Theoretical cosmology notes

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Contents

Contents of the course

We start with a derivation of the Friedmann eqs. from the Einstein equations.

We will then discuss the properties of the CMB, deriving the spectrum, and then CMB anisotropies.

Then we will discuss star and structure formation, about the nonlinear evolution of perturbations. We will use the path-integral approach to classical field theory. We will also discuss weak gravitational lensing in the universe.

We will use some smart nonlinear approximations: the Zel'dovich approximation and the adhesion approximation.

We will use an "effective Planck constant" instead of $\hbar$: it will be a parameter which can be fit in our model.

As for references: there are handwritten notes by the professor in the Dropbox folder (for access to the folder, write to the professor). Also, there notes by a student from the previous years, in Italian [Nat17], which are to be used with caution as they contain some errors.

0.1 Friedmann equations: a brief overview

Thursday 2020-3-12, compiled 2020-05-21

Throughout the course, we set $\hbar = c = k_B = 1$.

In the previous course we used the approximate symmetries of the universe to write the FLRW line element:

$$ds^2 = -\,dt^2 + a^2(t)\,d\sigma^2 \,, \tag{1}$$

do note that we switch signature from the previous course: now we use the mostly plus one. The spatial part is defined by

$$d\sigma^2 = \widetilde{g}_{ij}\,dx^i\,dx^j \,, \tag{2}$$

where $\widetilde{g}_{ij}$ is the maximally symmetric metric tensor in a 3D space. There are only 3 maximally symmetric 4D spacetimes: Minkowski, dS and AdS.

Since we have maximal symmetry, the Riemann tensor is

$$R_{ijkl} = k\Big(\widetilde{g}_{ik}\widetilde{g}_{jl} - \widetilde{g}_{il}\widetilde{g}_{jk}\Big)\,. \tag{3}$$

We can use spherical coordinates:

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2\,d\Omega^2 \,, \tag{4}$$

and we can define the coordinate χ by

$$d\chi = \frac{dr^2}{\sqrt{1-kr^2}}\,. \tag{5}$$

The Einstein equations read

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \tag{6}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is its trace, the scalar curvature, while $T_{\mu\nu}$ is the stress energy momentum tensor.

In cosmology we assume to have the SEMT of a perfect fluid. Really, we have particles, between which there is vacuum.

We need to use the Weyl tensor, which describes the parts of the Riemann tensor which are not in the traces. "The real world" is only described by the Weyl tensor, but in cosmology we make a great approximation in ignoring it.

What we do is to insert an ansatz for the metric tensor, which we use to derive the Christoffel symbols, and from these we write the Riemann tensor. Doing it the other way around, starting from the source SEMT, is very difficult.

Claim 0.1.1. *The Christoffel symbols for the FLRW metric are:*

$$\Gamma^t_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\dot{a}a}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 a\dot{a} & 0 \\ 0 & 0 & 0 & r^2 a\dot{a}\sin^2\theta \end{bmatrix} \tag{7a}$$

$$\Gamma^r_{\mu\nu} = \begin{bmatrix} 0 & \dot{a}/a & 0 & 0 \\ \dot{a}/a & \frac{kr}{(1-kr^2)} & 0 & 0 \\ 0 & 0 & (kr^2-1)r & 0 \\ 0 & 0 & 0 & (kr^2-1)r\sin^2\theta \end{bmatrix} \tag{7b}$$

$$\Gamma^\theta_{\mu\nu} = \begin{bmatrix} 0 & 0 & \dot{a}/a & 0 \\ 0 & 0 & 1/r & 0 \\ \dot{a}/a & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{7c}$$

$$\Gamma^\varphi_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & \dot{a}/a \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & \cos\theta/\sin\theta \\ \dot{a}/a & 1/r & \cos\theta/\sin\theta & 0 \end{bmatrix} . \tag{7d}$$

In order to calculate these, we can make use of certain simplifications: the FLRW metric is diagonal, and it does not depend on φ.

Notice that the spatial Christoffel symbols are nonzero even in Minkowski ($k = 0$, $\dot{a} = \ddot{a} = 0$): why is this? This is because we are using curvilinear coordinates, the Christoffel symbols express the *extrinsic* curvature, not the *intrinsic* curvature; they are not tensors, so they can be zero in a reference and nonzero in another.

In general, the Riemann tensor is given by

$$R^\mu_{\nu\rho\sigma} = -2\left(\Gamma^\mu_{\nu[\rho,\sigma]} + \Gamma^\alpha_{\nu[\rho}\Gamma^\mu_{\sigma]\alpha}\right), \tag{8}$$

where commas denote coordinate derivation, and square square brackets denote antisymmetrization (for clarification on this notation Wikipedia does a good job [19]).

The Ricci tensor is given by the contraction of the Riemann tensor along its first and third component:

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = -2\left(\Gamma^\alpha_{\mu[\alpha,\nu]} + \Gamma^\beta_{\mu[\alpha}\Gamma^\alpha_{\nu]\beta}\right) \tag{9a}$$

$$= \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} . \tag{9b}$$

A great simplification comes from the fact that, for the FLRW metric, the Ricci tensor is diagonal.[1]

[1] If there are a certain number of coordinates the metric is independent of, the Ricci tensor has very few nonzero components [Win96]. This is not enough to prove that the Ricci tensor must be diagonal for this metric, however in the specific case of FLRW this is the case anyways.

Claim 0.1.2. *The components of the Ricci tensor are:*

$$\begin{aligned} R_{tt} &= -3\partial_t\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 &\quad (10a)\\ &= -3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2\right) &\quad (10b)\\ &= -3\frac{\ddot{a}}{a}, &\quad (10c) \end{aligned}$$

$$\begin{aligned} R_{rr} = \partial_t&\left(\frac{\dot{a}a}{1-kr^2}\right) + \partial_r\left(\frac{kr}{1-kr^2}\right) - \partial_r\left(\frac{kr}{1-kr^2}\right) - 2\partial_r\left(\frac{1}{r}\right)\\ &+ \frac{\dot{a}a}{1-kr^2}3\frac{\dot{a}}{a} + \frac{kr}{1-kr^2}\left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) &\quad (11a)\\ &- 2\frac{\dot{a}}{a}\frac{\dot{a}a}{1-kr^2} - \left(\frac{kr}{1-kr^2}\right)^2 - 2\left(\frac{1}{r}\right)^2\\ = &\frac{\ddot{a}a + \dot{a}^2}{1-kr^2} + 3\frac{\dot{a}^2}{1-kr^2} + 2\frac{k}{1-kr^2} - 2\frac{\dot{a}^2}{1-kr^2} &\quad (11b)\\ = &\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2}, &\quad (11c) \end{aligned}$$

$$\begin{aligned} R_{\theta\theta} = r^2&\partial_t(a\dot{a}) + \partial_r\Big((kr^2-1)r\Big) - \partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right)\\ &+ 3\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Big(\Gamma^r_{rr} + 2\Gamma^\theta_{r\theta}\Big) - 2\Big(\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r}\Big) - \frac{\cos^2\theta}{\sin^2\theta} &\quad (12a)\\ = r^2&\Big(\ddot{a}a + \dot{a}^2\Big) + 3kr^2 - 1 + \frac{1}{\sin^2\theta} + r^2\dot{a}^2 - kr^2 - \frac{\cos^2\theta}{\sin^2\theta} &\quad (12b)\\ = r^2&\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big), &\quad (12c) \end{aligned}$$

$$\begin{aligned} R_{\varphi\varphi} &= \partial_\alpha\Gamma^\alpha_{\varphi\varphi} - \partial_\varphi\Gamma^\alpha_{\alpha\varphi} + \Gamma^\alpha_{\varphi\varphi}\Gamma^\beta_{\alpha\beta} - \Gamma^\beta_{\varphi\alpha}\Gamma^\alpha_{\varphi\beta} &\quad (13a)\\ &= r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big). &\quad (13b) \end{aligned}$$

The Ricci scalar then comes out to be

$$\begin{aligned} R = g^{\mu\nu}R_{\mu\nu} &= 3\frac{\ddot{a}}{a} + \frac{1-kr^2}{a^2}\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2}\\ &\quad + \frac{1}{a^2r^2}r^2\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) + \frac{1}{a^2r^2\sin^2\theta}r^2\sin^2\theta\Big(\ddot{a}a + 2\dot{a}^2 + 2k\Big) &\quad (14a)\\ &= 3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}a + 2\dot{a}^2 + 2k}{a^2} &\quad (14b) \end{aligned}$$

$$= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right]. \tag{14c}$$

The dimensions of the Ricci scalar are those of a length to the -2.

The stress energy tensor is the functional derivative of everything but the curvature in the action with respect to the metric: if our Lagrangian is

$$L = L_g + L_{\text{fluid}}, \tag{15}$$

where the gravitational Lagrangian is $L_g = M_P^2 R/2$ (and $M_P = 1/\sqrt{8\pi G}$ in natural units is the reduced Planck mass) then

$$T_{\mu\nu} \stackrel{\text{def}}{=} -\frac{2}{\sqrt{-g}} \frac{\delta L_{\text{fluid}}}{\delta g^{\mu\nu}}. \tag{16}$$

Discuss why this is equivalent to "flux of momentum component μ across a surface of constant x^ν".

We use perfect fluids: they have a stress-energy tensor like

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + p g^{\mu\nu}, \tag{17}$$

where u^μ is the 4-velocity of the fluid element. It is diagonal *in the comoving frame*, in which $u^\mu = \left(1, \vec{0}\right)$.

If we are not comoving, we have additional heat transfer off diagonal terms (this is discussed in my thesis [Tis19, section 4.2]).

If we take the covariant divergence of the Einstein tensor $G_{\mu\nu}$ we get zero; so the stress energy tensor must also have $\nabla_\mu T^{\mu\nu} = 0$. This is *not* a conservation equation.

In SR we had an equation like $\partial_\mu T^{\mu\nu}$: this *was* a conservation equation, a local one. In GR we also need Killing vectors in order to actually have conserved quantities. In cosmology we do not have symmetry with respect to time translation, so there is no timelike Killing vector ξ_μ such that $\xi_\nu \nabla_\mu T^{\mu\nu}$ represents the conservation of energy.

This equation, $\nabla_\mu T^{\mu\nu}$ follows from the fact that our fluid follows its equations of motion.

Let us explore the meaning of these equations: if, in the equation $0 = \nabla_\mu T^\mu_0$, we find

$$0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \tag{18a}$$

$$= -\dot{\rho} - 3H\left(\rho + P\right). \tag{18b}$$

For example consider radiation: $P = \rho/3$. This means that $\dot{\rho} = -4H\rho$: so, as the Hubble parameter increases, the radiation density decreases.

The other two Friedmann equations can be derived from the time-time and space-space components on the Einstein equations: we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right) \tag{19a}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{19b}$$

The space-space equation is not a dynamical equation, since it contains no second time derivatives: it is a *constraint* on the evolution of the system.

However, the three Friedmann equations are not independent: the time-time one can be found from the other two.

A useful theorem is the fact that for a maximally symmetric space the Ricci tensor must be given by

$$\widetilde{R}_{\alpha\beta} = 2k\widetilde{g}_{\alpha\beta}\,. \tag{20}$$

We can write the stress energy tensor as

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}\,, \tag{21}$$

where $h_{\mu\nu}$ is the projection tensor onto the spacelike subspace $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$.

This is more physically meaningful.

Tomorrow we will start the discussion on the CMB.

Chapter 1

The CMB

Today we discuss the CMB. This is discussed in the book Modern Cosmology [Dod03, chapter 3], we will follow the professor's notes, which are available in the Dropbox. Friday 2020-3-13, compiled 2020-05-21

A note: in these lectures a dot will refer to conformal time derivatives only, if we differentiate with respect to cosmic time we shall write the derivative explicitly. Let us suppose we have some particle species interacting, such as $1 + 2 \leftrightarrow 3 + 4$.

The variation in time of the abundance of particle type 1, (which is given by the density times a volume: $n_1 a^3$) is given by the difference of the particles which are created and destroyed. We write the formula first, and then explain it: this is given by

$$\begin{aligned} a^{-3}\frac{\mathrm{d}(n_1 a^3)}{\mathrm{d}t} &= \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1}\left[\prod_{i=2}^{4}\int \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}\right]\times \\ &\times (2\pi)^4 \delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\delta(E_1+E_2-E_3-E_4)\times \\ &\times |\mathcal{M}|^2\Big[f_3 f_4 (1\pm f_1)(1\pm f_2) - f_1 f_2 (1\pm f_3)(1\pm f_4)\Big] \end{aligned} \tag{1.1a}$$

where:

1. the delta functions account for momentum and energy conservation: energy is *not conserved* in general in cosmology, *but* we can use the equivalence principle to go to a reference frame which is locally Minkowski: in our description of an instantaneous process such as this, the deviations from this frame are negligible.

2. $\mathcal{M}$ is the *invariant scattering amplitude* between the initial and final states. In Quantum Field Theory, in order to describe a process one assumes that the particles involved usually evolve according to their free Hamiltonians, while their interaction is described by an interaction Hamiltonian $\hat{H}_{\text{int}}$ — we put ourselves in the interaction picture, so that the free time evolution is factored out. The matrix element of the time evolution between the initial state $|12\rangle$ and the final state $|34\rangle$ is called the S-matrix element:

$$S_{12,34} = \lim_{t\to\infty} \langle 12| U(t) |34\rangle \,, \tag{1.2}$$

where t is assumed to go to infinity since we are considering asymptotic initial and final states. This is in general hard to express, but it can be formally written as a

Dyson series in terms of the interaction Hamiltonian. If we only consider this series to first order we get (roughly speaking):

$$S_{12,34} \sim \underbrace{\langle 12|34\rangle}_{=0} + \langle 12|\, \hat{H}_{\text{int}}\, |34\rangle \,. \tag{1.3}$$

The invariant scattering amplitude is then defined so that

$$\left|S_{12,34}\right|^2 = (2\pi)^4 \delta^{(4)}(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu)|\mathcal{M}|^2 \,. \tag{1.4}$$

Do note that $\mathcal{M}$ is an amplitude, while $|\mathcal{M}|^2$ is a probability. If we integrate across all of momentum space, as we are doing, we get the total probability of the process happening. For more details, see Peskin [Pes19, sections 7.2, 7.3].

Also, note that in our model the processes are always time-reversible, so $\mathcal{M}_{12\to 34} = \mathcal{M}_{34\to 12}$.

3. The f_i are the phase space distributions of the different species: the terms including these account for the quantum statistics, we use $-$ for fermions and $+$ for bosons. The factors $1 \pm f_i$ account for the quantum phenomena of Bose enhancing and Pauli blocking [Nor28]. We include them here, but we will usually neglect them.

4. The 2π-s account for the normalization of the deltas: if we were to discretize phase space we would not need them, but when we move from discrete sums (and Kronecker deltas) to integrals (and Dirac deltas) we need to divide by h^3 for each integral over momentum; in our units however $\hbar = 1$, therefore $h = 2\pi$.

5. The energy of each particle species is given by $E^2 = p^2 + m^2$. Why are there $2E$ factors in the denominators? In principle, we should integrate in $\mathrm{d}^4 p$, however we work *on shell*: a priori, the particle does whatever it wants, however solutions to the equations of motion are (very much) preferred in the path integral. Because of this, it is an excellent approximation to just impose this condition: we do

$$\int \mathrm{d}^3 p \int_0^\infty \delta(E^2 - p^2 - m^2) = \int \mathrm{d}^3 p \int_0^\infty \frac{\delta(E - \sqrt{p^2 + m^2})}{2E} \,, \tag{1.5}$$

so we include the $2E$ term in the denominator: the integral over the energies has already been performed, and by E we always mean $E(\vec{p})$.

If there is no interaction, $n_1 \propto a^{-3}$, as we expect for nonrelativistic matter.

We are integrating over the momenta of all the particles except for 1 in order to account for all the possible ways the reaction could happen.

Something like:
The term for particle 1, E_1, has a different origin: the time is related to the proper time by p^0, which is E_1. The factor 2 is included for symmetry, it is indifferent if we include it or not since we can normalize the helicities g_i.
was mentioned during lecture, but why should we not integrate over $\mathrm{d}^4 p_1$ and apply the same reasoning as the other particles?

