

Gravitational physics notes

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0.1 Introduction

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Office hour: check by email.

Reading material: slides (to be used as an index of what is treated in the course), Hobson, Michele Maggiore.

This is a general class on gravitational physics and GW, it does not really follow any textbook: the field is young so there is not really a textbook.

The slides will be provided before lectures. There will be no home assignments.

The idea is that formulas are important, detailed calculations and derivations are not.

The target is to be able to read a research paper on GW and understand it. We will not go into very much detail on any topic: the program of the class is very large.

For the exam: it is a discussion of a GW paper (about 25 min), plus theoretical questions — focusing on the physical meaning, not on tedious derivations. It usually takes a bit less than an hour. The paper is optional.

Off session exams are OK, best if with several people (2-5 people).

Please fill out the questionnaire on the course before taking the exam.

0.1.1 Topics

Understanding what GW are: how they are described, how they are generated, what is their physical effect.

Some astrophysical and cosmological GW courses. The professor's background is more in experimental physics than in astrophysics and cosmology.

Interactions of GW with light and matter: ideas, techniques, experiments to detect GWs, especially GW interferometers.

Analysis of GW signals.

What we can learn from GW, overview of the most significant recent papers.

What follows is a long, somewhat divulgative introduction.

Einstein thought their detection impossible. Now we can not only *detect* them, we can actually *observe* them.

They are a test of GR in *extreme* conditions, where the weak-field approximation does not apply.

We can derive properties of matter in these extreme conditions, such as the equation of state for a neutron star.

GWs are “ripples” in the metric of spacetime, described by a quadrupole formula: the quadrupole is

$$Q_{jk} = \int \rho x_j x_k d^3x , \quad (1)$$

and then the perturbation propagates like

$$h_{jk} = \frac{2}{r} \frac{d^2 Q_{jk}}{dt^2} . \quad (2)$$

What generates GWs are non-spherically symmetric perturbations: by Jebsen-Birkhoff, if we have spherical symmetry there is no perturbation in the vacuum metric.

They “stretch” space by squeezing one direction and stretching a perpendicular one.

The typical relative scale of these perturbations is

$$\frac{\Delta L}{L} \sim 10^{-21} , \quad (3)$$

which is *really small*: if we multiply it by the radius of the Earth’s orbit we get a length on the order of the size of an atom.

An interesting thing which could emit in the ~ 1 Hz range are extreme Mass Ratio inspirals: we have what is effectively a test particle in a strong gravitational field.

We have different kinds of interferometers: for now we have used ground interferometers, there are also space detectors like LISA, Pulsar Timing Arrays at higher frequencies, and inflation probes (?).

In binary systems, we have different stages in the pulsation: an almost stationary one, the inspiral, the coalescence, and finally the ringdown.

In the early years, it was thought that GWs might be a coordinate artifact which could be “gauged away”.

In 1959, Joseph Weber proposed a “resonant bar” detector. These are based upon a sound principle: one of the last ones was AURIGA, the issue was that the sensitivity was insufficient and they are only sensitive in a specific frequency range.

GWs were detected indirectly using Hulse-Taylor pulsars: they measured the energy loss of a binary pulsar-NS system, which implied the loss of energy through gravitational wave emission.

The famous graph is not a fit line, it is the prediction based upon the measured orbital parameters.

Now we use laser interferometers: they are broad-band (a couple orders of magnitude, from 10 Hz to 1 kHz), they are inherently differential (as opposed to the single-mode excitation of a resonant bar).

We can use Fabry-Perrot cavities in order to amplify effective length. There is also a power recycling mirror in order for the light not to go back to the laser: modern lasers are on the order of 100 kW, so there is a huge amount of power circulating in the cavities.

We can plot the sensitivity of the interferometers. On the x axis we put the frequency of the incoming wave. On the y axis we put the amplitude spectral density $h(f)$, which is measured in $\text{Hz}^{-1/2}$.

Why?

The curve describes where the noise dominates. We can plot both the theoretical sensitivity and the measured one.

The signal comes out buried in noise, we must extract it in some way, like by correlating to a standard test signal.

