EC4333 HW 2

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Answer 1

(a) We observe that:

$$u(w) = a - be^{\gamma w}$$

$$u'(w) = \frac{\mathrm{d}u(w)}{\mathrm{d}w} = -b\gamma e^{\gamma w} < 0$$

A utility function exhibits non-satiability. Since $b > 0$ and $\gamma > 0$, $u'(w) < 0$ for all w. This means that the investor's utility decreases as wealth increases, which implies that the investor is satiable, not non-satiable.

$$u''(w) = \frac{\mathrm{d}^2u(w)}{\mathrm{d}w^2} = -b\gamma^2 e^{\gamma w} < 0$$

A utility function exhibits risk aversion if the utility function is concave or the second derivative of the utility function is negative ($u''(w) < 0$). Since $b > 0$ and $\gamma > 0$, we have $u''(w) < 0$ for all w. Thus, the utility function is concave, indicating the investor is risk-averse.

(b) We compute the Arrow Pratt measures of risk aversion as follows:

- Absolute Risk Aversion:
$$R_A(w) = -\frac{u''(w)}{u'(w)} = -\frac{-b\gamma^2 e^{\gamma w}}{-b\gamma e^{\gamma w}} = -\gamma < 0$$
This negative absolute risk aversion indicates that the utility function is behaving unusually. Typically, risk-averse investors have positive ARA. The negative value here reflects that the utility function decreases with wealth (as noted earlier), which implies that more wealth makes the investor feel worse—an unusual scenario in standard economic models.

- Relative Risk Aversion:
$$R_R(w) = w \cdot A(w) = -\gamma w < 0$$
Since the relative risk aversion is also negative and grows (in magnitude) with wealth, the investor becomes increasingly unwilling to take on risk with higher wealth. However, the negative sign again reflects unconventional behavior for a typical risk-averse individual.

Answer 2

(a) The probability of obtaining the first head on the n-th trial requires:

- The first $n-1$ trials to be tails, each with probability $1-p$.
- The n-th trial to be a head, with probability p.

Since the trials are independent, the probability mass function (PMF) of the random variable N, the number of trials required to get the first head, is:

$$P(N = n) = p(1-p)^{n-1}, \quad n = 1, 2, 3, \ldots$$

This is the PMF of a geometric distribution with success probability p.

(b) Given:

- Reward: $w = p^{-n}$.

- Utility function: $u(w) = \ln(w) = \ln(p^{-n})$.

We simplify the utility function:

$$u(w) = \ln\left(p^{-n}\right) = -n\ln(p).$$

Now, the expected utility $E(u(x))$ is:

$$\mathbb{E}(u(x)) = \sum_{n=1}^{\infty} u(w) \cdot P(N=n)$$

Substitute $u(w) = -n\ln(p)$ and $P(N=n) = p(1-p)^{n-1}$:

$$\mathbb{E}(u(x)) = \sum_{n=1}^{\infty} (-n\ln(p)) \cdot p(1-p)^{n-1}$$

$$\implies \mathbb{E}(u(x)) = -\ln(p)\sum_{n=1}^{\infty} np(1-p)^{n-1}$$

This is a standard result for the expectation of a geometric distribution with success probability p. We derive it as follows. Let:

$$S = \sum_{n=1}^{\infty} n(1-p)^{n-1}$$

To evaluate this, differentiate the geometric series sum $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ with respect to x:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) \implies \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

Now, substitute $x = 1-p$:

$$\sum_{n=1}^{\infty} n(1-p)^{n-1} = \frac{1}{p^2}$$

We substitute the above result into the expression for $\mathbb{E}(u(x))$:

$$\mathbb{E}(u(x)) = -p\ln(p) \cdot \frac{1}{p^2} = -\frac{\ln(p)}{p}$$

Answer 3

(a) Given:

- If healthy: Wealth $w_H = 1000000$.
- If injured: Wealth $w_I = 10000$.
- Probability of staying healthy: $P_H = 0.9$.
- Probability of injury: $P_I = 0.1$.
- Utility function: $u(w) = \ln(w)$.