Typically we have kinetic equilibrium, as long as the scattering time is very short with respect to the Hubble time. So, we use the conventional distribution functions for Bose-Einstein or Fermi-Dirac gases,

$$f_{\mathrm{BE/FD}} = \left(\exp\left(\frac{E-\mu}{T}\right) \pm 1\right)^{-1}, \tag{1.6}$$

where the sign is a $-$ for Bose Einstein statistics, while a $+$ for Fermi-Dirac statistics. Here, E is the energy of the particle, μ is its chemical potential, which quantifies how much the energy changes if we add particles, while T is the temperature.

For the nonrelativistic particles (all of them, except the photons[1], much less than the mass of the lightest particles involved, electrons.) we have $E - \mu \gg T$. If f becomes very small, then we can drop the terms $(1 \mp f_i)$.[2] This is the Boltzmann limit.

In theory we could not do this for photons, in practice we do it and the magnitude of the error is the same as the ratio $\zeta(3) \approx 1.2$ to 1, where ζ is the Riemann zeta function.

Then, our distributions will be given by

$$f(E) = e^{\mu/T} e^{-E/T}\,. \tag{1.8}$$

So the phase space distribution term is

$$\begin{aligned} f_1 f_2 - f_3 f_4 &= \exp\left(\frac{-(E_1+E_2)+(\mu_1+\mu_2)}{T}\right) - \exp\left(\frac{-(E_3+E_4)+(\mu_3+\mu_4)}{T}\right) &\text{(1.9a)}\\ &= \exp\left(-\frac{E_1+E_2}{T}\right)\left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T}\right), &\text{(1.9b)} \end{aligned}$$

where we used the fact that $E_1 + E_2 = E_3 + E_4$ by energy conservation.

What do the chemical potentials look like? If we were to enforce the Saha condition $\mu_1 + \mu_2 = \mu_3 + \mu_4$, which is equivalent to chemical equilibrium , then we would get precisely zero for our variation of species number. This makes sense: if there is equilibrium, the number densities of the species do not change. So, we cannot restrict ourselves to chemical equilibrium if we want to describe recombination.

We can get relations for the chemical potentials starting from the number densities of the species which are already present: the mean number density of species i is given by

$$n_i = \frac{g_i}{(2\pi)^3} \int \mathrm{d}^3 p\, f(p) = g_i e^{\mu_i/T} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T}\,, \tag{1.10}$$

where g_i is the number of helicity states of the particle (see the notes for the course in fundamentals of astrophysics and cosmology for more details [TM20b]).

[1] As we will discuss in more detail later, the recombination temperature is around 0.3 eV

[2] What do these terms look like? We have

$$1 \mp f_i = 1 \mp \frac{1}{e^{(E-\mu)/T} \pm 1} = \frac{e^{(E-\mu)/T} \pm 1 \mp 1}{e^{(E-\mu)/T} \pm 1} = \frac{1}{1 \pm e^{-(E-\mu)/T}}\,. \tag{1.7}$$

So, we can define

$$n_i^{(0)} = n_i\Big|_{\mu_i=0} \implies e^{\mu_i/T} = \frac{n_i}{n_i^{(0)}}\,. \tag{1.11}$$

These quantities can be estimated if the particles are either very relativistic ($E_i \approx p_i$) or very non-relativistic ($E_i \approx m_i + p^2/2m_i$): in the second case we have[3]

$$n_i^{(0)} = g_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T} \approx \frac{g_i}{(2\pi)^3} e^{-m_i/T} \int \mathrm{d}^3 p\, e^{-\frac{p^2}{2m_i T}} \tag{1.13a}$$

$$= \frac{g_i}{(2\pi)^3} e^{-m_i/T} \sqrt{(2\pi)^3 (m_i T)^3} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}\,, \tag{1.13b}$$

while in the very relativistic case we find

$$n_i^{(0)} = g_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{-E_i/T} \approx \frac{g_i}{(2\pi)^3} \int \mathrm{d}^3 p\, e^{-p_i/T} \tag{1.14a}$$

$$= \frac{g_i}{(2\pi)^3} 4\pi T^3 \int e^{-x} x^2\, \mathrm{d}x = \frac{g_i}{\pi^2} T^3\,. \tag{1.14b}$$

For photons, the correct expression accounting for their quantum mechanical statistics would be this one, multiplied by the Riemann zeta function calculated at 3:

$$\zeta(3) = \sum_{n\in\mathbb{N}} \frac{1}{n^3} \approx 1.2\,, \tag{1.15}$$

so we are wrong by about 20 % in neglecting it.

Inserting these $n_i^{(0)}$ we get the simpler expression

$$e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} = \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\,. \tag{1.16}$$

We can define the thermally-averaged cross section to encompass all the terms which do not depend on the number densities at a specific time:

$$\langle \sigma v \rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_{i=1}^{4} \int \frac{\mathrm{d}^3 p}{2E_i} e^{-(E_1+E_2)/T} (2\pi)^4 \delta^{(4)}\left(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu\right) |\mathcal{M}|^2\,, \tag{1.17}$$

so the final equation is

$$a^{-3} \frac{\mathrm{d}}{\mathrm{d}t} = \langle \sigma v \rangle\, n_1^{(0)} n_2^{(0)} \left[\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right]. \tag{1.18}$$

[3] We need to use a gaussian integral: in general, we have the theorem

$$\int \mathrm{d}^n x \exp\left(-\frac{1}{2} A_{ij} x_i x_j\right) = \sqrt{\frac{(2\pi)^n}{\det A}}\,, \tag{1.12}$$

and in our case $A_{ij} = (m_i T)^{-1} \delta_{ij}$, so $\det A = (m_i T)^{-3}$.

The left hand side is typically of the order $\sim n_1/t \sim n_1 H$, since the order of magnitude of the age of the universe is the Hubble time H^{-1}.

So, as the universe ages, the combination on the RHS must be "squeezed to zero": this is equivalent to the Saha equation, since it means that

$$e^{(\mu_1+\mu_2)/T} = e^{(\mu_3+\mu_4)/T} \implies \mu_1 + \mu_2 = \mu_3 + \mu_4 \,. \tag{1.19}$$

This is also called "chemical equilibrium" by particle physicists, or nuclear statistical equilibrium by people studying Big Bang Nucleosynthesis.

1.1 Hydrogen recombination

The process is

$$e^- + p \leftrightarrow H + \gamma \,, \tag{1.20}$$

so the Saha equation yields

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \,, \tag{1.21}$$

and charge neutrality implies $n_e = n_p$, *not* $n_e^{(0)} = n_p^{(0)}$.

At this stage in evolution, there are already some Helium nuclei, but we ignore them.

We define the ionization fraction

$$X = \frac{n_e}{n_e + H} \,. \tag{1.22}$$

This then yields

$$\frac{1 - X_e^n}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_e} \right)^{3/2} \exp(\epsilon_0 / T) \,, \tag{1.23}$$

where $\epsilon_0 = m_p + m_e - m_H = 13.6\,\text{eV}$ is the ionization energy of Hydrogen.

Then, we get that the temperature of recombination is $T_{\text{rec}} \approx 0.3\,\text{eV}$.

The evolution of the ionization fraction is

$$\frac{\mathrm{d}X_e}{\mathrm{d}t} = (1 - X_e)\beta(T) - X_e^2 n_b \alpha^{(2)}(T) \,, \tag{1.24}$$

where we defined the ionization rate

$$\beta(T) = \langle \sigma v \rangle \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T} \,, \tag{1.25}$$

and the recombination rate $\alpha^{(2)} = \langle \sigma v \rangle$.

The value of this can be solved numerically: the difference between this and the Saha equation is not great in the prediction in the recombination redshift; however the prediction of the residual ionized hydrogen is different: there is much more than Saha would predict.

The universe gets reionized at $z \gtrsim 6$; this is still under discussion.

There are many ingredients in the interaction of the universe. We are interested in the photons: we want to predict the anisotropies in the CMB. There is a dipole due to the movement of the solar system through the CMB. Now, we want to see what our predictions are if we subtract this.

[Scheme of the interactions.]

The metric interacts with everything, photons interact with electrons through Compton scattering, electrons interact with protons through Coulomb scattering, dark energy, dark matter and neutrinos interact only with the metric.

Instead of Compton scattering, we use its nonrelativistic limit which applies here.

Scattering between electrons and protons is suppressed since protons are very massive. The other terms in the universe affect the geometry and we could see them through this.

There are models which include DM-DE coupling, and quintessence models, and models in which dark energy clusters.

We are not going to consider these.

We go back to first principles:

$$\hat{\mathbb{L}}[f] = \hat{\mathbb{C}}[f] , \tag{1.26}$$

where $f = f(x^\alpha, p^\alpha)$, however actually we do not have that much freedom in the phase space distribution. If there are no collisions: $\hat{\mathbb{L}}[f] = 0$, which is equivalent to

$$\frac{\mathrm{D}f}{\mathrm{D}\lambda} = 0 , \tag{1.27}$$

where λ is the affine parameter.

In the nonrelativistic case,

$$\hat{\mathbb{L}} = \frac{\partial}{\partial t} + \dot{x} \cdot \nabla_x + \dot{v} \cdot \nabla_v = \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_x + \frac{F}{m} \cdot \nabla_v , \tag{1.28}$$

while in the GR case we need to account for the geodesic equation: and we write

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\lambda} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma , \tag{1.29}$$

where the affine parameter λ has the dimensions of a mass, in order to have dimensional consistency.

Then, the Liouville operator is

$$\hat{\mathbb{L}} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \stackrel{\text{def}}{=} \frac{\mathrm{D}}{\mathrm{D}\lambda} . \tag{1.30}$$

This is a total derivative in phase space with respect to the affine parameter.

In the FLRW background, $f = f(|p|, t)$ and

$$\hat{\mathbb{L}} = E \frac{\partial f}{\partial t} - \frac{\dot{a}}{a} |p|^2 \frac{\partial f}{\partial E} , \tag{1.31}$$

so if we define the number density

$$n(t) = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, f(E,t)\,, \tag{1.32}$$

so if we integrate over momenta we get

$$\int \frac{\mathrm{d}^3 p}{E} \hat{\mathbb{L}}[f]\,, \tag{1.33}$$

we find the equation from before:

$$\dot{n} + 3\frac{\dot{a}}{a} n\,, \tag{1.34}$$

??? to check

1.1.1 Metric perturbations

We want to write the most general perturbed cosmological metric solution to the Einstein Equations, starting from the FLRW metric: it will look like $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{FLRW}} + h_{\mu\nu}$, for a small perturbation $h_{\mu\nu}$. Recall that the FLRW metric, in the zero-curvature case, is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2 \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j\ . \tag{1.35}$$

We will neglect spatial curvature because its effect on the *geometry* of the universe is negligible; however we can see its effect in the dynamics.

In general this perturbation will be gauge-dependent, and we want to write only a number of independent components corresponding to the number of physical degrees of freedom. This is done via the scalar-vector-tensor decomposition [Ber00, section 2.1].

We decompose the components of $h_{\mu\nu}$ as:

$$h_{00} = -2\Phi\,, \qquad h_{0i} = a\omega_i \qquad h_{ij} = a^2\Big(-2\Psi\delta_{ij} + \chi_{ij}\Big)\,. \tag{1.36}$$

So, we have distinguished two scalar degrees of freedom Φ and Ψ, plus a vector ω_i and a traceless tensor χ_{ij}, which satisfies $\chi_i^i = 0$. We can assume χ_{ij} to be traceless since we can absorb any variations of the trace of the spatial part of the metric into Ψ; this takes away one degree of freedom from the 6 of the tensor.

This means we have $2 + 3 + 6 - 1 = 10$ degrees of freedom, the right amount before accounting for gauge.

Now, we make use of the Helmholtz decomposition: any vector field x^i can be written as $x^i = x^i_\perp + x^i_\parallel$, where $x^i_\perp$ is solenoidal ($\nabla_i x^i_\perp = 0$), while $x^i_\parallel$ is irrotational ($\epsilon^{ijk}\nabla_i x_j = 0$).

This means that we can split the three degrees of freedom of the vector field into one in $x^i_\parallel$ — since it is irrotational, it can be written as the gradient of a scalar function —, and two in $x^i_\perp$, since it is a 3D vector field for which one component is determined. Explicitly, this is

$$\omega_i = \underbrace{\partial_i \omega}_{\text{1 scalar dof}} + \underbrace{\omega^i_\parallel}_{\text{2 vector dof}}\,, \tag{1.37}$$

We have several names for a vector field x^i satisfying $\nabla_i x^i$: it can be called solenoidal (since the magnetic field, in a solenoid but also anywhere else, satisfies this property), divergence-free, or transverse. The last one can be figuratively interpreted to say that the vector is "orthogonal to the gradient operator"; more formally, in Fourier space the condition reads $k_i \widetilde{x}^i = 0$.

We can do a similar thing for the tensor perturbation, but it is slightly more complicated: it it written in the form

$$\chi_{ij} = \underbrace{\chi_{ij}^{\parallel}}_{\text{1 scalar dof}} + \underbrace{\chi_{ij}^{\perp}}_{\text{2 vector dof}} + \underbrace{\chi_{ij}^{T}}_{\text{2 tensor dof}} , \tag{1.38}$$

where the derivative combination is dictated by the condition of χ_{ij} being traceless. The tensor $\chi^{\parallel}$ contains only one degree of freedom, since it can be written as

$$\chi_{ij}^{\parallel} = \left(\nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \phi_S , \tag{1.39}$$

while $\chi_{ij}^{\perp}$ contains two degrees of freedom, since it can be written as the symmetrized gradient of a vector field S^T:

$$\chi_{ij}^{\perp} = 2 \nabla_{(i} S_{j)}^T , \tag{1.40}$$

where the vector field is divergence-free: $\nabla_j S_T^j = 0$.

The last term is fully transverse: its divergence vanishes, $\nabla_i \chi_T^{ij} = 0$. This term then contains two physical degrees of freedom, since it starts out from five — it is a symmetric spatial tensor with zero trace — but we impose 3 equations.

Let us enumerate the degrees of freedom we have, distinguishing them into "modes" based on how they transform:

1. The **tensor mode** is given by χ_{ij}^T, the part of the spatial metric perturbation which cannot be obtained as a gradient. It has two degrees of freedom, and it transforms as a spin-2 field (which means it is unchanged by rotations of angle π about its axis). This corresponds to gravitational radiation. It is gauge-invariant.

2. The **vector mode** is given by the transverse (divergence-free) vectors $\omega_i^{\parallel}$ and S_i^T. It has four degrees of freedom, two for each transverse vector. Two of these degrees of freedom are gauge, two are physical and correspond to the phenomenon of gravitomagnetism, which causes Lense-Thirring precession.[4] They transform like a spin-1 field: they must be rotated by 2π to come back to the starting point.

3. The **scalar mode** is given by Φ, Ψ, ω and ϕ_S. It has four degrees of freedom, two of which are physical and two of which are gauge. They transform as a spin-0 field (so they are invariant under rotations).

[4] See the General Relativity notes [TM20a, equation 341]: in order to derive the Lense-Thirring precession effect we must have a perturbation in the g_{03} component of the metric, corresponding to the vector ω_i in our notation.

We have separated the perturbations based on how they transform under the group of rotations. If we stick to linear perturbation theory, they are decoupled: they evolve independently.

As we discussed, we can set 2 scalar fields and 2 vector fields to zero by our gauge choice: we choose the Poisson gauge, in which we set ϕ_S, ω to zero for the scalars, and S_i^T to zero for the vectors.

So, the full metric we get looks like:

$$\boxed{\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + 2a\omega_i\,\mathrm{d}x^i\,\mathrm{d}t + a^2\left(e^{-2\Psi}\delta_{ij} + \chi_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right),} \tag{1.41}$$

where ω_i is a transverse vector: $\nabla^i\omega_i = 0$; while in χ_{ij} we only have the tensor mode, so the tensor is traceless and transverse: we have $\chi_i^i = 0 = \nabla^i\chi_{ij}$.

This is explained in more detail in the class by Nicola Bartolo ("early Universe"). These are GR perturbations.

We can compare the physical spacetime and the idealized FLRW metric. We need to do this since we cannot solve the EFE if there is no symmetry. So, we say that spacetime is *close* to the idealized spacetime.

Our next goal will be to solve the geodesic eqs. for the motion of the particles in this gauge.

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We come back to the discussion from last time, about the Boltzmann equation in a perturbed universe.

