# Simulating a Shared Register in a System that Never Stops Changing\*

Hagit Attiya\* Hyun Chul Chung<sup>†</sup> Faith Ellen<sup>‡</sup> Saptaparni Kumar<sup>†</sup>

Jennifer L. Welch<sup>†</sup>

August 26, 2018

#### **Abstract**

Simulating a shared register can mask the intricacies of designing algorithms for asynchronous message-passing systems subject to crash failures, since it allows them to run algorithms designed for the simpler shared-memory model. Typically such simulations replicate the value of the register in multiple servers and require readers and writers to communicate with a majority of servers. The success of this approach for static systems, where the set of nodes (readers, writers, and servers) is fixed, has motivated several similar simulations for dynamic systems, where nodes may enter and leave. However, existing simulations need to assume that the system eventually stops changing for a long enough period or that the system size is bounded.

This paper presents the first simulation of an atomic read/write register in a crash-prone asynchronous system that can change size and withstand nodes continually entering and leaving. The simulation allows the system to keep changing, provided that the number of nodes entering and leaving during a fixed time interval is at most a constant fraction of the current system size. The simulation also tolerates node crashes as long as the number of failed nodes in the system is at most a constant fraction of the current system size.

## 1 Introduction

Simulating a shared read/write register is a way to mask the intricacies of designing algorithms for asynchronous message-passing systems subject to crash failures, since it allows them to run algorithms designed for the simpler shared-memory model. Typically, such simulations replicate the value of the register in multiple servers and require readers and writers to communicate with a majority of servers.

Most of the work in this area has focused on simulating atomic shared registers. For example, the ABD simulation [5] replicates the value of the register in server nodes. It assumes that a majority of the server nodes do not fail. Consider the simplified case of a single writer and a single reader. To write the value v, the writer sends v, tagged with a sequence number, to all servers and waits for acknowledgements from a

<sup>\*</sup>A preliminary version of this paper appeared in DISC 2015 [6]. This work is supported by the Israel Science Foundation (grants 1227/10 and 1749/14), by the Natural Science and Engineering Research Council of Canada, and by the US National Science Foundation grants 0964696 and 1526725.

<sup>&</sup>lt;sup>†</sup>Hyun Chul Chung is currently working at Epoch Labs, Inc. Austin, TX USA.

<sup>\*</sup>Department of Computer Science, Technion

<sup>&</sup>lt;sup>†</sup>Department of Computer Science and Engineering, Texas A&M University

<sup>&</sup>lt;sup>‡</sup>Department of Computer Science, University of Toronto

majority of them. Similarly, to read, the reader contacts all servers, waits to receive values from a majority of them, and then returns the value with the highest sequence number. This approach can be extended to the case of multiple writers and multiple readers by having each operation consist of a read phase, used by a writer to determine its sequence number and used by a reader to obtain the return value, followed by a write phase, used by a writer to disseminate the value (and sequence number) and used by a reader to announce the sequence number of the value it is about to return [12].

The success of this approach for static systems, where the set of readers, writers, and servers is fixed, has motivated several similar simulations for *dynamic* systems, where nodes may enter and leave. Change in system composition due to nodes entering and leaving is called *churn*. However, existing simulations of atomic registers rely either on the assumption that churn eventually stops for a long enough period (e.g., DynaStore [2] and RAMBO [13]) or on the assumption that the system size is bounded (e.g., [7]). See Section 5 for a detailed discussion of related work.

In this paper, we take a different approach: we allow churn to continue forever, while still ensuring that read and write operations complete and nodes can join and leave the system. Our churn model puts an upper bound on the number of nodes that can enter or leave during any time interval of a certain length. The upper bound is a constant fraction of the number of nodes that are present in the system at the beginning of the time interval. So, as the system size grows, the allowable number of changes to its composition grows as well. Similarly, as the system size shrinks, the allowable number of changes shrinks.

In more detail, our churn model relies on a parameter D of the system model, which is an upper bound, unknown to the nodes, on the delay of any message (between nodes that have not crashed). It is important to note that we set no lower bound on the delay of messages, so consensus cannot be solved in this model, even in the static case with no nodes entering or leaving and the possibility of one node crashing. We assume that, in any time interval of length D, the number of nodes that can enter or leave the system is at most a constant fraction,  $\alpha$ , of the number of nodes in the system at the beginning of the interval. The constant  $\alpha$  is known to all nodes. For example, if the churn rate is  $\alpha = 0.05$  and, at time t, the system contains 100 nodes, at most 5 nodes can enter or leave the system within the interval [t, t + D].

We believe ours is an appealing and a reasonable churn model. For instance, if each node has the same probability of leaving in a time interval, then the number of leaves is expected to be a fixed fraction of the total number of nodes. (See [11] for a discussion of churn behavior in practice.)

Our algorithm tolerates crash failures of nodes, as well as churn. In the preliminary version of this paper [6], we assumed that the number of nodes that crash is bounded above by a fixed constant, f, independent of the system size. Here, we only require that the number of crashed nodes is always at most a constant fraction,  $\Delta$ , of the system size. The constant  $\Delta$  is known to all nodes. As a consequence, if the number of crashed nodes at some point in an execution is  $\Delta$  times the system size, then no nodes can leave. To make the model more dynamic, we allow, but do not require, the adversary (who is responsible for crashing nodes) to notify a node, p, which has not crashed, that a crashed node, q, has left. Then p sends a leave message on behalf of node q (analogously to [2] and [13]). Such forced leaves are counted as part of the churn.

Our algorithm, called CCREG (for *Continuous Churn Register*), is intuitive, combining the simple static algorithm for multiple readers and multiple writers outlined above with a joining protocol and careful estimations of the number of nodes from which responses should be received for joining, reading, and writing. In order to join, a newly entered node announces its entry, waits to receive sufficiently many acknowledgements, and then announces it has joined. Once a node has joined, it can perform reads and writes. A node leaves the system by announcing its departure. Each node maintains a set of changes to the composition of the system, based on the announcements of nodes entering, joining and leaving. This information is also propagated through appropriate echo messages and by having each node append the set of changes it has

seen to its messages that echo enter announcements.

When a node first receives an acknowledgement of its entrance announcement from a node that has already joined, it calculates the number of acknowledgements it needs to join as a fraction (depending on  $\alpha$  and  $\Delta$ , which it knows) of the number of nodes it believes are in the system. To ensure that information about the system composition is propagated properly, it is crucial that before a node joins, it gets at least one acknowledgement from a node whose information is up to date. This is ensured by requiring that that the number of acknowledgements it receives before joining is sufficiently large, so that at least one of them is from a node that has been in the system for sufficiently long. The number of necessary acknowledgements must also be small enough to ensure that the node will eventually receive enough acknowledgements.

Each reader and writer keeps track of the number of nodes that have joined, but not left. We call these *members*. The read and write phases of operations wait for responses from a constant fraction of the nodes believed to be members. As in the joining protocol, the number of responses must be small enough so that termination is guaranteed. To prove CCREG is atomic, we consider two cases: If a read occurs shortly after a write, then we must ensure that the sets of nodes contacted by the two operations are intersecting. This is analogous to the situation in the static, majority algorithm. If operations are farther apart in time, then, as in the join protocol, we ensure that information about writes to the register is propagated properly.

Our churn model has the pleasing property that it is algorithm-independent: It only refers to nodes that enter or leave irrespective of whether an entered node completes the join protocol.

## 2 Model

We consider an asynchronous message-passing system, with nodes running *client* threads (*reader* threads or *writer* threads) and *server* threads. Nodes do not have clocks, so they cannot determine the current time, nor directly measure how much time has elapsed since some event. Each node runs exactly one server thread, at most one reader thread, and at most one writer thread.

Nodes can enter and leave the system during an execution. We model this behavior by assuming the existence of an adversary that generates  $\mathsf{ENTER}(p)$  and  $\mathsf{LEAVE}(p)$  signals to indicate that p should enter or leave; only p experiences these signals (with one possible exception explained below). For each node p, there is at most one  $\mathsf{ENTER}(p)$  signal and at most one  $\mathsf{LEAVE}(p)$  signal. Thus a node that leaves the system cannot re-enter the system. (This restriction is easy to remove by giving a new name to a node that wants to re-enter.)

We say that a node is *present* at time t if it has entered the system (i.e., an  $\mathsf{ENTER}(p)$  signal has occurred) but has not left by time t (i.e., no  $\mathsf{LEAVE}(p)$  signal has occurred so far). We let N(t) denote the number of servers whose nodes are present at time t; N(t) is called the *system size*. We assume that there are always at least  $N_{min}$  servers whose nodes are present in the system, i.e., at all times t,  $N(t) \geq N_{min}$ . Let  $S_0$  denote the set of nodes that are present initially, i.e., at time 0. Note that  $|S_0| = N(0)$ .

Nodes are subject to crash failures. The adversary generates a CRASH(p) signal at node p to indicate that p has crashed. There can be at most one CRASH(p) signal for each node p. A crashed node does not take any more steps, does not send any more messages, and no more messages are delivered to it. We say that a node is *active* at time t if it is present at time t and has not crashed by time t.

Nodes communicate through a broadcast service that provides a mechanism to send the same message to all nodes in the system. If a server wants to send a message to a single client, it can do so by broadcasting the message and indicating that the message should be ignored by the other clients. Message delays are bounded above by a system parameter, D, that is unknown to the nodes. In more detail, a message that is broadcast by a node p at time t is guaranteed to arrive at each node  $q \neq p$  within D units of time, provided

that q is active throughout the interval [t,t+D]. If q is active for some, but not all, of [t,t+D], then q might or might not receive the message. Nodes that enter after time t+D do not (directly) receive the message. A message can take a different amount of time to reach different nodes. All messages broadcast by p are received by q in the same order in which p sent them. In addition to the maximum transmission delay, p includes the maximum time for handling the message at both the sender and the receiver. There is no lower bound on the actual length of time it takes for a message to be transmitted, nor on the amount of time to perform local computation at a node (i.e., they could take an arbitrarily small amount of time). An execution in which all messages satisfy these constraints is called p

Since there is no bound on the ratio between the fastest and slowest messages, the system is essentially asynchronous and consensus cannot be solved in our system model. In fact, any problem that can be solved in our model, even if the bound D is known by all nodes, can be solved in the standard asynchronous message passing model, where there is no upper bound on message delivery time. To see why, consider an algorithm  $\mathbb A$  that is designed to work when there is a known upper bound D on message delivery time. Now consider any execution e of this algorithm in the standard asynchronous message passing model. We compress it into an execution e' by changing the real time of the occurrence of the  $i^{th}$  event to be  $1-2^{-i}$ . Then every message that is received by a node in e is received within time D=1 of when it was sent in e'. Moreover, if, in the original execution, messages are received along a link in the order they were sent, then the same is true in this timed execution. Since e' is a valid execution of  $\mathbb A$  in our model with D=1, e is a valid execution of  $\mathbb A$ , with each node having the same local execution history in both executions.

