Compact Object Astrophysics

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Introduction

Tuesdays and Wednesday at 14.30 PM in room P1A, Paolotti building. 22 people.

This course overlaps with "Computational Astrophysics" by professor Mapelli.

The examination is an oral one, done either online or live.

We start with a brief overview of the final fates of massive stars. We have white dwarfs, neutron stars and black holes under the category of "compact objects", but white dwarfs are not really that compact.

We then discuss accretion onto compact objects, and neutron stars. An open question: what is the EOS of ultradense neutron matter?

"Accretion power in astrophysics", "The physics of Compact Objects", "Astrofisica Relativistica I & II", "Astrofisica delle Alte Energie".

Chapter 1

Compact objects

1.1 A journey into the life of a massive star

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Stars whose mass is $M \gtrsim 8M_\odot$ go supernova at the end of their life. During their lifetime, hydrogen fuses through two channels: the p-p chain and the CNO cycle.

In the CNO cycle, four protons turn into a ^{4}He nuclide, two positrons, two electron neutrinos using heavier nuclides as catalysts.

The critical temperature above which the CNO cycle dominates is around $T_c \sim 2 \times 10^7$ K. For the Sun, less than 8 % of energy production is through the CNO cycle.

When the temperature of the core reaches a value around $1 \div 2 \times 10^8$ K and the density is around $10^8 \div 10^9\,\mathrm{g/cm^3}$, helium starts to burn in the 3α process, becoming ^{12}C. The Q-value here is around 7.27 MeV.

As soon as we have carbon, this can fuse with an α particle giving rise to a nucleus of oxygen with $Q \approx 7.16\,\mathrm{MeV}$. This oxygen can further catch an α particle, making a ^{20}Ne nuclide.

Then, we have a temperature around 5×10^8 K and a density around $3 \times 10^6\,\mathrm{g/cm^3}$. Carbon starts to fuse with itself, making sodium, magnesium, and more neon (plus an α particle).

If you want carbon burning to proceed in a steady way, it must occur in a nondegenerate electron gas. This occurs only if the star is quite massive, more than $8M_\odot$. Otherwise, it is an explosive process.

Now the core temperature reaches 10^9 K. The energy of a typical photon is quite high, $h\nu \sim k_B T \sim 100\,\mathrm{keV}$. Suppose there are neon nuclei in the core (this will be the case since they are a product of fusion).

It is not hard for a Neon to lose an α particle through photodissociation, this produces an Oxygen. The energy required for this is of the order 4.7 MeV, at the high energy tail we have a few photons at this energy.

This is the "neon burning phase", after which we have oxygen and magnesium. Oxygen is the next candidate for nuclear burning, and after a further contraction the star starts burning it. It fuses with itself to produce ^{28}Si plus an α, or ^{32}S.

Sulfur cannot fuse with itself, the potential barrier is too high. Through successive α captures, the star synthesizes elements in the "iron peak": iron, nickel, cobalt.

The core tries to contract in the attempt to get them to burn, but they have the maximum possible binding energy per nucleon. So, the contraction continues.

If the mass of the contracting core exceeds the Chandrasekhar limit, it cannot become an electron-degenerate object. The mass of the core is always in excess of this limit mass for the stars which are massive enough to reach this stage of stellar burning.

Iron is photodissociated to make helium nuclei first, then bare protons, electrons and neutrons. Protons and electrons can combine into neutrons. The core becomes more and more neutrons rich, but the reaction also produces neutrinos, which can fly away.

The *neutron* degeneracy pressure can stop the collapse in certain cases: this is how a neutron star is formed. The threshold between neutron stars and black holes is hard to determine, but generally speaking with $8M_\odot < M < 25M_\odot$ a neutron star is formed, while for larger masses the core collapses further to form a black hole.

The freefall velocity is a significant fraction of the speed of light. What are the statistics? how many NS and BH are there in our galaxy?

We can model the distribution of star masses in our galaxy with the distribution, the IMF, as a Salpeter IMF,[1] which is given by

$$N(m) \propto m^{-\alpha}\,, \tag{1.1.1}$$

where $\alpha \approx 2.35$. Then, we can calculate the number of stars which have more than $8M_\odot$ by integrating: we find something proportional to $8^{-1.35}$, while the number of stars which have more than $25M_\odot$ we get something proportional $25^{-1.35}$. These will give us the amount of compact objects. The proportionality constant depend on the minimum and maximum mass of stars, but we can calculate the ratio of the two without concern for it. We find

$$\frac{N_{BH}}{N_{NS} + N_{BH}} = \left(\frac{8}{25}\right)^{-1.35} \approx 0.2\,. \tag{1.1.2}$$

The present rate of supernova explosions in the galaxy is around 1 per century, or $10^{-2}\,\text{yr}^{-1}$. In the age of the galaxy (around $10^{10}\,\text{yr}$), we will then have had around 10^8 compact objects.

How much can we trust this figure? Kind of, the true number is closer to 10^9, about 1 % of the number of stars in the galaxy.

The galaxy roughly looks like a cylinder with radius $R \sim 60\,\text{kpc}$ and height $H \sim 1\,\text{kpc}$. Its volume will then be $V = 2\pi R^2 H \approx 10^{13}\,\text{pc}^3$. Then, the number density of compact objects is around $0.1\,\text{pc}^{-3}$.

The typical separation between them will be something like 20 pc. Compact objects are close, common! Beware!

The closest compact object we know of is a neutron star 60 pc away: this is on the same order of magnitude.

Compactness The gravitational radius characterizing an object is

$$R_g = \frac{GM}{c^2}\,, \tag{1.1.3}$$

[1] See the evil organization in Mission Impossible.

while the Schwarzschild radius is $2R_g$. For the Sun, this is approximately 1.5 km. It's small.

The value R_g/R is 0.5 for black holes, 0.15 for neutron stars, 10^{-4} for white dwarfs.

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As we said earlier, massive stars go type-2 supernova: this corresponds to Core-Collapse.

Compact objects are quite common in the galaxy.

A compact object is one for which the ratio of the gravitational radius $R_g = GM/c^2$ is comparable to the radius of the true object. For a white dwarf, the ratio is of the order of 10^3.

We then need GR in order to deal with them. Let us quickly go over exact solutions of the Einstein Field Equations.

This lecture, we consider the vacuum Schwarzschild solution. The most general line element which is spherically symmetric (invariant under spatial rotations) must be made up of elements which are themselves invariant under spatial rotations. We will use spherical coordinates: r, θ, φ, t.

In flat spacetime, the line element reads

$$ds^2 = -c^2\, dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right) , \tag{1.1.4}$$

and our Schwarzschild solution will need to reduce to this in some limit.

The spatial line element is given by

$$d\vec{r} \cdot d\vec{r} = dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right) = g_{ij}\, dx^i\, dx^j \ . \tag{1.1.5}$$

Then, the most general spherically symmetric line element will read

$$ds^2 = F(r,t)\, dt^2 + M(r,t)\, dr^2 + G(r,t)\, dr\, dt + C(r,t) r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right) , \tag{1.1.6}$$

however, we can redefine the radial coordinate in order to remove the function multiplying the angular term, so we get

$$ds^2 = F\, dt^2 + M\, dr^2 + G\, dr\, dt + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right) . \tag{1.1.7}$$

We can also introduce a new time variable:

$$dt' = dt + \psi(r,t)\, dr \ . \tag{1.1.8}$$

If $\psi = rG/F$, then we remove the mixed term, and then we are left with the expression

$$ds^2 = -B(r,t)\, dt^2 + A\, dr^2 + r^2\, d\Omega^2 \ . \tag{1.1.9}$$

However, we have not yet determined the two functions, and we have not said anything about the Einstein Field Equations, which are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \,. \tag{1.1.10}$$

In vacuo, the stress-energy tensor vanishes. The curvature scalar must vanish (we can show this by contracting the EFE with the inverse metric), so the equations reduce to $R_{\mu\nu} = 0$. We restrict ourselves to the static case.

The linearly independent components of the Ricci tensor read

$$R^0_0 = \frac{B''}{2AB} - \frac{A'B'}{4A''B} - \frac{B'^2}{4AB^2} + \frac{B'}{rAB} = 0 \tag{1.1.11}$$
$$R^1_1 = \frac{B''}{2AB} - \frac{A'B'}{4A''B} - \frac{B'^2}{4AB^2} + \frac{A'}{rA^2} = 0 \tag{1.1.12}$$
$$R^2_2 = \frac{1}{4rA}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{r^2}\left(\frac{1}{A} - 1\right). \tag{1.1.13}$$

Computing $R^0_0 - R^1_1 = 0$ we find

$$\frac{1}{rA}\left(\frac{B'}{B} + \frac{A'}{A}\right) = 0 \tag{1.1.14}$$
$$\frac{\mathrm{d}\log(AB)}{\mathrm{d}r} = 0\,, \tag{1.1.15}$$

so AB is constant. Without losing generality we can take $A = 1/B$, since if this is not the case we can just rescale the radial or temporal coordinate until it is.

Then, we can compute

$$R^2_2 = \frac{B}{2r}\left(\frac{B'}{B} + \frac{B'}{B}\right) + \frac{1}{r^2}(B - 1) = 0 \tag{1.1.16}$$
$$B' + \frac{B}{r} - \frac{1}{r} = 0 \tag{1.1.17}$$
$$\frac{\mathrm{d}}{\mathrm{d}r}(rB) = 1\,, \tag{1.1.18}$$

so $rB(r) = r + C$, or equivalently

$$B(r) = \frac{C}{r} + 1\,. \tag{1.1.19}$$

After this, we can already substitute into the metric:

$$\mathrm{d}s^2 = -\left(1 + \frac{C}{r}\right)\mathrm{d}t^2 + \frac{1}{1 + C/r}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{1.1.20}$$

For any value of C, $B \to 1$ as $r \to \infty$: the metric reduces to the flat one asymptotically. Right now C is an arbitrary constant, however in the weak field limit it is known that

$$g_{00} = -\left(1 + 2\frac{\phi}{c^2}\right), \tag{1.1.21}$$

where $\phi = -GM/r$ is the Newtonian gravitational field. Equating this expression to the one for g_{00}, we find

$$g_{00} = -\left(1 - \frac{2GM}{rc^2}\right) \implies C = -\frac{2GM}{c^2}\,. \tag{1.1.22}$$

The constant M in the classical case is the mass of the source, however we are computing a vacuum solution. This is the mass we would compute if we were to measure the orbits of objects around the compact object.

This is then surely a *gravitational* mass, is it also an *inertial* mass? Can we show this in GR?

Then, we can write the Schwarzschild metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right) \mathrm{d}r^2 + r^2 \,\mathrm{d}\Omega^2 \,. \tag{1.1.23}$$

This is derived by assuming time-independence, however the result is the same even in the time-dependent case by the Jebsen-Birkhoff theorem (which we will prove in a moment). The element g_{rr} diverges as $r \to R_g = 2GM/c^2$, however this does not represent any physical divergence: the scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \propto r^{-6}$ does not diverge there.

There are coordinates which do not diverge near the horizon: one classical choice employs the "tortoise" coordinates, which are the same for r, θ, φ as the Schwarzschild ones, while the time becomes (setting $G = c = 1$)

$$t = t' - 2M \log\left(1 - \frac{r'}{2M}\right) . \tag{1.1.24}$$

Substituting this into the metric yields (dropping the primes for clarity):

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \frac{4M}{r} \mathrm{d}r \,\mathrm{d}t + \left(1 + \frac{2M}{r}\right) \mathrm{d}r^2 + r^2 \,\mathrm{d}\Omega^2 \,. \tag{1.1.25}$$

There is no pathology at $r = 2M$ anymore, so it was not a physical divergence. The temporal coefficient g_{00} is the same: it can be shown that it is an invariant under coordinate transformations.

If we take two points which are very close along a particle trajectory, they must be separated by an interval $\mathrm{d}s^2 < 0$.

If we consider a (nongeodesic!) radial path described by $r(t)$, we can compute the corresponding line element by neglecting the angular part:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \frac{4M}{r} \mathrm{d}r \,\mathrm{d}t + \left(1 + \frac{2M}{r}\right) \mathrm{d}r^2 \tag{1.1.26}$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = -\left(1 - \frac{2M}{r}\right) + \frac{4M}{r}\frac{\mathrm{d}r}{\mathrm{d}t} + \left(1 + \frac{2M}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 . \tag{1.1.27}$$

Now, the question we ask is: is it possible for the particle trajectory to be timelike or lightlike ($\mathrm{d}s^2 \leq 0$) and outgoing ($\mathrm{d}r/\mathrm{d}t > 0$) under these conditions? If this is the case, the signs of the three terms read

$$\underbrace{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}_{<0?} = -\left(1 - \frac{2M}{r}\right) \underbrace{+\frac{4M}{r}\frac{\mathrm{d}r}{\mathrm{d}t}}_{>0} \underbrace{+\left(1 + \frac{2M}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2}_{>0} , \tag{1.1.28}$$

so we can see that the equality can be satisfied (a positive number cannot equal a negative one!) as long as the first term on the right-hand side is negative, which means $r > 2M$. If $r \leq 2M$, on the other hand, this cannot be the case: a radial trajectory below the horizon *cannot* be outward.

