The Fav-Jerry Distribution: Another Member in the Lindley Class with Applications

Divine-Favour N. Ekemezie[1] and Okechukwu J. Obulezi[2]

[1,2]Department of Statistics, Faculty of Physical Sciences, Nnamdi Azikiwe University, Awka, Nigeria

Corresponding Author: Okechukwu J. Obulezi ; **email: oj.obulezi@unizik.edu.ng**

Abstract

In this paper, we designed another one-parameter distribution using a mixture of exponential and gamma distributions. This new distribution is unique among other members of the Lindley class because the qunatile function has a closed-functional form hence lending itself to analytical study. This distribution is named Fav-Jerry after the names of the authors. The statistical properties and point estimation using some non-bayesian methods were studied. We deploy tow real datasets to demonstrate the usefulness of the new model. The real data applications using data sets on mortality rate and failure rate in a particular airplane showed that the proposed model fits well compared to its competitors, therefore, the Fav-Jerry distribution is superior to Two parameter Chris-Jerry(TPCJ),Chris-Jerry, Exponentiated Inverted Exponential distribution, and Weibull distributions and then parametric plots showing the histogram, CDF, survival and TTT plots gotten from both data sets are displayed.

Keywords— Estimations, Real-life data, Monte-Carlo simulation, Lindley class of distributions, plots, Fav-Jerry distribution, goodness fit, Bayesian Estimation

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1 Introduction

Lindley [9] introduced a one parameter distribution using mixing proportion $\frac{\theta}{\theta+1}$ of two components of exponential distribution with scale parameter θ and gamma with shape parameter 2 and scale parameter θ. Within the last two decades and inevitably beyond, researchers have piqued interest in getting more effective and elastic modelling path for probability distributions that will defy the constant use of the standard probability distributions. In this fashion as Lindley distribution, Shanker and Shukla [20] created the Ishita distribution having merged the exponential distribution with θ as its scale parameter and gamma distribution with $(3, \theta)$ as its shape and scale parameters respectively, and with a mixing proportion of $\frac{\theta^3}{\theta^3+2}$. The Akash distribution which was postulated by Rama [15] is a one parameter distribution gotten from the combination of the exponential distribution with θ as its scale parameter and gamma distribution having its shape and scale parameters as 3, and θ respectively. Its mixing proportion is $\frac{\theta^2}{\theta^2+2}$. Shanker [17] proposed the Komal distribution, a convex combination of exponential (θ) and gamma $(2, \theta)$ with mixing proportions $\frac{\theta(\theta+1)}{\theta^2+\theta+1}$ and $\frac{1}{\theta^2+\theta+1}$. Odom and Ijomah [12] created the Odoma distribution from three components of exponential, gamma and gamma distributions, having a scale parameter of θ and then gamma distributions having the shape parameters of $(3, 5)$. The mixing proportions used in this distribution are $\frac{\theta^5}{\theta^5+\theta^3+6}$ and $\frac{\theta^3}{\theta^5+\theta^3+1}$. KK [7] derived the one parameter Pranav distribution that consists of the exponential and gamma distributions both having a scale parameter of θ and the gamma distribution having a shape parameter of 4. The mixing proportions used in this distribution are $\frac{\theta^4}{\theta^4+6}$ and $\frac{6}{\theta^4+6}$. Shanker [21] developed the Rani distribution having merged the exponential distribution with θ as its scale parameter and gamma distribution with $(5, \theta)$ as its shape and scale parameters respectively, then its mixing proportion is $\frac{\theta^5}{\theta^5+24}$, the Sujatha distribution was designed by Shanker [22] and it consists of the exponential and gamma distributions both having a scale parameter of θ and the gamma distribution having the shape parameter of 2. The mixing proportions used in this distribution are $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$. The Doje distribution studied by Oramulu et al. [14], a single parameter distribution from the two-components mixture of exponential with scale parameter θ and gamma with shape parameter 7 and scale parameter θ has a mixing proportion, $\frac{\theta^6}{\theta^6+720}$. Shanker [18] created the Rama distribution having merged the exponential distribution with θ as its scale parameter and gamma distribution with $(3, \theta)$ as its shape and scale parameters respectively, then its mixing proportion is $\frac{\theta^3}{\theta^3+6}$. The Aradhana distribution which was postulated by Shanker [16], a one parameter distribution gotten from the combination of the exponential distribution with θ as its scale parameter and gamma distribution having its shape and scale parameters as 2, and θ respectively. Its mixing proportion is $\frac{1}{\theta+1}$. Shanker [19] introduced the Shanker distribution having merged the exponential distribution with θ as its scale parameter and gamma distribution with 2, and θ as its shape and scale parameters respectively, then its mixing proportion is $\frac{\theta^2}{\theta^2+1}$. Mbegbu and Echebiri [11] derived the one parameter Juchez distribution that consists the exponential and gamma distributions both having a scale parameter of θ and the gamma distribution having a shape parameter of 2 and another gamma distribution with shape parameter 4. The mixing proportions used in this distribution are $\frac{\theta^3}{\theta^3+\theta^2+6}, \frac{\theta^2}{\theta^3+\theta^2+6}$ and $\frac{6}{\theta^3+\theta^2+6}$. The Ram Awadh distribution, credit to KK [8] consists the exponential and gamma distributions both having a scale parameter of θ and the gamma distribution having the shape parameter of 6 and corresponding scale parameter θ. The mixing proportions used in this distribution are $\frac{\theta^6}{\theta^6+120}$ and $\frac{120}{\theta^6+120}$. The design of Remkan distribution was studied by Uwaeme and Akpan [26] it consists of the exponential and gamma distributions both having a scale parameter of θ and the gamma distribution having the shape parameter of 3 and corresponding scale parameter θ and another gamma distribution with shape parameter and scale as $(4, \theta)$ respectively. The mixing proportions used in this distribution are $\frac{\theta}{\theta+2\theta+6}$, $\frac{2\theta}{\theta+2\theta+6}$ and $\frac{6}{\theta+2\theta+6}$. The design of Copoun distribution was studied by Uwaeme, Akpan, and Orumie [25] it consists of the exponential distribution having θ as its scale parameter and gamma distribution with shape parameter and scale as $(4, \theta)$ respectively. The mixing proportions used in this distribution are $\frac{\theta}{(\phi+\theta)}$ and $\frac{\theta}{(\theta+\theta)}$ and then there is the introduction of the Chris-Jerry distribution by Onyekwere and Obulezi [13] through the combination of the exponential and gamma distributions both having a scale parameter θ and the gamma distribution having the shape parameter of 3. Its mixing proportion is $\frac{\theta}{\theta+2}$ all these and lots more are probability distributions that were coined from the discovery of the Lindley approach.