$$\mathbb{E}(u(w)) = 0.9\ln(1000000) + 0.1\ln(10000)$$
$$\ln(1000000) = \ln\left(10^6\right) = 6\ln(10), \quad \ln(10000) = \ln\left(10^4\right) = 4\ln(10)$$
$$\implies \mathbb{E}(u(w)) = 0.9 \cdot 6\ln(10) + 0.1 \cdot 4\ln(10) = 5.8\ln(10) \approx 13.35499$$

(b)

If Desmond buys insurance for G, his wealth in case of injury becomes:

$$w_I' = 10000 + 990000 = 1000000.$$

His wealth in both states is now $1000000 - G$. The utility in both cases becomes:

$$u(w) = \ln(1000000 - G).$$

Thus, his expected utility with insurance is:

$$E(u(w)) = \ln(1000000 - G).$$

To find the maximum G, equate the expected utility with insurance to the expected utility without insurance:

$$\ln(1000000 - G) = (5.8)\ln(10). \implies 1000000 - G = 10^{5.8} \approx 630957.3$$

$$\implies 1000000 - G = 630957.3 \implies G = 1000000 - 630957.3 = 369042.7$$

Thus, the maximum price G that Desmond would be willing to pay for the insurance is:

$$G = 369042.7\,\text{SGD}.$$

Answer 4

We need to show that:

$$\int_{w}^{t} [F(s) - G(s)]\,\mathrm{d}s\Big|_{t=\underline{w}}^{t=\overline{w}} = 0$$

$$\int_{w}^{t} [F(s) - G(s)]\,\mathrm{d}s\Big|_{t=\underline{w}}^{t=\overline{w}} = \int_{w}^{t} [(1 - G(s)) - (1 - F(s))]\,\mathrm{d}s\Big|_{t=\underline{w}}^{t=\overline{w}}$$

$$= \int_{w}^{t} [1 - G(s)]\,\mathrm{d}s\Big|_{t=\underline{w}}^{t=\overline{w}} - \int_{w}^{t} [1 - F(s)]\,\mathrm{d}s\Big|_{t=\underline{w}}^{t=\overline{w}}$$

Now we show the following: For a random variable X having a CDF F, the expected value of the random variable is given by:

$$\mathbb{E}_F(X) = \int_{\underline{w}}^{\overline{w}} [1 - F(s)]\,\mathrm{d}s$$

(Fun fact: this is called the Darth Vader rule)

Proof. First we note that since X is a non negative random variable, we can set:

$$\underline{w} = 0$$

It is possible to X have a 0 density for some other positive interval too, but we need to have the density only over the non negative domain. The expected value of a continuous random variable X with cumulative distribution function (CDF) $F(x)$ is defined as:

$$\mathbb{E}_F(X) = \int_{\underline{w}}^{\overline{w}} x f(x)\,\mathrm{d}x,$$

where $f(x) = F'(x)$ is the probability density function (PDF) of X. To derive the given expression, we use integration by parts. Let:

$$u = x \quad \text{and} \quad \mathrm{d}v = f(x)\,\mathrm{d}x.$$

$$\implies \mathrm{d}u = \mathrm{d}x \quad \text{and} \quad v = F(x).$$

Applying the integration by parts formula:

$$\mathbb{E}_F(X) = \int_{\underline{w}}^{\overline{w}} x f(x)\,\mathrm{d}x = \int_{\underline{w}}^{\overline{w}} u\,\mathrm{d}v = uv\Big|_{\underline{w}}^{\overline{w}} - \int_{\underline{w}}^{\overline{w}} v\,\mathrm{d}u,$$

$$\implies \mathbb{E}_F(X) = [xF(x)]_{\underline{w}}^{\overline{w}} - \int_{\underline{w}}^{\overline{w}} F(x)\,\mathrm{d}x.$$

Since $F(x) \to 0$ as $x \to \underline{w}$ and $F(x) \to 1$ as $x \to \overline{w}$, the boundary term becomes:

$$[xF(x)]_{\underline{w}}^{\overline{w}} = \overline{w} = \overline{w} - 0 = \overline{w} - \underline{w} = \int_{\underline{w}}^{\overline{w}} 1\,\mathrm{d}x$$

