Theoretical cosmology notes

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Contents

Contents of the course

We start with a derivation of the Friedmann eqs. from the Einstein equations.

We will then discuss the properties of the CMB, deriving the spectrum, and then CMB anisotropies.

Then we will discuss star and structure formation, about the nonlinear evolution of perturbations. We will use the path-integral approach to classical field theory. We will also discuss weak gravitational lensing in the universe.

We will use some smart nonlinear approximations: the Zel'dovich approximation and the adhesion approximation.

We will use an "effective Planck constant" instead of $\hbar$: it will be a parameter which can be fit in our model.

As for references: there are handwritten notes by the professor in the Dropbox folder (for access to the folder, write to the professor). Also, there notes by a student from the previous years, in Italian [Nat17], which are to be used with caution as they contain some errors.

0.1 Friedmann equations: a brief overview

Thursday 2020-3-12, compiled 2020-03-19

In the previous course we used the approximate symmetries of the universe to write the FLRW line element:

$$ds^2 = -\,dt^2 + a^2(t)\,d\sigma^2 \,, \tag{1}$$

do note that we switch signature from the previous course: now we use the mostly plus one. The spatial part is defined by

$$d\sigma^2 = \widetilde{g}_{ij}\,dx^i\,dx^j \,, \tag{2}$$

where $\widetilde{g}_{ij}$ is the maximally symmetric metric tensor in a 3D space. There are only 3 maximally symmetric 4D spacetimes: Minkowski, dS and AdS.

Since we have maximal symmetry, the Riemann tensor is

$$R_{ijkl} = k\left(\widetilde{g}_{ik}\widetilde{g}_{jl} - \widetilde{g}_{il}\widetilde{g}_{jk}\right). \tag{3}$$

We can use spherical coordinates:

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\,, \tag{4}$$

and we can define the coordinate χ by

$$\mathrm{d}\chi = \frac{\mathrm{d}r^2}{\sqrt{1-kr^2}}\,. \tag{5}$$

The Einstein equations read

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \tag{6}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is its trace, the scalar curvature, while $T_{\mu\nu}$ is the stress energy momentum tensor.

In cosmology we assume to have the SEMT of a perfect fluid. Really, we have particles, between which there is vacuum.

We need to use the Weyl tensor, which describes the parts of the Riemann tensor which are not in the traces. "The real world" is only described by the Weyl tensor, but in cosmology we make a great approximation in ignoring it.

What we do is to insert an ansatz for the metric tensor, which we use to derive the Christoffel symbols, and from these we write the Riemann tensor. Doing it the other way around, starting from the source SEMT, is very difficult.

Claim 0.1.1. *The Christoffel symbols for the FLRW metric are:*

$$\Gamma^t_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\dot{a}a}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 a\dot{a} & 0 \\ 0 & 0 & 0 & r^2 a\dot{a}\sin^2\theta \end{bmatrix} \tag{7}$$

$$\Gamma^r_{\mu\nu} = \begin{bmatrix} 0 & \dot{a}/a & 0 & 0 \\ \dot{a}/a & \frac{kr}{(1-kr^2)} & 0 & 0 \\ 0 & 0 & (kr^2-1)r & 0 \\ 0 & 0 & 0 & (kr^2-1)r\sin^2\theta \end{bmatrix} \tag{8}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & \dot{a}/a & 0 \\ 0 & 0 & 1/r & 0 \\ \dot{a}/a & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{9}$$

$$\Gamma^{\varphi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & \dot{a}/a \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & \cos\theta/\sin\theta \\ \dot{a}/a & 1/r & \cos\theta/\sin\theta & 0 \end{bmatrix}. \tag{10}$$

In order to calculate these, we can make use of certain simplifications: the FLRW metric is diagonal, and it does not depend on φ.