Now, the motivation for this work is to generate probability model with a better goodness of fit to data and provide some kind of simplistic and docile mathematical resolve. It is notable that the proposition in this article has a quantile function with a closed functional form. This lends this distribution to many analytical tasks while providing flexibility in application compared to some popular models in the Lindley class. In the end, this distribution can be extended to more complex mathematical structures to accommodate data emanating from the complex global activities.

In this paper, we introduced a one-parameter distribution with a probability distribution function (pdf) given as

$$g(t; \psi) = \frac{\psi}{\psi^2 + 2} \left(2 + \psi^3 t\right) e^{-\psi t}; \quad t > 0, \psi > 0. \tag{1}$$

We refer the distribution as Fav-Jerry (FJ) distribution, devised from the names of the authors. The pdf was obtained by combining the exponential and gamma distributions. The exponential distribution has a scale parameter ψ, gamma distribution with its shape parameter as 2, and scale parameter as ψ. The formulation is $g(t; \psi) = p\exp(t, \psi) + (1 - p)\text{gamma}(t, 2, \psi)$, where $p = \frac{2}{\psi^2+2}$ is the mixing proportion. The corresponding cumulative distribution function (cdf) to eq. 1 is

$$G(t; \psi) = 1 - \left\{1 + \frac{\psi^3 t}{\psi^2 + 2}\right\} e^{-\psi t}. \tag{2}$$

Theorem 1 (Quantile Function)**.** Let $T \sim$ Fav-Jerry(ψ), inverting the cdf in eq. 2 for $G(t; \psi) = q$ we obtain

$$x_q = -\frac{1}{\psi} - \frac{2}{\psi^3} - \frac{1}{\psi} W\left\{-\frac{(\psi^2+2)(1-q)e^{-\frac{\psi^2+2}{\psi^2}}}{\psi^2}\right\}; \text{provided} \quad q \in (0, 1) \tag{3}$$

where $W(.)$ is the Lambert W function due to Corless et al. [4].

Proof. Let $q = G(t;\psi)$, then it easily follows that $(\psi^2+2)(1-q) = (\psi^2+2+\psi^3 t)e^{-\psi t}$. Assign $Z(t) = -(\psi^2+2+\psi^3 t)$, this yields $-\psi t = \frac{Z(t)+\psi^2+2}{\psi^2}$ so that $-\frac{(\psi^2+2)(1-q)e^{-\frac{\psi^2+2}{\psi^2}}}{\psi^2} = \frac{Z(t)}{\psi^2}e^{\frac{Z(t)}{\psi^2}}$. Taking Lambert W function, the rest is trivial. □

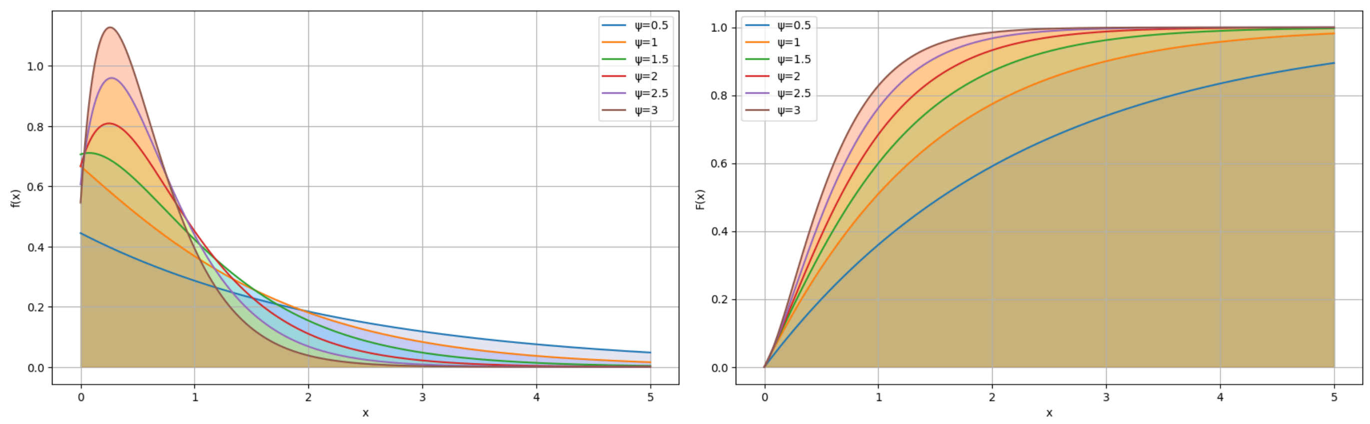
The survival and failure rate functions are obtained from eq. 1 and 2 as

$$s(t;\psi) = 1 - G(t;\psi) = \left(1 + \frac{\psi^3 t}{\psi^2+2}\right)e^{-\psi t}, \tag{4}$$

and

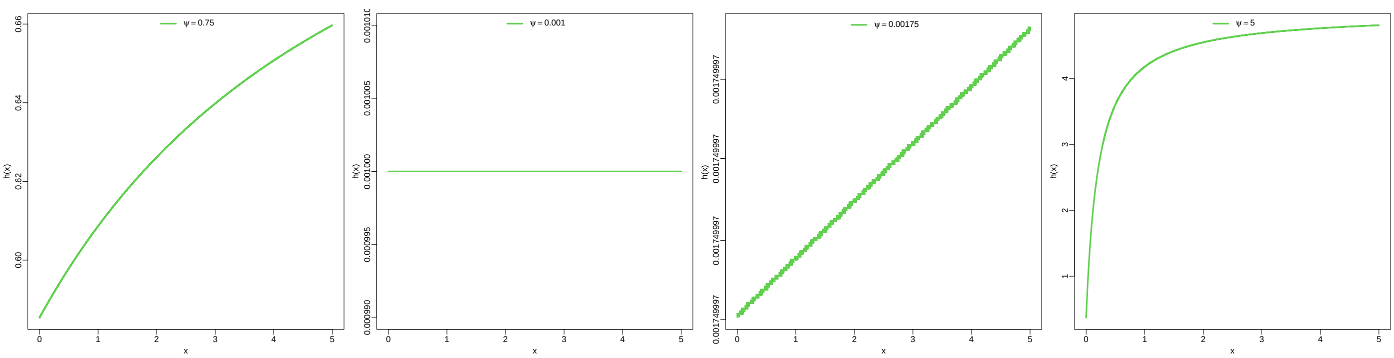
$$h(t;\psi) = \frac{g(t;\psi)}{s(t;\psi)} = \frac{\psi(2+\psi^3 t)}{\psi^2+2+\psi^3 t}, \tag{5}$$

respectively.