This is what "horizon" means: it is a *semi-permeable* membrane, particles can surpass it only in one direction.

Jebsen-Birkhoff This theorem states that the Schwarzschild solution also describes the spacetime around an object in the spherically-symmetric but time-*dependent* case. Let us give a sketch of its proof, omitting some tedious calculations. If we write out the components of the Ricci tensor, we find something in the form

$$R^0_0 = R^0_0\Big|_{\text{static}} + \dot{A}(\dots) \tag{1.1.29}$$

$$R^1_1 = R^1_1\Big|_{\text{static}} + \dot{A}(\dots)\,, \tag{1.1.30}$$

while R^2_2 and R^3_3 are the same. Also, the term R^1_0 does not vanish unlike the static case, and is equal to

$$R^1_0 = -\frac{\dot{A}}{rA^2} = 0\,, \tag{1.1.31}$$

so $\dot{A} = 0$: the equations then are the same as the static case ones! This, however, is not the end, since now the equation

$$\frac{A'}{A} + \frac{B'}{B} = 0 \tag{1.1.32}$$

is not necessarily solved by $\log A = -\log B$, since a prime denotes a *partial* derivative with respect to r, so in general we will have $\log A + \log B = f(t)$, some generic function of time. The metric will then read

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) f(t)\,\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \tag{1.1.33}$$

but we can simply rescale the time coordinate to $t \to \sqrt{f}t$ in order for this to reduce to the usual expression. This theorem was originally discovered by the Norwegian physicist Jebsen, and only later popularized in a textbook by Birkoff [JR05].

The source of the geometry can change in time while leaving the outside spacetime unperturbed, however this holds only as long as the variation remains spherically symmetric: collapse, expansion or pulsation. Any asymmetry can lead to the emission of gravitational radiation.

"Schwarzschild" means "Black Shield": no relation though, it is the name of a German scientist. We have discussed the properties of this metric.

Now, let us consider **geodesic motion** in the vacuum Schwarzschild spacetime.

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We cannot choose an arbitrary radius for an orbit around a Black Hole: there is an Innermost Stable Circular Orbit, and this has a direct impact on the accretion efficiency.

We will use Latin letters for spacetime indices. The four-velocity, as usual, is $u^i = \mathrm{d}x^i \big/ \mathrm{d}\tau$, where $\mathrm{d}\tau$ is the proper time. Also, for massive particles we also have the four-momentum $p^i = mu^i$.

Analogously to classical mechanics, the Lagrangian of a free particle is

$$L = -\sqrt{-g_{ij}p^i p^j} = -m\sqrt{-g_{ij}u^i u^j}\,, \tag{1.1.34}$$

and the corresponding Euler-Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial u^i}\right) - \frac{\partial L}{\partial x^i} = 0\,, \tag{1.1.35}$$

however we will not study these in the general case, it is too complicated. It can be shown that along a geodesic the Lagrangian itself is conserved: $L = \text{const}$. Therefore, $g_{ij}u^i u^j$ is also a constant, a negative one since the velocity is timelike. We can then set it to -1 by reparametrizing, so we will have $u^2 = -1$ and $p^2 = -m^2$.

Suppose that the there is a certain coordinate x^k such that

$$\frac{\partial g_{ij}}{\partial x^k} = 0\,, \tag{1.1.36}$$

that is, the metric does not change along the x^k direction. Therefore,

$$\frac{\partial L}{\partial x^k} = 0\,, \tag{1.1.37}$$

which tells us that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial u^k}\right) = 0\,, \tag{1.1.38}$$

meaning that this cyclic variable gives us a conserved quantity. If we do the calculation, the constant comes out to be

$$m g_{kj} u^j = p_k = \text{const}\,. \tag{1.1.39}$$

This is the case in Schwarzschild spacetime: the metric only depends on r and θ, while t and φ are cyclic. Therefore, we will have two constants of motion: in terms of the momentum, p_t and p_φ.

We can denote them as $E = -p_t$ and $L = p_\varphi$. Let us start from L:

$$L = g_{\varphi k}p^k = g_{\varphi\varphi} m \frac{\mathrm{d}x^\varphi}{\mathrm{d}\tau} \tag{1.1.40}$$

$$= r^2 \sin^2\theta\, m \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \tag{1.1.41}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{L}{mr^2 \sin^2\theta}\,. \tag{1.1.42}$$

Now, geodesic motion is in general planar since the metric is spherically symmetric: we do not lose any generality by setting $\theta = \pi/2$, so we can simplify $\sin^2\theta = 1$.

In general, even in Netwonian mechanics, when discussing orbits we can decompose the velocity into the radial and perpendicular direction. The angular momentum, in classical Newtonian mechanics, has a modulus $\left|\vec{L}\right| = |\vec{r} \wedge m\vec{v}| = mrv_\varphi = mr^2\dot{\varphi}$. This is the same relation we have found here.

The energy of an object with four-momentum p_k as measured by an observer with four-velocity u^k is given by $E = -p_k u^k$. We choose an observer which is static with respect to the coordinates:[2] then, we have $\mathrm{d}t/\mathrm{d}\tau = \gamma/\sqrt{-g_{tt}}$, so

$$E = -g_{tt} m \frac{\mathrm{d}x^t}{\mathrm{d}\tau} = +\left(1 - \frac{2M}{r}\right)\frac{m}{\sqrt{1-v^2}}\frac{1}{\sqrt{-g_t t}} \tag{1.1.43}$$

$$= \sqrt{1 - \frac{2M}{r}} m\gamma \tag{1.1.44}$$

$$\approx m\left(1 - \frac{M}{r} + \frac{v^2}{2}\right) \tag{1.1.45}$$

so in the Newtonian limit ($M \ll r$, $v \ll 1$) this is regular expression for the conservation of energy, gravitational plus kinetic.

Now we have the tools to study geodesic motion: let us write down $p^2 = -m^2$ explicitly,

$$g_{tt}\left(p^t\right)^2 + g_{rr}\left(p^r\right)^2 + \underbrace{g_{\theta\theta}\left(p^\theta\right)^2}_{=0 \text{ if } \theta \equiv \pi/2} + g_{\varphi\varphi}\left(p^\varphi\right)^2 = -m^2\,, \tag{1.1.46}$$

and if we substitute we must be careful: the constants of motion are related to the *covariant* components of the quantities, so we have

$$g_{tt}\left(g^{tt}p_t\right)^2 + g_{rr}\left(g^{rr}p_r\right)^2 + g_{\varphi\varphi}\left(g^{\varphi\varphi}p_\varphi\right)^2 = -m^2 \tag{1.1.47}$$

$$g_{tt}\left(g^{tt}\right)^2 E^2 + g_{rr}\left(p^r\right)^2 + g_{\varphi\varphi}\left(g^{\varphi\varphi}\right)^2 L^2 = -m^2 \tag{1.1.48}$$

$$g^{tt}E^2 + g^{rr}\left(p_r\right)^2 + g^{\varphi\varphi}L^2 = -m^2 \tag{1.1.49}$$

$$-\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(p^r\right)^2 + \frac{L^2}{r^2} = -m^2 \tag{1.1.50}$$

$$-\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1} m^2\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{L^2}{r^2} = -m^2\,, \tag{1.1.51}$$

which is an ODE for the single function $r(\tau)$: we can integrate this to find the shape of the trajectory. We now divide everything by m^2, and define the specific energy and angular momentum $\epsilon = E/m$ and $\ell = L/m$:

$$\left(1 - \frac{2M}{r}\right)^{-1}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \epsilon^2\right] + \frac{\ell^2}{r^2} = -1\,. \tag{1.1.52}$$

[2] Its four velocity will be given by $k^i = (1/\sqrt{-g_{tt}}, \vec{0})$ by normalization.

We can then express this as

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -\left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)+\epsilon^2\,, \tag{1.1.53}$$

which can give us $r(\tau)$ or $r(\varphi)$ in a rather simple way numerically, however the answer is not analytic: it is given by an elliptic integral.

We can study it analytically by looking at the turning points, those with $\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0$. These will tell us about the shape of the orbit. They obey the equation

$$0 = -\left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)+\epsilon^2\,, \tag{1.1.54}$$

which means

$$\frac{\ell^2}{r^2} = \left(\epsilon^2-1+\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-1}\,, \tag{1.1.55}$$

which has the solutions

$$l_\pm = \pm r\left[\left(\epsilon^2-1+\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)\right]^{1/2}\,. \tag{1.1.56}$$

Let us define $\lambda = \ell/2M$ and $x = r/2M$. These are dimensionless in our units. Also, we define $\Gamma = \epsilon^2 - 1$. They satisfy

$$\lambda_\pm = \pm x\sqrt{\left(\Gamma+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)^{-1}} = \pm x\sqrt{\frac{x\Gamma+1}{x-1}}\,. \tag{1.1.57}$$

We want to draw these as curves in the x, λ plane. We fix a value of ℓ, which means we have chosen λ. This depends on the initial conditions of the motion of the particle. If this λ is a λ_+, then x can move in all the region $x(\lambda_+) < x < \infty$.

This is the relativistic analog of the hyperbolic trajectories in the Keplerian case.

Let us try to compute $\frac{\mathrm{d}\lambda_\pm}{\mathrm{d}x}$: this yields

$$2x^2\Gamma + x - 3x\Gamma - 2 = 0 \tag{1.1.58}$$

$$\Gamma\left(2x^2-3x\right) + x - 2 = 0 \tag{1.1.59}$$

$$\Gamma = \frac{2-x}{2x^2-3x}\,. \tag{1.1.60}$$

This means we have a single curve, which will relate Γ and x. The physical cases are only the ones with $\Gamma > -1$. The minimum is at $x = 3$, corresponding to $r = 6M$, where we have $\Gamma = -1/9$. This corresponds to $\epsilon^2 - 1$: so, $\epsilon = \sqrt{2}\times 2/3$.

For $-1/9 < \Gamma < 0$ we have two choices for x, for $\Gamma > 0$ we have only one.

Coming back to the $\lambda_\pm$ curves, we can have $\lambda_\pm$ as two distinct curves, and there is a region around $\lambda = 0$ for which the particle will cross the horizon.

This allows us to compute the cross-section for gravitational capture, which is nonvanishing *even if* we neglect the size of the black hole.

Insert plots

Tuesday 2020-10-13, compiled 2020-11-04

We were discussing geodesics in the external Schwarzschild solution. We wrote down the locus of the inversion point $\dot{r} = 0$: this is

$$\lambda_{\pm} = \pm x \sqrt{\frac{\Gamma x + 1}{x - 1}} . \tag{1.1.61}$$

1. For $\Gamma > 0$ there is one extremum;
2. for $-1/9 < \Gamma < 0$ there are two extrema;
3. for $-1 < \Gamma < -1/9$ there are no extrema.

Let us consider the two-extrema case. Then, there is an intersection with the x axis at $x = -1/\Gamma$ for the $\lambda_{\pm}$. Then the curve looks like a doorknob: it is a closed curve.

In this case, we are considering *bound states*: the particle can start on the LHS of the curve, and if this is the case it must reach the inversion point and then go back and eventually cross the horizon. If it is in the "doorknob region", then it is trapped there.

These orbits are similar to the Newtonian elliptic ones, but there are differences. These are radially bound, however they are not in general periodic: they will precess.

We can also have circular orbits, both stable and unstable.

In the no-extrema case there are no orbits. The limiting case $\Gamma = -1/9$ is interesting: the minimum becomes an inflection point, with zero second derivative. This happens at $x = 3$. This is the ISCO: the Innermost Stable Circular Orbit.

The corresponding energy is given by $\epsilon^2 - 1 = -1/9$: $\epsilon = 2\sqrt{2}/3$, so

$$E_{\text{ISCO}} = \frac{2\sqrt{2}}{3} mc^2 . \tag{1.1.62}$$

If we can bring a particle from infinity to the ISCO (which is what happens as an accretion disk is formed) it can release an energy equal to $mc^2 - E_{\text{ISCO}} = (1 - 2\sqrt{2}/3)mc^2$. This is why an accretion disk can produce a large amount of energy.

The efficiency can then be computed as

$$\text{efficiency} = \frac{mc^2 - 2\sqrt{2}mc^2/3}{mc^2} \approx 6\,\% . \tag{1.1.63}$$

This is a *very large* efficiency: an order of magnitude more than the efficiency of nuclear burning.