What we will do in the first lessons:

1. A quick review of GR;
2. linearization and GW in free space;
3. the physical effect of GW: free falling reference frames, detector frame;
4. GW sources : binary systems, multipole expansion and quadrupole approximation, GW back reaction: energy & momentum loss, Hulse-Taylor pulsar;
5. GW sources: a rotating rigid body.

Friday
2020-3-13,
compiled
2020-03-20

0.2 A quick review of GR

We start from special relativity. The “old” way to do transformations are galilean transformations: in 2D they are

$$t' = t \quad (4a)$$

$$x' = x - vt. \quad (4b)$$

There are issues with these, such as the invariance of the speed of light. So, we use Lorentz transformations:

$$ct' = \gamma(ct - \beta x) \quad (5a)$$

$$x' = \gamma(x - \beta ct), \quad (5b)$$

where $\beta = v/c \leq 1$, c being the speed of light, and $\gamma = 1/\sqrt{1 - \beta^2} \geq 1$.

These preserve the spacetime interval

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2. \quad (6)$$

The intervals between two events can be spacelike ($\Delta s^2 > 0$), null ($\Delta s^2 = 0$) or timelike ($\Delta s^2 < 0$).

We can express this using an infinitesimal time interval

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu , \quad (7)$$

where we use Einstein summation convention. We are going to use the mostly plus metric convention.

We can define the differential *proper time* along a curve, by

$$c^2 d\tau^2 = -ds^2 = c^2 dt^2 (1 - \beta^2) = \frac{c^2}{\gamma^2} dt^2 , \quad (8)$$

which means that $d\tau = dt / \gamma$. We can use this as a *covariant* parametrization of a spacetime curve.

For curved spacetime, we model it as a 4D semi-Riemannian manifold with signature (1,3). Since it is a manifold, the parametrization of points in spacetime must be a homeomorphism, and we ask for the *transition maps* between two regions of spacetime to be infinitely differentiable. The set of local charts is called an atlas. The charts are maps from \mathbb{R}^4 to the manifold.

The metric is a function of the point at which we are, and (the way it changes) describes the local geometry of the manifold. Only the symmetric part of the metric appears in the spacetime interval, therefore we say that the metric is always symmetric without losing any generality.

The metric is a bilinear form at each point of the manifold, and it transforms as a (0,2) tensor. The components of this tensor in our chosen reference frame are $g_{\mu\nu}$. The choice of coordinates is arbitrary and tricky.

In a neighborhood of a point we can always choose a reference frame (Riemann normal coordinates) such that $g_{\mu\nu} = \eta_{\mu\nu}$, and $g_{\mu\nu,\alpha} = 0$ (partial derivatives calculated *at that point*), but the second derivatives $g_{\mu\nu,\alpha\beta}$ cannot all be set to zero.

Vectors in a manifold are defined in the tangent space *at a point*. Formally, we define curves parametrically as $X^\mu(\lambda)$.

Then, we define the tangent vector to the curve as the *directional derivative* operator along the curve:

$$\vec{\partial}(f) = \left. \frac{df}{d\lambda} \right|_C = \frac{\partial f}{\partial x^\mu} \frac{dX^\mu}{d\lambda} , \quad (9)$$

for any scalar field f . The motivation for this definition, as opposed to just taking the tangent vector to the curve, is the fact that there is no *intrinsic* way to do that.

If we define a curve using a coordinate as a parameter, with the other coordinates staying constant along the curve, this is called a *coordinate curve*.

Vectors defined at different points are in different spaces, we cannot compare them directly.

Tangent vectors to coordinate lines are called coordinate basis vectors $e_{(\mu)}$, where μ is not a vector index but instead it spans the basis vectors. So, any vector can be written as a linear combination as $\vec{v} = v^\mu e_\mu$. We also have $e_\mu \cdot e_\nu = g_{\mu\nu}$, so, in order to find the components of the scalar product $v \cdot w$ we need to do $v^\mu w^\nu g_{\mu\nu}$.

