Theoretical physics notes

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Contents

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General information

Written & oral exam.

As opposed to what was done in the physics curriculum, here there will be no grade truncation: we can grade a grade up to a 30 with the written exam only.

The suggested book is D'Auria & Trigiante [DT11]. For the second part it is also useful to have a look at Mandl & Shaw [MS10].

Live question time in Zoom at half past 11 on Mondays.

Things which will be taken for granted: four-vectors, Lorenz and Poincaré groups, basics of QM, basics of linear operators.

Contents

This course will deal with the basics of Relativistic Quantum Field Theory.

We will discuss the Lagrangian formalism for a Classical Field Theory. We will quantize these theories using canonical quantization, specifically for a scalar, a Dirac fermion, and a vector boson.

Then, we will introduce interactions in our Lagrangian: we will use the S-matrix expansion, and Feynman diagrams.

Chapter 1

Relativistic wave equations

1.1 The nonrelativistic wave equation

We will review the derivation of the nonrelativistic Schrödinger equation. We find it starting from the correspondence principle: we start from the expression of the energy

$$E = \frac{p^2}{2m} + V(x)\,, \tag{1.1}$$

and substitute the energy with $E \to i\partial_t$, the momentum with $\vec{p} \to -i\vec{\nabla}_x$ and the position with the position operator $\vec{x}$, all acting on the wavefunction. With this we get

$$i\frac{\partial\psi}{\partial t} = \left(\frac{-\nabla^2}{2m} + V(\vec{x})\right)\psi(\vec{x},t)\,. \tag{1.2}$$

We still need to assign a meaning to the wavefunction: this is given by the Bohr condition, which tells us that the probability density of finding the particle in a specific region is

$$\rho(\vec{x},t) = \left|\psi(\vec{x},t)\right|^2 \geq 0\,. \tag{1.3}$$

This probability density must be normalized as an initial condition:

$$\mathbb{P}(t_0) = \int_{\mathbb{R}^3} \mathrm{d}^3x\, \rho(\vec{x},t_0) = 1\,, \tag{1.4}$$

and we wish to show that it will also be normalized at later times:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3x\, \frac{\partial}{\partial t}\left|\psi(\vec{x},t)\right|^2 \tag{1.5a}$$

$$= \int_{\mathbb{R}^3} \mathrm{d}^3x \left(\psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi\right). \tag{1.5b}$$

Using the Schrödinger equation we can substitute in the expression for the derivative of the wavefunction:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{\psi^*\frac{1}{i}\left(-\frac{\nabla^2}{2m} + V\right)\psi - \frac{1}{i}\psi\left(-\frac{\nabla^2}{2m} + V\right)\psi^*\right\} \tag{1.6a}$$

$$= \frac{i}{2m} \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \nabla^2 \psi - 2m\psi^* V\psi - \psi \nabla^2 \psi^* + 2m\psi V \psi^* \right\}, \tag{1.6b}$$

and we use the fact that

$$\psi^* V \psi = \psi V \psi^* = \left(\psi^* V \psi\right)^*, \tag{1.7}$$

which is true since V is a symmetric operator: it has real eigenvalues. This allows us to simplify the terms which include V, and we find:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = \frac{i}{2m} \int_{\mathbb{R}^3} \mathrm{d}^3x \left\{ \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right\} \tag{1.8a}$$

$$= \frac{i}{2m} \int_{\mathbb{R}^3} \mathrm{d}^3x \, \nabla_{\vec{x}} \cdot \left[\psi^* \vec{\nabla} \psi - \psi \left(\vec{\nabla} \psi^* \right) \right], \tag{1.8b}$$

where we integrated by parts[1] so we can define

$$\vec{j}(\vec{x}, t) = -\frac{i}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right), \tag{1.11}$$

so that our equation now reads

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}t} = -\int_{\mathbb{R}^3} \mathrm{d}^3x \, \vec{\nabla}_x \cdot \vec{j} = \int_{\partial \mathbb{R}^3} \vec{j} \cdot \hat{n} \, \mathrm{d}^2x = 0, \tag{1.12}$$

since the wavefunction is integrable: that is, it goes to zero *quickly* as $|\vec{x}| \to \infty$. Therefore, $\left|\vec{j}\right| \to 0$ as $|\vec{x}| \to \infty$. For a more detailed explanation, see the Quantum Mechanics notes by Manzali [Man19, page 147].

So, if the probability is equal to one at a certain time than it keeps being equal to one.

We can express this as a differential equation for the integrand: the *continuity equation*,

$$\frac{\partial}{\partial t} \left| \psi(\vec{x}, t) \right|^2 + \vec{\nabla} \cdot \vec{j} = 0. \tag{1.13}$$

Let us now consider the way to solve the free Schrödinger equation:

$$i \frac{\partial \psi}{\partial t} = -\frac{\nabla^2 \psi}{2m}. \tag{1.14}$$

[1] The calculation, expressed using index notation (and the Einstein summation convention) for clarity, is as follows:

$$\psi^* \partial_i \partial^i \psi = \partial_i \left(\psi^* \partial^i \psi \right) - (\partial_i \psi^*)(\partial^i \psi) \tag{1.9}$$

and similarly for the other term. The terms which come out as the products of two gradients, $(\partial_i \psi^*)(\partial^i \psi)$, are equal for both the terms, so they simplify. Then, we are left with

$$\psi^* \partial_i \partial^i \psi - \psi \partial_i \partial^i \psi^* = \partial_i \left(\psi^* \partial^i \psi - \psi \partial^i \psi^* \right). \tag{1.10}$$

We start from an ansatz of the equation being factorizable: $\psi(\vec{x},t) = \chi(t)\varphi(\vec{x})$. So, we get

$$i\frac{\partial \psi_0}{\partial t} = \varphi(\vec{x}) i\frac{\partial \chi}{\partial t} \tag{1.15}$$

on the LHS, and

$$H_0(\psi) = -\chi(t)\frac{\vec{\nabla}^2}{2m}\varphi(\vec{x}) \tag{1.16}$$

on the RHS. Dividing both by $\psi = \chi\varphi$ we get

$$i\frac{1}{\chi}\frac{\partial \chi}{\partial t} = -\frac{1}{\varphi}\frac{\vec{\nabla}^2}{2m}\varphi\,, \tag{1.17}$$

and since these are dependent only on time (for the LHS) and only on position (for the RHS) they must be separately constant: let us call their value E. Therefore, we can integrate them to get

$$\frac{\partial \chi}{\partial t} = -iE\chi \implies \chi(t) = \chi(0)\exp(-iEt) \tag{1.18}$$

and

$$\nabla^2\varphi = -2mE\varphi \implies \varphi(\vec{x}) = \varphi(0)\exp\Big(i\vec{k}\cdot\vec{x}\Big)\,. \tag{1.19}$$

Here , $\vec{k}$ is a 3D vector such that $\left|\vec{k}\right|^2 = 2mE$.

This is called the *dispersion relation*. So, the full solution, which is called a *monochromatic* solution, is

$$\psi(\vec{x},t) = \exp\left(-i\Big(Et - \vec{k}\cdot\vec{x}\Big)\right), \tag{1.20}$$

where $\left|\vec{k}\right|^2 = 2mE$.

The general solution will be a continuous superposition of solutions of this form:

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\,\widetilde{\varphi}(\vec{k})\,\exp\left(-i\Big(\omega_k - \vec{k}\cdot\vec{x}\Big)\right)\Big|_{\omega_k = \frac{\left|\vec{k}\right|^2}{2m}}\,. \tag{1.21}$$

Our conventions for the Fourier transform are:

$$\varphi(\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\,\widetilde{\varphi}(\vec{k})\exp\Big(-i\vec{k}\cdot\vec{x}\Big)\widetilde{\varphi}(\vec{x}) \quad = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\,\varphi(\vec{k})\exp\Big(i\vec{k}\cdot\vec{x}\Big), \tag{1.22}$$

so we use the symmetric definition. Other conventions have factors $(2\pi)^{-3}$ on one side and nothing on the other; it is the same but the we must be consistent.

It is a theorem that $\left|\varphi\right|^2 = \left|\widetilde{\varphi}\right|^2$, where the square norm of φ, $\left|\varphi\right|^2$, is just the integral of $\varphi^*\varphi$ over all 3D space.

The 3D dirac delta function is defined as

$$\delta^3(\vec{x}-\vec{y}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k \exp\left(-i\vec{k}\cdot(\vec{x}-\vec{y})\right), \tag{1.23}$$

and the 3D delta in the momentum space is perfectly analogous.

The Schrödinger equation is manifestly *non relativistic*: we started from the nonrelativistic expression $E = p^2/2m + V$, so we should expect so. In the differential equation we have a second spatial derivative and a first temporal derivative: there is no way to write such an equation covariantly.

This kind of law of physics is only invariant under *galilean transformations*, which do not change time.

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1.2 Conventions

1.2.1 Natural units

Two constants which often come up in theoretical physics are Planck's constant $\hbar = h/2\pi \approx 6.582 \times 10^{-22}\,\mathrm{MeV/Hz}$ and the speed of light $c \approx 2.997 \times 10^8\,\mathrm{m/s}$. They are used to convert quantities which are *equivalent*: a length is equivalent to a time interval if light passes through that length in that time interval in a vacuum; an energy is equivalent to an angular velocity if a photon with that angular velocity has that energy.

So, we can express lengths in "seconds $\times c$", energies in "kilograms times c^2" or "Hertz times $\hbar$", and so on. Since this is very convenient, we go one step further and do not write the c and the $\hbar$. This allows us to not worry about the number of times these should appear in a formula.

Formally, we do this by imposing the conditions $\hbar = c = 1$, where 1 is adimensional. Then, we can select a common unit and use it for everything: a common choice is the electronVolt (or its multiples).

Some examples of physical quantities in natural units:

1. masses and energies are both measured in eV (for masses, we should multiply by c^2);
2. linear momenta $p = mv$ are measured in eV (times c);
3. angular momenta $L = r \wedge p$ are adimensional (they could be expressed in units of $\hbar$);
4. times and lengths are both measured in eV^{-1}.

This shortens formulas, but it obscures their dimensionality. Thankfully we can always reinsert the necessary cs and $\hbar$s by dimensional analysis.

1.2.2 Relativistic notation

A contravariant vector is denoted by writing its components,

$$v^\mu = \begin{bmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{bmatrix}, \tag{1.24a}$$

and examples of these include the position 4-vector $x^\mu = (t, \vec{x})$, and the 4-momentum $p^\mu = (E, \vec{p})$.

We shall use the worse convention for the metric, that is, the mostly negative $(+,-,-,-)$ one. This allows us to obtain covariant vectors as

$$x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x})\,. \tag{1.25}$$

The derivative operator, instead, is naturally covariant:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \vec{\nabla})\,. \tag{1.26}$$

1.3 The Klein Gordon equation

We shall use the correspondence principle, as we did for the Schrödinger equation; however this time we will apply it to a relativistic particle. Its 4-momentum has a modulus square equal to the square of its mass: M^2, since it is a relativistic invariant and we may compute it in any reference frame we like. In a generic frame, it is

$$M^2 = p^\mu p_\mu = p^\mu \eta_{\mu\nu} p^\nu = E^2 - \left|\vec{p}\right|^2, \tag{1.27}$$

which allows us to write the *dispersion relation*

$$E^2 = \vec{p}^2 + M^2\,. \tag{1.28}$$

This is *quadratic* in the energy, as opposed to the nonrelativistic expression $E = m + mv^2/2$.

Applying the correspondence principle, we find

$$-\partial_t^2 \varphi(\vec{x}, t) = \left(-\nabla^2 + M^2\right)\varphi(\vec{x}, t) \tag{1.29a}$$

$$0 = \left[\left(\partial_t^2 - \nabla^2\right) + M^2\right]\varphi(\vec{x}, t) \tag{1.29b}$$

$$= \left[\Box + M^2\right]\varphi(\vec{x}, t)\,, \tag{1.29c}$$

where we defined $\Box = \partial^\mu \partial_\mu = \partial_t^2 - \nabla^2$. Another way to see this is that, compactly stated, the correspondence principle is $p_\mu = i\partial_\mu$,[2] so from $p^2 = M^2$ we get

$$\left[\Box + M^2\right]\varphi(\vec{x}, t) = 0\,. \tag{1.30}$$

[2] So we get $E = p_0 = p^0 = i\partial_t$, while $\vec{p} = p^i = -p_i = -i\vec{\nabla} = -i\partial_i$.

This is the *free Klein-Gordon relativistic equation*.

Let us check its covariance *a posteriori*, even though it is guaranteed since we started from a covariant relation. Both $M^2 = p^\mu p_\mu$ and $\Box = \partial^\mu \partial_\mu$ are scalars.

What about the wavefunction φ? It should transform under a Lorentz transformation $x \to x' = \Lambda x$ as

$$\varphi'(x') = \varphi(x)\,. \tag{1.31}$$

This amounts to saying that it is a *scalar* function. With these constraints, we can say that the Klein-Gordon equation is *covariant*; moreover, since it is a scalar equation it is actually *invariant*. If we apply a Lorentz transformation, we find that indeed

$$\Big[\Box' + M'^2\Big]\varphi'(x') = \Big[\Box + M^2\Big]\varphi(x)\,, \tag{1.32}$$

since $\Box$, M^2 and $\varphi(x)$ are scalars; therefore covariance is proven. If this is zero in an inertial reference frame, it is also zero in any other inertial reference frame.

1.3.1 Continuity equation

Now, let us seek a probability current for the KG equation as we did with the Schrödinger equation. Let us multiply on the left the KG equation by the conjugate of the wavefunction, φ^*. We shall use this equation and its conjugate:

$$\varphi^*\Big[\Box + M^2\Big]\varphi = 0 \tag{1.33a}$$

$$\varphi\Big[\Box + M^2\Big]\varphi^* = 0\,, \tag{1.33b}$$

these both hold since $\Box$ and M^2 are real. If we subtract one of these from the other, the mass terms simplify and we get

$$0 = \varphi^*\Box\varphi - \varphi\Box\varphi^* \tag{1.34a}$$

$$= \varphi^*\partial^\mu\partial_\mu\varphi - (\varphi^* \leftrightarrow \varphi)\,, \tag{1.34b}$$

where the notation $-(\varphi^* \leftrightarrow \varphi)$ means that we are subtracting the same thing which appears before, but written swapping φ and φ^*. We can expand the derivatives:

$$0 = \partial^\mu\Big(\varphi^*\partial_\mu\varphi\Big) - \big(\partial^\mu\varphi^*\big)\Big(\partial_\mu\varphi\Big) - (\varphi^* \leftrightarrow \varphi) \tag{1.35a}$$

$$= \partial^\mu\Big(\varphi^*\partial_\mu\varphi\Big) - (\varphi^* \leftrightarrow \varphi) \tag{1.35b}$$

$$= \partial^\mu\Big(\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*\Big) \overset{\text{def}}{=} -2i\partial^\mu j_\mu\,, \tag{1.35c}$$

The product of the gradients is symmetric under interchange of φ and φ^*, since the metric is constant.

where we defined the 4-current

$$j_\mu = \frac{i}{2}\varphi^*\partial_\mu\varphi - (\varphi^* \leftrightarrow \varphi)\,, \tag{1.36}$$

so we can write the conservation equation $\partial^\mu j_\mu = \partial_\mu j^\mu = 0$ in 3-vector form as:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j}, \tag{1.37}$$

where

$$\rho = \frac{i}{2}\varphi^* \partial_t \varphi - (\varphi^* \leftrightarrow \varphi) \qquad \text{and} \qquad \vec{j} = -\frac{i}{2}\varphi^* \partial^i \varphi - (\varphi^* \leftrightarrow \varphi)\,. \tag{1.38}$$

Minus sign since we want the contravariant components.

We can integrate the continuity equation over all 3D space to obtain a conserved quantity:

$$\int_{\mathbb{R}^3} \mathrm{d}^3x\, \rho \overset{\text{def}}{=} Q\,, \tag{1.39}$$

which is actually constant since

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3x\, \partial_t \rho = -\int_{\mathbb{R}^3} \mathrm{d}^3x\, \vec{\nabla} \cdot \vec{j} = -\int_{S^2_\infty} \vec{j} \cdot \hat{n}\, \mathrm{d}A \to 0\,. \tag{1.40}$$

Used the divergence theorem, the surface S^2_∞ is a sphere with diverging radius.

We call this conserved quantity a "charge". There is an issue: in the Schrödinger case the quantity called ρ was positive by definition; now instead ρ and Q are not necessarily positive.

This can be proven as follows: suppose ρ was positive for a certain wavefunction φ. The conjugate wavefunction φ^* is also a solution to the KG equation, and for it the density will be negative, since permuting φ and φ^* is equivalent to changing the sign of j_μ.

So, we cannot use the Bohr ansatz, interpreting Q as a probability.

1.3.2 Solutions to the free KG equation

Let us forget about the physical interpretation for a while, and discuss the general solutions of the KG equation. We can decompose the wavefunction $\varphi(x)$ in terms of its Fourier transform $\widetilde{\varphi}(k)$:

$$\varphi(x) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4k\, e^{-ik^\mu x_\mu} \widetilde{\varphi}(k)\,. \tag{1.41}$$

In order for this to be covariant, if φ is a scalar then $\widetilde{\varphi}$ also must be. The argument of the exponential is also a scalar.

The volume form in the momentum space d^4k is a scalar: under a Lorentz transformation it transforms as

$$\mathrm{d}^4k' = |\det \Lambda|\, \mathrm{d}^4k\,, \tag{1.42}$$

so it does not change since $|\det \Lambda| = 1$ for Lorentz transformations.

Claim 1.3.1. *The inverse Fourier transform reads*

$$\widetilde{\varphi}(k) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4x\, e^{ik^\mu x_\mu} \varphi(x)\,. \tag{1.43}$$

Proof. We take the transform of the antitransform:

$$\widetilde{\varphi}(k) = \frac{1}{(2\pi)^2}\int \mathrm{d}^4x\,\varphi(x)e^{ik^\mu x_\mu} \tag{1.44a}$$

$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4x\,e^{ik^\mu x_\mu}\int \mathrm{d}^4k'\,e^{-ik'^\mu x_\mu}\varphi(k') \tag{1.44b}$$

$$= \int \mathrm{d}^4k'\left[\frac{1}{(2\pi)^4}\int \mathrm{d}^4x\,e^{-i(k'^\mu-k^\mu)x_\mu}\right]\varphi(k') \tag{1.44c}$$

$$= \int \mathrm{d}^4k'\,\delta^{(4)}(k-k')\varphi(k') = \varphi(k)\,, \tag{1.44d}$$

where we used the definition of the 4D Dirac δ function (here in position space, the definition in momentum space is perfectly analogous):

$$\delta^{(4)}(x^\mu) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,e^{-ik_\mu x^\mu}\,, \tag{1.45}$$

and its main property:

$$\int \mathrm{d}^4x\,\delta^{(4)}(x)f(x) = f(0)\,. \tag{1.46}$$

□

Now, to solve the KG equation we insert the Fourier expression of the wavefunction into it:

$$0 = \left[\Box + M^2\right]\varphi(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^2}\left[-k^2+M^2\right]e^{-ik^\mu x_\mu}\widetilde{\varphi}(k)\,, \tag{1.47}$$

The $\Box$ operator acts in position space, so it has no effect on $\widetilde{\varphi}(k)$: it applies only to the exponential, yielding $-k^2 = (-ik^\mu)(-ik_\mu)$.

and since the integral must be zero the integrand must be zero as well. This is because the monochromatic waves are a basis for the Hilbert space, and the Fourier transform is an isomorphism of Hilbert spaces, which maps the zero function to the zero function.

So, we can use the ansatz $\widetilde{\varphi}(k) = \delta(k^2-M^2)\widetilde{f}(k)$, where $\widetilde{f}(k)$ is a generic scalar function. If $\widetilde{\varphi}(k)$ is written in this way, it automatically satisfies the KG equation.

Now, recall that for a Dirac delta function applied to a generic function $f(x)$, whose zeroes are enumerated by the index i (that is, $f(x_i) = 0$ for all i between 1 and N) the following property holds:

$$\delta(f(x)) = \sum_{i=1}^{N}\frac{\delta(x-x_i)}{|f'(x_i)|}\,. \tag{1.48}$$

We can apply this property to $\delta(k^2-M^2)$. First of all, since $k^2 = k_0^2 - \left|\vec{k}\right|^2$, we can write this expression as $\delta(k_0^2-\omega_k^2)$, by defining ω_k as the positive root:

$$\omega_k = +\sqrt{|k|^2+M^2}\,. \tag{1.49}$$

Now, we apply the δ function property:

$$\delta(k^2 - M^2) = \delta(k_0^2 - \omega_k^2) = \frac{\delta(k_0 - \omega_k)}{|2k_0|} + \frac{\delta(k_0 + \omega_k)}{|2k_0|} \tag{1.50a}$$

$$= \frac{\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)}{2\omega_k}\,, \tag{1.50b}$$

where we substituted $|k_0| = \omega_k$, which holds both if $k_0 = \omega_k$ and if $k_0 = -\omega_k$. This finally gives us

$$\widetilde{\varphi}(k) = \frac{1}{2\omega_k}\big(\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)\big)\widetilde{f}(k)\,, \tag{1.51}$$

which we can insert this into the Fourier transform of $\varphi(x)$:

$$\varphi(x) = \frac{1}{(2\pi)^2}\int \mathrm{d}^4k\, \widetilde{\varphi}(k) e^{-ik^\mu x_\mu} \tag{1.52a}$$

$$= \frac{1}{(2\pi)^2}\int \frac{\mathrm{d}^4k}{2\omega_k}\big(\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)\big)e^{-ik^\mu x_\mu}\widetilde{f}(k) \tag{1.52b}$$

$$= \frac{1}{(2\pi)^2}\int \frac{\mathrm{d}^3k}{2\omega_k}\Big[e^{-i\omega_k x_0}e^{i\vec{k}\cdot\vec{x}}\widetilde{f}(\omega_k, \vec{k}) + e^{i\omega_k x_0}e^{i\vec{k}\cdot\vec{x}}\widetilde{f}(-\omega_k, \vec{k})\Big] \tag{1.52c}$$

We integrate in $\mathrm{d}k^0$ to get rid of the δ functions.

$$= \frac{1}{(2\pi)^2}\int \frac{\mathrm{d}^3k}{2\omega_k}\Big[e^{-ik^\mu x_\mu}\widetilde{f}(k^\mu) + e^{ik^\mu x_\mu}\widetilde{f}(-k^\mu)\Big]_{k_0=\omega_k}\,, \tag{1.52d}$$

where we indicate $k^\mu\big|_{k_0=\omega_k} = (\omega_k, \vec{k})$.

We used the fact that, in the Fourier transform integral, the terms $e^{i\vec{k}\cdot\vec{x}}\widetilde{f}(\vec{x})$ and $e^{-i\vec{k}\cdot\vec{x}}\widetilde{f}(-\vec{x})$ are equivalent: this is because, since we are integrating over all of 3D space, any contributions which are *odd* in $\vec{k}$ will not affect the total integral, therefore we can only consider the even part of the integrand.

Now, in order to simplify the notation we define

$$a(k) = \frac{\widetilde{f}(k)}{\sqrt{2\pi}\sqrt{2\omega_k}} \qquad \text{and} \qquad b(k) = \frac{\widetilde{f}(-k)}{\sqrt{2\pi}\sqrt{2\omega_k}}\,, \tag{1.53}$$

which are arbitrary like the initial function $\widetilde{f}$, however they are connected since $a(k) = b^*(-k)$. So, the final solution reads:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\Big[a(k)e^{-ik\cdot x} + b^*(k)e^{ik\cdot x}\Big]_{k_0=\omega_k} \tag{1.54a}$$

$$= \varphi_+(x) + \varphi_-(x)\,. \tag{1.54b}$$

Recall that all ks appearing in the expression are to be interpreted as $(\omega_k, \vec{k})$. The part dependent on $a(k)$ is conventionally called the *positive energy* solution φ_+, while the part depending on $b^*(k)$ is the *negative energy* solution φ_-.

This is because, as the energy of a wavefunction φ is computed by $E = i\partial_t\varphi$, we have

$$E(\varphi_+) = i\partial_0\left\{\frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}a(k)e^{-ik^\mu x_\mu}\right\} = i(-i)\omega_k\varphi_+ = \omega_k\varphi_+ \tag{1.55a}$$

$$E(\varphi_-) = i\partial_0 \left\{ \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k}{\sqrt{2\omega_k}} b^*(k) e^{ik^\mu x_\mu} \right\} = i(i)\omega_k \varphi_- = -\omega_k \varphi_- \,, \tag{1.55b}$$

so φ_+ has a positive energy while φ_- has a negative one.

This is the main difference between the Schrödinger and KG equations.

The solution to the KG equation is not explicitly covariant, but all the steps preserved covariance so the final solution is still covariant.

The KG equation is real, since $\Box$ is a real operator and M^2 is real, so it will admit real solutions. In order to find these we impose $\varphi = \varphi^*$.

Claim 1.3.2. *$\varphi = \varphi^*$ implies $a = b$.*

Proof. We write only the argument of the integrals for simplicity:

$$\varphi \sim a e^{-ikx} + b^* e^{ikx} \tag{1.56a}$$

$$\varphi^* \sim a^* e^{ikx} + b e^{-ikx} \,, \tag{1.56b}$$

so if $\varphi = \varphi^*$ we must identify these component by component, so we must have $a = b$, and $a^* = b^*$. □

Then, the most general real solution to the KG equation is

$$\varphi_{\mathbb{R}}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k}{\sqrt{2\omega_k}} \left[a(k) e^{-ik\cdot x} + a^*(k) e^{ik\cdot x} \right]_{k_0=\omega_k} . \tag{1.57}$$

Claim 1.3.3. *For a real KG solution, the function $a(k)$ can be written as*

$$a(k) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 x}{\sqrt{2\omega_k}} \left(\omega_k \varphi(x) + i\partial_0 \varphi(x) \right) e^{ik\cdot x} \Big|_{k_0=\omega_k} . \tag{1.58}$$

Proof. The solution is found by direct substitution of $\varphi_{\mathbb{R}}$ into the expression for a in order to verify it; the operations are all reversible so we can use the derivation backwards or forwards equivalently. We find

$$a \overset{?}{=} \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 x}{2\omega_k} \left[\omega_k \left(a e^{-ik\cdot x} + a^* e^{ik\cdot x} \right) + i\partial_0 \left(a e^{-ik\cdot x} + a^* e^{ik\cdot x} \right) \right] e^{ik\cdot x} \Big|_{k_0=\omega_k} \tag{1.59a}$$

$$= \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 x}{2\omega_k} \left[\left(\omega_k + i(-ik_0) \right) a^{-ik\cdot x} + \left(\omega_k + i(ik_0) \right) a^* e^{ik\cdot x} \right] e^{ik\cdot x} \Big|_{k_0=\omega_k} \tag{1.59b}$$

$$= \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 x}{2\omega_k} \left[2\omega_k a^{-ik\cdot x} \right] e^{ik\cdot x} \Big|_{k_0=\omega_k} \tag{1.59c}$$

Used the fact that $\omega_k = k_0$, and $i^2 = -1$.

$$= \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 x\, e^{i\vec{k}\cdot\vec{x}} \left[\frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k\, e^{-i\vec{k}\cdot\vec{x}} a \right] = a \,. \tag{1.59d}$$

3D Fourier inverse and direct transform. We simplified two factors of $e^{ik_0 x^0}$ and $e^{-ik_0 x^0}$ since they are equal everywhere ($k_0 = \omega_k$).