1.2 Schwarzschild internal solution

Now we try to solve the EFE under spherical symmetry *with* a source:

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 8\pi T_{ij} , \tag{1.2.1}$$

with a metric

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,d\Omega^2 \,. \tag{1.2.2}$$

Birkhoff does not apply here: we are *assuming* stationarity. We will use a very simple T_{ij}: a perfect fluid, where

$$T_{ij} = (\rho + P)u_i u_j + P g_{ij} \,, \tag{1.2.3}$$

where the energy density is given by $\rho = \rho_0(1+\epsilon)$: ρ_0 is the rest energy density of the fluid, while ϵ accounts for thermal motion and other kinds of internal energy.

We will assume that the fluid is at rest with respect to the interior: we want to find an analog of the hydrostatic equilibrium equation. So, $dr = d\theta = d\varphi = 0$, while only $dt \neq 0$. This means that only $u^0 \neq 0$, and normalization requires $g_{ij}u^i u^j = -1$: this means that $u^0 = 1/\sqrt{B}$, and the whole vector reads $u^i = \delta^i_0/\sqrt{-g_{00}}$.

Let us then write the nonzero mixed components of the stress-energy tensor:

$$T^0_0 = -\rho \qquad\qquad T^1_1 = T^2_2 = T^3_3 = P \,, \tag{1.2.4}$$

since if we have one upper and one lower index the normalization in $u^i u_j$ simplifies.

Let us also skip the computations: the Einstein tensor reads

$$G^0_0 = \frac{1}{A}\left(\frac{1}{r^2} - \frac{A'}{rA}\right) - \frac{1}{r^2} = -8\pi\rho \tag{1.2.5}$$

$$G^1_1 = \frac{1}{A}\left(\frac{1}{r^2} + \frac{B'}{rB}\right) - \frac{1}{r^2} = 8\pi P \tag{1.2.6}$$

$$G^2_2 = G^3_3 = -\frac{1}{2A}\left[\frac{A'B'}{2AB} + \left(\frac{B'}{B}\right)^2 - \frac{B'}{Br^2} + \frac{A'}{r^2 A} - \frac{B''}{2B}\right] = 8\pi P \,. \tag{1.2.7}$$

We need to solve for ρ, P, A and B: the equations of motion of the particles, $\nabla_i T^{ij} = 0$, are a consequence of the Einstein equations.

We have three equations for four variables: we need an additional one, typically we combine them with an *equation of state* for P: a simple case, a *barotropic EOS*, is in the form $P = P(\rho)$.

The first equation can be written as

$$\frac{1}{A} - \frac{rA'}{A^2} = 1 - 8\pi r^2 \rho \,, \tag{1.2.8}$$

which we can integrate to find

$$\frac{r}{A} = r - \int_0^r 8\pi \tilde{r}^2 \rho \, d\tilde{r} \,. \tag{1.2.9}$$

We introduce the quantity

$$m(r) = \int_0^r 4\pi r^2 \rho \, dr \,, \tag{1.2.10}$$

so that

$$A(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} . \tag{1.2.11}$$

The problem is that $m(r)$ is *not* the total mass-energy enclosed in the region below r: we are not integrating with respect to a covariant volume form.

With the second two equations, we end up with

$$\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r} = -\frac{1}{P+\rho}\frac{\mathrm{d}P}{\mathrm{d}r} . \tag{1.2.12}$$

We have essentially solved for B. We can also manipulate the equations to find:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{m(r)\rho}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi r^3 P}{m(r)}\right)\left(1 - \frac{2m(r)}{r}\right)^{-1} , \tag{1.2.13}$$

the **Tolman-Oppenheimer-Volkov** (TOV) equation. This reduces to the hydrostatic equilibrium equation in weak gravity.

The three other terms are basically three relativistic corrections, from velocity and radius: the first are applied based on whether the gas is relativistic *in its own rest frame*; if this is the case, then $P \sim \rho$. The first term is a local correction, the second one is a global one; the third term on the other hand is the usual GR correction, based on r_{Schw}/r.

We can also write

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho . \tag{1.2.14}$$

This equation, the TOV one and the equation of state form a closed system for m, ρ and P.

As for boundary conditions, we require that $m(0) = 0$ and $P(R) = 0$. This is hard to do numerically: it is a boundary value problem, not an initial condition one.

Finally, we discuss the meaning of $m(r)$. The total M is given by

$$M = \int_0^R 4\pi r^2 \rho \, \mathrm{d}r , \tag{1.2.15}$$

where R is the radius of the star. This is *not* the total mass, since $\mathrm{d}V$ is *not* $4\pi r^2 \, \mathrm{d}r$. The *proper* three-volume element is instead given by $\mathrm{d}V = 4\pi r^2 \sqrt{A(r)} \, \mathrm{d}r$, since the true radial distance is calculated through the metric: $\mathrm{d}s^2 = A(r) \, \mathrm{d}r^2$ if the measured length is radial.

We call M_* the mass which is calculated including the $\sqrt{A}$ term. Their difference is

$$M - M_* = \int_0^R 4\pi r^2 \rho \left(1 - \sqrt{A}\right) \mathrm{d}r . \tag{1.2.16}$$

If $r \gg 2m$ (the weak-field limit) then we can approximate $A = 1 - 2m/r$:

$$M - M_* \approx \int_0^R 4\pi r^2 \rho \left(1 - 1 - \frac{2m}{r}\right) \mathrm{d}r \tag{1.2.17}$$

$$\approx -\int_0^R \underbrace{4\pi r^2 \rho \,\mathrm{d}r}_{\mathrm{d}m} \frac{m}{r} \tag{1.2.18}$$

$$\approx -\int_M \frac{Gm}{r} \,\mathrm{d}m = E_g\,. \tag{1.2.19}$$

So, in the weak field limit, the mass defect is given by the gravitational potential energy! This is a physical consequence of the *nonlinearity* of the EFE.

The electromagnetic field behaves linearly at low energies, while in QED we can see nonlinearities.

1.3 The Kerr solution

Wednesday 2020-10-14, compiled 2020-11-04

Some history: the Schwarzschild was found in 1915 while S. was serving in the army; the second exact solution was found in 1963 by a PhD student in New Zeland [Ker63]. In Cartesian coordinates the metric is extremely ugly; nowadays we use the coordinates defined by Boyer and Lyndquist:

$$x = \sqrt{r^2 + a^2} \sin\theta \sin\varphi \tag{1.3.1}$$

$$y = \sqrt{r^2 + a^2} \sin\theta \cos\varphi \tag{1.3.2}$$

$$z = r\cos\theta\,. \tag{1.3.3}$$

These are then not *spherical* but *spheroidal* coordinates. Constant-r spheroids are oblate ellipsoids in the z direction.

The Kerr solution, for who wants to see the computation, is derived in "The mathematical theory of Black Holes" by Chandrasekhar [Cha98]. The metric reads

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{aA}{\Sigma}\right)\sin^2\theta\,\mathrm{d}\varphi^2 - \frac{2A}{\Sigma}\mathrm{d}t\,\mathrm{d}\varphi\,, \tag{1.3.4}$$

where:

$$\Delta = r^2 - 2Mr + a^2 \qquad \Sigma = r^2 + a^2\cos^2\theta \qquad A = 2Mar\sin^2\theta\,, \tag{1.3.5}$$

while $a = J/M$ is the *specific angular momentum* of the black hole.

It describes the spacetime outside an axially symmetric, stationary body. In the Schwarzschild spacetime we have identified an horizon at $r = 2M$ by the properties that $g_{00} \to 0$ and $g_{rr} \to \infty$.

What about Kerr? Are there horizons? In this case, the two conditions do not happen in the same place. The region $g_{00} = 0$ is known as the *limit of staticity*: if it is the case, an observer's worldline cannot be both *timelike* and *stationary* (meaning time-directed).

The condition $g_{00} > 0$ is equivalent to $\Sigma < 2Mr$. Now, consider the $\mathrm{d}r^2$ coefficient: Σ/Δ: this is positive always.

If we are in the region $\theta = \pi/2$, at $g_{00} > 0$ we can still have the metric's signature be $-+++$, since we have the $\mathrm{d}t\,\mathrm{d}\varphi$ term.

As long as $a\,\mathrm{d}\varphi > 0$, that term in the metric is negative, allowing the signature of the metric to be preserved.

This means that **the particle must be co-rotating** with the black hole if it is in the region $g_{00} > 0$.

As long as this is the case, however, a particle can remain at fixed r and even escape. This is, then, *not* an horizon.

The horizon, instead, is found when g_{rr} diverges: this is equivalent to $\Delta \to 0$, which means

$$r^2 - 2Mr + a^2 = 0 \implies r = M \pm \sqrt{M^2 - a^2}\,. \tag{1.3.6}$$

Both of these radii correspond to an horizon. Let us denote $r_+ = r_H$, since the inner horizon cannot really affect any observations.

In order for these two solutions to be real, we must have $a < M$. There is an horizon as long as $a/M < 1$.

If $a > M$, we have a **naked singularity**, since the singularity at $\Sigma = 0$ is still there. This singularity, in any case, is not shaped like a point: we only reach it along the equatorial plane.

Penrose proposed the cosmic **censorship hypothesis**: the universe is a "prude", it always hides singularities with horizons. There are good theoretical reasons to believe that this is verified.

Where is the limit of staticity? The equation is

$$r^2 + a^2\cos^2\theta - 2Mr = 0\,, \tag{1.3.7}$$

which is solved by

$$r_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{1.3.8}$$

There are two of these surfaces as well, and we consider the outer one as before: $r_E = r_+$. If the horizon exists, then this region also exists.

This region is called the **ergosphere**, and the region between $r_H < r < r_E$ is called the **ergoregion**.

Insert figure for the shape of the region

The name comes from the fact that we can extract rotational energy from the BH. "Ergo" means energy.

There has been a long debate about whether the Penrose process actually occurs in a realistic astrophysical setting: the consensus is that the trajectory a particle must take in order for this to happen is way too peculiar.

Note that in this case we also have the cyclic coordinates φ and t. We have two constants of motion like in Schwarzschild.

It can be shown that there exists a third constant of motion, beyond E and L_z: Q, called Carter's constant.

For motion in the equatorial plane we can write the expression

$$E = \frac{r^{3/2} - 2r^{1/2} \pm a M^{1/2}}{r^{3/2}\left(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}\right)^{1/2}}\,, \tag{1.3.9}$$

where $\pm$ refers to whether the particle moves along a prograde or retrograde trajectory.

The expression for the last stable circular orbit is

$$r_{LS} = M\Big[3 + z_2 \mp \big[(3 - z_1)(3 + z_1 + 2z_2)\big]^{1/2}\Big]\,, \tag{1.3.10}$$

check equation

where

$$z_1 = 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/3}\left[\left(1 + \frac{a}{M}\right)^{-1/3} + \left(1 - \frac{a}{M}\right)^{-1/3}\right] \tag{1.3.11}$$

$$z_2 = \left(3\frac{a^2}{M^2} + z_1^2\right)^{1/2}\,. \tag{1.3.12}$$

This reduces to $r_{LS} = 6M$ in the $a = 0$ case, since then we have $z_1 = z_2 = 3$.

What happens for an extreme Kerr BH, with $a = M$? Then $z_1 = 1$, $z_2 = 2$: so,

$$r_{LS} = M\Big[3 + 2 \mp \sqrt{2 \times 8}\Big] = M[5 \mp 4]\,, \tag{1.3.13}$$

which yields $r_{LS} = M$ in the corotating case, and $r_{LS} = 9M$ in the counter-rotating case.

I am writing LS for last-stable, the professor uses MS for marginally-stable

.

We expect that the efficiency of a Kerr BH in the extraction of energy from matter will be higher than the Schwarzschild solution. Let us use the expression we found for the specific energy E, with respect to the parameter $x = r/M$:

$$E = \frac{x^{3/2} - 2x^{1/2} \pm a/M}{x^{3/2}\sqrt{x^{3/2} - 3x^{1/2} \pm 2a/M}}\,, \tag{1.3.14}$$

which in the extreme case becomes

$$E = \frac{x^{3/2} - 2x^{1/2} \pm 1}{x^{3/2}\sqrt{x^{3/2} - 3x^{1/2} \pm 2}}\,, \tag{1.3.15}$$

so we can take the limit : in the direct case, sending $x \to 1$ we get $E = 1/\sqrt{3} \approx 0.577$.

The efficiency is given by

$$\eta = \frac{E_\infty - E}{E_\infty} = 1 - 1/\sqrt{3} \approx 42\,\%\,. \tag{1.3.16}$$

This is a huge amount of energy: we do not expect real black holes to be extreme. There are estimates of the spins of the black holes. What is found is that they seem to cover the whole range $0 < a/M < 1$.

