240 - HWK 1

Homework 1

1. $$R(t) = \frac{F_0}{\tau} \exp\left[- \frac{t - t_{SN}}{\tau} \right] \Theta(t - t_{SN})$$ (event rate)

Heaviside step

- t_{SN} : Supernovae explosion time
- τ : neutrino signal exponential decay
- F_0 : detector property

a. $\mathcal{L}(\underline{d} \mid H_0, \underline{\theta})$ with : $\underline{d}$ = ~~[illegible]~~ no. in each bin (0 or 1)

$\underline{\theta} = F_0, \tau, t_{SN}$

In interval : $\langle n \rangle = R(t) \cdot \Delta t = \lambda$ (model)

Each interval is small and within each interval we can model the probability that we got 1 or 0 neutrinos using poisson statistics and then ~~[illegible]~~ take products to get the probability we got our exact data (this exact "sequence") overall :

$$\mathcal{L} = \prod_i^{n_{bins}} P(k_i \mid \lambda_i) \quad \text{with} \quad k_i = 0 \text{ or } 1 \text{ neutrino}$$

$$= \prod_i^{n_{bins}} \frac{\lambda_i^{k_i} e^{-\lambda_i}}{k_i!} \qquad \left[\lambda_i = R(t_i) \Delta t \right]$$

Δt constant

- since $k_i! = 0!$ or $1! = 1$ for both we drop it

$$\mathcal{L} = \prod_i^{n_{bins}} \lambda_i^{k_i} e^{-\lambda_i} \qquad \left[n_{bins} = {}^{t_{SN}}/\Delta t \right]$$

$$\ln \mathcal{L} = \sum_i^{n_{bins}} \ln\left(\lambda_i^{k_i} e^{-\lambda_i} \right)$$

$(t_i \neq t_1, t_2, \ldots t_N$, it is a "constant":
$t_i = t_{SN} + i\Delta t)$

$$\ln \mathcal{L} = \sum_i^{n_{bins}} \left[k_i \ln \lambda_i - \lambda_i \right]$$

$$= \sum_i^{bins} \left[k_i \ln \left(R(t_i) \Delta t \right) - R(t_i) \Delta t \right]$$

$$= \sum_i^{bins} \left[k_i \ln R(t_i) + k_i \ln \Delta t - R(t_i) \Delta t \right]$$

We can do this by pretending to multiply $R(t_i)\Delta t$ by $\frac{1\,\text{sec}}{1\,\text{sec}}$ to keep dimensions correct.

Treating $\ln(\Delta t)$ as an additive constant, lets say A, and splitting sum:

$$\ln \mathcal{L} = \sum_i^{N} \left[\ln R(t_i) + A - R(t_i)\Delta t \right] + \sum_j^{n_{bins}-N} \left[-R(t_j)\Delta t \right]$$

(should've switched i & j index. Sub clarity since $t_i = t_1, t_2, \ldots, t_N$ now!)

a. $$= NA - N\Delta t \sum_i^{N} R(t_i) + \sum_i^{N} \ln R(t_i) - \sum_j^{n_{bins}-N} R(t_j)\Delta t$$

Since NA is an additive constant, we chose to ignore it.

b. Taking the limit of $\Delta t \to 0$: $(n_{bins} \to \infty)$

$$\ln \mathcal{L} = \sum_i \ln R(t_i) - \int_0^\infty R(t)\,dt + NA$$

$$\cdot \int_0^\infty R(t)\,dt = \int_{t_{SN}}^\infty \frac{F_0}{\tau} \exp\left[-\frac{t - t_{SN}}{\tau} \right] dt$$

Due to heaveside step function $\Theta(t - t_{SN})$

$$= \frac{F_0}{\tau}\left[\frac{1}{-1/\tau}\exp\left[\frac{-t+t_{BW}}{\tau}\right]\right]_{t_{BW}}^{\infty}$$

$$= -F_0\{0-1\} = F_0$$

$$\therefore \ln \mathcal{L} = \sum_i^N \ln R(t_i) - F_0 + NA$$

~~Maximum likelihood value for F_0:~~

$$\sum_i^N \ln R(t_i) = \sum_i^N \left[\ln(F_0/\tau) - \frac{t_i - t_{BW}}{\tau} + \ln(\Theta(t_i - t_{BW}))\right]$$

$$= N\ln(F_0/\tau) - \sum_i^N\left[\frac{t_i - t_{BW}}{\tau} - \ln(1)\right] \quad \{t_i \geq t_{BW}\}$$

$$= N\ln(F_0/\tau) - \sum_i^N \frac{t_i - t_{BW}}{\tau}$$

$$\therefore \ln \mathcal{L} = -\sum_i^N\left[\frac{t_i - t_{BW}}{\tau}\right] + N\ln(F_0/\tau) - F_0 + NA$$

c. Maximum likelihood of F_0 :

$$\frac{\partial \ln \mathcal{L}}{\partial F_0} = N \cdot \frac{1}{F_0/\tau} \cdot \frac{1}{\tau} - 1 = 0$$

$$\Rightarrow F_0^{max} = N$$

[maximum values for $\mathcal{L}$ and $\ln \mathcal{L}$ are identical – monotonic function)

