Radiative processes in astrophysics notes

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Introduction

The professor, Roberto Turolla, will follow the pdf of the book by Rybicki and Lightman [RL79] on his screen. It is available for free.

Understanding radiative processes is fundamental for the analysis of several phenomena: for example, in the Crab nebula the main process is Synchrotron radiation, in the Coma cluster we have Bremsstrahlung, in Cygnus-X1 we have Compton scattering.

Even in the era of multimessenger astrophysics, most of the information we receive still comes from electromagnetic radiation. The required background for this course is classical EM, special relativity and the basics of atomic structure.

The exam is an oral one. The lectures will be recorded and put on the Moodle until the emergency ends, every Wednesday and Thursday. The duration of recorded lectures will be shorter than the duration of the lectures we would have in the classroom.

Chapter 1

Fundamentals of radiative transfer

1.1 Basic properties of the EM spectrum

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Electromagnetic radiation can be decomposed into a spectrum according to either the frequency ν or the wavelength λ; these are connected by $c = \lambda\nu$, where c is the speed of light.

Sometimes we give the energy of the photons, which can be found using Planck's constant h: $E = h\nu$.

We conventionally divide the spectrum into bands: γ-rays, X-rays, ultraviolet light, visible light, infrared radiation, radio band.

1.1.1 The radiative flux

Let us consider an area element $\mathrm{d}A$, through which radiation passes for a time $\mathrm{d}t$: the energy will be proportional to both $\mathrm{d}A$ and $\mathrm{d}t$, so we define the radiative flux F by saying that the energy is equal to $F\,\mathrm{d}A\,\mathrm{d}t$. Of course, we need to account for orientation: if the surface is not perpendicular to the source the energy is less.

Let us consider a pointlike source, and draw two spherical surfaces of radii r_1 and r_2 along which we compute the flux: if there is no energy loss we must have

$$F(r_1)A_1\,\mathrm{d}t = F(r_2)A_2\,\mathrm{d}t \tag{1.1.1a}$$

$$F(r_1)4\pi r_1^2 = F(r_2)4\pi r_2^2 \tag{1.1.1b}$$

$$F(r_1) = F(r_2)\frac{r_2^2}{r_1^2}\,. \tag{1.1.1c}$$

This tells us that **the flux emitted from a pointlike source decreases like r^{-2}**.

The flux of energy is a measure of all the energy which passes through the surface; however we can get a more detailed description. We cannot consider photons at a specific frequency: the set has measure 0, we must use the language of probability densities. We look at a "pencil" of radiation: all the radiation coming from a solid angle $\mathrm{d}\Omega$ over an area $\mathrm{d}A$ and carried by photons of frequencies between ν and $\nu + \mathrm{d}\nu$.

Note that the solid angle $\mathrm{d}\Omega$ depends on both angles we use to describe a direction (typically θ, φ).

So, we define the *specific intensity of brightness* I_ν by

$$\mathrm{d}E \stackrel{\text{def}}{=} I_\nu \,\mathrm{d}A \,\mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}\nu \ . \tag{1.1.2}$$

This will depend on position (where we put the detector area with respect to any sources) and on direction (where we look).

We usually neglect the time-dependence. The units of this quantity are those of energy per unit time, area, frequency, solid angle.

How do we account for the direction? The differential flux for radiation coming with an angle θ to the normal is proportional to the dot product of the normal and the incidence direction:

$$\mathrm{d}F_\nu = I_\nu \cos(\theta) \,\mathrm{d}\Omega \ , \tag{1.1.3}$$

so the total net flux is

$$F_\nu = \int I_\nu \cos(\theta) \,\mathrm{d}\Omega \ . \tag{1.1.4}$$

This is about energy, but we can define the momentum flux per unit time per unit area (which is the pressure) with the same procedure; we get an additional factor of $\cos\theta$ since $\vec{p}$ is a vector, and we are interested in its component along the normal of the surface. So, the global formula for this pressure is

$$P_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \,\mathrm{d}\Omega \ . \tag{1.1.5}$$

These are *moments*: in general, a moment for a direction-dependent quantity like the intensity is something in the form

$$n\text{-th moment} = \int I_\nu \cos^n\theta \,\mathrm{d}\Omega \ . \tag{1.1.6}$$

