A more Flexible Lomax Distribution: Characterization, Estimation, Group Acceptance Sampling Plan and Applications

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Abstract

The aim of this paper is threefold. First, we developed a novel distribution, termed the Odd Perks-Lomax (OPL) distribution, derived from the Odd Perks-G class of distributions. Second, we studied various properties of the OPL distribution. Third, we designed a group acceptance sampling plan for truncated life tests based on this new distribution. The OPL distribution is a modification of the classical Lomax distribution, incorporating two additional parameters. Hypothetical plots of the hazard and density functions indicate that the OPL distribution is flexible and capable of modeling various data types. The model parameters were estimated using the maximum likelihood approach. We designed group acceptance sampling plans (GASPs) based on truncated life tests and used simulation studies, along with empirical examples, to demonstrate the model's effectiveness in quality assessment of products. To determine the suitability of the OPL distribution for modeling data, we applied it to two datasets: one on engineering records and the other on medical records. The results show that the proposed OPL distribution outperforms the classical Weibull and Exponentiated Weibull distributions. The maximum likelihood estimates of the parameters based on these datasets were reintroduced into the designed GASPs model to further demonstrate its applicability in statistical quality control.

Keywords— Odd Perks Lomax distribution, group acceptance sampling plan, maximum likelihood estimation,

quantile function, quality, producer's risk, consumer's risk, lifetime data.

1 Introduction

A product's quality is of utmost importance to both consumers and producers. Companies, industries, and organizations are often recognized for the quality of their products. Consequently, quality control (QC) methods play a central role in most business investigations Tripathi, Dey, and Saha [40], Mussa and Rosen [27], Aslam and Jun [6], and Sitkin, Sutcliffe, and Schroeder [37]. QC involves a systematic approach by organizations, industries, and companies to ensure their products meet improved quality standards Juran [20]. This is achieved by implementing benchmarks for quality assurance and testing products to identify statistically significant variations. QC methods vary depending on the product or industry. For example, in food and drug manufacturing, QC includes chemical and microbiological testing to ensure products do not harm consumers.

The evolution of QC has expanded from adjusting production to standardized models to meet consumer requirements to encompassing manufacturing processes and application across diverse service sectors. Statistical quality control methods, including process control, control charts, product quality control, and acceptance sampling, are adopted by organizations, companies, and industries to regulate the quality of products, procedures, and services. These methods involve statistically analyzing data collected during production or service delivery to identify and eliminate sources of variability and ensure consistent adherence to quality standards Naz et al. [29].

Recently, a good number of institutions have taken into consideration the implementation of statistical quality control (SQC) procedures and the decisive importance of enhancing their competitiveness in the labor market. The use of the SQC techniques assures great profit. Starting from enhancing the consistency of a product by observing and controlling the quality of the production process, businesses can ensure that their products meet the same high standards every time. This consistency not only improves the overall quality of the products but also builds trust with customers, who know they can rely on the business to deliver a consistent product Hoe and Mansori [18] and Mascarenhas, Kesavan, and Bernacchi [23].

The lifespan of electronic accessories cannot be pre-determined Neely [30], Sinclair and Zairi [36], Milakovich [24], and Mitra [25]. In recent times, much attention in the statistical literature has been on modifying, measuring, and observing the quality of its processes. We can only detect this trend with the assistance of several distributions in statistics. Acceptance sampling plan (ASP) is an approach to provide a high level of product sustainability, especially in product batches. It involves analyzing products in batches to determine their behavior as a whole, with the essential factor being attaining a certain standard for such product(s) Banihashemi, Nezhad, and Amiri [8], Ermer and Ploss [14], Jongenburger et al. [19], and Gonzales-Barron and Cadavez [17]. ASP methods simply involve a random sampling of the readily available product being selected and then these sample product(s) are tested. An acceptance or rejection module is then coined from the statistical distribution in justification of an establishment in the case of failure of the time distribution of such tools. Overall, this method of sampling is best used when trying to decide between accepting or rejecting a batch of products but it does not accurately determine the whole estimation. What this entails is, that the manufacturer distributes some samples from his batch of products to consumers and if the consumer notices a lower rate of imperfection, then the consumer will approve the entire product. Fundamentally, the sole purpose is to affirm if the defective items fall within the acceptance limit, and if the batch fails to meet the criteria, the entire lot is tagged unacceptable and will be rejected. In the use of the quality control method, a double sampling approach implies selecting two different samples from a whole and then analyzing them to checkmate the pre-established quality criterion Broadhurst et al. [10], Reitsma et al. [34], and Moser and Korstjens [26].

Noting the existence of several acceptance sampling plans, such as the single, double, and group sampling plan, we will be diverting our attention to the group sampling plan. Group acceptance sampling plans (GASP) are a statistical method used in quality control essentially to ascertain the acceptance or rejection level of a batch of services based on the results of a sample taken from that batch. There is, however, a difference between a GASP and an acceptance sampling plan (ASP) in terms of the way samples are taken and analyzed Ameeq et al. [5].

This research is driven by the continuous need to improve the performance of existing distributions, in particular, the need to advance the study of quality assessment of products from various manufacturers as the attention of customers continues to redirect towards products with competitive advantage. Recently, Naz et al. [29] designed

GASP under flexible new Kumaraswamy exponential distribution; Facchinetti, Osmetti, and Magagnoli [15] proposed a single acceptance sampling plan for generalized beta distribution; Alsultan [4] proposed a GASP design for an extended odd Weibull exponential distribution; Al-Omari and Ismail [32] studied gamma Lindley distribution based on GASP; Algarni [3] developed a GASP for a new compounded three-parameter Weibull distribution while Ahmed, Ali, and Yousof [2] designed a single acceptance sampling plan for a novel G family of distributions.

This article is therefore motivated by the desire to advance the Lomax distribution by introducing an additional parameter that will account for the shape of the distribution, making the new model more flexible. Specifically,

1. The additional parameter(s) does not prevent the tractability of the model and its characteristics.
2. We design a GASP under truncated life tests based on the proposed OPL distribution.
3. The new model is unique in providing a better goodness of fit than some known classical models such as the Weibull and the Exponentiated Weibull.
4. The probability density function (pdf) of the proposed model has a reversed bathtub shape, L-shape, and strictly decreasing (positively skewed) shape. These varying shapes demonstrate the wider applicability of the distribution as compared to the parent distribution.
5. In the end, the OPL distribution maintains the heavy-tailness and more heavier than the Lomax distribution using the extreme value theorem Beirlant, Matthys, and Dierckx [9] and Zhao et al. [42].

The subsequent sections of this article are organized in the following order; Section 2 introduces the OPL distribution, stating its distribution function, density function, and visualization of the functions. In Section 3, we provide the characteristics of the distribution. Section 5 gives a detailed GASP design based on the OPL distribution. Section 6 dwells on the application of OPL distribution to lifetime data, and lastly, in Section 7, we conclude the article.