Notice that the spatial Christoffel symbols are nonzero even in Minkowski ($k = 0$, $\dot{a} = \ddot{a} = 0$): why is this? This is because we are using curvilinear coordinates, the Christoffel symbols express the *extrinsic* curvature, not the *intrinsic* curvature; they are not tensors, so they can be zero in a reference and nonzero in another.

In general, the Riemann tensor is given by

$$R^{\mu}_{\nu\rho\sigma} = -2\left(\Gamma^{\mu}_{\nu[\rho,\sigma]} + \Gamma^{\alpha}_{\nu[\rho}\Gamma^{\mu}_{\sigma]\alpha}\right), \tag{11}$$

where commas denote coordinate derivation, and square square brackets denote antisymmetrization (for clarification on this notation Wikipedia does a good job [19]).

The Ricci tensor is given by the contraction of the Riemann tensor along its first and third component:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = -2\left(\Gamma^{\alpha}_{\mu[\alpha,\nu]} + \Gamma^{\beta}_{\mu[\alpha}\Gamma^{\alpha}_{\nu]\beta}\right) \tag{12}$$

$$= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}. \tag{13}$$

A great simplification comes from the fact that, for the FLRW metric, the Ricci tensor is diagonal.[1]

Claim 0.1.2. *The components of the Ricci tensor are:*

$$R_{tt} = -3\partial_t\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \tag{14}$$

$$= -3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2\right) \tag{15}$$

[1] If there are a certain number of coordinates the metric is independent of, the Ricci tensor has very few nonzero components [Win96]. This is not enough to prove that the Ricci tensor must be diagonal for this metric, however in the specific case of FLRW this is the case anyways.

$$= -3\frac{\ddot{a}}{a}\,, \tag{16}$$

$$\begin{aligned}
R_{rr} &= \partial_t\left(\frac{\dot{a}a}{1-kr^2}\right) + \partial_r\left(\frac{kr}{1-kr^2}\right) - \partial_r\left(\frac{kr}{1-kr^2}\right) - 2\partial_r\left(\frac{1}{r}\right) \\
&\quad + \frac{\dot{a}a}{1-kr^2}3\frac{\dot{a}}{a} + \frac{kr}{1-kr^2}\left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) \qquad (17) \\
&\quad - 2\frac{\dot{a}}{a}\frac{\dot{a}a}{1-kr^2} - \left(\frac{kr}{1-kr^2}\right)^2 - 2\left(\frac{1}{r}\right)^2 \\
&= \frac{\ddot{a}a + \dot{a}^2}{1-kr^2} + 3\frac{\dot{a}^2}{1-kr^2} + 2\frac{k}{1-kr^2} - 2\frac{\dot{a}^2}{1-kr^2} \qquad (18) \\
&= \frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2}\,, \qquad (19)
\end{aligned}$$

$$\begin{aligned}
R_{\theta\theta} &= r^2\partial_t(a\dot{a}) + \partial_r\Big((kr^2-1)r\Big) - \partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) \\
&\quad + 3\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Big(\Gamma^r_{rr} + 2\Gamma^\theta_{r\theta}\Big) - 2\Big(\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}\Big) - \frac{\cos^2\theta}{\sin^2\theta} \qquad (20) \\
&= r^2\Big(\ddot{a}a + \dot{a}^2\Big) + 3kr^2 - 1 + \frac{1}{\sin^2\theta} + r^2\dot{a}^2 - kr^2 - \frac{\cos^2\theta}{\sin^2\theta} \qquad (21) \\
&= r^2\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big)\,, \qquad (22)
\end{aligned}$$

$$\begin{aligned}
R_{\varphi\varphi} &= \partial_\alpha\Gamma^\alpha_{\varphi\varphi} - \partial_\varphi\Gamma^\alpha_{\alpha\varphi} + \Gamma^\alpha_{\varphi\varphi}\Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\varphi\alpha}\Gamma^\alpha_{\varphi\beta} \qquad (23) \\
&= r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big)\,. \qquad (24)
\end{aligned}$$

The Ricci scalar then comes out to be

$$\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= 3\frac{\ddot{a}}{a} + \frac{1-kr^2}{a^2}\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2} \\
&\quad + \frac{1}{a^2r^2}r^2\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) + \frac{1}{a^2r^2\sin^2\theta}r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) \qquad (25) \\
&= 3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{a^2} \qquad (26) \\
&= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right]. \qquad (27)
\end{aligned}$$

The dimensions of the Ricci scalar are those of a length to the -2.