This is because $g_{\mu\nu} dx^\mu dx^\nu = ds \cdot ds = (dx^\mu e_\mu) \cdot (dx^\nu e_\nu)$.

An orthonormal basis in one for which $e_\mu \cdot e_\nu = \eta_{\mu\nu}$.

Dual basis vectors e^μ are defined by $e^\mu e_\nu = \delta^\mu_\nu$.

We write a co-vector (or dual vector) as a linear combination of these: $v = v_\mu e^\mu$.

Then, we can raise and lower indices like

$$g_{\mu\nu} v^\mu w^\nu = v \cdot w = v_\mu e^\mu \cdot w^\nu e_\nu = v_\mu w^\nu \delta^\mu_\nu = v_\mu w^\mu. \quad (10)$$

The inverse metric is defined as $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$.

Tensors are geometrical objects which belong to the dual space to the cartesian product of n copies of the tangent space and m copies of the dual tangent space. The type of such a tensor is then said to be (n, m) , and its rank is $n + m$. This definition means that the tensor is a *multilinear* transformation.

Once we have a coordinate system, we can move to another via a coordinate transformation

$$x'^\mu = x'^\mu(x^\mu) \implies dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu. \quad (11)$$

A scalar does not transform: $\phi(x) = \phi'(x')$. A vector's component do transform: we find the transformation law by imposing $v = v'$ in components. This works both for covariant and contravariant vectors, and we find that these transform using either the Jacobian of the transformation or its inverse.

In order to compute derivatives we need to compare vectors in different tangent spaces: we need to “connect” infinitesimally close tangent spaces, and the tool to do so is indeed called a connection, or covariant derivative.

For a scalar field S the covariant derivative is $\nabla_\alpha S = \partial_\alpha S$.

A torsionless manifold is one in which

$$[\nabla_\mu, \nabla_\nu] S = 0, \quad (12)$$

for a scalar field S . This means that

$$\nabla_{[\mu} \nabla_{\nu]} S = \nabla_{[\mu} \partial_{\nu]} S = \partial_{[\mu} \partial_{\nu]} S - \Gamma_{[\mu\nu]}^\alpha \partial_\alpha S = 0 \implies \Gamma_{[\mu\nu]}^\alpha = 0. \quad (13)$$

Parallel transport: intuitively, we move along a curve and keep the angle with respect to the tangent vector constant. Formally, if u^μ is the tangent vector to the curve and V^μ is the vector we want to transport, we set $u^\mu \nabla_\mu V^\nu = 0$.

The Riemann tensor is defined as the commutator of the covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R_{\beta\mu\nu}^\alpha V^\beta, \quad (14)$$

and it can be expressed in terms of the Christoffel symbols as

$$R_{\nu\rho\sigma}^\mu = -2 \left(\Gamma_{\nu[\rho\sigma]}^\mu + \Gamma_{\nu[\rho}^\beta \Gamma_{\sigma]\beta}^\mu \right). \quad (15)$$

Geodesics: they are “the straightest possible path between two points”. They stationarize the proper length. Formally, they are curves whose tangent vector is parallel-transported along the curve.

We actually do not need to say that the derivative of the tangent vector with respect to the parameter is zero: it can be nonzero, as long as it is parallel to the tangent vector.

So, we could say that

$$h_{\nu\rho} \left(u^\mu \nabla_\mu u^\nu \right) = 0, \quad (16)$$

?

The path that a massive particle follows in the absence of external forces is a geodesic. We can describe the separation between two particles which follow geodesics: this is described by the equation of geodesic deviation. We take a geodesic x^μ and another $y^\mu = x^\mu + \zeta^\mu$, with ζ^μ being (at least initially) small.