Thus:

$$\mathbb{E}_F(X) = \int_{\underline{w}}^{\overline{w}} 1 \,\mathrm{d}x - \int_{\underline{w}}^{\overline{w}} F(x) \,\mathrm{d}x = \int_{\underline{w}}^{\overline{w}} (1 - F(x)) \,\mathrm{d}x$$

$$\implies \mathbb{E}_F(X) = \int_{\underline{w}}^{\overline{w}} (1 - F(s)) \,\mathrm{d}s$$

This completes the proof. $\square$

We can now use this in our original equation:

$$\int_{w}^{t} [F(s) - G(s)] \,\mathrm{d}s \bigg|_{t=\underline{w}}^{t=\overline{w}} = \int_{w}^{t} [1 - G(s)] \,\mathrm{d}s \bigg|_{t=\underline{w}}^{t=\overline{w}} - \int_{w}^{t} [1 - F(s)] \,\mathrm{d}s \bigg|_{t=\underline{w}}^{t=\overline{w}}$$

$$= \left(\int_{w}^{\overline{w}} [1 - G(s)] \,\mathrm{d}s - \int_{w}^{\underline{w}} [1 - G(s)] \,\mathrm{d}s \right) - \left(\int_{w}^{\overline{w}} [1 - F(s)] \,\mathrm{d}s - \int_{w}^{\underline{w}} [1 - F(s)] \,\mathrm{d}s \right)$$

$$= \left(\int_{w}^{\overline{w}} [1 - G(s)] \,\mathrm{d}s + \int_{\underline{w}}^{w} [1 - G(s)] \,\mathrm{d}s \right) - \left(\int_{w}^{\overline{w}} [1 - F(s)] \,\mathrm{d}s + \int_{\underline{w}}^{w} [1 - F(s)] \,\mathrm{d}s \right)$$

$$= \int_{\underline{w}}^{\overline{w}} [1 - G(s)] \,\mathrm{d}s - \int_{\underline{w}}^{\overline{w}} [1 - F(s)] \,\mathrm{d}s = \mathbb{E}_G(X) - \mathbb{E}_F(X) = 0$$

The above follows from what we are given: $\mathbb{E}_F(X) = \mathbb{E}_G(X)$.

Answer 5

Investment A follows a uniform distribution on the interval $[0, 1]$. The CDF $F_A(w)$ is:

$$F_A(w) = \begin{cases} 0, & \text{if } w < 0, \\ w, & \text{if } 0 \le w \le 1, \\ 1, & \text{if } w \ge 1. \end{cases}$$

The CDF $F_B(w)$ is given piecewise as:

$$F_B(w) = \begin{cases} \sqrt{w}, & \text{if } 0 \le w \le 0.16, \\ 0.5, & \text{if } 0.16 < w \le 0.6, \\ 0.75, & \text{if } 0.6 < w \le 0.8, \\ 0.9, & \text{if } 0.8 < w \le 1, \\ 1, & \text{if } w \ge 1. \end{cases}$$

(a)

$$\text{FOSD:} \quad F_B(w) \le F_A(w) \quad \text{for all } w \in [0, 1], \text{ with strict inequality for some } w.$$

We compare the two CDFs and observe that:

$$w = 0.3 \implies F_A(w) = 0.3 < 0.5 = F_B(w)$$

$$w = 0.6 \implies F_A(w) = 0.6 > 0.5 = F_B(w)$$

Since $F_B(w) < F_A(w)$ for some values and $F_B(w) > F_A(w)$ for some values, no investment will FOSD the other. Thus, FOSD is in conclusive in this case. This can also be seen through the plots of the CDF:

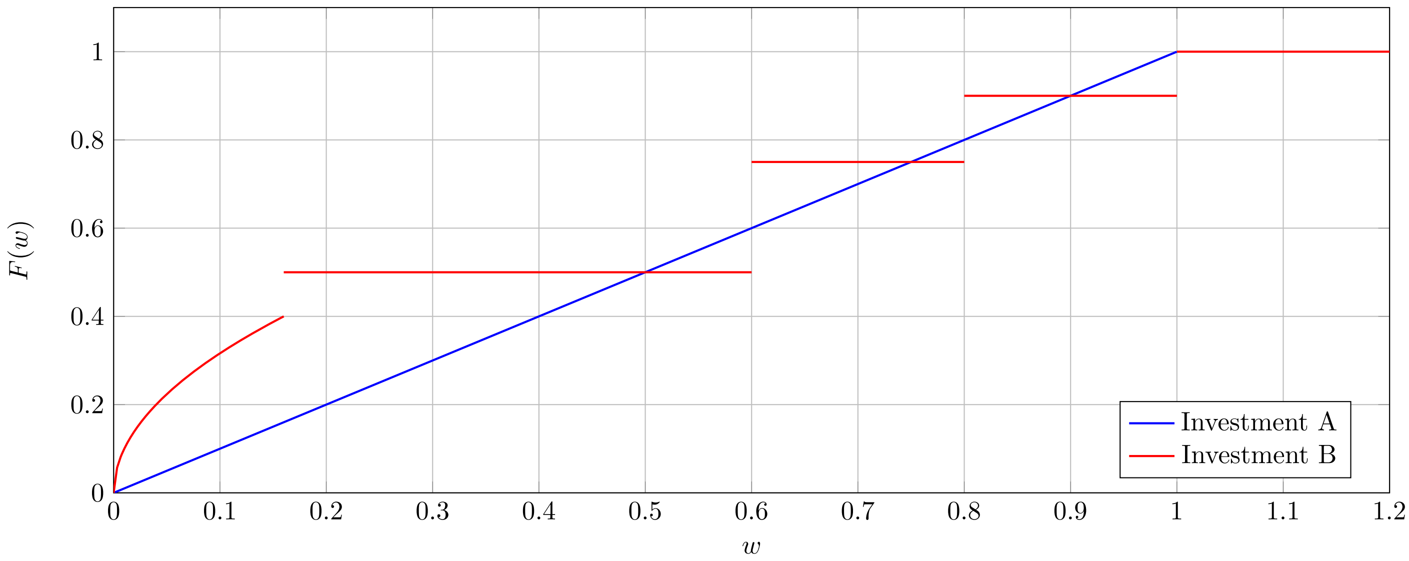

Figure 1: CDFs of Investment A and Investment B

(b) For SOSD, we need to compare the integrals of the CDFs up to each point w:

$$\int_0^x F_A(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^x F_B(t)\,\mathrm{d}t \quad \text{for various values of } x.$$

The integral of the CDF of Investment A from 0 to x is:

$$\int_0^x F_A(t)\,\mathrm{d}t = \int_0^x t\,\mathrm{d}t = \left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{2}.$$

Now, we compute the integral of $F_B(t)$ over the relevant intervals.
1. For $0 \le x \le 0.16$:

$$\int_0^x F_B(t)\,\mathrm{d}t = \int_0^x \sqrt{t}\,\mathrm{d}t = \left[\frac{2}{3}t^{3/2}\right]_0^x = \frac{2}{3}x^{3/2}.$$

2. For $0.16 < x \le 0.6$:

$$\int_0^x F_B(t)\,\mathrm{d}t = \int_0^{0.16} \sqrt{t}\,\mathrm{d}t + \int_{0.16}^x 0.5\,\mathrm{d}t.$$

Thus total area is:

$$\frac{2}{3}(0.16)^{3/2} + 0.5 \times (x - 0.16).$$

3. For $0.6 < x \le 0.8$:

$$\int_0^x F_B(t)\,\mathrm{d}t = \int_0^{0.16} \sqrt{t}\,\mathrm{d}t + \int_{0.16}^{0.6} 0.5\,\mathrm{d}t + \int_{0.6}^x 0.75\,\mathrm{d}t.$$

Thus total area is:

$$\frac{2}{3}(0.16)^{3/2} + 0.5 \times (0.6 - 0.16) + 0.75 \times (x - 0.6).$$

4. For $0.8 < x \le 1$:

$$\int_0^x F_B(t)\,\mathrm{d}t = \frac{2}{3}(0.16)^{3/2} + 0.5 \times (0.6 - 0.16) + 0.75 \times (0.8 - 0.6) + 0.9 \times (x - 0.8).$$