The stress energy tensor is the functional derivative of everything but the curvature in the action with respect to the metric: if our Lagrangian is

$$L = L_g + L_{\text{fluid}} \,, \tag{28}$$

where the gravitational Lagrangian is $L_g = M_P^2 R/2$ (and $M_P = 1/\sqrt{8\pi G}$ in natural units is the reduced Planck mass) then

$$T_{\mu\nu} \overset{\text{def}}{=} -2\frac{\delta L_{\text{fluid}}}{\delta g^{\mu\nu}} \,. \tag{29}$$

Discuss why this is equivalent to "flux of momentum component μ across a surface of constant x^ν".

We use perfect fluids: they have a stress-energy tensor like

$$T^{\mu\nu} = \left(\rho + P\right) u^\mu u^\nu + p g^{\mu\nu} \,, \tag{30}$$

where u^μ is the 4-velocity of the fluid element. It is diagonal *in the comoving frame*, in which $u^\mu = \left(1, \vec{0}\right)$.

If we are not comoving, we have additional heat transfer off diagonal terms (this is discussed in my thesis [Tis19, section 4.2]).

If we take the covariant divergence of the Einstein tensor $G_{\mu\nu}$ we get zero; so the stress energy tensor must also have $\nabla_\mu T^{\mu\nu} = 0$. This is *not* a conservation equation.

In SR we had an equation like $\partial_\mu T^{\mu\nu}$: this *was* a conservation equation, a local one. In GR we also need Killing vectors in order to actually have conserved quantities. In cosmology we do not have symmetry with respect to time translation, so there is no timelike Killing vector ξ_μ such that $\xi_\nu \nabla_\mu T^{\mu\nu}$ represents the conservation of energy.

This equation, $\nabla_\mu T^{\mu\nu}$ follows from the fact that our fluid follows its equations of motion.

Let us explore the meaning of these equations: if, in the equation $0 = \nabla_\mu T^\mu_0$, we find

$$\begin{aligned} 0 &= \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda & (31) \\ &= -\dot{\rho} - 3H\left(\rho + P\right) . & (32) \end{aligned}$$

For example consider radiation: $P = \rho/3$. This means that $\dot{\rho} = -4H\rho$: so, as the Hubble parameter increases, the radiation density decreases.

The other two Friedmann equations can be derived from the time-time and space-space components on the Einstein equations: we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) \tag{33}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,. \tag{34}$$

The space-space equation is not a dynamical equation, since it contains no second time derivatives: it is a *constraint* on the evolution of the system.

However, the three Friedmann equations are not independent: the time-time one can be found from the other two.

Exercise: calculate the Christoffel symbols for the FLRW metric, for any k.

Exercise: calculate the Ricci tensor and curvature scalar.

A useful theorem is the fact that for a maximally symmetric space the Ricci tensor must be given by

$$\widetilde{R}_{\alpha\beta} = 2k\widetilde{g}_{\alpha\beta}\,. \tag{35}$$

We can write the stress energy tensor as

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}\,, \tag{36}$$

where $h_{\mu\nu}$ is the projection tensor onto the spacelike subspace $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$.

This is more physically meaningful.

Tomorrow we will start the discussion on the CMB.

Chapter 1

The CMB

Friday 2020-3-13, compiled 2020-03-19

Today we discuss the CMB. This is discussed in the book Modern Cosmology [Dod03], we follow the professor's notes.