We can choose a coordinate system in which $\Gamma_{\nu\rho}^\mu = 0$. So,

$$\left. \frac{d^2 x^\mu}{du^2} \right|_P = 0, \quad (17)$$

$$\left(\frac{d^2 y^\mu}{du^2} + \Gamma_{\nu\rho}^\mu \frac{dy^\nu}{du} \frac{dy^\rho}{du} \right) \Big|_P = 0, \quad (18)$$

where u is the tangent vector to the geodesics. We approximate the Christoffel symbols to first order as

$$\Gamma_{\nu\rho}^\mu \Big|_Q = \zeta^\alpha \partial_\alpha \Gamma_{\nu\rho}^\mu. \quad (19)$$

If we subtract the two, we get

$$\ddot{\zeta}^\mu + \left(\partial_\alpha \Gamma_{\nu\rho}^\mu \right) \dot{x}^\nu \dot{x}^\rho \zeta^\alpha = 0, \quad (20)$$

but the first term is not an intrinsic derivative: that would be given by

$$\frac{D^2 \zeta^\mu}{Du^2} = \frac{d}{du} \left(\dot{\zeta}^\mu + \Gamma_{\nu\rho}^\mu \dot{\zeta}^\nu \dot{x}^\rho \right) = \ddot{\zeta}^\mu + \left(\partial_\alpha \Gamma_{\nu\rho}^\mu \right) \zeta^\alpha \dot{x}^\nu \dot{x}^\rho, \quad (21)$$

which means that

$$0 = \frac{D^2 \zeta^\mu}{Du^2} + \left(\partial_\alpha \Gamma_{\nu\rho}^\mu - \partial_\rho \Gamma_{\nu\alpha}^\mu \right) \zeta^\alpha \dot{x}^\mu \dot{x}^\nu = \frac{D^2 \zeta^\mu}{Du^2} + R_{\nu\rho\sigma}^\mu u^\nu u^\rho \zeta^\sigma. \quad (22)$$

The gravitational field is described by the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (23)$$

Remember that the version of the slides which appears during the lesson might be updated.

There might be some disruption in late March because the professor is having a child.

Monday
2020-3-16,
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0.3 Linearized GR

Let us assume that our metric tensor is almost flat: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$.

This is a coordinate dependent statement: however the physical situation is clear — almost flat spacetime — and the way we will proceed is to expand in h .

Do note that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$.

Choosing a reference in which these components are small limits our gauge freedom: we will be able to do only transformations which preserve the condition.

We will be able to do global Lorentz transformations:

$$g' = \Lambda \Lambda g, \quad (24)$$

therefore

$$g' = \Lambda \Lambda (\eta + h) = \eta + \Lambda \Lambda h, \quad (25)$$

so the flat metric does not change, while h changes to $h' = \Lambda \Lambda h$. We are omitting indices, they are in the usual positions.

A more general class of transformation is given by those which can be expressed as

$$x^\mu \rightarrow x'^\mu + \xi^\mu, \quad (26)$$

so the Jacobian will look like

$$\delta^\mu_\nu + \partial_\nu \xi^\mu, \quad (27)$$

and we ask that $\partial \xi$ is small — formally, first order in $h_{\mu\nu}$. This yields

$$g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - 2\partial_{(\mu} \xi_{\nu)}. \quad (28)$$

For an alternative reference on this derivation, see the notes on General Relativity in the course by Marco Peloso [TM20, section 10].

Now, we wish to linearize the Riemann tensor: we start from the Christoffel symbols. We get

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (2g_{\rho(\mu,\nu)} - g_{\mu\nu,\rho}) \quad (29a)$$

$$= \frac{1}{2} (\partial_\mu h^\sigma_\nu + \partial_\nu h^\sigma_\mu + \partial^\sigma h_{\mu\nu}), \quad (29b)$$

and now in the Riemann tensor $R = \partial\Gamma + \Gamma\Gamma$ the $\Gamma\Gamma$ terms are second order in h , so we ignore them. Then, we get the simplified expression

$$R^\sigma_{\mu\nu\rho} = \frac{1}{2} (\partial_\nu \partial_\mu h^\sigma_\rho + \partial_\rho \partial^\sigma h_{\mu\nu} - \partial_\nu \partial^\sigma h_{\mu\rho} - \partial_\rho \partial_\mu h^\sigma_\nu), \quad (30)$$

so the Ricci tensor — which we will set to zero in the vacuum — will be

$$R_{\mu\nu} = R^\sigma_{\mu\nu\sigma} = \frac{1}{2} (\partial_\nu \partial_\mu h + \square h_{\mu\nu} - \partial_\nu \partial_\sigma h^\sigma_\mu - \partial_\sigma \partial_\mu h^\sigma_\nu), \quad (31)$$

and the Ricci scalar is

$$R = \eta^{\mu\nu} R_{\mu\nu} = \square h - \partial_\nu \partial_\sigma h^{\sigma\nu}. \quad (32)$$