The below plots show the integrals of the CDFs:

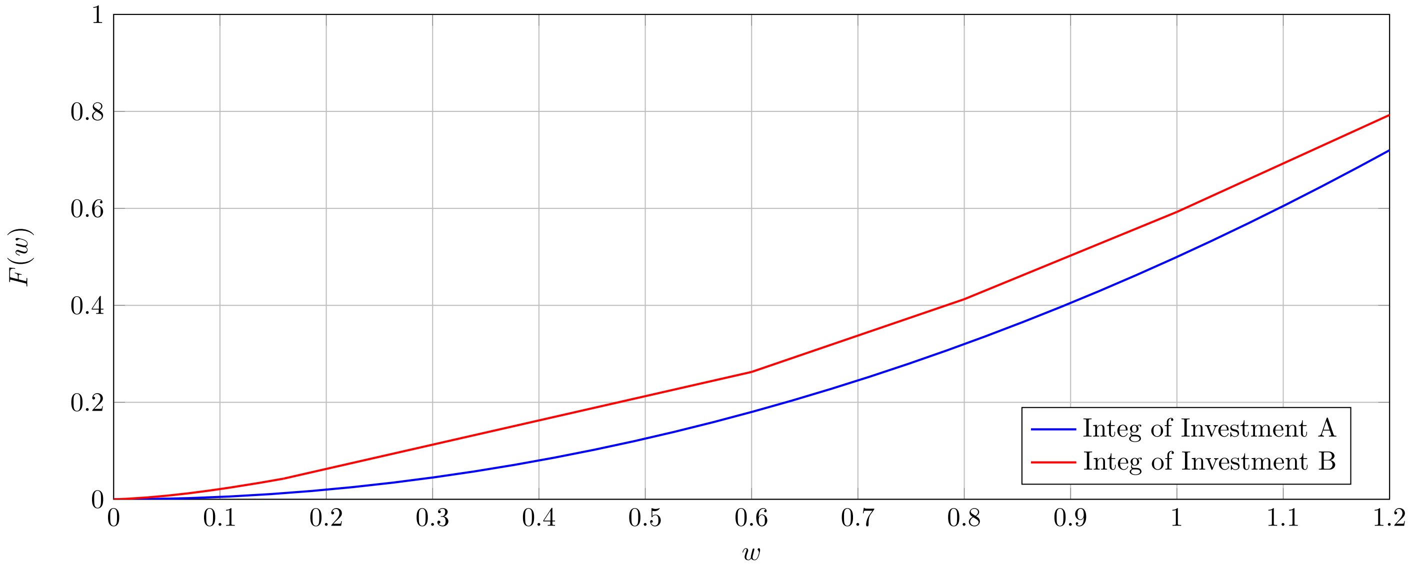

Figure 2: Integral of CDFs of Investment A and Investment B

We can also tabulate the cumulative integrals at key points to facilitate comparison (in addition to the graph above):

x	$\int_0^x F_A(t)\,\mathrm{d}t$	$\int_0^x F_B(t)\,\mathrm{d}t$
0.16	0.0128	0.04267
0.6	0.18	0.26267
0.8	0.32	0.41267
1.0	0.50	0.59267

Since the cumulative integral of $F_B(t)$ is consistently larger than that of $F_A(t)$ at all points (as seen in the graph), Investment A SOSD-dominates Investment B. This means Investment A is preferred under SOSD criteria (thus requiring risk aversion in addition to local non satiation).

Answer 6

(a)
The Taylor expansion of a function $u(w)$ around the point w_0 is given by:

$$u(w) = u(w_0) + u'(w_0)(w - w_0) + \frac{u''(w_0)}{2!}(w - w_0)^2 + \frac{u'''(w_0)}{3!}(w - w_0)^3 + \mathcal{O}((w - w_0)^4)$$

where $u'(w_0)$, $u''(w_0)$, and $u'''(w_0)$ are the first, second, and third derivatives of $u(w)$ evaluated at w_0.