A note: in these lectures a dot will refer to conformal time derivatives only, if we differentiate with respect to cosmic time we shall write the derivative explicitly. Let us suppose we have some particle species interacting, such as $1 + 2 \leftrightarrow 3 + 4$.

The variation in time of the abundance of particle type 1, (which is given by the density times a volume: $n_1 a^3$) is given by the difference of the particles which are created and destroyed. We write the formula first, and then explain it: this is given by

$$
\begin{aligned}
a^{-3}\frac{\mathrm{d}(n_1 a^3)}{\mathrm{d}t} &= \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \left[\prod_{i=2}^{4} \int \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i} \right] \times \\
&\quad \times (2\pi)^4 \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\delta(E_1 + E_2 - E_3 - E_4) \times \\
&\quad \times |\mathcal{M}|^2 \Big[f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \Big]
\end{aligned}
, \tag{1.1}
$$

where:

1. the delta functions account for momentum and energy conservation: energy is *not conserved* in general in cosmology, *but* we can use the equivalence principle to go to a reference frame which is locally Minkowski: in our description of an instantaneous process such as this, the deviations from this frame are negligible.

2. $\mathcal{M}$ is the invariant scattering amplitude between the initial and final states.

3. The f_i are the phase space distributions of the different species: the terms including these account for the quantum statistics, we use $-$ for fermions and $+$ for bosons. Bose statistics enhance the process, Pauli statistics block it.

4. The 2π-s account for the normalization of the deltas: if we were to discretize phase space and use Kronecker deltas we would not need them.

5. The energy of each particle species is given by $E = \sqrt{p^2 + m^2}$. Why are there $2E$ factors in the denominators? In principle, we should integrate in $\mathrm{d}^4 p$, however we work *on shell*. A priori, the particle does whatever it wants, however solutions to the equations of motion are preferred in the path integral. So, we impose this condition: we do

$$\int \mathrm{d}^3 p \int_0^\infty \delta(E^2 - p^2 - m^2) = \int \mathrm{d}^3 p \int_0^\infty \frac{\delta(E - \sqrt{p^2 + m^2})}{2E}\,, \tag{1.2}$$

so we include the term in the denominator.

Clarify definition of $\mathcal{M}$.

If there is no interaction, $n_1 \propto a^{-3}$.

We set $\hbar = c = k_B = 1$.

The term for particle 1, E_1, has a different origin: the time is related to the proper time by p^0, which is E_1. The factor 2 is included for symmetry, it is indifferent if we include it or not since we can normalize the helicities g_i.

Typically we have kinetic equilibrium, if the scattering time is very short with respect to the Hubble time. So, we use

$$f_{\mathrm{BE/FD}} = \left(\exp\left(\frac{E - \mu}{T} \right) \pm 1 \right)^{-1}\,, \tag{1.3}$$

where the sign is a $-$ for Bose Einstein statistics, while a $+$ for Fermi-Dirac statistics.

For the nonrelativistic particles (all of them, except the photons) we have $E - \mu \gg T$. If f becomes very small, then we can drop the terms $(1 \pm f_i)$. This is the Boltzmann limit.

In theory we could not do this for photons, in practice we do it and the magnitude of the error is the same as the ratio $\zeta(3) \approx 1.2$ to 1.

Then, our distributions will be given by

$$f(E) = e^{\mu/T} e^{-E/T}\,. \tag{1.4}$$

So the phase space distribution term is

$$\exp\left(-\frac{E_1 + E_2}{T} \right) \left(e^{(\mu_3 + \mu_4)/T} - e^{(\mu_1 + \mu_2)/T} \right)\,, \tag{1.5}$$

where we used the fact that $E_1 + E_2 = E_3 + E_4$ by energy conservation, as we said. If we enforced the Saha condition, chemical equilibrium $\mu_1 + \mu_2 = \mu_3 + \mu_4$, then we get precisely zero: the number densities of the species are constant.