These are frequency dependent; the corresponding *grey* (that is, frequency-integrated) quantities are in the form

$$F = \int F_\nu \,\mathrm{d}\nu \ . \tag{1.1.7}$$

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1.2 Radiative energy density

We define the energy density per unit solid angle, $u_\nu(\Omega)$ by: $\mathrm{d}E = u_\nu(\Omega) \,\mathrm{d}V \,\mathrm{d}\Omega \,\mathrm{d}\nu$. This is the differential amount of energy in the volume $\mathrm{d}V$, carried by radiation coming from the solid angle $\mathrm{d}\Omega$ which has energies between ν and $\nu + \mathrm{d}\nu$.

We consider a cylinder for our volume, its axis being aligned with the direction the radiation is coming from. Its volume can be expressed as $\mathrm{d}V = \mathrm{d}A \, c \,\mathrm{d}t$, where $\mathrm{d}t$ is the time taken by light to cross the height of the cylinder.

We can also express the differential energy using the definition of the specific intensity: then, we can compare the following two equations:

$$\begin{aligned} \mathrm{d}E &= u_\nu(\Omega) c \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}\nu && \text{(1.2.1a)} \\ &= I_\nu \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}\nu \;, && \text{(1.2.1b)} \end{aligned}$$

which allows us to conclude that $u_\nu = I_\nu / c$. Also, if we want to get the total energy density we just need to integrate over the volume of the whole sphere:

$$u_\nu = \frac{1}{c} \int I_\nu(\Omega) \, \mathrm{d}\Omega \stackrel{\text{def}}{=} \frac{4\pi}{c} J_\nu \,, \tag{1.2.2}$$

where J_ν is the *mean intensity*: $J_\nu = \langle I_\nu \rangle_\Omega$.

We can also integrate over frequencies to get the total energy density:

$$u = \int u_\nu \, \mathrm{d}\nu = \frac{4\pi}{c} \int J_\nu \, \mathrm{d}\nu \;. \tag{1.2.3}$$

1.2.1 Isotropic radiation field

An isotropic radiation field is one for which the specific intensity does not depend on angles. Let us start from the definitions of u_ν and P_ν:

$$\begin{aligned} u_\nu &= \int \frac{I_\nu}{c} \, \mathrm{d}\Omega = \frac{4\pi J_\nu}{c} && \text{(1.2.4a)} \\ P_\nu &= \int \frac{I_\nu}{c} \cos^2\theta \, \mathrm{d}\Omega = \int \frac{I_\nu}{c} \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \;, && \text{(1.2.4b)} \end{aligned}$$

and let us use the assumption that I_ν does not depend on Ω: so we can bring it out of the integrals, to find

$$\begin{aligned} u_\nu &= 4\pi \frac{I_\nu}{c} && \text{(1.2.5a)} \\ P_\nu &= -2\pi \frac{I_\nu}{c} \int \cos^2\theta \, \mathrm{d}\cos\theta = 2\pi \frac{I_\nu}{c} \int_{-1}^{1} x^2 \, \mathrm{d}x && \text{(1.2.5b)} \\ &= \frac{4\pi}{3} \frac{I_\nu}{c} \,, && \text{(1.2.5c)} \end{aligned}$$

The differential is negative, but we swap the integration bounds.

which gives us the result we sought:

$$P_\nu = \frac{u_\nu}{3} \,. \tag{1.2.6}$$

1.2.2 Specific intensity along a ray

We wish to see how the specific intensity I_ν changes along a beam of light rays. Let us consider two positions $1, 2$ along the beam, separated by a distance R. Then, by definition we will have, for $i = 1, 2$:

$$\mathrm{d}E_i = I_{\nu,i} \, \mathrm{d}A_i \, \mathrm{d}t_i \, \mathrm{d}\Omega_i \, \mathrm{d}\nu_i \;. \tag{1.2.7}$$

First, we make the assumption of the gravitational field being weak: therefore time dilation is negligible, so $\mathrm{d}t_1 = \mathrm{d}t_2$ and $\mathrm{d}\nu_1 = \mathrm{d}\nu_2$. Now, we ask these two expressions to describe the same beam: the same photons will pass through $\mathrm{d}A_1$ and $\mathrm{d}A_2$. Therefore, by conservation of energy, $\mathrm{d}E_1 = \mathrm{d}E_2$.