□

Claim 1.3.4. *For a complex KG solution, the functions $a(k)$ and $b^*(k)$ can be written as*

$$a(k) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3x}{\sqrt{2\omega_k}} \left(\omega_k \varphi(x) + i\partial_0 \varphi(x)\right) e^{ik\cdot x}\Big|_{k_0=\omega_k} \tag{1.60a}$$

$$b^*(k) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3x}{\sqrt{2\omega_k}} \left(\omega_k \varphi(x) - i\partial_0 \varphi(x)\right) e^{-ik\cdot x}\Big|_{k_0=\omega_k} . \tag{1.60b}$$

Proof. The derivation is the same as the real-valued solution case. The b^* terms simplify if there is a plus in front of the $i\partial_0$ term, if instead we have a minus the a terms simplify; everything else is precisely the same. □

Claim 1.3.5. *Given two real solutions to the KG equation, φ_1 and φ_2, one can always write a complex solution $\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$.*

Then, the functions a and b^* for the complex solution can be written in terms of the a_1 and a_2 for the real solution as:

$$a = \frac{a_1 + ia_2}{\sqrt{2}} \tag{1.61a}$$

$$b = \frac{a_1 - ia_2}{\sqrt{2}} . \tag{1.61b}$$

Proof. We write out the complex function:

$$\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \left[\frac{a_1 + ia_2}{\sqrt{2}} e^{-ik\cdot x} + \frac{a_1^* + ia_2^*}{\sqrt{2}} e^{ik\cdot x}\right] , \tag{1.62}$$

so, since $b^* = \left(a_1^* + ia_2^*\right)/\sqrt{2}$, we can get b by conjugating,

$$b = \frac{a_1 - ia_2}{\sqrt{2}} , \tag{1.63}$$

while a can be directly read off the expression. □

Monday 2020-3-16, compiled 2020-06-01

1.3.3 The Klein-Gordon equation in the presence of an external electromagnetic field

A complex scalar solution to the Klein-Gordon equation can represent a charged relativistic spin-0 particle. Since the particle is charged, it is interesting to study its interaction with the electromagnetic field.

We will discuss only the interaction of the particle with an externally determined field, that is, we will not discuss how the particle influences the field around it.

In order to describe the electric and magnetic fields in a covariant way we use the four-vector $A_\mu = (A_0, \vec{A})$, such that $\vec{E} = -\vec{\nabla} A_0 - \partial_0 \vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, or, in covariant terms,

$$F^{\mu\nu} = 2\partial^{[\mu} A^{\nu]} , \tag{1.64}$$

where the antisymmetric field-strength tensor $F^{\mu\nu}$ encodes both the electric and magnetic fields.

Definition 1.3.1. *The **minimal coupling** to an external electromagnetic field is obtained by substituting in the wave equation the partial derivative ∂_μ with the* covariant derivative

$$\mathrm{D}_\mu \overset{def}{=} \partial_\mu + iqA_\mu\,, \tag{1.65}$$

where q is the electric charge of the particle.

Inserting this, the coupled Klein-Gordon equation reads

$$\left[\mathrm{D}^\mu \mathrm{D}_\mu + M^2\right]\varphi(x) = 0\,. \tag{1.66}$$

Recall that the momentum is defined as $p_\mu = i\partial_\mu$, so this substitution is equivalent to changing the momentum to $i(\partial_\mu + iqA_\mu) = p_\mu - qA_\mu$.

This is the same minimal coupling ansatz which is used in nonrelativistic quantum mechanics: there, the minimally-coupled Schrödinger equation reads

$$E\psi = \left[\frac{\left(\vec{p} - q\vec{A}\right)^2}{2M} + qA_0\right]\psi\,, \tag{1.67}$$

in which we can see the two contributions: the three-momentum $\vec{p}$ is exchanged for $\vec{p} - q\vec{A}$, while the energy E is exchanged for $E - qA_0$ (and the term is brought on the other side of the equation for convenience).

Let us make the terms in the minimally coupled Klein-Gordon equation explicit:

$$\left[\left(\partial^\mu + iqA^\mu\right)\left(\partial_\mu + iqA_\mu\right) + M^2\right]\varphi = 0 \tag{1.68a}$$

$$\left[\Box + iqA^\mu\partial_\mu + iq\partial_\mu A^\mu - q^2A^2 + M^2\right]\varphi = 0 \tag{1.68b}$$

$$\left[\Box + 2iqA^\mu\partial_\mu + iq\left(\partial_\mu A^\mu\right) - q^2A^2 + M^2\right]\varphi = 0\,, \tag{1.68c}$$

Do note that anything on the right acts on anything on the left: the application of the derivatives looks like $\mathrm{D}^\mu\left(\mathrm{D}_\mu(\varphi)\right)$. So, the term $\partial_\mu A^\mu$ should be read as $\partial_\mu\left(A^\mu\varphi\right)$, which we expanded into $\left(\partial_\mu A^\mu\right) + A^\mu\partial_\mu$ — all acting on φ, but the term $\left(\partial_\mu A^\mu\right)$ is just multiplied by φ, it does not perform a differentiation.

The Coulomb potential

The 4-potential is not physical: if we change it by $A_\mu \to A'_\mu = A_\mu + \partial_\mu\Lambda$ for some scalar function Λ it does not change the resulting measurable electric and magnetic fields. Therefore, while still retaining full generality in our description of physical systems we can impose certain conditions, such as $\partial_\mu A^\mu = 0$. This condition, known as the *Coulomb gauge*, does not actually fix all of the gauge freedom — that is, even by imposing this we still do not have a one-to-one correspondence between the physical fields and the 4-potentials: after

imposing it, we can still perform gauge transformations where the function Λ is harmonic, that is, it satisfies $\Box\Lambda = 0$.

This condition is convenient since it allows us to get rid of a term in the KG equation, so we impose it.

Antimatter: an interpretation for the negative-energy solution

Let us consider a simple case: the monochromatic solution to the KG equation, $\varphi_{\pm}(x) = e^{\mp ik\cdot x}$, coupled to the EM field. Let us suppose that the 4-momentum $k^\mu = (\omega_k, \vec{k})$ corresponding to the two solutions is fixed, that is, the positive energy solution has $E\varphi^+ = \omega_k$ while the negative energy solution has $E\varphi^- = -\omega_k$, with the same ω_k and the same $\vec{k}$.

Splitting the energy and momentum terms, the equation reads:

$$\left[\left(E - qA_0\right)^2 + \left(\vec{p} - q\vec{A}\right)^2 + M^2\right]\varphi^{\pm}(x) = 0\,, \tag{1.69}$$

where $E = i\partial_t$ while $\vec{p} = -i\vec{\nabla}$.

Then, we can write the two equations like

$$\left[\left(\omega_k - qA_0\right)^2 + \left(\vec{k} - q\vec{A}\right)^2 + M^2\right]\varphi^{+}(x) = 0 \tag{1.70a}$$

$$\left[\left(\omega_k + qA_0\right)^2 + \left(\vec{k} + q\vec{A}\right)^2 + M^2\right]\varphi^{-}(x) = 0\,, \tag{1.70b}$$

where we used the fact that $(-\omega_k - qA_0)^2 = (\omega_k + qA_0)^2$, and similarly for the other term.

So, we can see that the two equations can be transformed into each other also by swapping the charge, $q \to -q$.

The interpretation for this is that the KG equation describes two degrees of freedom, which we call a *particle* (with $E > 0$) and an *antiparticle* (with $E < 0$).

The particle and antiparticle have the same mass, but their charge is opposite. In a certain sense, the existence of antimatter is a prediction of relativistic quantum mechanics.

1.3.4 The nonrelativistic limit

As a consistency check for our relativistic equation, we wish to verify that the nonrelativistic limit of the KG equation is the Schrödinger equation.

In the nonrelativistic limit the mass of the particle M is much larger than its momentum p, so we can expand:

$$E = \sqrt{|p|^2 + M^2} = M\sqrt{1 + \frac{|p|^2}{M^2}} = M + \underbrace{\frac{|p|^2}{2M}}_{E_k} + \mathcal{O}\left(\frac{|p|^4}{M^3}\right). \tag{1.71}$$

In quantum mechanics the time evolution of a particle depends on its energy, so if $M \gg E_k$ the evolution due to the mass will be much faster than that due to the kinetic energy: so, we factor it out by

$$\varphi(t, \vec{x}) = e^{-iMt}\varphi'(t, \vec{x})\,, \tag{1.72}$$

where the evolution of φ' will look like $e^{-iE_k t}\varphi'$.

Now, let us expand the derivative terms in the KG equation (in the Coulomb gauge, so we do not worry about $\partial_\mu A^\mu$):

$$\mathrm{D}^\mu \mathrm{D}_\mu = \left(\partial^0 + iqA^0\right)\left(\partial_0 + iqA_0\right) + \left(\partial^i + iqA^i\right)\left(\partial_i + iqA_i\right) \tag{1.73a}$$

$$= \left(\partial_0 + iqA_0\right)^2 - \left(\partial_i + iqA_i\right)^2 \tag{1.73b}$$

$$= \left(\partial_0 + iqA_0\right)^2 - \left(\vec{\nabla} - iq\vec{A}\right)^2, \tag{1.73c}$$

where we used the fact that $\vec{A} = A^i$, while $\vec{\nabla} = \partial_i = -\partial^i$.

In the full KG equation applied to the decomposed field $\varphi = e^{-iMt}\varphi'$ we move the momentum terms to the right and keep the mass and energy on the left:

$$\left[\left(\partial_0 + iqA_0\right)^2 + M^2\right] e^{-iMt}\varphi' = \left(\nabla - iq\vec{A}\right)^2 e^{-iMt}\varphi' \tag{1.74a}$$

$$\left[\partial_0^2 + 2iqA_0\partial_0 - q^2A_0^2 + M^2\right] e^{-iMt}\varphi' = e^{-iMt}\left(\nabla - iq\vec{A}\right)^2 \varphi'. \tag{1.74b}$$

The term e^{-iMt} is constant with respect to the spatial coordinates

The time-derivative on the right acts both on the mass term and on the wavefunction φ'. So, we get[3]

$$e^{-iMt}\left(\nabla - iq\vec{A}\right)^2 \varphi' = \tag{1.77a}$$

$$= e^{-iMt}\left[\partial_0^2 - M^2 - 2iM\partial_0 + 2iqA_0(-iM) + iq(\partial_0 A_0) + 2iqA_0\partial_0 - q^2A_0^2 + M^2\right]\varphi'$$

$$= \left[\partial_0^2 - 2iM\partial_0 + 2qA_0M + iq(\partial_0 A_0) + 2iqA_0\partial_0 - q^2A_0^2\right]\varphi' = \left(\nabla - iq\vec{A}\right)^2\varphi', \tag{1.77b}$$

and now we must discuss which terms we can discard.

The coefficient before the term $\propto \partial_0 A_0$is reported by the professor as 2, but I and several other people find only one of those terms.

On the right hand side we have terms of the order of the mass and ones of the order of the energy: we wish to keep only the former, since as we assumed they dominate the latter. Specifically, terms without explicit dependence on the mass are of the order of the kinetic

[3] It is really easy when doing these computations to forget some terms. A way I've found helpful is to write down the full expression,

$$\left(\partial_0 + iqA_0\right)\left(\partial_0 + iqA_0\right)\left(e^{-iMt}\varphi'\right), \tag{1.75}$$

give binary labels (000, 001... until 111) to each term and work through all 2^3 of them. Notice the expansion of the term

$$\partial_0\partial_0\left(e^{-iMt}\varphi'\right) = \partial_0\left(-iMe^{-iMt}\varphi' + e^{-iMt}\partial_0\varphi'\right) = -iMe^{-iMt}\partial_0\varphi' - M^2e^{-iMt}\varphi' + e^{-iMt}\partial_0^2\varphi - iMe^{iMt}\varphi'. \tag{1.76}$$

energy:[4] then, we remove and get precisely the minimally coupled Schrödinger equation:

$$\left[-2iM\partial_0 + 2q_0A_0M\right]\varphi' = \left(\vec{\nabla} - iq\vec{A}\right)^2\varphi' \tag{1.78a}$$

$$i\partial_0\varphi' = \left[-\frac{\left(\vec{\nabla} - iq\vec{A}\right)^2}{2M} + q_0A_0\right]\varphi' \,. \tag{1.78b}$$

This is precisely the equation we wrote above: the nonrelativistic Schrödinger equation for a charged spin-0 particle in an electromagnetic field.

1.3.5 Summary

1. The dispersion relation $E^2 = p^2 + m^2$ is a quadratic relation, the positive root is called $\omega_p = \sqrt{p^2 + m^2}$ and the energy can be $E = \pm\omega_p$.
2. A solution to the KG equation has two independent terms φ^+ and φ^-, which can be interpreted as a particle-antiparticle pair, where φ^+ has charge q while φ^- has charge $-q$.

   An antiparticle has the same quantum numbers, but the opposite charge.
3. There is no way to define a conserved (positive definite) probability for the KG equation, but the nonrelativistic limit of the KG equation is consistent with nonrelativistic quantum mechanics.

Claim 1.3.6. *The continuity equation for the coupled KG equation is given by*

$$\partial_\mu j^\mu_{EM} = 0 \,, \tag{1.79}$$

where

$$j^\mu_{EM} = \frac{1}{2}\left(i\varphi^*\partial^\mu\varphi - qA^\mu\varphi^*\varphi\right) - c.\ c.\,. \tag{1.80}$$

Proof. Similarly to what was done before, we notice that

$$\varphi^*\left(\mathrm{D}^\mu\mathrm{D}_\mu + M^2\right)\varphi = \varphi^*\left[\Box + 2iqA^\mu\partial_\mu + iq\left(\partial_\mu A^\mu\right) - q^2A^2 + M^2\right]\varphi = 0 \tag{1.81}$$

can be added to the negative of its complex conjugate, which reads

$$\varphi\left[\Box - 2iqA^\mu\partial_\mu - iq\left(\partial_\mu A^\mu\right)\right]\varphi^* = 0 \,, \tag{1.82}$$

[4] This is imprecise, we can do better: we distinguish the terms with time derivatives and those without. Those with time derivatives are applied to φ', which by construction evolves with $\omega = E_k$, therefore they are negligible. The other terms are $q^2A_0^2$, which is the electric potential energy, and $2iq\partial_0A_0$, which is proportional to the electric field: we assume the electric field to be nonrelativistic, that is, of intensity comparable to the kinetic energy of the particle, and much smaller than its mass.

to yield

$$0 = \varphi^*\Box\varphi + 2iqA_\mu\varphi^*\partial^\mu\varphi + iq\left(\partial_\mu A^\mu\right)\varphi\varphi^* + \\ - \varphi\Box\varphi^* + 2iqA_\mu\varphi\partial^\mu\varphi^* + iq\left(\partial_\mu A^\mu\right)\varphi\varphi^* \tag{1.83a}$$

The q^2A^2 and M^2 terms cancel since they are real

$$= \partial^\mu\left(\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*\right) + 2iq\left[A_\mu\left(\varphi^*\partial^\mu\varphi + \varphi\partial^\mu\varphi^*\right) + \varphi\varphi^*\partial_\mu A^\mu\right] \tag{1.83b}$$

$$= \partial^\mu\left[\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^* + 2iqA_\mu\varphi\varphi^*\right], \tag{1.83c}$$

so the expression inside the brackets is the conserved current, up to a multiplicative factor of $i/2$. □

1.3.6 The Klein paradox

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The fact that it was not possible to derive a conserved charge for the KG equation is not just a mathematical inconvenience: we will use a gedankenexperiment to show the physical consequences of this.

Consider the scattering of a particle, which is described by a pure plane wave, on a an electromagnetic potential step. Suppose we are in a frame in which $qA^\mu = (V(x), \vec{0})$; if we are not in such a frame we can always perform a boost so that we are. Also suppose that the potential looks like a step: $V(x) = V_0[x \geq 0]$. Here "x" refers to a 1d spatial coordinate.

Now, suppose that we have an incoming wave with energy ω and momentum k_x in the $x < 0$ region. Upon impact, there will in general be a reflected wave in the $x < 0$ region and a transmitted wave in the $x > 0$ region. We will call the former φ_1 and the latter φ_2.

We can decompose both of these into a time and space exponential: $\varphi_i(t, x) = e^{-i\omega t}\chi_i(x)$, for $i = 1, 2$.

Now, the incoming wave contributes to the global wavefunction with an exponential $e^{ik_x x}$ in the $x < 0$ region. In the same region we can find the reflected wave, which has the opposite momentum; also, its amplitude is less than that of the incoming wave. We write its contribution to χ_1 as $re^{-ik_x x}$, where r is the reflection coefficient.

By a similar reasoning, in the $x > 0$ region we will have a contribution $te^{ik'_x x}$: the direction of propagation is the same as the incoming wave, the momentum might be different. In the end, our wavefunction looks like

$$\varphi(t, x) = \varphi_1[x \leq 0] + \varphi_2[x \geq 0] \tag{1.84a}$$

$$= e^{-i\omega t}\left[\chi_1[x \leq 0] + \chi_2[x \geq 0]\right] \tag{1.84b}$$

$$= e^{-i\omega t}\left[\left(e^{ik_x x} + re^{-ik_x x}\right)[x \leq 0] + te^{ik'_x x}[x \geq 0]\right]. \tag{1.84c}$$

This wavefunction will need to satisfy the KG equation everywhere, however the equation has a different form in the two regions: if $\mathrm{D}^\mu = \partial^\mu + iqA^\mu$, then we can say that the two equations are

$$\left[\partial^\mu\partial_\mu + M^2\right]\varphi_1 = 0 \tag{1.85a}$$

$$\left[\mathrm{D}^\mu \mathrm{D}_\mu + M^2\right]\varphi_2 = 0\,, \tag{1.85b}$$

so we can insert the solutions into the equations: for the region without potential we get

$$\left[\partial^\mu\partial_\mu + M^2\right]\varphi_1 = \left[+\partial_0^2 - \partial_x^2 + M^2\right]\left[e^{-i\omega t}\left(e^{ik_x x} + re^{-ik_x x}\right)\right] \tag{1.86a}$$

$$= (-i\omega)^2\varphi_1 + M^2\varphi_1 - e^{-i\omega t}\partial_x^2\left(e^{ik_x x} + re^{-ik_x x}\right) \tag{1.86b}$$

$$= -\omega^2\varphi_1 + M^2\varphi_1 - e^{-i\omega t}\left((ik_x)^2 e^{ik_x x} + r(-ik_x)^2 e^{-ik_x x}\right) \tag{1.86c}$$

$$= \left[-\omega^2 + M^2 + k_x^2\right]\varphi_1\,, \tag{1.86d}$$

notice that even though the function has components with momentum in either direction it still is an eigenfunction of the operator $\Box$! For the other region we can use the fact we discussed earlier: with the minimal substitution, the momentum becomes $p^\mu = i\partial^\mu \to i\mathrm{D}^\mu = p^\mu - qA^\mu$, so in our case we will have $p^0 \to \omega_k - qA^0 = \omega_k - V_0$, while $p^i \to \vec{k}'$, since $\vec{A} = 0$. We can substitute the energy directly since the potential is constant, so there is no $\partial_0 V_0$ term. Then, the rest of the calculation is perfectly analogous, with k'_x instead of k_x.

So, we have the two relations

$$-\omega_k^2 + k_x^2 + M^2 = 0 \qquad \text{and} \qquad -\left(\omega_k - V_0\right)^2 + k_x'^2 + M^2 = 0\,. \tag{1.87}$$

We choose the solutions in which the wave is propagating towards increasing x: so we get

$$k_x = \sqrt{\omega_k^2 - M^2} \qquad \text{and} \qquad k'_x = \sqrt{\left(\omega_k - V_0\right)^2 - M^2}\,. \tag{1.88}$$

Now we must make these consistent with each other, by imposing that at the border the function be $\mathcal{C}^1$.

Why do we fix only the first derivative? Probably something to do with the fact that the KG eq is second order, so we would need $\Box = \mathrm{D}^\mu\mathrm{D}_\mu$ at the boundary, which cannot be the case.

We can work with the spatial components χ_i, since the time evolution factors. So, for continuity we get that $\chi_1(0) = \chi_2(0)$ implies:

$$1 + r = t\,, \tag{1.89}$$

while for differentiability we get that $\partial_x\chi_1(0) = \partial_x\chi_2(0)$ implies:

$$ik_x - rik_x = itk'_x \implies \frac{k'_x}{k_x} = \frac{1-r}{t}\,. \tag{1.90}$$

Now, what do we know of these variables? Well, from the dispersion relations if we fix the mass M, the energy ω and the potential V_0 we have k_x and k'_x. Then, from the two equations we have found we can calculate r and t; their expressions are exactly those found in the nonrelativistic case. Since $1 - r = 2 - t$ we have

$$\frac{k'_x}{k_x} = \frac{2}{t} - 1 \implies \frac{k'_x + k_x}{k_x} = \frac{2}{t} \implies t = \frac{2k_x}{k'_x + k_x} \qquad \text{and} \qquad r = t - 1 = \frac{k_x - k'_x}{k'_x + k_x}\,. \tag{1.91}$$

Now, we shall discuss the probability currents for the two regions: the formula is always

$$j^\mu = \frac{i}{2}\varphi^* \partial^\mu \varphi + \frac{q}{2} A^\mu \varphi^* \varphi + \text{c. c.}\,, \tag{1.92}$$

but in the first region the potential is identically zero.

One non-obvious thing to consider is that, in general, we could have k'_x be either real or imaginary, since while $\omega^2 - M^2$ must be > 0 there is no such constraint on $(\omega - V_0)^2 - M^2$. If we conjugate an exponential as $\left(e^{iz}\right)^*$, the result is e^{-iz^*} for a general complex z. The calculation is as follows:

$$\rho_1 \overset{\text{def}}{=} j^0_{(1)} = \frac{i}{2} e^{i\omega t} \partial^0 e^{-i\omega t} |\chi_1|^2 + \text{c. c.} = \frac{2}{2}\omega |\chi_1|^2 \tag{1.93a}$$

The χ_1 is not time-dependent, we get two terms by adding the conjugate. We get a minus sign from lowering the index

$$j_1 \overset{\text{def}}{=} j^x_{(1)} = -\frac{i}{2}\left(e^{ik_x x} + re^{-ik_x x}\right)\partial_x \left(e^{ik_x x} + re^{-ik_x x}\right) + \text{c. c.} \tag{1.93b}$$

$$= -\frac{2}{2} i(ik_x)\left(e^{ik_x x} + re^{-ik_x x}\right)\left(e^{ik_x x} - re^{-ik_x x}\right) = k_x\left(1 - |r|^2\right) \tag{1.93c}$$

$$\rho_2 \overset{\text{def}}{=} j^0_{(2)} = \frac{i}{2} t^* e^{i\omega t} e^{-ik'^*_x x} t(-i\omega) e^{-i\omega t} e^{ik'_x x} + \frac{1}{2} V_0 \varphi^* \varphi + \text{c. c.} \tag{1.93d}$$

$$= (\omega - V_0)|t|^2 e^{i\left(k'_x - k'^*_x\right)x} \tag{1.93e}$$

$$j_2 \overset{\text{def}}{=} j^x_{(2)} = -\frac{i}{2} t^* e^{i\omega t} e^{-ik'^*_x x} (ik'_x) e^{-i\omega t} e^{ik'_x x} + \text{c. c.} \tag{1.93f}$$

$$= \frac{1}{2}|t|^2 e^{i\left(k'_x - k'^*_x\right)x} k'_x + \text{c. c.} \tag{1.93g}$$

$$= \frac{k'_x + k'^*_x}{2}|t|^2 e^{i\left(k'_x - k'^*_x\right)x}\,. \tag{1.93h}$$

We can define the reflection and transmission coefficients, which are probabilities:

$$\mathcal{R} = \frac{j_{\text{in}} - j_1}{j_{\text{in}}} = |r|^2 \qquad \text{and} \qquad \mathcal{T} = \frac{j_2}{j_{\text{in}}} = \frac{k'_x + k'^*_x}{2k_x}|t|^2 e^{i(k'_x - k'^*_x)x}\,, \tag{1.94}$$

where $j_{\text{in}} = k_x$ is the current density corresponding to the incoming wave alone: it is calculated as the current density we would measure in region 1 if we did not have the reflected wave component:

$$j_{\text{in}} \overset{\text{def}}{=} j^x_{(1)}\Big|_{\chi_1 = e^{ik_x x}} = -\frac{i}{2}\left(e^{ik_x x}\right)\partial_x\left(e^{ik_x x}\right) + \text{c. c.} \tag{1.95a}$$

$$= -\frac{2}{2} i(ik_x) e^{ik_x x} e^{ik_x x} = k_x\,. \tag{1.95b}$$

Now, we have three possible cases for the behavior of k'_x: it is

1. real if $\omega_k - M > V_0$: this is the nonrelativistic case;
2. imaginary if $|\omega_k - V_0| < M$: this is the intermediate case;
3. real again if $\omega_k + M < V_0$: this is the fully relativistic case.

This is shown in figure 1.1. The NR approximation can only be applied in the first case, and in the second case only as long as $V_0 < \omega_k$: the case $\omega_k - M < V_0 \ll \omega_k$ is nonrelativistic quantum tunneling.

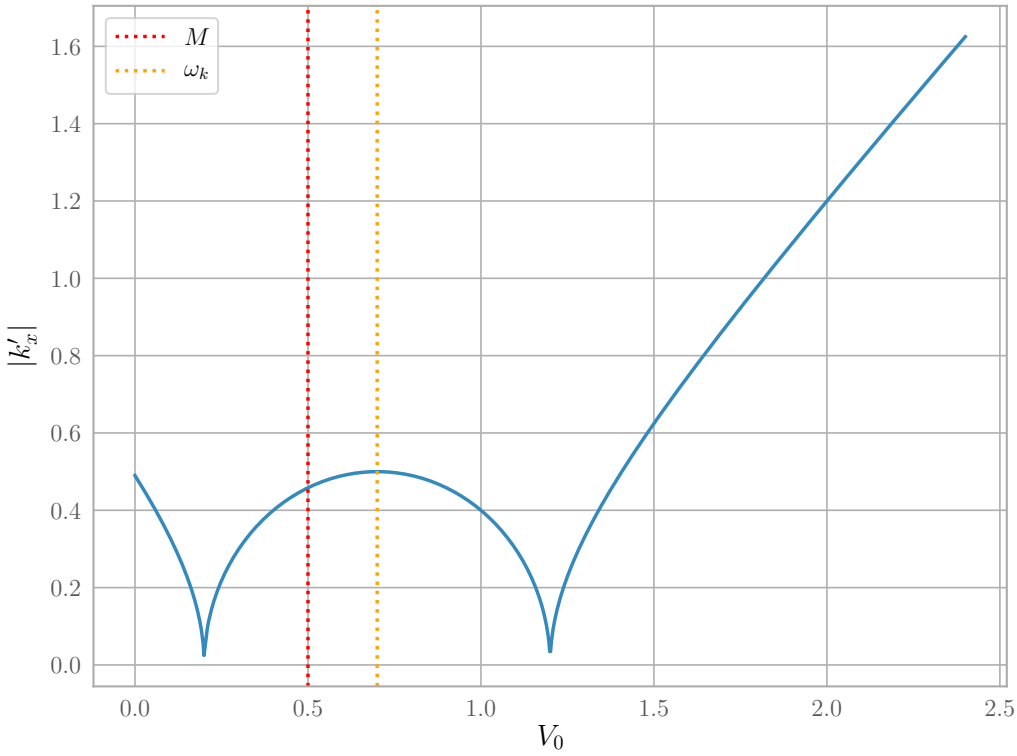

Figure 1.1: An illustration of how k'_x changes as V_0 increases. We have fixed the mass M and the the energy ω_k to 0.5 and 0.7 respectively. The regions are separated by the critical points of $|k'_x|$, at $V_0 = 0.2$ and $V_0 = 1.2$ respectively. Do note that this is a plot of the *absolute value* of k'_x: in the intermediate region it is purely imaginary.