Maximum likelihood value of τ:

$$\frac{\partial \ln \mathcal{L}}{\partial \tau} = N \cdot \frac{1}{F_0/\tau} \cdot \frac{(-1)F_0}{\tau^2} + \sum_i^N \frac{(+1)(t_i - t_{SN})}{\tau^2} = 0$$

$$\Rightarrow \quad \sum_i^N \frac{t_i - t_{SN}}{\tau^2} = \frac{N}{\tau}$$

$$\Rightarrow \quad \underline{\tau_{max} = \frac{1}{N} \sum_i^N (t_i - t_{SN})}$$

Maximum likelihood value of t_{SN}:

$$\frac{\partial \ln \mathcal{L}}{\partial t_{SN}} = - \sum_i^N -\frac{1}{\tau} = 0$$

$$\Rightarrow \quad \underline{N/\tau = 0} \quad \Rightarrow \quad \underline{\tau \rightarrow \infty}$$

d. $$\ln \mathcal{L} = -\sum_i^N \left[\frac{t_i - t_{SN}}{\tau}\right] + N\ln(F_0/\tau) - F_0 + NA$$

$$\therefore \quad \mathcal{L} = C \exp\left[N\ln(F_0/\tau) - \sum_i^N \left[\frac{t_i - t_{SN}}{\tau}\right] - F_0\right]$$

- we package $e^{NA} = C$ and largely ignore it for simplicity

(plot on jupyter notebook) $t_{SN} \leq 0$ must be.

e. Marginalise over F_0 & τ:

$$\mathcal{L} \equiv \mathcal{L}(\underline{d} \mid H, F_0, t_{SN}, \tau)$$

Prior

$$\mathcal{L}(\underline{d} \mid H, t_{SN}) = \int_{\text{param space}} \mathcal{L}(\underline{d} \mid H, F_0, t_{SN}, \tau) P(F_0, \tau \mid H, t_{SN}) \, d\tau \, dF_0$$

Since the priors are uniform, we can take its out of the integral:

$$\mathcal{L}(\underline{d} \mid H, t_{SN}) = P(F_0, \tau / H, t_{SN}) \int_0^\infty C e^{N \ln F_0} e^{-F_0} dF_0 \int_0^\infty e^{-N \ln \tau} e^{-\frac{1}{\tau} \sum_i^N t_i - t_{SN}} d\tau$$

$$= PC \int_0^\infty F_0^N e^{-F_0} dF_0 \int_0^\infty \tau^{-N} e^{-\frac{1}{\tau} \sum_i^N t_i - t_{SN}} d\tau$$

- $\int_0^\infty F_0^N e^{-F_0} dF_0 = \Gamma(N+1) = ((N+1)-1)! = N!$ (standard integral)
- $\int_0^\infty \tau^{-N} e^{-\frac{1}{\tau} \sum_i^N t_i - t_{SN}} d\tau = \int_0^\infty \tau^{-N} e^{-\frac{K}{\tau}} d\tau$ $\quad (K = \sum_i^N t_i - t_{SN})$

$$= K^{1-N} \Gamma(N-1) = \left(\sum_i^N t_i - t_{SN}\right)^{1-N} (N-2)!$$ (looked up online)

$$\Rightarrow \mathcal{L}(\underline{d} \mid H, t_{SN}) = PC \, N! (N-2)! \left(\sum_i^N t_i - t_{SN}\right)^{1-N}$$

$$\langle \mathcal{L} \rangle = PC \, N! (N-2)! \left\langle \left(\sum_i^N t_i - t_{SN}\right)^{1-N} \right\rangle$$

2. Data model: $x_i = \mu_i S + n_i$

- n_i : independent gaussian variables noise, $N(0, \sigma_i^2)$ ← known
- μ_i : ~~un~~known signal

~~a. H_0 : S = 0 H_1 = S = 1~~

a. $H_0 : S = 0 \Rightarrow x_i = n_i$

$H_1 : S = 1 \Rightarrow x_i = \mu_i + n_i$

- $\mathcal{L}(\underline{d} | H_0) = \prod_{i=1}^{N} N(x_i ; 0, \sigma_i^2)$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-x_i^2/2\sigma_i^2}$$

- $\mathcal{L}(\underline{d} | H_1) = \prod_{i=1}^{N} N(x_i ; \mu_i, \sigma_i^2)$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(x_i-\mu_i)^2/2\sigma_i^2}$$

$$T(\underline{d}) = \frac{\mathcal{L}(\underline{d}|H_1)}{\mathcal{L}(\underline{d}|H_0)} = \prod_{i=1}^{N} \frac{\cancel{1/\sqrt{2\pi\sigma_i^2}}\, e^{-(x_i-\mu_i)^2/2\sigma_i^2}}{\cancel{1/\sqrt{2\pi\sigma_i^2}}\, e^{-x_i^2/2\sigma_i^2}}$$