(b) We now take the expectation of the Taylor approximation up to the third order:

$$\mathbb{E}(u(w)) = \mathbb{E}\left[u(w_0) + u'(w_0)(w - w_0) + \frac{u''(w_0)}{2!}(w - w_0)^2 + \frac{u'''(w_0)}{3!}(w - w_0)^3\right]$$

Using the linearity of expectation:

$$\mathbb{E}(u(w)) = u(w_0) + u'(w_0)\mathbb{E}(w - w_0) + \frac{u''(w_0)}{2!}\mathbb{E}\left[(w - w_0)^2\right] + \frac{u'''(w_0)}{3!}\mathbb{E}\left[(w - w_0)^3\right]$$

Assuming $w \sim N(\mu, \sigma^2)$, the following properties hold:

$$\begin{aligned}
\mathbb{E}(w) &= \mu, \\
\mathbb{E}(w^2) &= \mu^2 + \sigma^2 \\
\mathbb{E}(w^3) &= \mu^3 + 3\mu\sigma^2
\end{aligned}$$

The above can be trivially shown using MGFs of normal distribution and simply taking derivatives:

$$\mathbb{E}[e^{tX}] = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$$

$$\frac{\mathrm{d}\mathbb{E}[e^{tX}]}{\mathrm{d}t} = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\} \cdot (\mu + \sigma^2 t) \implies \left.\frac{\mathrm{d}\mathbb{E}[e^{tX}]}{\mathrm{d}t}\right|_{t=0} = \mu$$

$$\frac{\mathrm{d}^2\mathbb{E}[e^{tX}]}{\mathrm{d}t^2} = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\} \cdot (\mu^2 + \sigma^2 + 2\mu\sigma^2 t + \sigma^4 t^2) \implies \left.\frac{\mathrm{d}^2\mathbb{E}[e^{tX}]}{\mathrm{d}t^2}\right|_{t=0} = \mu^2 + \sigma^2$$

$$\frac{\mathrm{d}^3\mathbb{E}[e^{tX}]}{\mathrm{d}t^3} = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\} \cdot (2\mu\sigma^2 + 2\sigma^4 t + (\mu^2 + \sigma^2 + 2\mu\sigma^2 t + \sigma^4 t^2) \cdot (\mu + \sigma^2 t)) \implies \left.\frac{\mathrm{d}^3\mathbb{E}[e^{tX}]}{\mathrm{d}t^3}\right|_{t=0} = \mu^3 + 3\mu\sigma^2$$

Thus, we have:

$$\begin{aligned}
\mathbb{E}(w - w_0) &= \mathbb{E}(w) - w_0 = \mu - w_0, \\
\mathbb{E}\left[(w - w_0)^2\right] &= \mathbb{E}(w^2) - 2w_0\mathbb{E}(w) + w_0^2 = \sigma^2 + (\mu - w_0)^2, \\
\mathbb{E}\left[(w - w_0)^3\right] &= \mathbb{E}(w^3) - 3w_0\mathbb{E}(w^2) + 3w_0^2\mathbb{E}(w) - w_0^3 = \mu^3 + 3\mu\sigma^2 - 3w_0(\mu^2 + \sigma^2) + 3w_0^2\mu + w_0^3
\end{aligned}$$

Thus, the expected utility becomes:

$$\mathbb{E}(u(w)) = u(w_0) + u'(w_0)(\mu - w_0) + \frac{u''(w_0)}{2}\left[\sigma^2 + (\mu - w_0)^2\right] + \frac{u'''(w_0)}{6}\left[\mu^3 + 3\mu\sigma^2 - 3w_0(\mu^2 + \sigma^2) + 3w_0^2\mu + w_0^3\right]$$

Now, we replace w_0 with 0, as it is an arbitrary point:

$$\mathbb{E}(u(w)) = u(0) + u'(0)\mu + \frac{u''(0)}{2}\left(\sigma^2 + \mu^2\right) + \frac{u'''(0)}{6}\left(\mu^3 + 3\mu\sigma^2\right)$$

(c) Notice that the expected utility depends only on the mean μ and the variance σ^2:

$$\mathbb{E}(u(w)) = u(0) + u'(0)\mu + \frac{u''(0)}{2}\left(\sigma^2 + \mu^2\right) + \frac{u'''(0)}{6}\left(\mu^3 + 3\mu\sigma^2\right)$$

This shows that when wealth is normally distributed, the expected utility is determined entirely by the mean and variance of wealth, which motivates the use of mean-variance portfolio theory in Markowitz's framework.

