CPEN 400Q Lecture 23
Quantum channels and noise

Monday 8 April 2024

$$|+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\langle Y \rangle = \mathrm{Tr}(Y \rho) \qquad \frac{1}{8}\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 3i & -5i \\ 3i & -3i \end{pmatrix} = 0$$

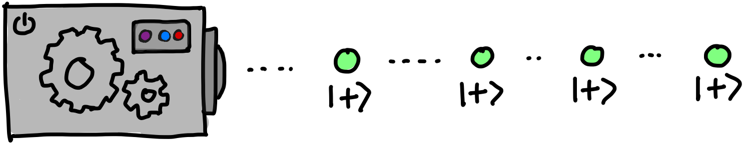
- Quiz 10 beginning of class today
- Literacy assignment 3 due Wednesday at 23:59
- Project due Friday at 23:59

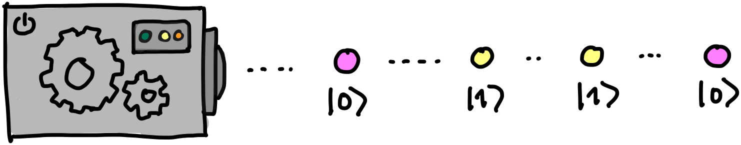
↳ to submit, add @glassnotes as collaborator on Github repo

$$H \cdot \rho \cdot H^\dagger = H \rho H$$

Last time

We introduced *mixed states*.





Mixed states are probabilistic mixtures of pure states.

	Pure state	Pure state ρ	Mixed state ρ
States	$\lvert\psi\rangle$	$\rho = \lvert\psi\rangle\langle\psi\rvert$	$\rho = \sum_i p_i \cdot \lvert\psi_i\rangle\langle\psi_i\rvert$
Ops.	$\lvert\psi'\rangle = U\lvert\psi\rangle$	$\rho' = U\rho U^\dagger$ $\lvert\psi'\rangle\langle\psi'\rvert = U\lvert\psi\rangle\langle\psi\rvert U^\dagger$	$\rho' = U\rho U^\dagger$
Meas.*	$\lvert\langle\varphi_i\lvert\psi\rangle\rvert^2$ $\langle\psi\rvert B\lvert\psi\rangle$	$Tr(\lvert\varphi_i\rangle\langle\varphi_i\rvert \cdot \lvert\psi\rangle\langle\psi\rvert)$ $Tr(B\rho)$	$Tr(P_i\,\rho)$ $Tr(B\rho)$

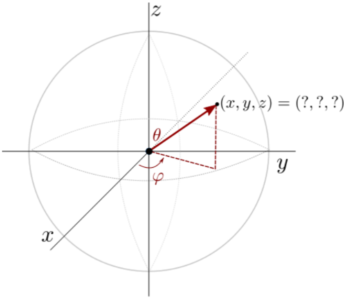
* where $\{\varphi_i\}$ form an orthonormal basis, and $\{P_i\}$ is a set of projectors $P_i = \lvert\varphi_i\rangle\langle\varphi_i\rvert$ or more generally a POVM ($\sum_i P_i = I$).

Last time

Using the mixed state measurement formalism, we can compute

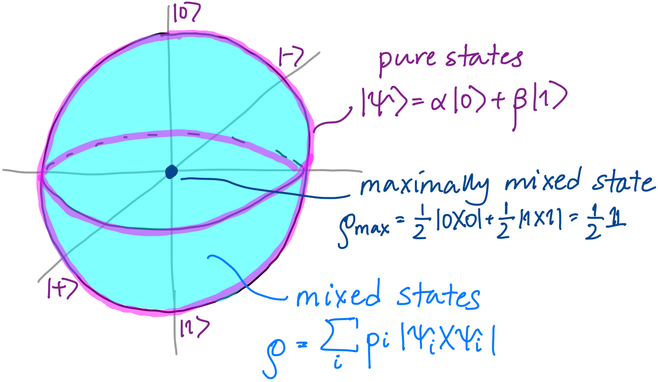
$$\langle X \rangle = \mathrm{Tr}(X \cdot \rho)$$
$$\langle Y \rangle = \mathrm{Tr}(Y \cdot \rho)$$
$$\langle Z \rangle = \mathrm{Tr}(Z \cdot \rho)$$

Write a function that uses quantum measurements to extract its Bloch vector, i.e., its coordinates in 3-dimensional space, as demonstrated by the following graphic:



Last time

I showed that Pauli expectation values for mixed states can produce things *inside the Bloch sphere*, but didn't really explain how...