This means that

$$I_{\nu,1}\,\mathrm{d}A_1\,\mathrm{d}\Omega_1 = I_{\nu,2}\,\mathrm{d}A_2\,\mathrm{d}\Omega_2 \,. \tag{1.2.8}$$

We can treat the photons' motion as time-reversal symmetric: so, whether they pass through $\mathrm{d}A_1$ or $\mathrm{d}A_2$ first is irrelevant. The linear scale of the differential area element is infinitesimal, while the separation between the two points, R, is macroscopic: so, we can consider all the photons which come through $\mathrm{d}A_1$ to be coming from a point source from position 2 and vice versa. This will define the angular size as seen from the "pointlike" position 2, $\mathrm{d}\Omega_2$.

Therefore, the differential solid angle will look like

$$\mathrm{d}\Omega_2 = \frac{\mathrm{d}A_1}{R^2}\,, \tag{1.2.9}$$

and we can apply the same reasoning reversing the photons' motion to find the same, alternate relation with $(1 \leftrightarrow 2)$. We can use this to write

$$I_{\nu,1}\frac{\mathrm{d}A_1}{\mathrm{d}\Omega_2} = I_{\nu,2}\frac{\mathrm{d}A_2}{\mathrm{d}\Omega_1} \tag{1.2.10a}$$

$$I_{\nu,1}R^2 = I_{\nu,2}R^2 \tag{1.2.10b}$$

$$I_{\nu,1} = I_{\nu,2}\,. \tag{1.2.10c}$$

This means that, under our assumptions, the specific intensity is conserved:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0\,, \tag{1.2.11}$$

where s is a parameter describing the light ray's trajectory. This is useful since, if the variation of the specific intensity is zero in a vacuum, then its variation in the presence of matter will only be due to transfer phenomena, and the sign of the variation will describe whether energy is being added or removed.

1.3 Radiative transfer

In general, as radiation passes through matter, its specific intensity changes. This is due to emission and absorption, but also to scattering, which preserves the total number of photons: even in the low-energy limit it can change the angular distribution of the radiation, and in general it also changes the energy of the photon.

1.3.1 Emission

Emission is a process through which photons are created. We can define the grey emission coefficient j and the monochromatic emission coefficient j_ν as:

$$\mathrm{d}E = j \,\mathrm{d}V \,\mathrm{d}\Omega \,\mathrm{d}t \tag{1.3.1a}$$

$$\mathrm{d}E = j_\nu \,\mathrm{d}V \,\mathrm{d}\Omega \,\mathrm{d}t \,\mathrm{d}\nu \ , \tag{1.3.1b}$$

they quantify the energy added to the radiation field per unit volume, solid angle (in order to account for the direction of emission) and unit time. For the monochromatic coefficient, we restrict ourselves to radiation emitted in the range from ν to $\nu + \mathrm{d}\nu$.

In the case of an isotropic emission we can integrate over the solid angle to find

$$P_\nu = 4\pi j_\nu \,, \tag{1.3.2}$$

the radiated power per unit volume and frequency.

Another useful concept is the emissivity ϵ_ν: it is the energy added to the radiation field per unit time, frequency and mass in the directions described by the solid angle $\mathrm{d}\Omega$. We express the infinitesimal mass as $\mathrm{d}m = \rho \,\mathrm{d}V$, so that in the case of isotropic emission we have

$$\mathrm{d}E = \epsilon_\nu \rho \,\mathrm{d}V \,\mathrm{d}t \,\mathrm{d}\nu \,\frac{\mathrm{d}\Omega}{4\pi} \,, \tag{1.3.3}$$

so the emissivity ϵ_ν and the emission coefficient j_ν are connected by

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi} \,. \tag{1.3.4}$$

We wish to describe the variation in specific intensity due to this emission. Let us consider a beam of cross section $\mathrm{d}A$ going through a length $\mathrm{d}s$, so that the volume it occupies is $\mathrm{d}V = \mathrm{d}A \,\mathrm{d}s$.

Now, if we compare the definitions of j_ν and I_ν we find that they differ by a factor $\mathrm{d}V \,/\, \mathrm{d}A = \mathrm{d}s$, the length of the beam cylinder we defined.