When can we drop some terms using a gauge transformation? We can do it for scalar and vector perturbations. We shall use the longitudinal, or Poisson gauge, in which the scalar perturbations are reduced to Φ and Ψ, so we can take the tensor terms to be traceless and covariantly constant.

We will not discuss vectors, since if they are zero at the beginning they cannot be generated, they stay at zero. In our natural units, the perturbations will be small: $\Phi \ll 1$ and $\Psi \ll 1$. Recall that we are neglecting spatial curvature.

Photons have $P^2 = 0$: so we can express this as

$$-(1+2\Phi)(p^0)^2 + p^2 = 0\,, \tag{1.42}$$

where p^2 is defined as $p^2 = g_{ij}P^iP^j$, since in our gauge choice the spatial part of the metric has a Kronecker delta this will only include the spatial parts. So, we get

$$P^0 = \frac{p}{\sqrt{1+2\Phi}} \approx p(1-\Phi)\,. \tag{1.43}$$

Now, we will write the Liouville operator by dividing through by P^0:

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\cdot\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}\,, \tag{1.44}$$

where we split the three-momentum into absolute value p and the unit vector $\hat{p}$, which has $\hat{p}^i = \hat{p}_i$ and $\delta_{ij}\hat{p}^i\hat{p}^j$. We have $\mathrm{d}x^i \big/ \mathrm{d}t = P^i/P^0$.

We are going to expand only to first order. Higher order are more important for small angular scales, and for secondary CMB anisotropies, these are interesting but we are not going to treat them.

To first order, the last term of the RHS is zero, since it is a product of two terms which are both zero to zeroth order (in an unperturbed universe the phase space distribution is perfectly isotropic and a particle keeps travelling in the same direction).

Now we define the amplitude A by $P^i = A\hat{p}^i$: now we will have

$$
\begin{aligned}
p^2 &= g_{ij}P^iP^j = a^2\delta_{ij}(1-2\Psi)\hat{p}^i\hat{p}^jA^2 && \text{(1.45a)}\\
&= a^2(1-2\Psi)A^2\,, && \text{(1.45b)}
\end{aligned}
$$

therefore, taking the square root and staying to first order we get

$$
A \approx p\frac{1+\Psi}{a}\,, \tag{1.46}
$$

so

$$
P^i = p\hat{p}^i\frac{1+\Psi}{a}\,, \tag{1.47}
$$

and the division by a can be interpreted as a redshift effect. Inserting this term we get

$$
\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{P^i}{P^0} = \hat{p}^i\frac{1+\Psi+\Phi}{a}\,, \tag{1.48}
$$

and we can notice that $\mathrm{d}x^i \big/ \mathrm{d}t$ multiplies the term $\partial f \big/ \partial x^i$, which is only nonzero to first order: so we must consider this term to zeroth order. So, we get

$$
\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}\,, \tag{1.49}
$$

and now we shall show that

$$
\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right), \tag{1.50}
$$

which will imply that

$$
\frac{\mathrm{D}f}{\mathrm{F}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right). \tag{1.51}
$$

We will use the geodesic equation for photons: it is enough to consider its zeroth component, which is

$$
\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta\,, \tag{1.52}
$$

which means that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\big(p(1+\Phi)\big) = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta\frac{1+\Phi}{p}\,, \tag{1.53}
$$

where we brought a P^0 from the left to the right side. This means that we have

$$(1-\Phi)\frac{\mathrm{d}p}{\mathrm{d}t} = p\frac{\mathrm{d}\Phi}{\mathrm{d}t} - \Gamma^0_{\alpha\beta}P^\alpha P^\beta \frac{1+\Phi}{p}\,, \tag{1.54}$$

and now we multiply both sides by $1+\Phi$, the inverse of $1-\Phi$ to linear order:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right) - \Gamma^0_{\alpha\beta}P^\alpha P^\beta \frac{1+2\Phi}{p}\,, \tag{1.55}$$

and now we have to start calculating the Christoffel symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)\,, \tag{1.56}$$

so we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\left(2g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}\right)\frac{P^\alpha P^\beta}{p}\,, \tag{1.57}$$

but g^{0i} are zero, since we are ignoring vector perturbations, and $g^{00} = -1+2\Phi$ (since it is the contravariant metric). Then we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Phi}{2}\left(2g_{0\alpha,\beta} - g_{\alpha\beta,0}\right)\frac{P^\alpha P^\beta}{p}\,, \tag{1.58}$$

and we also have

$$\frac{\partial g_{0\alpha}}{\partial x^\beta} = -2\frac{\partial\Phi}{\partial x^\beta}\delta_{\alpha 0}\,, \tag{1.59}$$

so we distinguish the components and find:

$$-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} = -\frac{\partial g_{00}}{\partial t}\frac{(P^0)^2}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^i P^j}{p} \tag{1.60a}$$

$$= 2\frac{\partial\Phi}{\partial t}p - a^2\delta_{ij}\left(-2\frac{\partial\Psi}{\partial t} + 2H(1-2\Psi)\right)\frac{P^i P^j}{p}\,, \tag{1.60b}$$

and we already have shown that $\delta_{ij}P^iP^j = p^2(1+2\Psi)/a^2$.

So on the whole we get

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = (-1+2\Phi)\left[-\frac{\partial\Phi}{\partial t}p - 2\frac{\partial\Phi}{\partial x^i}\frac{p\hat{p}^i}{a} + p\left(\frac{\partial\Psi}{\partial t} - H\right)\right]\,, \tag{1.61}$$

so putting everything together [extra passage] we get

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p\left(H - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i}\right)\,. \tag{1.62}$$

Now we need to choose how to perturb the photon distribution function. At zeroth order it is the Planckian:

$$f \approx \frac{1}{e^{p/T} - 1} \,. \tag{1.63}$$

In general we will have dependence on the position $\vec{x}$, the momentum $(p, \hat{p})$ and time t.

We do not observe spectral distortions in the CMB: it is always described by a Planckian, with anisotropies in the *temperature*. So, we parametrize it as

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left(\frac{p}{T(t)\big(1 + \Theta(\vec{x}, \hat{p}, t)\big)}\right) - 1\right]^{-1} , \tag{1.64}$$

where we assumed that $\Theta = \delta T / T$ does *not* depend on the momentum of the photon p: otherwise, we would have a spectral distortion. This is certainly true, at least to linear order.

So, we expand in Θ:

$$\begin{aligned} f &\approx \frac{1}{e^{p/T} - 1} + \left(\frac{\partial}{\partial T}\Big(\exp\big(p/T\big) - 1\Big)^{-1}\right) T\Theta & (1.65\text{a}) \\ &= f^{(0)} - p\frac{\partial f^{(1)}}{\partial p}\Theta \,, & (1.65\text{b}) \end{aligned}$$

since

$$T\frac{\partial f^{(0)}}{\partial T} = -p\frac{\partial f^{(0)}}{\partial p} \,. \tag{1.66}$$

At zeroth order we have

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0 \,, \tag{1.67}$$

since we do not have collision terms. We can write this differently using

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}t}\frac{\partial f^{(0)}}{\partial p} \,, \tag{1.68}$$

where we used the change in derivative variable from before. So we get

$$\left[-\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right]\frac{\partial f^{(0)}}{\partial p} = 0 \,, \tag{1.69}$$

which means $\dot{T}/T + \dot{a}/a = 0$, or $T \propto 1/a$, which is Tolman's law. Now, let us go to first order.

$$\frac{\mathrm{D}f}{\mathrm{D}t} = -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - p\frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} + Hp\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) + p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Psi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}\right] . \tag{1.70}$$

The final expression we get is

$$\frac{\mathrm{D}f}{\mathrm{D}t} = -p\frac{\partial f^{(0)}}{\partial p}\left[\underbrace{\frac{\partial \theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \theta}{\partial x^i}}_{\text{free-streaming}} + \underbrace{\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Phi}{\partial x^i}}_{\text{gravitational}}\right], \tag{1.71}$$

in which we can distinguish two terms which have to do with the propagation of the anisotropies from emission to now — this is not precise, but it is the reason we call them "free-streaming". The other two terms arise from the self-gravity of the matter appearing on the RHS of the Einstein equations.

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Now, we will discuss the collision terms. The interaction we wish to consider is Compton scattering, which has the form

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}') + \gamma(\vec{p}')\,, \tag{1.72}$$

and we are interested to see how this affects the momentum distribution of the photons.

The collision term in the equation

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \hat{\mathbb{C}}[f(\vec{p})] \tag{1.73}$$

reads

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 q'}{(2\pi)^3 2E_e(q')}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2(2\pi)^4 \\ \times \delta^{(3)}(\vec{p}+\vec{q}-\vec{p}'-\vec{q}')\delta(E(p)+E_e(q)-E(p')-E_e(q')) \\ \times \left[f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right]\end{aligned}\,. \tag{1.74a}$$

The factor $1/p$ at the start comes from the LHS, since we differentiated with respect to t and not λ on the LHS.

For the photons the energy is $E(p') = p'$, for the electrons instead $E_e(q) \approx m_e + q^2/2m_e$, since the electrons are nonrelativistic — the temperatures are of the order $0.3\,\mathrm{eV}$ at recombination, a vanishingly small fraction of the electrons' mass of $511\,\mathrm{keV}$.

So, we should perform all the integrations on the RHS: we start with the integration over q'. We get rid of a δ function and get

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{\pi}{2m_e^2}\frac{1}{p}\int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2 \\ \times \delta\left[p + \frac{q^2}{2m_e} - p' - \frac{\left(\vec{p}+\vec{q}-\vec{p}'\right)^2}{2m_e}\right] \\ \times \left[f_e(\vec{q}+\vec{p}-\vec{p}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right]\end{aligned}\,. \tag{1.75a}$$

For nonrelativistic Compton scattering we have

$$E_e(\vec{q}) - E_e(\vec{q}+\vec{p}-\vec{p}') = \frac{q^2}{2m_e} - \frac{\left(\vec{q}+\vec{p}-\vec{p}'\right)^2}{2m_q} \approx \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}\,, \tag{1.76}$$

which is true as long as $q \gg p, p'$. Also, the scattering is close to being elastic:

$$\frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e} \sim \frac{Tq}{m_e} \sim Tv_b \ll T\,, \tag{1.77}$$

since the velocity of the electrons is nonrelativistic.

So,

$$\frac{\Delta E_e}{E} \sim \frac{Tv_b}{Tc} \sim \frac{v_b}{c} \ll 1\,. \tag{1.78}$$

Let us motivate $q \ll p, p'$: we know that, since

$$\frac{q^2}{2m_e} \sim T\,, \tag{1.79}$$

we have $q \sim (m_e T)^{1/2}$, which is equal to

$$q \sim \left(\frac{m_e}{T}\right)^{1/2} T \gg T\,, \tag{1.80}$$

so $q \gg T \sim p$.

Now, the change to the electron kinetic energy is small so we can expand:

$$\delta\left[p + \frac{q^2}{2m_e} - p' - \frac{\left(\vec{q}+\vec{p}-\vec{p}'\right)^2}{2m_e}\right] \tag{1.81a}$$

$$\approx \delta(p-p') + \left[E_e(q') - E_e(q)\right] \times \frac{\partial}{\partial E_e(q')}\delta(p + E_e(q) - p' - E_e(q')) \tag{1.81b}$$

$$\approx \delta(p-p') + \frac{(\vec{p}-\vec{p})\cdot\vec{q}}{m_e}\frac{\partial\delta(\vec{p}-\vec{p}')}{\partial p'}\,, \tag{1.81c}$$

where we used the fact that

$$\frac{\partial(x-y)}{\partial x} = -\frac{\partial(x-y)}{\partial y}\,. \tag{1.82}$$

The derivative of the delta-function is defined as a functional yielding the derivative of the function it is integrated with.

This then gives us

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{\pi}{2m_e^2}\frac{1}{p}\int\frac{\mathrm{d}^3 q}{(2\pi)^3 2E_e(q)}\int\frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_e(p')}|\mathcal{M}|^2 \\ \times\left[\delta(p-p') + \frac{(\vec{p}-\vec{p})\cdot\vec{q}}{m_e}\frac{\partial\delta(\vec{p}-\vec{p}')}{\partial p'}\right]\left(f(\vec{p}') - f(\vec{p})\right)\end{aligned}\,. \tag{1.83a}$$

Now, it is a fact from QFT that we can compute

$$|\mathcal{M}|^2 = 6\pi\sigma_T m_e^2\left(1+\cos^2\theta\right)\,, \tag{1.84}$$

where $\cos\theta = \hat{p}\cdot\hat{p}'$.

For simplicity we replace this angle-dependent quantity with its angular average: the integral of the cosine gives us $1/3$, so we get a multiplier $6(1+1/3) = 8$:

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\pi\sigma_T m_e^2 \,. \tag{1.85}$$

Now, we insert this in the expression and integrate over q: this yields a mean velocity, and also we expand the phase space distributions to first order:

$$\begin{aligned}\hat{\mathbb{C}}[f(\vec{p})] = \frac{2\pi n_e \sigma_T}{p} \int \frac{\mathrm{d}^3 p'}{(2\pi)^3 p'} &\left[\delta(p-p')(\vec{p}-\vec{p}')\cdot\vec{v}_b \frac{\partial\delta(p-p')}{\partial p'}\right] \\ \times &\left[f^{(0)}(\vec{p}') - f^{(0)}(\vec{p}) - p'\frac{\partial f^{(0)}}{\partial p'}\theta(\vec{p}') + p\frac{\partial f^{(0)}}{\partial p}\theta(\vec{p})\right] .\end{aligned} \tag{1.86a}$$

[passages]

We do the angular integral, and simplify it by defining the *monopole* contribution:

$$\theta_0(\vec{x}, t) = \frac{1}{4\pi}\int \mathrm{d}\Omega'\, \theta(\vec{x}, \hat{p}', t)\,. \tag{1.87}$$

Then, finally, we integrate over p', which gives us the result

$$\hat{\mathbb{C}}[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\theta_0 - \theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right] . \tag{1.88}$$

The factor due to the electron spin, g_e, is accounted for in the electron momentum distribution f_e.

So, for the photons we have

$$\frac{\partial\theta}{\partial t}\frac{\hat{p}^i}{a}\frac{\partial\theta}{\partial x^i} - \frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T\left(\theta_0 - \theta + \hat{p}\cdot\vec{v}_b\right) . \tag{1.89}$$

Our last step is to move to conformal time η, defined by $\mathrm{d}\eta = \mathrm{d}t\,/a$; denoting derivatives with respect to conformal time with a dot (and multiplying everything by a) we get:

$$\dot{\theta} + \hat{p}^i\frac{\partial\theta}{\partial x^i} - \dot{\Psi} + \hat{p}^i\frac{\partial\Phi}{\partial x^i} = n_e\sigma_T a\left(\theta_0 - \theta + \hat{p}\cdot\vec{v}_b\right) . \tag{1.90}$$

Now, in order to solve this equation we perform a Fourier transform:

$$\theta(\vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}\widetilde{\theta}(\vec{k})\,, \tag{1.91}$$

and we define the *cosine* of the angle between the photon momentum $\vec{p}$ and the momentum $\vec{k}$: $\mu = \hat{k}\cdot\hat{p}$.

The optical depth is defined by

$$\tau(\eta) = \int_{\eta}^{\eta_0} \mathrm{d}\overline{\eta}\, a(\overline{\eta}) n_e \sigma_T\,, \tag{1.92}$$

and it is large at early times, small at late times since the density decreases. Its derivative is

$$\dot{\tau} = \frac{\mathrm{d}\tau}{\mathrm{d}\eta} = -n_e \sigma_T a \,, \tag{1.93}$$

which is a term in the Boltzmann equation! Substituting it, we get

$$\dot{\widetilde{\theta}} + ik\mu\widetilde{\theta} - \dot{\widetilde{\Psi}} + ik\mu\widetilde{\Psi} = -\dot{\tau}\left(\widetilde{\theta}_0 - \widetilde{\theta} + \mu\widetilde{v}_b\right). \tag{1.94}$$

It is an assumption we are making that the velocities are irrotational, so they can be expressed as a gradient: so, in Fourier space we get $\widetilde{\vec{v}}_b = \hat{k}\widetilde{v}_b$.

We still need to solve the Einstein equations in order to determine Φ and Ψ. In order to do this we need to describe all the component of the universe.