Answer 7

(a) For global minimum variance portfolio, we just want to find $w \in \mathbb{R}^4$ such that it solves the following optimisation problem:

$$\min_{w \in \mathbb{R}^4} \frac{1}{2} w^\top \Sigma w$$
$$\text{s.t.} \quad w^\top E = 1$$

As derived in the lectures, the solution to the above is obtained using the FOC of the Lagrangian, giving us:

$$w^{\text{GMV}} = \frac{1}{B}\Sigma^{-1}e; \quad \text{where} \quad B = e^\top \Sigma^{-1} e$$

After inverting the given matrix, we obtain:

$$\Sigma = \begin{pmatrix} 0.25 & -0.15 & 0.15 & 0.05 \\ -0.15 & 0.21 & -0.15 & -0.15 \\ 0.15 & -0.15 & 0.20 & 0.05 \\ 0.05 & -0.15 & 0.05 & 0.25 \end{pmatrix} \implies \Sigma^{-1} = \begin{pmatrix} 8.75 & 6.25000 & -2.5 & 2.5 \\ 6.25 & 27.08333 & 12.5 & 12.5 \\ -2.50 & 12.50000 & 15.0 & 5.0 \\ 2.50 & 12.50000 & 5.0 & 10.0 \end{pmatrix}$$

$$\implies B = e^\top \Sigma^{-1} e = 133.3333$$

$$\therefore w^{\text{GMV}} = \frac{1}{B}\Sigma^{-1}e = \frac{1}{133.3333}\Sigma^{-1}e = \begin{pmatrix} 0.1125 \\ 0.4375 \\ 0.2250 \\ 0.2250 \end{pmatrix}$$

For this portfolio, the expected return and variance are given by:

$$\mathbb{E}(R_p) = E(R)^\top w^{\text{GMV}} = 0.166875$$

$$\text{Var}(R_p) = \left(w^{\text{GMV}}\right)^\top \Sigma w^{\text{GMV}} = \frac{1}{B} = 0.0075$$

(b) For minimum variance portfolio, we just want to find $w \in \mathbb{R}^4$ such that it solves the following optimisation problem:

$$\min_{w\in\mathbb{R}^4} \frac{1}{2} w^\top \Sigma w$$

$$\text{s.t.} \quad w^\top E = 1 \; w^\top \mu = \mu_p = 0.22$$

As derived in the lectures, the solution to the above is obtained using the FOC of the Lagrangian, giving us:

$$w^{\text{MVP}} = \Sigma^{-1} \frac{(c - A\mu_p)e + (B\mu_p - A)\mu}{BC - A^2} \quad \text{where} \;\; A = \mu^\top\Sigma^{-1}e; \; B = e^\top\Sigma^{-1}e; \; C = \mu^\top\Sigma^{-1}\mu$$

$$A = \mu^\top\Sigma^{-1}e = 22.25$$
$$B = e^\top\Sigma^{-1}e = 133.3333$$
$$C = \mu^\top\Sigma^{-1}\mu = 3.734375$$

$$\therefore w^{\text{MVP}} = \Sigma^{-1} \frac{(c - A\mu_p)e + (B\mu_p - A)\mu}{BC - A^2} = \begin{pmatrix} 0.8802920 \\ 0.3211679 \\ -0.7211679 \\ 0.5197080 \end{pmatrix}$$

For this portfolio, the expected return and variance are given by:

$$\mathbb{E}(R_p) = E(R)^\top w^{\text{MVP}} = 0.22 \;\; \text{(as expected)}$$

$$\text{Var}(R_p) = \left(w^{\text{MVP}}\right)^\top \Sigma w^{\text{GMV}} = 0.1393431$$