The mean number density of species i is given by

$$n_i = g_i e^{\mu_i/T} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T} \,, \tag{1.6}$$

where g_i is the number of helicity states.

So, we find for the whole expression inside the brackets:

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \,, \tag{1.7}$$

so we can define the time-averaged cross section

$$\langle \sigma v \rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_i \int \frac{\mathrm{d}^3 p}{2E_i} \dots \,, \tag{1.8}$$

so the final equation is

$$a^{-3} \frac{\mathrm{d}}{\mathrm{d}t}(n_1 a^3) = \langle \sigma v \rangle \, n_1^{(0)} n_2^{(0)} \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \,, \tag{1.9}$$

and the left hand side is typically $\sim n_1/t \sim n_1 H$. So, the combination on the RHS must be "squeezed to zero" eventually, which is equivalent to the Saha equation.

This is basically saying that we eventually reach chemical equilibrium.

1.1 Hydrogen recombination

The process is

$$e^- + p \leftrightarrow H + \gamma \,, \tag{1.10}$$

so the Saha equation yields

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \,, \tag{1.11}$$

and charge neutrality implies $n_e = n_p$, *not* $n_e^{(0)} = n_p^{(0)}$.

At this stage in evolution, there are already some Helium nuclei, but we ignore them.

We define the ionization fraction

$$X = \frac{n_e}{n_e + H} \,. \tag{1.12}$$

This then yields

$$\frac{1 - X_e^n}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_e}\right)^{3/2} \exp(\epsilon_0/T)\,, \tag{1.13}$$

where $\epsilon_0 = m_p + m_e - m_H = 13.6\,\mathrm{eV}$ is the ionization energy of Hydrogen.

Then, we get that the temperature of recombination is $T_{\mathrm{rec}} \approx 0.3\,\mathrm{eV}$.

The evolution of the ionization fraction is

$$\frac{\mathrm{d}X_e}{\mathrm{d}t} = (1 - X_e)\beta(T) - X_e^2 n_b \alpha^{(2)}(T)\,, \tag{1.14}$$

where we defined the ionization rate

$$\beta(T) = \langle \sigma v \rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T}\,, \tag{1.15}$$

and the recombination rate $\alpha^{(2)} = \langle \sigma v \rangle$.

The value of this can be solved numerically: the difference between this and the Saha equation is not great in the prediction in the recombination redshift; however the prediction of the residual ionized hydrogen is different: there is much more than Saha would predict.

The universe gets reionized at $z \gtrsim 6$; this is still under discussion.

There are many ingredients in the interaction of the universe. We are interested in the photons: we want to predict the anisotropies in the CMB. There is a dipole due to the movement of the solar system through the CMB. Now, we want to see what our predictions are if we subtract this.

[Scheme of the interactions.]

The metric interacts with everything, photons interact with electrons through Compton scattering, electrons interact with protons through Coulomb scattering, dark energy, dark matter and neutrinos interact only with the metric.

Instead of Compton scattering, we use its nonrelativistic limit which applies here.

Scattering between electrons and protons is suppressed since protons are very massive. The other terms in the universe affect the geometry and we could see them through this.

There are models which include DM-DE coupling, and quintessence models, and models in which dark energy clusters.

We are not going to consider these.

We go back to first principles:

$$\hat{\mathbb{L}}[f] = \hat{\mathbb{C}}[f]\,, \tag{1.16}$$

where $f = f(x^\alpha, p^\alpha)$, however actually we do not have that much freedom in the phase space distribution. If there are no collisions: $\hat{\mathbb{L}}[f] = 0$, which is equivalent to

$$\frac{\mathrm{D}f}{\mathrm{D}\lambda} = 0\,, \tag{1.17}$$

where λ is the affine parameter.