Also, we need to determine the velocity of the baryons v_b; it is an assumption we are making that in order to preserve local as well as global neutrality the local velocities of baryons and leptons are equal.

1.2 Boltzmann equation for CDM

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[From yesterday: the distribution function for Compton scattering is general, as long as electrons and photons are in thermal equilibrium.]

Last time we derived the first order Boltzmann equation for the thermal anisotropy.

Now, however, we need to discuss CDM, and then we will discuss baryons..

For Dark Matter the collision term is negligible (since by the observations of galactic dynamics we know that DM is collisionless): so in the Boltzmann equation we have

$$\frac{\mathrm{D}f_{\mathrm{dm}}}{\mathrm{D}\lambda} = 0 \,. \tag{1.95}$$

The key fact is that DM is nonrelativistic. The calculation, except for this, is quite analogous. The momentum P^μ has to be normalized: $P^\mu P_\mu = g_{\mu\nu}P^\mu P^\nu = -m^2$, as opposed to 0, which we had with the photon. We define the norm of the three-momentum

$$p^2 = g_{ij}P^i P^j \,, \tag{1.96}$$

and with it the energy: $E = \sqrt{p^2 + m^2}$. Then, using our perturbed metric we get

$$P^0 = E(1 - \Phi) \qquad \text{and} \qquad P^i = p\hat{p}^i \frac{1+\Psi}{a} \,, \tag{1.97}$$

where $\hat{p}^i$ is a unit vector in the direction of p^i. This makes sense: the velocity, without perturbation, decays like a^{-1}.

Notice that we can make the derivative with respect to the affine parameter λ to one with respect to t, the factor $\mathrm{d}t/\mathrm{d}\lambda$ can be moved to the RHS and does not affect our discussion. The LHS of the Boltzmann equation (1.95) then reads

$$\frac{\mathrm{D}f_{\mathrm{dm}}}{\mathrm{D}t} = \frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\partial f_{\mathrm{dm}}}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f_{\mathrm{dm}}}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \underbrace{\frac{\partial f_{\mathrm{dm}}}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}}_{\text{higher order}} \,, \tag{1.98}$$

where the last term is second order since it is a product of factors which are both zero to zeroth order. With steps analogous to the massless case we find

$$0 = \frac{\partial f_{\mathrm{dm}}}{\partial t} + \frac{\hat{p}^i}{a} \frac{p}{E} \frac{\partial f_{\mathrm{dm}}}{\partial x^i} - \frac{\partial f_{\mathrm{dm}}}{\partial E} \left[\frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} \frac{p^2}{E} - \frac{p^2}{E} \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i p}{a} \frac{\partial \Phi}{\partial x^i} \right] . \tag{1.99}$$

The factor $\mathrm{d}a/\mathrm{d}t\ /a$ is just H, we write it this way in order not to overload the dot (which represents derivatives with respect to conformal time only in our notation).

In order to work with this, we shall consider only terms up to first order in p/E, which is justified since DM is non-relativistic. Also, we will take moments: in general, a moment is an integral of a function (with angular dependence) multiplied by some power of the cosine of the angle between the direction considered and some fixed other direction. In principle, the hierarchy goes to any order: in practice, we can usually truncate the hierarchy to some order according to some physical principle. Fortunately, DM behaves like a fluid, which allows us to work to first order only.

Are the NS equations not exact?

This approach is fully relativistic, but linear. In star formation, we use Newtonian approximation, but we account for nonlinearity. Having both is impossible.

The lowest (0th) moment is just given by the integral in $\mathrm{d}^3p\,/(2\pi)^3$ of the distribution, that is, over all angles: we find

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\mathrm{dm}} + \frac{1}{a} \frac{\partial}{\partial x^i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\mathrm{dm}} \frac{p \hat{p}^i}{E} \\ &\quad - \left[\frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} - \frac{\partial \Psi}{\partial t} \right] \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\partial f_{\mathrm{dm}}}{\partial E} \frac{p^2}{E} - \underbrace{\frac{1}{a} \frac{\partial \Phi}{\partial x^i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\partial f_{\mathrm{dm}}}{\partial E} \hat{p}^i p}_{\text{higher order}} . \end{aligned} \tag{1.100a}$$

We take out of the integral any term not depending on the angles in momentum space.

why is the last term higher order? There is no way to have a nonzero result to zeroth order in the integral by isotropy, and it is multiplied by a first-order perturbation.

Now, we introduce the dark matter density n_{dm} — for more details, see the notes for the course in Fundamentals of Astrophysics and Cosmology [TM20b, section 3.2], but do note that the number density defined there is global, while the one we are considering here is a local number density, which can be different for different points in spacetime. Also, we define its velocity v^i:

$$n_{\mathrm{dm}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\mathrm{dm}} \tag{1.101a}$$

$$v^i = \left\langle \frac{P^i}{E} \right\rangle_{\text{particle}} = \frac{1}{n_{\mathrm{dm}}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{\mathrm{dm}} \frac{p \hat{p}^i}{E} . \tag{1.101b}$$

Note that f_{dm} accounts for the number of degrees of freedom g_{dm}.

These are not covariant, right? Since n is the 0th component of a 4-vector...

In the first and second terms we can insert n_{dm} and v^i directly (to first order — we always do manipulations neglecting second order terms), while for the third term we need to integrate by parts, using $\mathrm{d}E/\mathrm{d}p = p/E$:[5] we obtain:

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\partial f_{\mathrm{dm}}}{\partial E} \frac{p^2}{E} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p \frac{\partial f_{\mathrm{dm}}}{\partial p} \tag{1.103a}$$

Changed the derivative of the distribution

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}p\, p^3 \left\langle \frac{\partial f_{\mathrm{dm}}}{\partial p} \right\rangle \tag{1.103b}$$

$$= -3 \frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}p\, p^2 \left\langle f_{\mathrm{dm}} \right\rangle = -3 n_{\mathrm{dm}}\,, \tag{1.103c}$$

which is a calculation discussed in the FAC course [TM20b] [I'll insert the section when I get to it].

This yields:

$$\frac{\partial n_{\mathrm{dm}}}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x^i} \left(n_{\mathrm{dm}} v^i \right) + 3 \left[\frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} - \frac{\partial \Psi}{\partial t} \right] n_{\mathrm{dm}} = 0\,, \tag{1.104}$$

which to 0th order yields

$$\frac{\partial n^{(0)}_{\mathrm{dm}}}{\partial t} + 3 \frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} n^{(0)}_{\mathrm{dm}} \iff a^3 n^{(0)}_{\mathrm{dm}} = \text{const}\,. \tag{1.105}$$

Now, we can write the full number density as

$$n_{\mathrm{dm}} = n^{(0)}_{\mathrm{dm}} \left[1 + \delta(\vec{x}, t) \right]\,, \tag{1.106}$$

where δ is the standard notation in cosmology for these kinds of dimensionless fractional perturbations.

Then, after some simplifications the evolution of this perturbation looks like

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial v^i}{\partial x^i} - 3 \frac{\partial \Psi}{\partial t} = 0\,, \tag{1.107}$$

which is our first order continuity equation.

This was for the first moment, for the next one we integrate after multiplying by $\hat{p}^j/E$. We neglect a term because it is too relativistic, that is, second order in p/E.

Integrating by parts, for a different term we get

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\partial f_{\mathrm{dm}}}{\partial p} \frac{p^2 \hat{p}^i}{E} = \int \frac{\mathrm{d}\Omega\, \hat{p}^j}{(2\pi)^3} \int \dots lni \qquad = -4 n_{\mathrm{dm}} v^i\,. \tag{1.108}$$

We finally obtain

$$\frac{\partial (n_{\mathrm{dm}} v^i)}{\partial t} + 4 \frac{1}{a} \frac{\mathrm{d}a}{\mathrm{d}t} n_{\mathrm{dm}} v^j + \frac{n_{\mathrm{dm}}}{a} \frac{\partial \Phi}{\partial x^i} = 0\,, \tag{1.109}$$

[5] This comes from $E = \sqrt{p^2 + m^2}$:

$$\frac{\partial E}{\partial p} = \frac{2p}{2\sqrt{p^2 + m^2}} = \frac{p}{E}\,. \tag{1.102}$$

so we factor out the background and get, to first order,

$$\frac{\partial v^j}{\partial t} + Hv^j + \frac{1}{a}\frac{\partial \Phi}{\partial x^j} = 0\,. \tag{1.110}$$

This is basically a linear Euler equation. In Fourier space, we get

$$\dot{\widetilde{\delta}} + ik\widetilde{v} - 3\dot{\widetilde{\psi}} = 0 \tag{1.111a}$$

$$\dot{\widetilde{v}} + \frac{\dot{a}}{a}\widetilde{v} + ik\widetilde{\Phi} = 0\,. \tag{1.111b}$$

1.3 Boltzmann equation for baryons

Which interaction are relevant? At recombination, electrons are tightly coupled with photons, while the interaction between protons and photons is suppressed by a factor $(m_e/m_p)^2$ with respect to to electron-photon scattering. We need to deal with Coulomb scattering between electrons and proton. So, we can say that the quantity

$$\frac{\rho_e - \rho_p^{(0)}}{\rho_e^{(0)}} = \rho_b\,, \tag{1.112}$$

is a perturbation. So we write two Boltzmann equations, for electrons and protons, assuming that they move with the same velocity. The collision term looks like

$$C_{e\gamma} = (2\pi)^4 \delta^{(4)}(p+q-p'-q') \frac{|\mathcal{M}|^2 \big[f_e(q')f_\gamma(p') - f_e(q)f_\gamma(p)\big]}{8E(p)E(p')E_e(q)E_e(q')}\,, \tag{1.113}$$

So the zeroth moment equation for electrons is the same as the one for Dark Matter: both of the terms in the RHS vanish in

$$\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i} + 3\left[H - \frac{\partial \Psi}{\partial t}\right] n_e = \left\langle C_{ep} \right\rangle + \left\langle C_{e\gamma} \right\rangle\,, \tag{1.114}$$

since they multiply antisymmetric terms in exchange of momenta [clear up].

For the first moment we sum the proton and electron equations multiplied by the respective charges. The symmetry of the momenta of electrons and protons cancels the term of Coulomb scattering in the first moment equation. So, we get

$$\frac{\partial v_b^j}{\partial t} + Hv_b^j + \frac{1}{a}\frac{\partial \Phi}{\partial x^j} = \frac{1}{\rho_b}\left\langle c_{e\gamma} q^j \right\rangle_{pp'q'q}\,, \tag{1.115}$$

but by total momentum conservation we can switch the momentum q^j in the equation to a p^j. We Fourier transform, and multiply by $\hat{k}^j$: then we get $\vec{p}\cdot\hat{k} = p\mu$, where $\mu = \cos\alpha$.

Two pages from Dodelson [Dod03].

In the perturbed Einstein Equations we will need the stress-energy tensor: it does not require the full phase space distribution, but only certain kinds of integrated information.

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With our approach, we did not assume that DE, DM and such are fluids: instead, we have shown it.

We can combine the equations for electrons and protons into a unique equation for the baryons:

$$m_p \frac{\partial}{\partial t}\left(n_b v_b^j\right) + 4Hm_p n_b v_b^j \ldots . \tag{1.116}$$

The quantity $-\langle c_{e\gamma} p\mu \rangle / \rho_b$ is similar to what we had for photons: a monopole term, the temperature anisotropy, and the Doppler term, which is called that since it involves a velocity. Then we get

$$-\langle c_{e\gamma} p\mu \rangle / \rho_b = \frac{n_e \sigma_T}{\rho_b} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p} \mu \left[\widetilde{\theta}_0 - \widetilde{\theta}(\mu) + \widetilde{v}_b \mu\right] \tag{1.117a}$$

$$= \frac{n_e \sigma_T}{\rho_b} \int_0^\infty \frac{\mathrm{d}p}{2\pi^2} p^4 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \mu \left[\widetilde{\theta}_0 - \widetilde{\theta}(\mu) + \widetilde{v}_b \mu\right], \tag{1.117b}$$

notice that the terms inside the bracket are independent of the modulus of p. Now we define the *dipole*:

$$\theta_1 = i \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \mu \theta(\mu), \tag{1.118}$$

so we finally get

$$\dot{\widetilde{v}}_b + \frac{\dot{a}}{a} \widetilde{v}_b + ik\widetilde{\Phi} = \dot{\tau} R \left(3i\widetilde{\theta}_1 + \widetilde{v}_b\right), \tag{1.119}$$

where we defined

$$R = \frac{3\rho_b^{(0)}}{4\rho_\gamma^{(0)}}. \tag{1.120}$$

In general we define the multipole moments as

$$\theta_\ell = \frac{1}{(-i)^\ell} \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} P_\ell(\mu) \theta(\mu), \tag{1.121}$$

where $P_e(\mu)$ are the Legendre polynomials.

Then we get

$$\dot{\widetilde{\theta}} + ik\mu\widetilde{\theta} - \dot{\widetilde{\Psi}} + ik\mu\widetilde{\Phi} = -\dot{\tau}\left[\widetilde{\theta}_0 - \widetilde{\theta} + \mu v_b - \frac{1}{2} P_2(\mu) \widetilde{\theta}_2\right], \tag{1.122}$$

where the addition of the quadrupole term accounts for (???), see the notes by Natale [Nat17].

Dropping the tildes, we get

$$\dot{\theta} + ik\mu\theta = \dot{\Psi} - ik\mu\Phi - \dot{\tau}\left[\theta_0 - \theta + \mu v_b - \frac{1}{2} P_2(\mu) \Pi\right] \tag{1.123a}$$

$$\Pi = \theta_2 - \theta_{P_2} + \theta_{P_0} \tag{1.123b}$$

$$\dot{\theta}_p + ik\mu\theta_p = -\dot{\tau}\left[-\theta_p + \frac{1}{2}\big(1 - P_2(\mu)\Pi\big)\right]\dots, \tag{1.123c}$$

anisotropy creates polarization, polarization does not affect polarization much.

Why do we not have a quadrupole term? Its integral is zero since it is multiplied by μ.

What about neutrinos? We can model the anisotropies in the cosmic neutrino background using Fermi-Dirac statistics, with

$$f = \frac{1}{\exp(d)}\dots, \tag{1.124}$$

so now repeating the considerations we made for the photons, we get

$$\dot{w}\dots. \tag{1.125}$$

1.4 Perturbing the Einstein Equations

Why do we consider only the scalar perturbations? Vectors do not obey dynamical equation but only conservation equations, and they die away quickly in the expansion. The Lense-Thirring effect is about the g_{0i} terms, and it is not governed by the (???) theorem, but we neglect it as well.