An issue regarding a wide-spread misconception: the parameter M in the interior Schwarzschild solution is the same as the M in the corresponding *exterior* solution. Can we apply the same kind of reasoning for Kerr? Is there an interior Kerr solution? We don't know, but most probably not. Many efforts were put into seeking it, and they all failed. The properties of the matter and radiation inside are weird.

The wide-spread misconception is to claim that the Kerr spacetime describes the spacetime around a rotating star. It sounds reasonable, but it's wrong. Numerically we can derive the true form of the spacetime outside something like a neutron star: it is very different from the Kerr spacetime.

A qualitative argument: a star will generally have a quadrupole moment and emit GWs, while Kerr does not. Kerr is a Petrov-type-B spacetime, which is *nonradiating*.

To clarify.

Next time, we will discuss Equations of state and degenerate gasses.

1.4 The equation of state and degenerate gasses

Tuesday 2020-10-20, compiled 2020-11-04

We move away from the relativistic realm, and treat the more classical Equation of State (EoS). In general P could be a function of ρ, μ, T and other variables. An often-used one is $P = P(\rho)$; also sometimes we use $u = u(\rho)$, where u is the internal energy density.

We will treat the equation of state of a completely degenerate gas.

Let us start for a very simple system: a **hydrogen plasma**. It is a collection of e^- and protons p.

We have complete collisional ionization for $T \gtrsim 10^5\,\mathrm{K}$. Under which conditions is this plasma relativistic or nonrelativistic? This is shown as the blue area in figure 1.1.

The electrons are surely relativistic if $k_B T \gtrsim m_e c^2$, which corresponds to $T \gtrsim 6 \times 10^9\,\mathrm{K}$. For the protons, an analogous equation yields $T \gtrsim 10^{13}\,\mathrm{K}$: these temperatures are basically never reached for realistic astrophysical scenarios.

In the region of $10^5\,\mathrm{K} \lesssim T \lesssim 10^9\,\mathrm{K}$ (for low densities), the plasma will be ionized but not relativistic.

The ideal gas law for the electrons reads $P_e = n_e k_B T$, for the ions $P_i = n_i k_B T$. However, electrons are much lighter than protons, so for the same forces they will accelerate more, so they will radiate away more energy.

Another complication is the following: consider a collection of N ionic species, then for each of these (labelled by k) we will have $P_{i,k} = n_{i,k} k_B T$. The total pressure can be calculated by summing over all of these, plus the electrons:

$$
\begin{aligned}
P &= n_e k_B T + \sum_k n_{i,k} k_B T = k_B T \left(n_e + \sum_k n_{i,k} \right) && (1.4.1) \\
&= \frac{n k_B T}{\mu}\,, && (1.4.2)
\end{aligned}
$$

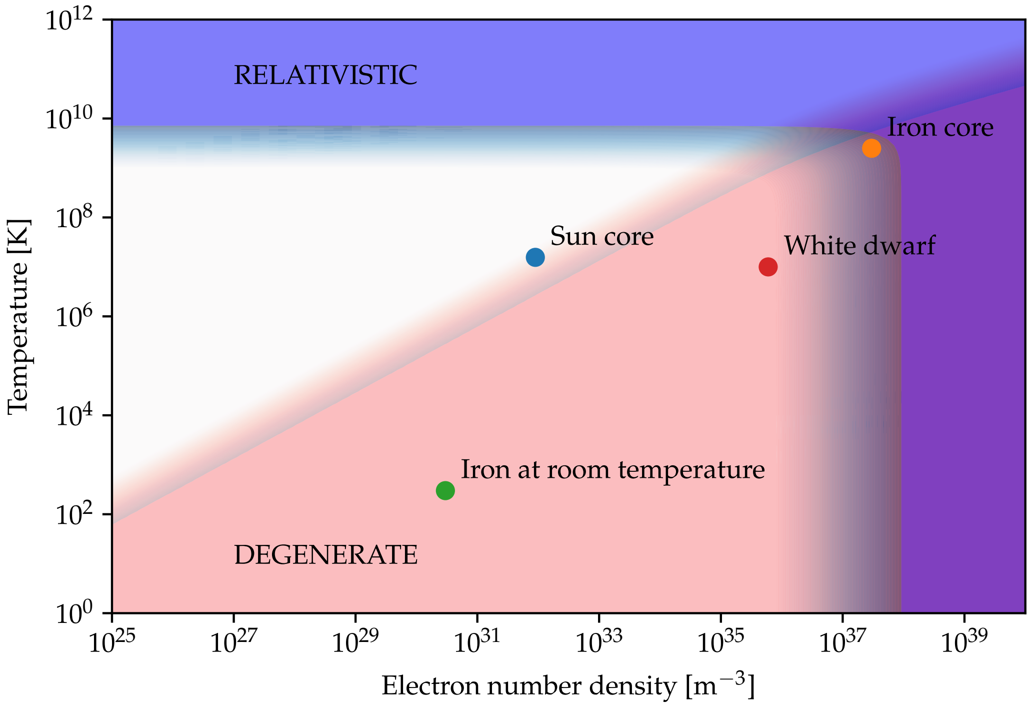

Figure 1.1:

where μ is the *mean molecular weight*, calculated through the total baryonic number density n:

$$\mu = \left[\frac{n_e}{n} + \frac{\sum_k n_{i,k}}{n}\right]^{-1} . \tag{1.4.3}$$

For a pure hydrogen plasma, we have one electron per proton, so we find

$$\mu_{\mathrm{H}} = [1+1]^{-1} = \frac{1}{2} . \tag{1.4.4}$$

For a pure helium plasma we have two electrons per Helium nucleus, which contains 4 baryons: so,

$$\mu_{^4\mathrm{He}} = \left[\frac{2}{4} + \frac{4}{4}\right]^{-1} = \frac{4}{3} . \tag{1.4.5}$$

This holds under the assumption that the plasma behaves like an ideal gas, and that the electrons and protons are nonrelativistic. There is a crucial reason why the first assumption fails: **degeneracy**.

Electrons obey Dirac statistics, and if the density is high enough they can behave very similarly to the $T = 0$ limit even for high temperatures. The region in which they behave like a fully degenerate gas is shown in pink in figure 1.1.

The distribution function for a system of fermions reads

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p} = \frac{2}{h^3}\underbrace{\left[\exp\left(\frac{E}{k_BT} - \alpha\right) + 1\right]^{-1}}_{f}. \tag{1.4.6}$$

Here, the *degneracy parameter* is $\alpha = \mu/k_BT$, where μ is the chemical potential. The energy is expressed as $E = \sqrt{m^2c^4 + p^2c^2}$.

If we want to compute α, we can just integrate:

$$n = \int_{\mathbb{R}^3} \frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p}\,\mathrm{d}^3p \tag{1.4.7}$$

$$= \frac{2}{h^3}\int_0^\infty \left[\exp\left(\frac{E}{k_BT} - \alpha\right) + 1\right]^{-1} 4\pi p^2\,\mathrm{d}p\,. \tag{1.4.8}$$

This integral can be computed for any value of α and T: we then find $n = n(\alpha, T)$. Inverting this relation, we find $\alpha = \alpha(n, T)$.

The logarithm of the absolute value of $\alpha k_BT = \mu$ can be plotted against $\log n$ for different values of the temperature T. For each T, there is a cusp: α changes sign. It can become very big and positive or very big and negative.

Do plot!

Let us start with the case $|\alpha| \gg 1$ and $\alpha < 0$: this corresponds to low n, high T. Then, the $-\alpha$ appearing in the distribution is large and positive: then, the distribution looks like $f \sim \exp\left(-\frac{E}{k_BT}\right)$, the Maxwell-Boltzmann distribution. The gas is behaving like an ideal gas.

Another option is $|\alpha| \gg 1$, $\alpha > 0$. This is the case in which we have low T, high n. Even in the $T \to 0$ limit, the product αk_BT stays finite: this is a function of n, and is called E_F.

The distribution then looks like $f \sim \left[\exp\big((E - E_F)/k_BT\big) + 1\right]^{-1} \to [E \leq E_F]$ in the limit $T \to 0$. (I use the Iverson bracket: [proposition] is 1 if the proposition is true, 0 if it is false).

The higher n is, the higher the T for which the behavior is close to the $T \to 0$ limit.

In the $T \to 0$ limit, we can do the integration analytically: this yields an explicit expression for n in terms of the Fermi momentum p_F corresponding to the Fermi energy E_F: inverting it we get

$$p_F = \sqrt[3]{\frac{3n}{8\pi}}h\,. \tag{1.4.9}$$

This momentum is a characteristic of n independently of the temperature: for $T > 0$ it will not be a hard limit anymore, but it is still a good descriptor of the Fermi gas.

We can write

$$E_F = \sqrt{m^2c^4 + p_F^2c^2} = \sqrt{1 + x_F^2}\,mc^2\,, \tag{1.4.10}$$

where $x_F = p_F/mc^2$. For a nonrelativistic particle distribution $x_F \ll 1$, so $E_F \approx mc^2 + x_F^2 mc^2/2$. The dependence on the number density of the kinetic part is $\sim x_F^2 \sim n^{2/3}$.

On the other hand, in the ultrarelativistic limit $E_F \approx x_F mc^2 \sim x_F \sim n^{1/3}$.

This is the reason why in figure 1.1 the pink boundary curves down in the blue (relativistic region).

It is useful to define this quantity in terms of proper density, in $\mathrm{g/cm^3}$, instead of number density.

We have

$$x_F = \frac{p_F}{mc^2} = \left(\frac{3h^3}{8\pi}\right)^{1/3} \frac{1}{mc^2} n^{1/3}\,; \tag{1.4.11}$$

if we multiply above and below by the mean baryon mass, we have

$$n_e = \frac{n m_b}{m_e m_b} = \frac{\rho}{m_e m_b}\,, \tag{1.4.12}$$

which gives us

$$x_F \sim 10^{-2} \left(\frac{\rho}{m_e}\right)^{1/3}. \tag{1.4.13}$$

Are we sure about this? I think I missed something.

The internal energy density u is computed in general as

$$u = \frac{2}{h^3} \int E(p) \frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p} \mathrm{d}^3p\,, \tag{1.4.14}$$

which in our case is

$$
\begin{aligned}
u &= \frac{2}{h^3} \int_0^{p_F} \sqrt{m^2c^4 + p^2c^2} 4\pi p^2 \,\mathrm{d}p && (1.4.15)\\
&= \frac{8}{h^3} \pi mc^2 (mc)^3 \int_0^{p_F} \sqrt{1+x^2}x^2 \,\mathrm{d}x && (1.4.16)\\
&= \frac{8\pi m^4 c^6}{h^3} \frac{x_F^4}{4} I(x_F)\,, && (1.4.17)
\end{aligned}
$$

where $I(x_F)$ can be computed analytically, but it is of order 1 as can be seen by the asymptotics of the integral.

The integral reads

$$\int_0^{p_F} \sqrt{1+x^2}x^2 \,\mathrm{d}x = \frac{1}{8}\left[x_F(1+2x_F)^2\sqrt{1+x_F^2} - \log\left(x_F + \sqrt{1+x_F^2}\right)\right]. \tag{1.4.18}$$

For the pressure we have a similar integral, which however is more complicated from the conceptual point of view.

The first law of thermodynamics states that

$$dU + p\,dV = 0\,, \tag{1.4.19}$$

as long as the transformation does not exchange heat with its surroundings. Note that $du = d(U/V) = V^{-1}\,dU - UV^{-2}\,dV$, which means that

$$\frac{dU}{V} = du + \frac{U}{V}\,dV\ . \tag{1.4.20}$$

Substituting into the first law of thermodynamics,

$$du + \frac{U}{V}\,dV + P\,dV = 0\,, \tag{1.4.21}$$

but since $V \propto 1/n$ we have

$$\frac{dV}{V} = -\frac{dn}{n}\,. \tag{1.4.22}$$

Substituting this in, we get

$$du - (P+u)\frac{dn}{n} = 0\,. \tag{1.4.23}$$

This means that

$$n\frac{du}{dn} = P + u\,, \tag{1.4.24}$$

which allows us to compute the pressure! The only step remaining is to replace $dn\,/n$ with an expression in terms of x_F, which is

$$\frac{dx_F}{x_F} = 3\frac{dn}{n}\,. \tag{1.4.25}$$

This finally yields

$$P = \frac{x_F}{3}\frac{du}{dx_F} - u\,. \tag{1.4.26}$$

Then we are almost done: we can compute the pressure with u, for which we have an analytic expression, and du/dx_F, which we can easily find since the original expression for u was an integral in dx_F, from which we can read off the integrand.