In the nonrelativistic case,

$$\hat{\mathbb{L}} = \frac{\partial}{\partial t} + \dot{x} \cdot \nabla_x + \dot{v} \cdot \nabla_v = \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_x + \frac{F}{m} \cdot \nabla_v\,, \tag{1.18}$$

while in the GR case we need to account for the geodesic equation: and we write

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma\,, \tag{1.19}$$

where the affine parameter λ has the dimensions of a mass, in order to have dimensional consistency.

Then, the Liouville operator is

$$\hat{\mathbb{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \stackrel{\text{def}}{=} \frac{\mathrm{D}}{\mathrm{D}\lambda}\,. \tag{1.20}$$

This is a total derivative in phase space with respect to the affine parameter.

In the FLRW background, $f = f(|p|, t)$ and

$$\hat{\mathbb{L}} = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}|p|^2 \frac{\partial f}{\partial E}\,, \tag{1.21}$$

so if we define the number density

$$n(t) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, f(E, t)\,, \tag{1.22}$$

so if we integrate over momenta we get

$$\int \frac{\mathrm{d}^3 p}{E} \hat{\mathbb{L}}[f]\,, \tag{1.23}$$

we find the equation from before:

$$\dot{n} + 3\frac{\dot{a}}{a} n\,, \tag{1.24}$$

??? to check

Now we use a perturbed FLRW metric, in the Poisson gauge.

$$ds^2 = -e^{2\Phi}\,dt^2 + 2a\omega_i\,dx^i\,dt + a^2\left(e^{-2\Psi}\delta_{ij} + \chi_{ij}\,dx^i\,dx^j\right), \tag{1.25}$$

where we neglect spatial curvature (which we will do from now on). This is because we would never be able to see the effect of spatial curvature in the geometry (although we could see it in the dynamics).

Let us describe the quantities we introduced. We have 10 degrees of freedom in the metric. We account for them like this: Φ and Ψ are scalar, ω is a vector, χ_{ij} is a tensor. This is explained in more detail in the class by Nicola Bartolo ("early Universe"). These are GR perturbations.

"Perturbation" means that we compare the physical spacetime and the idealized FLRW metric. We need to do this since we cannot solve the EFE if there is no symmetry. So, we say that spacetime is *close* to the idealized spacetime.

We need a map between the physical and idealized spacetimes: this is called a *gauge*. Perturbations are classified with respect to their effect on FLRW. In euclidean space we know scalars, vectors, tensors. The perturbations will behave as such, under a change of coordinates in the Cartesian space which is the 3D space-like slice of FLRW.

ω_i carries a 3D vector index. χ_{ij} contains the off-diagonal perturbations. Let us start with ω_i. In principle: Helmholtz's theorem says that we can decompose $\omega_i = \omega_{i,\text{transverse}} + \partial_i\omega$. We say that the part we are interested in is the transverse one, but we still have a gradient.

We choose our χ such that $\chi^i_j = \chi^i_{j,i} = 0$. χ could also contain vectors (objects with a vector index), as long as they are divergenceless.

It can also contain tensors: these are GWs.

So, we have 3 scalars (a, Ψ, Φ), 3 components of a transverse vector (ω_i), plus the divergence ω, while in the tensor we have $6 - 3 - 1 = 2$ degrees of freedom.

So we have 10 total degrees of freedom. The reason we do this is that the degrees of freedom obey independent eqs. of motion.

There is gauge ambiguity in our problem: we could change the mapping between physical space and FLRW. We could do a change such as $x^\mu \to x'^\mu = x^\mu +$ a perturbation.

This is explained better in the notes called "GR perturbations".

Gauge freedom allows us to drop 2 of the 4 scalars we have.

We use Poisson or longitudinal gauge, which is sometimes incorrectly called "Newtonian gauge" even though it is not Newtonian.

Our next goal will be to solve the geodesic eqs. for the motion of the particles in this gauge.

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We come back to the discussion from last time, about the Boltzmann equation in a perturbed universe.