Tensor perturbations are gravitational waves. They evolve separately from the scalar perturbations: we decompose the anisotropy into

$$\theta^T(k,\mu,\phi) = \theta^T_+(k,\mu)\left(1-\mu^2\right)\cos(2\phi) + \theta^t_\times(k,\mu)(1-\mu^2)\,, \tag{1.126}$$

and both of the polarizations ($\epsilon = +, \times$) satisfy

$$\dot{\theta}^T_\epsilon + ik\mu\theta^T_\epsilon + \frac{1}{2}\dot{h}_\epsilon = \dot{\tau}\left[\theta^T_\epsilon - \frac{1}{10}\theta^T_{\epsilon,0} - \frac{1}{7}\theta^T_{\epsilon,2} - \frac{3}{7}\theta^T_{\epsilon,4}\right]. \tag{1.127}$$

Let us then perturb Einstein's equations: we use conformal time, so the metric (in Euclidean flat 3-space) is

$$\mathrm{d}s^2 = a^2\left[-(1+2\Phi)\,\mathrm{d}\eta^2 + (1-2\Psi)\,\mathrm{d}\ell^2\right]. \tag{1.128}$$

We compute the Christoffel symbols, and from these we get the Ricci tensor, the Ricci scalar and the Einstein tensor. It is generally much more convenient to write the equations with mixed indices

$$G^\mu_\nu = 8\pi G T^\mu_\nu\,. \tag{1.129}$$

To first order we get

$$\Gamma^0_{00} = \frac{\dot{a}}{a} + \dot{\Phi} \tag{1.130a}$$

$$\Gamma^i_{0j} = \left(\frac{\dot{a}}{a} - \dot{\Psi} \right) \delta^i_j \tag{1.130b}$$

$$\Gamma^0_{ij} = \left[\frac{\dot{a}}{a} (1 - 2\Phi - 2\Psi) - \dot{\Psi} \right] \delta_{ij} \tag{1.130c}$$

$$\Gamma^0_{0i} = \partial_i \Phi \tag{1.130d}$$

$$\Gamma^i_{00} = \partial^i \Phi \tag{1.130e}$$

$$\Gamma^i_{jk} = -\partial_j \Psi \delta^i_k - \partial_k \Psi \delta^i_j + \partial^i \Psi \delta_{jk} \,, \tag{1.130f}$$

so the Einstein tensor reads:

$$G^0_0 = \frac{1}{a^2} [\dots] \,, \tag{1.131}$$

Often people make the quasi-static approximation, in which we neglect the time-derivatives of the potentials. We do not do that here. The stress-energy tensor has the following 00 component:

$$T^0_0 = -\sum_i g_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_i(p) f_i(\vec{x}, \vec{p}, t) \,. \tag{1.132}$$

For nonrelativistic matter, $E_i \sim m_i$: then we get

$$T_0^{0,\mathrm{dm}} = -\rho_{\mathrm{dm}} (1 + \delta) \,. \tag{1.133}$$

For the photons we have

$$T_0^{0,(\gamma)} = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \theta \right] = -\rho_\gamma [1 + 4\Theta_0] \,, \tag{1.134}$$

and for the neutrinos we get the exact same result, substituting the variables:

$$T_0^{0,(\nu)} = -\rho_\nu [1 + 4 w_0] \,. \tag{1.135}$$

Dark Energy is usually considered to be smooth. Then we have, for the 00 EFE:

$$\nabla^2 \Psi - 3 \frac{\dot{a}}{a} \left(\dot{\Psi} + \Phi \frac{\dot{a}}{a} \right) = 4\pi G a^2 \left(\rho_{\mathrm{dm}} \delta + \rho_b \delta_b + 4 \rho_\gamma \Theta_0 + 4 \rho_\nu w_0 \right) . \tag{1.136}$$

Recall that we want to solve for Ψ and Φ: we have a lot of redundancy in the EFE. Another convenient independent equation is the traceless part of the ij components. The simplest way to do these calculations is to use projectors in Fourier space.

On the left hand side we get

$$\left(\hat{k}_i \hat{k}^j - \frac{1}{3} \delta^i_j \right) \hat{G}^i_j = \left(\hat{k}_i \hat{k}^j - \frac{1}{3} \delta^j_i \right) \frac{k^i k_j}{a^2} (\Phi - \Psi) \tag{1.137a}$$

$$= \frac{2}{3} \frac{k^2}{a^2} (\Phi - \Psi) \,. \tag{1.137b}$$

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On the Right Hand Side, instead, since the stress energy tensor is given by

$$T^i_j = \sum_{\text{all species } a} g_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{p^i p_j}{E_a(p)} f_a(\vec{x}, \vec{p}, t) \tag{1.138}$$

we obtain

$$\left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^i_j \right) \widetilde{T}^i_j = \sum_a g_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{p^2\mu^2 - p^2/3}{E_a(p)} f_a(\vec{p}) \,, \tag{1.139}$$

where μ is the cosine of the angle between $\hat{k}$ and $\hat{p}$.

But we know that

$$\mu^2 - \frac{1}{3} = \frac{2}{3} P_2(\mu) \,, \tag{1.140}$$

where P_2 is the second Legendre polynomial; recall that these Legendre polynomials are precisely those used in the definition of the multipole moments θ_k [definition ref]. Using this, we get

$$\left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta^i_j \right) \widetilde{T}^i_j = -2 \int_0^\infty \frac{\mathrm{d}p\, p^2}{2\pi^2} \frac{\partial f^{(0)}}{\partial p} p^2 \int_{-1}^1 \frac{\mathrm{d}\mu}{2} \frac{2}{3} P_2(\mu) \theta(\mu) \tag{1.141a}$$

$$= 2\frac{2}{3}\widetilde{\theta}_2 \int_0^\infty \frac{\mathrm{d}p\, p^2}{2\pi^2} p^2 \frac{\partial f^{(0)}}{\partial p} = -\frac{8}{3}\rho_\gamma \widetilde{\theta}_2 \,, \tag{1.141b}$$

we have integrated by parts, and the quadrupole contribution has vanished by the orthogonality property of the Legendre polynomials.

[why is it $\rho_{\gamma,0}$? we are not computing it now, right?] We are writing ρ_γ and the such we really mean $\rho_\gamma^{(0)}$, an average at monopole order.

In a universe without neutrinos, we would have no difference between Φ and Ψ.

What would be the geometric meaning of the difference between Φ and Ψ?

So, if we collect terms we find that the whole Einstein equation reads:

$$k^2(\Phi - \Psi) = -32\pi G a^2 \underbrace{\left[\rho_\gamma \widetilde{\theta}_2 + \rho_\nu \widetilde{w}_2 \right]}_{\rho_r \theta_{r,2}} \,. \tag{1.142}$$

Since we have added the neutrino contribution, we are accounting for all of radiation. See Dodelson [Dod03, page 99].

The Einstein 00 equation (1.136) can be written in Fourier space as

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Psi} + \frac{\dot{a}}{a}\Phi \right) = -4\pi G a^2 \left(\rho_m \delta_m + 4\rho_r \widetilde{\theta}_{r,0} \right) , \tag{1.143}$$

where

$$\rho_m \delta_m = \rho_m \delta + \rho_b \delta_b \qquad \text{and} \qquad \rho_r \theta_{r,0} = \rho_\gamma \theta + \rho_\nu w \,. \tag{1.144}$$

Dark Matter and baryons do not contribute to the quadrupole (at first order).

Recall that the EFE are not independent: we could also have chosen the ${}^0_i$ one: those components of the stress-energy tensor are given by

$$T^0_i = \sum_\alpha a\rho_\alpha \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p_i f_\alpha(\vec{x}, \vec{p}, t)\,, \tag{1.145}$$

so with reasoning similar to what was done before we find that the ${}^0_i$ equation reads

$$\dot{\Psi} + aH\Phi = -\frac{4\pi G a^2}{ik}\left[\rho_m v_m - 4i\rho_r \theta_{r,1}\right], \tag{1.146}$$

where

$$\rho_m v_m = \rho_{\mathrm{dm}} v + \rho_b v_b \qquad \text{and} \qquad \rho_r \theta_{r,1} = \rho_g \theta_1 + \rho_\nu w_1\,. \tag{1.147}$$

In this case then we would find

$$k^2\Psi = -4\pi G a^2 \left[\rho_m \delta_m + 4\rho_r \theta_{r,0} + \frac{32H}{k}\left(i\rho_m v_m + 4\rho_r \theta_{r,1}\right)\right]. \tag{1.148}$$

Tensor models

Now that we have found the equation of motion of the scalar field, let us discuss tensor modes: we can find the equation of motion for these starting from the transverse traceless part of the ${}^i_j$ equation. We set $\chi_{ij} = 2h_{ij}$, and with this we can write the traceless Christoffel symbols:

$$\Gamma^0_{ij} = \frac{1}{2}\dot{h}_{ij} + \frac{\dot{a}}{a}h_{ij} \tag{1.149a}$$

$$\Gamma^i_{0j} = \frac{1}{2}\dot{h} \tag{1.149b}$$

$$\Gamma^i_{jk} = h^i_{(j,k)} - \frac{1}{2}h_{jk}{}^{,i} \tag{1.149c}$$

$$, \tag{1.149d}$$

which yield the traceless Einstein tensor

$$G^i_j = \frac{\dot{a}}{a}\dot{h}^i_j - \frac{1}{2}\nabla^2 h^i_j + \frac{1}{2}\ddot{h}^i_j\,. \tag{1.150}$$

In order to get the correct contribution on the RHS we need to project it: we apply the projection operator

$$\mathcal{P}^{k\,j}_{\ i\,l} = \mathcal{P}^k_i \mathcal{P}^j_l - \frac{1}{2}\mathcal{P}^k_l \mathcal{P}^j_i\,, \tag{1.151}$$

where

$$\mathcal{P}^i_j = \delta^i_j - \hat{k}^i \hat{k}_j\,. \tag{1.152}$$

The work [CM05] is mentioned, but the work only the two-index $\mathcal{P}$ is defined (in position space and not in momentum space, but it's the same) (eq. 13); but the four-index combination is not explicitly written…

What is usually done is to completely neglect the right hand side of the EFE (since it is second order in v/c?), to get the homogeneous expression

$$\ddot{h}_{ij} + 2\frac{\dot{a}}{a}h_{ij} - \nabla^2 h_{ij} = 0\,. \tag{1.153}$$

We can express the tensor perturbation in Fourier space as

$$h_{ij} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, e^{-\vec{k}\cdot\vec{x}} h_{ij}(\vec{k}, t)\,, \tag{1.154}$$

where $h_{ij}(\vec{k}, t)$ can be decomposed into the two basis polarizations $+$ and $\times$ so that our equation will read

$$\ddot{h}_\epsilon + 2\frac{\dot{a}}{a}\dot{h}_\epsilon + k^2 h_\epsilon = 16\pi G \pi_\epsilon\,, \tag{1.155}$$

where $\epsilon = +, \times$ (see Pritchard & Kamionkowski [PK05]). Here π_ϵ is the traceless tensor part of the stress-energy tensor, which contains a negligible contribution from photons and a contribution from neutrinos which has a non-negligible effect on gravity waves; the latter leads to

$$\ddot{h}_\epsilon + 2\frac{\dot{a}}{a}\dot{h}_\epsilon + k^2 h_\epsilon = -24 f_\nu(\eta)\left(\frac{\dot{a}}{a}\right)^2 \int_0^\eta \mathrm{d}\widetilde{\eta}\, K(k(\eta - \widetilde{\eta}))\dot{h}_\epsilon(\widetilde{\eta})\,, \tag{1.156}$$

where $f_\nu = p_\nu^{(0)}/p^0$ and

$$K(s) = -\frac{\sin s}{s^3} - \frac{3\cos s}{s^4} + \frac{3\sin s}{s^5}\,. \tag{1.157}$$

This is a damping, not a source term

Is it? as long as $s < 5$, sure, but is that guaranteed? around $s = 5.5$ the function changes sign…

Solutions to the tensor perturbation equations are given by

$$h_{\mathrm{rad}}(\eta) = h(0)\ldots \tag{1.158a}$$
$$h_{\mathrm{mat}}(\eta) = 3h(0)\ldots\,. \tag{1.158b}$$

Initial conditions

Plot: comoving scale versus log of scale factor. From Lucchin-Coles? [CL02].

We can say that the universe starts at the end of inflation.

Let us put the curvature k back into the equations we found. At early times, but after inflation, (???) are outside the horizon so we have $k\eta \ll 1$ and $\dot{\tau} \to \infty$, so the initial conditions look like

$$\dot{\theta}_0 = \dot{\Psi} \qquad \dot{w}_0 = \dot{\psi} \tag{1.159a}$$

$$\dot{\delta} = 3\dot{\psi} \qquad \dot{\delta}_b = 3\dot{\psi}\,; \tag{1.159b}$$

these equations assume that all the multipoles from the dipole onward are suppressed.

This is called the "separate universe" approximation. Locally, the universe looks like a separate FRLW curved universe.

Clarify

Also, the dipole must be much smaller than the monopole, and we must have the velocity of the baryons v_b be equal to $-3i\theta$ because of tight coupling.

In this limit, photons behave like a fluid, and this fluid has the same behavior as the e^- fluid.

We assume that the neutrino quadrupole is negligible. We are considering a radiation-dominated epoch, so we neglect matter.

We have then $\Psi = \Phi$, and

$$3\frac{\dot{a}}{a}\left(\dot{\Psi} + \frac{\dot{a}}{a}\Phi\right) = -16\pi G a^2 \rho_r \Theta_{r,0}\,, \tag{1.160}$$

which becomes

$$\frac{\dot{\Phi}}{\eta} + \frac{1}{\eta^2}\Phi = -\frac{16\pi G}{3}\rho_r a^2 \left(\frac{\rho_\gamma}{\rho_r\theta_0 + \frac{\rho_\nu}{\rho_r}w_0}\right). \tag{1.161}$$

Then the phase space distribution looks like

$$f_\nu = \frac{\rho_\nu}{\rho_\gamma + \rho_\nu}\,, \tag{1.162}$$

since $a \propto \eta$ and $\rho_r \propto a^{-4}$ and $\rho_r = \rho_\gamma + \rho_\nu$.

If we assume $\dot{\theta}_0 = \dot{w}_0 = \dot{\Phi}$. This finally yields

$$\ddot{\Phi}\eta + 4\dot{\Phi} = 0\,, \tag{1.163}$$

which has a vanishing solution, and a constant solution. We choose the latter.

We require the perturbation to be isentropic (see [CL02]). This yields

$$\delta_m = 3\theta_0 = \frac{3}{4}\delta_r\,. \tag{1.164}$$

Why do we assume this? If the perturbation is sourced by a single scalar field, then we can only consider a single mode: the isentropic mode.

With these, the 00 EFE becomes

$$\Phi = -2\left[(1 - f_\nu)\theta_0 + f_\nu w_0\right], \tag{1.165}$$

where we must have $f_\nu = \text{const}$.

Adiabaticity implies

$$\phi = -2\theta_0 \implies \delta = -\frac{3}{2}\Phi\,. \tag{1.166}$$

If we had kept the neutrino quadrupole, we would have gotten

$$\Phi = \Psi \left(1 + \frac{2f_\nu}{5} \right) . \tag{1.167}$$

f_ν is a constant at any order: it is a number fixed by the number of neutrino species. Friday 2020-4-3, compiled 2020-05-21

We work in the tight coupling limit.

[around equation 39]

We drop the quadrupole term.

The term multiplying $\dot{\tau}$ in eq 40 goes to infinity like $1/\dot{\tau}$: we need to replace it by a finite contribution. Thursday 2020-4-9, compiled 2020-05-21

We still need information about $\dot{\Psi}$ and Φ. This should be provided by the EFE. We approximate them as being independent of θ_0 and θ_1.

The trick to get θ_0 is to differentiate eq 39 with respect to η, and then take $\dot{\theta}_1$ from the other eq.

So we get eq 42: it describes the evolution of θ_0, on its right hand side we have a forcing term which we denote by $F(k, \eta)$. The factor $R/(1+R)$ is the ratio of the baryon energy density to the total energy density.

This is solved numerically, we want a simple analytic solution.

We basically have a damped wave equation, with a forcing term [43]. This is solved, as usual, by solving the homogeneous equation first. We solve the homogeneous equation with the use of an integrating factor.

People sometimes drop the viscous term: this is a good approximation if we are within the "sound horizon" (which we shall see shortly).

We replace the c_s, which is dependent on time in principle, by its integral in η. (?) The sound horizon is given by

$$r_\eta = \, . \tag{1.168}$$

Once we have solved the homogeneous equation, we use the Green function method to solve the inhomogeneous equation.

We use the cosine instead of the sine in order to account for the boundary conditions.

The wavenumber corresponding to equality is:

$$k_{\text{eq}} = 0.073 \text{Mpc}^{-1} \Omega_{0m} h^2 \, . \tag{1.169}$$

So we have that $\theta_0 \sim \cos$, while $\theta_1 \sim \sin$: they are in opposition of phase in the tight coupling limit.

Anisotropic stress means neutrinos. The perturbations enter the horizon in the radiation dominated epoch as long as…

Meszaros effect: perturbations which enter the horizon during radiation dominance are damped.

The gravitational perturbation is constant outside the horizon, it is damped under radiation dominance.

We now add the quadrupole, but we neglect the octupole.

In order to solve these equations, we can apply the same procedure as before, or we can use a trick: we get an approximate dispersion relation, compared to the previous solution we get an imaginary term, which correspond to damping in θ_0 and θ_1.

λ_D tells us how small a perturbation's wavelength can be if it is damped.

1.4.1 Free-streaming

We wish to describe what we expect to see. We discovered we can have anisotropies, so we can have hot and cold spots in the radiation content of the universe.

For photons, the distance in comoving time $\Delta\eta$ and the comoving distance ΔL are equal, since $\mathrm{d}s = 0$.

The quantity $\eta_0 - \eta_*$ is usually called *lookback time*. A multipole of order ℓ is maximized when ℓ is of the order $1/\theta$, where $\theta = k/(\eta_0 - \eta_*)$.

