{8\pi G}{3} \frac{\pi^2}{30} g_* T^4 \sim \frac{T_f^4}{M_P^2} \,. \tag{1.1.32}$$

There is model dependence here, in g_*! If we go BDSM (beyond de standard model) it could change. We are giving a very rough estimate with the Planck mass. This then means $H_f \sim T_f^2/M_p$. So,

$$\log \left(\frac{H_f}{T_f} \right) \approx \log \frac{T_f}{M_p} \,. \tag{1.1.33}$$

Typically, models predict

$$10^{-5} < \frac{T_f}{M_p} < 1 \,, \tag{1.1.34}$$

but this is not set in stone, we could have different predictions as well.

Now, let us assume that $w_i \sim -1$, something like a cosmological constant. Then, the prefactor is of the order 1, so the bound is $N \gtrsim 60 \div 70$ as was mentioned before.

We now discuss the causal structure of the FLRW metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right]. \tag{1.1.35}$$

Let us express this in different coordinates: we introduce χ, so that

$$r = S_k(\chi) = \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1 \end{cases}. \tag{1.1.36}$$

This allows the term $\mathrm{d}r^2/(1-kr^2)$ to become simply $\mathrm{d}\chi^2$.

Also, we introduce conformal time: $\mathrm{d}\eta = \mathrm{d}t/a(t)$, so that the metric becomes

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + S_k^2(\chi)\,\mathrm{d}\Omega^2\right]. \tag{1.1.37}$$

The meaning of χ is still a comoving distance. However, the interesting thing is that this metric is conformally related to ("is a time-dependent rescaling of") the Minkowski metric (if we consider radial motion, at least), we say that it is *conformally flat*.

In these coordinates, light propagates at 45° in the (η, χ) plane.

Then, we can draw a diagram for the horizon problem in these coordinates: the Big Bang singularity looks like a straight line at constant η. The last-scattering surface is also a straight line at constant η. We can then draw a past light-cone from a point in the last-scattering surface. Inflation pushes the BB surface back in conformal time, so that light has more time to propagate.

We can show (exercise) that the conformal time at the end of inflation looks like $\eta \propto 2/(1+3w_i)a^{2/(1+3w)}H_*^{-1}$.

So, if $w < -1/3$, we are good.

We can only detect correlations in the CMB up to the quadrupole, since the dipole is correlated with the Earth's motion... Roughly, this means that we can only see correlations on the scale of $\sim r_H$, corresponding to 90° separation, instead of being able to see them on the scale of 180°.

Now, we move to the flatness problem. The HBB model is not intrinsically flawed, however the shortcomings we are discussing tell us that the initial conditions which would be required in order to yield the current universe would be very specific.

We should set initial conditions which are homogeneous and isotropic, with very specific small fluctuations. Inflation provides a dynamical solution to these problems, which is an attractor towards these initial conditions.[3]

[3] See Hossenfelder [Hos19] for a critical discussion of this fine-tuning problem.

The first Friedmann equation reads

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{1.1.38}$$

which we can express through $\Omega = \rho/\rho_c$, where $\rho_c = 3H^2/(8\pi G)$:

$$1 - \Omega = \frac{k}{a^2 H^2} = k r_H^2(t)\,, \tag{1.1.39}$$

so if Ω differs from unity even by a small amount, this difference increases with time.

At 95 % CL, we know that $|\Omega - 1| = |\Omega_k| < 0.4\,\%$, so the universe we observe is consistent with flatness.

Specifically, in the Planck epoch we will have

$$\Omega(t_{\text{Pl}}) - 1 \approx (\Omega_0 - 1) \times 10^{-60}\,, \tag{1.1.40}$$

so $\left|\Omega(t_{\text{Pl}}) - 1\right| < 10^{-62}$.

What can be shown is also that

$$\left(\Omega^{-1} - 1\right)\rho a^2 = \text{const}\,. \tag{1.1.41}$$

For times before the matter-radiation equivalence $\rho \propto a^{-4}$, so $\rho(t) = \rho_{\text{eq}}(a_{\text{eq}}/a)^4$. Also, during matter domination up to now (neglecting the cosmological constant)

$$\rho_0 = \rho_{\text{eq}}\left(\frac{a_{\text{eq}}}{a_0}\right)^3, \tag{1.1.42}$$

therefore

$$(\Omega^{-1} - 1)\left(\frac{a_{\text{eq}}}{a}\right)^4 a^2 \rho_0 \left(\frac{a_0}{a_{\text{eq}}}\right)^3 \frac{1}{\rho_0 a_0^2} = (\Omega_0^{-1} - 1) \tag{1.1.43}$$

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)\frac{a^2}{a_{\text{eq}} a_0} \tag{1.1.44}$$

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)(1 + z_{\text{eq}})\frac{a^2}{a_0^2}\,, \tag{1.1.45}$$

since $1 + z_{\text{eq}} = a_0/a_{\text{eq}}$. Also, we can approximate $a/a_0 \sim T_0/T_{\text{Pl}}$ by extending Tolman's law. Then,

$$\Omega^{-1} - 1 = (\Omega_0^{-1} - 1)\underbrace{(1 + z_{\text{eq}})}_{\sim 10^4}\underbrace{\frac{T_0^2}{T_{\text{Pl}}^2}}_{\sim 10^{-64}}\,. \tag{1.1.46}$$

This proves the relation we wrote earlier. It is an extreme extrapolation to go back to the Planck time, but even if we only went back to Big Bang nucleosynthesis ($\sim 1\,\text{MeV}$) we would get

$$\left|\Omega(t_{\text{BBN}}) - 1\right| < 10^{-18}\,. \tag{1.1.47}$$

How does inflation solve the problem? Recall that $\Omega - 1 = kr_H^2$, and inflation is by definition a time in which r_H decreases. At the end of inflation, $\Omega - 1$ is very close to 0, meaning that r_H is small, but at the start of inflation it could have been relatively far from 1.

Next week we will discuss the proper mechanism of this process. During an inflationary phase, $a(t) \approx \exp(Ht)$. So, as long as H is approximately constant, we have

$$r_H^2 = \frac{1}{a^2 H^2} \propto \frac{1}{a^2}\,. \tag{1.1.48}$$

Then, we have

$$\frac{|\Omega - 1|_{t_f}}{|\Omega - 1|_{t_i}} \sim \left(\frac{a_i}{a_f}\right)^2 \sim \exp(-2N)\,. \tag{1.1.49}$$

This means that, with very broad possible initial conditions, we find $\Omega - 1$ very close to zero at the end of inflation.

A De-Sitter phase is a reference example of a possible inflationary stage. It would correspond to $\rho = \text{const}$, $w = -1$: in general, since the "curvature energy density" scales like a^{-2}, curvature becomes negligible.

Note, however, that this is an unrealistic example: it does not include any method for the inflation to end.

Bibliography

[Hos19] S. Hossenfelder. "Screams for Explanation: Finetuning and Naturalness in the Foundations of Physics". In: *Synthese* (Sept. 3, 2019). ISSN: 0039-7857, 1573-0964. DOI: 10.1007/s11229-019-02377-5. arXiv: 1801.02176. URL: http://arxiv.org/abs/1801.02176 (visited on 2020-09-14).