Nonrelativistic case

Now, k_x and k'_x are both real, so $k'_x = k'^*_x$, which means that the probability currents read:

$$\rho_1 = \omega|\chi_1|^2 \qquad j_1 = k_x\left(1 - |r|^2\right) \tag{1.96a}$$

$$\rho_2 = (\omega - V_0)|t|^2 \qquad j_2 = k'_x|t|^2\,, \tag{1.96b}$$

and, since r and r are both (real and) < 1 these quantities are all positive. Also, we must have $k'_x < k_x$, so the quantities

$$\mathcal{R} = |r|^2 \qquad \text{and} \qquad \mathcal{T} = \frac{k'_x}{k_x}|t|^2 \tag{1.97}$$

are both < 1. So, it will be consistent if we impose (?) a probabilistic Born-like interpretation in which these coefficients are reflection and transmission probabilities, with $\mathcal{R} + \mathcal{T} = 1$.

Intermediate case

Now, we have $|\omega_k - V_0| < M$, so, while $k_x > 0$,

$$k'_x = \pm i\sqrt{M^2 - (\omega_k - V_0)^2}\,. \tag{1.98}$$

In principle we could have both $+$ or $-$, but we select the positive imaginary because it is the physical in which the wave travels rightward.

Now, in our computation of the probability density ρ_2 we will have a factor

$$\exp\big(i(k'_x - k'^*_x)x\big) = \exp\Big(\pm 2i^2|k'_x|x\Big) = \exp\Big(\mp 2|k'_x|x\Big)\,, \tag{1.99}$$

while for the probability current j_2 we will get a factor $k'_x + k'^*_x = 0$ multiplying everything.

So, in the end we get

$$\rho_1 = \omega|\chi_1|^2 \qquad j_1 = k_x\Big(1 - |r|^2\Big) \tag{1.100a}$$

$$\rho_2 = (\omega - V_0)|t|^2 e^{-2|k'_x|x} \qquad j_2 = 0\,. \tag{1.100b}$$

Notice that in the intermediate region we can have both $V_0 < \omega_k$ and $V_0 > \omega_k$.

In this case, we get for the reflection and transmission probabilities

$$\mathcal{R} = |r|^2 = \left|\frac{k_x - ik'_x}{k_x + ik'_x}\right| = \left|\frac{z}{z^*}\right|^2 = 1\,, \tag{1.101}$$

The ratio of z and z^* is only a phase

while $\mathcal{T} \propto j_2 = 0$.

So, in this case the particle is certainly reflected; this is compatible with the probabilistic interpretation $\mathcal{R} + \mathcal{T} = 1$. However, ρ_2 is not positive defined so it cannot be interpreted as a probability density in general.

If $V_0 < \omega_k$ we still get the vanilla-QM result of $\rho_2 \sim e^{-2|k'_x|x}$: the particle penetrates the classically forbidden region. This effect is suppressed as V_0 increases.

Fully relativistic case

This is the situation which is completely outside the realm of classical QM description, in which we expect to see relativistic effects. We have $V_0 > \omega_k + M$.

Now

$$k'_x = \pm\sqrt{(\omega_k - V_0)^2 - M^2} \in \mathbb{R}\,, \tag{1.102}$$

so the densities are

$$\rho_1 = \omega|\chi_1|^2 \qquad j_1 = k_x\Big(1 - |r|^2\Big) \tag{1.103a}$$

$$\rho_2 = (\omega - V_0)|t|^2 < 0 \qquad j_2 = k'_x|t|^2\,, \tag{1.103b}$$

where the sign of j_2 is determined by that of k'_x, which is not necessarily positive. Actually, it must be negative: to see this, we can calculate the group velocity, which would be the global

velocity of the wavepacket we are approximating. We can calculate ω_k from the dispersion relation (1.87) to get

$$v_G = \frac{\partial \omega_k}{\partial k'_x} = \frac{\partial}{\partial k'_x}\left(V_0 + \sqrt{k'^2_x + M^2}\right) = \frac{k'_x}{\sqrt{k'^2_x + M^2}} = \frac{k'_x}{\omega_k - V_0}\,, \tag{1.104}$$

where we used the dispersion relation again in the last step. Now, for the packet to propagate forward we need $v_G > 0$, but since the denominator is negative this means $k'_x < 0$.

Then, we will have

$$r = \frac{k_x - k'_x}{k_x + k'_x} \implies \mathcal{R} = \left|\frac{k_x - k'_x}{k_x + k'_x}\right|^2 > 1\,, \tag{1.105}$$

while the other coefficient is

$$\mathcal{T} = \frac{k'_x}{k_x}|t|^2 = \frac{k'_x}{k_x}\left|\frac{2k_x}{k'_x + k_x}\right|^2 = k'_x \frac{4k_x}{(k'_x + k_x)^2} < 0\,. \tag{1.106}$$

So, we can still maintain the condition $\mathcal{R} + \mathcal{T} = 1$, but reflection is "more than certain"? Surely this is not the correct physical interpretation.

This is the heart of the Klein-Gordon paradox.

What is actually happening is that in our description we are fixing the number of particles to be one; a proper ("grand-canonical" instead of "canonical", in statistical mechanics terms) description of the situation would allow us to see that the energy in the system is enough to create real particle-antiparticle pairs.

The antiparticles gain energy under the potential (since they perceive it as $-V_0$, having a negative charge), so they are transmitted, while the particles are reflected. On average, then, more than one particle is reflected.

In order to properly study these extreme conditions, we need the new paradigm of *Quantum Field Theory*.

1.4 The Dirac Equation

Saturday 2020-3-28, compiled 2020-06-01

The Klein Gordon equation, which we discussed in the previous lectures, has two main issues:

1. It is a second order equation, since we started from the relativistic dispersion relation $E^2 = M^2 + |\vec{p}|^2$, so it admits a negative as well as a positive energy solution: $E = \pm\omega_p$.
2. The charge associated with its 4-current density J^μ is not positive definite:

$$Q = \int \mathrm{d}^3x\, J^0(\vec{x}, t) \tag{1.107}$$

   can be negative.

In order to clarify the problem with these, we discussed the Klein Paradox, in which there is a violation of unitarity for a scattering process: the reflection probability was > 1 while the transmission probability was negative.

Dirac, in 1928, tried a different approach.

1.4.1 Historical derivation of the Dirac equation

We want to build an equation in the form

$$i\frac{\partial}{\partial t}\psi = H_D\psi\,, \tag{1.108}$$

for some Dirac Hamiltonian H_D, which we require to be first-order in the space derivatives and the mass, so that our equation can be covariant: this in general will be written as

$$H_D = -i\vec{\alpha}\cdot\vec{\nabla} + \beta M = \vec{\alpha}\cdot\left(-i\vec{\nabla}\right) + \beta M\,, \tag{1.109}$$

for some yet to be determined 4 coefficients $\vec{\alpha}$ and β.

Why are we introducing the i? This formulation is equivalent to saying that the energy E is given by $\vec{\alpha}\cdot\vec{p} + \beta M$, since $\vec{p} = -i\vec{\nabla}$.

We are assuming the coefficients to be constant, we shall see that this hypothesis works out. Also, as we shall these cannot be numbers, but instead they are matrices: for now, all this means is that we need to be careful when manipulating them, since in general they will not commute.

In order to determine $\vec{\alpha}$ and β we impose the conditions:

1. the Dirac Hamiltonian H_D is hermitian, since it needs to describe the physical property of energy, therefore it is an observable;
2. if ψ solves the Dirac equation, then it also solves the KG equation.

Actually, KG solutions also Dirac solutions in general.

The first condition can be written as $H = H^\dagger$, but to find out what it means for the coefficients $\vec{\alpha}$ and β we need to know what $\vec{\nabla}^\dagger$: since we know that momentum $\vec{p} = -i\vec{\nabla}$ is self adjoint, we must have $\vec{p}^\dagger = (-i)^\dagger\vec{\nabla}^\dagger = i\vec{\nabla}^\dagger \overset{!}{=} -i\vec{\nabla} = \vec{p}$, which means that $\vec{\nabla}^\dagger = -\vec{\nabla}$. So, we get

$$H^\dagger = (-i)^\dagger\vec{\alpha}^\dagger\cdot\vec{\nabla}^\dagger + \beta^\dagger M^\dagger \tag{1.110a}$$

$$= -i\alpha^*\cdot\vec{\nabla} + \beta^* M\,, \tag{1.110b}$$

since M is real.[5] Therefore, the coefficients must satisfy $\vec{a}^* = \vec{\alpha}$ and $\beta^* = \beta$ themselves.

The other condition, consistency with the KG equation, means that when we square the Dirac time derivative operator

$$\frac{\partial}{\partial t} = -iH_D \tag{1.111}$$

we should get the KG square time- derivative operator

$$\frac{\partial^2}{\partial t^2} = \nabla^2 + M^2\,. \tag{1.112}$$

[5] In the professor's notes this is done explicitly by integrating by parts: this method can be adapted to look like that, since one can use integration by parts to show that for any wavefunctions ϕ and ψ we have $\langle\phi|\nabla\psi\rangle + \langle\nabla\phi|\psi\rangle = 0$. Then, the only difference between the approaches is whether we only integrate by parts to get the adjoint of ∇ or the whole of H_D.

So, we find

$$\nabla^2 + M^2 \stackrel{!}{=} (-iH_D)^2 = -H_D^2 \tag{1.113a}$$
$$= -\big(-i\alpha_i\nabla_i + \beta\big)\Big(-i\alpha_j\nabla_j + \beta\Big) \tag{1.113b}$$
$$= \alpha_i\alpha_j\nabla_i\nabla_j + i\big(\alpha_i\beta + \beta\alpha_i\big)\nabla_i + \beta^2 \tag{1.113c}$$
$$= \frac{1}{2}\Big\{\alpha_i, \alpha_j\Big\}\nabla_i\nabla_j + i\big\{\alpha_i, \beta\big\}\nabla_i + \beta^2\,, \tag{1.113d}$$

where we introduced the anticommutator bracket notation:

$$\{A, B\} = AB + BA \qquad \text{while} \qquad [A, B] = AB - BA\,. \tag{1.114}$$

So, in order for the equations to be equivalent we need to impose

$$\frac{1}{2}\Big\{\alpha_i, \alpha_j\Big\} = \delta_{ij} \qquad \big\{\alpha_i, \beta\big\} = 0 \qquad \beta^2 = M^2\,. \tag{1.115}$$

This means α_i and β cannot be real or complex numbers: for complex numbers x, y we have $\big\{x, y\big\} = 2xy$, since for them multiplication is commutative.

Specifically, they must be matrices in an N-dimensional vector space, called a spinorial space. So, α_i and β are $N \times N$ matrices, while the wavefunction ψ is an N dimensional vector (spinor). N is yet to be determined.

They must be Hermitian matrices, since they satisfy $\alpha_i^\dagger = \alpha_i$ and $\beta^\dagger = \beta$.

We must have $\alpha_i^2 = \frac{1}{2}\{\alpha_i, \alpha_i\} = \mathbb{1}$, where $\mathbb{1}$ is the N dimensional identity matrix. Also, $\beta^2 = \mathbb{1}$. If we write α_i and β in diagonal form, which we can do since they are Hermitian, we get their real eigenvalues on the diagonal. If we square them we get the identity, therefore they must all be ± 1.

Now, let us show that these are traceless: call c_μ one of the four $\big(\beta, \alpha_i\big)$. Then, we know that $\Big\{c_\mu, c_\nu\Big\} = 2\delta_{\mu\nu}$. So, we do the following manipulation: take c_μ and c_ν, with $\nu \neq \mu$:

$$\mathrm{Tr}\Big(c_\mu\Big) = \mathrm{Tr}\Big(c_\mu c_\nu c_\nu\Big) \tag{1.116a}$$
$$= \mathrm{Tr}\Big(-c_\nu c_\mu c_\nu\Big) \tag{1.116b}$$
$$= \mathrm{Tr}\Big(-c_\mu c_\nu c_\nu\Big) \tag{1.116c}$$
$$= -\,\mathrm{Tr}\Big(c_\mu\Big)\,, \tag{1.116d}$$

$c_\nu^2 = \mathbb{1}$

$\Big\{c_\mu, c_\nu\Big\} = 0$, so $c_\mu c_\nu = -c_\nu c_\mu$

Cyclic property of the trace, and linearity of the trace

so $\mathrm{Tr}\Big(c_\mu\Big) = -\,\mathrm{Tr}\Big(c_\mu\Big) \implies \mathrm{Tr}\Big(c_\mu\Big) = 0$. Since the trace is the sum of the eigenvalues, which are ± 1, the dimension must be even.

So, our simplest guess will be $N = 2$: a basis for Hermitian complex matrices is $\{\mathbb{1}, \sigma_i\}$ where σ_i are the Pauli matrices. However, $\mathbb{1}$ is not traceless: so our c_μ can only be written as $c_\mu = \vec{c}\cdot\vec{\sigma}$.

Now, our c_μ must be linearly independent over $\mathbb{C}$: if they were not, say because $c_\mu = z c_\nu$, with $\mu \neq \nu$ and $z \in \mathbb{C}$, then the anticommutation would read

$$\Big\{c_\mu, c_\nu\Big\} = \{z c_\nu, c_\nu\} = z\mathbb{1} \neq 0\,, \tag{1.117}$$

a contradiction. We only have three linearly independent Pauli matrices σ_i to express four c_μ, so this cannot work.

Do however note that if $M = 0$ then we do not need the β matrix since the term multiplying it goes to zero, so the only matrices we need are the three α_i: in the massless case, then, we can use the Pauli matrices, setting $\sigma_i = \alpha_i$ satisfies all our requirements, since $\left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}$.

The next possible dimensionality is $N = 4$, and we can use the following matrices:

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}. \tag{1.118a}$$

These are self adjoint since the Pauli matrices are, manifestly traceless, and they satisfy the anticommutation relations.

Proof. We expand the relation $\left\{\alpha_i, \alpha_j\right\}$: if we write $\alpha_i\alpha_j$ we get

$$\alpha_i\alpha_j = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} = \begin{bmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{bmatrix}, \tag{1.119a}$$

therefore

$$\left\{\alpha_i, \alpha_j\right\} = \begin{bmatrix} \left\{\sigma_i, \sigma_j\right\} & 0 \\ 0 & \left\{\sigma_i, \sigma_j\right\} \end{bmatrix} = \left\{\sigma_i, \sigma_j\right\}\mathbb{1}_4, \tag{1.120a}$$

but we know that $\left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}$, so we have verified the anticommutation relations between the α_i.

To see that $\beta^2 = \mathbb{1}_4$ is immediate since it is already diagonal, so we only need to show $\left\{\alpha_i, \beta\right\} = 0$. We have

$$\alpha_i\beta = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{1.121a}$$

$$\beta\alpha_i = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \tag{1.121b}$$

so the sum $\alpha_i\beta + \beta\alpha_i$ is zero, which proves our statement. □

The γ matrices

It is then conventional to define the matrices

$$\gamma^0 = \beta = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} \qquad \text{and} \qquad \gamma^i = \beta\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \tag{1.122a}$$

which obey the following properties:

1. $\gamma^0 = (\gamma^0)^\dagger$, while $\gamma^i = -(\gamma^i)^\dagger$;
2. $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, specifically $(\gamma^0)^2 = \mathbb{1}_4$ and $(\gamma^i)^2 = -\mathbb{1}_4$.

We can then interpret this collection of matrices as a 4-vector. We can rewrite the Dirac equation using them: we multiply it by β to get

$$i\partial_0\psi = \left(-i\alpha_i\partial_i + \beta M\right)\psi \tag{1.123a}$$
$$i\beta\partial_0\psi = \left(-i\beta\alpha_i\partial_i + \beta^2 M\right)\psi \tag{1.123b}$$
$$i\gamma^0\partial_0\psi = \left(-\gamma^i\partial_i + \beta^2 M\right)\psi \tag{1.123c}$$
$$\left(i\gamma^\mu\partial_\mu - M\right)\psi = 0\,. \tag{1.123d}$$

This looks simple, but recall that γ^μ are 4×4 matrices and ψ is a 4D spinor: if we write this explicitly using the spinor indices, for which we use the Hebrew letters א nd ב (aleph and beth[6]), we get

$$\left(i\gamma^\mu_{\aleph\beth} - M\delta_{\aleph\beth}\right)\psi_\beth = 0\,, \tag{1.124}$$

which is a set of 4 equations, indexed by the free index א. Since the γ matrices are not diagonal, the components $\psi_\beth$ are mixed in the equation.

If we introduce the slashed notation $\not{p} \overset{\text{def}}{=} \gamma^\mu p_\mu$.

With this notation, the Dirac equation reads

$$\boxed{\left(i\not{\partial} - M\right)\psi = 0\,.} \tag{1.125}$$

Representations of γ-matrices

The way we defined the γ matrices was arbitrary, and in fact there are other possible equivalent definitions which are also physically useful.

In general, for any unitary matrix C we can move to the new representation

$$\widetilde{\gamma}^\mu = C^{-1}\gamma^\mu C\,. \tag{1.126}$$

How do we prove that the physical results are the same? The Dirac equation is in the form $A\psi = 0$, and if we show that for an equation in this form $C^{-1}AC$ is as good as A then we are done: in $A = i\not{\partial} - M\mathbb{1}$ the $M\mathbb{1}$ term is unaffected since it commutes with the matrix, the derivative is unaffected since the C matrix is assumed to be constant. But then, if we want to use $\widetilde{A} = C^{-1}AC$, we will find solutions to $C^{-1}ACC^{-1}\psi = \widetilde{A}\widetilde{\psi}$, where $\widetilde{\psi} = C^{-1}\psi$: we will see later that the physical observables corresponding to spinors are related to their contractions, and indeed if we take the square modulus after applying a unitary transformation it is unchanged.

[6] Since this notation will only be used here, it does not matter if it is a little inconvenient. I did not use greek or latin letters since they both already have meaning in the conventional notation.

The Dirac representation we gave before is usually used when dealing with the non-relativistic limit, while the one we now show is called the relativistic, or Weyl, or chiral, representation. We use the matrix which is known in quantum information as the Hadamard gate:

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{1.127a}$$

we can move to the representation

$$\gamma^\mu = \left(\begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \right). \tag{1.128a}$$

Proof. We multiply, writing 1 for $\mathbb{1}$ and 0 for the 0 matrix for simplicity: for γ^0 we get

$$C^{-1}\gamma^0 C = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{1.129a}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1.129b}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{1.129c}$$

while for γ^i we have

$$C^{-1}\gamma^i C = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{1.130a}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_i & \sigma_i \\ -\sigma_i & \sigma_i \end{bmatrix} \tag{1.130b}$$

$$= \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}. \tag{1.130c}$$

□

The Dirac equation is consistent with the Klein Gordon equation: we can multiply it by the conjugate of the Dirac operator, $-(i\not{\partial} + M)$, to get

$$-\left(i\not{\partial} + M\right)\left(i\not{\partial} - M\right)\psi = 0 \tag{1.131a}$$

$$\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + M^2\right) = 0 \tag{1.131b}$$

$$\left(\Box + M^2\right)\psi = 0\,, \tag{1.131c}$$

which is equivalent to the Klein-Gordon equation since when we compute $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu$ we only have the symmetric part, since the partial derivatives commute, and $\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$.

The Klein-Gordon equation is then diagonal in the spinorial space: by this manipulation we have shown that the KG operator is proportional to the identity.

1.4.2 Covariance of the Dirac Equation

Sunday 2020-3-29, compiled 2020-06-01

How do these new spinors transform under Lorentz transformations?

We suppose that for a general Lorentz transformation Λ we will have something in the form

$$\psi'(x') = S(\Lambda)\psi(x)\,, \tag{1.132}$$

where $S(\Lambda)$ is a 4×4 matrix, belonging to the *spinorial representation of the Lorentz group*. We do this because if we were to simply impose $\psi'(x') = \psi(x)$ we get a contradiction, so the equation is not covariant.

Now, let us see what we must require of the matrices $S(\Lambda)$ so that the Dirac equation holds in the new frame as well as in the old. When we do the computation, recall that the γ^μ matrices are *not* a 4-vector, despite the position of the index: they need not be transformed using a Λ matrix; better put, they are a set of 4 Lorentz scalars

So, we must have

$$\Big(i\partial\!\!\!/' - M\Big)\psi'(x') = \Big(i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu - M\Big)S(\Lambda)\psi(x) \tag{1.133a}$$

$$= S(\Lambda)\Big[i\Lambda_\mu{}^\nu S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\partial_\nu - M\Big]\psi(x) \tag{1.133b}$$

$$= S(\Lambda)\big[i\partial\!\!\!/ - M\big]\psi(x)\,, \tag{1.133c}$$

where we used the following:

1. the matrix $S(\Lambda)$ is constant with respect to the spatial coordinates, since the Lorentz transformation is fixed, so we can bring it outside of the derivative ∂_ν;
2. the matrices γ^μ and $S(\Lambda)$ do not commute *a priori*, so if we wish to bring the equation in the form $S(\Lambda)\times\big[\text{old-coordinates Dirac eq}\big]$ we cannot simply commute them, instead we multiply by $\mathbb{1} = SS^{-1}$ on the left;
3. we impose the condition

$$\Lambda_\mu{}^\nu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) \overset{!}{=} \gamma^\nu \tag{1.134}$$

   in order to find the expression we need in order for the equation to be covariant, since as long as the matrix $S(\Lambda)$ is nondegenerate the equation $S(\Lambda)\big[i\partial\!\!\!/ - M\big]\psi = 0$ has the same solutions as the Dirac equation.

If we multiply by an inverse Lorentz matrix on both sides we can bring this equation into the form

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu\,, \tag{1.135}$$

so we can say that the Dirac equation is covariant as long as we find some matrices $S(\Lambda)$ satisfying these conditions. Do note that while this looks like a vector transformation law, the γ^μ matrices do not transform under Lorentz transformations: what we are stating is that it is equivalent to apply a Lorentz matrix to them and to transform them as spinorial matrices.

In a way, we are asking the transformation laws for spinors and vectors to be compatible.

Explicit realization of the spinorial representation

In order to do this, we consider infinitesimal Lorentz transformations,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}\,, \tag{1.136}$$

where $\omega_{\mu\nu} = \omega_{[\mu\nu]}$ is an antisymmetric tensor.[7]

A rank-2 antisymmetric tensor in 4 dimensions has 6 degrees of freedom: these physically correspond to three rotations and three boosts.

A rather general ansatz looks like:

$$S(\Lambda) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\,, \tag{1.137}$$

where $\Sigma^{\mu\nu}$ is the set of the generators of the spinorial representation of the Lorentz group. For fixed μ and ν, these are matrices in the spinor space: if we write all the indices explicitly, they look like $\Sigma^{\mu\nu}_{\aleph\beth}$.

This means that to each possible basis Lorentz transformation of spacetime (think boost or rotation) we are associating a 4×4 basis spinor transformation matrix. This is what finding a representation of the group means: for each element of the Lorentz group we are finding a transformation matrix.

Do note, however, that we are only working at linear order in $\omega_{\mu\nu}$, so we are not looking yet at a representation of the whole group, instead we are only considering elements which are close to the identity. So, we can insert this expression for $S(\Lambda)$ into the relation between the $S(\Lambda)$ and Λ, equation (1.135), using the fact that to first order $S = \mathbb{1} + \epsilon$ is the inverse of $S^{-1} = \mathbb{1} - \epsilon$ we have, to first order in ω:

$$\left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)\gamma^{\mu}\left(\mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}\right) = \gamma^{\mu} + \omega^{\mu}{}_{\nu}\gamma^{\nu} \tag{1.138a}$$

$$\gamma^{\mu} + \frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\gamma^{\mu} - \frac{i}{2}\gamma^{\mu}\omega_{\alpha\beta}\Sigma^{\alpha\beta} = \gamma^{\mu} + \omega_{\sigma\nu}\eta^{\mu\sigma}\gamma^{\nu} \tag{1.138b}$$

$$-\frac{i}{2}\omega_{\rho\sigma}\left[\gamma^{\mu}, \Sigma^{\rho\sigma}\right] = \omega_{\sigma\nu}\eta^{\mu[\sigma}\gamma^{\nu]} \tag{1.138c}$$

$$\left[\gamma^{\mu}, \Sigma^{\rho\sigma}\right] = i\left(\eta^{\mu\rho}\gamma^{\sigma} - \eta^{\mu\sigma}\gamma^{\rho}\right), \tag{1.138d}$$

where we antisymmetrized the indices in $\eta^{\mu\rho}\gamma^{\sigma}$ since they are multiplied by the antisymmetric tensor $\omega_{\rho\sigma}$, so any symmetric part would not contribute to the equation. We also used the fact that $\omega_{\rho\sigma}$ is proportional to the identity in the spinor space.

Claim 1.4.1. *This is satisfied by*

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}] = \frac{1}{2}\sigma^{\mu\nu}\,, \tag{1.139}$$

Proof. We plug the definitions in:

$$\left[\gamma^{\mu}, \Sigma^{\rho\sigma}\right] = \frac{i}{4}\left(\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma} - \gamma^{\mu}\gamma^{\sigma}\gamma^{\rho} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu} + \gamma^{\sigma}\gamma^{\rho}\gamma^{\mu}\right) \tag{1.140a}$$

[7] The fact that ω must be antisymmetric may be derived by imposing the condition $\eta_{\mu\nu} = \Lambda_{\mu}{}^{\alpha}\Lambda_{\nu}{}^{\beta}\eta_{\alpha\beta}$, with the perturbed Λ we wrote above.

$$i\left(\eta^{\mu\rho}\gamma^{\sigma}-\eta^{\mu\sigma}\gamma^{\rho}\right)=\frac{i}{2}\left(\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}+\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}-\gamma^{\mu}\gamma^{\sigma}\gamma^{\rho}-\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\right), \tag{1.140b}$$

where we used the anticommutation relations $\{\gamma^{\mu},\gamma^{\nu}\}=2\eta^{\mu\nu}$. Then, we must check whether these two expression are equal, that is, whether

$$\frac{1}{2}\left(\mu\rho\sigma-\mu\sigma\rho-\rho\sigma\mu+\sigma\rho\mu\right)\stackrel{?}{=}\mu\rho\sigma+\rho\mu\sigma-\mu\sigma\rho-\sigma\mu\rho \tag{1.141a}$$

$$=\{\mu,\rho\}\sigma-\{\mu,\sigma\}\rho\,, \tag{1.141b}$$

where we only write the indices of the γs for clarity.