2 The Specification of the OPL Distribution

Elbatal et al. [13] proposed the odd-perks-G class of distributions with probability density function (pdf) and cumulative distribution function (cdf) given respectively as

$$f(x)=\frac{\beta\theta\left(1+\beta\right)g\left(x;\delta\right)e^{\theta\left[\frac{G(x;\delta)}{\bar{G}(x;\delta)}\right]}}{\bar{G}\left(x;\delta\right)^{2}\left[1+\beta e^{\theta\left[\frac{G(x;\delta)}{\bar{G}(x;\delta)}\right]}\right]^{2}},\quad x>0,\quad \theta>0, \tag{1}$$

and

$$F(x)=1-\frac{1+\beta}{1+\beta e^{\theta\left[\frac{G(x;\delta)}{\bar{G}(x;\delta)}\right]}}, \tag{2}$$

where $G(.)$ and $g(.)$ are the cdf and pdf of any baseline distribution with δ as the parameter vector of the baseline distribution. The Lomax distribution or Shifted Pareto distribution was proposed by Lomax [22] with pdf and cdf respectively given as

$$g(x;\alpha,\lambda)=\frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)};\quad x>0, \tag{3}$$

and

$$G(x;\alpha,\lambda)=1-\left(\frac{\lambda}{x+\lambda}\right)^{\alpha}, \tag{4}$$

with $\lambda>0$ being the scale parameter and $\alpha>0$ is the shape parameter. Lomax distribution also known as Pareto Type II distribution is a heavy-tail model useful in business, economics, actuarial science, queueing theory, and Internet traffic modeling, see Balakrishnan, Johnson, and Kotz [7], Chen et al. [11]. Plug in equations (3) and (4) into equations (1) and (2), we obtain a new distribution called the Odd Perks-Lomax distribution with pdf and cdf respectively given as follows:

$$f(x;\theta,\beta,\lambda,\alpha)=\frac{\theta\alpha\beta(\beta+1)\left(1+\frac{x}{\lambda}\right)^{\alpha-1}e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}{\lambda\left\{1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right\}^{2}};\quad x>0,\quad \theta,\alpha,\beta,\lambda>0, \tag{5}$$

and

$$F(x;\theta,\beta,\lambda,\alpha)=1-\frac{1+\beta}{1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}. \tag{6}$$

In modeling lifetime data, the survival function and hazard rate function are essentially used to depict the reliability and extinction of the exposed population. The survival function is $s(x)=1-cdf$, hence

$$s(x)=\frac{1+\beta}{1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}, \tag{7}$$

and the hazard rate function

$$h(x)=\frac{\theta\alpha\beta\left(1+\frac{x}{\lambda}\right)^{\alpha-1}e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}{\lambda\left\{1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right\}}. \tag{8}$$

The asymptotic behavior of the hazard function of OPL distribution is such that $\lim_{x\to 0}h(x)=\frac{\theta\alpha\beta}{\lambda(1+\beta)}$ and $\lim_{x\to\infty}h(x)=\infty$. These show that the OPL hazard function is monotonically increasing.

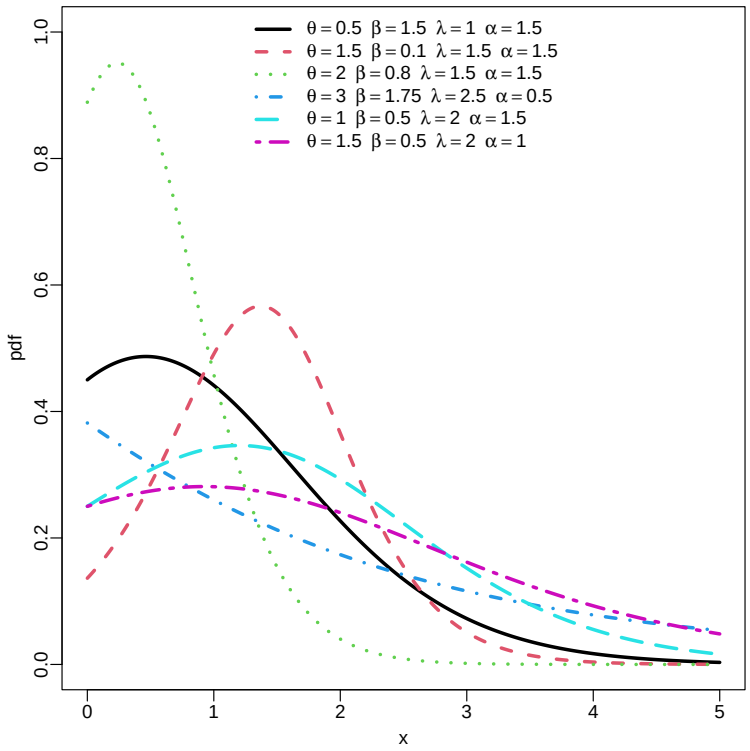


Figure 1: pdf of OPL $(\theta,\beta,\lambda,\alpha)$

The hypothetical plots in figure 1 show that the OPL distribution has a reverse bathtub shape, L-shape and strictly decreasing (positively skewed) shape. Figures 2 and 3 are the 3D plots of OPL. The plots of the hazard function shown in figures 4, 5, 6 have a J-shape, strictly increasing and L-shape. The surface plots in figures 2 and 3 which represents pdf and $h(x)$ has some resemblance. Both plots are skewed and provide visual evidence that the OPL distribution can model failure rate data.

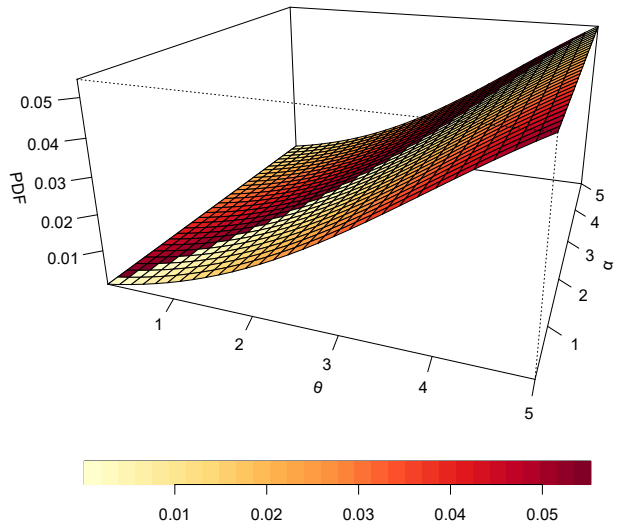


Figure 2: pdf 3D plot of OPL $(\theta, \beta, \lambda, \alpha)$

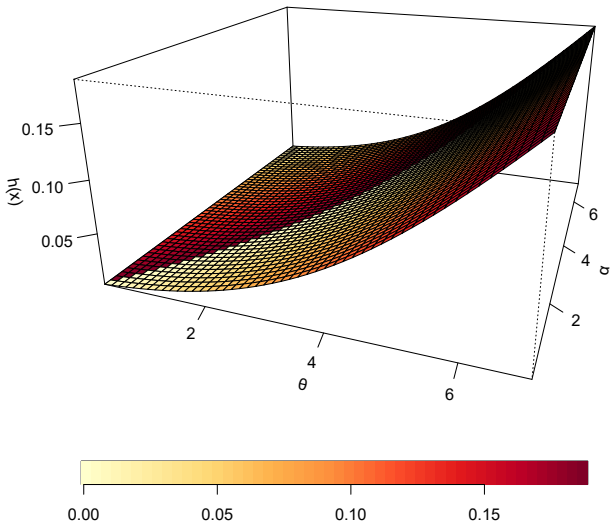


Figure 3: 3D hazard function of OPL $(\theta, \beta, \lambda, \alpha)$

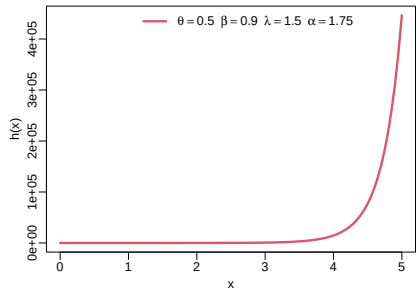


Figure 4: hazard function of OPL $(\theta, \beta, \lambda, \alpha)$

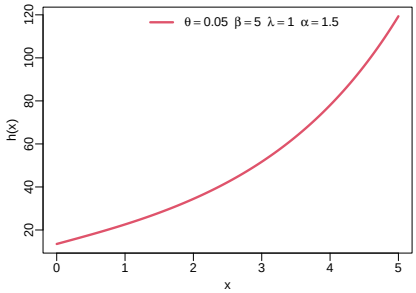


Figure 5: hazard function of OPL $(\theta, \beta, \lambda, \alpha)$

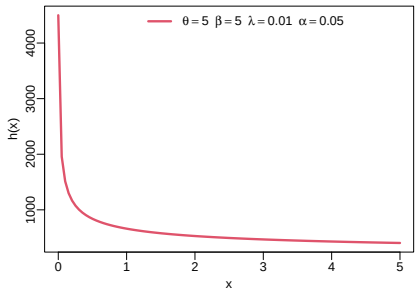


Figure 6: hazard function of OPL $(\theta, \beta, \lambda, \alpha)$

3 Characteristics of the OPL Distribution

In this section, we discuss the basic mathematical properties of the new distribution.