We define a new temporary function $\widetilde{S}$.

The most important equation is equation 48.

If we were to drop all the dependence on μ and integrate $\widetilde{S}$, we get the spherical Bessel functions.

The visibility function is defined as

Friday 2020-4-10, compiled 2020-05-21

$$g(\eta) = -\dot{\tau}e^{-\tau}\,, \tag{1.170}$$

this is normalized to one when integrated in $\mathrm{d}\eta$, and it has a peak at last scattering. This function will be later approximated as a δ function of $t - t_{\text{last scattering}}$.

Terms in equation [48]:

1. Sachs-Wolfe term
2. Velocity doppler term;
3. Integrated Sachs-Wolfe.

This is the main equation we use to study the CMB spectrum.

1.5 The angular power spectrum

Several telescopes have been launched in order to study the CMB. We need to map the whole sky, to a good resolution, and to a good spectral resolution. Some ones were COBE, WMAP. Now we have Planck.

The temperature is

$$T(\hat{x}, \hat{p}, \eta) = T(\eta)\left[1 + \theta(\hat{x}, \hat{p}, \eta)\right]\,, \tag{1.171}$$

and then we write θ as a sum of harmonic coefficients:

$$\theta(\hat{x}, \hat{p}, \eta) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}(\hat{x}, \eta) Y_{\ell m}(\hat{p})\,, \tag{1.172}$$

where the spherical harmonics are orthonormal.

We can recover the harmonic coefficients by Fourier transforming; at this step we can set $\vec{x} = 0$, not before.

The $\ell = 0$ contribution is not included: it is the monopole, it cannot be distinguished from a renormalization of the temperature background.

The dipole is telling us about the motion of the Earth with respect to to the CMB, so usually people start at 2. We then stop at the resolution of the experiment.

We have $\langle a_{\ell m} \rangle = 0$, however the angular average of

$$\left\langle a_{\ell m} a^*_{\ell' m'} \right\rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell \,. \tag{1.173}$$

This comes from the orthonormality, and the requirement of statistical isotropy. We account for the complications arising from galaxy emission.

$$C_\ell = \frac{2}{\pi} \int_0^\infty \mathrm{d}k \, k^2 P(k) \left| \frac{\theta_\ell(k)}{\delta(k)} \right|^2 , \tag{1.174}$$

where δ is the dark matter density perturbation,

$$\left\langle \delta(\vec{k}) \delta(\vec{k}') \right\rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') P(k) \,, \tag{1.175}$$

where $P(k)$ is defined by this equation, it is the power since it is quadratic.

An interesting effect is the fact that

$$(\theta_0 + \Phi)(?\eta_*) = \frac{1}{3} \dots . \tag{1.176}$$

Now we discuss the Sachs-Wolfe C_ℓ.

The simplest approximation is $P(k) \sim k^n$, a power low, where n is called the *spectral index*. For $n = 1$ we have lots of simplifications.

Thursday 2020-4-16, compiled 2020-05-21

Plot: figure 1 from [Col+19], the power spectrum. Error bars seem small, but consider the factor $\ell(\ell + 1)$ (?).

The resolution could go to $\ell = 3000$.

Temperature anisotropies map, also from Planck: the grey part is reconstructed from galactic absorption. Helpix: software to reconstruct the whole sky, which allows us to calculate the power spectrum.

We can include the polarization, to distinguish E and B-modes. B-modes are divergence-free, E-modes are curl-free.

We would like to take a statistical average over all observers, so we must average over the angles.

We define the quantity $\zeta = -\frac{3}{2}\Phi$, which is gauge invariant and connected to inflation.

Sachs-Wolfe, Doppler term, integrated Sachs-Wolfe.

We have

$$\left\langle \delta(\vec{k}) \delta(\vec{k}') \right\rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') P(k) \,, \tag{1.177}$$

where we have plus since we are computing the average of $\delta\delta$ instead of $\delta^*\delta$.

The fluctuation of CDM, δ, is proportional to $k^2\Phi$. This is conventional.

A powerlaw is scale-invariant: if we multiply by λ the powerlaw is multiplied by λ^n, but it is not distorted.

1.5.1 The free-streaming term

Dark energy changes Φ and Ψ, making them decrease. Friday 2020-4-17, compiled 2020-05-21

There are some additional notes on gravitational dynamics.

We have effects due to curvature: normalization, tilt, tensor modes. Tensor modes describe primordial gravitational waves, a stochastic GW background, which as opposed to the CMB has no reason to be thermal.

These GWs are radiation, so they dilute like a^{-4}. The scalar perturbation and primordial gravitational waves mix. There is no way to tell them apart.

We can characterize them by an integrated Sachs-Wolfe effect.

An interesting place to find topics for a discussion for the exam: Hu's website.

Chapter 2

Large Scale Structure

2.1 Gravitational instability

We are going to describe the full nonlinearity in the Newtonian approximation.

We need to choose coordinates which absorb the expansion of the universe. These are given by $\vec{r} = a\vec{x}$. These are inertial with respect to the background FRW evolution (not with respect to the matter, which is not uniform). So, we have

$$\vec{w} = \dot{\vec{r}} = \frac{\dot{a}}{a}\vec{r} + a\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = H\vec{r} + \vec{v}\,, \tag{2.1}$$

where $\vec{v}$ is the peculiar velocity. We also define the gradient

$$\nabla_{\vec{r}} = \frac{\partial}{\partial \vec{r}} = \frac{1}{a}\frac{\partial}{\partial \vec{x}} = \frac{1}{a}\nabla_{\vec{x}}\,. \tag{2.2}$$

We can also take the total derivative of a generic function:

$$\frac{\mathrm{D}f(\vec{r},t)}{\mathrm{D}t} = \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \frac{\partial f}{\partial \vec{r}}\dot{\vec{r}}\,, \tag{2.3}$$

[stuff] which means that we have

$$\left.\frac{\partial f}{\partial t}\right|_{\vec{x}} = \left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + H(\hat{r}\cdot\nabla_r)f\,. \tag{2.4}$$

The equations describing the fluid then are

$$\left.\frac{\partial f}{\partial t}\right|_{\vec{r}} + \nabla_{\vec{r}}(\rho\vec{w}) = 0 \tag{2.5a}$$

$$\left.\frac{\partial \vec{w}}{\partial t}\right|_{\vec{r}} + (\vec{w}\cdot\nabla_{\vec{r}})\vec{w} = -\frac{1}{\rho}\nabla_{\vec{r}}p - \nabla_{\vec{r}}\Phi \tag{2.5b}$$

$$\nabla_{\vec{r}}^2\Phi = 4\pi G\rho\,, \tag{2.5c}$$

which are continuity, Euler and gravitational field evolution. We perturb: $\rho = \rho_b + \delta\rho$ and $\Phi = \Phi_b + \phi$. We do this so that

$$\nabla_r^2\Phi_b = 4\pi G\rho_b(t)\,, \tag{2.6}$$

and

$$\nabla_r^2 \phi = 4\pi G \delta\rho \,. \tag{2.7}$$

Let us open the continuity equation. We get

$$\frac{\partial \rho}{\partial t}_{\vec{x}} + 3H\rho + \frac{1}{a}\nabla_{\vec{x}}(\rho\vec{v}) = 0 \,. \tag{2.8}$$

The Euler equation, instead, gives

$$\left.\frac{\partial \vec{v}}{\partial t}\right|_{\vec{r}} + H\vec{v} + \frac{1}{a}\vec{v}\cdot\nabla_x \vec{v} = -\frac{1}{a\rho}\nabla_x p - \frac{1}{a}\nabla_x \phi \,, \tag{2.9}$$

since we are neglecting the spatial variation of the background.

Now we Fourier transform δ, $\vec{v}$ and ϕ.

If we linearize, we get

$$k^2 \phi_{\vec{k}} = -4\pi G a^2 \rho_b(t) \delta_{\vec{k}} \tag{2.10a}$$

$$\dot{\delta}_{\vec{k}} + \frac{i\vec{k}\cdot\hat{v}_{\vec{k}}}{a} = 0 \tag{2.10b}$$

$$\dot{\vec{v}}_{\vec{k}} + H\vec{v}_{\vec{k}} = -\frac{i\vec{k}}{a}\left(c_s^2 \delta_{\vec{k}} + \phi_{\vec{k}}\right) , \tag{2.10c}$$

where $c_s^2 = \partial p/\partial\rho$. In the absence of dissipation, vorticity is conserved along stream lines: this is a theorem from Newtonian mechanics. This means that

$$\dot{\vec{v}}_\perp + H\vec{v}_\perp = 0 \,, \tag{2.11}$$

so $\vec{v}_\perp \propto 1/a$. It is not possible a priori to generate vorticity in QFT (?)

We can combine our equations to get

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} + \left[\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_b\right]\delta_{\vec{k}} = 0 \,, \tag{2.12}$$

so the Hubble radius is crucial since it appears in the second term. We can define a comoving Jeans wavenumber by setting the bracket to zero:

$$k_J = \frac{a}{c_s}\sqrt{4\pi G\rho_b} \,. \tag{2.13}$$

For CDM the Jeans wavenumber goes to infinity. Even if the DM is no longer thermal, it still has memory of its initial condition through velocity dispersion.

We can define a mass from the wavenumber.

Take a sphere of radius half the wavelength: then we have

$$\frac{4\pi}{3}(\frac{\lambda}{2})^3 \rho = m \,. \tag{2.14}$$

We get two solutions: $\delta \propto t^{\alpha}$, with $\alpha = 2/3$ or $\alpha = -1$. Therefore,

$$v_{\vec{k}} \propto t^{\beta} \,, \tag{2.15}$$

where $\beta = 1/3$ or $\beta = -4/3$.

Do galaxies have vorticity? No, that's fake news.

Thursday 2020-4-23, compiled 2020-05-21

We are going to continue discussing cosmic structure formation.

We treat the dynamics of self-gravitating collisionless particles.

We will neglect pressure gradients since the pressure is generally very small.

We have a Eulerian approach: we take a LIF with respect to the background field.

We sometimes do not write the mass of the fluid particle: by the equivalence principle the inertial and gravitational masses are equal.

We have the relations $\vec{r} = a\vec{x}$ and

$$\ddot{r} = -\nabla_r \Phi \,, \tag{2.16}$$

where the gravitational field is defined by

$$\nabla_r^2 \Phi = 4\pi G \rho \,. \tag{2.17}$$

If we take the background potential

$$\Phi = \Phi_b = \frac{2}{3}\pi G \rho_b(t) r^2 \,, \tag{2.18}$$

we recover the Friedmann equation

$$\ddot{a} = -\frac{4}{3}\pi G \rho_b(t) a \,. \tag{2.19}$$

The derivative of $\vec{r}$ is given by

$$\dot{r} = \dot{a}x + a\dot{x} = Hr + v \,, \tag{2.20}$$

which is the Hubble law in a fully Newtonian setting.

We can write the particle Lagrangian: it looks like

$$\mathscr{L}' = \frac{m}{2}(a\dot{x} + \dot{a}x)^2 - m\Phi(\vec{x}, t) \,, \tag{2.21}$$

so we have three terms: $a^2\dot{x}^2$, $\dot{a}^2x^2$ and $2ax\dot{a}\dot{x}$.

We use the canonical transformation

$$\mathscr{L} = \mathscr{L}' - \frac{\mathrm{d}\psi}{\mathrm{d}t} \qquad \text{where} \qquad \psi = \frac{1}{2} m a \dot{a} x^2 \,, \tag{2.22}$$

using which we remove the mixed terms, so we get

$$\mathscr{L} = \frac{m}{2} a^2 \dot{x}^2 - m\phi \,, \tag{2.23}$$

where $\phi = \Phi - \Phi_b = \Phi + a\ddot{a}x^2/2$.

We have different kinds of problems: we have problems in which we give initial positions and velocities, and ones in which we have initial and final positions.

We can also have mixed problems, where we give the initial velocity and the final position.

What are they called?

Then we get the equation for ϕ:

$$\nabla_x^2 = 4\pi G a^2 \rho + 3a\ddot{a} = 4\pi G a^2 \rho_b \delta\,. \tag{2.24}$$

In order to write Hamilton's equations we need the conjugate momentum:

$$\vec{p} = \frac{\partial \mathscr{L}}{\partial \dot{\vec{x}}} = ma^2 \dot{x}\,, \tag{2.25}$$

so

$$\dot{p} = \frac{\partial \mathscr{L}}{\partial \vec{x}} = -m\nabla_x \phi\,, \tag{2.26}$$

and if we define $\vec{v} = a\dot{\vec{x}}$ then we have

$$\dot{\vec{p}} = \frac{\partial}{\partial t}(ma\vec{v}) = ma\dot{\vec{v}} + m\dot{a}\vec{v}\,, \tag{2.27}$$

which yields our Euler-Lagrange equations:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + H\vec{v} = -\frac{1}{a}\nabla\phi\,. \tag{2.28}$$

This equation is always true, it does not assume that the element of mass behaves like a fluid.

Hamilton's equations are written by writing the Hamiltonian:

$$\begin{aligned} \mathscr{H}(\vec{x}, \vec{p}, t) &= \vec{p} \cdot \dot{\vec{x}} - \mathscr{L} && \text{(2.29a)} \\ &= \frac{p^2}{2ma^2} + m\phi(\vec{x}, t)\,, && \text{(2.29b)} \end{aligned}$$

and Hamilton's equations read

$$\begin{aligned} \vec{p} &= ma^2\dot{\vec{x}} && \text{(2.30a)} \\ \dot{\vec{p}} &= -m\nabla_x\phi\,. && \text{(2.30b)} \end{aligned}$$

By Liouville's theorem for collisionless matter the phase-space density is conserved:

$$\mathrm{d}f = 0 \implies \frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial f}{\partial \vec{x}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} = 0\,. \tag{2.31}$$

This yields the Vlasov equation:

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2} \cdot \nabla f - m\nabla\phi \cdot \frac{\partial f}{\partial \vec{t}} = 0\,, \tag{2.32}$$

which must be solved together with the Poisson equation for ϕ.

Nobody has managed to solve it yet: it is very hard.

We cannot use the linear regime: stars and structures are highly nonlinear.

Jim Peebles for "reconstruction problem". There is an ambiguity in the trajectories. We can find information about these things in a dropbox folder by the professor.

We take *moments*:

$$\rho(\vec{x}, t) = \frac{m}{a^3} \int \mathrm{d}^3 p \, f(\vec{x}, \vec{p}, t) \,, \tag{2.33}$$

the first one is

$$\vec{v}(\vec{x}, t) = \frac{1}{ma} \frac{\int \mathrm{d}^3 p \, \vec{p} f(\vec{x}, \vec{p}, t)}{\int \mathrm{d}^3 p \, f(\vec{x}, \vec{p}, t)} \,, \tag{2.34}$$

the second one is

$$\Pi^{ij} = \frac{\left\langle p^i p^j \right\rangle}{m^2 a^2} - v^i v^j \tag{2.35a}$$

$$= \frac{1}{m^2 a^2} [\dots] \,, \tag{2.35b}$$

which represents a velocity dispersion. In general we are not able to drop this term.

We integrate the Vlasov equation over $\vec{p}$ and manipulate.

Collecting all the terms we get

$$\boxed{\frac{\partial v^i}{\partial t} + H v^i + \frac{1}{a} \left(v_j \partial^j \right) v^i = -\frac{1}{a} \partial^i \phi - \frac{1}{a\rho} \partial_j \left(\rho \Pi^{ij} \right) \,,} \tag{2.36}$$

so the Euler equation changed: we have an additional velocity dispersion term.

What do we do then? Kolmogorov's approach is through a hierarchy of moments, and it works well in the turbolent regime.

We can have an ansatz for Π^{ij}, for example we can say that it is diagonal: this specifically does not really work here.

People say that f can be split in two: a *coarse-grain* distribution and a *fine-grain* distribution. The coarse distribution is an average, the fine grain captures the details.

A system behaves like a fluid if there is scattering between the particles.

How do we define the scale? This is difficult, since CDM does not have a cross-section. The "physical truth" is provided by N-body simulations, we make approximations until they match the simulations.

This is also why we do not use multiple-scale models: they are complicated and difficult to tune.

Friday 2020-4-24, compiled 2020-05-21

2.2 The Zel'dovich approximation

So far we have discussed perturbation in two ways: initially, we had a contribution on the RHS of the EFE given by the perturbations.