We assume the set of nodes that are present does not change too quickly: There is a constant  $\alpha < 1$ , known to all nodes, called the *churn rate*, such that for all times t, at most  $\alpha \cdot N(t)$  nodes enter or leave (i.e, experience ENTER or LEAVE signals) during the interval [t, t+D]. This is a constraint on the adversary.

There is also a constant  $\Delta < 1$ , known to all nodes, called the *failure fraction*, such that, at any time t, at most  $\Delta \cdot N(t)$  of the nodes present at time t have crashed (i.e., have experienced a CRASH signal previously). This is another constraint on the adversary. If  $\Delta \cdot N(t)$  nodes are crashed at time t, then active nodes cannot leave the system.

At any time, the adversary may choose to let some of the crashed nodes leave the system. Since a crashed node p cannot take any actions on its own behalf, this is accomplished by the adversary generating a LEAVE(p) signal at an active node q. We call these forced leaves and they contribute to the churn quota. That is, for any time t, the sum of the number of ENTER and LEAVE signals (including a LEAVE(p) signal that occurs at a node other than p) that happen in [t,t+D] is at most  $\alpha \cdot N(t)$ . This mechanism works even when the adversary generates a LEAVE(p) signal at more than one active nodes. However all LEAVE(p) signals for a crashed node p contribute to only one forced leave. It is important to note that forcing crashed nodes to leave is entirely optional for the adversary. However, letting the adversary have this ability allows active nodes to leave the system or more crashes to be accommodated if  $\Delta \cdot N(t)$  nodes are crashed at time t, even if no more nodes enter after time t.

In this model, we want to simulate a multi-reader multi-writer atomic read-write register. A user application at each node determines when read and write operations are invoked at the node, subject to two constraints. The first constraint is that no read or write is invoked on a node unless the node is active and it is ready to accept operations. The node indicates that it is ready by broadcasting a *joined* message. The second constraint is that the user does not invoke a read or write operation at a node if there is a previous read or write operation that it has invoked but is not yet completed.

To sum up, the adversary determines when nodes enter, leave, and crash, the network (which is controlled by the adversary) determines when messages are delivered, and the users (which are also controlled by

<sup>&</sup>lt;sup>1</sup>This mechanism could reflect the output of some failure detection method that is outside the scope of our paper.

the adversary) determine when reads and writes are invoked, subject to the constraints discussed above. The algorithm running at the nodes is responsible for joining and generating responses to read and write invocations.

We consider an algorithm to be *correct* if every execution of the algorithm in the model just described satisfies the following conditions:

- Every active process that does not leave or crash eventually joins.
- Every read or write that is invoked at a process that does not leave or crash eventually completes.
- The read and write operations satisfy atomicity: there is an ordering of all completed reads and writes and some subset of the uncompleted writes such that every read returns the value of the latest preceding write and, if an operation  $op_1$  finishes before another operation  $op_2$  begins, then  $op_1$  is ordered before  $op_2$ .

# 3 The CCREG Algorithm

The algorithm combines a mechanism for tracking the composition of the system, with a simple algorithm, very similar to [12], for reading and writing the register, which associates a unique timestamp with each value that is written.

In order to track the composition of the system (Algorithm 1), each node p maintains a set of events,  $Changes_p$ , concerning the nodes that have entered the system. When a node q enters, it adds enter(q) to  $Changes_q$  and broadcasts an enter message requesting information about prior events. We say that q enters or the enter(q) event occurs at the time,  $t_q^e$ , when this broadcast is sent. When a node p finds out that q has entered the system, either by receiving this message or by learning indirectly from another node, it adds enter(q) to  $Changes_p$ . When q has received sufficiently many messages in response to its request, it knows relatively accurate information about prior events and the value of the register. (Setting the join bound, denoted by  $\gamma$ , on the number of messages that should be received is a key challenge in the algorithm.) When this happens, q adds join(q) to  $Changes_q$ , sets its  $is\_joined_q$  flag to true, and broadcasts a message saying that it has joined. We say that q joins or the join(q) event occurs at the time,  $t_q^j$ , when this broadcast is sent. When p finds out that q has joined, either by receiving this message or by learning indirectly from another node, it adds join(q) to  $Changes_p$ . When q leaves, it simply broadcasts a leave message or another node broadcasts a leave message on q's behalf. We say that the leave(q) event occurs at the time,  $t_q^j$ , when this broadcast is sent. When p finds out that q has left the system, either by receiving this message or by learning indirectly from another node, it adds leave(q) to leave

When a node p receives an enter message from a node q, it responds with an enter-echo message containing  $Changes_p$ , its current estimate of the register value (together with its timestamp),  $is\_joined_p$  (indicating whether p has joined yet), and q. When q receives an enter-echo in response (i.e., that ends with q), it increments its join-counter. The first time q receives such an enter-echo from a joined node, it computes  $join\_bound$ , the number of enter-echo messages it needs in response before it can join.

Once a node has joined, its reader and writer threads can handle read and write operations. A node is a *member* at time t if it has joined, but not left, by time t. Initially,  $Changes_p = \{enter(q) \mid q \in S_0\} \cup \{join(q) \mid q \in S_0\}$ , if  $p \in S_0$ , and  $\emptyset$  otherwise. A node p also maintains the set  $Present_p = \{q \mid enter(q) \in Changes_p \land leave(q) \notin Changes_p\}$  of nodes that p thinks are present, i.e., nodes that have entered, but have not left, as far as p knows. The client at node p maintains the derived variable  $Members_p = \{q \mid join(q) \in Changes_p \land leave(q) \notin Changes_p\}$  of nodes that p thinks are members.

## **Algorithm 1** CCREG—Common code managing the *Changes* variable, for node p.

#### **Local Variables:**

```
is\_joined // Boolean to check if p has joined the system; initially false join\_counter // for counting the number of enter-echo messages received by p; initially 0 join\_bound // if non-zero, the number of enter-echo p should receive before joining; initially 0 changes // set of enter, enter, enter, and enter initially enter (enter) | enter) | enter (enter) | enter) | enter (enter) | enter) | enter0 | enter1 | enter2 | enter3 | enter3 | enter4 | enter4 | enter5 | enter5 | enter6 | enter6 | enter6 | enter6 | enter7 | enter7 | enter8 | enter9 | enter
```

#### **Derived Variable:**

 $Present = \{q \mid enter(q) \in Changes \land leave(q) \not\in Changes\}$ 

```
When \langle "joined", q \rangle is received:
    When p receives Enter(p) signal:
 1: add enter(p) to Changes
                                                              16: add join(q) to Changes
2: bcast ("enter", p)
                                                              17: add enter(q) to Changes
                                                              18: bcast ("joined-echo", q)
    When \langle "enter", q \rangle is received:
3: add enter(q) to Changes
                                                                  When \langle "joined-echo", q \rangle is received:
4: bcast ("enter-echo", Changes,
                                                              19: add join(q) to Changes
                                                              20: add enter(q) to Changes
      (val, seq, id), is\_joined, q
    When \langle "enter-echo", C, (v, s, i), j, q \rangle
                                                                  When p receives Leave(q) signal:
                                                              21: bcast ("leave", q)
      is received:
5: if (s, i) > (seq, id) then
                                                              22: if p = q then
       (val, seq, id) := (v, s, i)
                                                              23:
                                                                     halt
7: Changes := Changes \cup C
8: if \neg is\_joined \land (p = q) then
                                                                  When \langle "leave", q \rangle is received:
      if (j = true) \land (join\_bound = 0) then
                                                              24: add leave(q) to Changes
9:
          join\_bound := \gamma \cdot |Present|
                                                              25: bcast ("leave-echo", q)
10:
       join\_counter++
11:
12:
      if join\_counter \ge join\_bound > 0 then
                                                                  When \langle"leave-echo", q \rangle is received:
                                                              26: add leave(q) to Changes
13:
          is\_joined := true
14:
          add join(p) to Changes
         bcast ("joined", p)
15:
```

The client thread treats read and write operations in a similar manner (Algorithm 2). We assume that the code segment that is executed in response to each event executes without interruption. Both operations start with a read phase, which requests the current value of the register, using a query message, followed by a write phase, using an update message. A write operation broadcasts the new value it wishes to write, together with a timestamp, which consists of a sequence number that is one larger than the largest sequence number it has seen and its id that used to break ties. A read operation just broadcasts the value it is about to return, keeping its sequence number. As in [5], write-back is needed to ensure the atomicity of read operations. Both the read phase and the write phase wait to receive sufficiently many response messages. (Again, setting the *quorum bound*, denoted  $\beta$ , on the number of messages that should be received is a key challenge in the algorithm.)

## $\overline{\textbf{Algorithm 2}}$ CCREG—Client code, for node p.

#### **Local Variables:**

temp // temporary storage for the value being read or written; initially 0 tag // used to uniquely identify read and write phases of an operation; initially 0  $quorum\_size$  // stores the quorum size for a read or write phase; initially 0  $heard\_from$  // the number of responses/acks received for a read/write phase; initially 0  $rp\_pending$  // Boolean indicating whether a read phase is in progress; initially false  $wp\_pending$  // Boolean indicating whether a write phase is in progress; initially false  $read\_pending$  // Boolean indicating whether a read is in progress; initially false  $write\_pending$  // Boolean indicating whether a write is in progress; initially false

#### **Derived Variable:**

 $Members = \{q \mid join(q) \in Changes \land leave(q) \notin Changes\}$ 

```
When READ is invoked:
                                                            Procedure BeginWritePhase()
30: read\_pending := true
                                                        47: if write_pending then
31: call BeginReadPhase()
                                                              val := temp
                                                        49:
                                                              seq++
    When WRITE(v) is invoked:
                                                        50:
                                                              id := p
32: write\_pending := true
                                                        51: if read_pending then
                                                              temp := val
33: temp := v
34: call BeginReadPhase()
                                                        53: bcast ("update", (temp, seq, id), tag, p)
                                                        54: quorum\_size := \beta |Members|
    Procedure BeginReadPhase()
                                                        55: heard\_from := 0
35: tag++
                                                        56: wp\_pending := true
36: bcast ("query", tag, p)
37: quorum\_size := \beta |Members|
                                                            When \langle "ack", wt, q \rangle is received:
38: heard\_from := 0
                                                        57: if wp\_pending \land (wt = tag) \land (g = p) then
39: rp\_pending := true
                                                        58:
                                                              heard\_from++
                                                        59:
                                                              if heard\_from \ge quorum\_size then
    When ("response", (v, s, i), rt, q)
                                                                 wp\_pending := false
                                                        60:
     is received:
                                                                 if read_pending then
                                                        61:
                                                                    read\_pending := false
40: if rp\_pending \land (rt = tag) \land (g = p) then
                                                        62:
      if (s,i) > (seq,id) then
                                                        63:
                                                                    RETURN temp
         (val, seq, id) := (v, s, i)
                                                                 if write_pending then
42:
                                                        64:
43:
      heard\_from++
                                                        65:
                                                                    write\_pending := false
      if heard\_from \ge quorum\_size then
                                                                    ACK
44:
                                                        66:
45:
         rp\_pending := false
46:
         call BeginWritePhase()
```

A client p maintains a sequence number, tag, which it increments at the beginning of each read phase. This is used to identify responses belonging to its current read or write phase.