3.1 Quantile function

The quantile function is a utility function that aids in many ways in understanding the behavior of a model. For instance, the invertibility of a model is fully dependent on the analytical form of the quantile function. Again, data generation is also a function of the quantile function. Similarly, the applicability of a given model in statistical quality control cannot be without measuring the mean lifetime or median of the product which assumes the distribution. To obtain this all-important function, let $u = F(x)$ in the cdf given in 6. Then,

$$u = 1 - \frac{1+\beta}{1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}, \tag{9}$$

so that

$$\frac{u+\beta}{(1-u)\,\beta} = e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}. \tag{10}$$

Taking the logarithm of both sides, we have

$$\log_e \frac{u+\beta}{(1-u)\,\beta} = \log_e e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}, \tag{11}$$

and this results to,

$$Q(u) = \lambda \left[\left\{ 1 + \frac{1}{\theta} \log_e \left(\frac{u+\beta}{(1-u)} \right) \right\}^{\frac{1}{\alpha}} - 1 \right], \quad \text{provided} \quad u \in (0,1)\,. \tag{12}$$

3.2 Moment

The rth crude moment of the OPL distribution is obtained as

$$\begin{aligned} \mu_r^{'} = \mathbb{E}X^r &= \int_0^\infty x^r f(x;\theta,\beta,\lambda,\alpha)\, dx = \int_0^\infty x^r \frac{\theta\alpha\beta(\beta+1)\left(1+\frac{x}{\lambda}\right)^{\alpha-1} e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}{\lambda\left\{1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right\}^2}\, dx \\ &= \int_0^\infty x^r \frac{\theta\alpha\beta(\beta+1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{\alpha-1} e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\left\{1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right\}^{-2} dx. \end{aligned} \tag{13}$$

Using general binomial expansion, we have

$$\left[1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right]^{-2} = \sum_{i=1}^{\infty}\binom{1+i}{i}\beta^i e^{i\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}, \tag{14}$$

$$e^{i\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}} = \sum_{j=1}^{\infty}\sum_{k=1}^{j}(-1)^{j-k}\frac{i^j\theta^j}{j!}\binom{j}{k}\left(1+\frac{x}{\lambda}\right)^{\alpha k}. \tag{15}$$

Plug in equation (11) into equation (10) will produce

$$\left[1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right]^{-2} = \sum_{i,j=1}^{\infty}\sum_{k=1}^{j}(-1)^{i+j-k}\frac{i^j\theta^j\beta^j}{j!}\binom{j}{k}\binom{1+i}{i}\left(1+\frac{x}{\lambda}\right)^{\alpha k}. \tag{16}$$

$$\mu_r^{'} = \frac{\theta\alpha\beta\,(1+\beta)}{\lambda}\sum_{i,j=1}^{\infty}\sum_{k=1}^{j}(-1)^{i+j-k}\frac{i^j\theta^j\beta^j}{j!}\binom{j}{k}\binom{1+i}{i}\int_0^\infty x^r\left(1+\frac{x}{\lambda}\right)^{\alpha k}\left(1+\frac{x}{\lambda}\right)^{\alpha-1} e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}. \tag{17}$$

Let $\phi = \frac{\theta\alpha\beta(1+\beta)}{\lambda}\sum\limits_{i,j=1}^{\infty}\sum\limits_{k=1}^{j}(-1)^{i+j-k}\frac{i^j\theta^j\beta^j}{j!}\binom{j}{k}\binom{1+i}{i}$ in equation (13), then

$$\mu_r^{'} = \phi\int_0^\infty x^r\left(1+\frac{x}{\lambda}\right)^{\alpha(1+k)-1} e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\, dx = \phi e^{-\theta}\int_0^\infty x^r\left(1+\frac{x}{\lambda}\right)^{\alpha(1+k)-1} e^{\theta\left(1+\frac{x}{\lambda}\right)^{\alpha}}\, dx, \tag{18}$$

Let $y = \theta\left(1+\frac{x}{\lambda}\right)^{\alpha}$, then

$$\mu_r^{'} = \phi e^{-\theta}\frac{\lambda^{r+1}}{\theta\alpha}\int_0^\infty\left[\left(\frac{-y}{\theta}\right)^{\frac{1}{\alpha}}-1\right]^r\left(\frac{-y}{\theta}\right)^k e^{-y}\, dy. \tag{19}$$

by applying binomial expansion on

$$\left[\left(\frac{-y}{\theta}\right)^{\frac{1}{\alpha}}-1\right]^r = \sum_{h=1}^{r}(-1)^{r-h}\binom{r}{h}\left(-\frac{y}{\theta}\right)^{\frac{h}{\alpha}}. \tag{20}$$

$$\mu_r^{'} = \phi e^{-\theta}\frac{\lambda^{r+1}}{\theta\alpha}\sum_{h=1}^{r}(-1)^{r-h}\left(-\frac{1}{\theta}\right)^{\frac{h}{\alpha}+k}\binom{r}{h}\Gamma\left[\frac{h}{\alpha}+k+1\right]. \tag{21}$$

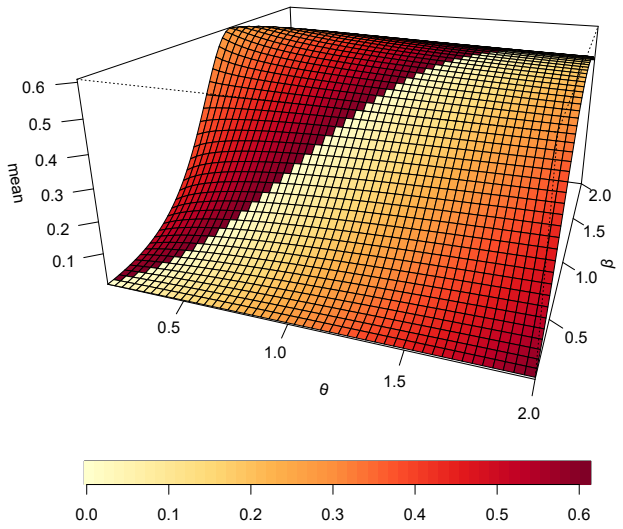


Figure 7: mean of OPL $(\theta, \beta, \lambda, \alpha)$

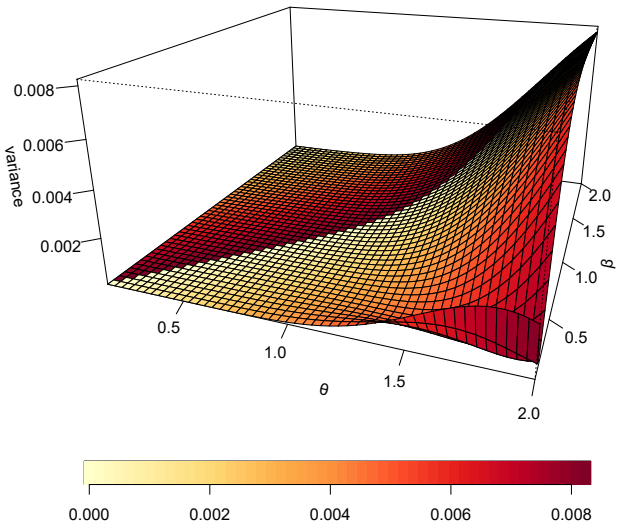


Figure 8: variance of OPL $(\theta, \beta, \lambda, \alpha)$

Figures 7, 8, 9, 10 are the 3D plots of the mean, variance, skewness, and kurtosis of OPL.