The difference between the specific intensities at the start and end of the cylinder would be zero without emission, now instead their difference can be calculated from the energy added; as we said most of the differentials simplify and we get that the variation of specific intensity is

$$\mathrm{d}I_\nu = j_\nu \,\mathrm{d}s \ . \tag{1.3.5}$$

1.3.2 Absorption

Absorption is described by a coefficient $\alpha_\nu > 0$, which is dimensionally an inverse length. The absorption law which defines the coefficient gives the decrease in radiative intensity for radiation of intensity I_ν crossing an absorbing medium of length $\mathrm{d}s$:

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu \,\mathrm{d}s \ . \tag{1.3.6}$$

Why should the variation in intensity be proportional to the intensity itself? We give a simple argument: let us assume that absorption is due to randomly absorbers with number density n and (frequency dependent) cross section σ_ν.

Let us consider our usual cylinder with cross sectional area $\mathrm{d}A$ and length $\mathrm{d}s$: the number of absorber in it will be $\mathrm{d}N = n\,\mathrm{d}A\,\mathrm{d}s$. The total effective cross section area presented for absorption will be $\sigma_\nu\,\mathrm{d}N$. The energy contained in photons in this cross sectional area will be lost: the energy lost $-\,\mathrm{d}I_\nu$ can be calculated as

$$-\,\mathrm{d}I_\nu\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu = I_\nu(\sigma_\nu n\,\mathrm{d}A\,\mathrm{d}s)\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \tag{1.3.7a}$$

$$\mathrm{d}I_\nu = -n\sigma_\nu I_\nu\,\mathrm{d}s\ , \tag{1.3.7b}$$

which is the relation written above, with $n\sigma_\nu = \alpha_\nu$. The number density is proportional to the mass density: $n\overline{m} = \rho$ where $\overline{m}$ is the average mass of a particle. Therefore, we can express α_ν as

$$\alpha_\nu = \rho\kappa_\nu = n\overline{m}\frac{\sigma_\nu}{\overline{m}}\,, \tag{1.3.8}$$

so we can see that κ_ν is a cross sectional area per unit mass. It is called the *mass absorption coefficient* or the *opacity*.

Conditions for validity This line of reasoning holds as long as the inter-absorber distances $d \sim n^{-1/3}$ are large compared to the linear scale of the cross section $\sigma_\nu^{1/2}$: we ask

$$\sigma_\nu^{1/2} \ll n^{-1/3}\,, \tag{1.3.9}$$

and also we must assume that the absorbers are independent and randomly distributed (at least locally). These assumptions are usually met in astrophysical systems.

1.3.3 The radiative transfer equation

We can account for both absorption and emission in a combined equation for the derivative with respect to the beam length travelled s of the specific intensity I_ν:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu\,, \tag{1.3.10}$$

and we can see that in the absence of emission and absorption I_ν is unchanged, as we have shown before. If j_ν and α_ν are known we can integrate this differential equation to find the specific intensity.

This will *not* be the case when we will include scattering: the scattering term will not just depend on I_ν but in its integral on the sphere, making this an integro-differential equation.

Solutions to the transfer equation in simple cases

If there is only emission, that is, only j_ν is nonzero, the intensity increases (linearly in s for constant j_ν):

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu \implies I_\nu = I_\nu(0) + \int_{s_0}^{s} j_\nu(\widetilde{s})\,\mathrm{d}\widetilde{s}\ . \tag{1.3.11}$$

If there is only absorption, that is, only α_ν is nonzero, then the intensity decreases (exponentially in s for constant j_ν):

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu \implies I_\nu = I_\nu(s_0) \exp\left(- \int_{s_0}^{s} \alpha_\nu(\widetilde{s}) \,\mathrm{d}\widetilde{s} \right) . \tag{1.3.12}$$

1.3.4 Optical depth and source function

The optical depth τ_ν is defined so that it increases by 1 when the intensity of light at frequency ν decreases e-fold: its differential is

$$\mathrm{d}\tau_\nu = \alpha_\nu \,\mathrm{d}s \ , \tag{1.3.13}$$

so that the solution in the absorption-only case reads $I_\nu \propto e^{-\int \mathrm{d}\tau} = e^{-\tau}$.