During radiation dominance there is not much need to go beyond linear theory.

Now we want to go full nonlinear.

We need to use a mixture of techniques: both the study of the 2-body problem and a continuous distribution of matter. This is what we do if we use numerical techniques.

On the other hand, we can make some smart approximations.

Last semester we mentioned the "spherical top-hat solution": we assume spherical symmetry, and work from there.

The Zel'dovich approximation gives an exact solution in the case of planar symmetry. We can also get exact solutions with cylindrical symmetry.

We come back to the fluid equations: continuity, Euler, Poisson on a FRLW background.

When you do linear perturbation theory, you perturb around the background value.

However, in doing this we are assuming that all the perturbations have the same weight: this is not necessarily true. We should define an internal hierarchy.

What we are doing is formal in the context of Lagrangian perturbation theory.

We define

$$\eta = \frac{\rho}{\rho_b} = 1 + \delta \tag{2.37a}$$

$$\vec{u} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}a} = \frac{\vec{v}}{a\dot{a}} \tag{2.37b}$$

$$\varphi = \frac{3t_*^2}{2a_*^3}\phi\,. \tag{2.37c}$$

This is because he wanted dependence on the growth factor. In EDS this is the scale factor, in general this could be a different thing.

We are referring to a more general background: there can be a growth suppression factor, such that the density perturbation δ does not scale with a but instead based on some function of a. We should use this function and not the scale factor in general. In the matter-filled universe we are considering right now this is the same.

The new equations are

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}a} + \frac{3}{2a}\vec{u} = -\frac{3}{2a}\nabla\varphi \tag{2.38a}$$

$$\frac{\mathrm{D}\eta}{\mathrm{D}a} + \eta\nabla\cdot\vec{u} = 0 \tag{2.38b}$$

$$\nabla^2\varphi = \frac{\delta}{a}\,, \tag{2.38c}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}a} = \frac{\partial}{\partial a} + \vec{u}\cdot\nabla \tag{2.39}$$

is the convective derivative.

We have chosen this variable $\vec{u}$ precisely because we want something which is almost constant in linear theory: we have $\delta \sim t^{2/3}$ and $v \sim t^{1/3}$ and $\phi \sim$ const.

So, we get

$$\frac{\partial \vec{u}}{\partial a} = 0 \,, \tag{2.40}$$

so then we can approximate:

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}a} = 0 \,. \tag{2.41}$$

At the linear level, $\vec{u} = -\nabla \varphi$. The argument is that we can *extend* the linear result beyond the linear level. This is because the velocity goes like k, while the potential goes like δ / k^2.

The system becomes more and more linear as we increase the scale.

Why don't we try to solve the equation by neglecting the nonlinearity in the velocity?

Then the equation system reads:

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}a} = 0 \tag{2.42a}$$

$$\frac{\mathrm{D}\eta}{\mathrm{D}a} + \eta \nabla \cdot \vec{u} = 0 \,. \tag{2.42b}$$

This can be solved exactly, as we will now see.

This is describing the inertial motion of particles with no external force acting on them.

It is important to say that this is a total derivative: the variation is zero along the trajectories of the particles.

The velocity of a particle which has a velocity $\vec{u}_0$ at a position $\vec{q}$ is preserevd. We can integrate the position straightforwardly, since the relation we get is linear.

This, however, is in our weird variables: the real motion is more complicated, but we can map the new variables to the new ones.

We can solve the continuity equation in different ways: we can divide by η to get logarithmic derivatives, so that

$$\frac{\mathrm{D} \log \eta}{\mathrm{D}a} = -\nabla \cdot \vec{u} \,. \tag{2.43}$$

when we integrate, though, we should follow the trajectory:

$$\eta(\vec{x}, a) = \eta_0(\vec{q}) \exp\left\{ -\int_{a_0}^{a} \mathrm{d}a' \, \nabla \cdot \vec{u}\big[\vec{x}(\vec{q}, a',), a'\big] \right\} . \tag{2.44}$$

This calculation is done in 'The large scale structure of the universe' by Peebles [Pee80].

Also, we can use an approach in which we use the Jacobian of the change of coordinates between Lagrangian and Eulerian.

The tensor

$$D_{ij} = \frac{\partial^2 \varphi_0}{\partial q_i \partial q^j} \tag{2.45}$$

is called the deformation tensor. The matrix

$$\frac{\partial x^i}{\partial q_j} = \delta^i_j - a \frac{\partial^2 \varphi_0}{\partial q_i \partial q^j} \tag{2.46}$$

defines the transformation between the old and new coordinates. The eigenvalues of the transformation tensor are local.

It can be shown (Doroshevick 1970) that there is a 92 % probability that at least one eigenvalue is positive.

This is computed using the assumption that φ_0 has a gaussian distribution.

This is important since we can write

$$\eta(\vec{x}, a) = \frac{1}{\prod_{i=1}^{3}(1 - \lambda_i(\vec{q})a)} \,. \tag{2.47}$$

Pancakes in terms of shapes, of course, not in terms of ingredients.

The jacobian of the change of coordinates is ill defined if two particles collide.

At that point, we get nonlinearities, which *are* structure formation. This is a sort of non-gaussianity we get from this approximation.

Thursday 2020-4-30, compiled 2020-05-21

Today we will improve on the Zel'dovich approximation with the adhesion approximation.

We see plots from a 1992 paper from the professor.

We are taking collisionless particles in 1D. The Zel'dovich approximation breaks down since the particles going through each other does not make sense: they will have gravitational interactions in that case.

We will discuss the difference, in structure formation, between "pancakes" and "filaments".

The issue seems to be that, after the trajectories of the particles cross the matrix of transformation between Eulerian and Lagrangian coordinates becomes singular.

How do we get extended structures? By continuity of the eigenvalues.

We cannot really estimate the global size of the structures here, they are transient anyway.

2.3 The adhesion approximation

We add a fictitious term for the gravitational "sticking" of particles into pancakes or filaments.

We start from the equations

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}\tau} = \nu \nabla^2 \vec{u} \tag{2.48}$$

$$\frac{\mathrm{D}\eta}{\mathrm{D}\tau} = -\eta \nabla \cdot \vec{u} \,. \tag{2.49}$$

We have basically added a kinematic viscosity term to the Navier-Stokes equations. The coefficient ν is called the kinematic viscosity coefficient, which has dimensions $[\nu] = L^2/T$.

This is completely artificial, but we do it to model the effect of gravitational interactions.

We are basically modelling gravity as a local effect.

What we obtain is called the 3D Burgers' equation. We can obtain turbulence here, but the system is solvable.

Is the system solvable because at small scales the vortices die out?

Let us assume that the velocity is irrotational: $\vec{u} = \nabla\Phi$. The quantity Φ is called a velocity potential.

Substituting in, we get

$$\frac{\partial\Phi}{\partial\tau} + \frac{1}{2}(\nabla\Phi)^2 = \nu\nabla^2\Phi\,, \tag{2.50}$$

which is called the Bernoulli equation.

To see that these are equivalent: if we take the gradient, we get

$$\frac{1}{2}\partial^i\left(\partial_j\Phi\partial^j\Phi\right) = \partial^j\Phi\partial_i\partial_j\Phi = u^j\partial_j u^i\,. \tag{2.51}$$

The other derivatives commute, so if we take the gradient of the equation we get back Euler's equation.

The issue is the fact that this is a nonlinear equation.

We do the Hopf-Cole transformation to solve the Burgers equation. So, we define

$$\Phi = -2\nu\log U\,, \tag{2.52}$$

where U is called the "expotential". This simplifies the equation into the Fokker-Planck equation:

$$\frac{\partial U}{\partial\tau} = \nu\nabla^2 U\,. \tag{2.53}$$

This is a linear diffusion equation. It models heat transfer, but it is also the Schrödinger equation after a Wick[1] rotation.

This is a parabolic equation. The Poisson equation, for instance, is elliptic: it only requires spatial initial conditions. Hyperbolic equations include the wave equation: they require initial conditions.

Parabolic equations require both: they imply infinite-speed transmission of signal.

We use the ansatz

$$U(\vec{x},\tau) = f(\tau)g(\vec{x})\,, \tag{2.54}$$

and we substitute

$$f(\tau) = e^{E\tau}\,, \tag{2.55}$$

so that we need to solve

$$Eg(\vec{x}) = \nu\nabla^2 g(\vec{x})\,. \tag{2.56}$$

[1] Pronounced "Vick", not "Uick".

In Fourier space, then,

$$E_k g_{\vec{k}} = -\nu k^2 g_{\vec{k}} \,. \tag{2.57}$$

So, a general solution will be a superposition:

$$U(\vec{x}, \tau) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-\nu k \tau^2} g_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \,. \tag{2.58}$$

The standard thing to do to get some boundary conditions is to look at a kernel of the function, its impulse response.

We get

$$K(\vec{x}, \tau | \hat{q}, 0) \overset{\tau \to 0}{\to} \delta(\vec{x} - \vec{q}) \,. \tag{2.59}$$

In order to get this, we need to set $g_{\vec{k}} = e^{-i\vec{k}\cdot\vec{q}}$. This yields

$$\begin{aligned} K(\vec{x}, \tau | \vec{q}, 0) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-\nu k^2 \tau} e^{i\vec{k}\cdot(\vec{x} - \vec{q})} &(2.60)\\ &= (4\pi\nu\tau)^{-3/2} \exp\left(-\frac{(\vec{x} - \vec{q})^2}{4\nu\tau} \right) . &(2.61) \end{aligned}$$

To get the initial conditions $U_0(\vec{q})$ we use the Chapman-Kolmogorov equation:

$$U(\vec{x}, \tau) = \int \mathrm{d}^3 q \, U_0(\vec{q}) K(\vec{x}, \tau | \vec{q}, 0) \,. \tag{2.62}$$

This is a consequence of Bayes's theorem: we can marginalize over a parameter like

$$\mathbb{P}(A) = \int \mathrm{d}B \, \mathbb{P}(B) \mathbb{P}(A|B) \,. \tag{2.63}$$

Inverting the definition of U, we have

$$U_0(\vec{q}) = e^{-\Phi_0(\vec{q})/2\nu} = e^{\phi_0/2\nu} \,, \tag{2.64}$$

and then we get

$$U(\vec{x}, \tau) = \int \frac{\mathrm{d}^3 q}{(4\pi\nu\tau)^{3/2}} e^{-S(\vec{x}, \vec{q}, \tau)/2\nu} \,. \tag{2.65}$$

The Bernoulli equation is the same as the Hamilton-Jacobi equation for the action.

Here,

$$S(\vec{x}, \vec{q}, \tau) = \frac{(\vec{x} - \vec{q})^2}{2\tau} - \phi_0(\vec{q}) \,, \tag{2.66}$$

and then we only need to recover the velocity: we need to differentiate and get

$$\vec{u}(\vec{x}, \tau) = -2\nu \frac{\nabla U}{U} = \frac{\int \mathrm{d}^3 q \, \frac{\vec{x} - \vec{q}}{\tau} e^{-S(\vec{x}, \vec{q}, \tau)/2\nu}}{\int \mathrm{d}^3 q \, e^{-S(\vec{x}, \vec{q}, \tau)/2nn}} \,. \tag{2.67}$$

We are interested in small values for ν. This parameter regulates the thickness of our pancakes, since it gives us a length scale for the derivative in the "Navier-Stokes" equation.

In this limit, the exponential is very peaked, and we can consider only the absolute minima of the action, just like in the path integral.

If we perform this approximation, we get

$$U(\vec{x}, \tau) = e^{-S(\vec{x}, \vec{q}, \tau)/2\nu} \sum_s j_s(\vec{x}, \vec{q}, \tau)\,, \tag{2.68}$$

where we expanded the action around the minimum to get

$$j_S = \left[\det \frac{\partial^2 S}{\partial q_i \partial q_j} \right]^{-1/2}. \tag{2.69}$$

Using the decaying mode is a mathematical trick.

Thursday 2020-5-7, compiled 2020-05-21

Understand this better.

As we saw last time, the length in ν can be interpreted as the thickness of the pancakes.

We have gotten an equation with no nonlinear terms: the Fokker-Planck equation,

$$\frac{\partial U}{\partial \tau} = \nu \nabla^2 U\,. \tag{2.70}$$

If we change τ to $i\tau$, this is the Schrödinger equation. We solve it by separation of variables.

In Fourier space, we get

$$E_k g_{\vec{k}} = -\nu k^2 g_{\vec{k}}\,. \tag{2.71}$$

This is a linear ODE, a general solution will be a superposition of eigenstates:

$$U(\vec{x}, \tau) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-\nu k^2 \tau} g_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\,, \tag{2.72}$$

and we just need to impose boundary conditions.

We look at the kernel: the "impulse response" of our function. We call it

$$K(\hat{x}, \tau | \hat{q}, 0)\,, \tag{2.73}$$

which is indeed a conditional probability amplitude: as $\tau \to 0$ it approaches $\delta(\hat{x} - \hat{q})$.

This is a probability, not a probability amplitude.

We can integrate to find it: it is

$$K(\hat{x}, \tau | \hat{q}, 0) = (4\pi\nu\tau)^{-3/2} \exp\left(-\frac{(\hat{x} - \hat{q})}{4\nu\tau} \right). \tag{2.74}$$

We find this by marginalizing.

This yields an equation which is basically the Hamilton-Jacobi one,

$$\frac{\partial S}{\partial \tau} + \frac{1}{2} (\nabla S)^2 = 0\,. \tag{2.75}$$

This yields an expression for the full (Eulerian) velocity. We are interested in the limit $\nu \to 0$. Only the minima of the action matter, and we can perform Gaussian integrals. We expand S around the absolute minima: we find

$$U(\hat{x}, \tau) = (4\pi\nu\tau)^{-3/2} e^{-S/2\nu} \int \mathrm{d}^3\delta q \exp\left(-\frac{1}{4\nu} \frac{\partial^2 S}{\partial q_i q^j \delta q^i \delta q_j^2} \right) \tag{2.76}$$

$$= \det \dots . \tag{2.77}$$

So, we are saying that the velocity is given by

$$\vec{u}(\vec{x}, \tau) = \sum_s \frac{\vec{x} - \vec{q}_s}{\tau} w_s(\vec{x}, \vec{q}, \tau) . \tag{2.78}$$

It is a superposition of different solutions, weighted by the w_s.

2.4 Schrödinger equation approach for LSS formation

Madelung proposed a hydrodynamical approach to QM: a decomposition

$$\psi(\vec{x}, t) = R(\vec{x}, t) e^{-\frac{S(\vec{x}, t)}{i\hbar}} , \tag{2.79}$$

which we can substitute into the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\frac{\nabla^2}{2m} \psi + V\psi , \tag{2.80}$$

which yields two separate equations for the real and imaginary parts: we get

$$-\nabla \left(\frac{R^2}{2m} \nabla S \right) = \frac{\partial}{\partial t} R^2 , \tag{2.81}$$

so if we define the usual probability current

$$j = \frac{\hbar}{2mi} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right) = \frac{R^2}{2m} \nabla S , \tag{2.82}$$

so we get the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{v} \right) = 0 \tag{2.83}$$

for the imaginary part of the equation.

For the real part of the equation, instead, we find

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 = -(V + Q) , \tag{2.84}$$

where

$$Q(\vec{x}, t) = -\frac{\hbar^2}{2m}\frac{1}{R}\nabla^2 R \tag{2.85}$$

a part we must include in the potential to account for quantum effects. This is the effectively the Bernoulli equation!

After the Madelung transformation, then, we get hydrodynamic equations. We cna also do this backward: we start from the Bernoulli equation and continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{v}\right) = 0 \tag{2.86}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\nabla \phi\right)^2 = -V\,, \tag{2.87}$$

and associate it with Schrödinger, using ν instead of $\hbar$. We then get Schrödinger, with a different potential:

$$i\nu \frac{\partial \Psi}{\partial t} = -\frac{\nu^2}{2}\nabla^2 \Psi + \left(V + \frac{\nu^2}{2}\frac{\nabla^2 R}{R}\right)\Psi\,. \tag{2.88}$$

We apply this procedure in cosmology: we get

$$\frac{\partial \eta}{\partial \tau} + \nabla \cdot \left(\eta \vec{u}\right) = 0 \tag{2.89}$$

$$\frac{\partial \Phi}{\partial \tau} + \frac{1}{2}(\nabla \Phi)^2 = -\frac{3}{2\tau}\left(\Phi + \varphi\right) \tag{2.90}$$

$$\nabla^2 \varphi = \frac{\delta}{\tau}\,. \tag{2.91}$$

Our variables are $\vec{u}$, η and φ, to be considered as a varies.

