

# **CPEN 400Q Lecture 03**

## **Measurement I (projective measurements)**

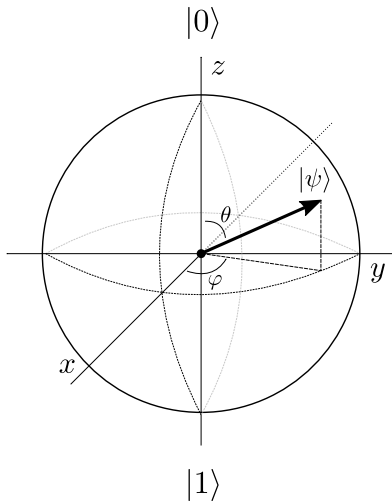
Monday 13 January 2024

# Announcements

- Quiz 1 today
- Tomorrow's tutorial: first hands-on assignment (for submission). **New room: CEME 1215**
- Assignment 1 due Sunday 26 Jan at 23:59

## Last time

We saw the most general single-qubit state parametrization, and how it can be represented in 3D space on the Bloch sphere write out params



## Last time

We learned about the three Pauli rotations

	Math	Matrix	Code	Special cases ( $\theta$ )
$RZ$	$e^{-i\frac{\theta}{2}Z}$	$\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$	<code>qml.RZ</code>	$Z(\pi), S(\pi/2), T(\pi/4)$
$RY$	$e^{-i\frac{\theta}{2}Y}$	$\begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$	<code>qml.RY</code>	$Y(\pi)$
$RX$	$e^{-i\frac{\theta}{2}X}$	$\begin{pmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$	<code>qml.RX</code>	$X(\pi), SX(\pi/2)$

## Last time

We interpreted unitary operations as rotations of the state vector on the Bloch sphere.

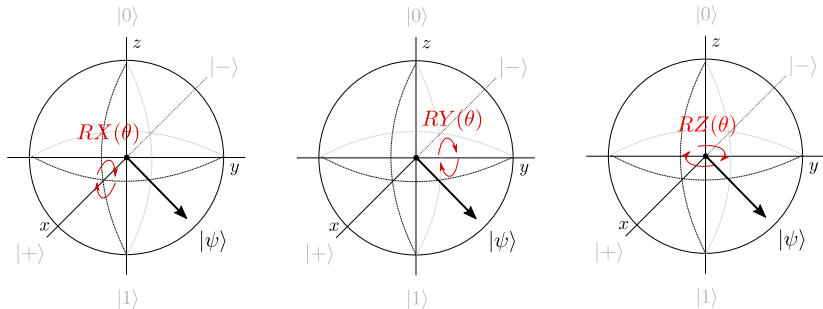


Image credit: Codebook node I.6

## Last time

We left off with a few exercises.

**Exercise 1:** design a quantum circuit that prepares

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5}{4}}|1\rangle$$

**Exercise 2:** In PennyLane, implement the circuit below



Run your circuit with two different values of  $\theta$  and take 1000 shots. How does  $\theta$  affect the measurement outcome probabilities?

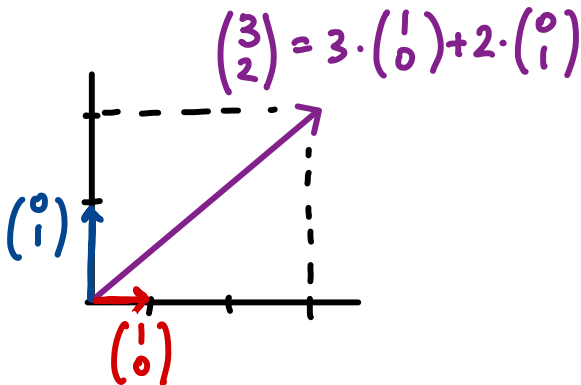
# Learning outcomes

- compute the inner product between two quantum states
- perform a projective measurement
- distinguish between global phase and relative phase
- measure a qubit in different bases

# Inner products

We can now create any single-qubit quantum state: how do we *compare* them?

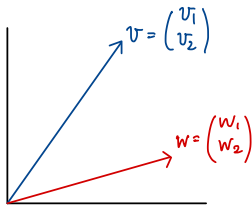
For intuition, consider a classical vector space.





# Inner products

We can define an **inner product** between two vectors to quantify much overlap they have.



## Inner products

Take just one of these representations:

The inner product in Hilbert space is defined as

To avoid cumbersome notation define the **bra** (*braket* notation):

**Exercise:** compute the inner product of the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with itself.

## Inner products

**Exercise:** compute the inner product between all possible combinations of  $|0\rangle$  and  $|1\rangle$ .

$\langle 0 0\rangle$	
$\langle 0 1\rangle$	
$\langle 1 0\rangle$	
$\langle 1 1\rangle$	

## Orthonormal bases

For a single qubit, a pair of states that are **normalized** and **orthogonal** constitute an **orthonormal basis** for the Hilbert space.