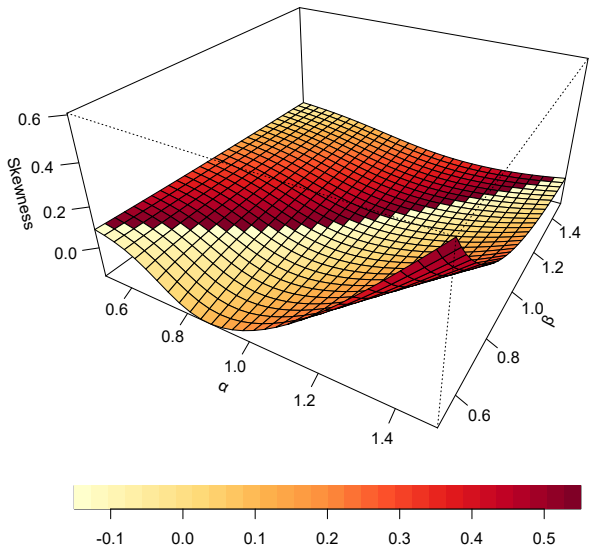


Figure 9: skewness of OPL $(\theta, \beta, \lambda, \alpha)$

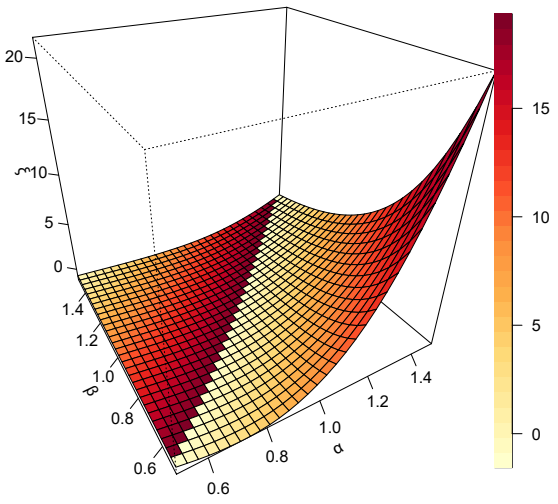


Figure 10: kurtosis of OPL $(\theta, \beta, \lambda, \alpha)$

Figures 7, 8, 9 and 10 represent the mean, variance, skewness and kurtosis of the OPL distribution. Figure 9 shows that the distribution is right-skewed while 10 shows that the distribution is leptokurtic.

3.3 Moment Generating Function

The moment-generating function $M_X(t)$ (if it exists) is useful in obtaining successive moments of a random variable X. Where the random variable is continuous and assumes a distribution with density $f(x)$, its moment-generating function is generally defined as $M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x)\, dx$.

Theorem 1. Suppose $X \sim f(x)$ is as defined in equation (5), then

$$M_X(t) = \phi \frac{\lambda}{\theta\alpha} e^{-(\lambda t+\theta)} \sum_{r=1}^{\infty} \frac{\lambda^r t^r}{r!} \left(-\frac{1}{\theta}\right)^{k+\frac{r}{\alpha}} \Gamma\Big[\frac{r}{\alpha} + k + 1\Big], \tag{22}$$

where $\phi = \frac{\theta\alpha\beta(1+\beta)}{\lambda} \sum\limits_{i,j=1}^{\infty} \sum\limits_{k=1}^{j} (-1)^{i+j-k} \frac{i^j \theta^j \beta^j}{j!} \binom{j}{k} \binom{1+i}{i}$.

Proof. Replace x^r with e^{tx} in equation (18) to have

$$M_X(t) = \phi e^{-\theta} \int_0^{\infty} e^{tx} \left(1 + \frac{x}{\lambda}\right)^{\alpha(1+k)-1} e^{\theta\left(1+\frac{x}{\lambda}\right)^{\alpha}} \, dx, \tag{23}$$

Let $-y = \theta \left(1 + \frac{x}{\lambda}\right)^{\alpha}$, then

$$M_X(t) = \phi e^{-\theta} \int_0^{\infty} e^{t^{\theta}\left[\lambda\left(-\frac{y}{\theta}\right)-\lambda\right]} \left(-\frac{y}{\theta}\right)^{(1+k)-\frac{1}{\alpha}} e^{-y\frac{-\lambda}{\theta\alpha}\left(-\frac{y}{\theta}\right)^{\frac{1}{\alpha}-1}} \, dy, \tag{24}$$

which yields

$$M_X(t) = \phi \frac{\lambda}{\theta\alpha} e^{-\lambda t+\theta} \sum_{r=1}^{\infty} \frac{\lambda^r t^r}{r!} \left(-\frac{1}{\theta}\right)^{k+\frac{r}{\alpha}} \int_0^{\infty} y^{k+\frac{r}{\alpha}} e^{-y} \, dy. \tag{25}$$

Therefore, the moment-generating function of the OPL distribution goes thus;

$$M_X(t) = \phi \frac{\lambda}{\theta\alpha} e^{-\lambda t+\theta} \sum_{r=1}^{\infty} \frac{\lambda^r t^r}{r!} \left(\frac{-1}{\theta}\right)^{k+\frac{r}{\alpha}} \Gamma\left[\frac{r}{\alpha} + k + 1\right]. \tag{26}$$

□

4 Maximum Likelihood Estimation

Given that $X \sim$ OPL $(\theta, \beta, \lambda, \alpha)$, we can estimate the parameters $(\theta, \beta, \lambda, \alpha)$ by first expressing the likelihood function $L(\xi|X)$ as

$$L(\xi|X) = \frac{\theta^n \alpha^n \beta^n (1+\beta)^n}{\lambda^n} e^{\theta \sum\limits_{i=1}^{n}\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}} \prod_{i=1}^{n} \left(1 + \frac{x}{\lambda}\right)^{\alpha-1} \left[1 + \beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right]^{-2}, \tag{27}$$

and ξ represents a vector of the parameters $(\theta, \beta, \lambda, \alpha)$. The log-likelihood function can be derived by taking the natural logarithm of the equation (27) which is given as follows

$$\begin{aligned} \ell = {} & n\ln\theta + n\ln\alpha + n\ln\beta + n\ln(1+\beta) - n\ln\lambda + \theta\sum_{i=1}^{n}\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\} + (\alpha-1)\sum_{i=1}^{n}\ln\left(1+\frac{x}{\lambda}\right) \\ & - 2\sum_{i=1}^{n}\ln\left[1+\beta e^{\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}\right]. \end{aligned} \tag{28}$$

Next, we take the partial derivative of ℓ with respect to the parameters $(\theta, \beta, \lambda, \alpha)$.

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\left[\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right] - 2\beta\sum_{i=1}^{n}\frac{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1}{\beta + e^{-\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}, \tag{29}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \frac{n}{1+\beta} - 2\sum_{i=1}^{n}\frac{1}{\beta + e^{-\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}, \tag{30}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \theta\sum_{i=1}^{n}\left(1+\frac{x}{\lambda}\right)^{\alpha}\ln\left(1+\frac{x}{\lambda}\right) + \sum_{i=1}^{n}\ln\left(1+\frac{x}{\lambda}\right) - 2\theta\beta\sum_{i=1}^{n}\frac{\left(1+\frac{x}{\lambda}\right)^{\alpha}\ln\left(1+\frac{x}{\lambda}\right)}{\beta + e^{-\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}, \tag{31}$$

and

$$\frac{\partial \ell}{\partial \lambda} = -\frac{n}{\lambda} - \frac{\theta\alpha}{\lambda^2}\sum_{i=1}^{n} x\left(1+\frac{x}{\lambda}\right)^{\alpha-1} - \frac{\alpha-1}{\lambda}\sum_{i=1}^{n}\frac{x}{x+\lambda} + \frac{2\theta\alpha\beta}{\lambda^2}\sum_{i=1}^{n}\frac{x\left(1+\frac{x}{\lambda}\right)^{\alpha-1}}{\beta+e^{-\theta\left\{\left(1+\frac{x}{\lambda}\right)^{\alpha}-1\right\}}}. \tag{32}$$

Equations (20), (21), (27), and (28) do not possess closed-form solution. This means that their implementations are better achieved through numerical iteration, see Team [39] for R package.

