Towards 2-derivators for formal ∞-category theory

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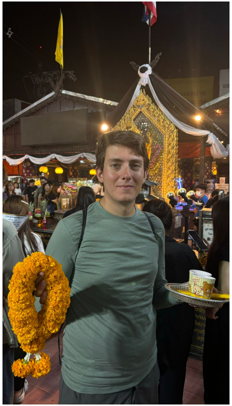
Outline

Hi there!

Info about myself:

- PhD in Mathematics at Macquarie University in Sydney (CoACT).
- Italian, still living in Sydney. Thanks for accommodating the time difference!
- Currently on the job market. Photo of me praying Ganesh for a job on the side.
- Mathematically, I'm interested in category theory and its connections to algebraic topology (especially to homotopy theory).
- Outside of mathematics, I have a wide variety of interests, many of them loosely connected by a curiosity about the human condition.

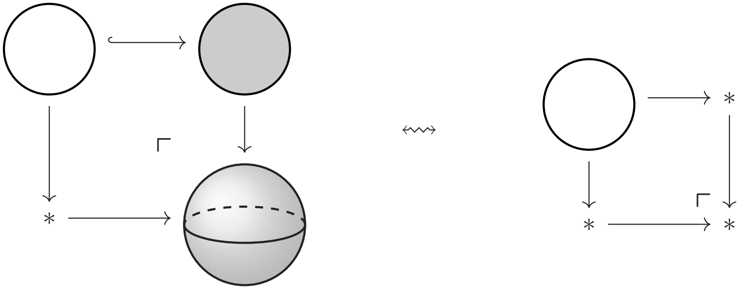


This talk presents research conducted during my PhD at the Centre of Australian Category Theory under the supervision of Dominic Verity (photo on the side), to whom I owe much for the many fruitful and insightful discussions.

Moreover, this is just the beginning of the story; there is still much to be done. It is a promising area of research: the first paper on this topic has just been published by *Advances in Mathematics*.

Why derivators?

Consider the cellular structure of the 2-sphere (left) and a diagram where we have replaced the span $* \leftarrow S^1 \rightarrow D^2$ with the weakly equivalent span $* \leftarrow S^1 \rightarrow *$ (right).



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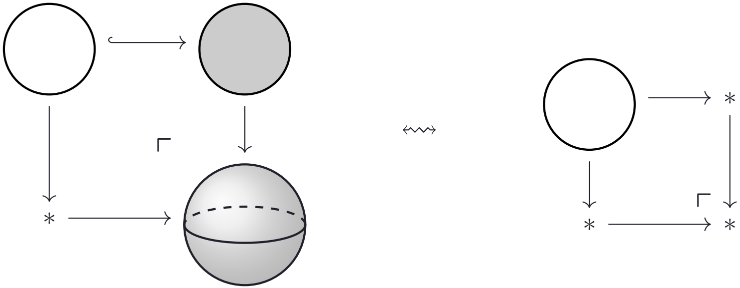

We know from undergrad topology that $S^2 \not\simeq *$ (e.g. because $\pi_2(S^2) \cong \mathbb{Z}$):

(co)limits are not well-behaved homotopically!

This example shows that the colimit functor *at the level of spaces* doesn't descend to a functor

$$\mathrm{Ho}(\mathrm{Spaces}^{\bullet\leftarrow\bullet\rightarrow\bullet}) \to \mathrm{Ho}(\mathrm{Spaces}).$$

However, we do have the usual (ill-behaved) colimit functor *at the level of homotopy categories*

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but when we consider the homotopy categories of all diagrams categories simultaneously this becomes part of the data of a derivator. This is the right level of generality to ensure that limits and colimits behave well homotopically.

Derivators

So far we've seen that:

- there is an assignment $I \mapsto \mathrm{Ho}(\mathcal{M}^I)$, where I is a category serving as the diagram shape and $\mathcal{M}$ is some suitable "category of spaces with a notion of weak equivalence",
- there are functors $\mathrm{Ho}(\mathcal{M}^I) \to \mathrm{Ho}(\mathcal{M})^I$ for every I,
- (co)limits have to be indexed in $\mathrm{Ho}(\mathcal{M}^I)$ and not in $\mathrm{Ho}(\mathcal{M})^I$.