This yields

$$\begin{aligned} P &= \frac{8\pi m^4 c^5}{h^3}\left[\frac{1}{3}x_F^3\sqrt{1+x_F^2} - \frac{1}{8}\left(x_F(1+2x_F)^2\sqrt{1+x_F^2} - \log\left(x_F + \sqrt{1+x_F^2}\right)\right)\right] && (1.4.27) \\ &= \frac{m^4c^5\pi}{h^3}\left[x_F\sqrt{1+x_F^2}\left(\frac{8}{3}x_F^2 - \left(1+2x_F^2\right)\right) + \log\left(x_F + \sqrt{1+x_F^2}\right)\right] && (1.4.28) \end{aligned}$$

$$= \frac{m^4 c^5 \pi}{h^3} \left[x_F \sqrt{1+x_F^2} \left(\frac{2}{3} x_F^2 - 1 \right) + \log\left(x_F + \sqrt{1+x_F^2} \right) \right] . \tag{1.4.29}$$

Wednesday 2020-10-21, compiled 2020-11-04

About the differences between the Kerr spacetime and the spacetime outside a rotating star: the spacetime outside a rotating body is indeed *not* Kerr.

Kerr is Petrov type D, which means nonradiative: we would need to compute the Weyl invariants of the metric. If we do the calculation, we find that the quadrupole moment of Kerr is $Q = J/M$. If we compute the invariant telling us whether a spacetime is radiative or not, we find that it is nonradiative iff $Q = J/M$. The Hartle-Thorne approximation allows us to work up to a certain value of Ω/Ω_k, close to the mass shedding limit.

The thing we find is that in general for a star the quadrupole is $Q \neq Q_{\text{Kerr}}$, however for small values of the rotation the spacetimes converge.

A reference for this: Berti et al. [Ber+05].

To leading order, deviations from type-D are driven by deviations of Q from Q_{Kerr}.

Back to what we were saying: we have an explicit expression for the pressure of a degenerate electron gas.

Let us consider the ultrarelativistic limit first, $x_F \gg 1$:

$$P \approx \frac{\pi m^4 c^5}{h^3} \left[x_F^2 \frac{2}{3} x_F^2 + \log \dots \right] \tag{1.4.30}$$

$$\approx \frac{\pi m^4 c^5}{h^3} \frac{2}{3} x_F^4 \propto n^{4/3} \propto \rho^{4/3} . \tag{1.4.31}$$

In the opposite limit, $x_F \ll 1$, we would need to expand up to fifth order to see through all the cancellations: skipping all that mess, we find

$$P \approx \frac{8}{15} \frac{m^4 c^5}{h^3} x_F^5 \propto n^{4/3} \propto \rho^{5/3} . \tag{1.4.32}$$

The results are similar: in both cases, $P \propto \rho^\gamma$, with $\gamma = 5/3$ and $4/3$ respectively.

This allows us to compute the maximum mass that a spherical equilibrium configuration can reach if it is supported by the degeneracy pressure of the electron gas alone: the **Chandrasekhar mass**. This is the maximum mass of a white dwarf.

We need to start with the **Lane-Emden equation**, which is also due to work by Chandrasekhar.

We have a spherical star with no nuclear burning. The only forces are due to gravity and pressure. This is a good description of a white dwarf. Sometimes white dwarfs can have some burning on their surface if they are in binary systems, if they accrete fresh mass.

The equation of hydrostatic equilibrium reads

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G m(r) \rho}{r^2} , \tag{1.4.33}$$

where $m(r)$ is the mass contained within a shell of radius r:

$$m(r) = \int_0^r 4\pi r^2 \rho \, \mathrm{d}r \qquad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho . \tag{1.4.34}$$

We want to couple these two equations in order to get a single one: we will find a second-order equation. We start from

$$\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -Gm \tag{1.4.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\frac{\mathrm{d}m}{\mathrm{d}r} = -4\pi G r^2 \rho \tag{1.4.36}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho\,. \tag{1.4.37}$$

We also need to specify an equation of state, which we will have in the form $P(\rho)$: we assume $P = K\rho^\gamma$, a **polytropic** EoS, with constant γ. Sometimes this is also written through $\gamma = 1 + 1/n$, where n is called the polytropic index.

A convenient way to solve the equation is to substitute $\rho = \lambda\phi^n$: then, $P = \lambda^{1+1/n}K\phi^{n+1}$. Then, if we assume that the function ϕ is dimensionless and such that $\phi(0) = 1$, we have $\lambda = \rho_c$.

We need to compute

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(K\lambda^{1+1/n}\phi^{n+1}\right) \tag{1.4.38}$$

$$= K\lambda^{1+1/n}(n+1)\phi^n\frac{\mathrm{d}\phi}{\mathrm{d}r}\,. \tag{1.4.39}$$

Then, the equation reads

$$K\lambda^{1/n}(n+1)\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -4\pi G\lambda\phi^n \tag{1.4.40}$$

$$a^2\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -\phi^n\,, \tag{1.4.41}$$

where

$$a^2 = \frac{K(n+1)\lambda^{-1+1/n}}{4\pi G}\,. \tag{1.4.42}$$

The physical dimensions of a are those of a length. We then introduce a new radial coordinate $\xi = r/a$, and we have the right amount of as on the left-hand side to adimensionalize everything:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) = -\phi^n\,, \tag{1.4.43}$$

which is the usual formulation of the Lane-Emden equation.

What are the boundary conditions we need to set? If we assume that $\rho(r=0) = \rho_c$, we can set $\lambda = \rho_c$ and $\phi(0) = 1$.

We also need to set $\mathrm{d}P/\mathrm{d}r = 0$ at $r = 0$, which also means that $\mathrm{d}\phi/\mathrm{d}\xi = 0$ there as well.

n	ξ_1
0	$\sqrt{6}$
$3/2$	3.65
5	∞

Figure 1.2: First zero crossing as a function of n.

Is that not achieved through an argument of differentiability?

After prescribing these conditions, we can solve the equation: the solution will depend on n. We have analytical solutions for $n = 0, 1, 5$ (note that however n is not necessarily an integer).

The radius of the star described by the equation is given by the first zero: ξ_1. As a function of n,

Let us calculate the total mass of the star: it is given by an integral we can simplify inserting the Lane-Emden equation:

$$
\begin{aligned}
M &= \int_0^{\xi_1} 4\pi\lambda\phi^n \xi^2 a^3 \, \mathrm{d}\xi && (1.4.44)\\
&= -4\pi\lambda a^3 \int_0^{\xi_1} \frac{\xi^2}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) \mathrm{d}\xi && (1.4.45)\\
&= -4\pi\lambda a^3 \, \xi^2 \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\bigg|_0^{\xi_1} && (1.4.46)\\
&= -4\pi\lambda a^3 \xi_1^2 \left.\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right|_{\xi_1}, && (1.4.47)
\end{aligned}
$$

where $\mathrm{d}\phi/\mathrm{d}\xi$ at ξ_1 is necessarily negative, since ξ_1 is the first zero-crossing while $\phi = 1$ at $\xi = 0$.

We want expressions for R and M in terms of λ, the central density, and the polytropic index.

Now, disregarding the constant numbers we have $M \propto \lambda a^3$, while $a^2 \propto \lambda^{-1+1/n}$, therefore $a^3 \propto \lambda^{3(-1+1/n)/2}$.

This yields a dependence of the mass M on the central density $\rho_c = \lambda$ as follows:

$$
M \propto \lambda^{-\frac{3}{2}+\frac{3}{2n}+1} = \lambda^{-\frac{1}{2}+\frac{3}{2n}}. \qquad (1.4.48)
$$

On the other hand, the radius scales like

$$
R = a\xi_1 \propto \lambda^{-\frac{1}{2}+\frac{2}{n}}. \qquad (1.4.49)
$$

The polytropic index for a nonrelativistic gas is given by $1 + 1/n = 5/3$, meaning $n = 3/2$; for an ultrarelativistic gas we have $1 + 1/n = 4/3$, meaning $n = 3$.

Then, we want to find solutions with varying ρ_c: let us start with relatively-small central densities, $\rho_c < 10^6\,\mathrm{g/cm^3}$, and with the star being supported by electron degeneracy pressure, in the nonrelativistic case. Then, we increase the central density.

In this case we have $M \propto \rho_c^{1/2}$. Increasing the central density increases the mass, in a polynomial fashion. On the other hand, $R \propto \rho_c^{-1/6}$. The radius decreases as we increase the central density.

Increasing ρ_c, sooner or later we move towards the relativistic region of the equation of state: then, we need to change n from $3/2$ to 3: this means $M \propto \rho_c^0 = \text{const}$ and $R \propto \rho_c^{-1/3}$. $M \propto \rho_c^0$ means, qualitatively, that "$\rho_c \propto M^\infty$": the central density is extremely dependent on the mass, and it will diverge with an asymptote. Then, we cannot go beyond this threshold, which we can compute using the first zero of the Lane-Emden equation:

$$M_{\text{Ch}} = -4\pi \left[\frac{K(n+1)\rho_c^{-1+1/n}}{4\pi G} \right]^{3/2} \rho_c \left(\xi^2 \frac{\mathrm{d}\phi}{\mathrm{d}\xi} \right) \bigg|_{\xi_1} , \tag{1.4.50}$$

which we can evaluate with the correct numbers for the constants $M_{\text{Ch}} = 5.81/\mu_e^2 M_\odot \approx 1.44 M_\odot$ in the case of a pure Helium composition.

So we need to be careful when applying this to the iron cores of supernovae!

1.5 Bondi accretion

Tuesday 2020-10-27, compiled 2020-11-04

We now start discussing the interaction of a compact object with its environment. Black holes are intrinsically black, however we can see them through their interactions with the surrounding medium.

A neutron star might have a temperature of the order $T \sim 10^6 \div 10^7\,\text{K}$; we know that the flux is given by $F = \sigma T^4$, so the emitted luminosity is calculated by

$$L = 4\pi R^2 \sigma T^4 \sim 10^{31}\,\text{erg/s}\,, \tag{1.5.1}$$

which comes out to be *very small*, due to the fact that the radius is very small compared to a star. For comparison, the Sun has a luminosity of around 100 times more.

A compact object's gravitational pull, however, induces *accretion*: material falls onto the object, heating up. We will consider spherical accretion, the simplest geometrical situation. It is not fully realistic, however it can be used to model the real case. We assume to have a point-like object of mass M, and we make some other assumptions:

1. spherical symmetry;
2. stationarity;
3. the accreting matter can be treated as a gas: this is not obvious, since it means that collisions are efficient enough to couple the temperatures of the gas, or equivalently the mean free path λ_c should be much smaller than the radius r;[3]
4. the accreting matter is a perfect gas which can be treated adiabatically;

[3] This is not easily verified in astrophysical scenarios, magnetic fields make it easier for it to happen.

5. the accreting matter is non-self-gravitating: the total mass of gas surrounding the BH, m, is much smaller than M;

6. we will use Newtonian gravity, although the general-relativistic way to treat this is not very hard.[4]

Under these hypotheses, the problem is called **Bondi-Hoyle accretion**, and old problem in astrophysics, dating back to the fifties [Bon52]. The final treatment of accretion onto a BH was done in 1991 [NTZ91].

This is very much a toy model: adiabaticity precludes the production of radiation. However, we shall see how the conditions might be relaxed.

The first equation is the continuity equation, whose physical meaning is rest mass conservation. The velocity field is denoted as $u(r)$. The quantity $4\pi r_2^2 \rho(r_2) u(r_2)$ is the rate of mass crossing the surface of radius r_2. If there are no sources nor sinks of mass, this expression should be equal for another choice of radius r_1. Then, in general we can write

$$4\pi r^2 \rho(r) u(r) = \dot{M} = \text{const} . \tag{1.5.2}$$

The Euler equation, coming from the conservation of momentum, reads

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \text{force per unit mass} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} . \tag{1.5.3}$$

The total time derivative of the velocity can be written as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \underbrace{\frac{\partial u}{\partial t}}_{=0} + \frac{\partial u}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial u}{\partial r} u \tag{1.5.4}$$

by stationarity. Then the Euler equation reads

$$u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0 , \tag{1.5.5}$$

which we can integrate: it becomes

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}P}{\rho} - \frac{GM}{r} = \text{const} , \tag{1.5.6}$$

which is Bernoulli's theorem, the conservation of energy. It is slightly different from the usual form because we are considering a compressible flow; this is a gas, and it definitely can be compressed. This is our second conservation law. We need a third equation: an equation of state, which we can derive from our assumption of adiabaticity, $P = K\rho^\Gamma$, a polytropic EoS.

Luckily, we have managed to integrate already, so we do not have ODEs anymore: we are left with a simple algebraic system. If we introduce the isentropic speed of sound,[5]

[4] For the GR and non-ideal fluid version see my Bachelor's thesis [Tis19].