5 Design of the GASP using OPL Distribution

The GASP oversees the sampling plans of different products in groups. This is used when there is confusion in inspecting each product individually or when there is evidence of a risk of lot-to-lot variation Ram [33]. A GASP is considered truncated if the life tests assume the product follows a certain probability distribution Ahmadi Nadi and Sadeghpour Gildeh [1] and Nwankwo et al. [31]. For tests like these, the sample size is equal to the group number. Multiple works on GASP have been put forward to house different lifetime distributions. There is a difference between ASP and GASP in that the latter evaluates a batch based on a random sample while the former divides batches into subgroups and examines the combined sample from each of the groups.

The chief reason for using GASP is to escalate productivity and reduce the cost of the quality control process. We use groups to check each item instead of in batches, which saves time and cost by allowing representative sample items to be tested, thus providing insight into the overall quality control. Another incentive contributed by Al-Nasser et al. [28] for using GASP is to have a low chance of accepting a batch with highly defective products. Samples with many flaws can be rejected by testing the minute items. Additionally, GASP can assist in elevating confidence in the quality of the products that an organization produces, which is important for developing customer loyalty.

Given that μ is the actual median of the lifetime of a product, assume that μ_0 is the theoretical median, provided that it follows an OPL distribution. We design a sampling plan so that the true median life of items in a lot μ exceeds the theoretical median μ_0. Here, the decision rule is to accept the batch if there is sufficient evidence that $\mu \geq \mu_0$ at some levels of consumer's and producer's risks; otherwise, the whole lot is rejected.

Under the truncated life tests, we develop the following GASP;

1. The first decision is on the appropriate number of groups g, and then we assign r units for each group, hence the sample size is $n = r \times g$ for the lot.
2. Secondly, we decide on the acceptance number (or action limit) ω for a group, and the termination time of the experiment t_0.
3. With the choice of the number of groups g, we simultaneously conduct experiments recording the number of failures for each group.
4. The lot is rejected at any point when the number of failures is greater than the acceptance number ω.

When the group size $r = 1$, the sample size n equals the number of groups g, and the sampling strategy becomes the single sampling plan. Suppose the lifetime of a product assumes the OPL distribution, with cdf and pdf stated in equations (6) and (5) respectively which are dependent on α, β, λ and θ, the binomial distribution can be used to derive the operating characteristic (OC) curve of a sampling plan, provided the batch size is large enough and the objective is either to accept or reject the lot.

It is traditional to follow this sequence of analytics; define

$$m = \left[1 + \frac{1}{\theta}\ln\left(\frac{0.5+\beta}{0.5\beta}\right)\right]^{\frac{1}{\alpha}} - 1, \tag{33}$$

from the quantile function expressed in eq. 12, where $m = \frac{\mu}{\lambda}$ and

$$\mu = \lambda\left[\left\{1 + \frac{1}{\theta}\ln\left(\frac{p+\beta}{(1-p)\,\beta}\right)\right\}^{\frac{1}{\alpha}} - 1\right], \tag{34}$$

where $\mu = \lambda m$ and $t_0 = a\mu_0$ which is the true average/median lifetime. The probability of failure is given as

$$p = F(t_0) = 1 - \frac{1+\beta}{1+\beta e^{\theta\left\{\left(1+\frac{a\times m}{\frac{\mu}{\mu_0}}\right)^{\alpha}-1\right\}}}. \tag{35}$$

This re-designation is coined from the cdf in equation (6). The acceptance of the probability lot is mathematically expressed as;

$$P_{\alpha}(p) = \left[\sum_{i=0}^{\omega}\binom{r}{i}p^{i}(1-p)^{r-1}\right]^{g}. \tag{36}$$

For the consumer's average risk, we obtain

$$P_{o}(p_1|\mu/\mu_o = r_1) = \left[\sum_{i=0}^{\omega}\binom{r_1}{i}p^{i}(1-p)^{r_1-1}\right]^{g} \leq \beta^{*}, \tag{37}$$

and for the producer's average risk, we get

$$P_{o}(p_2|\mu/\mu_o = r_2) = \left[\sum_{i=0}^{\omega}\binom{r_2}{i}p^{i}(1-p)^{r_2-1}\right]^{g} \geq 1-\alpha^{*}. \tag{38}$$

Note, that the average ratio of the consumer's risk is denoted by r_1 and the producer's average risk is denoted by r_2, hence;

$$p_1 = 1 - \frac{1+\beta}{1+\beta e^{\theta\{(1+a\times m)^{\alpha}-1\}}}, \tag{39}$$

and

$$p_2 = 1 - \frac{1+\beta}{1+\beta e^{\theta\left\{\left(1+\frac{a\times m}{r_2}\right)^{\alpha}-1\right\}}}. \tag{40}$$

see the illustration of this design in tables 1 and 2.

Table 1: GASP for OPL($\theta = 3, \beta = 2.0$ and $\alpha = 1.75$) showing minimum g and ω

		$r = 5$						$r = 10$					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β^*	$\frac{\mu}{\mu_0}$	g	ω	$L(p)$	g	ω	$L(p)$	g	ω	$L(p)$	g	ω	$L(p)$
0.25	2	930	4	0.950411	0	4	0.00000	46	5	0.956430	8	6	0.953884
	6	11	2	0.988999	2	2	0.985479	2	2	0.979954	1	3	0.989508
	10	3	1	0.976909	1	1	0.970989	1	1	0.968266	1	2	0.983530
	14	3	1	0.987948	1	1	0.984617	1	1	0.983045	1	2	0.993401
0.10	2	0	4	0.000000	0	4	0.000000	391	6	0.966383	41	7	0.968616
	6	17	2	0.983050	4	2	0.971170	3	2	0.970082	2	3	0.979125
	10	5	1	0.961812	4	2	0.993203	3	2	0.992728	1	2	0.983530
	14	5	1	0.979993	2	1	0.969471	2	1	0.966378	1	2	0.993401
0.05	2	0	4	0.000000	0	4	0.000000	508	6	0.956545	54	7	0.958873
	6	22	2	0.978120	5	2	0.964093	4	2	0.960309	2	3	0.979125
	10	6	1	0.954351	5	2	0.991512	4	2	0.990316	2	2	0.967331
	14	6	1	0.976040	2	1	0.969471	2	1	0.966378	2	2	0.986845
0.01	2	0	4	0.000000	0	4	0.000000	0	4	0.000000	427	8	0.974010
	6	34	2	0.966387	7	2	0.950094	15	3	0.987218	3	3	0.968852
	10	34	2	0.992359	7	2	0.988136	6	2	0.985510	2	2	0.967331
	14	9	1	0.964277	3	1	0.954558	6	2	0.994440	2	2	0.986845

Defining the parameters under the GASP design where Tables 1 and 2 are illustrated, with the specific values of $\xi = (3, 2.0, 1.75)$ and $(1.5, 1.25, 0.15)$, respectively $\beta^* = (0.25, 0.10, 0.05, 0.10)$, $r_2 = (2, 6, 10, 14)$, with $a_1 = (0.5, 1)$ and $r = (5, 10)$, are respectively, having r_1 as 5, r_2 with the ratio $(2, 6, 10, 14)$, and r is the specific number of items representing each group $(5, 10)$.