Now, the field equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx \frac{1}{2} \left[\partial_\nu \partial_\mu h + \square h_{\mu\nu} - \partial_\nu \partial_\sigma h_\mu^\sigma - \partial_\sigma \partial_\mu h_\nu^\sigma - \eta_{\mu\nu} (\square h - \partial_\nu \partial_\sigma h^{\sigma\nu}) \right] = -\frac{1}{M_P^2} T_{\mu\nu}. \quad (33)$$

Now, we define the trace reverse perturbation as

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad (34)$$

so the doubly-trace reverse is the perturbation itself, and $\bar{\bar{h}} = -h$.

Now we shall use Lorenz gauge¹

We shall use our gauge freedom. How does the trace-reverse perturbation transform?

$$\bar{h}'^{\mu\rho} = h^{\mu\rho} - 2\partial^{(\mu} \xi^{\rho)} - \frac{1}{2} \eta^{\mu\rho} (h - 2\partial_\sigma \xi^\sigma) \quad (35a)$$

$$= \bar{h}^{\mu\rho} - 2\partial^{(\mu} \xi^{\rho)} + \eta^{\mu\rho} \partial_\sigma \xi^\sigma. \quad (35b)$$

The derivatives of the new and old perturbations differ by

$$\partial_\rho \bar{h}'^{\mu\rho} - \partial_\rho \bar{h}^{\mu\rho} = -\square \xi^\mu, \quad (36)$$

so we can choose a variable change ξ^μ such that $\partial_\rho \bar{h}'^{\mu\rho} = 0$.

Then, the linearized field equations will read

$$\square \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} - \partial_\nu \partial_\rho \bar{h}_\mu^\rho - \partial_\mu \partial_\rho \bar{h}_\nu^\rho = -2 \frac{T_{\mu\nu}}{M_P^2}, \quad (37)$$

which finally yields

$$\square \bar{h}_{\mu\nu} = -\frac{2T_{\mu\nu}}{M_P^2}. \quad (38)$$

Comments on the linearized equations Since we expanded in $\eta_{\mu\nu}$, the quantities $h_{\mu\nu}$ have a geometric meaning but we are treating them as 16 scalar fields.

When we look at the geodesic equations, we get a prediction of the gravity having no effect on matter. We are treating gravity as a linear theory, so we have the superposition principle.

We are ignoring the physical principle that “gravity gravitates”: curvature of spacetime is associated to a SEMT in a nonlinear way.

¹ This is not the same as Lorentz, after whom Lorentz covariance is called.

GWs in empty space We set $T_{\mu\nu}$ to zero, so we get

$$\square \bar{h}_{\mu\nu} = 0, \quad (39)$$

which is the usual wave equation, with solutions $\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\lambda x^\lambda}$.

In general $A_{\mu\nu}$ is symmetric, constant, complex. k_λ is constant and real, and we must have

$$\eta^{\rho\sigma} k_\rho k_\sigma A_{\mu\nu} e^{ik \cdot x} \implies k^2 = 0. \quad (40)$$

In order to have these conditions, we must still have

$$\partial_\mu \bar{h}^{\mu\nu} = A^{\mu\nu} k_\mu e^{ik \cdot x} = 0 \implies A^{\mu\nu} k_\nu = 0. \quad (41)$$

The conjugate of the wave equation also holds, so after our manipulations we will always be able to take the real part.

We still have gauge freedom: we can perform transformations if they satisfy $\square \xi^\mu = 0$.