Friday 2020-5-8, compiled 2020-05-21

Notice that ν from now is not the viscosity!

We have the equations we wrote last time using

$$\eta = \frac{\rho}{\rho_b} = 1 + \delta\,, \tag{2.92}$$

and $\vec{u} = \mathrm{d}\vec{x}/\mathrm{d}\tau$, and $\tau = a(t)$.

So, we define the wavefunction:

$$\Psi = Re^{i\Phi/\nu} = (1+\delta)^{1/2} e^{i\Phi/\nu}\,. \tag{2.93}$$

With the Madelung procedure we get a Schrödinger equation with a correction to the potential:

$$i\nu \frac{\partial \Psi}{\partial t} = -\frac{\nu^2}{2}\nabla^2 \Psi + \left(V + \frac{\nu^2}{2}\frac{\nabla^2 R}{R}\right)\Psi\,. \tag{2.94}$$

Then, we take

$$V = \frac{3}{2\tau}\left(\Phi + \varphi\right)\,, \tag{2.95}$$

so that we get the Poisson equation in the form

$$\nabla^2\left(V + \frac{3i\nu}{4\tau}\log\left(\frac{\Psi}{\Psi^*}\right)\right) = -\frac{3}{2\tau^2}\left(|\Psi|^2 - 1\right). \tag{2.96}$$

Can we drop the quantum term in the Schrödinger equation? kind of, we can say that it is quadratic in ν... This is what is done in the literature, anyway.

Then, we get the quantum term in the regular Poisson equation:

$$\frac{\partial \Phi}{\partial \tau} + \frac{1}{2}(\nabla\Phi)^2 = -V - \frac{\nu^2}{2}\frac{\nabla^2 R}{R}. \tag{2.97}$$

Also, let us set the potential $V = 0$: then we will have the quantum mechanics of free particles.

So then, we get

$$i\nu\frac{\partial \Psi}{\partial \tau} + \frac{\nu^2}{2}\nabla^2\Psi = 0. \tag{2.98}$$

We solve this equation by performing a Wick rotation and using the results we obtained in the adhesion approximation.

The similarity is to the Fokker-Planck equation: the only difference is that we have to map $\tau \to i\tau$, that is, we perform a Wick rotation. So, we take everything we discussed for the solution to the Fokker-Planck equation and adapt it here.

So, we consider a Green's function as

$$\Psi(\vec{x}, \tau) = \int G(\vec{x}, \tau, \vec{q}, 0)\Psi_i(\vec{q})\, \mathrm{d}^3 q\ , \tag{2.99}$$

so we can use the Feynman formula:

$$G(\vec{x}, \tau, \vec{q}, 0) = (2\pi i\nu\tau)^{-3/2}\exp\left(\frac{i}{\nu}\frac{(\vec{x}-\vec{q})^2}{2\tau}\right), \tag{2.100}$$

and then from the Madelung transformation we get

$$\Psi(\vec{x}, \tau) = (2\pi i\nu\tau)^{-3/2}\int \mathrm{d}^3 q\left(1 + \delta_i(\vec{q})\right)^{1/2}\exp\left(\frac{i}{\nu}\left[\frac{(\vec{x}-\vec{q})^2}{2\tau} + \Phi_i(\vec{q})\right]\right). \tag{2.101}$$

Now, we apply the saddle-point approximation: this integral, kind of like Faynman's path-integral, is dominated by the regions in which the exponentials interfere constructively, which are defined by

$$\nabla_q\left[\frac{(\vec{x}-\vec{q})^2}{2\nu\tau} + \frac{\Phi_i(\vec{q})}{\nu}\right] = 0. \tag{2.102}$$

Now we are going to directly get both velocity and density. We assume that $\delta_i = 0$. Then we find

$$\vec{x} = \vec{q}_s - \tau\nabla_q\varphi_0(q_s). \tag{2.103}$$

As $\nu \to 0$, we find

$$\Psi(\vec{x}, \tau) \approx \eta^{1/2}(\vec{q}_s) e^{\frac{i}{\nu}\chi(\vec{q}_s)} \left[\det \left(\delta_{ij} - \tau \frac{\partial^2 \varphi_0}{\partial q_i \partial q_j} \right) \right]^{1/2}, \tag{2.104}$$

where

$$\chi = \frac{(\vec{x} - \vec{q})^2}{2\tau} - \varphi_0(\vec{q}) \,. \tag{2.105}$$

This looks similar to Zeldovich, but suppose that we had different stationary points: then, we get interference!

In the laminar regime we get

$$\eta = \Psi^* \Psi \sim \overset{-1}{\det} \left(\delta_{ij} - \tau \frac{\partial^2 \varphi_0}{\partial q_i \partial q_j} \right) . \tag{2.106}$$

Sandro Winberger is in Parma, he knows how to do this stuff.

2.5 Statistics of the Large-Scale Structure

We want to understand the statistics of the distribution in space of structures. We can compare the models of LSS formation in terms of statistical properties of the distribution of galaxies, not in terms of geography.

We only have one universe, and we see only a part of it.

The person who introduced the fair sample hypothesis is Birkhoff.

2.6 Statistical methods

Thursday 2020-5-14, compiled 2020-05-21

The **Fair Sample Hypothesis**: we see it from Peebles [Pee80].

In cosmology we do observations, not experiments: we cannot do spatial averages, only angular averages.

Now, the statement is that the statistical properties of the universe are spatially uncorrelated on large enough scales.

Now we move to the lectures by Martinez.

The probability density functional is unaffected by translations. We only can probe a specific realization of the field $\psi(\vec{x})$, but in theory we discuss the probability $p[\psi(\vec{x})]$. We hope that what we observe is large enough to be a statistical sample.

In the CMB we have *cosmic variance*, due to the fact that we are a single observer. We assume the ergodic theorem: ensemble averages and spatial averages are equivalent. The theorem holds for Gaussian random fields with a continuous power spectrum.

Correlation functions: we take the joint probability, given two volumes dV_1 and dV_2 at a distance r, of finding a galaxy in each. This will be

$$dP(r) = \overline{n}^2 \big(1 + \xi(r)\big) \, dV_1 \, dV_2 \,. \tag{2.107}$$

The correction $\xi(r)$ is precisely the two-point correlation function.

Now we move to the notes by U. Natale.

We look at conditional probability: note the typos in the notes there.

We can get the mean number of galaxies in a certain sphere. So, we get a fractal structure of dimension $3 - \gamma$.

We can consider *three-point* correlation functions, beyond two-point ones. We can express this with respect to two-point ones, plus a "connected part" ζ.

The connected part must go to zero when any of the lengths go to zero. By the definitions we gave, we can equate ξ with $\left\langle \delta_g(\vec{x}) \right\rangle \delta_g(\vec{x} + \vec{r})$.

We can go to Fourier space. Then, we define

$$\left\langle \delta_{k_1} \delta_{k_2} \right\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) P_m(k) \,. \tag{2.108}$$

We get an ultraviolet divergence, but it is not an issue: we are describing cosmological structures, so it is fine that we are not able to describe small structures.

Friday 2020-5-15, compiled 2020-05-21

Yesterday we defined correlation functions starting from probabilities.

We want to connect them to the density perturbations.

We define the mean density $\langle \rho \rangle = n$. To order two then we can define

$$\xi(r) = \left\langle \Big(\rho(x + r) - \langle \rho \rangle \Big) \Big(\rho - \langle \rho \rangle \Big) \right\rangle / n^2 \,, \tag{2.109}$$

which is defined so that

$$\left\langle \rho(x + r) \rho(x) \right\rangle = n^2 \big(1 + \xi(r) \big) \,. \tag{2.110}$$

Then we define

$$\delta P = n^3 \delta V_1 \delta V_2 \delta V_3 \big(1 + \xi(r_a) + \xi(r_b) + \xi(r_c) + \zeta(r_a, r_b, r_c) \big) \,. \tag{2.111}$$

We assume the galaxy distribution to be connected to the (dark) matter density by

$$\delta_g(\vec{x}) = b \delta(\vec{x}) \,, \tag{2.112}$$

which is ok to large enough scales, typically tens of Mpc. The function $\xi(s)$ has a peak around 100 Mpc, due to *baryon acoustic oscillations*.

The power spectrum is easier to work with experimentally, since its values are independent for different k, while the correlation function values at different radii are correlated.

2.7 Path integrals in cosmology

The approach here is very different from the one in field theory. We do not start from an action principle here.

Consider the space of square-integrable 3D functions. We consider them at fixed time, even though it does not really make sense since we can only observe our past light-cone.

We require the existence of a basis ϕ_n such that

$$\int \mathrm{d}^3 x\, \phi_m(x)\phi_n(x) = \delta_{mn}\,, \tag{2.113}$$

and we also ask that they are complete:

$$\sum_n \phi_n(x)\phi_m(y) = \delta^{(3)}(x-y)\,. \tag{2.114}$$

A functional F takes a function and returns a real or complex number. If we write it with respect to the basis components of a function, we can express F as a (real, or complex number valued) function of an infinite number of variables.

For instance, we could have

$$F_1[q] = \sum_n q_n f_n \tag{2.115}$$

$$F_2[q] = \sum_{mn} k_{mn} q_m q_n\,, \tag{2.116}$$

where q_n are the basis components of a function q.

We can discretize the space, so that the functional is given by its contribution to each small space box.

We can write a functional as a series in the form

$$F[q] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_i \mathrm{d}x_i\, q(x_i)\,, \tag{2.117}$$

for instance we can use the exponential. In that case the expression is

$$e^{\int f(x)q(x)\mathrm{d}x} = \sum_n \frac{1}{n!} \left[\int f(x)q(x)\,\mathrm{d}x\right]^n, \tag{2.118}$$

which can be extended replacing $\int f(x)q(x)\,\mathrm{d}x$ with anything else.

Thursday 2020-5-21, compiled 2020-05-21

We want to define the notion of a **functional derivative**.

The first order in η the difference of the functional applied to q and $q+\eta$ is defined to be

$$F[q+\eta] - F[q] = \int \frac{\delta F}{\delta q(y)} \eta(y)\,\mathrm{d}y\ . \tag{2.119}$$

We can interpret this as

$$\frac{\delta F}{\delta q(y)} = \lim_{\substack{\tau\to 0\\ q_i \to y}} \frac{1}{\tau} \frac{\partial \hat{F}}{\partial q_i}\,, \tag{2.120}$$

where we compute the functional using infinitesimal "cells" q_i of volume τ in coordinate space.

This behaves like a derivative, and we have

$$\frac{\delta q(x)}{\delta q(y)} = \delta(x-y)\,. \tag{2.121}$$

An example: if $q(x)$ is a 1d function, define

$$F_n[q] = \int \prod_i \mathrm{d}x_i \, q(x_i) f(x_1, \dots, x_n) \,, \tag{2.122}$$

where f is a symmetric function of the coordinates. Then,

$$\frac{\delta F_n}{\delta q(y)} = n \int \cdots \int f(x_1, \dots, x_{n-1}, y) q(x_1) \dots q(x_{n-1}) \, \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \,. \tag{2.123}$$

We can extend this to

$$\frac{\delta^m F_n}{\delta q(y_1) \dots^m} q(y_m) = \frac{n!}{(n-m)!} \int \cdots \int f(x_1, \dots, x_{n-m}, y_1, \dots y_m) \, \mathrm{d}x_1 \dots \mathrm{d}x_{n-m} \,. \tag{2.124}$$

Now, **linear functional transformations** Consider a mapping $q(x) \to q'(x)$ in the form

$$q(x) = \int \mathrm{d}y \, K(x,y) q'(y) \,. \tag{2.125}$$

If this is invertible, we can write

$$q'(x) = \int \mathrm{d}y \, K^{-1}(x,y) q(y) \,. \tag{2.126}$$

Often in field theory $K(x,y)$ can be written as $K(x-y)$ because of translation invariance. If these are compatible, then we must have

$$\int \mathrm{d}y \, K(x,y) K^{-1}(y,z) = \delta(x-z) \,. \tag{2.127}$$

We can do **Legendre Transforms** in this context as well: if we define

$$p(y) = \frac{\delta F[q]}{\delta q(y)} \,, \tag{2.128}$$

then we can write

$$G[p] = F[q] - \int q(x) p(x) \, \mathrm{d}x \,. \tag{2.129}$$

This has a different sign than what we do classically, $H = p \cdot \dot{q} - L$. This has to do with the signature of the metric in QFT, and Wick rotations. In dealing with probability densities we must be general.

We can also do **functional integration**: it is formally an integration over an infinite number of variables.

How do we do changes of variable? We will need the determinant of the matrix K giving the change of variables. We only consider *linear* ones.

We find that, for a path integral with a quadratic and a linear term,

$$\mathscr{Z}[b] = \int \mathcal{D}q \exp\left(-\frac{1}{2} (B, K^{-1}, b) \right) . \tag{2.130}$$

2.8 Spherical top-hat model

We start off with the discussion of curved models, which is the same as in the FAC course.

We consider an Einstein-de Sitter universe, and a sphere with $\Omega_p(t_i) = 1 + \delta_i$.

The density profile is considered to be in the shape of a top-hat, which is flat in the center and at the boundaries, shaped like a plateau. We want to have zero peculiar velocity at the beginning; pure Hubble flow. Initially we have $\delta = \delta_+ + \delta_-$, and we must also have $\frac{2}{3}\delta_+ = \delta_-$.

We find

$$\rho_P(t_m) = \rho_c(t_i)\Omega_P(t_i)\left[\frac{\Omega_P(t_i) - 1}{\Omega_P(t_i)}\right]^3 , \tag{2.131}$$

where t_m is the turnaround time.

With these hypotheses, we find that the ratio of the inner to the outer density is around 5.6:

$$\frac{\rho_P}{\rho_C} = \left(\frac{3\pi}{4}\right)^2 . \tag{2.132}$$

If we used linear theory instead of the correct theory we would have found 2.07 for the ratio: the result is very different.

The total energy of our top-hat is given by

$$E_{\text{eq}} = -\frac{1}{2}\frac{3}{5}\frac{GM^2}{R_{\text{eq}}} . \tag{2.133}$$

At turn-around $T = 0$, so at that point

$$E_m = -\frac{3}{5}\frac{GM^2}{R_m} , \tag{2.134}$$

and as long as the energy is conserved, we find that $R_m = 2R_{\text{eq}}$. This means that the volume at turnaround is 8 times that of equilibrium.

Therefore, we can find what the ratio of the perturbation density to the background one is of 180 at collapse. If we still were to use linear theory at this time we would get 1.686.

Now we introduce the *mass function of cosmic structure*, which is defined as

$$n(M) = \frac{\mathrm{d}N}{\mathrm{d}M} , \tag{2.135}$$

the number of objects per unit volume with mass between M and $M + \mathrm{d}M$.

Then we take a filter, we consider the probability that at a certain point the perturbation's magnitude is larger than 1.686, which is equivalent to saying that it has virialized.

Then, the ansatz is

$$n(M)M\,\mathrm{d}M = \rho_m\big(\mathbb{P}_{>\delta_c} - \mathbb{P}_{>\delta_c}(M + \mathrm{d}M)\big) = \rho_m\left|\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}M}\right|\mathrm{d}M , \tag{2.136}$$

but the probability depends on the mass only through the variance of the distribution of density: then we write

$$n(M)M\,\mathrm{d}M = \rho_m \left|\frac{\mathrm{d}\mathbb{P}_{>\delta_c}}{\mathrm{d}\sigma_M}\right| \left|\frac{\mathrm{d}\sigma_M}{\mathrm{d}M}\right| \mathrm{d}M\ , \tag{2.137}$$

and finally we get

$$n(M) = \frac{2}{\sqrt{\pi}}\frac{\rho_m}{M_*^2}\alpha\left(\frac{M}{M_*}\right)^{\alpha-2}\exp\left(-\left(\frac{M}{M_*}\right)^{2\alpha}\right)\ , \tag{2.138}$$

which is equivalent to the Schecter luminosity function, where $\alpha = 1/2$ — that is, with white noise.

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