We can now use the identity:

$$[\mu,\rho\sigma]=\mu\rho\sigma-\rho\sigma\mu \tag{1.142a}$$

$$=\mu\rho\sigma+\rho\mu\sigma-\rho\mu\sigma-\rho\sigma\mu \tag{1.142b}$$

Added and subtracted

$$=\{\mu,\rho\}\sigma-\rho\{\mu,\sigma\}\,. \tag{1.142c}$$

Note that in our convention there is no division by 2 in the commutator and anticommutator. Then, we can recognize these in the initial claim: we find

$$\frac{1}{2}\left(\mu\rho\sigma-\mu\sigma\rho-\rho\sigma\mu+\sigma\rho\mu\right)=\frac{1}{2}\Big([\mu,\rho\sigma]-[\mu,\sigma\rho]\Big) \tag{1.143a}$$

$$=\frac{1}{2}\Big(\{\mu,\rho\}\sigma-\rho\{\mu,\sigma\}-\{\mu,\sigma\}\rho+\sigma\{\mu,\rho\}\Big) \tag{1.143b}$$

$$=\{\mu,\rho\}\sigma-\{\mu,\sigma\}\rho\,, \tag{1.143c}$$

since the anticommutators $\{\mu,\sigma\}$ are proportional to the metric $\eta^{\mu\sigma}$, which is proportional to the identity in the spinorial space, which commutes with the gamma matrices, so $\{\mu,\sigma\}$ commutes with any gamma matrix. □

Up until now we have worked "near the identity" of our transformation group; if we want to extrapolate these results to general transformations we may use the exponential map, which gives us relations in the form

$$S(\Lambda)=\exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) \tag{1.144a}$$

$$S^{-1}(\Lambda)=\exp\left(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right). \tag{1.144b}$$

1.4.3 Dirac conjugate spinor

We want to define a notion of a conjugate spinor, starting from what we know about hermitian conjugate operators: we start by taking the Hermitian conjugate of the Dirac equation,

$$\Big[\left(i\partial\!\!\!/-M\right)\psi\Big]^{\dagger}=-i\psi^{\dagger}\left(i\gamma^{\mu\dagger}\overleftarrow{\partial}_{\mu}+M\right), \tag{1.145}$$

where the result follows from the fact that the adjoint of a product is the product of the reverse-ordered adjoints, while the notation $\overleftarrow{\partial}_\mu$ means that the derivative operator is acting on what is on its left.

In order to simplify this, we employ the following facts: $(\gamma^0)^2 = \mathbb{1}$, and

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu \,. \tag{1.146}$$

Proof. For γ^0 the result is immediate, since it is self adjoint: $\gamma^0 = \gamma^{0,\dagger}$, so the expression reduces to $\left(\gamma^0\right)^3 = \left(\gamma^0\right)^2 \left(\gamma^0\right) = \gamma^0$.

Then let us consider γ^i: since they are block matrices of the Pauli matrices, which are self-adjoint and which have determinant -1, we have:

$$\gamma^{i\dagger} = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}^\dagger = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix} = -\gamma^i \,, \tag{1.147a}$$

therefore we need to prove that $\gamma^0 \gamma^i \gamma^0 = -\gamma^i$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{bmatrix} \tag{1.148a}$$

$$= \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix} . \tag{1.148b}$$

□

With these results, we can use the manipulation

$$\psi^\dagger \gamma^{\mu\dagger} \overleftarrow{\partial}_\mu = \psi^\dagger \gamma^0 \gamma^0 \gamma^{\mu\dagger} \gamma^0 \gamma^0 \overleftarrow{\partial}_\mu \tag{1.149a}$$

$$= \psi^\dagger \gamma^0 \gamma^\mu \overleftarrow{\partial}_\mu \gamma^0 \,, \tag{1.149b}$$

which holds since the matrix γ^0 is constant. We also define $\overline{\psi} = \psi^\dagger \gamma^0$, so that the whole equation reads

$$\overline{\psi}\left(i\overleftarrow{\partial\!\!\!/} + M\right) = 0 \,, \tag{1.150}$$

where we removed the γ^0 at the right, since the equation multiplied by it is equivalent to the one which is not.

Now, in order to see how $\overline{\psi}$ transforms under a Lorentz transformation we use the following facts:

1. $\psi'(x') = S(\Lambda)\psi(x)$ and
2. $\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$.

Proof. Let us first work to first order: we know that $S(\Lambda)$ is written as

$$S(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) \approx \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\,, \tag{1.151}$$

so its adjunct is

$$S^\dagger(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu\dagger}\right) \approx \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu\dagger}\,, \tag{1.152}$$

so when we multiply it from the left and right by γ^0 it goes through everything, and we are left with

$$\gamma^0\Sigma^{\mu\nu\dagger}\gamma^0 \stackrel{?}{=} \Sigma^{\mu\nu}\,, \tag{1.153}$$

since the expression for $S^{-1}(\Lambda)$ differs from that of $S(\Lambda)$ only for a sign in the exponent, but we picked up a sign when taking the adjunct. Recall that the explicit expression of the Σ matrices is

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\,, \tag{1.154}$$

so we can show directly that

$$\gamma^0\Sigma^{\mu\nu\dagger}\gamma^0 = \frac{i}{4}\gamma^0\gamma^{\mu\dagger}\gamma^{\nu\dagger}\gamma^0 - \frac{i}{4}\gamma^0\gamma^{\nu\dagger}\gamma^{\mu\dagger}\gamma^0 \tag{1.155a}$$

$$= \frac{i}{4}\gamma^0\gamma^{\mu\dagger}\gamma^0\gamma^0\gamma^{\nu\dagger}\gamma^0 - \frac{i}{4}\gamma^0\gamma^{\nu\dagger}\gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 \tag{1.155b}$$

$$= \frac{i}{4}[\gamma^\mu\gamma^\nu] = \Sigma^{\mu\nu}\,. \tag{1.155c}$$

We have worked to first order until now, but the result we found actually works up to any order, since in the full expression of the exponential we will have terms proportional to

$$\frac{1}{n!}\left(\omega_{\mu\nu}\Sigma^{\mu\nu\dagger}\right)^n\,, \tag{1.156}$$

so if we multiply on both sides by γ^0 we get something like (we omit the indices for simplicity)

$$\frac{1}{n!}\omega^n\gamma^0\Sigma^{\dagger n}\gamma^0 = \frac{1}{n!}\omega^n\gamma^0\Sigma^\dagger\gamma^0\gamma^0\Sigma^\dagger\ldots\gamma^0\gamma^0\Sigma^\dagger\gamma^0 = \frac{\omega^n}{n!}\prod_n\left(\gamma^0\Sigma^\dagger\gamma^0\right) = \frac{\omega^n}{n!}\Sigma^n\,, \tag{1.157}$$

so the reasoning extends to arbitrary orders in the expansion of the exponential. □

Using these results, we can show that the conjugate spinor $\overline{\psi} = \psi^\dagger\gamma^0$ transforms like:

$$\overline{\psi}'(x') = \psi'^\dagger\gamma^0 \tag{1.158a}$$

$$= \left(S(\Lambda)\psi\right)^\dagger\gamma^0 \tag{1.158b}$$

$$= \psi^\dagger S^\dagger(\Lambda)\gamma^0 \tag{1.158c}$$

$$= \psi^\dagger\gamma^0\gamma^0 S^\dagger(\Lambda)\gamma^0 \tag{1.158d}$$

$$= \overline{\psi}S^{-1}(\Lambda) \tag{1.158e}$$

$$. \tag{1.158f}$$

1.4.4 Continuity equation

We can apply the same strategy we used in the KG case: if we multiply the Dirac equation by $\overline{\psi}$ on the left and the conjugate equation by ψ on the right, we get

$$\overline{\psi}(i\partial\!\!\!/ - M)\psi = 0\overline{\psi}\left(i\overleftarrow{\partial\!\!\!/} + M\right)\psi \qquad = 0\,. \tag{1.159}$$

if we sum them, the $M\overline{\psi}\psi$ terms cancel, so we have

$$0 = \overline{\psi}\left(i\overleftarrow{\partial\!\!\!/} + i\partial\!\!\!/\right)\psi \tag{1.160a}$$

$$= i\Big((\partial_\mu\overline{\psi})\gamma^\mu\psi + \overline{\psi}\gamma^\mu\partial_\mu\psi\Big) \tag{1.160b}$$

$$= \partial_\mu\left(\overline{\psi}\gamma^\mu\psi\right), \tag{1.160c}$$

so we have the conserved current $\overline{\psi}\gamma^\mu\psi = J^\mu$. Its corresponding charge density is $\rho = J^0 = \overline{\Psi}\gamma^0\psi = \psi^\dagger\psi$.

This looks like a positive definite conserved charge, so it could be interpreted as a probability. In fact, this is not the case, as we shall see in the future.

1.4.5 Bilinear forms and Lorentz transformations

Our observables can not still carry spinorial indices: all the spinorial indices must be saturated in the expression for an observable, since we can only observe scalars in the spin tensor algebra.

For instance, $\overline{\psi}\psi$ is the dot product of two spinors, so it carries no indices. It is a scalar under Lorentz transformations, since $\overline{\psi}$ transforms with a S^{-1} while ψ transforms with an S.

$\overline{\psi}\gamma^\mu\psi$ is instead a vector under Lorentz transformations: it transforms like

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = \overline{\psi}\Lambda^\mu{}_\nu\gamma^\nu\psi\,, \tag{1.161}$$

where we used the property shown in equation (1.135).

So, everything we are doing is covariant when we are discussing observables.

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1.4.6 General solution of the Dirac equation

We have imposed that solutions of the Dirac equation are solutions of the KG equation and vice-versa, so a general solution of the Dirac equation should look like those we found when discussing the Klein-Gordon equation:

$$\psi(x) = \psi_+(x) + \psi_-(x) \sim e^{-ikx}u(k) + v(k)e^{ikx}\Big|_{k_0=\omega_k}\,, \tag{1.162}$$

where the subscript $+$ or $-$ refers to the sign of the energy of the solution, and now $u(k)$ and $v(k)$ are spinors in momentum space. We use the $\sim$ sign since these are only plane wave solutions, a general solution will be an integral of these over momentum space.

If we apply the Dirac operator $i\not{\partial} - M$ to these we find

$$\big(i\not{\partial} - M\big)\psi_+ \sim e^{-ikx}(\not{k} - M)u(k) = 0 \tag{1.163a}$$

$$\big(i\not{\partial} - M\big)\psi_- \sim -e^{ikx}(\not{k} + M)v(k) = 0\,, \tag{1.163b}$$

so, since the exponentials are nonzero we can write these two equations as

$$\boxed{(\not{k} - M)u(k) = 0 \qquad \text{and} \qquad (\not{k} + M)v(k) = 0\,.} \tag{1.164}$$

Let us now assume that the particle we are considering is not massless, so we can go in its rest frame. If we were to consider a massless particle, we could work with the $N = 2$ Weyl spinors. Instead, we will need the $N = 4$ Dirac spinors.

In the rest frame $k^\mu = (M, \vec{0})$, so the two equations read

$$0 = (\not{k} - M)u(k) \tag{1.165a}$$

$$= \Big(\gamma^\mu k_\mu - M\Big)u(k) \tag{1.165b}$$

$$= \Big(\gamma^0 M - M\Big)u(k) = M\Big(\gamma^0 - \mathbb{1}\Big)u(k) \tag{1.165c}$$

$$= M\left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} - \begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix}\right)u(k) \tag{1.165d}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -2\mathbb{1} \end{bmatrix} Mu(k) = 0\,, \tag{1.165e}$$

so a generic solution looks like

$$u(M) = c\begin{bmatrix} \xi \\ 0 \end{bmatrix}. \tag{1.166a}$$

Note that we are using the Dirac representation, but if we were to choose a different one the spinors would look different. For $v(k)$ the equation looks like

$$M\Big(\gamma^0 + \mathbb{1}\Big)v(M) = \begin{bmatrix} 2\mathbb{1} & 0 \\ 0 & 0 \end{bmatrix} Mv(M) = 0\,, \tag{1.167a}$$

so

$$v(M) = c\begin{bmatrix} 0 \\ \xi \end{bmatrix}. \tag{1.168a}$$

Here ξ is a two-dimensional vector, while c is a normalization constant. We have two independent solutions for each case, so in total there are four independent ones. A basis we can choose is

$$u_r(M) = \sqrt{2M}\begin{bmatrix} \xi_r \\ 0 \end{bmatrix} \qquad \text{and} \qquad v_r(M) = \sqrt{2M}\begin{bmatrix} 0 \\ \xi_r \end{bmatrix}, \tag{1.169a}$$

where $r = 1, 2$ and $(\xi_r)^i = \delta^i_r$ are unit vectors, a basis for the 2D space.

If we choose this normalization, then we will have $\overline{u}_r(M)u_s(M) = 2M\delta_{rs}$ and $\overline{v}_r(M)v_s(M) = -2M\delta_{rs}$, while all the us and the vs are respectively orthogonal: $\overline{u}_r v_s = \overline{v}_r u_s = 0$.

This choice corresponds to having chosen to have the spin of the particle along the third axis, as we shall see shortly.

The fact that we were able to find a basis of four independent vectors for the solution means that the solution of the Dirac equation has four independent degrees of freedom, two of which have positive energy and two of which have negative energy.

We put ourselves in the rest frame: if we wish to compute the solutions $u(k)$ and $v(k)$ in a generic frame we need to perform a Lorentz boost from the rest frame. In order to study how this boost affects the spinor, we need to study the spinorial representation of the Lorentz boost, $S(\Lambda)$.

Instead of explicitly writing out the full representation of the Lorentz matrix we are interested in, which is long and complicated, we can use the following trick:

$$
\begin{aligned}
(\not{k} - M)(\not{k} + M) &= (\not{k} + M)(\not{k} - M) && \text{(1.170a)}\\
&= \gamma^\mu \gamma^\nu k_\mu k_\nu - M^2 && \text{(1.170b)}\\
&= \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} k_\mu k_\nu - M^2 = k^2 - M^2 = 0\,, && \text{(1.170c)}
\end{aligned}
$$

since $\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$, and the square of the 4-momentum always corresponds to the mass of the particle under our assumptions (of working on-shell).

This allows us to quickly prove that the ansatz

$$
u(k) = C(\not{k} + M)u(M) \tag{1.171}
$$

satisfies the KG equation (with positive energy), since

$$
(\not{k} - M)u(k) = C\left(k^2 - M^2\right)u(M) = 0\,, \tag{1.172}
$$

which follows from the identity (1.170a) which we just proved.

But then we could use something which is not $u(M)$ for our rest-frame solution, right?

Similarly, for the negative-energy case we have the solution

$$
v(k) = C(\not{k} - M)v(M)\,, \tag{1.173}
$$

which will satisfy $(\not{k} + M)v(k) = 0$.

The constant C is for normalization, and we choose it such that the normalization is the same as in the rest frame: so, the identities to be satisfied are

$$
\begin{aligned}
\overline{u}_r(k)u_s(k) &= 2M\delta_{rs} && \text{(1.174a)}\\
\overline{v}_r(k)v_s(k) &= -2M\delta_{rs} && \text{(1.174b)}\\
\overline{u}_r(k)v_s(k) &= \overline{v}_r(k)u_s(k) = 0\,. && \text{(1.174c)}
\end{aligned}
$$

Claim 1.4.2. *The final result we get from this manipulation is*

$$u_r(k) = \frac{(\not{k} + M)}{\sqrt{2M(\omega_k + M)}} u_r(M) = \begin{bmatrix} \xi_r \sqrt{\omega_k + M} \\ \frac{\vec{k} \cdot \vec{\sigma}}{\sqrt{\omega_k + M}} \xi_r \end{bmatrix} \tag{1.175a}$$

$$v_r(k) = \frac{(-\not{k} + M)}{\sqrt{2M(\omega_k + M)}} v_r(M) = \begin{bmatrix} \frac{\vec{k} \cdot \vec{\sigma}}{\sqrt{\omega_k + M}} \xi_r \\ \xi_r \sqrt{\omega_k + M} \end{bmatrix} . \tag{1.175b}$$

The first solution is the positive-energy one, the second is the negative-energy one. Notice that the expression $\vec{k} \cdot \vec{\sigma}$ yields a 2×2 complex matrix, which is applied to the vector ξ_r.

We prove the second equality for the positive energy solution.

Proof. We begin by writing out the operator

$$\not{k} = \gamma^\mu \eta_{\mu\nu} k^\nu = \begin{bmatrix} \mathbb{1}\omega_k & 0 \\ 0 & -\mathbb{1}\omega_k \end{bmatrix} - \begin{bmatrix} 0 & \vec{k} \cdot \vec{\sigma} \\ -\vec{k} \cdot \vec{\sigma} & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{1}\omega_k & -\vec{k} \cdot \vec{\sigma} \\ \vec{k} \cdot \vec{\sigma} & -\mathbb{1}\omega_k \end{bmatrix} . \tag{1.176a}$$

So, when we apply this (plus M times the identity) to the solution

$$u_r(M) = \sqrt{2M} \begin{bmatrix} \xi_r \\ 0 \end{bmatrix} , \tag{1.177a}$$

we get:

$$(\not{k} + M) u_r(M) = \sqrt{2M} \begin{bmatrix} (\omega_k + M)\xi_r \\ \left(\vec{k} \cdot \vec{\sigma}\right) \xi_r \end{bmatrix} , \tag{1.178a}$$

which we can divide by $\sqrt{2M(\omega_k + M)}$ to find the desired expression, equation (1.175a). □

Claim 1.4.3. *The conjugate spinors $\overline{u}$ and $\overline{v}$ in momentum space read respectively:*

$$\overline{u}_r(k) = \overline{u}_r(M) \frac{(\not{k} + M)}{\sqrt{2(\omega_k + M)}} = \begin{bmatrix} \xi_r^\top \sqrt{\omega_k + M}, & -\xi_r^\top \frac{\vec{k} \cdot \vec{\sigma}}{\sqrt{\omega_k + M}} \end{bmatrix} \tag{1.179a}$$

$$\overline{v}_r(k) = \overline{v}_r(M) \frac{(-\not{k} + M)}{\sqrt{2(\omega_k + M)}} = \begin{bmatrix} \xi_r^\top \frac{\vec{k} \cdot \vec{\sigma}}{\sqrt{\omega_k + M}}, & -\xi_r^\top \sqrt{\omega_k + M} \end{bmatrix} . \tag{1.179b}$$

Proof. Recall that $\overline{u} = u^\dagger \gamma^0$, and that in our representation

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} . \tag{1.180a}$$

We can then compute

$$\overline{u}_r(k) = \begin{bmatrix} \xi_r\sqrt{\omega_k+M} \\ \dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k+M}}\xi_r \end{bmatrix}^\dagger \gamma_0 \tag{1.181a}$$

$$= \begin{bmatrix} \xi_r^\top\sqrt{\omega_k+M}, & \xi_r^\top\dfrac{(\vec{k}\cdot\vec{\sigma})}{\sqrt{\omega_k+M}} \end{bmatrix}\gamma_0 \tag{1.181b}$$ $\vec{k}$ and ξ_r are real, $\sigma=\sigma^\dagger$.

$$= \begin{bmatrix} \xi_r^\top\sqrt{\omega_k+M}, & -\xi_r^\top\dfrac{(\vec{k}\cdot\vec{\sigma})}{\sqrt{\omega_k+M}} \end{bmatrix}. \tag{1.181c}$$

The computation for the negative energy solution is analogous. □

Claim 1.4.4. *We can derive the normalization conditions (1.174a) from the explicit expressions of the solutions.*

Proof. Writing out the multiplication explicitly for the real solutions we have:

$$\overline{u}_r(k)u_s(k) = \begin{bmatrix} \xi_r^\top\sqrt{\omega_k+M}, & -\xi_r^\top\dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k+M}} \end{bmatrix}\begin{bmatrix} \xi_s\sqrt{\omega_k+M} \\ \dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k+M}}\xi_s \end{bmatrix} \tag{1.182a}$$

$$= \xi_r^\top\sqrt{\omega_k+M}\xi_s\sqrt{\omega_k+M} - \xi_r^\top\frac{(\vec{k}\cdot\vec{\sigma})}{\sqrt{\omega_k+M}}\frac{(\vec{k}\cdot\vec{\sigma})}{\sqrt{\omega_k+M}}\xi_s \tag{1.182b}$$

$$= \delta_{rs}(\omega_k+M) - \delta_{rs}\frac{|k|^2}{\omega_k+M} \tag{1.182c}$$

$$= \delta_{rs}\frac{(\omega_k+M)^2-|k|^2}{\omega_k+M} \tag{1.182d}$$

$$= \delta_{rs}\frac{M^2+2\omega_kM+\omega_k^2-|k|^2}{\omega_k+M} = 2M\delta_{rs}\frac{\omega_k+M}{\omega_k+M}, \tag{1.182e}$$ Used $\omega_k^2-|k|^2=M^2$

where we applied the identity

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \left(\vec{a}\cdot\vec{b}\right)\mathbb{1} + i\left(\vec{a}\times\vec{b}\right)\cdot\vec{\sigma}\,, \tag{1.183}$$

which follows from the commutation and anticommutation relations

$$[\sigma_a,\sigma_b] = 2i\epsilon_{abc}\sigma_c \tag{1.184a}$$

$$\{\sigma_a,\sigma_b\} = 2\delta_{ab}\mathbb{1}\,. \tag{1.184b}$$

Now we can replicate the calculation for the negative energy solution:

$$\overline{v}_rv_s = \begin{bmatrix} \xi_r^\top\dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k+M}}, & -\xi_r^\top\sqrt{\omega_k+M} \end{bmatrix}\begin{bmatrix} \dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k+M}}\xi_s \\ \xi_s\sqrt{\omega_k+M} \end{bmatrix} \tag{1.185a}$$

$$= \delta_{rs}\left(\frac{|k|^2}{\omega_k + M} - \omega_k - M\right) \tag{1.185b}$$

$$= -2M\delta_{rs}\,, \tag{1.185c}$$

where we skipped some steps since we can recognize the opposite of the expression we found earlier, equation (1.182b).

For the mixed terms, instead, we get

$$\overline{v}_r u_s = \begin{bmatrix} \xi_r^\top \dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k + M}}, & -\xi_r^\top \sqrt{\omega_k + M} \end{bmatrix} \begin{bmatrix} \xi_s \sqrt{\omega_k + M} \\ \dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k + M}}\xi_s \end{bmatrix} \tag{1.186a}$$

$$= \xi_r^\top \frac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k + M}}\xi_s \sqrt{\omega_k + M} - \xi_r^\top \frac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k + M}}\xi_s \sqrt{\omega_k + M} = 0 = \overline{u}_r v_s\,. \tag{1.186b}$$

□

Claim 1.4.5. *The following identities hold:*

$$u_r^\dagger(k)u_s(k) = 2\omega_k\delta_{rs} \tag{1.187a}$$

$$v_r^\dagger(k)v_s(k) = 2\omega_k\delta_{rs} \tag{1.187b}$$

$$u_r^\dagger(k)v_s(-k) = v_r^\dagger(k)u_s(-k) = 0\,. \tag{1.187c}$$

Notice that now we have a dagger instead of a bar.

Proof. In order to see what these solutions are we need to compute $u_r^\dagger$: the difference between it and $\overline{u}_r$ is the lack of multiplication by γ^0, which in our representation means that the sign of the second component is not flipped. So, in the calculation at step (1.182b) we have instead

$$\xi_r^\top \sqrt{\omega_k + M}\xi_s \sqrt{\omega_k + M} - \xi_r^\top \frac{(\vec{k}\cdot\vec{\sigma})}{\sqrt{\omega_k + M}} \frac{(\vec{k}\cdot\vec{\sigma})}{\sqrt{\omega_k + M}}\xi_s = \delta_{rs}\frac{\omega_k^2 + M^2 + 2\omega_k M + |k|^2}{\omega_k + M} \tag{1.188a}$$

$$= 2\delta_{rs}\omega_k\,, \tag{1.188b}$$

where we applied a similar line of reasoning to the other proof. The negative sign makes it so instead of cancelling the ω_k^2 term we cancel the M^2 term.

For the negative energy solution we have basically the same thing. Let us consider the product of the negative and positive solutions: if we swap the sign we find that the result is nonzero since the terms in (1.186b) do not cancel anymore.

However, if we flip the sign of one of the two momenta the terms cancel. □

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So, we can write out the **general solution** for the Dirac equation: the spinor and its conjugate are

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \sum_{r=1}^{2} \left[c_r(k)u_r(k)e^{-ikx}d_r^*(k)v_r(k)e^{ikx}\right]_{\omega_k = k^0} \tag{1.189a}$$

$$\overline{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k}{\sqrt{2\omega_k}} \sum_{r=1}^{2} \left[d_r(k) \overline{v}_r(k) e^{-ikx} c_r^*(k) \overline{u}_r(k) e^{ikx} \right]_{\omega_k = k^0} . \tag{1.189b}$$

The functions c_r and d_r represent the two degrees of freedom of the positive and negative energy solution respectively.

The important thing to recall is

$$\psi_+ \sim cu \qquad \text{and} \qquad \psi_- \sim d^* v \,. \tag{1.190}$$

Note that c_r and d_r are scalar functions, while u_r and v_r are 4D spinors.

1.4.7 Energy projectors

We wish to distinguish positive and negative energy solutions: so, we define the projector onto the positive and negative energy subspaces,

$$\Lambda_\pm(k) = \frac{\pm \not{k} + M}{2M} \,. \tag{1.191}$$

These two are an incomplete set of orthogonal projectors.

Claim 1.4.6. *They satisfy*

$$\Lambda_\pm^2(k) = \Lambda_\pm(k) \tag{1.192a}$$
$$\Lambda_+(k) + \Lambda_-(k) = \mathbb{1} \tag{1.192b}$$
$$\Lambda_+(k)\Lambda_-(k) = 0 \tag{1.192c}$$
$$\mathrm{Tr}\left[\Lambda_\pm(k)\right] = 2 \,. \tag{1.192d}$$

Proof. Let us first establish what $\not{k}^2$ is equal to:

$$\not{k}^2 = \gamma^\mu k_\mu \gamma^\nu k_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} k_\mu k_\nu = \eta^{\mu\nu} k_\mu k_\nu = k^2 \mathbb{1}_4 \,. \tag{1.193}$$

So, we can compute

$$\Lambda_\pm^2(k) = \frac{1}{4M^2}\left[\not{k}^2 + M^2 \mathbb{1} \pm 2\not{k} M\right] \tag{1.194}$$
$$= \frac{1}{4M^2}[2M\mathbb{1} \pm 2\not{k} M] = \frac{\pm\not{k} + M}{2M} \,. \tag{1.195}$$

For the second property, we get

$$\Lambda_+ + \Lambda_- = \frac{+\not{k} + M - \not{k} + M}{2M} = \frac{2M}{2M} = \mathbb{1} \,. \tag{1.196}$$

For the third expression, we find

$$\Lambda_+\Lambda_- \propto (\not{k} + M)(\not{k} - M) = \not{k}^2 - M^2 = 0 \,. \tag{1.197}$$

For the fourth expression, instead, we find that since $\not{k}$ is traceless the trace is equal to that of $(M/2M)\mathbb{1}$, which is equal to 2 since spinorial matrices are 4-dimensional. □

We can apply these projectors to the solutions we know how to write from the general expression (1.175):

$$\Lambda_+ u_r(k) = \frac{C}{2M}(\not{k} + M)^2 u_r(M) = C(\not{k} + M)u_r(M) = u_r(k) \tag{1.198}$$

$$\Lambda_- u_r(k) = \frac{C}{2M}(-\not{k} + M)(\not{k} + M)u_r(M) = 0 \tag{1.199}$$

$$\Lambda_+ v_r(k) = \frac{C}{2M}(\not{k} + M)(\not{k} - M)u_r(M) = 0 \tag{1.200}$$

$$\Lambda_- v_r(k) = \frac{C}{2M}(-\not{k} + M)(-\not{k} + M)u_r(M) = v_r(k)\,. \tag{1.201}$$

These projectors can also be obtained from the vectors themselves: if a vector v is normalized to 1, the matrix $vv^\top$ is a projector onto the subspace of the vector.