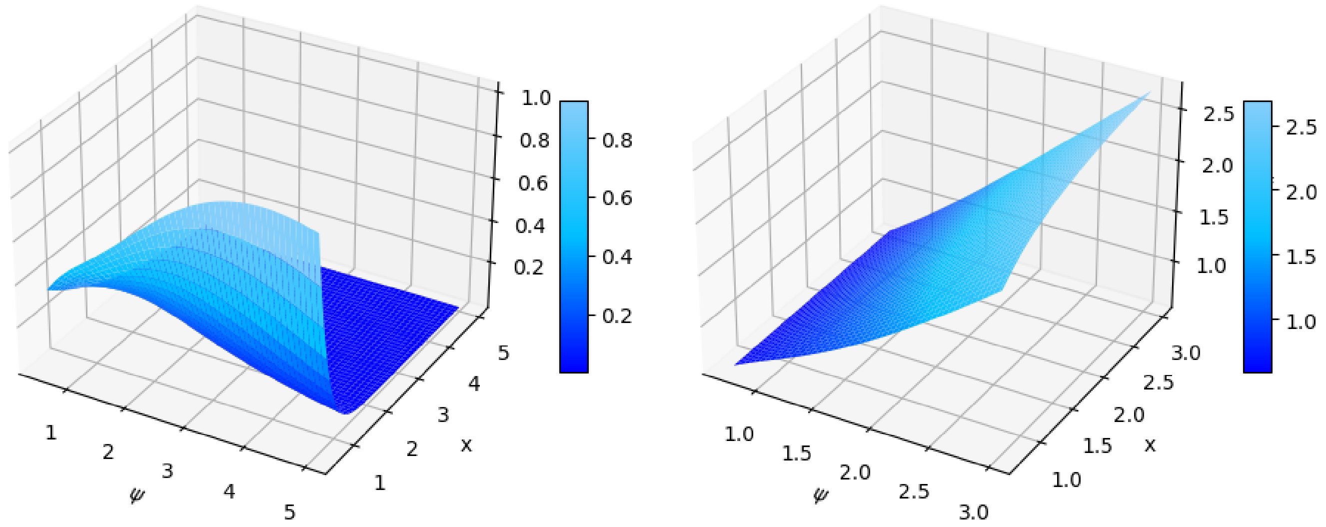
(a) pdf of Fav-Jerry (ψ) **(b)** cdf of Fav-Jerry (ψ)

Figure 1: visualization of the Fav-Jerry (ψ)



(a) hrf of Fav-Jerry (ψ) **(b)** hrf of Fav-Jerry (ψ) **(c)** hrf of Fav-Jerry (ψ) **(d)** hrf of Fav-Jerry (ψ)

Figure 2: visualization of the Fav-Jerry Distribution



(a) 3D pdf of Fav-Jerry (ψ) **(b)** 3D hazard function of Fav-Jerry (ψ)

Figure 3: 3D pdf and hazard function

Theorem 2 (Moment Generating and Characteristic Functions)**.** There are basically two reasons for moment generating function. First, the MGF of X gives us all moments of X . That is why it is called the moment generating function (MGF). Second, the MGF (if it exists) uniquely determines the distribution. That is, if two random variables have the same MGF, then they must have the same distribution. Thus, if you find the MGF of a random variable, you have indeed determined its distribution.

$$\begin{aligned}
M_x(t) = E(e^{tx}) &= \int_0^{\infty} e^{tx} \frac{\psi}{\psi^2+2}\left(2+\psi^3 x\right) e^{-\psi x} dt \\
&= \frac{\psi}{\psi^2+2}\left[2\int_0^{\infty} e^{-(\psi-t)x} dt + \psi^3 \int_0^{\infty} t e^{-(\psi-t)x} dt\right] \\
&= \frac{\psi}{\psi^2+2}\left[\frac{2\Gamma(1)}{(\psi-t)} + \frac{\psi^3}{(\psi-t)^2}\right] \\
M_x(t) &= \frac{\psi}{\psi^2+2}\left[\frac{2}{(\psi-t)} + \frac{\psi^3}{(\psi-t)^2}\right]
\end{aligned} \tag{6}$$

Similarly, the characteristic function does always exists.

$$\begin{aligned}
M_x(t) = E(e^{itx}) &= \int_0^{\infty} e^{itx} \frac{\psi}{\psi^2+2}\left(2+\psi^3 x\right) e^{-\psi x} dt \\
&= \frac{\psi}{\psi^2+2}\left[2\int_0^{\infty} e^{-(\psi-it)x} dt + \psi^3 \int_0^{\infty} it e^{-(\psi-it)x} dt\right] \\
&= \frac{\psi}{\psi^2+2}\left[\frac{2\Gamma(1)}{(\psi-it)} + \frac{\psi^3}{(\psi-it)^2}\right] \\
M_x(t) &= \frac{\psi}{\psi^2+2}\left[\frac{2}{(\psi-it)} + \frac{\psi^3}{(\psi-it)^2}\right]
\end{aligned} \tag{7}$$

Theorem 3 (sth crude Moment)**.** Let $T \sim$ Fav-Jerry (ψ), the *sth* crude moment is given as

$$\mu_s^{'} = \frac{\psi s!\left[2+\psi(s+2)(s+1)\right]}{(\psi^2+2)\psi^{s+1}}; \quad \text{for} \quad s = 1, 2, \cdots \tag{8}$$

Proof. The *sth* crude moment of a random variable T with a pdf $g(t;\psi)$, is mathematically defined thus, $\mu_s^{'} = \mathrm{E}(T) = \int\limits_0^{\infty} t^s g(t;\psi) dt = \frac{\psi}{\psi^2+2}\left\{2\int\limits_0^{\infty} t^s e^{-\psi t} dt + \psi^3 \int\limits_0^{\infty} t^{s+2} e^{-\psi t} dt\right\}$. The remainder easily follows. □

