General Relativity exercises

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We set $c = 1$.

1 Sheet 1

1.1 Lorentz transformations

1.1.1 Inverses

We can consider a Lorentz boost with velocity v in the x direction, and we look at its representation in the (t, x) plane (since the y and z directions are unchanged). Its matrix expression looks like:

$$\Lambda = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} , \tag{1}$$

where $\gamma = 1/\sqrt{1 - v^2}$. The inverse of this matrix can be computed using the general formula for a 2x2 matrix:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} . \tag{2}$$

The determinant of Λ is equal to $\gamma^2(1 - v^2) = 1$, therefore the inverse matrix is:

$$\Lambda = \begin{bmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{bmatrix} . \tag{3}$$

1.1.2 Invariance of the spacetime interval

Our Lorentz transformation is

$$\mathrm{d}t' = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{4a}$$
$$\mathrm{d}x' = \gamma(-v\,\mathrm{d}t + \mathrm{d}x) \tag{4b}$$
$$\mathrm{d}y' = \mathrm{d}y \tag{4c}$$
$$\mathrm{d}z' = \mathrm{d}z \tag{4d}$$

and we wish to prove that the spacetime interval, defined by $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is preserved: $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$. Let us write the claimed equality explicitly:

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \gamma(\mathrm{d}t - v\,\mathrm{d}x) \tag{5a}$$

1.1.3 Tensor notation pseudo-orthogonality

The invariance of the spacetime interval $\mathrm{d}s'^{\,2} = \mathrm{d}s^2$ can be also written as $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x'^{\,\mu}\,\mathrm{d}x'^{\,\nu}$. By making the primed differentials explicit we have:

$$\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho\,\mathrm{d}x^\rho\,\Lambda^\nu{}_\sigma\,\mathrm{d}x^\sigma\ , \tag{6}$$

but the dummy indices on the LHS can be changed to ρ and σ, so that both sides are proportional to $\mathrm{d}x^\rho\,\mathrm{d}x^\sigma$. Doing this we get:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = (\Lambda^\top)_\rho{}^\mu\eta_{\mu\nu}\Lambda^\nu{}_\sigma\,, \tag{7}$$

or, in matrix form, $\eta = \Lambda^\top\eta\Lambda$.

1.1.4 Explicit pseudo-orthogonality

For simplicity but WLOG we consider a boost in the x direction with velocity v and Lorentz factor γ. The matrix expression to verify is:

$$\begin{bmatrix}\gamma & -v\gamma \\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\gamma & -v\gamma \\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix} \tag{8a}$$

$$\begin{bmatrix}\gamma & -v\gamma \\ -v\gamma & \gamma\end{bmatrix}\begin{bmatrix}-\gamma & v\gamma \\ -v\gamma & \gamma\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix} \tag{8b}$$

$$\begin{bmatrix}-\gamma^2 + \gamma^2 v^2 & v\gamma^2 - v\gamma^2 \\ v\gamma^2 - v\gamma^2 & -v\gamma^2 + \gamma^2\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}, \tag{8c}$$

which by $\gamma^2 = 1/(1 - v^2)$ confirms the validity of the expression.

1.2 Muons

1.2.1 Nonrelativistic approximation

The survival probability is given by $\mathbb{P}(t) = \exp\left(-t/2.2 \times 10^{-6}\,\mathrm{s}\right)$. If the ground is $h = 15\,\mathrm{km}$ away, then the muon will reach it in $t = h/v = 15\,\mathrm{km}/(0.995c) \approx 5.03 \times 10^{-5}\,\mathrm{s}$, therefore $\mathbb{P}(t) \approx 1.2 \times 10^{-10}$.

1.2.2 Relativistic effects: ground perspective

The observer on the ground will see the muon having to traverse the whole $h = 15\,\mathrm{km}$, but the muon's time will be dilated for them by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$.

1.2.3 Relativistic effects: muons perspective

The muons in their system will observe lenght contraction, with respect to Lorentz boost, by a factor $\gamma_v \approx 10$: therefore the survival probability will be $\mathbb{P}(t) = \exp\left(-t/(\gamma_v \times 2.2 \times 10^{-6}\,\mathrm{s})\right) \approx 0.1$. This result is the same of the one predicted by ground observer, with respect to relativity principle.

1.3 Radiation

1.3.1 New angle

In the source frame the radiation velocity components are $u'_x = \cos\theta', u'_y = \sin\theta'$. From the composition of velocities we obtain:

$$u_y = \sin\theta = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\sin\theta'}{\gamma_v(1 + v\cos\theta')} \tag{9a}$$

$$u_x = \cos\theta = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma_v(\mathrm{d}x' + v\,\mathrm{d}t')}{\gamma_v(\mathrm{d}t' + v\,\mathrm{d}x')} = \frac{\cos\theta' + v}{1 + v\cos\theta'}\,, \tag{9b}$$

hence:

$$\frac{1}{\tan\theta} = \frac{\gamma_v}{\tan\theta'} + \frac{\gamma_v v}{\sin\theta'}\,. \tag{10}$$

1.3.2 Angle plot and relevant limits

See the jupyter notebook in the `python` folder for plots. For $v = 0$ we have $\theta = \theta'$ as we expected, while for $v = 1$, $\theta = 0$.