The server thread is simple (Algorithm 3). The nodes uses the variables val, seq, and id to store the latest value of the register it knows about (in val) and that value's associated timestamp (in seq and id). When the server receives an update message with a larger timestamp, it updates the value and the timestamp. (Note that timestamps, which consist of (seq, id) pairs, are ordered lexicographically.) When a server receives a query, it responds with the value and its timestamp.

## **Algorithm 3** CCREG—Server code, for node p.

When 
$$\langle$$
 "update",  $(v, s, i), wt, q \rangle$  is received:

70: **if** 
$$(s, i) > (seq, id)$$
 **then**

71: 
$$(val, seq, id) := (v, s, i)$$

72: **if** *is\_joined* **then** 

73: bcast $\langle$ "ack",  $wt, q \rangle$ 

74: bcast  $\langle$  "update-echo",  $(val, seq, id) \rangle$ 

## When $\langle$ "query", $rt, q \rangle$ is received:

75: **if**  $is\_joined$  **then** 

76: bcast
$$\langle$$
"response",  $(val, seq, id), rt, q \rangle$ 

When  $\langle$  "update-echo",  $(v, s, i) \rangle$  is received:

77: **if** 
$$(s, i) > (seq, id)$$
 **then**

78: 
$$(val, seq, id) := (v, s, i)$$

	system parai	derived algorithm parameters		
churn rate $(\alpha)$	failure fraction ( $\Delta$ )	minimum system size $(N_{min})$	join bound $(\gamma)$	quorum size $(\beta)$
0	0.33	N/A	N/A	0.665
0.01	0.26	7	0.67	0.684
0.04	0.06	9	0.72	0.737

Table 1: Sets of values for which the assumptions on system parameters  $(\alpha, \Delta, N_{min})$  and derived algorithm parameters  $(\gamma, \beta)$  are satisfied.

The correctness of CCREG relies on the following assumption about the churn rate

$$\alpha < 1 - 2^{-1/4} \approx 0.159$$
 (A)

and the following relation between the system parameters  $\alpha$ ,  $\Delta$  and  $N_{min}$ :

$$1 < \left( \left( 1 - \alpha \right)^3 - \Delta \left( 1 + \alpha \right)^3 \right) N_{min} \tag{B}$$

Note that Assumption (A) restricts the churn rate  $\alpha$  during an interval of length D to less than 16%. We believe this is reasonable, since D, the maximum message delay, will typically be quite small.

The algorithm parameters  $\gamma$  and  $\beta$  must satisfy the following constraints:

$$\gamma \ge \frac{1}{N_{min}(1-\alpha)^3} + (1+\Delta)\frac{(1+\alpha)^3}{(1-\alpha)^3} - 1 \tag{C}$$

$$\gamma \le \frac{(1-\alpha)^3}{(1+\alpha)^3} - \Delta \tag{D}$$

$$\beta \le \frac{(1-\alpha)^3}{(1+\alpha)^2} - \Delta(1+\alpha) \tag{E}$$

$$\beta > \frac{(1+\alpha)^5 - 1}{(1-\alpha)^4} \tag{F}$$

$$\beta > \frac{(1+\Delta)(1+\alpha)^3 - (1-\alpha)^3 + 1}{(2-2\alpha+\alpha^2)(1-\alpha)^2(1+\alpha)^{-2}}$$
 (G)

Table 1 gives a few sets of values for which the above assumptions are satisfied.

## 4 Correctness Proof

We will show that CCREG satisfies the three properties listed at the end of Section 2. Lemmas 1 through 8 are used to prove Theorem 9, which states that every node eventually joins, provided it does not crash or leave. Lemmas 11 through 13 are used to prove Theorem 14, which states that every operation invoked by a node that remains active eventually completes. Lemmas 15 through 18 are used to prove Theorem 21, which states that atomicity is satisfied.

Consider any execution. We use  $Changes_p^t$ ,  $Present_p^t$ , and  $Members_p^t$  to denote the sets  $Changes_p$ ,  $Present_p$ , and  $Members_p$ , respectively, at time t of the execution. We begin by bounding the number of nodes that enter during an interval of time and the number of nodes that are present at the end of the interval, as compared to the number present at the beginning.

**Lemma 1.** For all  $i \in \mathbb{N}$  and all  $t \geq 0$ , at most  $((1 + \alpha)^i - 1)N(t)$  nodes enter during (t, t + Di] and  $(1 - \alpha)^i N(t) \leq N(t + Di) \leq (1 + \alpha)^i N(t)$ .

*Proof.* The proof is by induction on i. For i=0 and all  $t\geq 0$ , (t,t+Di] is empty, and hence,  $0=((1+\alpha)^i-1)N(t)$  nodes enter during this interval and

$$N(t + iD) = N(t) = (1 + \alpha)^{i} N(t) = (1 - \alpha)^{i} N(t).$$

Now let  $i \geq 0$  and  $t \geq 0$ . Suppose at most  $((1 + \alpha)^i - 1)N(t)$  nodes enter during (t, t + Di] and  $(1 - \alpha)^i N(t) \leq N(t + Di) \leq (1 + \alpha)^i N(t)$ .

Let  $e \ge 0$  and  $\ell \ge 0$  be the number of nodes that enter and leave, respectively, during (t+Di, t+D(i+1)]. By the churn assumption,  $e+\ell \le \alpha N(t+Di)$ , so  $e,\ell \le \alpha N(t+Di) \le \alpha (1+\alpha)^i N(t)$ . The number of nodes that enter during (t,t+D(i+1)] is at most

$$((1+\alpha)^{i}-1)N(t)+e \le ((1+\alpha)^{i}-1)N(t)+\alpha(1+\alpha)^{i}N(t)=((1+\alpha)^{i+1}-1)N(t).$$

Hence,

$$N(t + D(i + 1)) \le N(t) + ((1 + \alpha)^{i+1} - 1)N(t) = (1 + \alpha)^{i+1}N(t).$$

Furthermore,

$$N(t + D(i + 1)) \ge N(t + Di) - \ell \ge N(t + Di) - \alpha N(t + Di) = (1 - \alpha)N(t + Di) \ge (1 - \alpha)^{i+1}N(t).$$

By induction, the claim is true for all  $i \in \mathbb{N}$ .

We are also interested in the number of nodes that leave during an interval of time. In the proof of the next lemma, the calculation of the maximum number of nodes that leave during an interval is complicated by the possibility of nodes entering during the interval, allowing additional nodes to leave.

**Lemma 2.** For  $\alpha > 0$ , all nonegative integers  $i \le -1/\log_2(1-\alpha)$  and all  $t \ge 0$ , at most  $(1-(1-\alpha)^i)N(t)$  nodes leave during (t,t+Di].

*Proof.* The proof is by induction on i. When i=0, the interval is empty, so  $0=(1-(1-\alpha)^0)N(t)$  nodes leave during the interval. Now let  $i\geq 0$ , let  $t\geq 0$ , and suppose at most  $(1-(1-\alpha)^i)N(t+D)$  nodes leave during (t+D,t+D(i+1)].

Let  $e \ge 0$  and  $\ell \ge 0$  be the number of nodes that enter and leave, respectively, during (t, t+D]. By the churn assumption,  $e+\ell \le \alpha N(t)$ , so  $\ell \le \alpha N(t)$  and  $N(t+D)=N(t)+e-\ell=N(t)+(\ell+e)-2\ell \le 0$ 

 $(1 + \alpha)N(t) - 2\ell$ . The number of nodes that leave during (t, t + D(i + 1)] is the number that leave during (t, t + D) plus the number that leave during (t + D, t + D(i + 1)], which is at most

$$\begin{array}{lcl} \ell + (1 - (1 - \alpha)^{i})N(t + D) & \leq & \ell + (1 - (1 - \alpha)^{i})[(1 + \alpha)N(t) - 2\ell] \\ & = & (1 - (1 - \alpha)^{i})(1 + \alpha)N(t) + (2(1 - \alpha)^{i} - 1)\ell \\ & \leq & (1 - (1 - \alpha)^{i})(1 + \alpha)N(t) + (2(1 - \alpha)^{i} - 1)\alpha N(t) \\ & = & (1 - (1 - \alpha)^{i+1})N(t). \end{array}$$

Note that  $2(1-\alpha)^i-1\geq 0$ , since  $i\leq -1/\log_2(1-\alpha)$ . By induction, the claim is true for all  $i\in\mathbb{N}$ .

Recall that a node is *active* at time t if it has entered by time t, but has not left or crashed by time t. The next lemma shows that some node remains active throughout any interval of length 3D.

**Lemma 3.** For every t > 0, at least one node is active throughout  $[\max\{0, t - 2D\}, t + D]$ .

*Proof.* Let S be the set of nodes present at time  $t' = \max\{0, t - 2D\}$ , so  $|S| = N(t') \ge N_{min}$ . By Lemma 1, at most  $((1+\alpha)^3-1)|S|$  nodes enter during (t',t+D], so there are at most  $(1+\alpha)^3|S|$  nodes present at time t+D and at most  $\Delta(1+\alpha)^3|S|$  nodes have crashed by time t+D. Assumption (A) implies that  $-1/\log_2(1-\alpha) \ge 4 \ge 3$ . So, by Lemma 2, at most  $(1-(1-\alpha)^3)|S|$  nodes leave during (t',t+D] and there are at least  $(1-\alpha)^3|S|$  nodes present at time t+D. Thus, at least

$$((1-\alpha)^3 - \Delta(1+\alpha)^3)|S| \ge ((1-\alpha)^3 - \Delta(1+\alpha)^3)N_{min}$$
(H)

nodes in S are active at time t+D. By Assumption (B),  $\left((1-\alpha)^3-\Delta(1+\alpha)^3\right)N_{min}>1$ , so at least one node in S is still active at time t+D.

We define the set of all enter, join, and leave events that occur during time interval I to be

$$\mathit{SysInfo}^I = \{enter(q) \mid t^e_q \in I\} \cup \{join(q) \mid t^j_q \in I\} \cup \{leave(q) \mid t^\ell_q \in I\}.$$

In particular,  $SysInfo^{[0,0]} = \{enter(q) \mid q \in S_0\} \cup \{join(q) \mid q \in S_0\}.$ 

Since a node p that is active throughout  $[t_p^e, t + D]$  directly receives all enter, joined, and leave messages broadcast during  $[t_p^e, t]$ , within D time, we have:

**Observation 4.** For every node p and all times  $t \ge t_p^e$ , if p is active at time t + D, then  $SysInfo^{[t_p^e,t]} \subseteq Changes_p^{t+D}$ .