Learning outcomes

- Define and apply quantum channels to qubit states
- Describe the effects of common noise channels
- Add noise to quantum circuits in PennyLane

Mixed states and the Bloch sphere

Recall ρ is Hermitian; Paulis are a basis.

$$\rho = a_I I + \underbrace{a_x X + a_y Y + a_z Z}_{\text{related to } \langle X\rangle, \langle Y\rangle, \langle Z\rangle}$$

ρ must have trace 1:

$$\begin{aligned} \mathrm{Tr}(\rho) &= \mathrm{Tr}(a_I I + a_x X + a_y Y + a_z Z) \\ &= a_I \cdot 2 \qquad \Rightarrow \quad a_I = \frac{1}{2} \\ &= 1 \end{aligned}$$

Trace out another Pauli:

$$\begin{aligned} \mathrm{Tr}(X\rho) &= \mathrm{Tr}(\cancel{a_I X} + a_x \cdot I + \cancel{a_y \underbrace{XY}_{iZ}} + \cancel{a_z \underbrace{XZ}_{-iY}}) \\ &= a_x \cdot 2 \\ &= \langle X \rangle \qquad \Rightarrow \quad a_x = \frac{1}{2}\langle X\rangle \end{aligned}$$

Re-express:

$$\rho = \frac{1}{2} I + \frac{\langle X\rangle}{2} X + \frac{\langle Y\rangle}{2} Y + \frac{\langle Z\rangle}{2} Z$$

$$= \frac{1}{2}\begin{pmatrix} 1+\langle Z\rangle & \langle X\rangle - i\langle Y\rangle \\ \langle X\rangle + i\langle Y\rangle & 1-\langle Z\rangle \end{pmatrix}$$

Exercise: As ρ is positive semidefinite, its eigvals are ≥ 0. What constraint does this put on $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$? (Hint: look at $\det(\rho)$)

$$\det(\rho) \geq 0$$

$$\det(\rho) = \frac{1}{2}\left(1 - \langle Z\rangle^2 - \langle X\rangle^2 - \langle Y\rangle^2\right) \geq 0$$

$$\Rightarrow \langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 \leq 1$$

Exercise: How many eigenvalues does ρ have if it is pure? What constraint does this put on $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$?

1 eigval

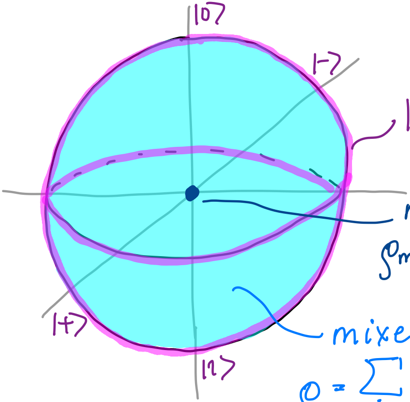
$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 & \\ 0 & \ddots & \\ & & 0 \end{pmatrix} \to \det(\;) = 0$$

other pure state: $|\psi\rangle\langle\psi| = U|0\rangle\langle 0|U^\dagger$

$\det(\rho) = 0$

$$\Rightarrow \langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 = 1$$

Mixed states and the Bloch sphere



pure states

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

maximally mixed state

$\rho_{max} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{1}$

mixed states

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Exercise: What happens if we apply a unitary U?

$$\rho' = U \rho U^\dagger \Rightarrow \det(\rho') = \det(\rho)$$

$$\det(\rho) = \frac{1}{2}(1 - \langle X \rangle^2 - \langle Y \rangle^2 - \langle Z \rangle^2)$$

To "get inside" the Bloch sphere, we have to apply a different kind of operation: a quantum channel.