1.3.3 Radiation speed invariance

Are the components of the velocity, which we called $\sin\theta$ and $\cos\theta$, actually normalized? Let us check:

$$\sin^2\theta + \cos^2\theta = \frac{(\frac{\sin\theta'}{\gamma_v})^2 + (\cos\theta' + v)^2}{(1 + v\cos\theta')^2} \tag{11a}$$

$$= \frac{(1 - v^2)\sin^2\theta' + \cos^2\theta' + v^2 + 2v\cos\theta'}{(1 + v\cos\theta')^2} \tag{11b}$$

$$= \frac{1 + v^2(1 - \sin\theta') + 2v\cos\theta'}{(1 + v\cos\theta')^2} = 1\,, \tag{11c}$$

therefore the square modulus of the speed of the radiation is still c, as we could have assumed earlier.

1.3.4 Isotropic emission

Since the angular distribution of emission varies when changing inertial reference, we might suppose that every system in relative motion respect to O with $v \neq 0$ observes nonisotropic emission.

This can be seen by noticing that for $v \simeq 1$ we have that in the observer system there is almost only emission at an angle $\theta = 0$. In general, since there is a Lorentz γ factor multiplying a function of the angle in the radiation emission frame O', the cotangent of the angle in the observation frame O must get larger and larger as the relative velocity v increases, therefore the radiation gets compressed towards angles with large cotangents: $\theta \sim 0$.

See the jupyter notebook in the `python` folder for interactive plots :)

2 Sheet 2

2.1 Constant acceleration

2.1.1 Coordinate velocity

We are given the position as a function of time,

$$x(t) = \frac{\sqrt{1 + \kappa^2 t^2} - 1}{\kappa}\,, \tag{12}$$

and we can directly compute its derivative

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\kappa t}{\sqrt{\kappa^2 t^2 + 1}}\,. \tag{13}$$

It is clear from the expression that $|v| < 1$ for all times, while v approaches 1 at positive temporal infinity and -1 at negative temporal infinity.

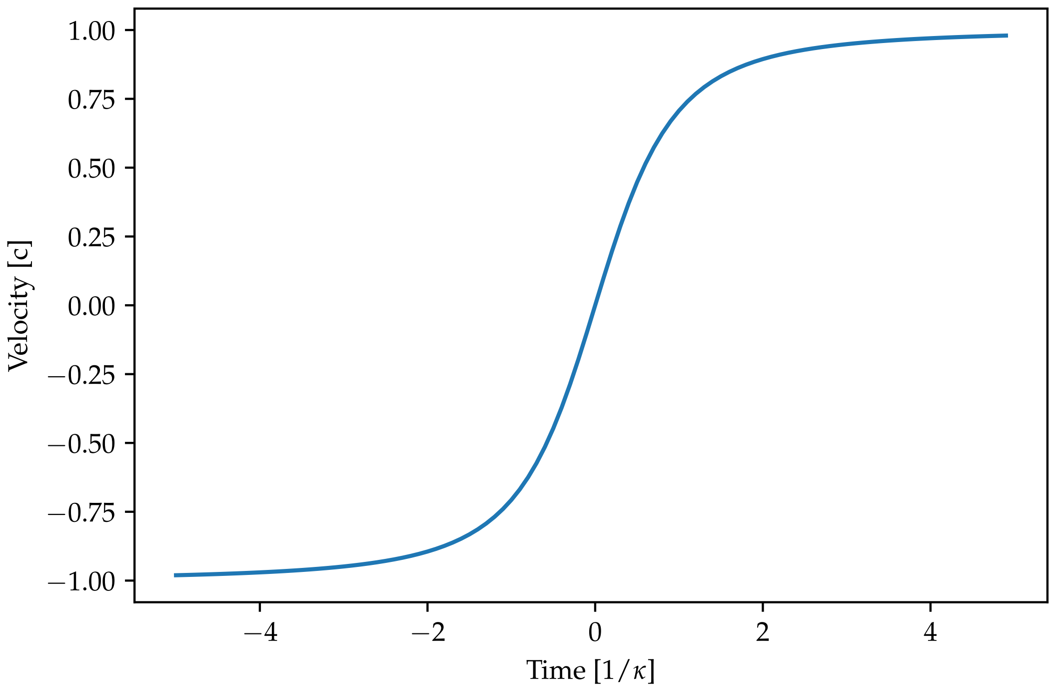

Figure 1: Velocity as a function of coordinate time t

2.1.2 Components of u^μ

The Lorentz factor γ is given by

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{\kappa^2 t^2}{\kappa^2 t^2+1}}} = \sqrt{\kappa^2 t^2 + 1}\,, \tag{14}$$

therefore the four-velocity is given by:

$$u^\mu = \begin{bmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix}. \tag{15}$$

2.1.3 Proper time

The relation between coordinate and proper time is given by the definition of the first component of the four-velocity: $u^0 = \mathrm{d}t/\mathrm{d}\tau = \gamma$, therefore $\mathrm{d}\tau = \mathrm{d}t\,/\gamma$. Integrating this relation we get:

$$\tau = \int \mathrm{d}\tau = \int \frac{\mathrm{d}t}{\gamma} = \frac{\operatorname{arcsinh}(\kappa t)}{\kappa}\,, \tag{16}$$

where the constant of integration is selected by imposing $t = 0 \iff \tau = 0$. Notice that, as we would expect, when expanding up to second order near $t = \tau = 0$ we have $t \sim \tau$, since in that region the velocity is much less than unity.