By assumption, for every node  $p \in S_0$ ,  $SysInfo^{[0,0]} \subseteq Changes_p^0$ , and hence Observation 4 implies:

**Observation 5.** For every node  $p \in S_0$ , if p is active at time  $t \ge 0$ , then  $SysInfo^{[0,\max\{0,t-D\}]} \subseteq Changes_p^t$ .

The purpose of Lemmas 6, 7, and 8 is to show that information about nodes entering, joining, and leaving is propagated properly, via the *Changes* sets.

**Lemma 6.** Suppose that, at time T'', a node  $p \notin S_0$  receives an enter-echo message from a node Q sent at time T' in response to an enter message from p. Let T be any time such that  $\max\{0,T''-2D\} \le T \le t_p^e$ . Suppose p is active at time T+2D and Q is active throughout [U,T+D], where  $U \le \max\{0,T''-2D\}$ . Then  $SysInfo^{(U,T)} \subseteq Changes_p^{T+2D}$ .

*Proof.* Consider any node r that enters, joins, or leaves at time  $\hat{t} \in (U,T]$ . Note that Q directly receives the announcement of this event, since Q is active throughout (U,T+D], which contains  $[\hat{t},\hat{t}+D]$ , the maximum interval during which the announcement message is in transit. We consider two cases, depending on the time, v, at which Q receives this message.

Case 1:  $v \leq T'$ . Since Q receives the enter message from p at T', information about this change to r is in  $Changes_Q^{T'}$ , which is part of the enter-echo message that Q sends to p at time T'. Thus, this information is in  $Changes_p^{T''} \subseteq Changes_p^{T+2D}$ .

Case 2: v > T'. Messages are not received before they are sent, so  $T' \ge t_p^e$ . Since  $v \le \hat{t} + D$ , it follows that  $v + D \le \hat{t} + 2D \le T + 2D$ . Thus [v, v + D] is contained in  $[t_p^e, T + 2D]$ . Immediately after receiving the announcement about r, node Q broadcasts an echo message in response. Since p is active throughout this interval, it directly receives this echo message.

In both cases, the information about r's change reaches p by time T+2D. it follows that  $SysInfo^{(U,T]} \subseteq Changes_p^{T+2D}$ .

**Lemma 7.** For every node p, if p is active at time  $t \geq t_p^e + 2D$ , then  $SysInfo^{[0,t-D]} \subseteq Changes_p^t$ .

*Proof.* The proof is by induction on the order in which nodes enter the system. If  $p \in S_0$ , then  $t_p^e = 0$ , so  $SysInfo^{[0,t-D]} \subseteq Changes_p^t$  follows from Observation 5.

Now consider any node  $p \notin S_0$  and suppose that the claim is true for all nodes that enter earlier than p. Suppose p is active at time  $t \ge t_p^e + 2D$ . By Lemma 3, there is at least one node q that is active throughout  $[\max\{0, t_p^e - 2D\}, t_p^e + D]$ . Node q receives an enter message from p at some time  $t' \in [t_p^e, t_p^e + D]$  and sends an enter-echo message back to p. This message is received by p at some time  $t'' \in [t', t' + D]$ .

sends an enter-echo message back to p. This message is received by p at some time  $t'' \in [t', t' + D]$ . If  $q \in S_0$ , then  $SysInfo^{[0,\max\{0,t'-D\}]} \subseteq Changes_q^{t'}$ , by Observation 5. If  $q \notin S_0$ , then  $0 < t_q^e \le \max\{0, t_p^e - 2D\}$ , so  $t_q^e \le t_p^e - 2D$ . Therefore  $t_q^e + 2D \le t_p^e \le t'$ . Since q entered earlier than p, it follows from the induction hypothesis that  $SysInfo^{[0,t'-D]} \subseteq Changes_q^{t'}$ . Thus, in both cases,  $SysInfo^{[0,\max\{0,t'-D\}]} \subseteq Changes_q^{t'}$ . At time  $t'' \le t$ , p receives the enter-echo message from q, so  $SysInfo^{[0,\max\{0,t'-D\}]} \subseteq Changes_p^{t'}$ .  $\subseteq Changes_p^{t'}$ .

Applying Lemma 6 with  $Q=q,\ U=\max\{0,t_p^e-D\},\ T=t_p^e,\ T'=t'$  and T''=t'' gives  $SysInfo^{(\max\{0,t'-D\},t_p^e]}\subseteq Changes_p^{t_p^e+2D}$ . Since  $t\geq t_p^e+2D$ ,  $Changes_p^{t_p^e+2D}$  is a subset of  $Changes_p^t$ . Observation 4 implies  $SysInfo^{[t_p^e,t-D]}\subseteq Changes_p^t$ . Hence,  $SysInfo^{[0,t-D]}\subseteq Changes_p^t$ .

**Lemma 8.** For every node  $p \notin S_0$ , if p joins at time  $t_p^j$  and is active at time  $t \ge t_p^j$ , then  $SysInfo^{[0,\max\{0,t-2D\}]} \subseteq Changes_p^t$ .

*Proof.* The proof is by induction on the order in which nodes join the system. Let  $p \notin S_0$  be a node that joins at time  $t_p^j \le t$  and suppose the claim holds for all nodes that join before p. If  $t \ge t_p^e + 2D$ , then the claim follows by Lemma 7. So, assume that  $t < t_p^e + 2D$ .

Before p joins, it receives an enter-echo message from a joined node in response to its enter message. Suppose p first receives such an enter-echo message at time t'' and this enter-echo was sent by q at time t'. Then  $t_p^e \le t' \le t'' \le t_p^j$ . If  $q \in S_0$ , then Observation 5 implies that  $SysInfo^{[0,\max\{0,t'-D\}]} \subseteq Changes_q^{t'}$ . Otherwise, by the induction hypothesis,  $SysInfo^{[0,\max\{0,t'-2D\}]} \subseteq Changes_q^{t'}$ , since q joined prior to p and is active at time  $t' \ge t_q^j$ . Note that  $Changes_q^{t'} \subseteq Changes_p^{t''} \subseteq Changes_p^{t''}$ . If  $t \le 2D$ , then  $\max\{0,t-2D\}=0$  and the claim is true. So, assume that t > 2D.

Let S be the set of nodes present at time  $\max\{0,t'-2D\}$ , so  $|S|=N(\max\{0,t'-2D\})$ . By Lemma 2 and Assumption (A), at most  $(1-(1-\alpha)^3)|S|$  nodes leave during  $(\max\{0,t'-2D\},t'+D]$ . Since  $t'' \leq t'+D$ , it follows that  $|Present_p^{t''}| \geq |S|-(1-(1-\alpha)^3)|S|=(1-\alpha)^3|S|$ . Hence, from lines 10 and 12 of Algorithm 1, p waits until it has received at least  $join\_bound = \gamma \cdot |Present_p^{t''}| \geq \gamma \cdot (1-\alpha)^3|S|$  enter-echo messages before joining.

By Lemma 1, the number of nodes that enter during  $(\max\{0,t'-2D\},t'+D]$  is at most  $((1+\alpha)^3-1)|S|$ . Thus, at time t'+D, there are at most  $(1+\alpha)^3|S|$  nodes present and at most  $\Delta(1+\alpha)^3|S|$  nodes are crashed. Hence, the number of enter-echo messages p receives before joining from nodes that were active throughout  $[\max\{0,t'-2D\},t'+D]$  is  $join\_bound$  minus the total number of enters, leaves and crashes, which is at least

$$\gamma \cdot (1 - \alpha)^{3} |S| - [((1 + \alpha)^{3} - 1)|S| + (1 - (1 - \alpha)^{3})|S| + \Delta(1 + \alpha)^{3} |S|]$$

$$= [(1 + \gamma)(1 - \alpha)^{3} - (1 + \Delta)(1 + \alpha)^{3}]|S|$$

$$\geq [(1 + \gamma)(1 - \alpha)^{3} - (1 + \Delta)(1 + \alpha)^{3}]N_{min}. \quad (I)$$

Rearranging Assumption (C), we get  $[(1+\gamma)(1-\alpha)^3-(1+\Delta)(1+\alpha)^3N_{min}]\geq 1$ , so expression (J) is at least 1. Hence p receives an enter-echo message at some time  $T''\leq t_p^j$  from a node q' that is active throughout  $[\max\{0,t'-2D\},t'+D]\supseteq [\max\{0,t'-2D\},t-D]$ . Let T' be the time that q' sent its enter-echo message in response to the enter message from p. Applying Lemma 6 with Q=q',  $U=\max\{0,t'-2D\}$ , and T=t-2D gives  $SysInfo^{(\max\{0,t'-2D\},t-2D]}\subseteq Changes_p^t$ . Thus  $SysInfo^{[0,t-2D]}=SysInfo^{[0,\max\{0,t'-2D\}]}\cup SysInfo^{(\max\{0,t'-2D\},t-2D]}\subseteq Changes_p^t$ .

Next we prove that every node that remains active sufficiently long after it enters succeeds in joining.

**Theorem 9.** Every node  $p \notin S_0$  that is active at time  $t_p^e + 2D$  joins by time  $t_p^e + 2D$ .

*Proof.* The proof is by induction on the order in which nodes enter the system. Let  $p \notin S_0$  be a node that enters at time  $t_p^e$  and is active at time  $t_p^e + 2D$ . Suppose the claim is true for all nodes that enter before p.

By Lemma 3, there is a node q that is active throughout  $[\max\{t_p^e-2D,0\},t_p^e+D]$ . If  $q\in S_0$ , then q joins at time 0. If not, then  $t_q^e\leq t_p^e-2D$ , so, by the induction hypothesis, q joins by time  $t_q^e+2D\leq t_p^e$ . Since q is active at time  $t_p^e+D$ , it receives the enter message from p during  $[t_p^e,t_p^e+D]$  and sends an enter-echo message in response. Since p is active at time  $t_p^e+2D$ , it receives the enter-echo message from q by time  $t_p^e+2D$ . Hence, by time  $t_p^e+2D$ , p receives at least one enter-echo message from a joined node in response to its enter message.

Suppose the first enter-echo message p receives from a joined node in response to its enter message is sent by node q' at time t' and received by p at time t''. By Lemma 8,  $SysInfo^{[0,\max\{0,t'-2D\}]} \subseteq Changes_{q'}^{t'}$   $\subseteq Changes_p^{t''}$ .

Let S be the set of nodes present at time  $\max\{0,t'-2D\}$ . Since  $t'' \le t' + D$ , it follows from Lemma 1 that at most  $((1+\alpha)^3-1)|S|$  nodes enter during  $(\max\{0,t'-2D\},t'']$ . Thus,  $|\mathit{Present}_p^{t''}| \le |S| + ((1+\alpha)^3-1)|S| = (1+\alpha)^3|S|$ . From line 10 in Algorithm 1, it follows that  $join\_bound \le \gamma \cdot (1+\alpha)^3|S|$ .