Table 2: GASP for OPL($\theta = 1.5, \beta = 1.25$ and $\alpha = 0.15$) showing minimum g and ω

		$r=5$						$r=10$					
		$a_1=0.5$			$a_1=1$			$a_1=0.5$			$a_1=1$		
β^*	$\frac{\mu}{\mu_0}$	g	ω	$L(p)$	g	ω	$L(p)$	g	ω	$L(p)$	g	ω	$L(p)$
0.25	2	0	4	0.000000	0	4	0.000000	0	4	0.000000	0	4	0.000000
	6	134	4	0.979249	44	4	0.955702	8	5	0.976212	8	6	0.970467
	10	16	3	0.982489	7	3	0.959839	4	4	0.982211	3	5	0.984310
	14	4	2	0.96401	7	3	0.981192	2	3	0.975690	2	4	0.975619
0.10	2	0	4	0.000000	0	4	0.000000	0	4	0.000000	0	4	0.000000
	6	222	4	0.965857	0	-	0.000000	13	5	0.961633	13	6	0.952453
	10	26	3	0.971701	73	4	0.980757	6	4	0.973436	5	5	0.973987
	14	6	2	0.951503	12	3	0.967974	3	3	0.963757	3	4	0.963652
0.05	2	0	4	0.000000	0	4	0.000000	0	4	0.000000	0	4	0.000000
	6	289	4	0.955783	0	-	0.000000	17	5	0.950127	54	7	0.976012
	10	33	3	0.964219	95	4	0.975031	7	4	0.969078	7	5	0.963772
	14	33	3	0.985670	15	3	0.960130	4	3	0.951970	4	4	0.951832
0.01	2	0	4	0.000000	0	4	0.000000	0	4	0.000000	0	4	0.000000
	6	0	-	0.000000	0	-	0.000000	81	6	0.972441	82	7	0.963801
	10	444	4	0.986465	146	4	0.961885	11	4	0.951839	25	6	0.983061
	14	51	3	0.977940	146	4	0.985816	11	4	0.983175	10	5	0.981962

6 Applications of OPL distribution to Lifetime Data

The first dataset in table 3 contains 63 observations representing the strengths of 1.5cm glass fibers, measured at the National Physical Laboratory, England, and studied by Smith and Naylor [38].

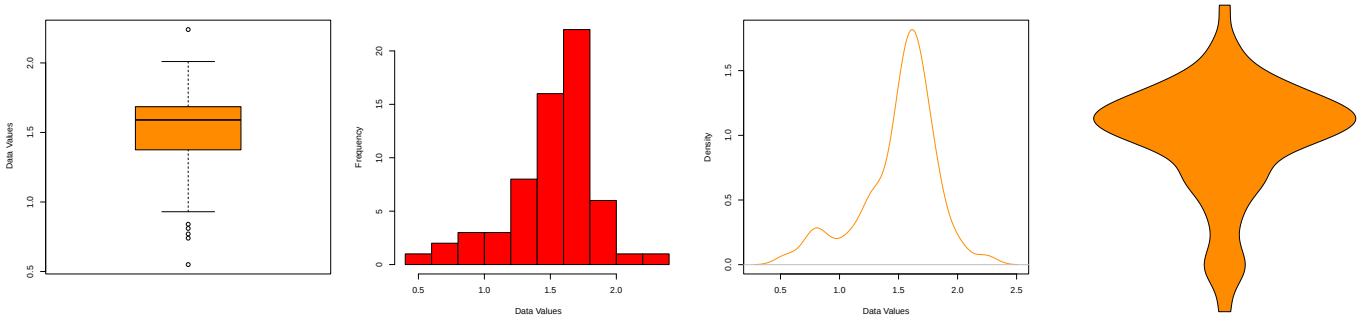
Table 3: Strengths of 1.5cm glass fibers, measured at the National Physical Laboratory, England

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2.00	0.74	1.04	1.27
1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42	1.5	1.54
1.6	1.62	1.66	1.69	1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.5	1.55	1.61	1.62	1.66
1.7	1.77	1.84	0.84	1.24	1.3	1.48	1.51	1.55	1.61	1.63	1.67	1.7	1.78	1.89	

We fit the OPL distribution to the strength of fiber data in table 3 and compare it with the Weibull distribution Fréchet [16], Rosin [35], and Weibull [41], and Exponential Weibull distribution Cordeiro, Ortega, and Lemonte [12].

Table 4: Summary Statistics for the Strength of Fiber data

σ^2	n	$\bar{x}$	sd	median	trimmed μ	MAD	min	max	range	skew	kurtosis	S_e
1	63	1.51	0.32	1.59	1.54	0.16	0.55	2.24	1.69	-0.88	0.8	0.04



(a) Boxplot **(b)** Histogram **(c)** Kernel density **(d)** Violin plot

Figure 11: Non-parametric plots of the strength of fiber data

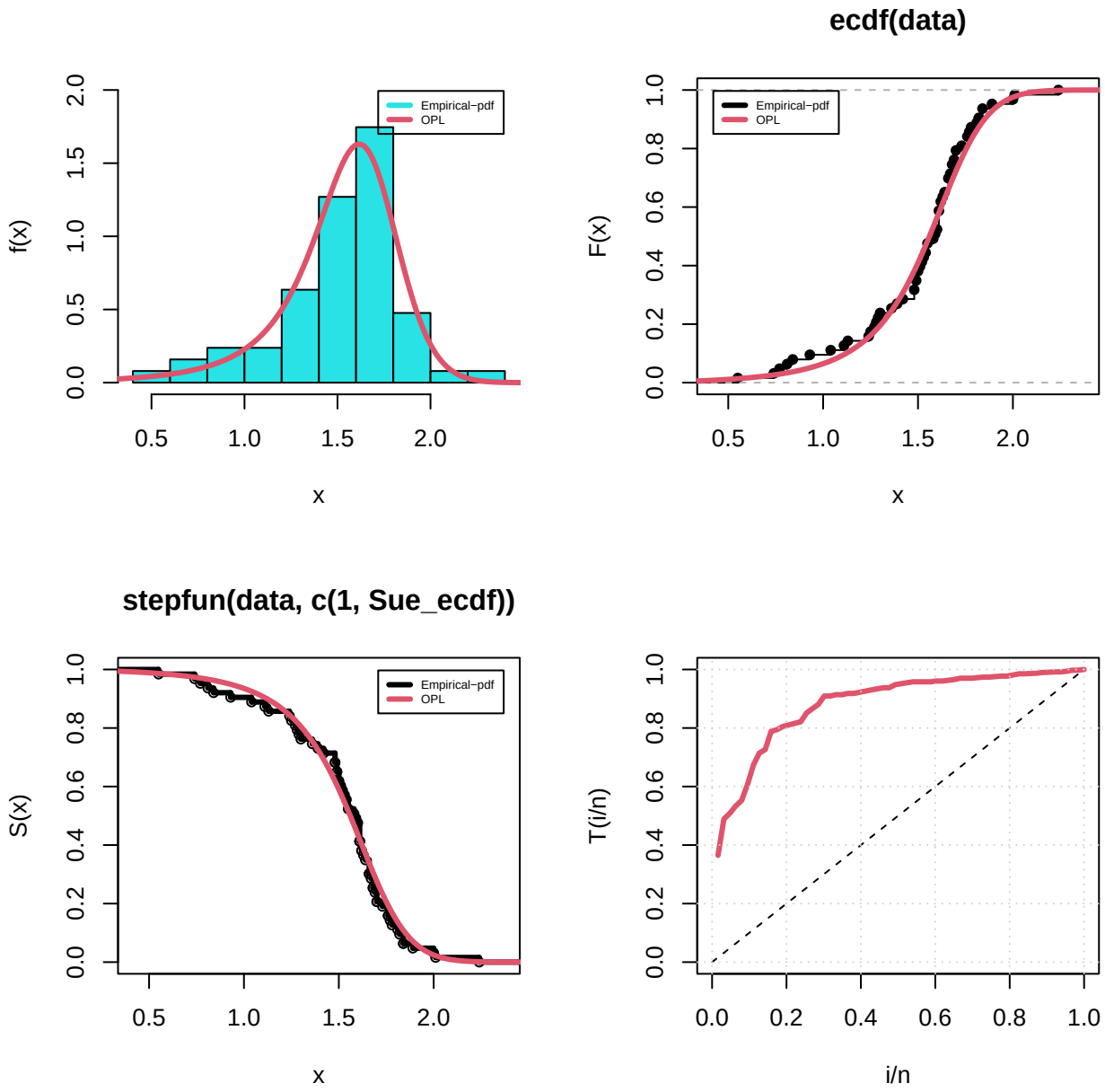


Figure 12: density, cdf, survival, and TTT plots for strength of fiber data

The measures of model fitness used are the Cramér von Misses statistic, the Anderson-Darling statistic, and the P-value of the Kolmogorov-Smirnov statistics. These show that OPL distribution fits the data better than the popular Weibull distribution used for modeling lifetime data with a p-value of 0.6052 way higher than the other two.