A quantum channel Φ *maps* states to other states.

$$\rho \longrightarrow \rho' = \Phi(\rho)$$

Channels are linear CPTP (completely positive, trace-preserving) maps characterized by a set of **Kraus operators** $\{K_i\}$,

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$$

where

$$\sum_i K_i^\dagger K_i = I$$

Example: a channel with a single Kraus operator is a unitary operation ("unitary channel"): $\mathcal{U}$.

$$\rho \to \rho' = \mathcal{U}(\rho) = U\rho U^\dagger$$

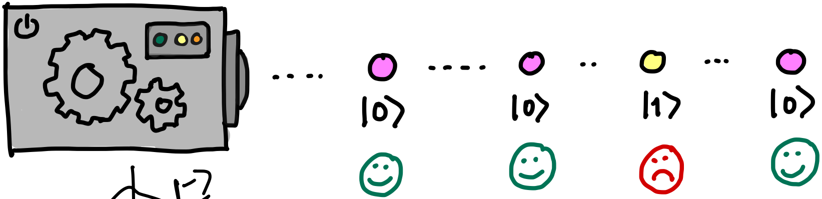
A channel's Kraus operators represent, loosely, a set of possible things that can happen to a system, including *errors*. We can use them to model noise in a system.

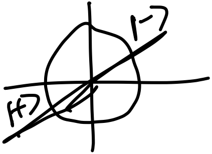
The bit flip channel

Suppose a "bit flip" (Pauli X) error occurs with probability p.

$$\mathcal{E}(\rho) = (1-p)\cdot\rho + p\cdot X\rho X$$

$$K_0 = \sqrt{1-p}\, I \qquad\qquad K_1 = \sqrt{p}\, X$$

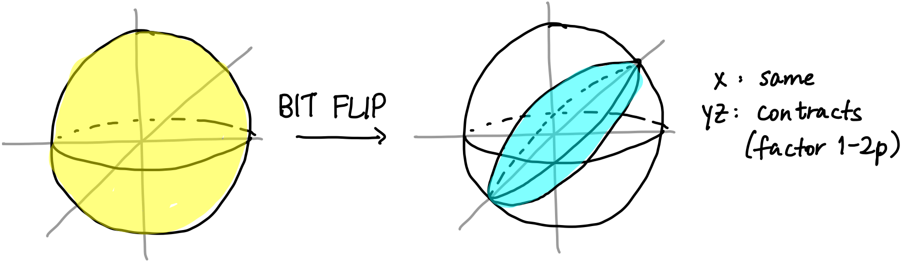




$$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \to \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \to -\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

We can visualize the effects of such a channel by observing how it deforms the Bloch sphere.

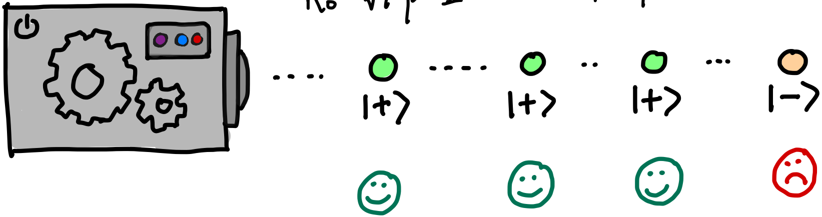


The phase flip channel

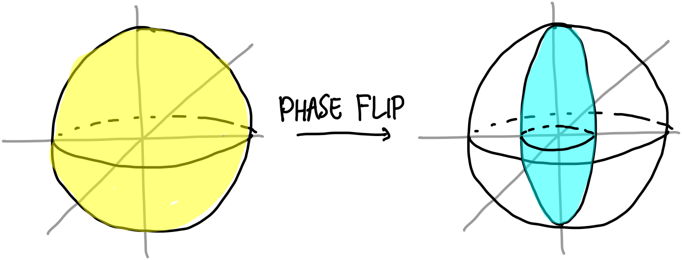
Suppose a “phase flip” (Pauli Z) error occurs with probability p.