We define

$$\xi^{\mu\nu} = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\rho} \partial_\rho \xi^\sigma, \quad (42)$$

and we will have $\square \xi^{\mu\nu} = 0$ since the D'Alembertian commutes with the other derivatives.

So, if $\bar{h}^{\mu\rho}$ satisfies the Vacuum field equations, then $\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \xi^{\mu\nu}$ also does.

So, we can use the 4 functions ξ^μ to set 4 constraints on $\bar{h}^{\mu\rho}$:

$$\bar{h}_{TT}^{0i} = 0 \quad (43a)$$

$$\bar{h}_{TT} = 0, \quad (43b)$$

which conveniently means that $\bar{h}_{\mu\nu} = h_{\mu\nu}$. This is called TT-gauge.

Combined with the Lorenz gauge, which says

$$0 = \partial_\rho \bar{h}_{TT}^{0\rho} = \partial_0 \bar{h}_{TT}^{00} = 0, \quad (44)$$

which means that the metric element is constant, so we can rescale time in order to set it to zero.

Also, the other Lorenz gauge conditions are

$$0 = \partial_\rho \bar{h}_{TT}^{j\rho} = \partial_i \bar{h}_{TT}^{ji}, \quad (45)$$

which means that, of the 6 potentially free components of the h^{ij} , we actually have only 2 degrees of freedom.

Now, if we assume $\vec{k} = k\hat{z}$ then we will have $k^\mu = (k, 0, 0, k)$; also, in the matrix $A_{\mu\nu}$ we will have $A^{ij}k_j = 0$, so $A^{i3} = 0$.

Then, in full generality under our gauge choices we shall have

$$\bar{h}_{TT}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{ik(t-z)}. \quad (46a)$$

For a generic direction of propagation, we can define a projector onto the direction orthogonal to the direction of propagation k_i : $P_{ij} = \delta_{ij} - k_i k_j$, so we will have

$$A_{TT}^{ij} = \left(P_k^i P_l^j - \frac{1}{2} P^{ij} P_{kl} \right) A^{kl}. \quad (47)$$

Friday
2020-3-20,
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0.4 The physical effects of gravitational waves

We want to discuss how we can build instruments which can detect gravitational waves.

An open question for decades was to see whether the effects of gravitational waves could be removed using a proper gauge choice. There was a conference in Chapel Hill (?) which showed examples of non-removable gravitational wave effects, such as the “beads on a stick”, which move and dissipate energy if a GW passes through them: a non removable effects.

What happens to free particles in the TT gauge? The geodesic equation for the spatial indices reads

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma_{\mu\nu}^i \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}, \quad (48)$$

where the parameter τ parametrize our curve. We assume that the particle starts out at rest: then its four-velocity is $dx^\mu/d\tau = (dx^0/d\tau, \vec{0})$. So, we get the simplification

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \left(\frac{dx^0}{d\tau} \right)^2, \quad (49)$$

and in linearized gravity

$$\Gamma_{00}^i \approx \frac{1}{2} (2\partial_0 h_0^i - \partial^i h_{00}) = 0 \quad (50)$$

if we use the TT gauge. This means that the derivative of the velocity is zero: so, the velocity of a stationary particle remains zero indefinitely. Let us consider geodesic deviation between two particles instead: say that the first particle has the geodesic $x(\tau)$ and the second is $x(\tau) + \xi(\tau)$. Their geodesic equations will read

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma_{\mu\nu}^\sigma(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \quad (51)$$

$$\frac{d^2 (x^\sigma + \xi^\sigma)}{d\tau^2} + \Gamma_{\mu\nu}^\sigma(x + \xi) \frac{dx^\mu}{d\tau} \frac{d(x^\nu + \xi^\nu)}{d\tau} = 0, \quad (52)$$

which we can expand to first order using the perturbation: $\Gamma_{\mu\nu}^\sigma(x + \xi) = \Gamma_{\mu\nu}^\sigma(x) + \partial_\gamma \Gamma_{\mu\nu}^\sigma \xi^\gamma$, using which and expanding we finally get