So, a useful distinction to make is based on the magnitude of τ, since it quantifies how much light can shine through a medium:

1. if $\tau \gg 1$ the medium is said to be *opaque* or *optically thick*;
2. if $\tau \ll 1$ the medium is said to be *transparent* or *optically thin*;
3. if $\tau \approx 1$ the medium is said to be *translucent*.

If we define the source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \,, \tag{1.3.14}$$

we can write the radiative transfer equation as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu \,. \tag{1.3.15}$$

Divided through by α_ν, used definition of τ_ν.

1.3.5 A formal solution of the radiative transfer equation

We can solve this equation by defining $Y_\nu = I_\nu e^{\tau_\nu}$, which obeys

$$\frac{\mathrm{d}Y_\nu}{\mathrm{d}\tau_\nu} = \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} e^{\tau_\nu} + I_\nu e^{\tau_\nu} \,, \tag{1.3.16}$$

so we can multiply the radiative transport equation by e^{τ_ν} to get

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} e^{\tau_\nu} = -I_\nu e^{\tau_\nu} + S_\nu e^{\tau_\nu} \tag{1.3.17a}$$

$$\frac{\mathrm{d}Y_\nu}{\mathrm{d}\tau_\nu} = S_\nu e^{\tau_\nu} \tag{1.3.17b}$$

$$Y_\nu(\tau_\nu) = Y_\nu(0) + \int_0^{\tau_\nu} S_\nu(\widetilde{\tau}_\nu) e^{\widetilde{\tau}_\nu} \,\mathrm{d}\widetilde{\tau}_\nu \tag{1.3.17c}$$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\widetilde{\tau}_\nu) e^{\widetilde{\tau}_\nu - \tau_\nu} \,\mathrm{d}\widetilde{\tau}_\nu \ , \tag{1.3.17d}$$

Divided through by e^{τ_ν}.

which has a direct intuitive meaning: the intensity at a certain point must be computed accounting for the initial one and emission all through the beam before the point we are considering, and each of these contributions to the emission is weighted by an exponential factor: the relevance of a term decreases if the optical distance increases.

If S_ν is a constant, we have the simplified expression

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right), \tag{1.3.18}$$

and we can see that for large optical depths the intensity is dictated purely by the source at that point, since $1 - e^{-\tau_\nu}$ approaches 0 for large τ_ν.

1.3.6 Thermal radiation

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Consider a blackbody enclosure at equilibrium at a certain temperature T. If we open a small hole in the enclosure and measure the radiation inside it, the intensity I_ν we will see will be isotropic, and only depending on the temperature T: let us call the (still unspecified) way this dependence look $I_\nu = B_\nu(T)$. The function $B_\nu(T)$ will be called the **blackbody function**.

Now, imagine we put a small chunk of material inside the blackbody enclosure. If this material is characterized by a source function $S_\nu = j_\nu/\alpha_\nu$, then the evolution of the intensity inside the cavity will be

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu\,. \tag{1.3.19}$$

If we wait for equilibrium, though, we must have $I_\nu \equiv B_\nu$, meaning that its derivative must be zero: this implies that $I_\nu = B_\nu = S_\nu$.

This is called **Kirkhoff theorem**: a chunk of material which can emit and absorb and which is placed in a blackbody enclosure then it must satisfy

$$B_\nu = \frac{j_\nu}{\alpha_\nu} = S_\nu\,. \tag{1.3.20}$$

What is the explicit expression of the blackbody function B_ν? From Bose-Einstein statistics we know that the phase space density of photons in thermal equilibrium is given by

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\,\mathrm{d}^3p} = \frac{2}{h^3}\frac{1}{\exp\left(\frac{h\nu}{k_BT}\right) - 1}\,, \tag{1.3.21}$$

where the 2 comes from the two polarizations of the photons, h^3 is the size of the phase space cell, while the exponential factor is the occupation number.

Bibliography

[RL79] G. B. Rybicki and A. P. Lightman. *Radiative Processes in Astrophysics*. John Wiley and Sons, 1979. ISBN: 978-0-471-82759-7. URL: http://www.bartol.udel.edu/~owocki/phys633/RadProc-RybLightman.pdf.