The inverse relation is given by $t = \sinh(\kappa\tau)/\kappa$. Using this, we can write:

$$x(t(\tau)) = \frac{\cosh(\kappa\tau) - 1}{\kappa}\,. \tag{17}$$

2.1.4 Four-acceleration

Now, we wish to compute the four-acceleration. There are many ways to approach this: an easy one is to simply find the explicit expression $u^\mu(\tau)$ and to differentiate it. The expression we get is:

$$a^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}u^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \sqrt{\sinh^2(\kappa\tau)+1} \\ \frac{\sqrt{\kappa^2t^2+1}\sinh(\kappa\tau)}{\sqrt{\sinh^2(\kappa\tau)+1}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}\kappa\sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa\cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

which is a bit unwieldy but it can be used to check two important facts: $a^\mu a_\mu = \text{const}$ and $a^\mu u_\mu = 0$. The first of the two is:

$$a^\mu a_\mu = -(a_0)^2 + (a_1)^2 = \kappa^2\cosh^2(\kappa\tau) - \frac{\kappa^2\sinh^2(2\kappa\tau)}{2\left(\cosh(2\kappa\tau)+1\right)} = \kappa^2\,, \tag{19}$$

which tells us that the constant acceleration $\sqrt{a^\mu a_\mu} = \kappa$.

Also, we verify the orthogonality to the four-velocity:

$$a^\mu u_\mu = -\frac{\sqrt{2}\kappa\sqrt{\sinh^2(\kappa\tau)+1}\sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} + \kappa\sinh(\kappa\tau)\cosh(\kappa\tau) = 0\,. \tag{20}$$

2.1.5 Local velocity & acceleration

We can apply a Lorentz boost corresponding to this velocity: it will be given by the matrix:

$$\begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

where v and γ are those found before. Without doing any calculations we could already say that the transformed velocity will be equal to the time-like unit vector, while the acceleration will be equal to κ times the unit x-directed vector.

The velocity becomes:

$$(u^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} \\ \kappa t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{22}$$

as we expected.

The acceleration instead becomes:

$$(a^\mu)' = \begin{bmatrix} \sqrt{\kappa^2 t^2 + 1} & -\kappa t & 0 & 0 \\ -\kappa t & \sqrt{\kappa^2 t^2 + 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}\kappa \sinh(2\kappa\tau)}{2\sqrt{\cosh(2\kappa\tau)+1}} \\ \kappa \cosh(\kappa\tau) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \kappa \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

2.2 Fixed target collision

2.2.1 Center of mass momenta

In the CoM frame, the momenta of the two protons are respectively $(E_p, \pm p, 0, 0)^\top = m_p(\gamma, \pm v, 0, 0)$, where $E_p^2 = m_p^2 + p^2$. The total CoM energy is $-(p_A^\mu + p_B^\mu)^2 = 2m_p^2$.

2.2.2 Center of mass velocity

The momentum of particle B will be given by $p^\mu = m_p u^\mu = (m_p\gamma, m_p\gamma v, 0, 0)^\top$. Therefore, $\gamma v = p/m_p$. Solving this we get:

$$v = \frac{p}{m_p}\sqrt{\frac{1}{(p/m_p)^2 + 1}} = \frac{p}{E_p}, \tag{24}$$

2.2.3 Lab frame momenta

The momentum of particle B in its own rest frame will just be $(m_p, 0, 0, 0)^\top$. The momentum of particle A instead will be given by a boost in the x direction with velocity $-v$:

$$(p_A^\mu)_{\text{lab}} = \begin{bmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_p \\ p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma E_p + v\gamma p \\ v\gamma E_p + \gamma p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_p \frac{1+v^2}{\sqrt{1-v^2}} \\ 2\gamma p \\ 0 \\ 0 \end{bmatrix}, \tag{25}$$

2.3 Weak field gravitational time dilation

2.3.1 Time dilation expression

We say: for the first pulse the rocket is h long, therefore it will take a time h to reach it. After Δt_A, the rocket's speed will be $v_1 = g\Delta t_A$. Therefore, if we boost, the second beam will only have a distance $h\sqrt{1 - v_1^2}$ to travel.

The time difference will therefore be

$$h - h\sqrt{1 - v_1^2} \approx h\left(\frac{v_1^2}{2}\right), \tag{26}$$