$$\frac{d^2 \xi^\sigma}{d\tau^2} + 2\Gamma_{\mu\nu}^\sigma \frac{dx^\mu}{d\tau} \frac{d\xi^\nu}{d\tau} + \xi^\gamma \partial_\gamma \Gamma_{\mu\nu}^\sigma \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}, \quad (53)$$

so if we restrict ourselves to only spatial components, and assume that the particles start out stationary we get

$$\frac{d^2 \xi^i}{d\tau^2} = -2\Gamma_{0\nu}^i \frac{dx^0}{d\tau} \frac{d\xi^\nu}{d\tau} + \xi^\gamma \partial_\gamma \Gamma_{00}^i \frac{dx^0}{d\tau} \frac{dx^0}{d\tau}, \quad (54)$$

so, using the expressions for the Christoffel symbols in the TT gauge we get

$$\frac{d^2 \xi^i}{d\tau^2} = -2c\Gamma_{0j}^i \frac{d\xi^j}{d\tau} = -c\partial_0 h^{ij} \frac{d\xi^j}{d\tau}, \quad (55)$$

so, parallel geodesics remain parallel: if the separation initially is stationary, it will remain so.

The issue is that in the TT gauge we are using a special set of coordinates which “follow” the gravitational wave. We see no change in coordinate distance since the coordinates are moving around with the gravitational wave: we did a coordinate change using ξ^μ satisfying $\square \xi^\mu = 0$, so the coordinates are harmonically moving, together with the GW.

It is like we defined wave-like coordinates, “gauging away” the wave-like motion.

We can overcome this issue by calculating *proper distances* instead of coordinate distances: for two space-like separated objects along the x axis we get

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + h_{\mu\nu}^{TT} dx^\mu dx^\nu. \quad (56)$$

Let us apply this to the case of a GW propagating along the z axis, for particles separated along the x axis. The full metric perturbation looks like

$$h_{TT}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{i\omega(t-z/c)}, \quad (57)$$

then the distance, in the case of an h_+ polarized wave, becomes

$$s = (x_1 - x_2) \sqrt{1 + h_+ \cos(\omega t)} \approx (x_1 - x_2) \left(1 + \frac{1}{2} h_+ \cos(\omega t) \right). \quad (58)$$

So, the fractional change is given by $h_+/2$.

For two general events separated by the 4-vector L :

$$s^2 = (\eta_{\mu\nu} + h_{\mu\nu}) L^\mu L^\nu \approx L \left(1 + \frac{1}{2} h_{ij} L^i L^j \right). \quad (59)$$

We would, however, like to work in coordinates. The useful frame to define is the *free falling frame*, whose coordinates are rigid and not perturbed by the GW.

In order to build such a frame, a local inertial frame which will be inertial, we will need to define 4 orthogonal vectors on the point P :

$$\eta_{\mu\nu} e_\alpha^\mu e_\beta^\nu = \eta_{\alpha\beta}. \quad (60)$$

Consider a geodesic through point P whose tangent vector at P is a unit vector \hat{n} . If it is spacelike, we parametrize it by s (defined with ds^2 from the metric), if it is timelike we parametrize it with τ (defined by $d\tau^2 = -ds^2$). We call λ a generic one of those.

Now, the coordinates of point Q are generically $\lambda\hat{n}$, if the geodesic starting with unit vector \hat{n} reaches Q when its parameter is λ .

We can reach almost every point this way, the points which are only connected through null geodesics can be reached by continuity, and in a small enough region the coordinates of a point Q are unique — that is, the geodesics do not cross.

In this frame, then, $g_{\mu\nu}(P) = \eta_{\mu\nu}(P)$; also, in the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0, \quad (61)$$

we have that the second derivatives are zero since $x^\mu(\lambda)$ is linear, so we must have $\Gamma_{\nu\rho}^\mu n^\nu n^\rho = 0$. This must be true for any unit vector, therefore we have $\Gamma_{\nu\rho}^\mu = 0$. The system giving $g_{\mu\nu,\rho}(P)$ from $\Gamma_{\nu\rho}^\mu$ is nondegenerate, so we also have $g_{\mu\nu,\rho}(P) = 0$.² These are called **Riemann normal coordinates**.