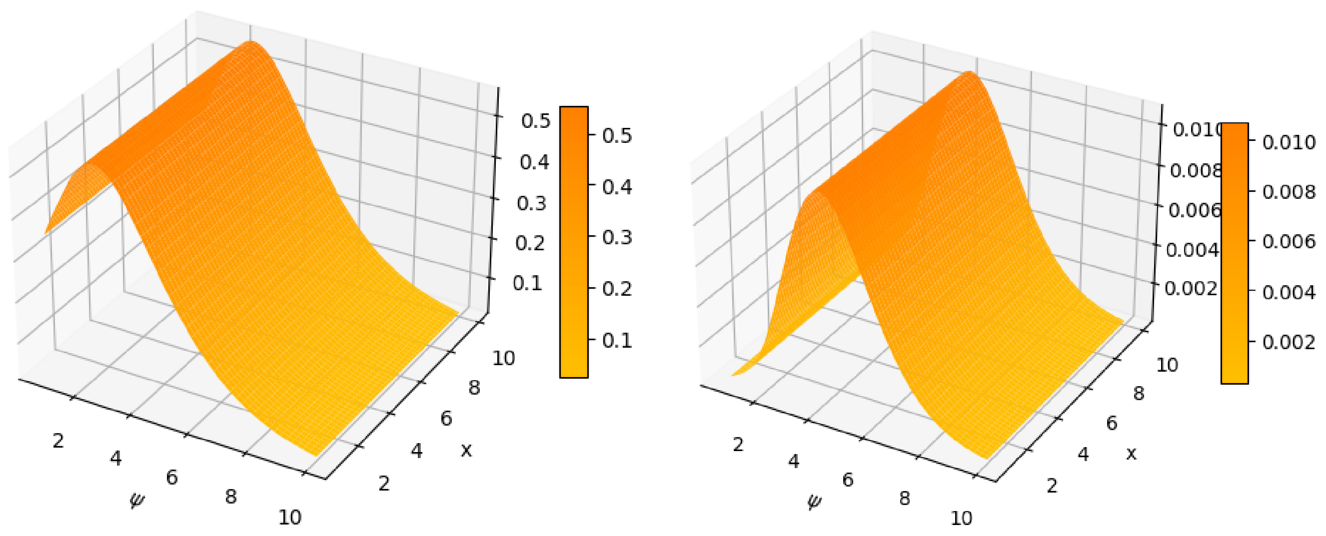
Corollary 3.1 (The Mean)**.** Let $X \sim$ Fab-Jerry (ψ),with $s = 1$, the mean goes thus;

$$\mu_1^{'} = \frac{\psi 1\,[2 + \psi(3)(2)]}{(\psi^2 + 2)\psi^2} \tag{9}$$

Theorem 4 (The shape of Fav-Jerry (ψ))**.** Suppose $T \sim$ Fav-Jerry (ψ), $\exists$ a unique number t_0 which makes the distribution uni-modal. This t_0 accounts for the shape of the distribution, and it is given as

$$t_0 = \frac{\psi^2 - 2}{\psi^3}; \text{where} \quad \psi \geq \sqrt{2}. \tag{10}$$

Proof. The maximum point occurs when $g^{''}(t) = 0$, so we differentiate the pdf in eq. 1 and equate the result to zero. This gives $t_0 = \frac{\psi^2-2}{\psi^3}$. Since $t_0 \geq 0$, we set the numerator to zero to obtain the value of ψ, which defines t_0, and the rest is trivial. □



(a) Mean of Fav-Jerry (ψ) **(b)** Variance of Fav-Jerry (ψ)

Figure 4: Measure of Central tendency and dispersion plots

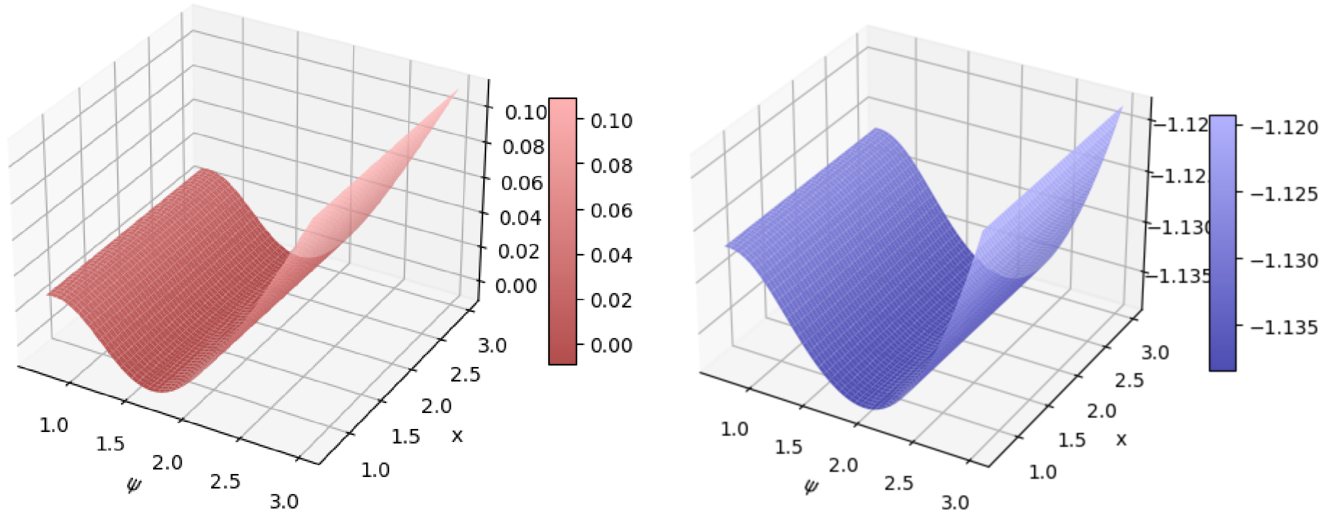
Theorem 5 (Shannon Entropy)**.** Let $T(t)$ be a signal produced from a $g(t;\psi)$, the average rate at which information is sourced Shannon [23] is defined thus

$$H(T) = -\sum_{i=1}^{n} g(t_i;\psi)\log\left(g(t_i;\psi)\right) = -\frac{\psi}{\psi^2 + 2}\sum_{i=1}^{n}(2 + \psi^3 t_i)e^{-\psi t_i}\left\{\log\psi - \log\left(\psi^2 + 2\right) + \log\left(2 + \psi^3 t_i\right) - \psi t_i\right\} \tag{11}$$

Proof. The proof of theorem 5 easily follows from substituting the pdf of the Fav-Jerry (ψ) distribution into $-\sum_{i=1}^{n} g(t_i;\psi)\log\left(g(t_i;\psi)\right)$. □