[5] There are other kinds of speed of sound, depending on the kind of perturbation.

$a^2 = (\partial P / \partial \rho)_s = k\Gamma\rho^{\Gamma-1}$ (calculated at constant entropy, but since we assumed adiabaticity this is not an additional assumption).

Also, the pressure differential reads $\mathrm{d}P = k\Gamma\rho^{-1}\,\mathrm{d}\rho$, so

$$\frac{1}{2}u^2 + \int \frac{k\Gamma\rho^{\Gamma-1}}{\rho}\,\mathrm{d}\rho - \frac{GM}{r} = \text{const} \tag{1.5.7}$$

$$\frac{1}{2}u^2 + \frac{a^2}{\Gamma-1} - \frac{GM}{r} = \text{const}\,. \tag{1.5.8}$$

This can be solved together with the continuity equation. What is this constant? We can calculate it for an arbitrary r, since it is always the same. We choose $r \to \infty$, so we get

$$\frac{1}{2}u^2 + \frac{a^2}{\Gamma-1} - \frac{GM}{r} = \frac{1}{2}u_\infty^2 + \frac{a_\infty^2}{\Gamma-1}\,, \tag{1.5.9}$$

where u_∞ and a_∞ are the velocity and speed of sound very far from the source. What can we say about them? A special case we can consider is $u_\infty = 0$: the gas at infinity is at rest, it has no bulk motion with respect to the black hole. We now need to replace ρ in the continuity equation with something in terms of the sound speed: we know that $a^2 = k\Gamma\rho^{\Gamma-1}$, so

$$\rho = \left(\frac{a^2}{k\Gamma}\right)^{1/(\Gamma-1)}\,. \tag{1.5.10}$$

We can calculate this at infinity: ρ_∞ is then given in terms of a_∞. Taking the ratio of the two expressions, we get

$$\frac{\rho}{\rho_\infty} = \left(\frac{a}{a_\infty}\right)^{2/(\Gamma-1)}\,. \tag{1.5.11}$$

Inserting this into the continuity equation we get

$$4\pi r^2 \rho_\infty \left(\frac{a}{a_\infty}\right)^{2/(\Gamma-1)} u = \dot{M}\,. \tag{1.5.12}$$

There are several constants appearing, but only the two variables $u(r)$ and $a(r)$. What can we say about the constants? $\dot{M}$ is called the *mass accretion rate*. The other two constants are a_∞ and ρ_∞. One might think that we would need to specify all three of the constants: this is, however, not the case. Only two of these constants are actually independent. Fixing a_∞ and ρ_∞ constrains $\dot{M}$ to a single value, an *eigenvalue* of the problem.

Let us select a fixed value of $r = \bar{r}$. Then, the two equations are just functions of u and a. The Euler equation looks like

$$\frac{1}{2}u^2 + \frac{a^2}{\Gamma-1} = \text{const}\,, \tag{1.5.13}$$

an ellipse (of which we consider only a quarter, with $u > 0$, $a > 0$). The continuity equation, instead, is in the form

$$u \propto \dot{M} a^{-\frac{2}{\Gamma-1}}\,. \tag{1.5.14}$$

This is a part of a hyperbola. The two may cross in zero, one or two points. In order for a solution to exist there needs to be at least one intersection. Changing $\dot{M}$ moves the hyperbola. We get a single $\dot{M}$ so that the two curves cross at a single point. For now, we can surely say that $\dot{M}$ cannot be chosen to have any value, since we must have at least an intersection.

We can apply a similar kind of reasoning by changing $\bar{r}$ instead of $\dot{M}$. We can connect these solutions: curves in the (u, a) plane, parametrized by $\bar{r}$. The intersections always lie on opposite sides of the $u = a$ line, which corresponds to the sonic condition. One solution is always supersonic, one is always subsonic.

The solution which is always supersonic has $u_\infty > a > 0$, which is not good for us. The subsonic solution might then work: however, if the central mass is a BH, then the speed at which the matter crosses the horizon is the speed of light, and surely $c > a$.[6]

This argument shows that we need a transsonic solution: we need a radius r_s at which the two curves are tangent to one another. If we fix this, we get two solutions, only one of which has $u_\infty = 0$. The opposite solution could describe a transsonic stellar wind. The consequence of this is that there is a single acceptable value for $\dot{M}$, providing us with a radius so that the two curves are tangent.

This yields a fixed $\dot{M}(\rho_\infty, a_\infty)$. We can get more information by going back to the differential form of the equations of motion:

$$4\pi r^2 \rho u = \dot{M} \tag{1.5.15}$$

$$u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0\,. \tag{1.5.16}$$

We can write the Euler equation as

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2} \tag{1.5.17}$$

$$u\frac{\partial u}{\partial r} = -\frac{a^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2}\,. \tag{1.5.18}$$

We can also find a differential form of the continuity equation:

$$r^2 \rho u = \frac{\dot{M}}{4\pi} \implies \frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{2u}{r} - \frac{u}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\,, \tag{1.5.19}$$

which we can substitute into the Euler equation:

$$-2\frac{u^2}{r} - \frac{u^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{a^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2} \tag{1.5.20}$$

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{2u^2}{r} - \frac{GM}{r^2} \tag{1.5.21}$$

$$\frac{\mathrm{d}\log\rho}{\mathrm{d}r} = \frac{1}{a^2 - u^2}\left[\frac{2u^2}{r} - \frac{GM}{r^2}\right], \tag{1.5.22}$$

[6] The largest physically possible sound of speed is $c/\sqrt{3}$, achieved for ultrarelativistic matter.

but we want there to exist a point at which $a = u$: therefore, the denominator vanishes, meaning that if we do not want the derivative of the log-density to diverge we must have

$$2u^2(r_s) = \frac{GM}{r_s}\,, \tag{1.5.23}$$

and $u(r_s) = a(r_s)$.

This ensures that there is no divergence at the transsonic point. This is a *regularity condition*, imposed at a *critical point*.

Wednesday 2020-10-28, compiled 2020-11-04

We have seen why $\dot{M}$ is fixed as long as a_∞ and ρ_∞ are. Let us then write the Bernoulli equation at the sonic radius:

$$\frac{1}{2}a_s^2 + \frac{a_s^2}{\Gamma - 1} - \frac{GM}{r_s} = \frac{a_\infty^2}{\Gamma - 1} \tag{1.5.24}$$

$$\frac{1}{2}a_s^2 + \frac{a_s^2}{\Gamma - 1} - 2a_s^2 = \frac{a_\infty^2}{\Gamma - 1}\,, \tag{1.5.25}$$

so we can calculate the sonic speed if we know a_∞:

$$a_s^2\left(\frac{1}{2} - 2 + \frac{1}{\Gamma - 1}\right) = \frac{a_\infty^2}{\Gamma - 1} \tag{1.5.26}$$

$$a_s = a_\infty\left(\frac{2}{5 - 3\Gamma}\right)^{1/2}\,, \tag{1.5.27}$$

so, since the proportionality constant is of order unity, we can say that the speed of sound at infinity and at the sonic point are of the same order of magnitude. We can also calculate the sonic radius:

$$r_s = \frac{GM}{2a_s^2} = \frac{GM}{2a_\infty^2}\left(\frac{5 - 3\Gamma}{2}\right) \sim \frac{GM}{a_\infty^2}\,. \tag{1.5.28}$$

This relates the chemical potential at infinity and the gravitational potential energy: when these contributions are of the same order of magnitude we find the sonic transition.

We are close to being able to write an explicit expression for the accretion rate $\dot{M}$; we only need to use the continuity equation:

$$\dot{M} = 4\pi r^2 \rho_\infty \left(\frac{a}{a_\infty}\right)^{2/(\Gamma-1)} u\bigg|_{r=r_s} \tag{1.5.29}$$

$$= 4\pi r_s^2 \rho_\infty \left(\frac{a_s}{a_\infty}\right)^{2/(\Gamma-1)} a_s \tag{1.5.30}$$

$$\propto \frac{M^2 \rho_\infty}{a_\infty^3}\,. \tag{1.5.31}$$

What happens when $\Gamma = 5/3$? it would seem that then the sonic speed diverges; this is not the case, as was clarified relatively recently.

What happens if the massive object is moving through the cloud, in which $u_\infty \neq 0$? We lose spherical symmetry, but we can still get a solution, in the form

$$\dot{M} \propto \frac{M^2 \rho_\infty}{(u_\infty^2 + a_\infty^2)^{3/2}}. \tag{1.5.32}$$

Numerically, we have

$$\dot{M} = 10^{11} \frac{\left(\frac{M}{M_\odot}\right)^2 n_\infty}{(u_{\infty,10}^2 + a_{\infty,10}^2)^{3/2}} \mathrm{g/s}, \tag{1.5.33}$$

where the speeds are measured in units of $10\,\mathrm{km/s}$, while the number density is in particles per cubic centimeter.

The equation we found earlier is

$$\frac{a^2 - u^2}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{2u^2}{r} - \frac{GM}{r^2}. \tag{1.5.34}$$

How do u and ρ depend on r, roughly speaking? Let us give a qualitative argument. First, we assume that $u \ll a$ (which also means $r \gg r_s$).

Then, everything on the right-hand side is approximately zero, while $a^2 - u^2 \approx a^2$: this means

$$\frac{a^2}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} \approx 0, \tag{1.5.35}$$

or, $\rho \equiv \rho_\infty$. This happens if we are far from the star, in the *hydrostatic region*. If $\rho \approx$ const, then by continuity $u \propto r^{-2}$.

Insert plot

In the opposite limit, we have $u \gg a$, $r \ll r_s$. Then, substituting in from the continuity we get

But u^2/r is *not* negligible compared to GM/r^2! This does not change the substance, the proportionality still works, however we must be careful.

$$-\frac{1}{\rho} u^2 \frac{\mathrm{d}\rho}{\mathrm{d}r} \approx -\frac{GM}{r^2} \tag{1.5.36}$$

$$\frac{1}{\rho} \left(\frac{1}{r^2 \rho}\right)^2 \frac{\mathrm{d}\rho}{\mathrm{d}r} \propto \frac{1}{r^2} \tag{1.5.37}$$

$$\frac{1}{\rho^3} \frac{\mathrm{d}\rho}{\mathrm{d}r} \propto r^2 \tag{1.5.38}$$

$$\frac{1}{\rho^2} \propto r^3, \tag{1.5.39}$$

therefore $\rho \propto r^{-3/2}$. This is the crucial aspect: in the supersonic region the density increases with decreasing r. We know that $u \propto 1/r^2\rho \propto r^{-1/2}$. This is the same thing we would have found by considering free fall:

$$\frac{1}{2}mv^2 = \frac{GMm}{r}\,. \tag{1.5.40}$$

The full GR treatment of spherical accretion onto a Schwarzschild BH is quite close to what we have found here [NTZ91, fig. 2, top left]. What is the emitted **luminosity**? We can express in terms of the *efficiency* η: $L = \eta \dot{M} c^2$.

Let us give a qualitative, Newtonian argument about the maximum accretion efficiency of a Neutron Star. The surface of the NS is rigid, when matter falls upon it it basically *stops*, so $u(r_*) = 0$, and let us assume that all the impact energy is radiated away. Then, the efficiency is the

$$\eta_{NS} = \frac{mc^2 - \left(mc^2 - GMm/r_*\right)}{mc^2} = \frac{GM}{c^2 r_*} = \frac{r_{\text{Schw}}}{2r_*}\,. \tag{1.5.41}$$

In GR this is slightly different, but not by much. Typical values are generally

$$\eta = \frac{3(M/M_\odot)\text{km}}{2 \times 10(R/R_\odot)} \approx 0.15\,. \tag{1.5.42}$$

With typical values, we get

$$L = 0.15 \times 10^{11} \times 9 \times 10^{20} \text{erg/s} \sim 10^{31}\,\text{erg/s}\,, \tag{1.5.43}$$

100 times lower than the solar luminosity. The average number density in interstellar space is of the order of $1\,\text{cm}^{-3}$, the region in which we live has a lower number density, one or two orders of magnitude less. It was swept away by multiple supernova events.

People have looked for the luminosity of NS accretion, without result.

NSs are strongly magnetized: radio pulsars have magnetic fields of the order of $10^7\,$T, magnetars reach $10^{10}\,$T. These magnetic fields can inhibit any accretion, through the *propeller effect*.

If we consider a rotating dipolar field,[7] it propels the particles away.

What about black holes? There is now no solid surface, and what we can do is to compare the typical timescales for production of radiation, τ_{rad}, and the typical dynamical (free-fall) timescale τ_{dyn}. If $\tau_{\text{rad}} \gg \tau_{\text{dyn}}$, then hardly any radiation will be produced before the plasma can fall in. This is what happens if we do the proper modelling of the flow. Typically, $10^{-8} \lesssim \eta \lesssim 10^{-2}$. This is at least 10 times larger.