Derivators

Let's formalize this:

- there is an assignment $I \mapsto \mathbb{D}(I)$, where I is a category serving as the diagram shape,
- there are functors $\mathrm{dia}_I \colon \mathbb{D}(I) \to \mathbb{D}(\mathbb{1})^I$ for every I,
- (co)limits have to be indexed in $\mathbb{D}(I)$ and not in $\mathbb{D}(\mathbb{1})^I$.

Definition

Given a suitable **Dia** $\subseteq$ **Cat**, a *prederivator* is a 2-functor $\mathbb{D} \colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$.

Definition

The *underlying diagram functors* $\mathrm{dia}_I \colon \mathbb{D}(I) \to \mathbb{D}(\mathbb{1})^I$ are the ones coming from the action of $\mathbb{D}$ on morphisms and the product-internal hom adjunction:

$$\dfrac{\dfrac{I \cong \mathbf{Dia}(\mathbb{1}, I) \xrightarrow{\mathbb{D}_{I,\mathbb{1}}} \mathbb{D}(\mathbb{1})^{\mathbb{D}(I)}}{\mathbb{D}(I) \times I \to \mathbb{D}(\mathbb{1})}}{\mathrm{dia}_I \colon \mathbb{D}(I) \to \mathbb{D}(\mathbb{1})^I}$$

Derivators

Definition

A prederivator $\mathbb{D}$ is called a *derivator* if the following axioms hold.

(Der 1) $\mathbb{D}(\varnothing) \cong \mathbb{1}$ and the canonical map $\mathbb{D}(\mathcal{I} \sqcup \mathcal{J}) \to \mathbb{D}(\mathcal{I}) \times \mathbb{D}(\mathcal{J})$ is an equivalence of categories for every I and J,

(Der 2) the functors $\mathrm{dia}_I\colon \mathbb{D}(I) \to \mathbb{D}(\mathbb{1})^I$ are conservative for all I,

(Der 3) every functor $u\colon I \to J$ induces an adjoint triple $u_! \dashv u^* \dashv u_*$ of *homotopy Kan extensions*,

(Der 4) homotopy Kan extensions are pointwise, i.e. the following squares

$$\begin{tikzcd} (u/k) \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{pt}"'] \arrow[dr, phantom, "\swarrow"] & J \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "k"'] & K \end{tikzcd} \qquad \begin{tikzcd} (k/u) \arrow[r, "\mathrm{pt}"] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\swarrow"] & \mathbb{1} \arrow[d, "k"] \\ J \arrow[r, "u"'] & K \end{tikzcd}$$

are exact for every $u\colon J \to K$ and every $k \in K$.

M. Groth, **Derivators, pointed derivators and stable derivators**, Algebraic & Geometric Topology 13 (2013) 313–374

Examples of derivators

- $\mathbf{Cat}(-, \mathcal{A})$ for a complete and cocomplete category $\mathcal{A}$,
- $\mathrm{Ho}(\mathcal{M}^{-})$ for a nice enough model category $\mathcal{M}$ (e.g. $\mathcal{D}(\mathcal{A}^{-})$),
- $\mathrm{h}(\mathcal{C}^{N-})$ for a complete and cocomplete quasi-category $\mathcal{C}$ (a model for ∞-categories),
- If $\mathbb{D}$ is a derivator so is $\mathbb{D}^{I} := \mathbb{D} \circ (- \times I)$, called the *shifted* derivator.

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Homotopy (co)limits as special cases

For a derivator associated with a model category $\mathcal{M}$, homotopy (co)limits can be characterized by the adjunction $\mathrm{pt}_! \dashv \mathrm{pt}^* \dashv \mathrm{pt}_*$ where $\mathrm{pt}_I : I \to \mathbb{1}$ is the unique morphism from the diagram shape to the terminal category.

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Definition

Strong derivators are the ones satisfying the following additional axiom:

(Der 5) $\mathrm{dia}_{\mathbb{2}} : \mathbb{D}^I(\mathbb{2}) \to (\mathbb{D}^I(\mathbb{1}))^{\mathbb{2}}$ is full and essentially surjective for every $I \in \mathbf{Dia}$.

Why 2-derivators?

Derivators sit between ∞-categories and their homotopy categories, capturing much of the homotopical information available in the former. But what if we are interested in studying ∞-categories themselves?

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Analytical approach: using models such as quasi-categories, complete Segal spaces, etc. (see, e.g., Joyal, Lurie, Rezk).