$$\mathcal{E}(\rho) = (1-p)\cdot\rho + p\cdot Z\rho Z$$

$$K_0 = \sqrt{1-p}\, I \qquad K_1 = \sqrt{p}\, Z$$



The phase flip channel

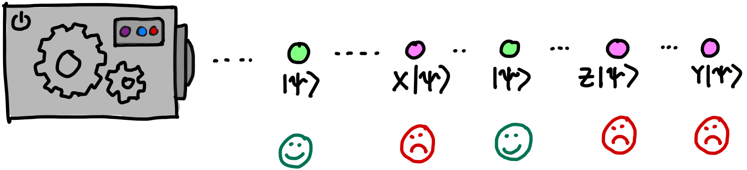


z : same
xy : contracts
(factor 1-2p)

The depolarizing channel

Suppose each Pauli error occurs with probability $p/3$. This is called the *depolarizing channel*.

$$\mathcal{E}(\rho) = (1-p)\cdot\rho + \frac{p}{3}\cdot X\rho X + \frac{p}{3} Y\rho Y + \frac{p}{3} Z\rho Z$$



The depolarizing channel

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$$

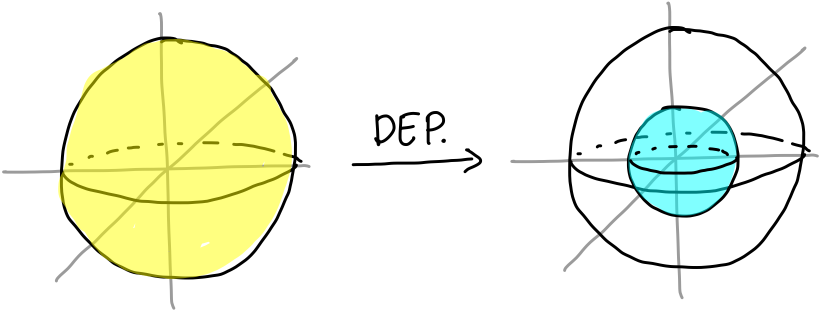
can also be written as

$$\mathcal{E}(\rho) = (1-p)\cdot\rho + p\cdot\frac{I}{2}$$

Exercise: show this!

Think of this as outputting ρ w/probability $1-p$, and maximally mixed state with probability p.

The depolarizing channel



Exercise: Suppose we prepare a system in the state

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

However, depolarization with strength $p = 0.02$ occurs. What is the probability of measuring (in the computational basis) and obtaining the $|0\rangle$ state as output?

Solution 1: solve by hand.

... too tedious, but you can evaluate

Solution 2: solve with PennyLane's ``default.mixed'' device!

Amplitude damping channel

Example: amplitude damping. $|1\rangle$ relaxes to $|0\rangle$ with probability p.

We use the inner product to tell us how close two pure states $|\psi\rangle$ and $|\phi\rangle$ were:

What can we do for mixed states?

How close are two mixed states σ, ρ?

One common metric is the **trace distance**:

$$\mathrm{Tr}(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1 = \frac{1}{2}\mathrm{Tr}\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)}$$

Value of trace distance is bounded by $0 \leq T(\rho, \sigma) \leq 1$, and *lower* trace distance is better.

Another is the **fidelity**:

$$F(\rho, \sigma) = \left(\mathrm{Tr} \sqrt{\sqrt{\rho}\, \sigma \sqrt{\rho}} \right)^2$$

Value of fidelity is bounded by $0 \leq F(\rho, \sigma) \leq 1$, and *higher* fidelity is better.

Exercise: Suppose both σ and ρ are pure states. What does the expression for fidelity reduce to?

Exercise: Suppose ρ is pure but σ is not. What does the expression for fidelity reduce to?

Example

Let's apply some simulated noise to the VQE problem from hands-on 4.

Next time

Last class:

- Discussion about current state of quantum computers

Action items:

1. Literacy assignment 3
2. Project code and report

Recommended reading:

- Quantum volume demo `https://pennylane.ai/qml/demos/quantum_volume.html`