We can add projectors together in order to get projectors onto larger subspaces.

Claim 1.4.7. *We can recover the projectors $\Lambda_\pm$ by the expressions:*

$$\Lambda_+(k) = \sum_r \frac{u_r(k)\overline{u}_r(k)}{2M} \tag{1.202}$$

$$\Lambda_-(k) = -\sum_r \frac{v_r(k)\overline{v}_r(k)}{2M}\,. \tag{1.203}$$

Proof. The calculation yields

$$u_r(k)\overline{u}_r(k) = \begin{bmatrix} \xi_r\sqrt{\omega_k + M} \\ \dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k + M}}\xi_r \end{bmatrix} \begin{bmatrix} \xi_r^\top\sqrt{\omega_k + M}, & -\xi_r^\top\dfrac{\vec{k}\cdot\vec{\sigma}}{\sqrt{\omega_k + M}} \end{bmatrix} \tag{1.204}$$

$$= \begin{bmatrix} \xi_r\xi_r^\top(\omega_k + M) & -\xi_r\xi_r^\top(\vec{k}\cdot\vec{\sigma}) \\ (\vec{k}\cdot\vec{\sigma})\xi_r\xi_r^\top & -\frac{(\vec{k}\cdot\vec{\sigma})\xi_r\xi_r^\top(\vec{k}\cdot\vec{\sigma})}{\omega_k + M} \end{bmatrix}, \tag{1.205}$$

and now we notice the fact that $\sum_r \xi_r\xi_r^\top = \mathbb{1}_2$, so if we sum over r we get

$$\sum_r u_r(k)\overline{u}_r(k) = \begin{bmatrix} \omega_k + M & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -\frac{|k|^2}{\omega_k + M} \end{bmatrix} \tag{1.206}$$

$$= \begin{bmatrix} \omega_k + M & -\vec{k}\cdot\vec{\sigma} \\ \vec{k}\cdot\vec{\sigma} & -\omega_k + M \end{bmatrix} \tag{1.207}$$

$$= \not{k} + M\,. \tag{1.208}$$

If we divide by $2M$ we get our result. The computation for v_r is similar. □

1.4.8 Spin operators

Recall, the spinorial representation of the Lorentz group is generated by the matrices $\Sigma^{\mu\nu}$, which can be calculated by

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\,. \tag{1.209}$$

We can use these to define, in the Dirac representation, the generators of boosts:

$$K_i = \Sigma^{i0} = -\frac{i}{2}\begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \tag{1.210}$$

and of rotations:

$$\Sigma_i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk} = \frac{1}{2}\begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}. \tag{1.211}$$

We can exponentiate these to get the representations of general boosts and rotations: a boost will be written as

$$S(\Lambda) = \exp\left(-\frac{i}{2}\omega_{0i}K_i\right), \tag{1.212}$$

while a rotation will be written as

$$S(\Lambda) = \exp\left(-\frac{i}{2}\theta_k\Sigma_k\right), \tag{1.213}$$

where we used the angle $\theta_k = \epsilon_{ijk}\omega^{ij}$. These Σ_k are called the spin operators. They result in an *internal rotation* of the spinor corresponding to a rotation of our coordinates.

In the rest frame of the particle, the basis states we chose are the eigenstates of Σ_3: We have

$$\Sigma_3 u_r(M) = \frac{1}{2}\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}\sqrt{2M}\begin{bmatrix} \xi_r \\ 0 \end{bmatrix} = \pm\frac{1}{2}u_R(M), \tag{1.214}$$

where we have either $+$ or $-$ depending on whether $r = 1$ or 2, respectively.

Also, the negative energy solutions are basis states: we get

$$\Sigma_3 v_r(M) = \frac{1}{2}\begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}\sqrt{2M}\begin{bmatrix} 0 \\ \xi_r \end{bmatrix} = \pm\frac{1}{2}v_r(M). \tag{1.215}$$

So, our particles have spin $1/2$. We can have the z component of the spin to be $\pm 1/2$, and positive and negative energy: in the end, these are 4 degrees of freedom.

Note that we have eigenstates of Σ_3 specifically because of the way we chose ξ_r: we can have other equivalent choices.

1.4.9 The helicity operator

Claim 1.4.8. *The spin projection along a given axis is not a good quantum number: it does not commute with the Hamiltonian, $[H, \Sigma_3] \neq 0$.*

Proof. What is the Dirac Hamiltonian? Recall that we started the whole discussion by requiring that the Dirac equation should have the form

$$i\partial_t\psi = H_D\psi, \tag{1.216}$$

and the ansatz we used was

$$H_D = \alpha \cdot \vec{p} + \beta M \,, \tag{1.217}$$

and we found that for a massive particle we can write α_i and β as 4-dimensional matrices, which in the Dirac representation are

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad \text{and} \qquad \beta = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} . \tag{1.218}$$

So, the Hamiltonian is a 4-dimensional matrix, which looks like

$$H_D = \begin{bmatrix} M & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -M \end{bmatrix} . \tag{1.219}$$

So, let us compute the commutator: we get

$$\left[\vec{\alpha} \cdot \vec{p} + \beta M, \Sigma_3\right] = \left[\vec{\alpha}, \Sigma_3\right] \cdot \vec{p} + M\left[\beta, \Sigma_3\right] . \tag{1.220}$$

We can just compute the two matrix commutators, since the momentum and the mass are fixed. We find, for each component:

$$[\alpha_i, \Sigma_3] = \frac{1}{2} \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & [\sigma_3, \sigma_i] \\ [\sigma_3, \sigma_i] & 0 \end{bmatrix} , \tag{1.221}$$

and we know that this is given by[8]

$$[\sigma_3, \sigma_i] = 2i\epsilon_{3ik}\sigma_k \,, \tag{1.222}$$

so in the full expression we find

$$(\sigma_a \otimes \sigma_b)[\alpha_i, \Sigma_3] = i\epsilon_{3ik}\alpha_k \,, \tag{1.223}$$

while β and Σ_3 are both diagonal, and therefore they commute. So, we can finally say that

$$[H_D, \Sigma_3] = i\epsilon_{3jk} p_j \alpha_k \,, \tag{1.224}$$

which is not zero *unless* $\vec{p}$ is oriented along the z axis. This, however, is not a covariant condition. □

So, we define the **helicity** as the projection of the spin along the direction of motion:

$$\Sigma_p = \frac{\vec{\Sigma} \cdot \vec{p}}{|p|} = \vec{\Sigma} \cdot \hat{p} \,. \tag{1.225}$$

[8] A more elegent way to deal with these computations is to intrpret them in terms of tensor products of Pauli matrices: we have, for instance, $\alpha_i = \sigma_x \otimes \sigma_i$, and there are explicit formulas for products in the form $(\sigma_a \otimes \sigma_b)(\sigma_c \otimes \sigma_d)$ [aga18].

The helicity is a good quantum number. It commutes with the Dirac Hamiltonian, whereas Σ_3 does not. So, the eigenvalue of Σ_p is conserved: this makes it a good quantum number, since it provides a reliable description of the state.

From the result we found before we can see that what we are basically doing here is calculating Σ_p along the momentum, which is equivalent (after a rotation of the axes) to setting $\vec{p} = |p|\hat{z}$. Then, from equation (1.224) we can see that Σ_p will commute with the Hamiltonian.

This is much simpler than the solution proposed in the professor's notes, but just as valid, I think: after all, we can choose the coordinates in which we perform the calculation as we wish since in the end we get a covariant result.

1.4.10 Pauli-Lubanski vector and helicity

We define the following pseudovector:[9]

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} p_\sigma \,, \tag{1.226}$$

where $J_{\nu\rho}$ is the total angular momentum tensor, which includes both an angular and spin component:

$$J_{\nu\rho} = L_{\nu\rho} + \Sigma_{\nu\rho} \qquad \text{where} \qquad L_{\nu\rho} = 2x_{[\nu} p_{\rho]} \,. \tag{1.227}$$

There are two terms in the expression for ω^μ, but the angular momentum one vanishes: we have

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} L_{\nu\rho} p_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Big(x_\nu p_\rho - x_\rho p_\nu\Big) p_\sigma = 0 \,, \tag{1.228}$$

since in both cases we have the contraction of two copies of the momentum, which are symmetric in their indices, with the antisymmetric tensor.

Now, let us move to the rest frame of the particle, so that $p^\mu = (M, \vec{0})$. Then, we will have

$$\omega^\mu = \frac{M}{2}\epsilon^{\mu\nu\rho 0}\Sigma_{\nu\rho} \,. \tag{1.229}$$

Therefore, this vector is purely spatial since setting $\mu = 0$ means the RHS vanishes: so, $\omega^0 = 0$. On the other hand, we have

$$\omega^i = \frac{M}{2}\epsilon^{0i\nu\rho}\Sigma_{\nu\rho} = \frac{M}{2}\epsilon^{ijk}\Sigma_{jk} = M\Sigma^i \,, \tag{1.230}$$

[9] Pseudovector means that it is even under parity transformations $P : \vec{x} \to -\vec{x}$.

More formally, we can say that it transforms under spatial rotations like a vector density: if $R \in O(3)$, a regular vector transforms like $v' = Rv$, but a pseudovector transforms like $w' = (\det R)Rw$. So, while a vector is flipped by a parity: $Pv = -v$ a pseudovector is not: $Pw = w$.

We can obtain pseudovectors by taking the cross product of regular vectors. The magnetic field $\vec{B}$ is an example of a pseudovector.

where we applied the definition of the spin generator Σ^i (1.211), and we moved from the four-dimensional Kronecker symbol to the three-dimensional one: we are allowed to move to three-vector indices since the case in which they are equal to zero is excluded by the first index being zero.

So, we can take the square modulus of ω^μ in the rest frame, and normalize it by M^2. Since this is a 4-vector this will be a covariant quantity:

$$\left.\frac{\omega_\mu \omega^\mu}{M^2}\right|_{\text{rest frame}} = \Sigma^i \Sigma_i = -\Sigma^i \Sigma^i = -\frac{1}{4}\begin{bmatrix} \sigma_i \sigma_i & 0 \\ 0 & \sigma_i \sigma_i \end{bmatrix} = -\frac{3}{4}\mathbb{1}\,. \tag{1.231}$$

Factor 3 since there is a sum over i.

This is consistent with the expression we know from quantum mechanics, that is, the eigenvalue of s^2 being $s(s+1)$ with $s = 1/2$.

What about the negative sign?

So, the PL vector gives us a way to define the spin of a particle in a relativistic manner.

We can use a unit vector n^μ which is orthogonal to the momentum of the particle, that is, such that $n^\mu p_\mu = 0$ and $n^\mu n_\mu = -1$, to define a spin in a generic direction:

$$\left.\frac{\omega^\mu n_\mu}{M}\right|_{\text{rest frame}} = -\vec{\Sigma}\cdot\vec{n}\,. \tag{1.232}$$

We can also define the helicity in this frame work, using the vector

$$n_p^\mu = \frac{1}{M}\begin{bmatrix} |p| \\ \omega_p \hat{p} \end{bmatrix}, \tag{1.233}$$

which has square modulus $(|p|^2 - \omega_p^2)/M^2 = -1$, and is orthogonal to the momentum: their product is $\propto |p|\omega_p - \omega_p|p| = 0$.

For some geometrical intuition: if we trace out a 2D spacetime diagram with the directions of time and $\hat{p}$, the momentum will be some timelike vector, while this n_p^μ will be its reflection with respect to the light-like axis.

With this, we recover

$$\left.\frac{\omega^\mu n_{\mu,p}}{M}\right|_{\text{rest frame}} = -\Sigma_p = -\vec{\Sigma}\cdot\hat{p}\,. \tag{1.234}$$

1.4.11 Chirality

We can define another matrix beyond the γ^μ: it is

$$\gamma_5 = -\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = +i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{1.235}$$

Note the sign convention: we choose $\epsilon^{0123} = +1$, which means that we get $\epsilon_{0123} = -1$, since we must lower an odd amount of spacelike indices.

Claim 1.4.9. *The γ^5 matrix is self adjoint:*

$$\gamma_5 = (\gamma_5)^\dagger , \tag{1.236}$$

it squares to the identity:

$$(\gamma_5)^2 = \mathbb{1} , \tag{1.237}$$

and it anticommutes with the other gamma matrices:

$$\{\gamma^\mu, \gamma_5\} = 0 . \tag{1.238}$$

Proof. We will need to move a few matrices around, so first let us establish that, since

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu} , \tag{1.239}$$

we have that the square of γ^0 is $\mathbb{1}$, while the square of γ^i is $-\mathbb{1}$.

On the other hand, we can switch the places of two different γ matrices as long as we switch the global sign. So, first of all let us show that the square of the γ_5 matrix is the identity: we write a number in place of the γ, for clarity.

$$(\gamma_5)^2 = i^2 01230123 = -01230123 \tag{1.240}$$

$$= (-)^4 00123123 = (-)^6 00112323 = (-)^7 00112233 \tag{1.241}$$

$$= -\mathbb{1}(-\mathbb{1})^3 = \mathbb{1} . \tag{1.242}$$

For the self-adjointness, we have

$$\gamma_5^\dagger = +\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^{\sigma\dagger}\gamma^{\rho\dagger}\gamma^{\nu\dagger}\gamma^{\mu\dagger} \tag{1.243}$$

$$= \frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu \tag{1.244}$$

$$= \frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma . \tag{1.245}$$

Here we used the fact that the Dirac matrices are self-adjoint, and that we can get from the configuration $\mu\nu\rho\sigma$ to $\sigma\rho\nu\mu$ in an even number of "hops". At each hop we can exchange the matrices by changing sign, like before, so there is no issue. The matrices we are exchanging are surely different since they are multiplied by the Kronecker symbol.

For the anticommutation: say we have the expression $\gamma^\mu\gamma_5$ for some μ and we want to bring the γ^μ to the other side. We can commute it with the Kronecker symbol and itself, and we need to anticommute it with three matrices different from itself. This gives us three minus signs, so the result has an opposite sign to what we started with: $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$, what we wanted to prove. □

In the Dirac representation,

$$\gamma_5 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} . \tag{1.246}$$

Claim 1.4.10. *The γ_5 matrix commutes with the spin tensor $\Sigma^{\mu\nu}$.*

Proof. Recall that $\Sigma^{\mu\nu} \propto [\gamma^\mu, \gamma^\nu]$. Then, as before writing the indices without the γs:

$$[\gamma_5, \Sigma^{\mu\nu}] \propto 5\mu\nu - 5\nu\mu - \mu\nu 5 + \nu\mu 5 \tag{1.247}$$

$$= -\mu 5\nu + \nu 5\mu + \mu 5\nu - \nu 5\mu = 0\,, \tag{1.248}$$

where we anticommuted γ_5 with γ^μ and γ^ν. □

The fact that these commute means that they can be simultaneously diagonalized.

This allows us to classify the eigenvectors by the eigenvalue f γ_5 instead of the spin.

The $N = 4$ spinorial representation is reducible: it is the sum of 2 2-dimensional representations.

The γ_5 matrix is called the **chirality** operator, using it we can define the projectors

$$P_{R,L} = \frac{\mathbb{1} \pm \gamma_5}{2}\,, \tag{1.249}$$

where the minus corresponds to the left projector, while the plus corresponds to the right.

Claim 1.4.11. *These are indeed a complete set of orthogonal projectors: they are idempotent, orthogonal and they sum to the identity.*

Also, they are self adjoint.

Proof. First, let us show idempotence; we denote either of L or R by x.

$$P_x^2 = \frac{1}{4}\left(1 \pm 2\gamma_5 + \gamma_5^2\right) = \frac{1}{4}(2\mathbb{1} \pm 2\gamma_5) = P_x\,. \tag{1.250}$$

Then, to see orthogonality:

$$P_L P_R = \frac{1}{4}(\mathbb{1} - \gamma_5)(\mathbb{1} + \gamma_5) = \frac{1}{4}\left(\mathbb{1}^2 - \gamma_5^2\right) = 0\,. \tag{1.251}$$

Finally, they sum to the identity:

$$P_L + P_R = \frac{1}{2}(\mathbb{1} - \gamma_5) + \frac{1}{2}(1 + \gamma_5) = \mathbb{1}\,. \tag{1.252}$$

They are self-adjoint because $\mathbb{1}$ and γ_5 are. □

We can then project any Dirac spinor into the left or right chiral subspaces by applying these matrices.

The projections onto the right-chiral subspace will have a $+1$ eigenvalue for γ_5, the projections onto the left-chiral subspace will have eigenvalue -1.

These chiral spinors are the irreducible $N = 2$ representations which constitute the full spinor.

For the massless case we could also have used a $N = 2$ representation for the full space, the Weyl spinor. This would have behaved like a left or right chiral spinor.

Finally, we can define the **conjugate chiral spinors**:

$$\overline{\psi}_L = \left(\psi_L^\dagger\right)\gamma^0 = \psi^\dagger P_L \gamma^0 = \overline{\psi} P_R \tag{1.253}$$

$$\overline{\psi}_R = \left(\psi_R^\dagger\right)\gamma^0 = \psi^\dagger P_R \gamma^0 = \overline{\psi} P_L\,, \tag{1.254}$$

where we used the fact that

$$\gamma^0(1 \pm \gamma_5) = (1 \mp \gamma_5)\gamma^0\,, \tag{1.255}$$

since the identity commutes with γ^0, while γ^5 anticommutes with it.

1.5 Dirac equation coupled to an external EM field

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We can use a complex solution to the Dirac equation to describe a spin $1/2$ particle. As we did before, we make the minimal coupling ansatz:

$$\partial_\mu \to \mathrm{D}_\mu = \partial_\mu + iqA_\mu\,, \tag{1.256}$$

so that our Dirac equation will read

$$(i\not{\mathrm{D}} - M)\psi = \left(i\not{\partial} - q\not{A} - M\right) = 0\,, \tag{1.257}$$

which in three-vector notation (after the computation in the equations (1.123)) reads

$$i\partial_0\psi = \left[i\vec{\alpha}\cdot\left(-\vec{\nabla} + iq\vec{A}\right) + \beta M + qA_0\right]\psi\,, \tag{1.258}$$

where we are using the definition $\partial_i = \vec{\nabla}$, while $A^i = \vec{A}$: the "natural" placement for the spatial index of the derivative is lower, while for other vectors it is upper.

1.5.1 Nonrelativistic limit of the Dirac equation

As we did in the case of the Klein-Gordon equation, we start out by factoring out the time-evolution: we define

$$\psi(\vec{x}, t) = e^{-iMt}\psi'(\vec{x}, t)\,, \tag{1.259}$$

which we can plug into the expression for the Dirac equation: we find

$$i\partial_0\left(e^{-iMt}\psi'\right) = \left[i\vec{\alpha}\cdot\left(-\vec{\nabla} + iq\vec{A}\right) + \beta M + qA_0\right]\left(e^{-iMt}\psi'\right) \tag{1.260}$$

$$e^{-iMt}\left(-i^2 + i\partial_0\right)\psi' = e^{-iMt}\left[i\vec{\alpha}\cdot\left(-\vec{\nabla} + iq\vec{A}\right) + \beta M + qA_0\right]\psi' \tag{1.261}$$

$$i\partial_0\psi' = \left[i\vec{\alpha}\cdot\left(-\vec{\nabla} + iq\vec{A}\right) + (\beta - \mathbb{1})M + qA_0\right]\psi'\,, \tag{1.262}$$

which we can write using the explicit spinorial expressions for the $\vec{\alpha}$ and β matrices; for clarity we also divide ψ' into two two-component spinors φ' and χ':

$$\begin{aligned} i\partial_0 \begin{bmatrix} \varphi' \\ \chi' \end{bmatrix} &= \begin{bmatrix} 0 & i\vec{\sigma}\cdot(-\vec{\nabla}+iq\vec{A}) \\ i\vec{\sigma}\cdot(-\vec{\nabla}+iq\vec{A}) & 0 \end{bmatrix} \begin{bmatrix} \varphi' \\ \chi' \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & -2M \end{bmatrix} \begin{bmatrix} \varphi' \\ \chi' \end{bmatrix} \\ &+ \begin{bmatrix} qA_0 & 0 \\ 0 & qA_0 \end{bmatrix} \begin{bmatrix} \varphi' \\ \chi' \end{bmatrix} . \end{aligned} \tag{1.263}$$

Then we can read off the two *coupled* equations for these 2D spinors:

$$i\partial_0\varphi' = i\vec{\sigma}\cdot\left(-\vec{\nabla}+iq\vec{A}\right)\chi' + qA_0\varphi' \tag{1.264}$$

$$i\partial_0\chi' = i\vec{\sigma}\cdot\left(-\vec{\nabla}+iq\vec{A}\right)\varphi' + (-2M+qA_0)\chi' . \tag{1.265}$$

Notice that the mass only appears in the second equation: so, we can apply the nonrelativistic approximation, which amounts to saying that the mass M is the largest energy at play; formally, this means

$$\left|qA_0\right| \ll M \qquad \text{and} \qquad \left|\frac{\partial_0\chi'}{\chi'}\right| \ll M , \tag{1.266}$$

so we remove those terms in the second equation; if we bring the term $2M\chi'$ to the left hand side we get:

$$\chi' = \frac{i}{2M}\vec{\sigma}\cdot\left(-\vec{\nabla}+iq\vec{A}\right)\varphi' , \tag{1.267}$$

so we have an explicit constraint on the value of χ' if we know φ'.

We do not know *a priori* the asymptotic relation of χ' to φ', so we only made cancellations in terms which were comparable since they were all applied to χ'.

Now, then we have found that, since $\chi' \sim \varphi'/M$, the magnitude of φ' is much larger than that of χ'.

We can substitute what we found for χ' into the first equation: we find

$$i\partial_0\varphi' = \frac{1}{2M}\left[i\vec{\sigma}\cdot\left(-\vec{\nabla}+iq\vec{A}\right)\right]^2\varphi' + qA_0\varphi' \tag{1.268}$$

$$= qA_0\varphi' - \frac{1}{2M}\left[-\vec{\sigma}\cdot\vec{\nabla}+iq\vec{\sigma}\cdot\vec{A}\right]^2\varphi' . \tag{1.269}$$

This is the *Pauli equation coupled to an external magnetic field*, the generalization of the Schrödinger equation to spin $1/2$ charged particles.

We can make it more explicit by squaring the differential operator.

The sign convention in the notes is weird, in that a 3D vector written as $\vec{x}$ does not consistently mean neither x^i nor x_i. Instead, we have $\vec{\nabla} = \partial_i$ and $\vec{A} = A^i$.

When we write out a scalar product we imply that one index should be upper and one should be lower, if this is not the case already one of them must be raised or lowered. Once we are in component notation, though, we can move indices around as we like.

So, explicitly we have

$$\left(-\vec{\sigma}\cdot\vec{\nabla}+iq\vec{\sigma}\cdot\vec{A}\right)^2\varphi' = \left(-\sigma^i\partial_i+iq\sigma^iA^i\right)^2\varphi' \tag{1.270}$$

$$= \left(-\sigma^i\partial_i-iq\sigma^iA_i\right)^2\varphi' \tag{1.271}$$

$$= \left(\sigma_i\partial_i+iq\sigma_iA_i\right)^2\varphi' \tag{1.272}$$

$$= \left(\sigma_i\partial_i+iq\sigma_iA_i\right)\left(\sigma_j\partial_j+iq\sigma_jA_j\right)\varphi' \tag{1.273}$$

$$= \left(\sigma_i\partial_i+iq\sigma_iA_i\right)\left(\sigma_j\left(\partial_j\varphi'\right)+iq\sigma_jA_j\varphi'\right) \tag{1.274}$$

$$= \sigma_i\sigma_j\left[\partial_i\partial_j+iq\left(\partial_iA_j+A_j\partial_i+A_i\partial_j\right)-q^2A_iA_j\right]\varphi', \tag{1.275}$$

and now we notice that all the terms except for ∂_iA_j are symmetric in ij: so, we split the product $\sigma_i\sigma_j$ into its symmetric and antisymmetric parts. This yields

$$\sigma_i\sigma_j = \frac{1}{2}\left\{\sigma_i,\sigma_j\right\}+\frac{1}{2}\left[\sigma_i,\sigma_j\right] = \delta_{ij}+i\epsilon_{ijk}\sigma_k\,, \tag{1.276}$$

so we find

$$\left(-\vec{\sigma}\cdot\vec{\nabla}+iq\vec{\sigma}\cdot\vec{A}\right)^2\varphi' = \left(\delta_{ij}+i\epsilon_{ijk}\sigma_k\right)\left[\partial_i\partial_j+iq\left(\partial_iA_j+A_j\partial_i+A_i\partial_j\right)-q^2A_iA_j\right]\varphi' \tag{1.277}$$

$$= \left[\partial_i\partial_i+iq(\partial_iA_i+2A_i\partial_i)-q^2A_iA_i\right]\varphi'+\left[i^2q\partial_iA_j\epsilon^{ijk}\sigma_k\right]\varphi' \tag{1.278}$$

$$= \left[\vec{\nabla}^2-iq\left(\vec{\nabla}\cdot\vec{A}+2\vec{A}\cdot\vec{\nabla}\right)-q^2\vec{A}^2+\vec{\sigma}\cdot\vec{B}\right]\varphi' \tag{1.279}$$

$$= \left[-\vec{\nabla}+iq\vec{A}\right]^2\varphi+q\vec{\sigma}\cdot\vec{B}\varphi'\,, \tag{1.280}$$

where we used the fact that $\partial_iA_j\epsilon_{ijk} = \frac{1}{2}F_{ij}\epsilon_{ijk} = B_k$, and in the last step we set $\nabla\cdot\vec{A} = 0$, the Coulomb gauge condition.

Notice that we had to raise the indices of both $\vec{A}$ and $\vec{B}$, so we had to switch the sign of those terms.

Then, we can finally write the full Pauli equation:

$$i\partial_0\varphi' = \left\{-\frac{1}{2M}\left[-\vec{\nabla}+iq\vec{A}\right]^2+qA_0-\frac{q}{2M}\vec{\sigma}\cdot\vec{B}\right\}\varphi'\,, \tag{1.281}$$

which differs from the minimally-coupled Schrödinger equation by the spin coupling to the magnetic field. This new term in the Hamiltonian is

$$H_{\text{dip}} = -\frac{q}{2M}\vec{\sigma}\cdot\vec{B} = -\vec{\mu}_s\cdot\vec{B}\,, \tag{1.282}$$

where we introduced the **intrinsic magnetic moment**

$$\vec{\mu}_s = \frac{q}{2M}\vec{\sigma} = \frac{q}{M}\vec{\Sigma}^{(3)}\,. \tag{1.283}$$

In nonrelativistic quantum mechanics this term is introduced by hand: if the wavefunction is a scalar there is no way for this term to come about. Once we start describing it as a spinor, though, we can see where the term comes from.