Synthetic (or *formal*) approach: viewing ∞-categories as objects inside $(\infty, 2)$-categories (see, e.g., Riehl and Verity's theory of ∞-cosmoi).

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Idea: An ∞-cosmos (more generally, a nice enough $(\infty, 2)$-category) gives rise to a *2-derivator*.

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Idea: An ∞-cosmos (more generally, a nice enough $(\infty, 2)$-category) gives rise to a *2-derivator*.

So, what axioms should a 2-derivator satisfy?

Domain

Up to now, two domains have been explored:

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1) $\mathbf{Dia} \subseteq \mathbf{2\text{-}Cat}$,
2) $\mathbf{Dia} \coloneqq h_{**}(\mathbf{sSet}\text{-}\mathrm{Cat})$,

where h_{**} is induced by $h \colon \mathbf{sSet} \to \mathbf{Cat}$ using change of base twice.

In other words, in the second case, we consider the 3-category with objects small simplicially enriched categories and homs $\mathbf{Dia}(\mathcal{A}, \mathcal{B})$ defined to be the homotopy 2-category of the simplicially enriched category $[\mathcal{A}, \mathcal{B}]$. Spelling this out:

i) the 1-cells of $\mathbf{Dia}$ are simplicially enriched functors,
ii) the 2-cells of $\mathbf{Dia}$ are simplicially enriched natural transformations,
iii) the 3-cells of $\mathbf{Dia}$ are homotopy classes of formal composites of modifications.

Domain

Domain 1)

- ✓ A direct extension of **Dia** for derivators.
- ✓ Since it doesn't require simplicial categories, it is more straightforward to work with.
- ✗ It doesn't capture more advanced information, such as that related to monadicity.

Domain 2)

- ✓ More general, it contains Domain 1)
- ✓ Can be used to address monadicity (work in progress).
- ✗ More complicated, it requires simplicial categories.

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However this doesn't work because

- ✗ the examples we examine often involve weak structures and fibrant or cofibrant replacements,
- ✗ we are unable to characterize equivalences in a componentwise fashion.

We instead use **GRAY**, the (large) Gray-category of 2-categories whose 1-cells are 2-functors, 2-cells are *pseudonatural* transformations, and 3-cells are modifications.

Brief digression on Gray-categories

Definition

A **Gray**-category is a category enriched over the monoidal category $\mathbf{Gray} = (\mathbf{2\text{-}Cat}_0, \otimes, \mathbb{1})$.

$$\mathbb{2} \times \mathbb{2} \quad = \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \mathbb{2} \otimes \mathbb{2} \quad = \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

The monoidal category **Gray** is both symmetric and closed.

Definition

If $\mathcal{A}, \mathcal{B} \in$ **Gray**, we will denote their internal hom with respect to this structure as $[\mathcal{A}, \mathcal{B}]_p$. This is the 2-category of strict 2-functors, *pseudonatural transformations* and modifications between $\mathcal{A}$ and $\mathcal{B}$.

The key example: enriched model categories

Let $\mathcal{M}$ be a combinatorial $\mathbf{sSet}_{\mathrm{Joyal}}$-model category. For every small 2-category $\mathcal{J}$, we consider the diagram category $[\mathcal{J}, \mathcal{M}]$. The latter is again $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched and admits both the projective and the injective enriched model structure. Define

$$\begin{aligned} \mathbb{D}_{\mathcal{M}} \colon \mathbf{Dia}^{\mathrm{op}} &\longrightarrow \mathbf{GRAY} \\ \mathcal{I} &\mapsto h_*[\mathcal{I}, \mathcal{M}]^{\mathrm{proj}}_{cf} \\ (\mathcal{I} \xrightarrow{g} \mathcal{J}) &\mapsto h_*([\mathcal{J}, \mathcal{M}]^{\mathrm{proj}}_{cf} \xrightarrow{-\circ g} [\mathcal{I}, \mathcal{M}]^{\mathrm{proj}}_{f} \xrightarrow{C} [\mathcal{I}, \mathcal{M}]^{\mathrm{proj}}_{cf}) \end{aligned}$$

and similarly for the higher cells.

2-prederivators

What kind of functor should a 2-prederivator be?

Derivators are strict 2-functors, so we might consider defining a 2-derivator as a strict 3-functor; nevertheless, in the examples, we see that a weaker kind of functor is needed.

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Definition

A *2-prederivator* is a *trihomomorphism* $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{GRAY}$.