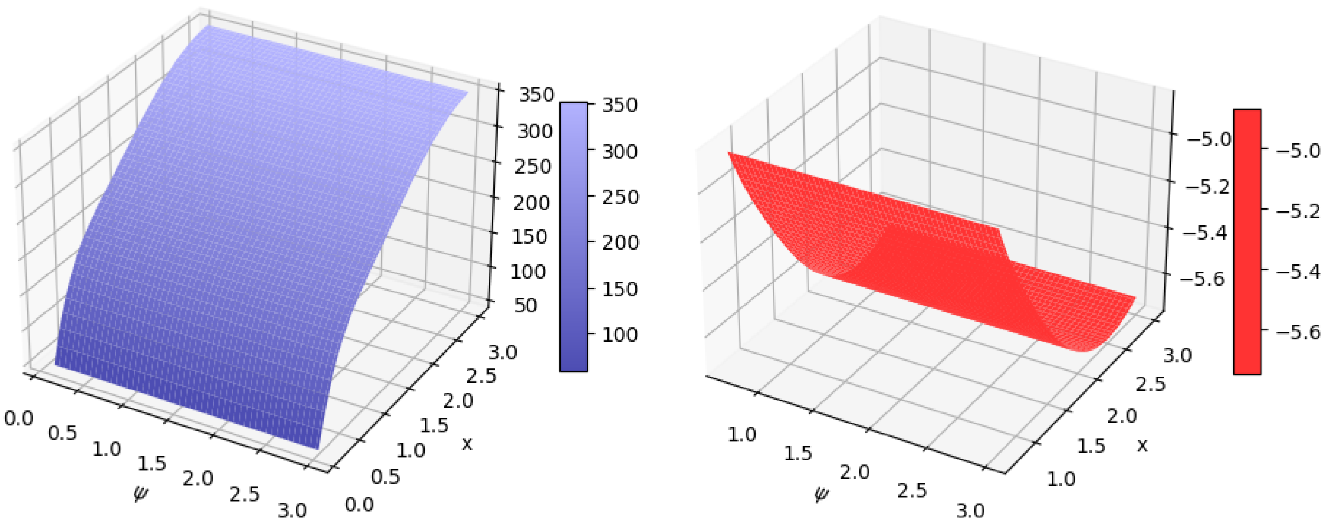
2 Non-Bayesian Point Estimation

This section studies the various non-bayesian estimation methods namely the maximum likelihood, the ordinary least squares, the weighted ordinary least squares, the maximum product spacing, the cramér von mises, anderspn-darling, right-tailed anderson-darling and the percentile estimation procedures.



(a) Skewness of Fav-Jerry (ψ) **(b)** Kurtosis of Fav-Jerry (ψ)

Figure 5: Measure of Asymmetry and Peakedness plots



(a) $m(x)$ of UL Fav-Jerry (ψ) **(b)** Rény entropy of Fav-Jerry (ψ)

Figure 6: Measure of additional life and Information Loss plots

2.1 Maximum Likelihood Estimation

For $T_1, T_2, \cdots, T_n$ samples of size n with joint pdf $g(t_i;\psi)$, the likelihood function of the parameter ψ can be expressed as

$$L(t_i;\psi) = \frac{\psi^n}{(\psi^2+2)^n} e^{-\psi \sum\limits_{i=1}^{n} t_i} \prod_{i=1}^{n} (2+\psi^3 t_i) \tag{12}$$

Set $\xi = \ln\{L(t_i;\psi)\}$, then the log-likelihood of ψ is

$$\xi = n\ln\psi - n\ln(\psi^2+2) - \psi\sum_{i=1}^{n} t_i + \sum_{i=1}^{n} \ln(2+\psi^3 t_i) \tag{13}$$

The first total derivative yields the maximum likelihood estimate $\hat{\psi}$ of the parameter ψ given as

$$\frac{d\xi}{d\psi} = \frac{n}{\psi} - \frac{2n\psi}{\psi^2+2} - \sum_{i=1}^{n} t_i + 3\psi^3 \sum_{i=1}^{n} \frac{t_i}{2+\psi^3 t_i} \tag{14}$$

2.2 Maximum product spacing (MPS)

Employing the method introduced by [2], the derivative of the MPS of Fav-Jerry (ψ) distribution is obtained by maximizing the function

$$\daleth(\psi) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln g(\psi) \tag{15}$$

The estimator $\hat{\psi}$ for ψ is derived by finding the solution to the non-linear equation below

$$\frac{\partial}{\partial\psi}\daleth(\psi) = \frac{1}{n+2} \sum_{i=1}^{n+1} \frac{1}{g(\tau)} \left(\epsilon\left(t_{i:n}|\psi\right) - \epsilon\left(t_{i:n}|\psi\right)\right) = 0$$

where

$$\epsilon\left(t_{i:n}|\psi\right) = \frac{te^{-\psi t}\left[\left(\psi^2+2\right)^2 - \psi^2\left(4 - 2\psi t - \psi^3 t\right)\right]}{\left(\psi^2+2\right)^2}, \tag{16}$$

This obtained by differentiating the cdf of Fav-Jerry (ψ) with respect to ψ. MPS aids in Bayesian inference and numerical analysis to reduce the correlation between parameters and enhance the accuracy of parameter estimates and convergence of numerical algorithms.

2.3 Ordinary Least Squares (OLS)

Given that

$$E\left(G_{t:n}|\psi\right) = \frac{i}{n+1} \quad \text{and} \quad V\left(G_{t:n}|\psi\right) = \frac{i\left(n-i+1\right)}{\left(n+1\right)^2\left(n+2\right)} \tag{17}$$

[24] proposed that the least squares estimate of $\hat{\psi}_{\text{OLS}}$ of τ is obtained by minimizing the function

$$T(\psi) = \sum_{i=1}^{n} \left(G\left(t_{i:n}|\psi\right) - \frac{i}{n+1}\right)^2$$

differentiating partially yields

$$\sum_{i=1}^{n} \left(G\left(t_{i:n}|\psi\right) - \frac{i}{n+1}\right) \epsilon\left(t_{i:n}|\psi\right) = 0 \tag{18}$$

2.4 Weighted Least Squares (WLS)

Similarly, the weighted least squares estimate $\hat{\psi}_{\text{WLS}}$ for Fav-Jerry (ψ) distribution parameter ψ is achieved by minimizing the function $\omega(\psi)$ with respect to ψ

$$\omega(\psi) = \underset{(\psi)}{\arg\min} \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[G(t_{i:n}|\psi) - \frac{i}{n+1}\right]^2. \tag{19}$$

resolving partially, we obtain the following non-linear equation

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[G(t_{i:n}|\psi) - \frac{i}{n+1}\right] \epsilon(t_{i:n}|\psi) = 0 \tag{20}$$

where $\epsilon(t_{i:n}|\psi)$ is as defined in equation (23)

2.5 Cramér-von-Mises estimation (CVM)

The method for Cramér-von-Mises estimates of $\hat{\psi}_{\text{CVME}}$, of the Fav-Jerry (ψ) distribution parameter ψ, is defined as

$$C(\psi) = \arg\min_{(\psi)} \left\{ \frac{1}{12n} + \sum_{i=1}^{n} \left[G(t_{i:n}|\psi) - \frac{2i-1}{2n} \right]^2 \right\}. \tag{21}$$

The estimates are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} \left(G(t_{i:n}|\psi) - \frac{2i-1}{2n} \right) \epsilon(t_{i:n}|\psi) = 0$$

2.6 Anderson-Darling Estimation (AD)

The Anderson-Darling estimate $\hat{\psi}$ for Fav-Jerry (ψ) distribution with parameter ψ is obtained by minimizing the function $\vartheta(\psi)$ with respect to ψ

$$\vartheta(\psi) = \arg\min_{(\psi)} \sum_{i=1}^{n} (2i-1) \Big\{ \ln G(t_{i:n}|\psi) + \ln \Big[1 - G(t_{n+1-i:n}|\psi) \Big] \Big\}. \tag{22}$$