Table 5: Model fitness and adequacy measures for the strength of Fiber data

Dist.	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value
OPL	13.28	32.23	32.92	40.80	35.60	0.0815	0.4770	0.0961	0.6052
Weibull	15.21	34.57	34.77	38.86	36.26	0.2427	1.3331	0.1708	0.0507
ExpWe	14.68	35.35	35.76	41.78	37.88	0.20	1.1118	0.1462	0.1351

Table 6: MLEs of the parameters of the fitted distributions using the strength of fiber data

Dist.	α	β	θ	λ
OPL	5.5043	0.0327	0.0944	1.6677
Weibull	5.5807	1.6172		
ExpWe	0.6713	7.2843	1.7181	

Table 7: GASP for OPL($\theta = 0.0944, \beta = 0.0327$ and $\alpha = 5.5043$) showing minimum g and ω for the Strength of Fiber data

	r=5							r=10					
	$a_1 = 0.5$				$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β^*	$\frac{\mu}{\mu_0}$	g	ω	$L(P)$	g	ω	$L(P)$	g	ω	$L(P)$	g	ω	$L(P)$
0.25	2	0	4	0.000000	1	1	0.991379	489	2	0.979275	1	1	0.964919
	6	161	1	0.996079	1	0	0.980894	39	1	0.995748	1	0	0.962152
	10	10	0	0.957596	1	0	0.990260	5	1	0.957596	1	0	0.980615
	14	10	0	0.970534	1	0	0.993592	5	1	0.970534	1	0	0.987047
0.10	2	0	4	0.000000	2	1	0.982833	811	2	0.965863	1	1	0.964919
	6	266	1	0.993530	1	0	0.980894	65	1	0.992924	1	0	0.962152
	10	266	1	0.998010	1	0	0.990260	65	1	0.997818	1	0	0.980615
	14	15	0	0.956128	1	0	0.993502	8	0	0.953273	1	0	0.987047
0.05	2	0	4	0.000000	0	4	0.982833	0	4	0.000000	1	1	0.964919
	6	347	1	0.991568	0	4	0.980894	84	1	0.990865	1	0	0.962152
	10	347	1	0.997404	0	4	0.990260	84	1	0.997181	1	0	0.980615
	14	347	1	0.99876	0	4	0.993502	84	1	0.998653	1	0	0.987047
0.01	2	0	4	0.000000	0	4	0.974360	0	4	0.000000	2	2	0.994335
	6	532	1	0.987101	0	4	0.962152	129	1	0.986005	1	0	0.962152
	10	532	1	0.996023	0	4	0.980615	129	1	0.995674	1	0	0.980615
	14	532	1	0.998102	0	4	0.987047	129	1	0.997933	1	0	0.987047

The second data set is on the remission times (in months) of a sample of 128 bladder cancer patients studied by Kayid [21].

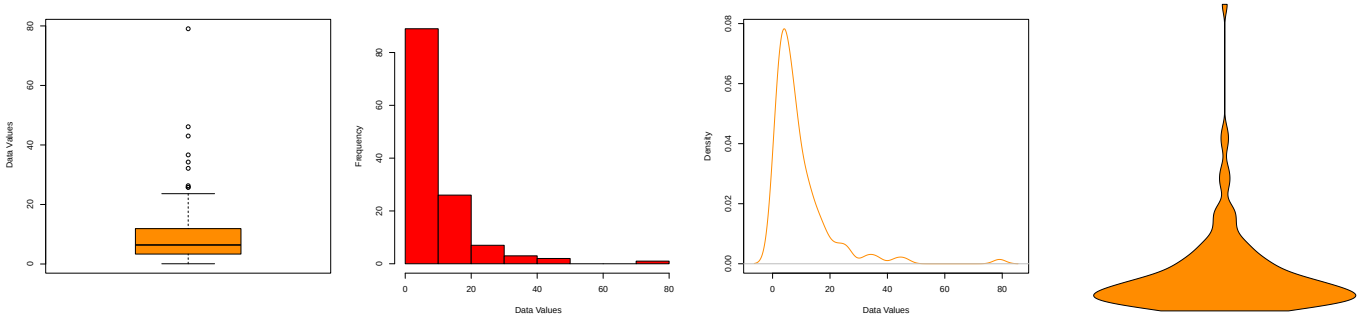
Table 8: The remission times (in months) of a sample of 128 bladder cancer patients

3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.5	4.98
6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28
9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25
8.37	12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.00	3.36	6.93	8.65	12.63	22.69				

Table 9: Summary Statistics for the Bladder Cancer data

σ^2	n	$\bar{x}$	sd	median	trimmed μ	MAD	min	max	range	skew	kurtosis	S_e
1	128	9.36	10.51	6.39	7.42	5.46	0.08	79.05	78.97	3.25	15.19	0.93

Table 9 is a summary statistics of the bladder cancer data.



(a) Boxplot **(b)** Histogram **(c)** Kernel density **(d)** Violin plot

Figure 13: Non-parametric plots of the bladder Cancer data

The figures 13 represent the boxplot, histogram , kernel density and violin plots of the bladder cancer data. The insight from these non-parametric plots reveal that the data contains outlier and it is positively skewed. One can therefore deduce that since OPL distribution fits the data with a p-value of 0.9999, the distribution is very flexible in the face of decreasing failure rate data.

Table 10: Measures of Model Performance and Fitness for the Fitted Distributions Using the Bladder Cancer Data

Dist.	NLL	AIC	CAIC	BIC	HQIC	W^*	A^*	K-S	P-value
OPL	409.66	826.8597	827.1849	838.2678	821.4949	0.0144	0.0883	0.0288	0.9999
Weibull	414.08	832.7357	832.8317	838.4398	835.0533	0.1440	0.8603	0.0638	0.6741
ExpWe.	410.68	827.3551	827.5486	835.911	830.8315	0.0433	0.2857	0.0433	0.9703

Table 11: MLE of the Parameters for the Fitted Distributions Using the Bladder Cancer Data

Dist.	α	β	θ	λ
OPL	0.1651	0.0044	5.3263	0.0907
Weibull	1.0889	9.3785		
ExpWe	2.7938	0.6544	3.3471	