2-prederivators

Breaking this down, a *2-prederivator* $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{GRAY}$ consists of

- function $\mathrm{Ob}(\mathbf{Dia}) \to \mathrm{Ob}(\mathbf{GRAY})$
- 2-functors $\mathbf{Dia}(\mathcal{D}, \mathcal{C}) \to \mathbf{GRAY}(\mathbb{D}(\mathcal{C}), \mathbb{D}(\mathcal{D}))$,
- equivalences $f^*g^* \simeq (gf)^*$
- $\mathrm{id}_{\mathbb{D}(\mathcal{A})} \simeq (\mathrm{id}_{\mathcal{A}})^*$,
- associativity up to equivalence,
- coherences.

Axiom on (co)products

(HDer 1)

The canonical map

$$\mathbb{D}(\mathcal{I} \sqcup \mathcal{J}) \to \mathbb{D}(\mathcal{I}) \times \mathbb{D}(\mathcal{J})$$

is an equivalence of 2-categories for every $\mathcal{I}, \mathcal{J} \in$ **Dia**. In addition, $\mathbb{D}(\varnothing) \simeq \mathbb{1}$.

Axiom on componentwise equivalences

Definition

For any $\mathcal{C} \in \mathbf{Dia}$, the underlying diagram 2-functor

$$\mathrm{dia}_{\mathcal{C}} \colon \mathbb{D}(\mathcal{C}) \to [h_*\mathcal{C}, \mathbb{D}(\mathbb{1})]_p$$

is the output of the chain of transpositions

$$\cfrac{\cfrac{h_*\mathcal{C} \cong \mathbf{Dia}(\mathbb{1}, \mathcal{C}) \xrightarrow{\mathbb{D}_{\mathcal{C},\mathbb{1}}} [\mathbb{D}(\mathcal{C}), \mathbb{D}(\mathbb{1})]_p}{\mathbb{D}(\mathcal{C}) \otimes h_*\mathcal{C} \to \mathbb{D}(\mathbb{1})}}{\mathrm{dia}_{\mathcal{C}} \colon \mathbb{D}(\mathcal{C}) \to [h_*\mathcal{C}, \mathbb{D}(\mathbb{1})]_p}$$

(HDer 2)

The 2-functors $\mathrm{dia}_{\mathcal{C}} \colon \mathbb{D}(\mathcal{C}) \to [h_*\mathcal{C}, \mathbb{D}(\mathbb{1})]_p$ are conservative on 1-cells for every $\mathcal{C} \in \mathbf{Dia}$.

Existence of Kan extensions

Definition

A *biadjunction* $f \dashv_b u$ consists of 2-categories $\mathcal{A}$ and $\mathcal{B}$, 2-functors $f : \mathcal{B} \to \mathcal{A}$ and $u : \mathcal{A} \to \mathcal{B}$ and pseudonatural transformations $\eta\colon \mathrm{id}_{\mathcal{B}} \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow \mathrm{id}_{\mathcal{A}}$ such that there exist invertible modifications filling the triangles

$$\begin{tikzcd} f \arrow[r, Rightarrow, "f\eta"] \arrow[dr, equal] & fuf \arrow[d, Rightarrow, "\epsilon f"] \arrow[dl, phantom, "\cong"] \\ & f \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r, Rightarrow, "\eta u"] \arrow[dr, equal] & ufu \arrow[d, Rightarrow, "u\epsilon"] \arrow[dl, phantom, "\cong"] \\ & u \end{tikzcd}$$

(HDer 3)

Every $f : \mathcal{A} \to \mathcal{B}$ in **Dia** induces a biadjoint triple $f_! \dashv_b f^* \dashv_b f_*$

Kan extensions are pointwise

Definition

The *left Kan extension* of a $\mathcal{V}$-functor $G\colon \mathcal{A} \to \mathcal{B}$ along a $\mathcal{V}$-functor $K\colon \mathcal{A} \to \mathcal{C}$, if it exists, is a $\mathcal{V}$-functor $\operatorname{Lan}_K G\colon \mathcal{C} \to \mathcal{B}$ defined by the formula

$$\operatorname{Lan}_K G(c) = \operatorname{colim}(\mathcal{C}(K-, c), G)$$

Dually, the *right Kan extension* is defined by the formula

$$\operatorname{Ran}_K G(c) = \lim(\mathcal{C}(c, K-), G).$$

Definition

An *enriched profunctor* or *$\mathcal{V}$-profunctor* $W\colon \mathcal{A} \nrightarrow \mathcal{B}$ between the $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{V}$-functor $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$.