The **magnetic dipole moment** associated with the *orbital angular momentum*, as opposed to the spin, is defined as

$$\vec{\mu}_L = \frac{q}{2M}\vec{L}\,, \tag{1.284}$$

so in this case the ratio of magnetic moment to momentum is

$$\frac{\left|\vec{\mu}_L\right|}{\left|\vec{L}\right|} = \frac{q}{2M}\,, \tag{1.285}$$

while for the spin we defined (already in nonrelativistic QM)

$$\vec{\mu}_s = \frac{q}{2M}g_e\vec{\Sigma}^{(3)} \qquad \Longrightarrow \qquad \frac{\left|\vec{\mu}_s\right|}{\left|\vec{\Sigma}\right|} = \frac{q}{2M}g_e\,, \tag{1.286}$$

where we define the **electron gyromagnetic factor** g_e. This can be compared with the equation we found before for the intrinsic magnetic moment (1.283), to yield our prediction:

$$g_e = 2\,. \tag{1.287}$$

Claim 1.5.1. *We have the relation*

$$\Big[i\partial_\mu - qA_\mu, i\partial_\nu - qA_\nu\Big]\psi = -iqF_{\mu\nu}\psi\,. \tag{1.288}$$

Proof. We write the terms of the product out, antisymmetrizing everything:

$$\Big[i\partial_\mu - qA_\mu, i\partial_\nu - qA_\nu\Big]\psi = \tag{1.289}$$

$$= 2\left(i\partial_{[\mu}i\partial_{\nu]} - iq\left(\partial_{[\mu}A_{\nu]}\right) - iqA_{[\nu}\partial_{\mu]} - iqA_{[\mu}\partial_{\nu]} + q^2A_{[\mu}A_{\nu]}\right)\psi \tag{1.290}$$

$$= -2iq\partial_{[\mu}A_{\nu]}\psi = -iqF_{\mu\nu}\psi\,, \tag{1.291}$$

where we removed all the terms which were symmetric in $\mu \leftrightarrow \nu$.

Note that this is the commutator of the covariant derivatives on the manifold: it yields the Riemann tensor, which we then have shown to be given by the electromagnetic field-strength. □

Claim 1.5.2. *We have the relation*

$$\Sigma^{\mu\nu}F_{\mu\nu} = i\vec{\alpha}\cdot\vec{E} + \vec{\Sigma}\cdot\vec{B}\,, \tag{1.292}$$

where

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \qquad \textit{and} \qquad \vec{\Sigma} = \frac{1}{2}\begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}\,. \tag{1.293}$$

Proof. Recall the definitions of

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{1.294}$$

and of the the electric and magnetic fields in terms of the field-strength tensor:

$$E^i = F^{i0} = -F^{0i} = +F_{0i} \qquad \text{and} \qquad B^k = \frac{1}{2}F^{ij}\epsilon^{ijk}\,. \tag{1.295}$$

For the magnetic field we can also write the inverse expression:

$$F^{ij} = \epsilon^{ijk}B^k\,, \tag{1.296}$$

So, we distinguish the cases where $\mu = 0$ and $\mu = i$, a spatial 3D index.
In the first case, the index ν must be nonzero by antisymmetry, so we find

$$\Sigma^{0j}F_{0j} = \frac{i}{4}\left[\gamma^0, \gamma^j\right]E^j\,, \tag{1.297}$$

and since different γ matrices anticommute we can replace the commutator with twice the product:

$$\Sigma^{0j}F_{0j} = \frac{i}{2}\gamma^0\gamma^j E^j = \frac{i}{2}\gamma^0\gamma^0\alpha^j E^j = \frac{i}{2}\vec{\alpha}\cdot\vec{E}\,. \tag{1.298}$$

In the final expression we are summing over μ and ν, so the contribution will be twice this, since we need to account for the case where $\nu = 0$ as well as $\mu = 0$.

In the other case, we apply a similar reasoning; in this case we also need to recall the definition of the vector $\vec{\Sigma}$:

$$\Sigma^i = \frac{1}{2}\epsilon^{ijk}\Sigma^{jk}\,, \tag{1.299}$$

so we can substitute this into the expression and find:

$$\begin{aligned}\Sigma^{ij}F_{ij} &= \Sigma^{ij}F^{ij} &(1.300)\\ &= \Sigma^{ij}\epsilon^{ijk}B_k &(1.301)\\ &= 2\vec{\Sigma}^k B_k\,, &(1.302)\end{aligned}$$

which has an extra factor two... should figure out why this is the case

□

Claim 1.5.3. *We can also derive the prediction $g_e = 2$ from the full relativistic Dirac equation.*

Proof. We start by applying the operator $i\not{D} + M$ to the relativistic Dirac equation to an external electromagnetic field, just like what we did to recover the Klein-Gordon equation: we find

$$\left(-\not{D}^2 - M^2\right)\psi = 0 \tag{1.303}$$

$$\left[-\Big(\partial_\mu + ieA_\mu\Big)\gamma^\mu(\partial_\nu + ieA_\nu)\gamma^\nu - M^2\right]\psi = 0\,, \tag{1.304}$$

since $\mathrm{D}_\mu = \partial_\mu + ieA_\mu$. We have a product of gamma matrices: we can decompose it into its symmetric and antisymmetric parts, as

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] \tag{1.305}$$

$$= \eta^{\mu\nu} - 2i\Sigma^{\mu\nu}\,, \tag{1.306}$$

since

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]\,. \tag{1.307}$$

Therefore, we find

$$\left[\Big(\partial_\mu + ieA_\mu\Big)(\partial^\mu + ieA^\mu) - 2i\Sigma^{\mu\nu}\Big(\partial_\mu + ieA_\mu\Big)(\partial_\nu + ieA_\nu) - M^2\right]\psi = 0 \tag{1.308}$$

$$\left[-\mathrm{D}_\mu\mathrm{D}^\mu + 2i\Sigma^{\mu\nu}\mathrm{D}_{[\mu}\mathrm{D}_{\nu]} - M^2\right]\psi = 0\,, \tag{1.309}$$

and we can expand the antisymmetrized covariant derivative given what we know from equation (1.288):

$$-iqF_{\mu\nu}\psi = \Big[i\partial_\mu - qA_\mu, i\partial_\nu - qA_\nu\Big]\psi \tag{1.310}$$

$$= \Big[i\mathrm{D}_\mu, i\mathrm{D}_\nu\Big]\psi \tag{1.311}$$

$$= -2\mathrm{D}_{[\mu}\mathrm{D}_{\nu]}\psi \tag{1.312}$$

$$, \tag{1.313}$$

so we can write

$$\left[-\mathrm{D}_\mu\mathrm{D}^\mu + 2i\Sigma^{\mu\nu}\left(\frac{1}{2}iqF_{\mu\nu}\right) - M^2\right]\psi = 0 \tag{1.314}$$

$$\left[-\mathrm{D}_\mu\mathrm{D}^\mu - \Sigma^{\mu\nu}qF_{\mu\nu} - M^2\right]\psi = 0 \tag{1.315}$$

$$\left[-\mathrm{D}_\mu\mathrm{D}^\mu - g_e\frac{q}{2}\Sigma^{\mu\nu}F_{\mu\nu} - M^2\right]\psi = 0\,, \tag{1.316}$$

where $g_e = 2$. □

Chapter 2

Free field theories

2.1 Lagrangian and Hamiltonian formalisms

Saturday 2020-5-2, compiled 2020-06-01

The Lagrangian and Hamiltonian formalism can aid in the description of systems with either a finite of infinite number of degree of freedom.

2.1.1 Classical system with finite DoF: Lagrangian formalism

The usual example is a system of particles labelled by the index i with masses m_i, positions $q_i(t)$ and velocities $\dot{q}_i(t)$.

If the forces acting on the particles are **conservative**, we can express them in terms of a potential, which we assume not to depend on the velocities nor on time: $V(q_i)$. From now on when we write q or $\dot{q}$ we will mean the full vector of the positions or velocities.

If this is the case, the motion of the particles is described by Newton's equation:

$$m_i \ddot{q}_i = -\frac{\partial V}{\partial q_i} \,. \tag{2.1}$$

This can alternatively be described in terms of a function called the Lagrangian:

$$L(q, \dot{q}, t) \,, \tag{2.2}$$

which could depend on time, but we usually assume it to be independent of time explicitly, that is, $\partial_t L = 0$ (although the *total* derivative of the Lagrangian may be nonzero!).

The condition of the Lagrangian being explicitly time-independent is equivalent to the system of forces being conservative.

The Lagrangian can be a generic function of the positions and velocities of the particles, but in order to reproduce Newton's law we need it to be in the form

$$L = T - V = \frac{1}{2} m_i \dot{q}_i^2 - V(q_i) \,, \tag{2.3}$$

where a sum over the particles is implied in the kinetic energy term.

Typically problems in Lagrangian mechanics are given by fixing the boundary conditions as the initial and final position, as opposed to writing the initial values of position and velocity.

We want to find the physical trajectory the particle(s) will take corresponding to those initial and final conditions.[1]

This can be accomplished using Hamilton's principle of stationary action. We start by introducing the action:

$$S[q(t), t_{\text{in}}, t_{\text{fin}}] = \int_{t_{\text{in}}}^{t_{\text{fin}}} L(q(t), \dot{q}(t))\, \mathrm{d}t \ , \tag{2.4}$$

which depends on the full *path* $q(t)$: the integral is computed by evaluating the Lagrangian along it. Because of this the action S is called a *functional*, since it is a function of a function. The Lagrangian, on the other hand, is a regular function.

We then state the following:

Claim 2.1.1. *The path taken physically by the system satisfies the stationary action principle:*

$$\delta S = 0\,. \tag{2.5}$$

The variation is the functional derivative of S, meant to be a variation in the infinite-dimensional space of possible curves.

This is not a proven principle, but rather an axiom to be taken, analyzed and confronted with experiment.

The action principle, as stated, only guarantees that the action should be stationary: it could be a minimum or a maximum. There are considerations to be made about stability: a maximum-action path will solve the equations of motion and thus be physical, but it will be unstable. This is why the principle is often referred to as the minimum-action principle.

Let us calculate what the variation of the action means physically. We start by considering two nearby paths, γ and γ', such that

$$\gamma' = \gamma + \delta_0 \gamma\,, \tag{2.6}$$

so that the coordinates q at a time t are given by

$$q_\gamma(t) \qquad \text{and} \qquad q_{\gamma'}(t) = q_\gamma(t) + \delta_0 q_\gamma(t)\,, \tag{2.7}$$

and we impose that the variation of the path, $\delta_0 q_\gamma(t)$, is zero at the initial and final time, so that the boundary conditions are satisfied: $\delta_0 q_\gamma(t_{\text{in}}) = 0 = \delta_0 q_\gamma(t_{\text{fin}})$.

[1] There can be issues arising from this approach: for instance, the process for finding trajectories may find many equivalent ones, for example in situations with some symmetry or nontrivial topology. In the gravitational two-body problem we can ask what is the stationary action path needed in order to reach the antipodal point to the current one: there are infinite ones, corresponding to the choice of azimuthal angle of the initial velocity. However, this is not really a problem, since the approach we are about to introduce — Hamilton's principle of stationary action — just provides a formulation which eventually yields the same differential equations as the initial value problem; physically the velocity as well as the position of the particles is well determined in the initial moment.

The reason for the subscript 0 for the variation δ is that we consider *synchronous* variations: the difference between the perturbed and unperturbed trajectory, at a *fixed time*. Also, we take the synchronous variation of the action:

$$\begin{aligned}
\delta_0 S[q_\gamma] &= S[q_\gamma + \delta_0 q_\gamma] - S[q_\gamma] && (2.8)\\
&= \int_{t_{\text{in}}}^{t_{\text{fin}}} \delta_0 L(q_\gamma, \dot{q}_\gamma)\, \mathrm{d}t && (2.9)\\
&= \int_{t_{\text{in}}}^{t_{\text{fin}}} \left[\frac{\partial L}{\partial q} \cdot \delta_0 q + \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} \right] \mathrm{d}t && (2.10)\\
&= \int_{t_{\text{in}}}^{t_{\text{fin}}} \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right] \cdot \delta_0 q \, \mathrm{d}t \,, && (2.11)
\end{aligned}$$

since, as we are considering synchronous variations, δ_0 and ∂_t commute; also we neglected boundary terms in the integration by parts as they are set to zero by the fact that the variation of the path, $\delta_0 q_\gamma$, is zero at the boundaries.

Since this must hold for any variation of the path, by the fundamental lemma of the calculus of variation the integrand must vanish: this yields the Euler-Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} = 0 \,. \tag{2.12}$$

Note that there is **gauge freedom** in the choice of Lagrangian: an easy symmetry to see is the scaling one; the Lagrange equations of L and cL for $c \in \mathbb{R}$ are the same. Also, the equations for L and $L + \mathrm{d}F/\mathrm{d}t$ are the same: the action is the integral in time of the Lagrangian, therefore adding a total derivative to it shifts the action by a constant $\Delta F\big|_{\text{in}}^{\text{fin}}$, which vanishes when taking the variation.

2.1.2 Classical system with finite DoF: Hamiltonian formalism

We start off by defining the momenta: in component form, they are

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \,. \tag{2.13}$$

As long as the Lagrangian is well behaved[2] we can move between momenta and velocities, by using the inverse relation $\dot{q} = \dot{q}(q, p)$.

The Hamiltonian is then defined as

$$H(q, p) = \dot{q}(q, p) \cdot p - L(q, \dot{q}(q, p)) \,, \tag{2.15}$$

[2] The formal requirement is that the matrix

$$\frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}} \tag{2.14}$$

should be invertible.

and this procedure is called a *Legendre transform*. Note that since we assumed the Lagrangian to have no explicit dependence of time the Hamiltonian will not depend on time either. This is tied to the conservation of energy: if the Lagrangian is given by $T - V$ the Hamiltonian is given by $T + V$, and thus represents the total energy of the system. A theorem in Hamiltonian mechanics is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}\,, \tag{2.16}$$

so the variation of the energy in the evolution of the system (total derivative) is equal to the partial derivative of the Hamiltonian with respect to time.

Let us take the functional derivative of the Hamiltonian: using indices for the clarity of the tensorial structure, we find

$$\begin{aligned}
\delta H &= \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial q^i}\delta q^i && (2.17)\\
&= \left[\dot{q}^i + \frac{\partial \dot{q}^j}{\partial p_i}p_j - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial p_i}\right]\delta p_i + \left[\frac{\partial \dot{q}^j}{\partial q^i}p_j - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial q^i}\right]\delta q^i && (2.18)\\
&= \dot{q}^i\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i && (2.19)\\
&= \dot{q}^i\delta p_i - \dot{p}_i\delta q^i\,, && (2.20)
\end{aligned}$$

Used the fact that $p_i = \partial L \big/ \partial \dot{q}^i$.

Used the Euler-Lagrange equations.

so we can equate the first and last equations to find the coupled equations

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}\,.} \tag{2.21}$$

These are known as **Hamilton's equations of motion**, they are an equivalent formulation of Lagrange's ones but they may be more useful in certain contexts such as when performing numerical integration.

In the Hamiltonian contexts it is useful to define the Poisson bracket. If we define the operator ∇ as the derivative operator on the $2N$-dimensional q, p phase space, then the Poisson bracket is defined as

$$\begin{aligned}
\{f, g\} &= (\nabla_i f)\mathbb{J}_{ij}\Big(\nabla_j g\Big) && (2.22)\\
&= \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,, && (2.23)
\end{aligned}$$

where we defined the *symplectic unity* tensor

$$\mathbb{J} = \begin{bmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{bmatrix}. \tag{2.24}$$

With the aid of these, and defining the vector $X = \big[q, p\big]^\top$ we can write Hamilton's equations as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \{X, H\}\,, \tag{2.25}$$

or, more explicitly,

$$\dot{q} = \{q, H\} \qquad \text{and} \qquad \dot{p} = \{p, H\}\,. \tag{2.26}$$

The Poisson brackets between the coordinates in phase space are given by

$$\left\{q^i, q^j\right\} = 0 \qquad \left\{p_i, p_j\right\} = 0 \qquad \left\{q^i, p_j\right\} = \delta^i_j\,. \tag{2.27}$$

A generic function's variation in time is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}\,. \tag{2.28}$$

There is a clear analogy between the classical Poisson bracket and the commutator in quantum mechanics.

2.1.3 Classical system with infinite DoF

Sunday 2020-5-3, compiled 2020-06-01

We move from a system of particles to a field, basically. We make the following substitutions:

1. the coordinates $q_i(t)$ become a field, a function of the coordinates $\varphi(\vec{x}, t)$;
2. sums over the coordinates, $\sum_i$, become integrals in space: $\int \mathrm{d}^3x$;
3. the Lagrangian $L(q_i, \dot{q}_i)$ becomes a Lagrangian density $\mathscr{L}(\varphi, \partial_\mu\varphi)$.

The Lagrangian density is a function of position in space, we can recover the full Lagrangian by integrating:

$$L(t) = \int \mathrm{d}^3x\, \mathscr{L}(\varphi(\vec{x}, t), \partial_\mu\varphi(\vec{x}, t))\,. \tag{2.29}$$

The action is now a functional of the fields:

$$S[\varphi] = \int \mathrm{d}^4x\, \mathscr{L}(\varphi, \partial_\mu\varphi)\,, \tag{2.30}$$

where we integrate in spacetime since we must integrate in d^3x to recover the Lagrangian, and then in time to recover the action from the Lagrangian. Note that this yields a number as an output, and takes the whole field in spacetime as an input. As such, it is not a function of the spacetime coordinates.

The domain of integration is usually Minkowski spacetime with some boundary conditions at infinity: we assume that the fields and their derivatives vanish at infinity.

As in the finite DoF case, we stationarize the action and set $\delta S = 0$. What does it mean to vary the field? We assume the variation to be "synchronous", that is,

$$\varphi'(x) = \varphi(x) + \delta_0\varphi(x)\,, \tag{2.31}$$

so the point in spacetime is not affected: only the field changes.

The variation of the action is given by

$$\delta_0 S[\varphi] = S[\varphi + \delta_0\varphi] - S[\varphi] \tag{2.32}$$

$$= \int \mathrm{d}^4x \left\{ \mathscr{L}(\varphi + \delta_0\varphi, \partial_\mu\varphi + \delta_0\partial_\mu\varphi) - \mathscr{L}(\varphi, \partial_\mu\varphi) \right\} \tag{2.33}$$

$$= \int \mathrm{d}^4x \, \delta_0 \mathscr{L}(\varphi, \partial_\mu\varphi) \tag{2.34}$$

$$= \int \mathrm{d}^4x \left\{ \frac{\partial\mathscr{L}}{\partial\varphi}\delta_0\varphi + \frac{\partial\mathscr{L}}{\partial\partial_\mu\varphi}\delta_0\partial_\mu\varphi \right\} \tag{2.35}$$

$$= \int \mathrm{d}^4x \left\{ \frac{\partial\mathscr{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial\partial_\mu\varphi} \right\}\delta_0\varphi \,. \tag{2.36}$$

We integrated by parts, and the boundary term vanished because of our boundary conditions on the fields. Also, we assumed that $\delta_0\left(\mathrm{d}^4x\right) = 0$ in our second step.

In this case, like in the finite-DoF one, $\left[\partial_\mu, \delta_0\right] = 0$.

Since $\delta_0 S = 0$ must hold for all $\delta_0\varphi$, the rest of the integrand must be equal to zero: these are the Euler-Lagrange equations of the theory,

$$\frac{\partial\mathscr{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial\partial_\mu\varphi} = 0 \,. \tag{2.37}$$

Claim 2.1.2. *Two Lagrangians which are equal up to the divergence of a function of the field, $\partial_\mu k^\mu(\varphi)$, are equivalent (they have the same EL equations).*

Proof. We make this assumption, we can calculate the variation of the action we get after adding this term:

$$\Delta S = \int \mathrm{d}^4x \, \partial_\mu k^\mu \,, \tag{2.38}$$

so we can calculate this in a region whose radius we send to infinity. Properly speaking, our region should be in the form $I \times B_R$, the product of an interval in time an a spere in space. We then apply the divergence theorem: the field φ goes to zero at infinity, so as long as k^μ is regular it will tend to a constant value $k^\mu(0)$, of which we must take the flux on a symmetric region: it will be zero. □

2.1.4 Interlude: functional derivatives

We define a space φ of smooth functions:

$$\varphi = \left\{ f : \left| \begin{array}{ccc} M_4 & \longrightarrow & \mathbb{R} \text{ or } \mathbb{C} \\ x & \longmapsto & f(x) \end{array} \right. \right\}, \tag{2.39}$$

and define **functionals** to be maps from φ to $\mathbb{R}$ (or $\mathbb{C}$), so the space of functionals is the dual of φ: a functional F is in the form

$$F : \left| \begin{array}{ccc} \varphi & \longrightarrow & \mathbb{R} \text{ or } \mathbb{C} \\ f & \longmapsto & F[f] \end{array} \right. . \tag{2.40}$$

We are giving general definitions, but the case we will be interested in is usually the one of real-valued functionals.

We encountered two functionals so far: the Lagrangian is a functional of the field $\varphi(\vec{x}, \bar{t})$ for a fixed $\bar{t}$; the action is a functional of the field $\varphi(x^\mu)$.

We can vary a functional: if we are given a function $g \in \varphi$, we can select a small perturbation δg so that we can define

$$\delta S[g] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} S[g + \epsilon(\delta g)]\Big|_{\epsilon=0} = S[g + \delta g] - S[g]\Big|_{\text{first order}} . \tag{2.41}$$

The functional derivative of $F[g]$, which is denoted by $\delta f / \delta g$, is defined by the relation

$$\delta F = \int \frac{\delta F}{\delta g} \delta g \, \mathrm{d}x \, , \tag{2.42}$$

which should be interpreted as the functional analogue of the formula for the differential

$$\mathrm{d}F\,(x_i) = \sum_i \frac{\partial F}{\partial x_i} \, \mathrm{d}x_i \, . \tag{2.43}$$

Claim 2.1.3. *The functional derivative is indeed a derivative: it is linear*

$$\frac{\delta}{\delta g}\big(\alpha F + \beta G\big) = \alpha \frac{\delta F}{\delta g} + \beta \frac{\delta G}{\delta g} \, ; \tag{2.44}$$

zero for constants: $\delta C / \delta g = 0$ *if* $C[g] \equiv C \in \mathbb{R}$*; and finally it satisfies the Leibniz rule:*

$$\frac{\delta}{\delta g}(FG) = \frac{\delta F}{\delta g} G + F \frac{\delta G}{\delta g} \, . \tag{2.45}$$

Proof. First of all, what to we mean by sum and product of functionals? The intuitive definition to give is that the sum of two functionals $F[g]$ and $G[g]$ is the functional $F + G$ such that $(F + G)[g] = F[g] + G[g]$. For the product we adopt a similar definition.

Then, we get

$$\delta(\alpha F + \beta G)[g] = \big(\alpha F[g + \delta g] + \beta G[g + \delta g]\big) - \big(\alpha F[g] + \beta G[g]\big)\Big|_{\text{lin}} \tag{2.46}$$

$$= \alpha \, \big(F[g + \delta g] - F[g]\big)\Big|_{\text{lin}} + \beta \, \big(G[g + \delta g] - G[g]\big)\Big|_{\text{lin}} \tag{2.47}$$

$$= \alpha \delta F[g] + \beta \delta G[g] \, . \tag{2.48}$$

For a constant we get $C[g + \delta g] = C[g]$, so the differential is zero.

For the Leibniz rule, we have

$$\delta(FG)[g] = F[g + \delta g] G[g + \delta g] - F[g] G[g] \tag{2.49}$$

$$= , \tag{2.50}$$

To review... the definitions do not convince me

A reference: [ED11]. □

A possible functional we can apply to smooth functions is the *evaluation* functional: $F_{\mathbb{1}}[g] = g(y)$; we can represent it as an integral by introducing the delta function,

$$F_{\mathbb{1}}[g] = \int \delta(x-y)g(x)\,\mathrm{d}x\ . \tag{2.51}$$

This is sometimes referred to as the *identity functional*, since it does not alter the function beyond changing the name of the variable.

By the definition of the functional derivative, we find

$$F_{\mathbb{1}}[g] = \int \delta(x-y)g(x)\,\mathrm{d}x = \int \frac{\delta F_{\mathbb{1}}}{\delta g}g(x)\,\mathrm{d}x\ , \tag{2.52}$$

so we must identify

$$\frac{\delta F_{\mathbb{1}}}{\delta g(x)} = \frac{\delta g(y)}{\delta g(x)} = \delta(x-y)\,. \tag{2.53}$$

In the 4D case we have, analogously:

$$\frac{\delta\varphi(x^\mu)}{\delta\varphi(y^\mu)} = \delta^{(4)}(x^\mu - y^\mu)\,, \tag{2.54}$$

and if we fix one or more of the coordinates the dimension of the δ function diminishes accordingly.

Claim 2.1.4. *We can derive the Euler-Lagrange equations using functional derivatives.*

Add discussion of the formal stuff

2.1.5 Hamiltonian formulation

We define the **canonically conjugate momentum** as

$$\pi(\vec{x},t) = \frac{\delta L(t)}{\delta\dot{\varphi}_t(\vec{x})} = \frac{\partial\mathscr{L}}{\partial\partial_0\varphi(\vec{x})}\,, \tag{2.55}$$

so that we can introduce the Hamiltonian density via the Legendre transform:

$$\mathscr{H} = \pi(\vec{x},t)\partial_0\varphi(\vec{x},t) - \mathscr{L}(\varphi,\partial_\mu\varphi)\,, \tag{2.56}$$

and the Hamiltonian as its integral

$$H(t) = \int \mathrm{d}^3x\,\mathscr{H}\,. \tag{2.57}$$

The Hamiltonian is a functional of the fields π and φ:

$$H(t) = H[\pi_t,\varphi_t]\,. \tag{2.58}$$

Claim 2.1.5. *We can calculate Hamilton's equations from the functional variation of H: they read*

$$\dot{\varphi}(\vec{x},t) = \frac{\delta H}{\delta \pi_t(\vec{x})} \qquad \textit{and} \qquad \dot{\pi}(\vec{x},t) = -\frac{\delta H}{\delta \varphi_t(\vec{x})}\,. \tag{2.59}$$

We can also write the HE with the Poisson brackets: the definition must however be adapted; it will read

$$\{F,G\}_t = \int \mathrm{d}^3x \left(\frac{\delta F}{\delta \varphi_t(\vec{x})}\frac{\delta G}{\delta \pi_t(\vec{x})} - \frac{\delta F}{\delta \pi_t(\vec{x})}\frac{\delta G}{\delta \varphi_t(\vec{x})} \right), \tag{2.60}$$

which yields a function of the coordinates $\vec{y}$, since the functional derivatives are functions of $\vec{y}$ as well as of $\vec{x}$, and we marginalize over $\vec{x}$?

Then, Hamilton's equations will read

$$\dot{\varphi}_t = \{\varphi, H\}_t \qquad \text{and} \qquad \dot{\pi}_t = -\{\pi, H\}(t)\,. \tag{2.61}$$

The brackets of the fields φ and π are the same as in the finite-DoF case, substituting δ^i_j with $\delta^{(3)}(\vec{x}-\vec{y})$.

2.2 Nöther's theorem

Monday 2020-5-4, compiled 2020-06-01

We can relate symmetries and conserved quantities.