$$= \prod_{i=1}^{N} \frac{e^{-(x_i-\mu_i)^2/2\sigma_i^2}}{e^{-x_i^2/2\sigma_i^2}} = \prod_{i=1}^{N} \frac{e^{-x_i^2/2\sigma_i^2} \cdot e^{(2x_i\mu_i)/2\sigma_i^2} \cdot e^{-\mu_i^2/2\sigma_i^2}}{e^{-x_i^2/2\sigma_i^2}}$$

$$= \prod_{i=1}^{N} e^{-\mu_i^2/2\sigma_i^2} e^{2x_i\mu_i/2\sigma_i^2}$$

$$= \prod_{i=1}^{N} e^{\mu_i(2x_i - \mu_i)/2\sigma_i^2}$$

$$\ln(\Upsilon(\underline{d})) = \sum_{i=1}^{N} \ln\left(e^{\mu_i(2x_i-\mu_i)/2\sigma_i^2}\right)$$

$$= \underline{\underline{\sum_{i=1}^{N} \frac{1}{2\sigma_i^2} \mu_i(2x_i - \mu_i)}} = T'$$

$a = -\mu_i^2/2\sigma_i^2$

$b = \mu_i/\sigma_i^2$

b. Using the fact that the linear superposition of gaussian random variables is itself a gaussian random variable:

$Y = a + bX \quad (X \sim N(\mu, \sigma^2))$

$\langle x \rangle = a + b\mu, \ \sigma_x = b^2\sigma^2$

- $H_0: \ x_i \sim N(0, \sigma_i^2)$

Ignoring sum;

$$\text{mean} = -\frac{\mu_i^2}{2\sigma_i^2} + \frac{\mu_i}{\sigma_i^2} \cdot 0 = -\mu_i^2/2\sigma_i^2$$

$$\text{std.}^2 = \frac{\mu_i^2}{\sigma_i^4} \cdot \sigma_i^2 = \mu_i^2/\sigma_i^2$$

$$\Rightarrow \underline{\underline{T'_{H_0} \sim N\left(-\sum_i^N \frac{\mu_i^2}{2\sigma_i^2}, \sum_i^N \frac{\mu_i^2}{\sigma_i^2}\right)}}$$

- $H_1: \quad x_i \sim N(\mu_i, \sigma_i^2)$

Ignoring sum; $\quad \text{mean} = -\frac{\mu_i^2}{2\sigma_i^2} + \frac{\mu_i}{\sigma_i^2}\mu_i = \frac{\mu_i^2}{2\sigma_i^2}$

$$\text{std.}^2 = \frac{\mu_i^2}{\sigma_0^4} \cdot \sigma_0^2 = \frac{\mu_i^2}{\sigma_0^2}$$

$$\Rightarrow \underline{\underline{T'_{H_1} \sim N\left(\sum_i^N \frac{\mu_i^2}{2\sigma_0^2} , \sum_i^N \mu_i^2/\sigma_i^2 \right)}}$$

c. For a false positive probability:

$$\alpha = \int_{\eta}^{\infty} \frac{1}{\sqrt{2\pi \sum_0^N \mu_i^2/\sigma_i^2}} \exp\left[- \frac{(x + \sum_0^N \mu_0^2/2\sigma_0^2)^2}{2 \sum_i^N \mu_i^2/\sigma_i^2} \right] dx$$

For false negative probability:

$$\beta = \int_{-\infty}^{\eta} \frac{1}{\sqrt{2\pi \sum_i^N \mu_i^2/\sigma_i^2}} \exp\left[- \frac{(x - \sum_i^N \mu_i^2/2\sigma_0^2)^2}{2 \sum_i^N \mu_0^2/\sigma_i^2} \right] dx$$

To make this more tractable we can work with an SNR instead:

$$SNR = \frac{\langle T' \rangle_{H_1} - \langle T' \rangle_{H_0}}{\sigma_{T'}} = \frac{\sum_i^N \mu_i^2/2\sigma_i^2 + \sum_i^N \mu_0^2/2\sigma_0^2}{\sqrt{\sum_i^N \mu_0^2/\sigma_0^2}} = \underline{\sqrt{\sum_i^N \mu_i^2/\sigma_i^2}}$$

$$\Rightarrow \quad T'_{H_0} \sim N(0, 1) \quad \& \quad T'_{H_1} \sim N(SNR, 1)$$

So: $\alpha = \int_{\eta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$

$$\beta = \int_{-\infty}^{\eta} \frac{1}{\sqrt{2\pi}} e^{-(x-\mathrm{SNR})^2/2} \, dx \quad \Big\downarrow \; y = \mathrm{SNR} - x$$

$$= \int_{\mathrm{SNR}-\eta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

d. (see jupiter notebook)

e. $\alpha = SF(\eta) \Rightarrow \eta = ISF(\alpha)$

Find η for $\alpha = \beta = 0$ & $\alpha = \beta = 1$ & iterate between them to build the curve. ($\eta = \infty$ to $-\infty$ for $\alpha = 0$ to 1)

(see Jupiter notebook)

f. (see jupiter notebook)

For fixed $\alpha = 0.01$: $1-\beta = \begin{cases} \sim 0.74 & \text{(mine)} \\ \sim 0.07 & \text{(friends)} \end{cases}$

Their true positive rate is in general less than ours as seen from the two curves.