The estimates are obtained by solving the following sets of non-linear equations

$$\sum_{i=1}^{n} (2i-1) \left[\frac{\varepsilon(t_{i:n}|\psi)}{G(t_{i:n}|\psi)} - \frac{\varepsilon(t_{n+1-i:n}|\psi)}{1 - G(t_{n+1-i:n}|\psi)} \right] = 0 \tag{23}$$

where $\varepsilon(t.|\psi)$ is as defined in Equation (23)

2.7 Right-Tailed Anderson-Darling Estimation (RTAD)

The Right-Tailed Anderson-Darling estimates $\hat{\psi}_{\text{RTAD}}$ of the Fav-Jerry (ψ) distribution parameter ψ obtained by minimizing the function $Q(\psi)$ with respect to ψ

$$Q(\psi) = \arg\min_{(\psi)} \left\{ \frac{n}{2} - 2 \sum_{i=1}^{n} G(t_{i:n}|\psi) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \ln \Big[1 - G(t_{n+1-i:n}|\psi) \Big] \right\}. \tag{24}$$

The estimates can be obtained by solving the following set of non-linear equations

$$-2 \sum_{i=1}^{n} \frac{\varepsilon(t_{i:n}|\psi)}{G(t_{i:n}|\psi)} + \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[\frac{\varepsilon(t_{n+1-i:n}|\psi)}{1 - G(t_{n+1-i:n}|\psi)} \right] = 0 \tag{25}$$

where $\varepsilon(t.|\psi)$ is as defined in (23). Anderson-Darling right tailed method is best when the dataset is censored or truncated.

3 Bayesian Point Estimation

This section focuses on the Bayesian estimation of the unknown parameters in the Fav-Jerry (FJ) distribution. Various loss functions, including squared error, LINEX, and generalized entropy loss functions, can be employed for Bayesian parameter estimation. Specifically, we consider gamma prior for the parameter ψ with probability density function (pdf) as follows:

$$\pi(\psi) = \frac{\beta^\alpha}{\Gamma(\alpha)} \psi^{\alpha-1} \mathrm{e}^{-\beta\psi} \tag{26}$$

The posterior distribution is proportional to the product the likelihood function of the pdf $L(\psi \mid t_1, t_2, \cdots, t_n)$ and the prior $\pi(\psi)$. That is, $\pi(\psi) \propto L(\psi \mid t_1, t_2, \cdots ..., t_n)\pi(\psi)$. Substituting, this gives

$$\pi\left(\psi \mid t_1, t_2, \cdots, t_n\right) \propto \left[\prod_{i=1}^{n} \frac{\psi}{\psi^2 + 2} \left(2 + \psi^3 t\right) e^{-\psi t} \right] \frac{\beta^\alpha}{\Gamma(\alpha)} \psi^{\alpha-1} \mathrm{e}^{-\beta\psi}$$

Simplifying, we get:

$$\pi\left(\psi \mid t_1, t_2, \cdots, t_n\right) \propto \frac{\beta^\alpha}{\Gamma \alpha} \psi^{n+\alpha-1} \mathrm{e}^{-\psi\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t}_{\mathrm{i}}+\beta\right)} . \prod_{i=1}^n \frac{2+\psi^3 t}{\psi^2+2}$$

The normalize posterior distribution is obtained by integrating the unnormalized posterior over all possible values of ψ. Thus

$$\int_0^{\infty} \frac{\beta^\alpha}{\Gamma \alpha} \psi^{n+\alpha-1} \mathrm{e}^{-\psi\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}}+\beta\right)} \prod_{i=1}^n \frac{2+\psi^3 t}{\psi^2+2} d \psi$$

Let $\Phi=\prod\limits_{i=1}^n \frac{1}{\psi^2+2}$, simplifying, we get

$$\pi\left(\psi \mid x_1, x_2, \cdots, x_n\right) \propto \psi^{n+\alpha-1} \mathrm{e}^{-\psi\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}}+\beta\right)} \Phi \beta^\alpha . \prod_{i=1}^n\left(2+\psi^3 t\right)$$

The role of the shape parameter ψ of the prior is to influence the shape of the posterior distribution while β, the rate parameter is added to the sum of the observations $\sum_{i=1}^n x_i$, to influence the rate parameter of the exponential part of the posterior.

Given any function, such as $l(\phi)$ under the squared error loss (SEL) function, the Bayes estimator is given by

$$\hat{\phi}_{BE_{SEL}}=E[l(\phi) \mid \mathbf{x}]=\int_\phi l(\phi) \pi(\phi \mid x) d \phi. \tag{27}$$

SEL impacts underestimation and overestimation equally because it has an asymmetric loss function. In many real situations, both underestimation and overestimation can have serious implications. A proposed LINEX loss can be made in certain instances as an alternative to the SE loss given by

$$\left(l(\phi), \hat{l}(\phi)\right)=\mathrm{e}^{\{\hat{l}(\phi)-l(\phi)\}}-v\left(\hat{l}(\phi)-l(\phi)\right)-1.$$

where $v \neq 0$ is a shape parameter. Here $v>1$ suggests that an overestimation is more serious than an underestimation, and vice versa for $v<0$. Further v approaching zero replicates the SE loss function itself. One may refer to Varian [27] and Doostparast, Akbari, and Balakrishna [5] for more details in this regard. The BE of $l(\phi)$ under this loss can be derived as

$$\hat{\phi}_{BE_{LINEX}}=E\left[\mathrm{e}^{\{-v l(\phi)\}} \mid \mathbf{x}\right]=-\frac{1}{v} \log \left[\int_\phi \mathrm{e}^{\{-v l(\phi)\}} \pi(\phi \mid x) d \phi\right]. \tag{28}$$

Additionally, we take into account the general entropy loss (GEL) function suggested by Calabria and Pulcini [1], which is defined as follows.

$$\left(l(\phi), \hat{l}(\phi)\right)=\left(\frac{\hat{l}(\phi)}{l(\phi)}\right)^\psi-\psi \psi \log \left(\frac{\hat{l}(\phi)}{l(\phi)}\right)-1,$$

where the shape parameter $\psi \neq 0$ denotes a departure from symmetry. It views overestimation as more significant than underestimating when $\psi>0$ and the opposite is true when $\psi<0$. The Bayes estimator with regard to the GE loss function is given.

$$\hat{\phi}_{BE_{GEL}}=\left[E\left((l(\phi))^{-\psi} \mid \mathbf{x}\right)\right]^{-1 / \psi}=\left[\int_\phi(l(\phi))^{-\psi} \pi(\phi \mid x) d \phi\right]^{-1 / \psi}. \tag{29}$$

The estimations produced by (27), (28), and (29) can be seen to not be able to be transformed into closed-form expressions. We then use the Markov chain Monte Carlo (MCMC) approach to generate posterior samples and arrive at suitable BEs.