Definition 2.2.1. *A **symmetry** of the theory is the transformation of the fields and/or of the spacetime coordinates such that the equations of motion do not change.*

This is equivalent to saying that the action is stationary for the transformed fields iff it is stationary for the untransformed ones.

The Lagrangian's symmetries are different from those of the EoM: for instance, we have shown that $\mathscr{L} \to \mathscr{L} + \partial_\mu k^\mu$ preserves them, but the Lagrangian definitely changes.

Symmetries can be classified into **discrete** and **continuous** ones, depending on the group they correspond to. A *discrete* group is a group which is endowed with the discrete topology — the one in which all the subsets are open. A *continuous* group, for what we will need, is a Lie group: one which is described as a differentiable manifold.

The discrete groups we use are usually finite, but they can be inifinite: for instance, translations by 1 m along a certain axis are isomorphic to $\mathbb{Z}$. Continuous groups are always uncountably infinite, since at the very least they are described by one real parameter. Here, we are interested in continuous symmetries, since Nöther's theorem applies to them.

We classify these based on two characteristics: first, we have the **global** versus **local** symmetries. Local symmetries' parameters depend on the position in spacetime, global symmetries' parameters are constant.

Spacetime symmetries alter only the spacetime coordinates, **internal** symmetries alter the fields as well as the spacetime coordinates.

1. Lorentz transformations $x \to \Lambda x$ are global spacetime symmetries;
2. general-relativistic diffeomorphisms $x \to x'(x)$ are local spacetime symmetries;
3. leptonic and hadronic number symmetries and flavour symmetry are global internal symmetries;
4. gauge field symmetries like $A_\mu \to A_\mu + \partial_\mu \Lambda$ are local internal symmetries.

Nöther's theorem associates a conserved current and charge to every continuous symmetry of the action: the current is $j^\mu_{(a)}$, where the index μ is a spacetime one, while (a) labels the generators of the group of symmetry. This current will be conserved:

$$\partial_\mu j^\mu_{(a)} = 0\,. \tag{2.62}$$

A charge can be associated with this current:

$$Q_{(a)} = \int \mathrm{d}^3x\, j^0_{(a)}\,. \tag{2.63}$$

This charge is conserved: $\partial_t Q_{(a)} = 0$ (note that this is a partial derivative but it can also be written as a total one: the total charge only depends on time). This statement follows from the conservation $\partial_\mu j^\mu_{(a)} = 0$: if we integrate this law over a cylinder, the product of an interval in time by a spatial sphere with diverging radius, we get

$$\int \mathrm{d}^4x\, \partial_\mu j^\mu_{(a)} = \int \mathrm{d}t\, \mathrm{d}^3x \left(\partial_0 j^0_{(a)} + \partial_i j^i_{(a)}\right) = Q_{(a)}(t_{\text{fin}}) - Q_{(a)}(t_{\text{in}})\,, \tag{2.64}$$

since the integral of the divergence vanishes by the usual boundary conditions.

2.2.1 Nöther's theorem for internal global symmetries

An internal global symmetry leaves the spacetime coordinates unchanged, and changes the fields as

$$\varphi \to \varphi + \delta_0\varphi \qquad \text{where} \qquad \delta_0\varphi = \epsilon^{(a)} X_{(a)}(\varphi)\,. \tag{2.65}$$

Here $\epsilon^{(a)}$ is a vector of small constant parameters, while $X_{(a)}$ are the generators of the symmetry. The index a runs from 1 to n, the dimension of the symmetry group.

So, we can take the variation of the perturbed action $\delta S[\varphi + \delta_0\varphi]$ and set it to zero. An important note, though: the variation of the action must be zero *locally* as well as globally, so we can perform our integral in an arbitrary spacetime volume V. We find:

$$\delta S[\varphi + \delta_0\varphi]\Big|_V = \delta \int_V \mathrm{d}^4x\, \mathscr{L}\Big(\varphi + \delta_0\varphi, \partial_\mu\varphi + \partial_\mu\delta_0\varphi\Big) \tag{2.66}$$

$$= \int_V \mathrm{d}^4x \left(\frac{\partial\mathscr{L}}{\partial\varphi}\delta_0\varphi + \frac{\partial\mathscr{L}}{\partial\partial_\mu\varphi}\Big(\partial_\mu\delta_0\varphi\Big)\right) \tag{2.67}$$

$$= \int_V \mathrm{d}^4x \left(\frac{\partial\mathscr{L}}{\partial\varphi}\delta_0\varphi - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\partial_\mu\varphi}\right)\delta_0\varphi + \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\partial_\mu\varphi}\delta_0\varphi\right)\right). \tag{2.68}$$

Integrated by parts.

Unlike what we usually do when we derive the EL equations, we cannot neglect the boundary term: since we are integrating in an arbitrary volume, there is no reason why the variation of the field should vanish at the boundary ($\delta_0\varphi\big|_{\partial V} = 0$).

The terms which are multiplied by $\delta_0\varphi$ vanish by the regular Euler-Lagrange equations (which hold everywhere). Therefore, in order for the variation in the action to be zero, we must ask the extra term to vanish: this means

$$\int_V \mathrm{d}^4x\, \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} \delta_0 \varphi \right) = \epsilon^{(a)} \int_V \mathrm{d}^4x\, \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} X_{(a)} \varphi \right) = 0\,, \tag{2.69}$$

which must hold:

1. for any region V;
2. for any choice of $\epsilon^{(a)}$.

Therefore, the only possibility is that

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} X_{(a)}(\varphi) \right) = 0\,, \tag{2.70}$$

for all choices of a. This is a conservation equation just like the one we mentioned above: so, we define

$$j^\mu_{(a)} = \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} X_{(a)}(\varphi) \tag{2.71}$$

and we are done. The conserved charges will look like

$$Q_{(a)} = \int \mathrm{d}^3x\, j^0_{(a)} = \int \mathrm{d}^3x\, \frac{\partial \mathscr{L}}{\partial \partial_0 \varphi} X_{(a)}(\varphi)\,. \tag{2.72}$$

2.2.2 Nöther's theorem for spacetime global symmetries

Let us consider a more general formulation of the theorem: the statement is the same, but we generalize it to a symmetry which acts both on the spacetime coordinates and on the fields.

The variations will be

$$\delta x^\mu = \epsilon^{(a)} Y^\mu_{(a)} \tag{2.73}$$

$$\delta\varphi = \epsilon^{(a)} X_{(a)}(\varphi)\,. \tag{2.74}$$

Here, $X_{(a)}$ are the generators of the action of the group on φ, while $Y_{(a)}$ are the generators of its action on x^μ.

The variation is now δ instead of δ_0: it is not synchronous anymore. To be more explicit:

$$\delta\varphi = \varphi'(x') - \varphi(x) \tag{2.75}$$

$$= \varphi'(x') - \varphi(x') + \varphi(x') - \varphi(x) \tag{2.76}$$

Added and subtracted.

$$\approx \delta_0\varphi + (\partial_\mu\varphi)\delta x^\mu \,. \tag{2.77}$$

Here φ is a test field, but this reasoning holds for any function.

Note that, while the synchronous variation δ_0 commutes with ∂_μ, this is not the case for the total variation δ! In fact, we have

$$\Big[\partial_\mu, \delta\Big]\,\varphi = (\partial_\nu\varphi)\partial_\mu\delta x^\nu \,. \tag{2.78}$$

Also, in general if we transform the coordinates we are also modifying the volume element:

$$\mathrm{d}^4x' \approx \Big(1 + \partial_\mu\delta x^\mu\Big)\,\mathrm{d}^4x \,. \tag{2.79}$$

Therefore, we can write $\delta\Big(\mathrm{d}^4x\Big) = \partial_\mu\delta x^\mu\,\mathrm{d}^4x$.

Proof. This can be shown by using Liouville's formula [Win10, eq. 4.9]:

$$\det\exp(M) = \exp(\operatorname{Tr} M)\,. \tag{2.80}$$

The variation of the volume element is given, in full generality, by

$$\mathrm{d}^4x' = \left|\frac{\partial x'}{\partial x}\right|\mathrm{d}^4x \,, \tag{2.81}$$

where the Jacobian is given by:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\delta x^\mu \approx \exp(\partial_\nu\delta x^\mu) = \exp(M)\,. \tag{2.82}$$

So, we can apply the formula to get:

$$\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \exp(\operatorname{Tr} M) = \exp\big(\operatorname{Tr}(\partial_\nu\delta x^\mu)\big) \tag{2.83}$$

$$= \exp\Big(\partial_\mu\delta x^\mu\Big) \tag{2.84}$$

$$\approx 1 + \partial_\mu\delta x^\mu \,, \tag{2.85}$$

which proves our claim. One plus a small value is positive, so there is no issue with the modulus. □

So, let us get to the computation: as before, we integrate in a spacetime region V, and we find

$$\delta S = \int_V \delta\Big(\mathrm{d}^4x\Big)\mathscr{L} + \int \mathrm{d}^4x\,\delta\mathscr{L} \tag{2.86}$$

$$= \int_V \mathrm{d}^4x\left\{\partial_\mu\delta x^\mu\mathscr{L} + \delta_0\mathscr{L} + \Big(\partial_\mu\mathscr{L}\Big)\delta x^\mu\right\} \tag{2.87}$$

$$= \int_V \mathrm{d}^4x\, \partial_\mu \left((\mathscr{L}\delta x^\mu) + \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} \delta_0 \varphi \right) + \mathrm{LE} \times \delta_0 \varphi \,, \tag{2.88}$$

where the reasoning for the $\delta_0 \mathscr{L}$ part is the same as in the internal symmetry case; while the two new additional terms were combined into a single derivative.

As before, the Lagrange Equations vanish identically, and since we are integrating in a generic region V the rest of the integrand must be zero everything: so, we have found a conserved current!

It is, however, still written in terms of δ_0, while the variation of the field is not the synchronous one: so, we substitute the expression for $\delta_0 \varphi$ in terms of $\delta\varphi$. This yields

$$\delta S = \int_V \mathrm{d}^4x\, \partial_\mu \left(\mathscr{L}\delta x^\mu + \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} \left(\delta\varphi - \partial_\rho \varphi \delta x^\rho \right) \right) \tag{2.89}$$

$$= \int_V \mathrm{d}^4x\, \partial_\mu \left[\left(\mathscr{L}\delta^\mu_\rho - \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} \partial_\rho \varphi \right) \delta x^\rho + \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} \delta\varphi \right] \tag{2.90}$$

$$= \epsilon^{(a)} \int_V \mathrm{d}^4x\, \partial_\mu \left[\left(\mathscr{L}\delta^\mu_\rho - \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} \partial_\rho \varphi \right) Y^\rho_{(a)} + \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} X_{(a)} \right] , \tag{2.91}$$

which, as before, must hold for any volume V and for any $\epsilon^{(a)}$, therefore we get that the integrand must be conserved: our final result for the conserved current is

$$j^\mu_{(a)} = \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} \partial_\rho \varphi - \mathscr{L}\delta^\mu_\rho \right) Y^\rho_{(a)} - \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} X_{(a)} \,, \tag{2.92}$$

where we changed the global sign; this is purely conventional, as any scalar multiple of the conserved current is also conserved.

2.2.3 Application: Poincaré invariance

We want our theories to respect the principles of special relativity. So, the action must be a Lorentz scalar. Poincaré transformations are the most general ones in SR, they consist in combinations of translations (4 generators), Lorentz boosts (3 generators) and Lorentz rotations (3 generators).

Translations

Translations shift the coordinates by constant amounts, and should not alter the fields. We then have

$$\delta x^\mu = \epsilon^\mu = \epsilon^\nu \delta^\mu_\nu = \epsilon^{(\nu)} Y^\mu_{(\nu)} \tag{2.93}$$

$$\delta\varphi = 0 = X \,. \tag{2.94}$$

So, our conserved current reads

$$j^{\mu}_{(\nu),\text{transl}} = \frac{\partial L}{\partial \partial_\mu \varphi}\partial_\nu \varphi - \mathscr{L}\delta^{\mu}_{(\nu)} \stackrel{\text{def}}{=} \widetilde{T}^{\mu}_{\nu}\,, \tag{2.95}$$

which is called the canonical stress-energy-momentum tensor. Note that this tensor is not generally symmetric as written, although there are procedures to make it so.

The components $\widetilde{T}^0_\nu$ are called the *momentum density* associated to the field; their integrals over 3-space are the components of the total momentum P^μ. They are

$$P_\mu = \int \mathrm{d}^3x\, \widetilde{T}^0_\mu = \int \mathrm{d}^3x \left(\frac{\partial \mathscr{L}}{\partial \partial_0 \varphi}\partial_\mu \varphi - \mathscr{L}\delta^0_\mu\right), \tag{2.96}$$

so the 00 component P_0 is precisely the Hamiltonian, which is defined by the Legendre transform:

$$H = \int \mathrm{d}^3x\, \frac{\partial \mathscr{L}}{\partial \partial_0 \varphi}\partial_0 \varphi - \mathscr{L}\,. \tag{2.97}$$

Lorentz transformations

Lorentz rotations and boosts are generated by antisymmetric tensors $\omega_{\mu\nu}$. These have six degrees of freedom. They alter the fields as

$$x'^\mu = x^\mu + \omega^\mu_\nu x^\nu \tag{2.98}$$

$$\varphi'(x') = \varphi(x) - \frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\varphi(x)\,, \tag{2.99}$$

where $\Sigma_{\mu\nu}$ are the representations of the generators of Lorentz transformations for our field: for scalar fields they are zero, for a Dirac spinor we found that they can be expressed as

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\,. \tag{2.100}$$

So, we can express the transformations in our general form:

$$\delta x^\mu = \frac{1}{2}\omega^{\nu\rho}Y^\mu_{(\nu\rho)} \tag{2.101}$$

$$\delta\varphi = \frac{1}{2}\omega^{\mu\nu}X_{(\mu\nu)}\,, \tag{2.102}$$

where we included a $1/2$ in the coordinate transformations since we are summing over two antisymmetric indices, so we will have two copies of every term. In order for these expressions to be equal, we need to set:

$$\omega^\mu_\nu x^\nu = \frac{1}{2}\omega^{\nu\rho}Y^\mu_{(\nu\rho)} \tag{2.103}$$

$$\frac{-i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu} = \frac{1}{2}\omega^{\mu\nu}X_{(\mu\nu)}\,, \tag{2.104}$$

which is solved by

$$Y^{\mu}_{(\nu\rho)} = \left(\delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\nu\rho}\right)x^{\nu} \tag{2.105}$$

$$X_{(\rho\sigma)} = -i\Sigma_{\rho\sigma}\varphi\,. \tag{2.106}$$

Note that although we are using two indices for the bookkeeping of these transformations there are only six independent ones by antisymmetry.

The conserved currents are then given directly from the formula:

$$J^{\mu}_{(\rho\sigma)} = 2x_{[\rho}\widetilde{T}^{\mu}_{\sigma]} - i\frac{\partial\mathscr{L}}{\partial\partial_{\mu}\varphi}\Sigma_{\rho\sigma}\varphi \tag{2.107}$$

$$= L^{\mu}_{\rho\sigma} + S^{\mu}_{\rho\sigma}\,. \tag{2.108}$$

Add first computation step

So, we have both an "external" angular momentum term and a spin term — as long as the generators $\Sigma_{\mu\nu}$ are nonzero! The fact that a scalar field carries no spin can be read off from here.

2.3 Relativistic classical field theory

Friday 2020-5-8, compiled 2020-06-01

We want to build a relativistic field theory starting from a lagrangian density $\mathscr{L}$. What conditions should we impose on it?

1. The action $S = \int \mathrm{d}^4x\,\mathscr{L}$ must be a real functional, so $\mathscr{L}$ must be a real function.
2. The action S must be adimensional,[3] so the Lagrangian must have the dimensions of a length to the -4, which is equivalent to a mass to the 4th power.
3. The action S must be a Lorentz scalar, therefore (since d^4x is a scalar) the Lagrangian must also be a scalar.
4. We want second-order equations of motion, so we ask that $\mathscr{L} = \mathscr{L}(\varphi, \partial_{\mu}\varphi)$.

2.3.1 Real scalar field

We consider a real scalar field $\varphi(x)$. In analogy to classical theory, we choose the Lagrangian

$$\mathscr{L} = \underbrace{\frac{1}{2}\left(\partial_{\mu}\varphi\right)\left(\partial^{\mu}\varphi\right)}_{\text{kinetic term}} - \underbrace{\frac{1}{2}m^2\varphi^2}_{\text{mass term}}\,. \tag{2.110}$$

[3] Why? Well, the simplest argument I can think of is that is should have the same dimensionality it has in classical theory. Beyond that, when we write a path integral

$$K \sim \int \mathcal{D}x e^{-iS} \tag{2.109}$$

the action goes into the argument of the exponential, so it must be adimensional.

This is a *free particle* Lagrangian: it does not contain powers of φ which are higher than two. These higher terms would represent interactions. In order for our conditions to hold, the dimensions of both m and φ must be those of a mass.

The Euler-Lagrange equations read:

$$\left(\Box + m^2\right)\varphi = 0\,, \tag{2.111}$$

which is the Klein-Gordon equation. As we saw in section 1.3.2, the solutions to this equation are given by

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \left(a(k)e^{-ikx} + a^*(k)e^{ikx}\right)_{k^0=\omega_k}\,. \tag{2.112}$$

Let us also express this in the Hamiltonian formulation: the momentum is given by

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial \partial^0 \varphi} = \partial_0 \varphi\,, \tag{2.113}$$

and using it we can write the Hamiltonian density with the usual formula:

$$\mathscr{H} = \pi\partial_0\varphi - \mathscr{L} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\varphi\right)^2 + \frac{1}{2}m^2\varphi^2\,. \tag{2.114}$$

Note that this is positive definite: so the Hamiltonian, its integral ($H = \int \mathscr{H}\,\mathrm{d}^3x$) also is. We can write down the Hamilton equations and see that they are equivalent to the Lagrange ones: they read

$$\dot{\varphi}_t(\vec{x}) = \frac{\delta H}{\delta \pi_t(\vec{x})} = \pi = \partial_0\varphi \tag{2.115}$$

$$\dot{\pi}_t = -\frac{\delta H}{\delta \varphi_t} \tag{2.116}$$

$$= -\frac{1}{2}\frac{\delta}{\delta\varphi}\int \pi^2 + \left(\nabla\varphi\right)^2 + m^2\varphi^2\,\mathrm{d}^3x \tag{2.117}$$

$$= -\frac{\delta}{\delta\varphi}\int \mathrm{d}^3x \left[-\varphi\nabla^2\varphi + \frac{1}{2}m^2\varphi\right] \tag{2.118}$$

$$= \nabla^2\varphi - m^2\varphi\,, \tag{2.119}$$

where we used the fact that

$$\frac{1}{2}\frac{\delta}{\delta\varphi(x)}\int \left(\nabla\varphi(y)\right)^2\mathrm{d}^3y = \int \mathrm{d}^3y\,\frac{\delta\nabla\varphi(y)}{\delta\varphi(x)}\cdot\nabla\varphi(y) \tag{2.120}$$

$$= \int \mathrm{d}^3y\,\nabla\left(\frac{\delta\varphi(y)}{\delta\varphi(x)}\right)\cdot\left(\nabla\varphi(y)\right) \tag{2.121}$$

$$= -\int \mathrm{d}^3y\,\frac{\delta\varphi(y)}{\delta\varphi(x)}\nabla^2\varphi(y) \tag{2.122}$$

$$= -\int \mathrm{d}^3y\,\delta^{(3)}(x-y)\nabla^2\varphi(y) = -\nabla^2\varphi(x)\,. \tag{2.123}$$

By the KG equation, the result can also be written as $\partial_0^2\varphi$.

Let us now discuss the **Nöther currents** associated with the symmetries of this Lagrangian. It does not explicitly depend on space nor time, so it has **Poincaré invariance**: the associated canonical stress-energy-momentum tensor is given by

$$\widetilde{T}^\mu_\nu = \left(\partial^\mu\varphi\right)\left(\partial_\nu\varphi\right) - \delta^\mu_\nu\mathscr{L}\,. \tag{2.124}$$

Note that, in this case, the stress-energy tensor is symmetric ($\widetilde{T}^{\mu\nu} = \widetilde{T}^{\nu\mu}$). This is not a general fact, but it can be shown that the canonical stress-energy tensor can be made into a symmetric tensor in general: this is the so-called Belinfante-Rosenfeld stress-energy tensor.

The components of the total momentum are given by integrating $\widetilde{T}^0_\nu$ over 3-space: they are

$$P_0 = \int \widetilde{T}^0_0 = \int \mathrm{d}^3x\,\left(\partial_0\varphi\right)^2 - \mathscr{L} = \int \mathrm{d}^3x\,\mathscr{H} = H \tag{2.125}$$

$$P_i = \int \widetilde{T}^0_i = \int \mathrm{d}^3x\,\left(\partial^0\varphi\right)\left(\partial_i\varphi\right)\,. \tag{2.126}$$

Claim 2.3.1. *We can check explicitly that these currents are indeed conserved, that is, that $\partial_\mu\widetilde{T}^\mu_\nu = 0$ is an identity.*

Proof. The computation goes as such:

$$\begin{aligned}
\partial_\mu\widetilde{T}^\mu_\nu &= \partial_\mu\left(\partial^\mu\varphi\partial_\nu\varphi\right) - \partial_\nu\mathscr{L} && (2.127)\\
&= \partial_\mu\partial^\mu\varphi\partial_\nu\varphi + \partial^\mu\varphi\partial_\mu\partial_\nu\varphi - \frac{1}{2}\partial_\nu\left(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right) && (2.128)\\
&= \partial_\mu\partial^\mu\varphi\partial_\nu\varphi + \partial^\mu\varphi\partial_\mu\partial_\nu\varphi - \partial_\nu\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi\partial_\nu\varphi && (2.129)\\
&= (\Box\varphi + m^2\varphi)\partial_\nu\varphi = 0\,, && (2.130)
\end{aligned}$$

The 1/2 goes away since we are differentiating squares.

where derivatives are always intended to only act on what's directly in front of them, and where finally we recover the KG equation. □

The Lagrangian is also invariant under **Lorentz transformations**: since our field is a Lorentz scalar it will not change under them, therefore the generators X must be zero.

So, in equation (2.107) we will have $\Sigma_{\rho\sigma} = 0$, which means

$$J^\mu_{(\rho\sigma)} = 2x_{[\rho}\widetilde{T}^\mu_{\sigma]} = L^\mu_{\rho\sigma}\,, \tag{2.131}$$

The current density associated with the regular angular momentum. The integral of its $\mu = 0$ component in d^3x yields the total angular momentum.

Claim 2.3.2. *This is indeed a conserved current: $\partial_\mu J^\mu_{\rho\sigma} = 0$.*

Proof. We take the divergence explicitly:

$$\begin{aligned}
\partial_\mu J^\mu_{\rho\sigma} &= \partial_\mu\left(2x_{[\rho}\widetilde{T}^\mu_{\sigma]}\right) && (2.132)\\
&= 2\left(\partial_\mu x_{[\rho}\right)\widetilde{T}^\mu_{\sigma]} + 2x_{[\rho}\partial_\mu\widetilde{T}^\mu_{\sigma]} && (2.133)\\
&= 2\eta_{\mu[\rho}\widetilde{T}^\mu_{\sigma]} = \widetilde{T}_{[\rho\sigma]} = 0\,, && (2.134)
\end{aligned}$$

where we have used the fact that the canonical stress-energy tensor for a real scalar field is conserved: $\partial_\mu\widetilde{T}^\mu_\nu = 0$, and symmetric. □

2.3.2 Complex scalar field

Now, let us consider a complex scalar field, such that in general $\varphi \neq \varphi^*$ and $\varphi'(x') = \varphi(x)$ under Poincaré transformations.

Then, we can write our Lagrangian as

$$\mathscr{L} = \left(\partial^\mu \varphi^*\right)\left(\partial_\mu \varphi\right) - m^2 \left|\varphi\right|^2 , \tag{2.135}$$

where we dropped the $1/2$ factor from the real case. This is equivalent, since Lagrangians are defined up to a multiplicative factor, but there is a reason for it.

The complex field $\varphi_{\mathbb{C}}$ represents two real degrees of freedom:

$$\varphi_{\mathbb{C}}(x) = \frac{\varphi_{\mathbb{R},1}(1) + i\varphi_{\mathbb{R},2}}{\sqrt{2}} , \tag{2.136}$$

where we included a division by $\sqrt{2}$ for normalization purposes.

This is what makes us remove the $1/2$ factor: if we take real fields to be special cases of the complex one, for each one we consider we must divide by $\sqrt{2}$, and since the Lagrangian is quadratic this yields the global factor.

We can recover the two real DoF by either adding or subtracting φ and φ^*. In terms of the real fields we can rewrite the Lagrangian as

$$\mathscr{L} = \left(\partial_\mu \frac{\varphi_1 - i\varphi_2}{\sqrt{2}}\right)\left(\partial^\mu \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}\right) - m^2 \frac{\varphi_1 - i\varphi_2}{\sqrt{2}} \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \tag{2.137}$$

$$= \frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{1}{2}m^2 \varphi_1^2 - \frac{1}{2}m^2 \varphi_2^2 \tag{2.138}$$

$$= \mathscr{L}_1 + \mathscr{L}_2 . \tag{2.139}$$

Therefore, the two scalar degrees of freedom are fully decoupled: each has its own Lagrangian, and there is no interaction between them.

In order to derive the equations of motion we can differentiate with respect to either φ or φ^*. This yields the Klein-Gordon equation for φ^* and φ respectively.

So, both φ and φ^* satisfy the KG equation.

We have the general form of the solution in terms of the coefficients $a(k)$ and $b^*(k)$.

We can define the momenta

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial \partial_0 \varphi} = \partial_0 \varphi^* \tag{2.140}$$

$$\pi^*(x) = \frac{\partial \mathscr{L}}{\partial \partial_0 \varphi^*} = \partial_0 \varphi . \tag{2.141}$$

In order to fully describe the field, we need both the pair φ, π and φ^*, π^*. The Hamiltonian density is defined as usual: it is

$$\mathscr{H} = \pi^* \pi + \left(\nabla \varphi\right)^* \cdot \left(\nabla \varphi\right) + m^2 \varphi^* \varphi \geq 0 . \tag{2.142}$$

Claim 2.3.3. *The Hamilton equations read*

$$\dot{\varphi}^* = \frac{\delta H}{\delta \pi^*} = \pi = \partial_0 \varphi^* \tag{2.143}$$

$$\dot{\pi}^* = -\frac{\delta H}{\delta \varphi^*} = \nabla^2 \varphi - m^2 \varphi \,, \tag{2.144}$$

and their conjugate.

TODO: derive Poincaré currents, I suspect they look very similar to the real case.

Beyond Poincaré invariance, this Lagrangian has an additional internal symmetry: the $U(1)$ symmetry $\varphi \to \varphi e^{i\alpha}$.