The conditions on the metric and its derivatives only hold at the point. We can do slightly better with *Fermi normal coordinates*, where we require a gyroscope's angular momentum to be parallel-transported along the geodesics.

How do we make such a frame? we use free falling particles: we could put them in orbit. This is not actually that simple: consider *drag-free satellites*.

Consider a particle orbiting the Sun. The Sun's radiation pressure pushes the particle away from a geodesic.

Maybe we could put a thrusted spacecraft around our test mass: it balances the sun's radiation pressure; we measure the distance to the test mass without touching it, and then balance the thrusters by keeping at a constant distance from it.

This is the idea behind LISA. The distances between the spacecrafts should be about 5Gm apart. We do not measure the distances between the spacecrafts, but instead the distances between the test masses inside them, which are 2 kg, 4.6 cm side, gold-platinum shielded cubes: this is known as *drag-free* navigation.

The interferometric measurements have pico-meter (10^{-12} m) sensitivity. It takes about 30 s for light to move between the mirrors: this time-delayed interferometry needs special consideration.

We have a *gravitational reference sensor*, a cubic shell around the cube: we keep measuring the distances between the two. We also need to discharge the masses, otherwise electrostatic forces are too strong. Also, the thrusters need to be very weak, on the order of the μ N.

The LISA Pathfinder mission checked all of these boxes, except for time-delayed interferometry. It only used one spacecraft, and measured how well the drag-free navigation worked.

Using two masses, we need to account for the gravitational pull between them.

Even accounting for this by relaxing the acceleration precision requirement 10-fold, the results were exceptional.

² Do note that this reasoning works only at the point, since if we moved along a geodesic we do not have access to the other unit vectors anymore (in these coordinates).

Let us discuss the sources of noise: at high frequencies, inertia prevents a force from creating significant displacement. This applies to external forces, not to gravitational forces, since the latter are proportional to the mass. So, there is a limit at high frequencies because of our inability to measure that fast.

At low frequencies, it is easy to measure, but it is hard to verify whether the mass is indeed in free fall.

In the end, it was verified that we can do

$$s_a^{1/2} \leq 3 \times 10^{-14} \text{ m/s}^2 / \sqrt{\text{Hz}} \quad (62)$$

at 1 mHz. Solar radiation pressure is two orders of magnitude higher.

We can have issues with the parasitic coupling of the test mass to the spacecraft.

Let us now come back to theory by discussing the *proper detector frame*: coordinates defined by a rigid ruler. Rigid rulers do not really exist, but we can approximate it well enough. If the gravitational pull is small compared to the restoring forces in the ruler than its length will approximately not change.

Let us put ourselves in a free falling frame in Fermi local coordinates; in the origin, the metric is always flat. Let us expand to second order in the spatial coordinates

$$g_{\mu\nu}(x) \approx g_{\mu\nu}(0) + x^i \partial_i g_{\mu\nu} \Big|_{x=0} + \frac{1}{2} x^i x^j \partial_i \partial_j g_{\mu\nu} \Big|_{x=0} \quad (63)$$

$$= \eta_{\mu\nu} + \frac{1}{2} x^i x^j \partial_i \partial_j g_{\mu\nu} \Big|_{x=0}, \quad (64)$$

which we can rewrite in terms of the Riemann tensor by making use of the expression of the Riemann tensor in the LIF: we get

$$ds^2 \approx -c^2 dt^2 \left(1 + R_{0i0j} x^i x^j \right) - 2c dt dx^i \left(\frac{2}{3} R_{0ijk} x^j x^k \right) + dx^i dx^j \left(\delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l \right), \quad (65)$$

so the corrections are of the order $\mathcal{O}(r^2/L_B^2)$, where r^2 is the square distance from the origin, while L_B is the typical spatial scale of the variation of the metric, such that $R_{0ijk} = \mathcal{O}(L_B^{-2})$. This L_B is the wavelength of the GW if we are describing a GW.

So, this coordinate description works as long as the scale of the region we are describing is small compared to the wavelength of the GW.

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