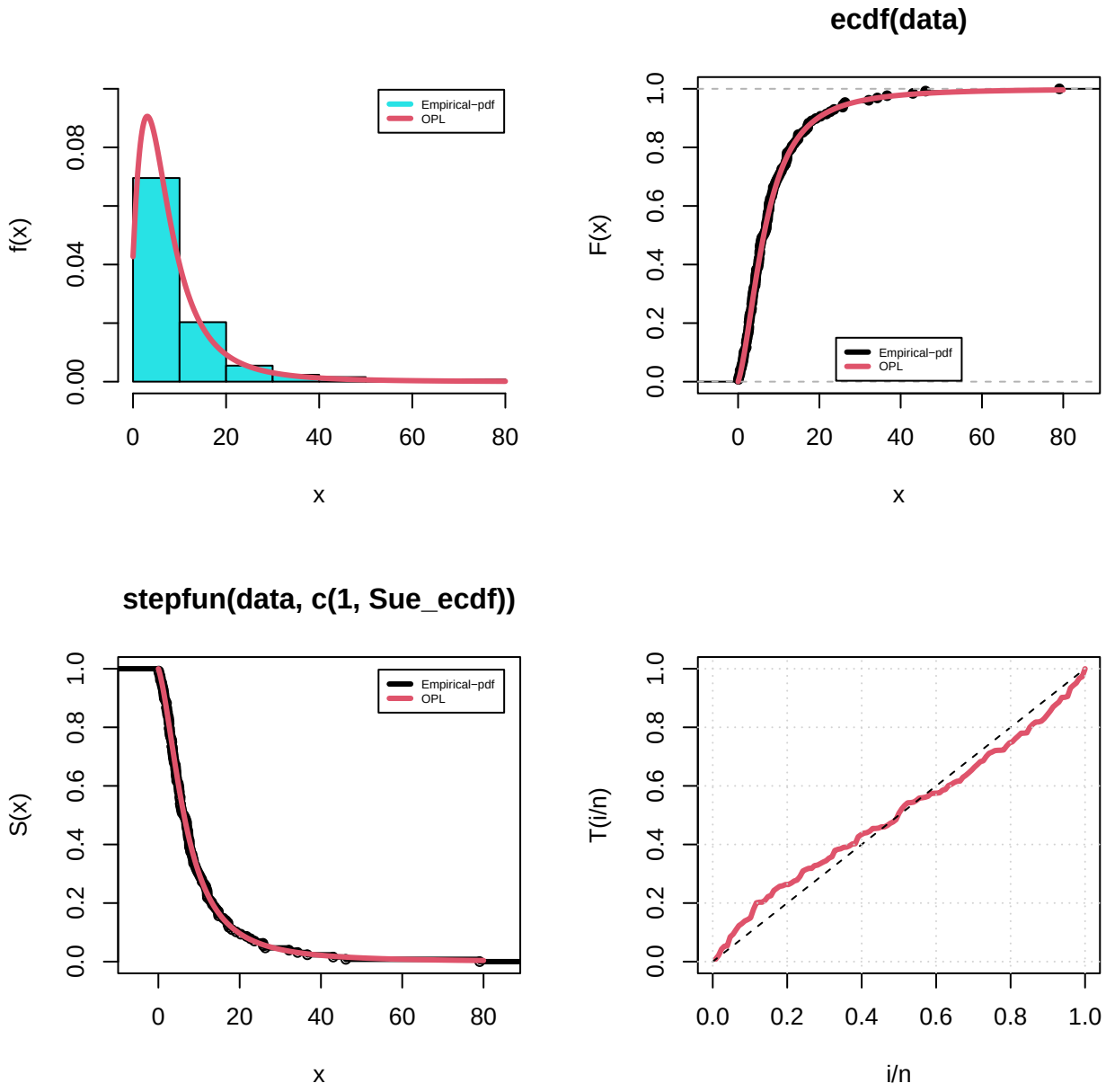


Figure 14: density, cdf, survival, and TTT plots for remission times 128 bladder cancer patients data

Figure 12 is the density, cdf, survival and TTT plots of the bladder cancer data. The TTT plots almost perfectly fits the true line. This implies a very accurate fit of the proposed OPL to the data.

Table 12: GASP for OPL($\theta = 0.1651, \beta = 0.0044$ and $\alpha = 5.3263$) showing minimum g and ω for the Bladder Cancer data

		r=5						r=10					
		$a_1 = 0.5$			$a_1 = 1$			$a_1 = 0.5$			$a_1 = 1$		
β	$\frac{\mu}{\mu_0}$	g	ω	L(p)	g	ω	L(p)	g	ω	L(p)	g	ω	L(p)
0.25	2	103	3	0.950236	44	4	0.965312	20	4	0.964952	3	5	0.952101
	6	4	1	0.973065	1	1	0.966625	3	2	0.994000	1	2	0.979882
	10	4	1	0.991296	1	1	0.89705	2	1	0.981395	1	1	0.958506
	14	4	1	0.995792	1	1	0.995191	2	1	0.991395	1	1	0.979916
0.10	2	0	4	0.000000	0	4	0.000000	141	5	0.977215	13	6	0.964866
	6	6	1	0.959870	4	2	0.991562	4	2	0.992008	1	2	0.979882
	10	6	1	0.986972	2	1	0.979516	2	1	0.981395	1	1	0.958506
	14	6	1	0.991601	2	1	0.990450	2	1	0.990859	1	1	0.979916
0.05	2	0	4	0.000000	0	4	0.000000	184	5	0.970371	16	6	0.956935
	6	31	2	0.994289	5	2	0.989464	6	2	0.988036	2	2	0.960169
	10	8	1	0.982667	2	1	0.979516	3	1	0.972223	1	1	0.958506
	14	8	1	0.991601	2	1	0.990405	3	1	0.986320	1	1	0.979916
0.01	2	0	4	0.000000	0	4	0.000000	282	5	0.954950	82	7	0.974884
	6	17	2	0.991354	7	2	0.9852080	8	2	0.984079	2	2	0.960169
	10	11	1	0.976245	3	1	0.969431	4	1	0.963137	2	2	0.992659
	14	11	1	0.988470	3	1	0.985643	4	1	0.981802	2	1	0.960235

Tables 1 and 2 are simulation results for the GASPs based on the formulated OPL distribution, while tables 7 and 12 are the empirical designed GASPs using the MLEs of the parameters of OPL distribution for the strength of fiber and bladder cancer data, respectively. Let's illustrate the hypothetical example. Using the empirical design for the strength of fiber data, for instance, we can illustrate the potency of these GASPs under the OPL model. If a producer claims that the supposed value of their item is $\mu_0 = 5000$ hours and that the lifetime of the items is OPL distributed with $\theta = 0.0944, \beta = 0.0327, \alpha = 5.5043$, the consumer's and the producer's risk are 0.25 and 0.05, respectively. If we further assume that the group size is $r = 10$, and the actual value of μ is 500 hours, the above information can be deployed in conducting a life test experiment for 1000 hours, to design a GASP for $a_1 = 0.5, \beta = 0.25$ and true median life $\frac{\mu}{\mu_0} = 10$. Using Table 7, the design parameters g and ω can be obtained as $g = 5$ and $\omega = 1$. As a result, a sample of size 50 (10×5) should be collected, and each of the 10 groups should get 5 units. If no more than one unit fails in any group before 1000 hours, the lot is eventually accepted; otherwise, the lot is rejected.

7 Conclusion

Truncated life tests abound in reality and it is usually because many test times exceed some item's life cycle. Therefore, subjecting such items to an extended period beyond their lifespan will ultimately be disruptive, and desired results won't be recorded. Such experiments where life tests are truncated are therefore advocated in the literature. In this article, we focused on designing a group acceptance sampling plan based on the truncated life tests for items whose median lifetime is Odd Perks-Lomax (OPL) distributed. We considered the basic assumptions of the generic design of the group acceptance sampling scheme such as the number of groups g and the acceptance number ω, which is obtained from finding a balance between the consumer's risk and the producer's risk. The above plan's initiative is targeted at attaining quality by developing a limit on the number of defective products that are found in a particular batch. After the analysis, using Weibull and Exponentiated-Weibull distributions for comparisons, it is concluded that the OPL distribution has a better quality index and this is represented in both tabular and graphical forms in this work.

Authors Contribution

Conceptualization:, D.N.E and O.J.O.; **Methodology:** O.J.O; **Software:** O.J.O; **Validation:** D.N.E., H.M.A and O.J.O.; **Formal Analysis:** D.N.E and O.J.O; **Investigation:** O.J.O.; **Resources:** F.M.A., F.H.R; **Data Curation:** H.M.A., F.H.R; **Writing – Original Draft Preparation:** D.N.E. and O.J.O.; **Writing – Review & Editing:** O.J.O.; **Visualization:** D.N.E; **Supervision:** F.M.A; **Project Administration:** F.M.A. and M.M.A; **Funding Acquisition:** F.M.A, H.M.A, F.H.R. and M.M.A.

Conflict of Interest

The authors declare that there is no conflict of interest.

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