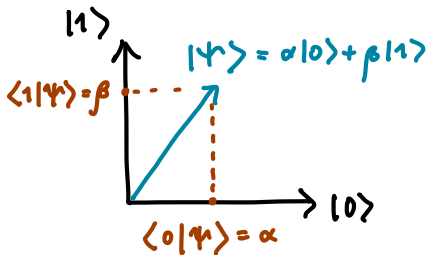
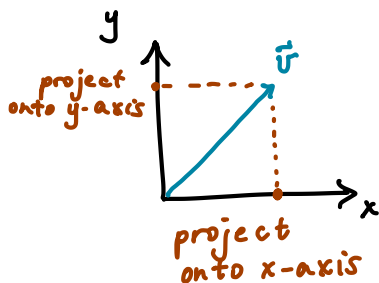
**Exercise:** do the states

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

form an orthonormal basis?

# Projective measurements

Measurement is performed with respect to a basis; we perform **projections** to determine the overlap with a given basis state.



(Image for expository purposes only!)

# Projective measurements

When we measure state  $|\varphi\rangle$  with respect to basis  $\{|\psi_i\rangle\}$ , the probability of obtaining outcome  $i$  is

If we observe outcome  $i$ , the system will be left in state  $|\psi_i\rangle$  after the measurement.

## Measurement in the computational basis

Let  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ .

Then if we measure  $|\psi\rangle$  in the computational basis,



## Measurement in the computational basis

**Exercise:** what are the measurement outcome probabilities if we measure

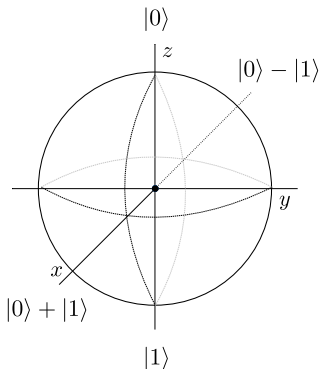
$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

in the computational basis?

# Measurement in the computational basis

These are *not* the same state: there is a relative phase.

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$



How to tell them apart?

# Basis rotations

Projective measurements can be performed with respect to any orthonormal basis. For example,  $\{|+\rangle, |-\rangle\}$ :

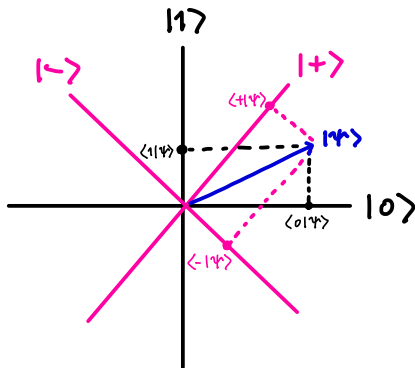


Image credit: Codebook node I.9

So far we've seen 3 ways of extracting information out of a QNode:

1. `qml.state()`
2. `qml.probs(wires=x)`
3. `qml.sample()`

But these return results of measurements in the computational basis; and most hardware only allows for this.

How can we measure with respect to *different bases*?

## Basis rotations

Use a basis rotation to “trick” the quantum computer.

Suppose we want to measure in the “Y” basis:

$$|p\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle), \quad |m\rangle = \frac{1}{\sqrt{2}} (|0\rangle - i|1\rangle).$$

Recall: unitary operations preserve length *and* angles between normalized quantum state vectors (prove on A1!)

There exists a unitary operation that will convert between this basis and the computational basis.

**Exercise:** determine a quantum circuit that sends

$$|0\rangle \rightarrow |p\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)$$

$$|1\rangle \rightarrow |m\rangle = \frac{1}{\sqrt{2}} (|0\rangle - i|1\rangle)$$

## Basis rotations

At the end of our circuit, apply the reverse (adjoint) of this transformation to rotate *back* to the computational basis.

If we measure and observe  $|0\rangle$ , we know the qubit was previously  $|p\rangle$  in the  $Y$  basis (analogous result for  $|m\rangle$ ).

# Adjoints

In PennyLane, we can compute adjoints of operations *and* entire quantum functions using `qml.adjoint`:

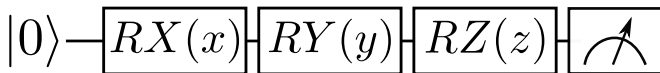
```
def some_function(x):  
    qml.RZ(Z, wires=0)  
  
def apply_adjoint(x):  
    qml.adjoint(qml.S)(wires=0)  
    qml.adjoint(some_function)(x)
```

`qml.adjoint` is a special type of function called a **transform**. We will cover transforms in more detail later in the course.



## Basis rotations: hands-on

Let's run the following circuit, and measure in the  $Y$  basis



Hands-on time...

## Next time

### Content:

- Mathematical representation of multi-qubit systems
- Multi-qubit gates
- Entanglement

### Action items:

1. Work on Assignment 1 (can do questions 1 & 5, 2ai,ii now)

### Recommended reading:

- Codebook modules IQC, SQ
- Nielsen & Chuang 4.3