By Lemma 2 and Assumption (A), at most  $(1-(1-\alpha)^3)|S|$  nodes leave during  $(\max\{0,t'-2D\},t'+D]$ . Also, by Lemma 1, at most  $(1+\alpha)^3|S|$  nodes are present in the system at t'+D and so at most  $\Delta(1+\alpha)^3|S|$  nodes are crashed at t'+D. Since  $t_p^e \leq t' \leq t_p^e + D$ , the nodes in S that do not leave during  $(\max\{0,t'-2D\},t'+D]$  and are not crashed at t'+D are active throughout  $[t_p^e,t_p^e+D]$  and send enter-echo messages in response to p's enter message. By time  $t_p^e+2D$ , p receives all these enter-echo

messages. There are at least  $|S| - (1 - (1 - \alpha)^3)|S| - \Delta(1 + \alpha)^3|S| = (1 - \alpha)^3|S| - \Delta(1 + \alpha)^3|S|$  such enter-echo messages. By Assumption (D),

$$\frac{(1-\alpha)^3}{(1+\alpha)^3} - \Delta \ge \gamma,$$

so the value of *join\_bound* is at most

$$\gamma \cdot (1+\alpha)^3 |S| \le \left(\frac{(1-\alpha)^3}{(1+\alpha)^3} - \Delta\right) \cdot (1+\alpha)^3 |S| = (1-\alpha)^3 |S| - \Delta(1+\alpha)^3 |S|.$$

Thus, by time  $t_p^e + 2D$ , the condition in line 12 of Algorithm 1 holds and node p joins.

Next, we show that all read and write operations terminate. Specifically, we show that the number of responses for which an operation waits is at most the number that it is guaranteed to receive.

Since enter(q) is added to  $Changes_p$  whenever join(q) is, we get the following observation.

**Observation 10.** For every time  $t \geq 0$  and every node p that is active at time t, Members $_p^t \subseteq Present_p^t$ .

Lemma 11 relates a node's current estimate of the number of nodes present to the number of nodes that were present in the system 2D time units earlier. Lemma 12 relates a node's current estimate of the number of nodes that are members to the number of nodes that were present in the system time 4D time units earlier. Lemma 11 is used in the proof of Lemma 13 and Lemma 12 is used in the proof of Theorem 21. The proofs of Lemmas 12 and 11 are very similar to each other and are thus presented together.

**Lemma 11.** For every node p and every time  $t \ge t_p^j$  at which p is active,

$$(1-\alpha)^2 \cdot N(\max\{0,t-2D\}) \leq |\textit{Present}_p^t| \leq (1+\alpha)^2 \cdot N(\max\{0,t-2D\}).$$

*Proof.* By Lemma 8,  $SysInfo^{[0,\max\{0,t-2D\}]} \subseteq Changes_p^t$ . Thus  $Present_p^t$  contains all nodes that are present at time  $\max\{0,t-2D\}$ , plus any nodes that enter in  $(\max\{0,t-2D\},t]$  which p has learned about, minus any nodes that leave in  $(\max\{0,t-2D\},t]$  which p has learned about. Then, by Lemma 1,  $|Present_p^t| \le N(\max\{0,t-2D\}) + ((1+\alpha)^2-1) \cdot N(\max\{0,t-2D\}) = (1+\alpha)^2 \cdot N(\max\{0,t-2D\})$ . Similarly, by Lemma 2 and Assumption (A),  $|Present_p^t| \ge N(\max\{0,t-2D\}) - (1-(1-\alpha)^2) \cdot N(\max\{0,t-2D\}) = (1-\alpha)^2 \cdot N(\max\{0,t-2D\})$ . □

**Lemma 12.** For every node p and every time  $t \ge t_p^j$  at which p is active,

$$(1-\alpha)^4 \cdot N(\max\{0,t-4D\}) \leq |\mathit{Members}_p^t| \leq (1+\alpha)^4 \cdot N(\max\{0,t-4D\}).$$

Proof. By Lemma 8,  $SysInfo^{[0,\max\{0,t-2D\}]}\subseteq Changes_p^t$  and, by Theorem 9, every node that enters by time  $\max\{0,t-4D\}$  joins by time  $\max\{0,t-2D\}$  if it is still active. Thus  $Members_p^t$  contains all nodes that are present at time  $\max\{0,t-4D\}$  plus any nodes that enter in  $(\max\{0,t-4D\},t]$  which p learns have joined, minus any nodes that leave in  $(\max\{0,t-4D\},t]$  which p learns have left. Then, by Lemma 1,  $|Members_p^t| \leq N(\max\{0,t-4D\}) + ((1+\alpha)^4-1) \cdot N(\max\{0,t-4D\}) = (1+\alpha)^4 \cdot N(\max\{0,t-4D\})$ . Similarly, by Lemma 2 and Assumption (A),  $|Members_p^t| \geq N(\max\{0,t-2D\}) - (1-(1-\alpha)^4) \cdot N(\max\{0,t-4D\}) = (1-\alpha)^4 \cdot N(\max\{0,t-4D\})$ .

The next lemma provides a lower bound on the number of nodes that will reply to an operation's query or update message.

**Lemma 13.** If node p is active at time  $t \geq t_p^j$ , then the number of nodes that join by time t and are still active at time t+D is at least  $\left\lceil \frac{(1-\alpha)^3}{(1+\alpha)^2} - \Delta(1+\alpha) \right\rceil \cdot |Present_p^t|$ .

*Proof.* By Lemma 2 and Assumption (A), the maximum number of nodes that leave during  $(\max\{0,t-2D\},t+D]$  is at most  $(1-(1-\alpha)^3)\cdot N(\max\{0,t-2D\})$ . By Lemma 1, at most  $((1+\alpha)^3-1)\cdot N(\max\{0,t-2D\})$  nodes enter during  $(\max\{0,t-2D\},t+D]$ . So, at most  $\Delta(1+\alpha)^3\cdot N(\max\{0,t-2D\})$  nodes are crashed by t+D. Thus, there are at least

$$N(\max\{0, t - 2D\}) - (1 - (1 - \alpha)^3) \cdot N(\max\{0, t - 2D\}) - \Delta(1 + \alpha)^3 \cdot N(\max\{0, t - 2D\})$$
$$= [(1 - \alpha)^3 - \Delta(1 + \alpha)^3] \cdot N(\max\{0, t - 2D\})$$

nodes that were present at time  $\max\{0,t-2D\}$  and are still active at time t+D. This number is bounded below by  $\left[\frac{(1-\alpha)^3}{(1+\alpha)^2} - \Delta(1+\alpha)\right] \cdot |\mathit{Present}_p^t| \text{ since, by Lemma 11, } N(\max\{0,t-2D\}) \geq |\mathit{Present}_p^t|/(1+\alpha)^2.$  By Theorem 9, all of these nodes are joined by time t.

**Theorem 14.** Every read or write operation invoked by a node that remains active completes.

*Proof.* Each operation consists of a read phase and a write phase. We show that each phase terminates within 2D time, provided the client does not crash or leave.

Consider a phase of an operation by client p that starts at time t. Every node that joins by time t and is still active at time t+D receives p's query or update message and replies with a response or ack message by time t+D. By Lemma 13, there are at least  $\left[\frac{(1-\alpha)^3}{(1+\alpha)^2} - \Delta(1+\alpha)\right] \cdot |Present_p^t|$  such nodes.

From Assumption (E) and Observation 10,

$$\left\lceil \frac{(1-\alpha)^3}{(1+\alpha)^2} - \Delta(1+\alpha) \right\rceil \cdot |\mathit{Present}_p^t| \geq \beta |\mathit{Present}_p^t| \geq \beta |\mathit{Members}_p^t| = quorum\_size_p^t.$$

Thus, by time t + 2D, p receives sufficiently many response or ack messages to complete the phase.  $\Box$ 

Now we prove atomicity of the CCREG algorithm. Let  $\mathcal{T}$  be the set of read operations that complete and write operations that execute line 53 of Algorithm 2. For any node p, let  $ts_p^t = (seq_p^t, id_p^t)$  denote the *timestamp* of the latest register value known to node p at time t. Note that new timestamps are created by write operations (on lines 49-50 of Algorithm 2) and are sent via enter-echo, update, and update-echo messages. Initially,  $ts_p^0 = (0, \bot)$  for all nodes p.

For any read or write operation o in  $\mathcal{T}$  by p, the *timestamp of its read phase*,  $ts^{rp}(o)$ , is  $ts^t_p$ , where t is the end of its read phase (i.e., when the condition on line 44 of Algorithm 2 evaluates to true). The *timestamp of its write phase*,  $ts^{wp}(o)$ , is  $ts^t_p$ , where t is the beginning of its write phase (i.e., when it broadcasts on line 53 of Algorithm 2). The *timestamp of a read operation* in  $\mathcal{T}$  is the timestamp of its read phase. The *timestamp of a write operation* in  $\mathcal{T}$  is the timestamp of its write phase.

Lemmas 15–18 show that write phase information propagates properly through the system. They are analogous to Observation 5 and Lemmas 6–8, which concern the propagation of information about *enter*, *join*, and *leave* events.

**Lemma 15.** If o is an operation in  $\mathcal{T}$  whose write phase w starts at  $t_w$ , node p is active at time  $t \geq t_w + D$ , and  $t_p^e \leq t_w$ , then  $ts_p^t \geq ts^{wp}(o)$ .

*Proof.* Since p is active throughout  $[t_w, t_w + D]$ , it directly receives the update message broadcast by w at time  $t_w$ . Hence, from lines 70–71 of Algorithm 3,  $ts_n^t \ge ts^{wp}(o)$ .

**Lemma 16.** Suppose a node  $p \notin S_0$  receives an enter-echo message at time t'' from a node q that sends it at time t' in response to an enter message from p. If o is an operation whose write phase w starts at  $t_w$ , p is active at time  $t \ge \max\{t'', t_w + 2D\}$ , and q is active throughout  $[t_w, t_w + D]$ , then  $ts_p^t \ge ts^{wp}(o)$ .

*Proof.* Since q is active throughout  $[t_w, t_w + D]$ , it receives the update message from w at some time  $\hat{t} \in [t_w, t_w + D]$ , so  $ts_q^{\hat{t}} \ge ts^{wp}(o)$ . At time  $t'' \le t$ , node p receives the enter-echo sent by node q at time t', so  $ts_p^{t} \ge ts_p^{t''} \ge ts_q^{t'}$ . If  $t' \ge \hat{t}$ , then  $ts_q^{t'} \ge ts_q^{\hat{t}}$ , so  $ts_p^{t} \ge ts^{wp}(o)$ . If  $\hat{t} > t'$ , then q sends an update-echo at time  $\hat{t} \le t_w + D$ , p receives it by time  $\hat{t} + D \le t_w + 2D \le t$ , and, thus,  $ts_p^{t} \ge ts_q^{\hat{t}} \ge ts^{wp}(o)$ .