Using the properties of the unit $\mathcal{V}$-category $\mathcal{I}$, a weight $\mathcal{A} \to \mathcal{V}$ is the same as a $\mathcal{V}$-functor $\mathcal{I}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$ or in other words a $\mathcal{V}$-profunctor $\mathcal{A} \nrightarrow \mathcal{I}$.

Definition

Assume that $\mathcal{V}$ has an initial object $\varnothing$ that is preserved on both sides by $\otimes$. Given a $\mathcal{V}$-profunctor $W\colon \mathcal{A} \nrightarrow \mathcal{B}$, we define its *collage* $\operatorname{coll}(W)$ to be the $\mathcal{V}$-category having as objects the coproduct $\operatorname{Ob}\mathcal{A} \sqcup \operatorname{Ob}\mathcal{B}$ and such that

$$\operatorname{coll}(W)(x,y) = \begin{cases} \mathcal{A}(x,y), & \text{if } x, y \in \mathcal{A} \\ \mathcal{B}(x,y), & \text{if } x, y \in \mathcal{B} \\ W(x,y), & \text{if } x \in \mathcal{B} \text{ and } y \in \mathcal{A} \\ \varnothing, & \text{otherwise} \end{cases}$$

where the composition comes from the ones in $\mathcal{A}$ and $\mathcal{B}$ and from the $\mathcal{V}$-functoriality of W.

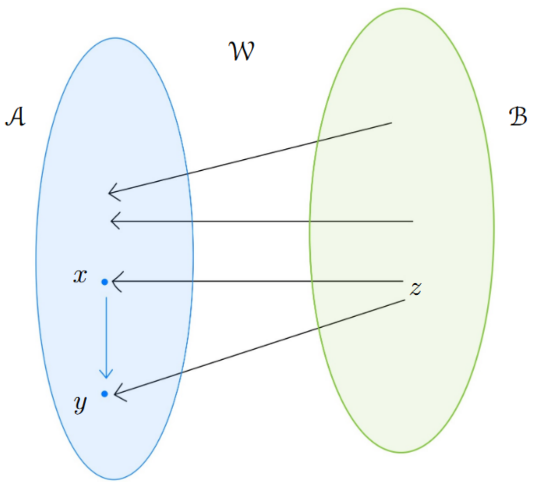

FIGURE 1: Collage of a profunctor

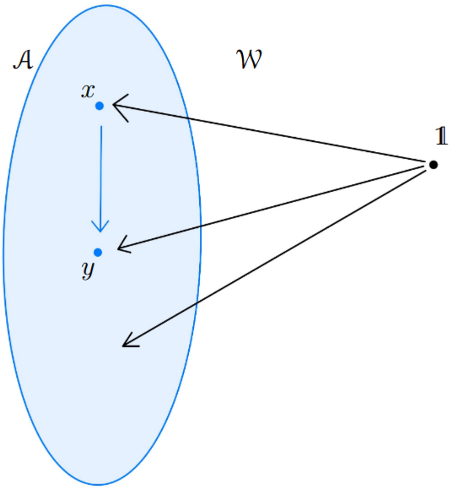

FIGURE 2: Special case for $\mathcal{B} = \mathbb{1}$: collage of a weight

Theorem

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a $\mathcal{V}$-functor and $W\colon \mathcal{A} \to \mathcal{V}$ a weight. The weighted limit $\lim(W, F)$ *exists if and only if the pointwise right Kan extension of F along the inclusion $i_{\mathcal{A}}^{W}\colon \mathcal{A} \hookrightarrow coll(W)$ exists. In this case it can be computed as*

$$\lim(W, F) \cong \mathrm{Ran}_{i_{\mathcal{A}}^{W}}(\bullet),$$

where $\bullet$ is the unique object of $\mathbb{1}$ in the collage. Dually, we can compute weighted colimits as left Kan extensions.