When can we drop some terms using a gauge transformation? We can do it for scalar and vector perturbations. We shall use the longitudinal, or Poisson gauge, in which the scalar perturbations are reduced to Φ and Ψ, so we can take the tensor terms to be traceless and covariantly constant.

We will not discuss vectors, since if they are zero at the beginning they cannot be generated, they stay at zero. In our natural units, the perturbations will be small: $\Phi \ll 1$ and $\Psi \ll 1$. Recall that we are neglecting spatial curvature.

Photons have $P^2 = 0$: so we can express this as

$$-(1+2\Phi)(p^-)^2 + p^2 = 0\,, \tag{1.26}$$

where p^2 is defined as $p^2 = g_{ij}P^iP^j$, since in our gauge choice the spatial part of the metric has a Kronecker delta this will only include the spatial parts. So, we get

$$P^0 = \frac{p}{\sqrt{1+2\Phi}} \approx p(1-\Phi)\,. \tag{1.27}$$

Now, we will write the Liouville operator by dividing through by P^0:

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\cdot\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}\,, \tag{1.28}$$

where we split the three-momentum into absolute value p and the unit vector $\hat{p}$, which has $\hat{p}^i = \hat{p}_i$ and $\delta_{ij}\hat{p}^i\hat{p}^j$. We have $\mathrm{d}x^i \big/ \mathrm{d}t = P^i/P^0$.

We are going to expand only to first order. Higher order are more important for small angular scales, and for secondary CMB anisotropies, these are interesting but we are not going to treat them.

To first order, the last term of the RHS is zero.

Figure out how this works

Now we define the amplitude A by $P^i = A\hat{p}^i$: now we will have

$$\begin{aligned} p^2 = g_{ij}P^iP^j &= a^2\delta_{ij}(1-2\Psi)\hat{p}^i\hat{p}^jA^2 \qquad (1.29)\\ &= a^2(1-2\Psi)A^2\,, \qquad (1.30) \end{aligned}$$

therefore, taking the square root and staying to first order we get

$$A \approx p\frac{1+\Psi}{a}\,, \tag{1.31}$$

so

$$P^i = p\hat{p}^i\frac{1+\Psi}{a}\,, \tag{1.32}$$

and the division by a can be interpreted as a redshift effect. Inserting this term we get

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{P^i}{P^0} = \hat{p}^i \frac{1+\Psi+\Phi}{a}, \tag{1.33}$$

and we can notice that $\mathrm{d}x^i \Big/ \mathrm{d}t$ multiplies the term $\partial f \Big/ \partial x^i$, which is only nonzero to first order: so we must consider this term to zeroth order. So, we get

$$\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}, \tag{1.34}$$

and now we shall show that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H - \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right), \tag{1.35}$$

which will imply that

$$\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left(H - \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right). \tag{1.36}$$

We will use the geodesic equation for photons: it is enough to consider its zeroth component, which is

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta, \tag{1.37}$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(p(1+\Phi)\big) = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta \frac{1+\Phi}{p}, \tag{1.38}$$

where we brought a P^0 from the left to the right side. This means that we have

$$(1-\Phi)\frac{\mathrm{d}p}{\mathrm{d}t} = p\frac{\mathrm{d}\Phi}{\mathrm{d}t} - \Gamma^0_{\alpha\beta}P^\alpha P^\beta \frac{1+\Phi}{p}, \tag{1.39}$$

and now we multiply both sides by $1+\Phi$, the inverse of $1-\Phi$ to linear order:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right) - \Gamma^0_{\alpha\beta}P^\alpha P^\beta \frac{1+2\Phi}{p}, \tag{1.40}$$

and now we have to start calculating the Christoffel symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\Big(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\Big), \tag{1.41}$$

so we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\Big(2g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\Big)\frac{P^\alpha P^\beta}{p}, \tag{1.42}$$

but g^{0i} are zero, since we are ignoring vector perturbations, and $g^{00} = -1 + 2\Phi$

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