**Lemma 17.** If o is an operation in  $\mathcal{T}$  whose write phase w starts at  $t_w$  and node p is active at time  $t \ge \max\{t_p^e + 2D, t_w + D\}$ , then  $ts_p^t \ge ts^{wp}(o)$ .

*Proof.* The proof is by induction on the order in which nodes enter the system. Suppose the claim is true for all nodes that enter earlier than p. If  $t_p^e \le t_w$ , which is the case for all  $p \in S_0$ , then the claim follows from Lemma 15. So, suppose that  $t_w < t_p^e$ .

By Lemma 3, there is at least one node q that is active throughout  $[\max\{0,t_p^e-2D\},t_p^e+D]$ . It receives an enter message from p at some time  $t'\in[t_p^e,t_p^e+D]$  and sends an enter-echo message containing  $ts_q^{t'}$  back to p. This message is received by p at some time  $t''\leq t'+D\leq t_p^e+2D\leq t$ , so  $ts_q^{t'}\leq ts_p^{t''}\leq ts_p^t$ .

The first case is when  $t_w \ge \max\{0, t_p^e - 2D\}$ . Since  $t_w + D < t_p^e + D$ , it follows that q is active throughout  $[t_w, t_w + D]$ . Furthermore,  $t \ge t_p^e + 2D \ge \max\{t'', t_w + 2D\}$ . Hence, Lemma 16 implies that  $ts_p^t \ge ts^{wp}(o)$ .

The second case is when  $t_w < \max\{0, t_p^e - 2D\}$ . Since  $t_w \ge 0$ , it follows that  $t_p^e - 2D > 0$ ,  $t_q^e \le \max\{0, t_p^e - 2D\} = t_p^e - 2D$ , and  $t_w < t_p^e - 2D \le t' - 2D$ , so  $t' \ge \max\{t_q^e + 2D, t_w + D\}$ . Note that q is active at time t' and q enters before node p, so, by the induction hypothesis,  $ts_q^{t'} \ge ts^{wp}(o)$ . Hence,  $ts_p^t \ge ts^{wp}(o)$ .

**Lemma 18.** If o is an operation in  $\mathcal{T}$  whose write phase starts at  $t_w$ , node  $p \notin S_0$  joins at time  $t_p^j$ , and p is active at time  $t \ge \max\{t_p^j, t_w + 2D\}$ , then  $ts_p^t \ge ts^{wp}(o)$ .

*Proof.* The proof is by induction on the order in which nodes enter the system. Suppose the claim is true for all nodes that join before p. If  $t \geq t_p^e + 2D$ , then the claim follows by Lemma 17. So, assume that  $t < t_p^e + 2D$ . If  $t_p^e \leq t_w$ , then the claim follows by Lemma 15. So, assume that  $t_w < t_p^e$ .

Before p joins, it receives an enter-echo message from a joined node in response to its enter message. Suppose p first receives such an enter-echo message at time t'' and this enter-echo was sent by q at time t'. Then  $t'' \le t_p^j \le t$  and  $ts_q^{t'} \le ts_p^{t''} \le ts_p^t$ .

Now we prove that p receives an enter echo message from a node q' that is active throughout  $[\max\{0,t'-2D\},t'+D]$ . Let S be the set of nodes present at time  $\max\{0,t'-2D\}$ , so  $|S|=N(\max\{0,t'-2D\})$ . By Lemma 2 and Assumption (A), at most  $(1-(1-\alpha)^3)|S|$  nodes leave during  $(\max\{0,t'-2D\},t'+D]$ . Since  $t'' \leq t'+D$ , it follows that  $|Present_p^{t''}| \geq |S|-(1-(1-\alpha)^3)|S|=(1-\alpha)^3|S|$ . Hence, from lines 10 and 12 of Algorithm 1, p waits until it has received at least  $join\_bound = \gamma \cdot |Present_p^{t''}| \geq \gamma \cdot (1-\alpha)^3|S|$  enter-echo messages before joining.

By Lemma 1, the number of nodes that enter during  $(\max\{0,t'-2D\},t'+D]$  is at most  $((1+\alpha)^3-1)|S|$ . Thus, at time t'+D, there are at most  $(1+\alpha)^3|S|$  nodes present and at most  $\Delta(1+\alpha)^3|S|$  nodes are crashed. Hence, the number of enter-echo messages p receives before joining from nodes that were active throughout  $[\max\{0,t'-2D\},t'+D]$  is  $join\_bound$  minus the total number of enters, leaves and crashes, which is at

least

$$\gamma \cdot (1 - \alpha)^{3} |S| - [((1 + \alpha)^{3} - 1)|S| + (1 - (1 - \alpha)^{3})|S| + \Delta(1 + \alpha)^{3} |S|]$$

$$\geq [(1 + \gamma)(1 - \alpha)^{3} - (1 + \Delta)(1 + \alpha)^{3}] N_{min}. \quad (J)$$

Rearranging Assumption (C), we get  $[(1+\gamma)(1-\alpha)^3-(1+\Delta)(1+\alpha)^3N_{min}]\geq 1$ , so expression (J) is at least 1. Hence p receives an enter-echo message at some time  $T''\leq t_p^j$  from a node q' that is active throughout  $[\max\{0,t'-2D\},t'+D] \supseteq [\max\{0,t'-2D\},t-D]$ . Let T' be the time that q' sent its enter-echo message in response to the enter message from p. Then  $ts_{q'}^{T'}\leq ts_p^{T''}\leq ts_p^{T}$ .

enter-echo message in response to the enter message from p. Then  $ts_{q'}^{T'} \leq ts_p^{T''} \leq ts_p^{T''} \leq ts_p^{T''} \leq ts_p^{T''}$ . Note that  $t_w < t_p^e \leq t'$  so  $t_w + D \leq t' + D$ . If  $t_w \geq \max\{0, t' - 2D\}$ , then q' is active throughout  $[t_w, t_w + D]$ . Since  $t \geq \max\{T'', t_w + 2D\}$ , it follows by Lemma 16 that  $ts_p^t \geq ts^{wp}(o)$ . So, assume that  $t_w < \max\{0, t' - 2D\}$ .

Since  $t_w \ge 0$ , it follows that  $t' > t_w + 2D$ . If  $q \in S_0$ , then  $t_q^e = 0 \le t_w$ , so, by Lemma 15,  $ts_q^{t'} \ge ts^{wp}(o)$ . If  $q \notin S_0$ , then, by the induction hypothesis,  $ts_q^{t'} \ge ts^{wp}(o)$ , since q joins at time  $t_q^j < t_p^j \le t'$ . Thus, in both cases,  $ts_p^t \ge ts^{wp}(o)$ .

Lemma 19 is the key lemma for proving atomicity of CCREG. It shows that for two non-overlapping operations in  $\mathcal{T}$ , the timestamp of the read phase of the latter operation is at least as big as the timestamp of the write phase of the former. Lemma 20 uses Lemma 19 to show that the timestamps of two non-overlapping operations respect real time ordering. Theorem 21 uses Lemmas 19 and 20 to complete the proof of atomicity.

**Lemma 19.** For any two operations  $op_1$  and  $op_2$  in  $\mathcal{T}$ , if  $op_1$  finishes before  $op_2$  starts, then  $ts^{wp}(op_1) \leq ts^{rp}(op_2)$ .

*Proof.* Let  $p_1$  be the node that invokes  $op_1$ , let w denote the write phase of  $op_1$ , let  $t_w$  be the start time of w, and let  $\tau_w = ts^{wp}(op_1) = ts^{t_w}_{p_1}$ . Similarly, let  $p_2$  be the node that invokes  $op_2$ , let r denote the read phase of  $op_2$ , let  $t_r$  be the start time of r, and let  $\tau_r = ts^{rp}(op_2) = ts^{t_r}_{p_2}$ .

Let  $Q_w$  be the set of nodes that  $p_1$  hears from during w (i.e., that sent messages causing  $p_1$  to increment  $heard\_from$  on line 58 of Algorithm 2) and  $Q_r$  be the set of nodes that  $p_2$  hears from during r (i.e., that sent messages causing  $p_2$  to increment  $heard\_from$  on line 43 of Algorithm 2). Let  $P_w = |Present_{p_1}^{t_w}|$  and  $M_w = |Members_{p_1}^{t_w}|$  be the sizes of the Present and Members sets belonging to  $p_1$  at time  $t_w$ , and  $P_r = |Present_{p_2}^{t_r}|$  and  $M_r = |Members_{p_2}^{t_r}|$  be the sizes of the Present and Present

## Case I: $t_r > t_w + 2D$ .

We start by showing there exists a node q in  $Q_r$  such that  $t_q^j \leq t_r - 2D$ . Each node  $q \in Q_r$  receives and responds to r's query, so it joined by time  $t_r + D$ . By Theorem 9, the number of nodes that can join during  $(t_r - 2D, t_r + D]$  is at most the number of nodes that can enter in  $(\max\{0, t_r - 4D\}, t_r + D]$ . By Lemma 1, the number of nodes that can enter during  $(\max\{0, t_r - 4D\}, t_r + D]$  is at most  $((1+\alpha)^5 - 1) \cdot N(\max\{0, t_r - 4D\})$ . By Lemma 12,  $N(\max\{0, t_r - 4D\}) \leq M_r/(1-\alpha)^4$ . From the code and Assumption (F), it follows that  $|Q_r| \geq \beta M_r > M_r(1+\alpha)^5 - 1)/(1-\alpha)^4 \geq (1+\alpha)^5 - 1) \cdot N(\max\{0, t_r - 4D\})$ , which is at most the number of nodes that can enter in  $(\max\{0, t_r - 4D\}, t_r + D]$ . Thus, there is a node  $q \in Q_r$  that joins by time  $t_r - 2D$ .

Suppose q receives r's query message at time  $t' \geq t_r \geq t_w + 2D$  If  $q \in S_0$ , then  $t_q^j = 0 \leq t_w$ , so, by Lemma 15,  $ts_q^{t'} \geq ts^{wp}(op_1) = \tau_w$ . Otherwise,  $q \notin S_0$ , so  $0 < t_q^j \leq t_r - 2D < t'$ . Since

 $t_w + 2D < t_r \le t'$ , Lemma 18 implies that  $ts_q^{t'} \ge ts^{wp}(op_1) = \tau_w$ . In either case, q responds to r's query message with a timestamp at least as large as  $\tau_w$  and, hence,  $\tau_r \ge \tau_w$ .

**Case II:**  $t_r \le t_w + 2D$ .

Let  $J = \{p \mid t_p^j < t_r \text{ and } p \text{ is active at time } t_r\} \cup \{p \mid t_r \leq t_p^j \leq t_r + D\}$ , which contains the set of all nodes that reply to r's query. By Theorem 9, all nodes that are present at time  $\max\{0, t_r - 2D\}$  join by time  $t_r$  if they remain active. Therefore all nodes in J are either active at time  $\max\{0, t_r - 2D\}$  or enter during  $(\max\{0, t_r - 2D\}, t_r + D]$ . By Lemma 1,  $|J| \leq (1 + \alpha)^3 N(\max\{0, t_r - 2D\})$ .