The luminosity produced by a solar-mass BH would be even lower. Isolated BHs are typically larger, but not by *that much*. The very massive BHs can only come from very massive stars in the universe, where the low metallicity allowed for low stellar winds and high remnant masses.

[7] It's kind of like making homemade mayonnaise…

1.6 Roche Lobe Overflow

What happens in a **binary system** with a Compact Object and another *donor* star?

They will revolve around their common center of mass. If the star is massive, then there will be a strong stellar wind. A fraction of the matter expelled will fall onto the CO, however this will not be a large fraction.

The other possibility is *Roche lobe overflow*.

The idea is: consider a test particle (or, really, a test fluid element, since we will consider gasses) moving under the action of two centrally condensed ("point-like") masses. This is basically a problem in celestial mechanics. Let us denote the two masses by $M_{1,2}$. They will orbit around a common center of mass, and by Kepler's third law we can link their orbital separation and the period of the motion:

$$4\pi^2 a^3 = G(M_1 + M_2)P^2 \,. \tag{1.6.1}$$

Solving the Roche problem means that we have to write down Newton's second law; it is convenient to do so in a system which is *co-rotating* with the two stars, and centered in the center of mass. Then, we will need to account for the fictitious forces for this noninertial reference system.

The Euler equation (we start with this directly, for a single test particle the effects are the same) will read

$$\frac{\partial \vec{v}}{\partial t} + \underbrace{\left(\vec{v} \cdot \vec{\nabla}\right)\vec{v}}_{\text{convective derivative}} = -\frac{\vec{\nabla} P}{\rho} - \underbrace{2\vec{\omega} \wedge \vec{v}}_{\text{Coriolis}} - \underbrace{\omega \wedge (\vec{\omega} \wedge \vec{r})}_{\text{centrifugal}} - \vec{\nabla}\phi_G \,. \tag{1.6.2}$$

We can write down a potential for the centrifugal term:

$$\phi_C = -\frac{1}{2}(\vec{\omega} \wedge \vec{r})^2 \,, \tag{1.6.3}$$

and we can then introduce an effective potential: the *Roche* potential, $\phi_R = \phi_C + \phi_G$.

This potential reads

$$\phi_R = \frac{GM_1}{|\vec{r} - \vec{r}_1|} + \frac{GM_2}{|\vec{r} - \vec{r}_2|} + \frac{1}{2}(\vec{\omega} \wedge \vec{r})^2 \,. \tag{1.6.4}$$

Add plot!

If we plot the contour lines of this potential, they are determined by the mass ratio $q = M_2/M_1$, while the linear scale of the system is only determined by a. Tuesday 2020-11-3, compiled 2020-11-04

Let us come back to the Roche potential. The equation of motion of a particle in the corotating frame is

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla} P - \vec{\omega} \wedge \vec{v} - \vec{\nabla}\phi_R \,, \tag{1.6.5}$$

where ϕ_R is the Roche potential:

there might be a wrong sign in the last lecture!

$$\phi_R = -\frac{GM_1}{|\vec{r}-\vec{r}_1|} - \frac{GM_2}{|\vec{r}-\vec{r}_2|} - \frac{1}{2}(\vec{\omega}\wedge\vec{r})^2 \,. \tag{1.6.6}$$

insert diagram for the quantities in the potential

This can be written in terms of the parameters $q = M_2/M_1$ and a.
If θ is the angle between $\vec{\omega}$ and $\vec{r}$ we can write

$$(\vec{\omega}\wedge\vec{r})^2 = \omega^2 r^2 \sin^2\theta \,. \tag{1.6.7}$$

Then,

$$\phi_R = -G(M_1+M_2)\left[+\frac{1}{2}\frac{\omega^2 r^2 \sin^2\theta}{G(M_1+M_2)} + \frac{GM_1}{G(M_1+M_2)|\vec{r}-\vec{r}_1|} + \frac{GM_2}{G(M_1+M_2)|\vec{r}-\vec{r}_2|}\right] \tag{1.6.8}$$

$$= -G(M_1+M_2)\left[\frac{1}{2}\frac{\omega^2 r^2 \sin^2\theta}{G(M_1+M_2)} + \frac{1}{(1+q)|\vec{r}-\vec{r}_1|} + \frac{q}{(1+q)|\vec{r}-\vec{r}_2|}\right] . \tag{1.6.9}$$

We can then use Cartesian coordinates centered in M_1: then,

$$\phi_R = -\frac{G(M_1+M_2)}{2a^3}\left[\frac{2a^3}{1+q}\frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{2qa^3}{1+q}\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} + \left[(x-x_{\mathrm{CM}})^2+y^2\right]\right] , \tag{1.6.10}$$

where

$$x_{\mathrm{CM}} = \frac{qa}{1+q} , \tag{1.6.11}$$

so if we rescale all the spatial coordinates by the major semi axis a we get

$$\phi_R = -\frac{G(M_1+M_2)}{2a}\left[\frac{2}{1+q}\frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{2q}{1+q}\frac{1}{\sqrt{(x-1)^2+y^2+z^2}} + \left[\left(x-\frac{q}{1+q}\right)^2+y^2\right]\right] . \tag{1.6.12}$$

If the stars are stationary, or in slow evolution. Then, their surfaces will be (at least in first approximation) at rest. Therefore, the derivative terms vanish, and we get

$$0 = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi_R \,, \tag{1.6.13}$$

but the surface is defined by $\vec{\nabla}P = 0$, which by this equation corresponds to $\vec{\nabla}\phi_R = 0$.
The star will then take the shape of an equi-Roche potential surface. What is this shape?

Make 2D plots, cuts of the surface

We have roughly spherical contours near the stars, and a figure-eight contour eventually. The center of this contour is called L_1, the first or "inner" Lagrange point.

Analogy with a dog food container for Roche Lobe overflow.

Is the overflow stable? It depends on whether the volume of the Roche lobe increases or decreases as the donor star loses mass: in the latter case.

We can introduce a characteristic **radius** of the Roche Lobe, calculated by

$$V_{\text{lobe}} = \frac{4}{3}\pi R_{\text{lobe}}^3 \,, \tag{1.6.14}$$

although the lobe is not a sphere this allows us to give a characteristic number. We can calculate

$$\frac{R_{\text{lobe}}}{a} = f(q) = \begin{cases} 0.38 + 0.2 \log q & 0.5 < q < 20 \\ 0.46 \left(\frac{q}{1+q}\right)^{1/3} & 0 < q < 0.5 \end{cases} \,, \tag{1.6.15}$$

and we can see that the dependence on q is quite weak.

However, the orbital separation a is itself a function of q, and the dependence $a(q)$ is much more relevant than the dependence of R_{lobe}/a.

In order to calculate $a(q)$, let us make some assumptions.

1. $M_1 + M_2 = \text{const}$. This is realistic.
2. The total angular momentum L_{tot} is a constant. This is a bit tricky: even a small amount of mass loss can result in high angular momentum loss.
3. $L_{\text{tot}} = L_{\text{orb}}$: all the angular momentum is orbital. We are neglecting the spins of the stars, and the angular momentum of the gas. This is realistic, since tidal forces move the configuration towards a tidally locked state.

The conservation of mass yields $M_1(1+q)$. The orbital angular momentum is written in terms of the distances of the stars from the center of mass:

$$L_{\text{orb}} = M_1 v_1 a_1 + M_2 v_2 a_2 \tag{1.6.16}$$

$$= M_1 a_1^2 \omega + M_2 a_2^2 \omega \,, \tag{1.6.17}$$

so if we place the origin of our coordinates in the center of mass we can write everything in terms of the orbital separation $a = a_1 + a_2$

So a is not the semimajor axis

$$-M_1 a_1 + M_2 a_2 = 0 \tag{1.6.18}$$

$$-(M_1 + M_2) a_1 + M_2 a = 0 \tag{1.6.19}$$

$$a_1 = \frac{qa}{1+q} \tag{1.6.20}$$

$$a_2 = \frac{a}{1+q}\,. \tag{1.6.21}$$

In terms of angular momentum we get

$$\left[M_1 \frac{q^2 a^2}{(1+q)^2} + M_2 \frac{a^2}{(1+q)^2}\right]\omega = \text{const} \tag{1.6.22}$$

$$M_1 a^2 \left[\frac{q^2}{(1+q)^2} + \frac{q}{(1+q)^2}\right]\omega = \text{const}\,, \tag{1.6.23}$$

which means

$$\frac{a^2 M_1 \omega q}{1+q} = \frac{a^2 M_1 2\pi q}{P(1+q)} = \text{const}\,, \tag{1.6.24}$$

since $\omega = 2\pi/P$, where P is the orbital period. We also know that $M_1 \propto (1+q)$, therefore

$$\frac{a^2 q}{(1+q)^2}\frac{1}{P}\,, \tag{1.6.25}$$

but also Kepler's third law tells us that $a^3 \propto P^2$: so, $P \propto a^{3/2}$. This yields

$$\frac{a^{2-3/2} q}{(1+q)^2} = \sqrt{a}\frac{q}{(1+q)^2} = \text{const}\,, \tag{1.6.26}$$

therefore

$$a \propto \frac{(1+q)^4}{q^2}\,. \tag{1.6.27}$$

Then,

$$\log R_2 = \log a + \log f\,, \tag{1.6.28}$$

and if we consider a variation of q, the variation of $\log R_2$ will be given by the sum of the variations of the two logarithms.

We have

$$\log a = 4\log\big(1+q\big) - 2\log q + \text{const} \tag{1.6.29}$$

$$\Delta \log a = 4\frac{\Delta q}{1+q} - 2\frac{\Delta q}{q} \tag{1.6.30}$$

$$= \frac{4q-2-2q}{(1+q)q}\Delta q = \frac{\Delta q}{q}\frac{q-1}{q+1}\,. \tag{1.6.31}$$

Then, the fractional variation is approximately

$$\frac{\Delta R_2}{R_2} = \Delta \log R_2 \approx \frac{\Delta q}{q}\frac{q-1}{q+1}\,, \tag{1.6.32}$$

since $\Delta \log f$ is small. As the star M_2 donates mass, M_2 decreases and M_1 increases. Then, $q = M_2/M_1$ decreases, therefore $\Delta q < 0$.

If we want the Roche lobe to shrink, we want $\Delta R_2 < 0$: so, we want $q - 1 > 0$. This means that **Roche lobe accretion is self-sustaining, or stable iff the donor star is larger than the receiver**. Including Δf, things don't change by much.

Is this the case in real systems? Usually the mass of the BH is of at least a few solar masses, so we need an unusually large companion. For example, the companion to the BH in Cygnus X-1 is a very large O-star.

1.7 Accretion disks

Now we will discuss what happens when this is indeed the case. What is the fate of the matter passing through the inner Lagrange point?

Let us change the perspective: we are not comoving with respect to the orbiting stars. Let us call b_1 the separation of the L_1 point from the center of star 1. This is slightly larger than R_1.

If we are stationary at the center the companion compact star will appear to rotate at a large velocity compared to us.

The components of this velocity can be decomposed into the parallel and perpendicular to the separation vector between the star, however the component $v_\parallel$ will be of the order of the speed of sound in the gas: roughly,

$$v_\parallel \approx c_s = \sqrt{\frac{\partial P}{\partial \rho}} = \sqrt{\frac{k_B T}{\mu m_p}} \approx 10 \sqrt{\frac{T}{10^4\,\text{K}}}\text{km/s}\,; \tag{1.7.1}$$

while the component $v_\perp$ will be large: of the order of

$$v_\perp \sim b_1 \omega\,, \tag{1.7.2}$$

and the distance b_1 will be roughly $b_1 \approx a(0.5 - 0.227 \log q)$, while by Kepler's third law

$$4\pi^2 a^3 = G(M_1 + M_2)P^2\,, \tag{1.7.3}$$

so

$$a \approx 3 \times 10^{11} \left(\frac{M_1}{M_\odot}\right)^{1/3} (1+q)^{1/3} \left(\frac{P}{1\,\text{d}}\right)^{2/3} \text{cm}\,, \tag{1.7.4}$$

while the angular velocity is

$$\omega = \frac{2\pi}{P} \approx 7 \times 10^{-5} \left(\frac{P}{1\,\text{d}}\right)^{-1} \text{rad/s}\,. \tag{1.7.5}$$

This means that the perpendicular velocity is approximately

$$v_\perp \approx 100 \left(\frac{M_1}{M_\odot}\right)^{1/3} (1+q)^{1/3} \left(\frac{P}{1\,\text{d}}\right)^{-1/3} \text{km/s}\,, \tag{1.7.6}$$

an order of magnitude more than the speed of sound. Wednesday 2020-11-4, compiled 2020-11-04

We were discussing the flow of plasma from the donor star to the compact object through the inner Lagrangian point. The velocities of the compact object in the frame of the gas are $v_\parallel \sim 10\,\text{km/s}$ and $v_\perp \sim 100\,\text{km/s}$, so we neglect $v_\parallel$.