4 Numerical Analysis

In this section, the performance of the Fav-Jerry (ψ) distribution is illustrated using two life data sets. The first set of information is a description of the infant mortality rate per 1,000 live births for a few chosen nations in 2021, as reported by a https://data.worldbank.org/indicator/SP.DYN.IMRT.IN This real data set is presented as

56	10	22	3	69	6	7	11	4	4	19	13	7	27
12	3	4	11	84	27	25	6	35	14	11	12	6	

Here, we compare the goodness of fit of the Fav-Jerry (ψ) distribution with the two-parameter Chris-Jerry (TPCJ) distribution by Chinedu et al. [3], Chris-Jerry (CJ) distribution by Onyekwere and Obulezi [13], Exponentiated Inverse Exponential (EIE) by [6], and Weibull distribution as shown in Table 1. The fitness metrics considered are the Negative log-likelihood (NLL), the Akaike information criterion (AIC), the corrected AIC (CAIC), the Bayesian information criterion (BIC), the Hannan–Quinn information criterion (HQIC), Anderson Darling (AD) and Cramér-von-Mises (CVM) statistics. The model with the lowest values of these metrics is chosen as the best performer.

Table 1: The Fitness Metrics and Performance Statistics for the Models using World Infant Mortality Rate per 1000 Live Birth Data

Dist	LL	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value	scale	shape
Fav-Jerry	-106.24	214.47	214.63	215.77	214.86	0.12	0.80	0.16	0.6008	0.053	-
TPCJ	-106.16	216.31	216.81	218.90	217.08	0.11	0.75	0.16	0.5345	399.51	0.06
Chris-Jerry	-112.39	226.77	226.93	228.07	227.16	0.17	1.10	0.28	0.0260	0.15	-
EIE	-103.88	211.76	212.26	214.36	212.54	0.08	0.50	0.17	0.4187	0.42	6.66
Weibull	-106.11	231.36	231.86	233.95	232.13	0.13	0.82	0.32	0.0084	0.90	8.90

Table 2: Estimating the parameter of Fav-Jerry distribution with different methods using the first data set

Methods	Estimate	Std. Error
MLE	0.01561	0.00317
MPS	0.014664	0.00299
LSE	0.015854	0.00885
WLSE	0.01551	0.00075
CVM	0.015964	0.008806
ADE	0.015537	0.003524
RTADE	0.01524	0.00446
Bayes	0.01852	0.00294

From the above estimations using the several method estimation approach, it is concluded that the best estimation method for estimating the parameter in Fav-Jerry distribution is WLSE. The reson for this choie is because WLSE has the least standard error value.

The second application is on the failure rate of air-condition system of an airplane studied by Linhart and Zucchini [10]. The data are as follows

3	5	5	13	14	15	22	22	23	30	36	39
44	46	50	72	79	88	97	102	139	188	197	210

Table 3: The Fitness Metrics and Performance Statistics for the Models using the data which indicate the failure times (in hour) of air-conditioning system of an airplane

Dist	LL	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value	scale	shape
Fav-Jerry	-123.86	249.72	249.90	250.89	250.03	0.02	0.23	0.0835	0.9961	0.02	$-$
TPCJ	-123.73	251.46	252.03	253.82	252.09	0.02	0.22	0.0808	0.9900	0.02	239.17
Chris-Jerry	-132.94	267.87	268.05	269.05	268.18	0.04	0.33	0.2757	0.05	0.0054	$-$
EIE	-123.4	250.79	251.36	251.14	251.41	0.02	0.18	0.09	0.9880	0.02	5.03
Weibull	-123.85	288.20	288.77	290.56	288.82	0.02	0.19	0.54	$1.25e^{-06}$	0.36	9.69

Table 4: Estimating the parameter of Fav-Jerry distribution with different methods using the second data set

Methods	Estimate	Std. Error
MLE	0.05324	0.01027
MPS	0.049513	0.00958
LSE	0.057119	0.02792
WLSE	0.0529	0.00206
CVM	0.057211	0.027721
ADE	0054555	0.011792
RTADE	0.05564	0.01586
Bayes	0.06285	0.00976

From the above estimations using the several method estimation approach, it is concluded that the best estimation method for estimating the parameter in Fav-Jerry distribution using the second data set is also WLSE. The reson for this choie is because WLSE has the least standard error value.