Let K be the set of all nodes that are present at time  $\max\{0,t_r-2D\}$  and do not leave or crash during  $(\max\{0,t_r-2D\},t_r+D]$ . Note that K contains all the nodes in  $Q_w$  that do not leave or crash during  $[t_w,t_r+D]\subseteq [\max\{0,t_r-2D\},t_r+D]$ . By Lemma 2 and Assumption (A), at most  $(1-\alpha)^3N(\max\{0,t_r-2D\})$  nodes leave during  $[\max\{0,t_r-2D\},t_r+D]$ . By Lemma 1, at most  $((1+\alpha)^3-1)N(\max\{0,t_r-2D\})$  nodes enter during  $[\max\{0,t_r-2D\},t_r+D]$ . So, at most  $\Delta(1+\alpha)^3N(\max\{0,t_r-2D\})$  nodes are crashed at  $t_r+D$ .

From the code,  $|Q_r| \ge \beta M_r$  and, by Lemma 12,  $M_r \ge (1-\alpha)^4 N(\max\{0,t_r-4D\})$ . So,

$$|Q_r| \ge \beta (1 - \alpha)^4 N(\max\{0, t_r - 4D\}).$$

Similarly,

$$|Q_w| \ge \beta M_w \ge \beta (1 - \alpha)^4 N(\max\{0, t_w - 4D\}).$$

Therefore, by the definition of K:

$$|K| \ge |Q_w| - (1 - (1 - \alpha)^3 + \Delta(1 + \alpha)^3)N(\max\{0, t_r - 2D\})$$

$$\geq (\beta(1-\alpha)^4 N(\max\{0, t_w - 4D\})) - (1 - (1-\alpha)^3 + \Delta(1+\alpha)^3) N(\max\{0, t_r - 2D\}). \tag{K}$$

Since  $t_r - t_w < 2D$ , it follows that  $\max\{0, t_r - 4D\} - \max\{0, t_w - 4D\} < 2D$ . By Lemma 1,  $N(\max\{0, t_r - 4D\}) \le (1 + \alpha)^2 \cdot N(\max\{0, t_w - 2D\})$ . Thus we can replace  $N(\max\{0, t_w - 4D\})$  in Formula (K) with  $(1 + \alpha)^2 \cdot N(\max\{0, t_r - 4D\})$  and get:

$$|Q_r| + |K| \ge \beta (1 - \alpha)^4 N(\max\{0, t_r - 4D\})$$

$$+ \beta (1 - \alpha)^4 (1 + \alpha)^{-2} N(\max\{0, t_r - 4D\})$$

$$- (1 - (1 - \alpha)^3 + \Delta (1 + \alpha)^3) N(\max\{0, t_r - 2D\})$$

$$= \beta (1 - \alpha)^4 (1 + \alpha)^{-2} (2 + 2\alpha + \alpha^2) N(\max\{0, t_r - 4D\})$$

$$- (\Delta (1 + \alpha)^3 - (1 - \alpha)^3 + 1) N(\max\{0, t_r - 2D\}).$$

By Lemma 1,  $N(\max\{0,t_r-4D\}) \geq (1-\alpha)^{-2}N(\max\{0,t_r-2D\})$ . Thus,

$$|Q_r| + |K| \ge \beta (1 - \alpha)^2 (1 + \alpha)^{-2} (2 + 2\alpha + \alpha^2) N(\max\{0, t_r - 2D\})$$

$$- (\Delta (1 + \alpha)^3 - (1 - \alpha)^3 + 1) N(\max\{0, t_r - 2D\})$$

$$= (\beta (1 - \alpha)^2 (1 + \alpha)^{-2} (2 + 2\alpha + \alpha^2) - (\Delta (1 + \alpha)^3 - (1 - \alpha)^3 + 1)) N(\max\{0, t_r - 2D\})$$

By Assumption (G), 
$$\beta(1-\alpha)^2(1+\alpha)^{-2}(2+2\alpha+\alpha^2) > (1+\Delta)(1+\alpha)^3 - (1-\alpha)^3 + 1$$
, so 
$$|Q_r| + |K| > (((1+\Delta)(1+\alpha)^3 - (1-\alpha)^3 + 1) - (\Delta(1+\alpha)^3 - (1-\alpha)^3 + 1)) \cdot N(\max\{0, t_r - 2D\})$$
$$= (1+\alpha)^3 N(\max\{0, t_r - 2D\})$$
$$\geq |J|.$$

This implies that K and  $Q_r$  intersect, since  $K, Q_r \subseteq J$ . For each node p in the intersection,  $ts_p \ge \tau_w$  when p sends its response to r and, thus,  $\tau_w \le \tau_r$ .

**Lemma 20.** For any two operations  $op_1$  and  $op_2$  in  $\mathcal{T}$  such that  $op_1$  finishes before  $op_2$  starts,

- (a)  $ts(op_1) \leq ts(op_2)$  if  $op_2$  is a read, and
- (b)  $ts(op_1) < ts(op_2)$  if  $op_2$  is a write.

*Proof.* By definition of the timestamp of an operation,  $ts(op_1) \le ts^{wp}(op_1)$  and  $ts^{rp}(op_2) \le ts(op_2)$ . By Lemma 19,  $ts^{wp}(op_1) \le ts^{rp}(op_2)$ , and thus part (a) follows. Part (b) follows from the observation that when  $op_2$  is a write,  $ts^{wp}(op_2) = ts^{rp}(op_2) + 1$ .

## **Theorem 21.** CCREG ensures atomicity.

*Proof.* We show that, for every execution, there is a total order on the set of operations in  $\mathcal{T}$  such that every read returns the value of the latest preceding write and, if an operation  $op_1$  finishes before another operation  $op_2$  begins, then  $op_1$  is ordered before  $op_2$ .

Before describing the total order, we first argue that each write operation in  $\mathcal{T}$  is assigned a unique timestamp. Recall that the timestamp of an operation op executed by a node p is the timestamp of op's write phase, which is the ordered pair consisting of the values of p's seq and id variables when p executes Line 53 for op. Note that the id variable is equal to p and the seq variable is set to one greater than the largest sequence value observed during op's read phase. These timestamps are unique because all nodes have unique ids, each node runs at most one write thread, and the writer remembers the last value that it assigned to seq.

Claim: Consider any read  $op_1$  in  $\mathcal{T}$ . If the timestamp of  $op_1$  is  $(0, \perp)$ , then  $op_1$  returns  $\perp$ . Otherwise there exists a write  $op_2$  in  $\mathcal{T}$  such that  $ts(op_1) = ts(op_2)$  and the value returned by  $op_1$  equals the value written by  $op_2$ .

This claim can be shown by a simple induction, based on the fact that every timestamp other than  $(0, \perp)$  ultimately comes from Lines 49-50 of Algorithm 2.

The total order is constructed as follows. First, order the write operations in order of their (unique) timestamps. Place each read with timestamp  $(0, \perp)$  at the beginning of the total order. Place every other read immediately following the write operation it reads from. Finally, reorder all reads that are between the same two writes (or are before the first write or after the last write) according to the times when they start. By the claim above, every read in the total order returns the value of the latest preceding write (or returns  $\perp$  if there is no preceding write).

The rest of the proof shows that the total order respects the real-time order of non-overlapping operations in the execution. Let  $op_1$  and  $op_2$  be two operations such that  $op_1$  finishes before  $op_2$  begins in the execution. We do the proof in the following cases:

• W-W: If  $op_1$  and  $op_2$  are both writes, then Lemma 20(b) implies that  $ts(op_1) < ts(op_2)$  and thus the construction orders  $op_1$  before  $op_2$ .

- W-R: Suppose  $op_1$  is a write and  $op_2$  is a read. By Lemma 20(a) and the construction,  $op_2$  is placed after the write  $op_3$  that  $op_2$  reads from. If  $ts(op_1) = ts(op_2)$  then  $op_1 = op_3$  and  $op_2$  is placed after  $op_1$ . If  $ts(op_1) < ts(op_2)$  then  $op_3$  is placed after  $op_1$  as  $ts(op_1) < ts(op_3)$  and thus  $op_2$  is placed after  $op_1$  in the total order.
- **R-W:** Suppose  $op_1$  is a read and  $op_2$  is a write. By Lemma 20(b),  $ts(op_1) < ts(op_2)$ . Now, either  $op_2$  is the first write in the execution and  $op_1$ 's timestamp is  $(0, \bot)$  or there exists another write  $op_3$  that  $op_1$  reads from. If  $op_1$ 's timestamp is  $(0, \bot)$  then the construction orders  $op_1$  before  $op_2$ . Otherwise, the construction orders  $op_3$  before  $op_2$ . Since  $op_1$  is ordered after  $op_3$  but before any subsequent write,  $op_1$  precedes  $op_2$  in the total order.
- **R-R:** Finally, suppose that  $op_1$  and  $op_2$  are both reads. By Lemma 20(a),  $ts(op_1) \le ts(op_2)$ . If  $op_1$  and  $op_2$  have the same timestamp, then they are placed after the same write (or before the first write) and the construction orders them based on their starting times. Since  $op_1$  completes before  $op_2$  starts, the construction places  $op_1$  before  $op_2$ . If  $op_2$  has a timestamp greater than that of  $op_1$ , then  $ts(op_2)$  cannot be  $(0, \bot)$  and so, there is a write operation  $op_3$  whose timestamp is greater than that of  $op_1$  and equal to that of  $op_2$ . The construction places  $op_1$  before  $op_3$  and  $op_2$  after  $op_3$ .

Thus CCREG ensures atomicity.

The CCREG algorithm violates atomicity if our churn assumption  $\alpha$  is violated. This is demonstrated by the following execution, in which large numbers of nodes enter and leave very quickly.

Let  $|S_0|=n$  and let p be some node in  $S_0$ . Suppose the following sequence of events occur before time D. First, a set of nodes, denoted  $S_{new}$ , enter the system, with  $|S_{new}|=m\gg n$ . All join-related messages between  $S_0-\{p\}$  and  $S_{new}\cup\{p\}$  take D time, while the rest are very fast. As a result, p is the first joined node that nodes in  $S_{new}$  hear from and they use p's estimate of the system size as being p to calculate the number of messages they should hear from before joining. Thus all nodes in  $S_{new}$  join before time p0 but no node in p0 other than p2 knows about p0 so far.

Second, immediately after joining, some node q in  $S_{new}$  invokes write(1). All write-related messages between  $S_0$  and  $S_{new}$  take D time, while the rest are very fast.  $S_{new}$  is sufficiently large that the write protocol completes for q based solely on hearing from nodes in  $S_{new}$ . Thus the write completes before time D but no node in  $S_0$  knows about the enters or the write so far.