Suppose we have a mass M, and a particle in a bound orbit around this mass. Its energy E and energy per unit mass ϵ will be

$$E = -\frac{GMm}{2a} \qquad \epsilon = -\frac{GM}{2a}\,, \tag{1.7.7}$$

while its specific angular momentum will be

$$\left(\frac{L}{m}\right)^2 = \ell^2 = (1-e^2)GMa\,. \tag{1.7.8}$$

Then, we can write the semimajor axis as

$$\frac{1}{a} = \frac{(1-e^2)GM}{\ell^2}\,, \tag{1.7.9}$$

so the specific energy will read

$$\epsilon = -\frac{GM(1-e^2)GM}{\ell^2} = -\frac{(GM)^2(1-e^2)}{2\ell^2}\,. \tag{1.7.10}$$

We can ask ourselves: what is the orbit which has the minimum energy ϵ_{min} at fixed ℓ? The only thing which can vary is the eccentricity e, so the minimum energy is attained for the circular orbit, with $e = 0$, where

$$\epsilon_{\text{min}} = -\frac{(GM)^2}{2\ell^2}\,. \tag{1.7.11}$$

The stream of gas will be subjected to frictional forces, which will dissipate energy, and since the energy of circular orbits is minimum this will circularize the orbit. We will discuss the timescale of this process later.

We can estimate the radius of circularization, R_{circ}: we know that for a circular orbit the angular momentum will reach its Keplerian value, L_k, and the velocity will reach its Keplerian value. This reads

$$v_K = \sqrt{\frac{GM}{R}}\,, \tag{1.7.12}$$

which comes from equating v^2/R and GM/R^2. The Keplerian (specific!) angular momentum reads

$$L_K = R v_K\,, \tag{1.7.13}$$

which we can compute at R_{circ}:

$$L_K(R_{\text{circ}}) = \sqrt{GMR_{\text{circ}}}\,. \tag{1.7.14}$$

If we fix the specific angular momentum ℓ of an incoming fluid parcel we can then determine the radius of its orbit, $R_{\text{circ}} = \ell^2/GM$.

We can compute this initial value of ℓ since the velocity of the fluid is given by the $v_\perp = \omega b_1$, and then $\ell = v_\perp b_1 = \omega b_1^2$. Then, finally, we have

$$R_{\text{circ}} = \frac{\omega^2 b_1^4}{GM} = \frac{4\pi^2 b_1^4}{GMP^2}\,. \tag{1.7.15}$$

In units of the orbital separation, and using Kepler's law

$$\omega^2 = \frac{G(M_1 + M_2)}{a^3} \tag{1.7.16}$$

we get

$$\begin{aligned} \frac{R_{\text{circ}}}{a} &= \frac{\omega^2 b_1^4}{GMa} && (1.7.17)\\ &= \frac{b_1^4 G(M_1+M_2)}{GMa^4} = \left(\frac{b_1}{a}\right)^4 (1+q)\,. && (1.7.18) \end{aligned}$$

Yesterday we found

$$\frac{b_1}{a} \approx 0.5 - 0.227\log q\,, \tag{1.7.19}$$

using which we get

$$\frac{R_{\text{circ}}}{a} \approx \big(0.5 - 0.227\log q\big)^4 (1+q)\,, \tag{1.7.20}$$

and we can also calculate the radius of the Roche lobe using a result from yesterday:

$$\frac{R_1}{a} = \begin{cases} 0.38 - 0.2\log q & 0.05 < q < 2 \\ \frac{0.426}{(1+q)^{1/3}} & q > 2 \end{cases}\,. \tag{1.7.21}$$

We can then see that R_{circ} is at least 10 times smaller than the radius of the lobe.

For a star we would need to ensure that $R_{\text{circ}} > R_*$, but for a compact object there are no issues.

There are three characteristic times:

$$t_{\text{dyn}} < t_{\text{rad}} < t_{\text{visc}}\,, \tag{1.7.22}$$

the dynamical, radiative and viscous timescale. Injection happens on a short t_{dyn} timescale, circularization happens on a longer t_{rad} timescale, shrinkage happens on an even longer t_{visc} timescale.

The true trajectory of a fluid element will be a spiral, which we can approximate with a succession of circles. This is how an accretion disk forms.

Since $M_{\text{disc}} \ll M_1$, the self-gravity of the accretion disk is negligible. Therefore, the azimuthal velocity of matter in the disk will closely match the Keplerian velocity

$$v_\phi = v_K = \sqrt{\frac{GM_1}{R}}\,. \tag{1.7.23}$$

We can already estimate the efficiency of the accretion process: the specific energy of the gas at the inner radius of the disk, R_{in}, which is the star radius for a NS and the ISCO for a BH. The specific energy is

$$\epsilon(R_{\text{in}}) = -\frac{GM_1}{R_{\text{in}}} + \frac{1}{2}v_K^2 = -\frac{1}{2}\frac{GM_1}{R_{\text{in}}}\,. \tag{1.7.24}$$

The variation of the energy can be calculated starting from infinity since $R_1 \gg R_{\text{in}}$:

$$\epsilon_\infty - \epsilon(R_{\text{in}}) = \frac{1}{2}\frac{GM_1}{R_{\text{in}}}\,, \tag{1.7.25}$$

therefore the luminosity of the disk will be

$$L_{\text{disc}} = \frac{1}{2}\frac{GM_1}{R_{\text{circ}}}\dot{M}c^2\,, \tag{1.7.26}$$

only half of the accretion luminosity, defined as

$$L_{\text{acc}} = \frac{GM_1}{R_{\text{in}}}\dot{M}c^2\,. \tag{1.7.27}$$

Now we want to make more detailed predictions. A key point is viscosity: friction between the various gas elements.

Let us consider two layers of the disk. They will have a macroscopic bulk motion, with $v_\phi = \Omega R$, superimposed with a microscopic motion which can be at very small, up to mesoscopic scales. We can have micro-scale motion of ions, but also medium-scale structures can form: turbulent eddies.

Suppose we have an eddy which starts in A, moves radially, and then dissipates in A'. Further, let us say that the length scale of its motion is λ, and its typical velocity is $\overline{v}$.

Its radius and velocity at A will be $R, \Omega(R) \times R$; at A' they will be $R + \lambda$ and still $\Omega(R) \times R$.

In terms of specific angular momentum, when the eddy dies it will dissipate angular momentum $(R + \lambda)R\Omega(R)$.

For an eddy moving from B to B' in the opposite direction we will have $R(R+\lambda)\Omega(R+\lambda)$. Since the motion is thermal, on average there will be as many particles going in both direction.

But there is more volume at higher R, so more matter!

Suppose that the height of the disk is H, then the mass carried by the eddies will be $H2\pi R\overline{v}\rho$.

Then, the variation of angular momentum will be

$$\frac{\Delta L}{\Delta t} = 2\pi R H \rho \overline{v}\big[R(R+\lambda)\Omega(R) - R(R+\lambda)\Omega(R+\lambda)\big] \tag{1.7.28}$$

$$\approx -2\pi R^2(R+\lambda)H\rho\overline{v}\frac{\mathrm{d}\Omega}{\mathrm{d}R}\lambda \tag{1.7.29}$$

$$\approx -2\pi R^3 H\rho\overline{v}\lambda\frac{\mathrm{d}\Omega}{\mathrm{d}R}, \tag{1.7.30}$$

and if we introduce the surface density of the disk:

$$\Sigma = \int_{-H/2}^{H/2} \rho\,\mathrm{d}z \approx \rho H \tag{1.7.31}$$

we can write this torque as

$$\frac{\Delta L}{\Delta t} = \tau \approx -2\pi R\Sigma(\overline{v}\lambda)R^2\frac{\mathrm{d}\Omega}{\mathrm{d}R}. \tag{1.7.32}$$

For a Keplerian accretion disk $\mathrm{d}\Omega/\mathrm{d}R < 0$, since $\Omega_K = \sqrt{GM/R}$.

This is the torque which the inner part of the disk exerts on the outer part, decelerating it. We can then introduce a function $G(R) = -\tau$, the torque exerted by the outer part of the disk on the inner part, accelerating it.

For a given layer rotating at $R\Omega(R)$, the layers above it will try to accelerate it, while the ones below it will try to decelerate it. What will be the net effect? It will be

$$G(R+\mathrm{d}R) - G(R) = \frac{\mathrm{d}G}{\mathrm{d}R}\,\mathrm{d}R\,. \tag{1.7.33}$$

These opposite effects will dissipate heat. The differential work dissipated will be given by

$$\mathrm{d}W = \tau\,\mathrm{d}\phi = \frac{\mathrm{d}G}{\mathrm{d}R}\,\mathrm{d}R\,\mathrm{d}\phi\,, \tag{1.7.34}$$

so the power will be

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}R}\,\mathrm{d}R\,\Omega\,. \tag{1.7.35}$$

Integrating to find the total power we get

$$\dot{E} = \int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}} \frac{\mathrm{d}G}{\mathrm{d}R}\Omega\,\mathrm{d}R\,, \tag{1.7.36}$$

but we can integrate by parts to find

$$\dot{E} = G\Omega\Big|_{R_{\mathrm{in}}}^{R_{\mathrm{out}}} - \int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}} G\frac{\mathrm{d}\Omega}{\mathrm{d}R}\,\mathrm{d}R\,, \tag{1.7.37}$$

so we can identify a global, *convective term*: the variation of $G\Omega$. On the other hand $G\,\mathrm{d}\Omega/\mathrm{d}R\,\mathrm{d}R$ is a local dissipation term.

Let us introduce the radiated power per unit area of the disk (which is positive, we leave the minus sign out):

$$D(R) = G\frac{\mathrm{d}\Omega}{\mathrm{d}R}\frac{1}{2\times 2\pi R\,\mathrm{d}R} = \frac{G}{4\pi R}\frac{\mathrm{d}\Omega}{\mathrm{d}R}\,. \tag{1.7.38}$$

Divided by 2 since the disk has two faces.

This is written as

$$D(R) = \frac{G}{4\pi R}\frac{\mathrm{d}\Omega}{\mathrm{d}R} = \frac{1}{2}R^2\overline{v}\lambda\Sigma\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2. \tag{1.7.39}$$

In order to dissipate energy the differential rotation $\mathrm{d}\Omega/\mathrm{d}R$ is crucial. We see next time that $\overline{v}\lambda = \nu$, the kinematic viscosity coefficient.

Bibliography

[Ber+05] Emanuele Berti et al. *Rotating Neutron Stars: An Invariant Comparison of Approximate and Numerical Spacetime Models*. Jan. 11, 2005. DOI: 10.1111/j.1365-2966.2005.08812.x/abs/. arXiv: gr-qc/0405146. URL: http://arxiv.org/abs/gr-qc/0405146 (visited on 2020-10-21).

[Bon52] H. Bondi. "On Spherically Symmetrical Accretion". In: *Monthly Notices of the Royal Astronomical Society* 112 (1952), p. 195. DOI: 10.1093/mnras/112.2.195. URL: http://adsabs.harvard.edu/abs/1952MNRAS.112..195B (visited on 2020-10-27).

[Cha98] Subrahmanyan Chandrasekhar. *The Mathematical Theory of Black Holes*. Clarendon Press, 1998. 676 pp. ISBN: 978-0-19-850370-5. Google Books: LBOVcrzFfhsC.

[JR05] Nils Voje Johansen and Finn Ravndal. *On the Discovery of Birkhoff's Theorem*. Version 2. Sept. 6, 2005. arXiv: physics/0508163. URL: http://arxiv.org/abs/physics/0508163 (visited on 2020-03-08).

[Ker63] Roy P. Kerr. "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics". In: *Physical Review Letters* 11.5 (Sept. 1, 1963), pp. 237–238. DOI: 10.1103/PhysRevLett.11.237. URL: https://link.aps.org/doi/10.1103/PhysRevLett.11.237 (visited on 2020-10-14).

[NTZ91] Luciano Nobili, Roberto Turolla, and Luca Zampieri. "Spherical Accretion onto Black Holes - A Complete Analysis of Stationary Solutions". In: *The Astrophysical Journal* 383 (Dec. 1, 1991), pp. 250–262. DOI: 10.1086/170781. URL: http://adsabs.harvard.edu/abs/1991ApJ...383..250N (visited on 2020-10-27).

[Tis19] Jacopo Tissino. "Relativistic Non-Ideal Flows". Laurea triennale. Sept. 9, 2019. URL: http://tesi.cab.unipd.it/63377/ (visited on 2020-03-04).