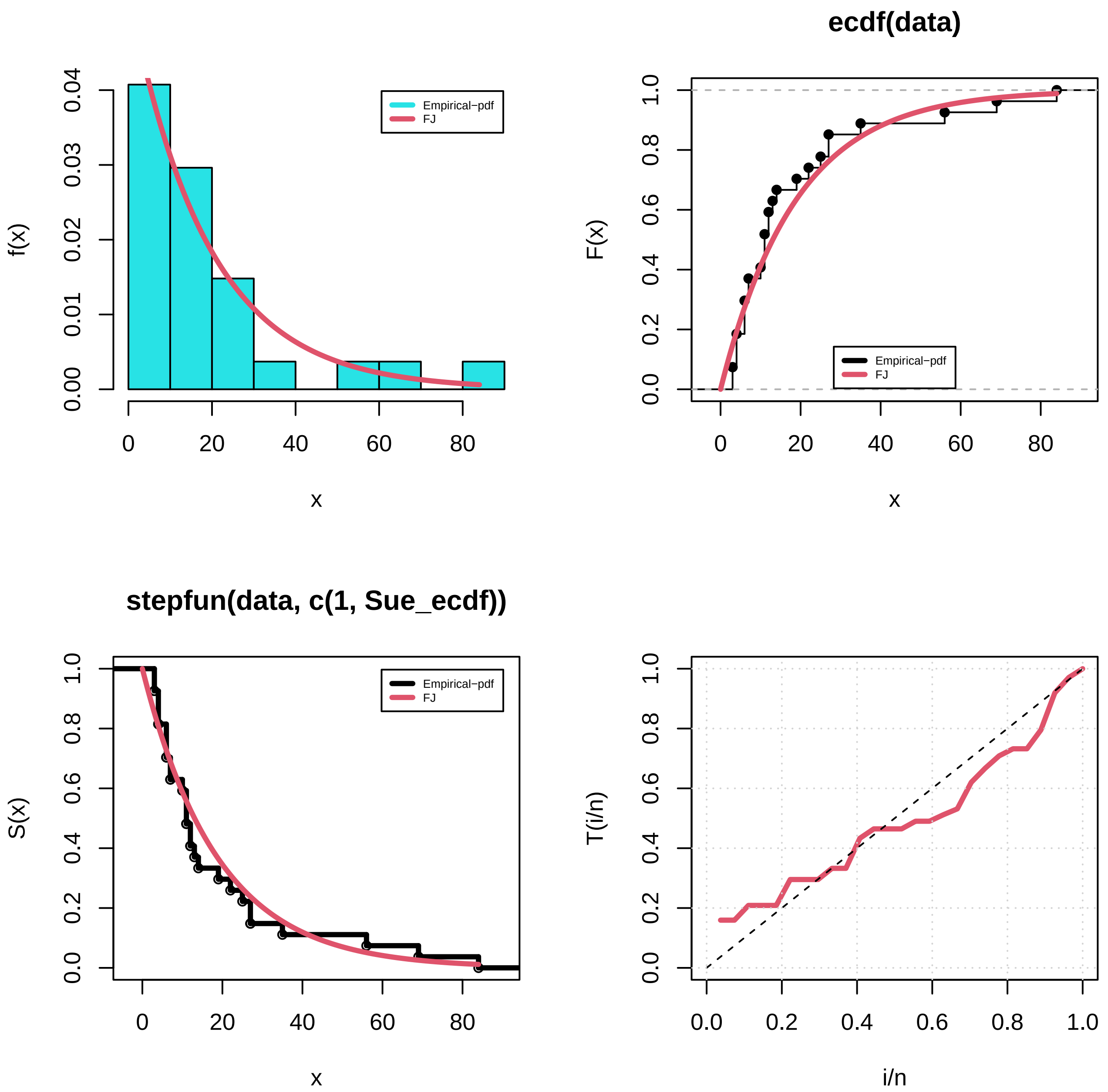


Figure 7: Histogram, CDF, Survival and TTT plots of Fav-Jerry using the Infant Mortality Data

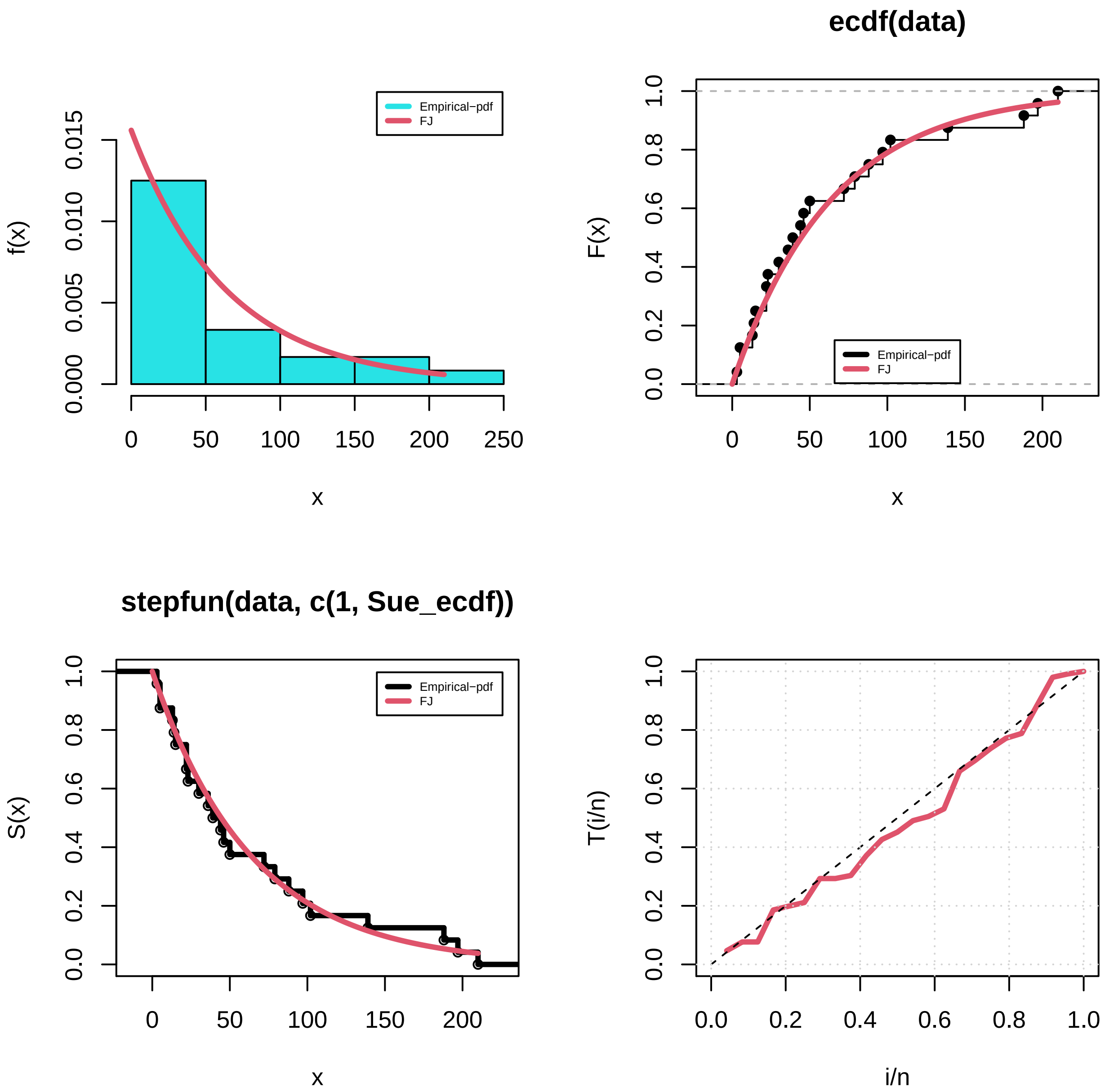


Figure 8: Histogram, CDF, Survival and TTT plots of Fav-Jerry using the Data on failure rate of Air Conditioning of an Airplane

5 Conclusion

A novel distribution named "Fav-Jerry distribution" was created in this paper. Several mathematical properties such as moment, quantile function, the shape the distribution takes, shanon entropy, including its mean, variance, kurtosis, skewness,(graphs included). Going further into this work, the writers discussed the maximum likelihood function. Using LL, AIC, BIC, K-S statistic, the test for the goodness of fit was conducted. From two data sets used to rum the analysis, it shows that the Fav-Jerry distribution performed better than Pranav, Shanker, Odoma, Rani, Rama, Juchez, Copoun, Ram Awadh distributions. Next off is the Bayesian point estimate which looks into the the various loss function of the distribution, which includes the squared error, LINEX and genererlized entropy loss function. The writers also used the Markov chain Monte Carlo (MCMC) approach to generate posterior samples and arrive at suitable BEs. Towards the conclusion of this work, the writer got the estimation of the parameter of this distribution using different methods from the data sets and they came to a conclusion that WLSE is the best approach to be taken for the distribution. The parametric plots containing the histogram, CDF, survival and TTT plots gotten from both data sets are displayed which show good fits to the two data sets.

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