Third, immediately after the write finishes, all the nodes in  $S_{new}$  leave. All leave-related messages between  $S_0$  and  $S_{new}$  take D time, while the rest are very fast. Thus no node in  $S_0$  knows about the enters, the write, or the leaves so far.

Finally, immediately after the leaves, node  $p' \neq p$  in  $S_0$  invokes a read. All read-related messages are very fast. Node p' uses its estimate of the system size as n to decide how many messages to wait for and is able to complete its read before time D by hearing only from nodes in  $S_0 - \{p\}$ . Since none of these nodes knows anything about the write, the read returns 0, which violates atomicity (as well as regularity).

## 5 Related Work

A simple simulation of a single-writer, multi-reader register in an asynchronous static network was presented by Attiya, Bar-Noy and Dolev [5], henceforth called the ABD simulation. Their paper also shows that it is impossible to simulate an atomic register in an asynchronous system if at least half of the nodes in the system can be faulty. It was followed by extensions that reduce complexity [4,8–10], support multiple writers [12],

or tolerate Byzantine failures [1, 3, 14, 16]. To optimize load and resilience, the simple majority quorums used in these papers can be replaced by other, more complicated, quorum systems (e.g., [15, 19]).

A survey of simulations of a atomic multi-writer, multi-reader register in a dynamic system with churn appears in [17]. The first such simulation was RAMBO [13]. Here, the notion of churn is abstracted by defining a sequence of quorum configurations. Each quorum configuration consists of a set S of nodes (which are called members) plus sets of read-quorums and write-quorums, each of which is a subset of S. The system supports reconfiguration, in which an older quorum configuration is replaced by a newer one. RAMBO consists of three protocols: joiner, reader-writer, and recon. The joiner protocol handles the joining of new nodes. The reader-writer protocol is responsible for executing the read and write operations using the read-quorums and write-quorums, as well as for performing garbage collection to remove old quorum configurations. The reads and writes are similar to the ABD simulation. The recon protocol handles quorum configuration changes to install new quorum systems. Reconfiguration is done in two parts: first, a member proposes a new quorum configuration. Second, these proposed configurations are reconciled by running an eventually-terminating distributed consensus algorithm (a version of the Paxos algorithm) among the members of the current quorum configuration. RAMBO requires intermittent periods of synchrony for the consensus to terminate. Reconfigurations can occur concurrently with reads and writes. The model does not differentiate between nodes that crash and nodes that leave the system. The algorithm guarantees atomicity of operations for all executions, even when there are arbitrary crashes (or leaves) and message loss. However, liveness is only ensured during periods when the system is sufficiently well-behaved with respect to synchrony, message loss, and churn.

DynaStore [2] simulates an atomic multi-writer, multi-reader register in a dynamic system, without using consensus. The set of nodes that are in the system is called a *view*. The nodes start with some default initial *view*. The nodes in the current view can propose the addition and removal of other nodes. The algorithm supports three types of operations: *read*, *write* and *reconfig*. Reads and writes are similar to the ABD simulation, with a read-phase followed by a write-phase. The reconfig operation starts with a phase in which information about the new view is sent to a majority of nodes in the old view. Then a read phase and a write phase are performed using the old view. DynaStore ensures atomicity for all executions. To ensure liveness of operations, the algorithm makes two assumptions. First, at any point in time, the number of crashed nodes and the number of nodes whose removal is pending (via reconfig) is a minority of the current view and of any pending future views. Second, it assumes that only a finite number of reconfiguration requests occur (i.e., churn is eventually *quiescent*).

Baldoni et al. [7] proposed a model in which the system size varies within a known range, that is, the nodes know an upper and a lower bound on the system size. Their churn model has a similar flavor to ours, since churn never stops and at most a *constant* fraction of nodes enter and leave periodically. The authors implement a regular register in an eventually synchronous system. The algorithm has three protocols: *join\_register*, *read*, and *write*. The join\_register module ensures that nodes join the system with sufficient knowledge about the system. The read and write protocols are similar to the ABD simulation. The simulation violates regularity if the churn assumption is violated. This can be shown with an argument similar to the one presented at the end of Section 4.

Baldoni et al. [7] also prove that it is impossible to simulate a regular register when there is no upper bound on message delay. In this case, it does not help for nodes to announce when they are leaving, since messages containing such announcements can be delayed for an arbitrarily long time. Thus, a node leaving is essentially the same as a crash. Their proof works by considering scenarios in which at least half of the nodes fail. Then they invoke the lower bound in [5], which shows that simulating a register is impossible unless fewer than half the nodes are faulty. Their proof can be adapted to hold when there is an unknown

upper bound, D, on the message delay and half the nodes can be replaced during any time interval of length D, provided that nodes are not required to announce when they leave. Thus, in our model, there must be an upper bound on the fraction of nodes that can crash during any time interval of length D. Also, it is necessary that either nodes announce when they leave or there is an upper bound on the fraction of nodes that can leave during this time interval.

We summarize the results of [13], [2] and [7] and compare them with our algorithm in Table 2. In [18] and [2], it is claimed that termination of operations cannot be guaranteed unless the churn eventually stops. This claim does not contradict our result due to differences in the churn models. One of the contributions of this paper is to point out that by making different, yet still reasonable, assumptions on churn it is possible to get a solution with different, yet still reasonable, properties and, in particular, to overcome the prior constraint that churn must stop to ensure termination of operations. That is, we are suggesting a different point in the solution space.

Algorithm	Consistency	Synchrony	Consensus	Algorithm-Independent	Tolerates	Failure/Leave
	Condition	Level	used	Churn Assumption	Continuous Churn	Model
RAMBO [13]	Atomic	Intermittently	Yes	No	No, needs periods	Leaves are same
		Asynchronous			of quiescence	as crashes
		and Synchronous				
DynaStore [2]	Atomic	Asynchronous	No	No	No, needs periods	Leaves are different
					of quiescence	from crashes
Baldoni	Regular	Eventually	No	Yes	Yes	Leaves are same
et al. [7]		Synchronous				as crashes
CCREG	Atomic	Asynchronous	No	Yes	Yes	Leaves are different
						from crashes

Table 2: Comparison of our algorithm to RAMBO [13], DynaStore [2], and Baldoni et al. [7]

## 6 Discussion

We have shown how to simulate an atomic read/write register in a crash-prone asynchronous system where nodes can enter and leave. The only assumptions are that the number of nodes entering and leaving during each time interval of length D is at most a constant fraction of the current system size and the number of failures at any given time is a constant fraction of the system size.

It would be nice to improve the constants for the churn rate and the maximum fraction of faulty nodes, perhaps with a tighter analysis. Proving lower bounds or tradeoffs on these parameters is an interesting avenue for future work. In fact, it might be possible to completely avoid the bound  $\alpha$  on the churn rate, by spreading out the handling of node joins and leaves: To ensure a minimal number of nonfaulty nodes, a node might need to obtain permission before leaving, similarly to joins. This may allow the algorithm to maintain safety even when the churn is high.

Because of the bounded churn, it may be possible to implement a failure detector to get rid of crashed nodes, rather than relying on an external component for this.

CCREG sends increasingly large Changes sets. The amount of information communicated might be reduced by sending only recent events, or by removing very old events. Another interesting research direction is to extend CCREG to tolerate more severe kinds of failures.

## References

- [1] I. Abraham, G. Chockler, I. Keidar, and D. Malkhi. Byzantine disk paxos: optimal resilience with Byzantine shared memory. *Dist. Comp.*, 18(5):387–408, 2006.
- [2] M. K. Aguilera, I. Keidar, D. Malkhi, and A. Shraer. Dynamic atomic storage without consensus. *J. ACM*, 58(2):7, 2011.
- [3] A. S. Aiyer, L. Alvisi, and R. A. Bazzi. Bounded wait-free implementation of optimally resilient byzantine storage without (unproven) cryptographic assumptions. In *Proceedings of 21st International Symposium on Distributed Computing*, pages 7–19, 2007.
- [4] H. Attiya. Efficient and robust sharing of memory in message-passing systems. *J. Alg.*, 34(1):109–127, Jan. 2000.
- [5] H. Attiya, A. Bar-Noy, and D. Dolev. Sharing memory robustly in message-passing systems. *J. ACM*, 42(1):124–142, Jan. 1995.
- [6] H. Attiya, H. C. Chung, F. Ellen, S. Kumar, and J. L. Welch. Simulating a shared register in an asynchronous system that never stops changing. In *Proceedings of 29th International Symposium on Distributed Computing*, pages 75–91, 2015.
- [7] R. Baldoni, S. Bonomi, A.-M. Kermarrec, and M. Raynal. Implementing a register in a dynamic distributed system. In *IEEE International Conference on Distributed Computing Systems*, pages 639–647, 2009.
- [8] P. Dutta, R. Guerraoui, R. R. Levy, and A. Chakraborty. How fast can a distributed atomic read be? In *Proceedings of the 23rd Annual ACM Symposium on Principles of Distributed Computing*, pages 236–245, 2004.
- [9] R. Guerraoui and R. Levy. Robust emulations of shared memory in a crash-recovery model. In *Proceedings of the International Conference on Distributed Computing Systems*, pages 400–407, 2004.
- [10] R. Guerraoui and M. Vukolić. Refined quorum systems. In *Proceedings of the 26th Annual ACM Symposium on Principles of Distributed Computing*, pages 119–128, 2007.
- [11] S. Y. Ko, I. Hoque, and I. Gupta. Using tractable and realistic churn models to analyze quiescence behavior of distributed protocols. In *IEEE Symposium on Reliable Distributed Systems*, pages 259–268, 2008.
- [12] N. A. Lynch and A. A. Shvartsman. Robust emulation of shared memory using dynamic quorum-acknowledged broadcasts. In *Proceedings of the 27th International Symposium on Fault-Tolerant Computing*, pages 272–281, 1997.
- [13] N. A. Lynch and A. A. Shvartsman. Rambo: A Reconfigurable Atomic Memory Service for Dynamic Networks. In *Proceedings of the 16th International Conference on Distributed Computing*, pages 173–190, 2002.
- [14] D. Malkhi and M. K. Reiter. Byzantine quorum systems. Dist. Comp., 11(4):203–213, 1998.

- [15] D. Malkhi, M. K. Reiter, A. Wool, and R. N. Wright. Probabilistic quorum systems. *Information and Computation*, 170(2):184–206, 2001.
- [16] J.-P. Martin, L. Alvisi, and M. Dahlin. Minimal byzantine storage. In *Proceedings of the 16th International Conference on Distributed Computing*, pages 311–325, 2002.
- [17] P. Musial, N. Nicolaou, and A. A. Shvartsman. Implementing distributed shared memory for dynamic networks. *Commun. ACM*, 57(6):88–98, 2014.
- [18] A. Spiegelman and I. Keidar. On liveness of dynamic storage. CoRR, abs/1507.07086, July 2015.
- [19] M. Vukolic. *Quorum Systems: With Applications to Storage and Consensus*. Synthesis Lectures on Distributed Computing Theory. Morgan & Claypool Publishers, 2012.