Definition

We define the *collage* $\mathrm{coll}(f, g)$ of the cospan $\mathcal{A} \xrightarrow{f} \mathcal{C} \xleftarrow{g} \mathcal{B}$ in $\mathcal{V}$-**Cat** as the $\mathcal{V}$-category with set of objects the coproduct $\mathrm{Ob}\,\mathcal{A} \sqcup \mathrm{Ob}\,\mathcal{B}$, hom-objects $\mathcal{A}(a, a')$ and $\mathcal{B}(b, b')$ between $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, $\mathcal{C}(fa, gb)$ from an object $a \in \mathcal{A}$ to an object $b \in \mathcal{B}$ and $\varnothing$ elsewhere.

In other words, $\mathrm{coll}(f, g) := \mathrm{coll}(\mathcal{C}(f-, g-))\colon \mathcal{B} \nrightarrow \mathcal{A}$

Kan extensions are pointwise

(HDer 4)

For every $f\colon \mathcal{A} \to \mathcal{B}$ in **Dia** and every object $b \in \mathcal{B}$ the diagrams

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, equal] & \mathrm{coll}(f, b) \arrow[d, "\pi_{\mathcal{B}}"] \\ \mathcal{A} \arrow[r, "f"'] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, equal] & \mathrm{coll}(b, f) \arrow[d, "\pi_{\mathcal{B}}"] \\ \mathcal{A} \arrow[r, "f"'] & \mathcal{B} \end{tikzcd}$$

are exact.

(Tentative) Definition

A 2-prederivator is called a *2-derivator* if it satisfies (HDer 1-4).

Axiom on strongness

Definition

A 2-functor is called *smothering* if it is surjective on objects, full on 1-cells and on 2-cells, and conservative on 2-cells.

Axiom on strongness

Definition

A 2-functor is called *smothering* if it is surjective on objects, full on 1-cells and on 2-cells, and conservative on 2-cells.

(HDer 5)

The following 2-functors are smothering for every $\mathcal{A} \in \mathbf{Dia}$:

1. $\mathrm{dia}_{\mathbf{Adj}} \colon \mathbb{D}^{\mathcal{A}}(\mathbf{Adj}) \to [\mathbf{Adj}, \mathbb{D}^{\mathcal{A}}(\mathbb{1})]_p$,
2. $\mathrm{dia}_{\mathbb{2}} \colon \mathbb{D}^{\mathcal{A}}(\mathbb{2}) \to [\mathbb{2}, \mathbb{D}^{\mathcal{A}}(\mathbb{1})]_p$,
3. $\mathrm{dia}_{\mathbb{I}} \colon \mathbb{D}^{\mathcal{A}}(\mathbb{I}) \to [\mathbb{I}, \mathbb{D}^{\mathcal{A}}(\mathbb{1})]_p$.

where **Adj** is the free living adjunction.

(Tentative) Definition

A 2-derivator is called *strong* if it satisfies (HDer 5).

Theorem (Di Vittorio)

Every combinatorial model category enriched in the category of simplicial sets equipped with the Joyal model structure gives rise to a 2-derivator.

N. Di Vittorio, **Towards 2-derivators for formal ∞-category theory**, Advances in Mathematics, Volume 485, February 2026, 110726.

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Theorem (Di Vittorio)

*If $\mathbb{D}$ is a 2-derivator so is $\mathbb{D}^I$, for every $I \in$ **Dia**.*

N. Di Vittorio, **Towards 2-derivators for formal ∞-category theory**, Advances in Mathematics, Volume 485, February 2026, 110726.

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- there is an obvious way to define a represented 2-derivator, but as it stands, it doesn't satisfy (HDer 2),
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Axioms:

- prove that $\mathbb{D}_{\mathcal{M}}$ is strong,
- generalize (HDer 4) to codiscrete cofibrations, reflecting that (Der 4) in derivator theory is equivalent to the base change axiom for Grothendieck (op)fibrations,
- understanding what additional axioms a 2-derivator should satisfy to be stable, and giving meaning to the concept of stability.

What is left to do?

Conjectures:

- The 2-prederivator $\mathbb{D}_{\mathcal{M}}$ can also be defined via the injective model structure. We conjecture this leads to a 2-prederivator equivalent to the one defined before.
- In derivator theory, Cisinski's result shows that the homotopy theory of simplicial sets can be recovered as the free completion of the point by homotopy colimits. A similar approach could be applied to 2-derivators to capture the category theory of ∞-categories.

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Medium to long term projects:

- study monadicity in the setting of 2-derivators,
- develop the theory of fibrations and fibred ∞-category theory in this setting,
- use 2-derivators to give a synthetic treatment of enriched ∞-category theory and higher algebra.