For this symmetry, we have

$$\delta x^\mu = 0 \tag{2.145}$$

$$\delta \varphi = i\alpha \varphi \,, \tag{2.146}$$

so the generators Y are zero, while

$$X_\varphi = i\varphi \qquad \text{and} \qquad X_{\varphi^*} = -i\varphi^* \,, \tag{2.147}$$

so our conserved current is

$$J^\mu = \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi} X_\varphi + \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi^*} X_{\varphi^*} \tag{2.148}$$

$$= i\left(\partial^\mu \varphi^*\right)\varphi - i\left(\partial^\mu \varphi\right)\varphi^* \tag{2.149}$$

$$= i\left(\varphi \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi\right) . \tag{2.150}$$

Claim 2.3.4. *This is indeed conserved.*

Proof. The computation reads:

$$\partial_\mu J^\mu \propto \partial_\mu \left(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*\right) \tag{2.151}$$

$$= \partial_\mu \varphi^* \partial^\mu \varphi + \varphi^* \Box \varphi - \partial_\mu \varphi^* \partial^\mu \varphi - \varphi \Box \varphi^* \tag{2.152}$$

$$= \varphi^* \Box \varphi - \varphi \Box \varphi^* \tag{2.153}$$

$$= \varphi^* m^2 \varphi - \varphi m^2 \varphi^* = 0 \,. \tag{2.154}$$

□

We will identify this with leptonic or hadronic number conservation: it will not represent the probability of finding a particle anymore, as it did for the KG equation.

Claim 2.3.5. *The Hamiltonian and total momentum for the real scalar field can be expressed in terms of the coefficients $a(k)$ as:*

$$H = \frac{1}{2} \int \mathrm{d}^3 k \, \omega_k (a^* a + a a^*) \tag{2.155}$$

$$= \int \mathrm{d}^3k\, \omega_k a^* a\,, \tag{2.156}$$

and

$$P^i = \frac{1}{2}\int \mathrm{d}^3k\, k^i (a^* a + a a^*) \tag{2.157}$$

$$= \int \mathrm{d}^3k\, k^i a^* a(k)\,. \tag{2.158}$$

Claim 2.3.6. *The total charge for the complex scalar field associated with the $U(1)$ symmetry is*

$$Q = q\int \mathrm{d}^3k\, (a^* a - b b^*) = q\int \mathrm{d}^3k\, (a^* a - b^* b)\,, \tag{2.159}$$

so the particle with negative energy has the opposite charge as the one with positive energy.

Claim 2.3.7. *A Lagrangian in the form*

$$\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2 \tag{2.160}$$

$$\mathscr{L}_i = \frac{1}{2}\left(\partial_\mu \varphi_i\right)\left(\partial^\mu \varphi_i\right) - \frac{1}{2}m^2\varphi_i^2 \tag{2.161}$$

has $O(2)$ symmetry in the fields φ_1 and φ_2.

Claim 2.3.8. *Consider the Lagrangian*

$$\mathscr{L} = \left(\partial_\mu \phi\right)^\dagger \left(\partial^\mu \phi\right) - m^2 \phi^\dagger \phi \tag{2.162}$$

$$\phi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix}, \tag{2.163}$$

where φ and χ are complex fields. It is symmetric under $\phi \to U\phi$, where $U \in U(2)$; and the current which is conserved due to this symmetry reads

$$J^\mu_\nu = i\phi^\dagger \sigma_\nu \partial^\mu \phi - i\phi \sigma_\nu \partial^\mu \phi^\dagger\,, \tag{2.164}$$

where $\sigma_\nu = (\mathbb{1}, \sigma_i)^\top$.

2.4 Canonical quantization

2.4.1 System with finite DoF

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The classical structure of such a system is completely determined by the equations of motion:

$$\dot{q} = \{q, H\} \tag{2.165}$$

$$\dot{p} = \{p, H\}\,, \tag{2.166}$$

and by the commutation relations:

$$\left\{q^i, q^j\right\} = 0 \tag{2.167}$$

$$\left\{p_i, p_j\right\} = 0 \tag{2.168}$$

$$\left\{q^i, p_j\right\} = \delta^i_j \,. \tag{2.169}$$

Note that these Poisson brackets are to be calculated at a fixed time.
We can **quantize** such a system by the following substitutions:

1. the coordinates q and p, which are functions of phase space in the classical case, become operators: X and P, which in the Heisenberg picture are functions of time;

2. the Poisson brackets $\{\cdot, \cdot\}$ become commutators:

$$\{a, b\} \to \frac{1}{i\hbar}[A, B] \,. \tag{2.170}$$

The Hamilton equations then read

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{1}{i\hbar}[X, H] \tag{2.171}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{i\hbar}[P, H] \,, \tag{2.172}$$

and the commutation relations are also generalized; the interesting one is the position-momentum commutator which now reads

$$\frac{1}{i\hbar}[X, P] = 1 \implies [X, P] = i\hbar \,. \tag{2.173}$$

Claim 2.4.1. *The Heisenberg and Schrödinger descriptions of QM are equivalent.*

Proof. To say that the two approaches are equivalent means that they yield the same exact predictions for any observation.

In the Schrödinger approach, the wavefunction evolves as $\left|\psi(t)\right\rangle = U(t)\left|\psi_0\right\rangle$ while the observables are stationary; in the Heisenberg approach the wavefunction is stationary while the operators evolve as $A(t) = U^\dagger A U$. So, the expectation value is

$$\left\langle A(t)\right\rangle_{\psi_0} \stackrel{?}{=} \left\langle A\right\rangle_{\psi(t)} \tag{2.174}$$

$$\left\langle \psi_0\right| A(t) \left|\psi_0\right\rangle \stackrel{?}{=} \left\langle \psi(t)\right| A \left|\psi(t)\right\rangle \tag{2.175}$$

$$\left\langle \psi_0\right| U^\dagger A U \left|\psi_0\right\rangle = \left\langle \psi_0\right| U^\dagger A U \left|\psi_0\right\rangle \,. \tag{2.176}$$

□

2.4.2 Field theory

The evolution of the fields is given in classical field theory as

$$\dot{\varphi} = \{\varphi, t\}\dot{\pi} \qquad\qquad = \{\pi, t\}\,, \tag{2.177}$$

and we have the commutation relations

$$\{\varphi(x), \varphi(y)\} = 0 \tag{2.178}$$
$$\{\pi(x), \pi(y)\} = 0 \tag{2.179}$$
$$\{\varphi(x), \pi(y)\} = \delta^{(3)}(x-y)\,, \tag{2.180}$$

all considered at constant time.

In order to **quantize** this system, we replace the fields φ and π by field operators $\hat{\varphi}$ and $\hat{\pi}$, and replace the Poisson brackets by commutators as in the finite-DoF case.

The equations of motion will then read

$$\frac{\mathrm{d}\hat{\varphi}}{\mathrm{d}t} = \frac{1}{i\hbar}[\hat{\varphi}, H] \tag{2.181}$$
$$\frac{\mathrm{d}\hat{\pi}}{\mathrm{d}t} = \frac{1}{i\hbar}[\hat{\pi}, H]\,, \tag{2.182}$$

and the commutation relations will read

$$[\hat{\varphi}(x), \hat{\pi}(y)] = i\hbar\delta^{(3)}(x-y)\,, \tag{2.183}$$

and the zero ones as usual between φ, φ and π, π. These commutators are always taken at constant time.

2.4.3 Canonical quantization of a scalar field

As we have seen before, the field and momentum of the free scalar field, which satisfies the Klein-Gordon equation, can be expressed as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\left[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\right]_{k^0=\omega_k} \tag{2.184}$$
$$\pi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}(-i\omega_k)\left[a(k)e^{-ikx} - a^\dagger(k)e^{ikx}\right]_{k^0=\omega_k}\,, \tag{2.185}$$

and we will now interpret this as an operator expression: even if we omit the hats, φ and π are intended to be operators. Therefore, a and $a^\dagger$ must also be.

Claim 2.4.2. *We can invert this relation to get a and $a^\dagger$ in terms of φ and π:*

$$a(k) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3x}{\sqrt{2\omega_k}}\left[\omega_k\varphi + i\pi\right]\, e^{ikx}\Big|_{k^0=\omega_k} \tag{2.186}$$
$$a^\dagger(k) = \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3x}{\sqrt{2\omega_k}}\left[\omega_k\varphi - i\pi\right]\, e^{-ikx}\Big|_{k^0=\omega_k} \tag{2.187}$$
$$. \tag{2.188}$$

Proof. Let us compute the interesting one:

$$\left[a(k), a^\dagger(p)\right] = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3x\,\mathrm{d}^3y}{2\omega_k}\left[\omega_k\varphi + i\pi, \omega_k\varphi - i\pi\right]e^{i(k-p)x} \tag{2.189}$$

$$= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3x\,\mathrm{d}^3y}{2\omega_k} 2i(-)\omega_k\left[\varphi, \pi\right]e^{i(k-p)x} \tag{2.190}$$

$$= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3x\,\mathrm{d}^3y}{2\omega_k} 2i\omega_k(-)i\delta^{(3)}(x-y)e^{i(kx-py)} \tag{2.191}$$

$$= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\, e^{i(k-p)x} \tag{2.192}$$

$$= \delta^{(3)}(k-p)\,. \tag{2.193}$$

We have used the fact that, by linearity and antisymmetry of the commutator:

$$\left[\omega_k\varphi + i\pi, \omega_k\varphi - i\pi\right] = \left[\omega_k\varphi, \omega_k\varphi\right] + \left[\omega_k\varphi, -i\pi\right] + \left[i\pi, \omega_k\varphi\right] + \left[i\pi, -i\pi\right] \tag{2.194}$$

$$= -2\omega_k i\left[\varphi, \pi\right]. \tag{2.195}$$

For the other two the computation is similar, and the difference comes about in the commutator step: for $[a, a]$ we get

$$\left[\omega_k\varphi + i\pi, \omega_k\varphi + i\pi\right] = 0\,, \tag{2.196}$$

and for $\left[a^\dagger, a^\dagger\right]$:

$$\left[\omega_k\varphi - i\pi, \omega_k\varphi - i\pi\right] = 0\,, \tag{2.197}$$

since any operator commutes with itself. □

This algebra is the same one we would have for an infinite number of decoupled harmonic oscillators. The operators a and $a^\dagger$ are called the annihilation and creation operators.

2.4.4 Number density operator

Since, as we saw, the algebra of our operators resembles that of a harmonic oscillator, we are justified in defining the number density operator

$$N(k) = a^\dagger(k)a(k)\,, \tag{2.198}$$

and the total number density,

$$N = \int \mathrm{d}^3k\, N(k)\,. \tag{2.199}$$

Claim 2.4.3. *Both N and $N(k)$ are self-adjoint: $N = N^\dagger$, and they satisfy the following commutation relations:*

$$\left[N(k), a(p)\right] = -a(k)\delta^{(3)}(\vec{p} - \vec{k}) \tag{2.200}$$

$$\left[N(k), a^\dagger(p)\right] = +a^\dagger(k)\delta^{(3)}(\vec{p}-\vec{k}) \tag{2.201}$$
$$\left[N, a(p)\right] = -a(p) \tag{2.202}$$
$$\left[N, a^\dagger(p)\right] = +a^\dagger(p)\,. \tag{2.203}$$

There is a typo in the notes by the professor: the argument of a and $a^\dagger$ for the local commutators is k, not p.

Proof. The computation goes as follows; I use subscripts as $a(k) = a_k$ because it makes reading the expressions easier, I think:

$$\left[N_k, a_p\right] = N_k a_p - a_p N_k \tag{2.204}$$
$$= a_k a_k^\dagger a_p - a_p a_k a_k^\dagger \tag{2.205}$$
$$= -a_k a_p a_k^\dagger + a_k a_k^\dagger a_p \tag{2.206}$$
$$= -a_k\left[a_p, a_k^\dagger\right] \tag{2.207}$$
$$= -a_k\delta^{(3)}(p-k)\,. \tag{2.208}$$

Commuted a_p and a_k — their commutator is zero.

Used relation (2.193).

The other computation is similar:

$$\left[N_k, a_p^\dagger\right] = N_k a_p^\dagger - a_p^\dagger N_k \tag{2.209}$$
$$= a_k a_k^\dagger a_p^\dagger - a_p^\dagger a_k a_k^\dagger \tag{2.210}$$
$$= -a_p^\dagger a_k a_k^\dagger + a_k a_p^\dagger a_k^\dagger \tag{2.211}$$
$$= -\left[a_p, a_k^\dagger\right] a_k^\dagger \tag{2.212}$$
$$= -a_k^\dagger\delta^{(3)}(p-k)\,. \tag{2.213}$$

while for the total number operator we only need to integrate over d^3k:

$$\int \mathrm{d}^3k\left[N(k), a(p)\right] = -\int \mathrm{d}^3k\, a(k)\delta^{(3)}(\vec{p}-\vec{k}) \tag{2.214}$$
$$\left[N, a(p)\right] = -a(p)\,. \tag{2.215}$$

□

2.4.5 Hamiltonian and momentum density operators

The Hamiltonian density for a real scalar field is given by

$$\mathscr{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2\,, \tag{2.216}$$

and the total Hamiltonian is given by its integral over 3D space, which is given by

$$H = \int \frac{\mathrm{d}^3k}{2}\omega_k\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]. \tag{2.217}$$

This can be derived without commuting a and $a^\dagger$ anywhere; we can now get the desired expression in terms of the number operator by inserting a commutator:

$$H = \int \frac{\mathrm{d}^3k\,\omega_k}{2}\left[2a^\dagger a + \left[a, a^\dagger\right]\right] \tag{2.218}$$

$$= \int \mathrm{d}^3k\,\omega_k N(k) + \int \frac{\mathrm{d}^3k\,\omega_k}{2}\delta^{(3)}(k-k)\,, \tag{2.219}$$

so the second term is proportional to $\omega_0\delta^{(3)}(0)$, which diverges. The first term, on the other hand, makes perfect sense: we count how many particles are at each energy, and add their energies to get the total.

Is the divergence a problem? Not really: Hamilton's equations only depend on the derivatives of H, so a constant is not an issue. Intuitively, we can imagine a regular quantum harmonic energy at each (continuous!) value of k, so for each of those we will have a ground energy $\omega\hbar/2$.

For the momentum we must integrate the T^{0i} components of the stress-energy tensor over d^3x: so, we find

$$P^i = \int \mathrm{d}^3x\,\pi(x)\partial_i\varphi\,, \tag{2.220}$$

and for π and φ we must use the operator expressions in momentum space: so, we find

$$\begin{aligned} P^i &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3x \left[\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}(-i\omega_k)\left(a(k)e^{-ikx} - a^\dagger(k)e^{ikx}\right)\right]\times \\ &\quad\times\left[\int \frac{\mathrm{d}^3k'}{\sqrt{2\omega_{k'}}}\left(-ik'_i a(k')e^{-ik'x} + ik'_i a^\dagger(k')e^{ik'x}\right)\right] \end{aligned} \tag{2.221}$$

$$\begin{aligned} &= \int \mathrm{d}^3k'\,\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\,e^{ik'x}\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\,e^{-ikx}\times \\ &\quad\times\left[\frac{-i\omega_k}{\sqrt{2\omega_k}}\left(a(k) - a^\dagger(-k)\right)\frac{1}{\sqrt{2\omega_{k'}}}\left(+ik'_i a(-k') + ik'_i a^\dagger(k')\right)\right] \end{aligned} \tag{2.222}$$

$$= \int \frac{\mathrm{d}^3k}{2}\left[-i\left(a(k) - a^\dagger(-k)\right)\left(ik_i a(-k) + ik_i a^\dagger(k)\right)\right] \tag{2.223}$$

$$= \int \frac{\mathrm{d}^3k}{2}k_i\left\{\left(a(k) - a^\dagger(-k)\right)\left(a(-k) + a^\dagger(k)\right)\right\} \tag{2.224}$$

$$= \int \frac{\mathrm{d}^3k}{2}k_i\left\{\underbrace{a(k)a(-k)}_{\text{odd}} + a(k)a^\dagger(k) - \underbrace{a^\dagger(-k)a(-k)}_{\text{change } k\to -k} - \underbrace{a^\dagger(-k)a^\dagger(k)}_{\text{odd}}\right\} \tag{2.225}$$

$$= \int \frac{\mathrm{d}^3k}{2}k^i\left\{a(k)a^\dagger(k) + a^\dagger(k)a(k)\right\}. \tag{2.226}$$

Now, as before we have the integral of $a^\dagger a/2 + aa^\dagger/2$, and we can swap them getting a term proportional to $\delta^{(3)}(0)$. This physically means that the vacuum has an infinite momentum, which we can ignore.

First thing: in the professor's notes the integral is reported as being over d^3x, but it is not!

Second thing: now the term being integrated is $k_i\delta^{(3)}(k-k)$, which diverges, sure, but it is also odd... the momentum of the vacuum is "infinite" but also zero by symmetry? i guess it doesn't really matter either way.

If we perform this operation, we get

$$P_i = \int \mathrm{d}^3k\, k_i N(k)\,. \tag{2.227}$$

2.4.6 Normal ordering

This sort of procedure is very common in QFT, so it has been given a name: the **normal ordering** of operators is the process of reordering them such that the creation operators $a^\dagger$ are on the *left* while the annihilation operators a are on the *right*. This corresponds physically to the choice of the positive energy.

The notation we will use for the normal-ordering of a product of operators Q is $N[Q]$. One can also find the notation $:Q:$ used to mean the same thing. As an example, consider:

$$N[a^\dagger(m)a(k_2)a^\dagger(k_3)a(k_4)] = a^\dagger(m)a^\dagger(k_3)a(k_2)a(k_4)\,. \tag{2.228}$$

This formalizes what we were doing before:

$$N\left[\frac{a_k^\dagger a_k + a_k a_k^\dagger}{2}\right] = a_k^\dagger a_k\,. \tag{2.229}$$

Since the order of the product is fixed inside the normal ordering, if we work inside the normal ordering all the creation and annihilation operators commute with each other, for our **scalar bosonic theory**.

Taking the normal ordering means we are fixing the energy of the vacuum.

Taking the normal ordering does not alter the harmonic oscillator algebra.

2.4.7 Fock space

Up until now wee have been writing operators without being really clear about what space they are acting on. The space we need is called the **Fock space**: it is a space containing multiparticle states, and we will now construct it.

We start from a vacuum state $|0\rangle$, which must satisfy

$$\forall k : a(k)\,|0\rangle = 0\,, \tag{2.230}$$

which implies $N(k)\,|0\rangle = 0$. This means that there are no particles.

Do we assume uniqueness? What do we do if there are several vacua?

The Fock space is made up of all the states we can get by repeatedly applying the creation operators $a^\dagger(k)$ for different values of k. Physically, each of these operators adds a particle to the state.

If we have a set of N momenta k_ℓ, and we want the state describing the presence of n_ℓ particles for each, we use the state

$$|n_1 \dots n_N\rangle \propto \left(a^\dagger(k_1)\right)^{n_1} \dots \left(a^\dagger(k_N)\right)^{n_N} |0\rangle \ . \tag{2.231}$$

The labels n_ℓ are the *occupation numbers*, indicating the number of particles in each momentum "slot". The proportionality sign is there because we have not yet decided on the normalization we want for our states.

Let us then consider the properties of the states of this Fock space. For starters, the vacuum $|0\rangle$ obeys

$$N|0\rangle = H|0\rangle = \vec{p}\,|0\rangle = 0\,, \tag{2.232}$$

which is consistent with there being no particles, since we have normalized the vacuum energy to zero.

Now, our one-particle state will look like

$$\big|1(p)\big\rangle = C a_p^\dagger |0\rangle \ , \tag{2.233}$$

for some constant C.

Claim 2.4.4. *This satisfies:*

$$N\big|1(p)\big\rangle = 1\big|1(p)\big\rangle \tag{2.234}$$
$$H\big|1(p)\big\rangle = \omega_p\big|1(p)\big\rangle \tag{2.235}$$
$$\vec{p}\,\big|1(p)\big\rangle = \vec{p}\,\big|1(p)\big\rangle \ . \tag{2.236}$$

Proof. We will need the properties outlined in the equations (2.200). The computation is as follows:

$$Na^\dagger(p)|0\rangle = \Big(Na^\dagger(p) - a^\dagger(p)N\Big)|0\rangle \tag{2.237}$$
We used the fact that $N|0\rangle = 0$.
$$= \Big[N, a^\dagger(p)\Big]|0\rangle \tag{2.238}$$
$$= a^\dagger(p)|0\rangle = \big|1(p)\big\rangle \ . \tag{2.239}$$

For the energy the computation is similar:

$$Ha^\dagger(p)|0\rangle = \int \mathrm{d}^3k\,\omega_k N(k) a^\dagger(p)|0\rangle \tag{2.240}$$
$$= \int \mathrm{d}^3k\,\omega_k \Big[N(k), a^\dagger(p)\Big]|0\rangle \tag{2.241}$$
$$= \int \mathrm{d}^3k\,\omega_k a^\dagger(k)\delta^{(3)}(p-k)|0\rangle \tag{2.242}$$
$$= \omega_p a^\dagger(p)|0\rangle \ . \tag{2.243}$$

For the momentum the steps are almost identical. □

One can construct an n-particle state (where all the particles have the same momentum) similarly, by applying $a^\dagger$ n times: then the eigenvalues of N, H and $\vec{p}$ will be multiplied by n.

Also, we can have states with many particles with different momenta: the energy, momentum and number will be the sum of the individual ones.

2.4.8 Spin-statistics connection

The procedure we have employed so far implicitly assumed we were using **Bose-Einstein** statistics, which is consistent with the fact that our field is spin-0.

This can be seen by the fact that in our Fock space we are allowing more than one particle to have the same momentum; also, the state vector does not change sign when we swap the particles, since $a_p^\dagger$ and $a_k^\dagger$ commute.

2.4.9 Normalization of Fock states

We start by setting $\langle 0|0\rangle = 1$. What do we do for states with stuff in them? If we were to set $|1\rangle = a_k^\dagger |0\rangle$, we would have an issue: if we wanted to compute the norm we would have

$$\big\langle 1(k)\big|1(p)\big\rangle = \langle 0|\, a(k)a^\dagger(p)\, |0\rangle = \delta^{(3)}(k-p)\,, \tag{2.244}$$

and we know that the Dirac delta is not covariant. We can fix this issue: if we impose

$$\big|1(k)\big\rangle = (2\pi)^{3/2}\sqrt{2\omega_k}a_k^\dagger\, |0\rangle\ , \tag{2.245}$$

so that when we compute the modulus we get

$$\big\langle 1(k)\big|1(p)\big\rangle = (2\pi)^3 2\omega_k\, \langle 0|\, a(k)a^\dagger(p)\, |0\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(k-p)\,. \tag{2.246}$$

Claim 2.4.5. *This is covariant.*

Proof. This is explained very well in Peskin's textbook [Pes19, section 3.5].

We want to show that $\omega_k\delta^{(3)}(\vec{k}-\vec{p})$ is a Lorentz invariant (also a Poincarè invariant, but translation invariance is manifest). In order to do so, we must perform a boost: the system is spherically symmetric, so we choose arbitrarily to perform a boost in the direction z, with parameter β.

Then, we know that since p^μ is a contravariant 4-vector it transforms like

$$E' = \gamma(E - \beta p_z) \tag{2.247}$$
$$p_x' = p_x \tag{2.248}$$
$$p_y' = p_y \tag{2.249}$$
$$p_z' = \gamma(p_z - \beta E)\,. \tag{2.250}$$

In the expression $\delta^{(3)}(\vec{k}-\vec{p})$ the only thing which changes is p_z. The triple delta function is the tensor product of three deltas, but two of them are unchanged in the transformation. So, we must check how $\delta(k_z - p_z)$ transforms: recall that

$$\delta\big(f(x)\big) = \frac{1}{\big|\mathrm{d}f/\mathrm{d}x\big|}\delta(x)\,, \tag{2.251}$$

as long as $f(x)$ has a single zero at $x = 0$, which holds in our case since Lorentz transforms are linear.

The conversion factor between the δs will be given by the inverse of the absolute value of

$$\frac{\partial p_z'}{\partial p_z} = \frac{\partial}{\partial p_z}\big(\gamma(p_z + \beta E)\big) \tag{2.252}$$

$$= \gamma\left(1 + \beta\frac{\partial E}{\partial p_z}\right) \tag{2.253}$$

$$= \gamma\left(1 + \beta\frac{p_z}{E}\right) \tag{2.254}$$

$$= \frac{\gamma\big(E + \beta p_z\big)}{E} = \frac{E'}{E}\,, \tag{2.255}$$

which means that the product $E\delta^{(3)}(\vec{k} - \vec{p})$ is covariant. □

This makes sense: the covariant integration element is $\mathrm{d}^3p\,/2E$, so if we want to apply the δ linear functional to a Lorentz scalar function $f(k)$ we can do it simply as

$$\int \frac{\mathrm{d}^3p}{2E} 2E\delta^{(3)}(p - k) f(k)\,, \tag{2.256}$$

which yields an invariant number.

2.4.10 Field operators and particle interpretation

Up until now we have been using the operators a and $a^\dagger$ in momentum space, since their physical interpretation is more intuitive. Let us now consider how to interpret the field operators φ and π.

Recall, the real scalar field is given in general by

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right) = \varphi_+(x) + \varphi_-(x)\,. \tag{2.257}$$

Claim 2.4.6. *We have the following transition amplitudes:*

$$\langle 0|\, \varphi_+(x)\, \big|1(p)\big\rangle = e^{-ipx} \tag{2.258}$$

$$\big\langle 1(p)\big|\, \varphi_-(x)\, |0\rangle = e^{ipx}\,. \tag{2.259}$$

Proof. In order to compute this we substitute in directly:

$$\langle 0|\, \varphi_+(x)\, \big|1(p)\big\rangle = \langle 0|\, \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} e^{-ikx} a(k)\, \big|1(p)\big\rangle \tag{2.260}$$

$$= \langle 0|\, \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} e^{-ikx} a(k) (2\pi)^{3/2} \sqrt{2\omega_p} a^\dagger(p)\, |0\rangle \tag{2.261}$$

$$= \langle 0|0\rangle \int \mathrm{d}^3k\, e^{-ikx} \delta^{(3)}(k - p) = e^{-ipx}\,, \tag{2.262}$$

where we used the fact that $a_k a_p^\dagger\, |0\rangle = \left[a_k, a_p^\dagger\right] |0\rangle$.

For the other one the computation is similar: we work from the other direction. □

The interpretation for this is that the field operator φ_+ can annihilate a particle with generic momentum p at the point x in space, since when we apply it to $\big|1(p)\big\rangle$ the result is proportional to the vacuum. Similarly, φ_- can create a particle with completely undetermined momentum at the position x.

2.4.11 Covariant commutators

In order to perform canonical quantization we started with the commutators at equal time between the fields φ and π. Are the expressions for these covariant?

We start by defining

$$\begin{aligned} D(x-y) &= \big[\varphi(x), \varphi(y)\big] && (2.263)\\ &= \big[\varphi_+(x) + \varphi_-(x), \varphi_+(x) + \varphi_-(x)\big] && (2.264)\\ &= \big[\varphi_+(x), \varphi_-(x)\big] + \big[\varphi_-(x) + \varphi_+(x)\big] && (2.265)\\ &= D_+(x-y) + D_-(x-y)\,. && (2.266) \end{aligned}$$

The dependence of D would generally be on x and y separately, but if we assume translational invariance (and we always do) this reduces to $x-y$. We have used the fact that $\varphi_\pm$ commute with themselves at different points: this can be computed as

$$\big[\varphi_-(x), \varphi_-(y)\big] = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \int \frac{\mathrm{d}^3p}{\sqrt{2\omega_p}} \big[\hat{a}(k), \hat{a}(p)\big] e^{-ikx} e^{-ipy} = 0\,, \tag{2.267}$$

since $\big[\hat{a}(k), \hat{a}(p)\big] = 0$. The computation for φ_+ is analogous.

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