CPEN 400Q Lecture 19
Intro to quantum error correction

Wednesday 19 March 2025

Announcements

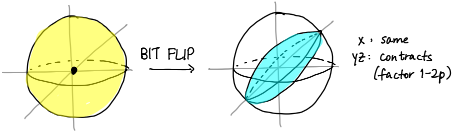
- Quiz 9 Monday
- Sign up for project presentations, and final oral interview (Canvas calendar)
- A3 due Tuesday 25 March 23:59

Last time

We expressed a noisy processes as quantum channels.

Bit flip channel

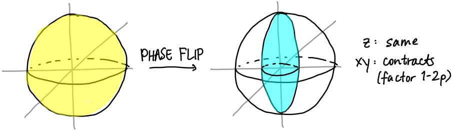
$$\Phi(\rho) = (1-p)\rho + pX\rho X$$



$$|0\rangle \longleftrightarrow |1\rangle$$

Phase flip channel

$$\Phi(\rho) = (1-p)\rho + pZ\rho Z$$

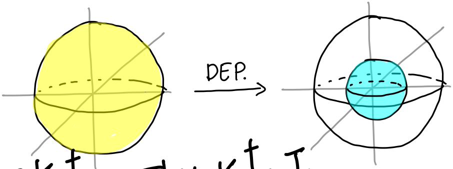


$$|+\rangle \longleftrightarrow |-\rangle$$

Last time

Depolarizing channel

$$\Phi(\rho) = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$$



$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger \qquad \sum_i K_i K_i^\dagger = I$$

Amplitude damping channel

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

$$\hookrightarrow \sqrt{p}\,|0\rangle\langle 1|$$

Last time

We compared density matrices using two distance measures.

Trace distance

$$T(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1 = \frac{1}{2}\mathrm{Tr}\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)}$$

Fidelity

$$F(\rho, \sigma) = \left[\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right]^2$$

We simulated these processes and computed distance measures using PennyLane's ''`default.mixed`'' device.

Learning outcomes

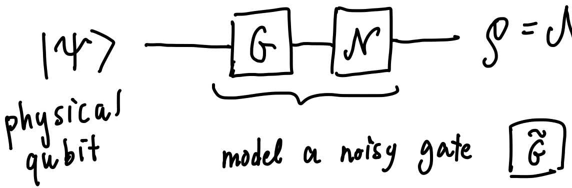
Module 5 (NEW!): Introduction to quantum error correction

Today:

1. Design circuits to correct bit flips using a simple quantum error correcting code (repetition code)
2. Apply logical operations to encoded qubits
3. Correct bit and phase flip errors with the 9-qubit code

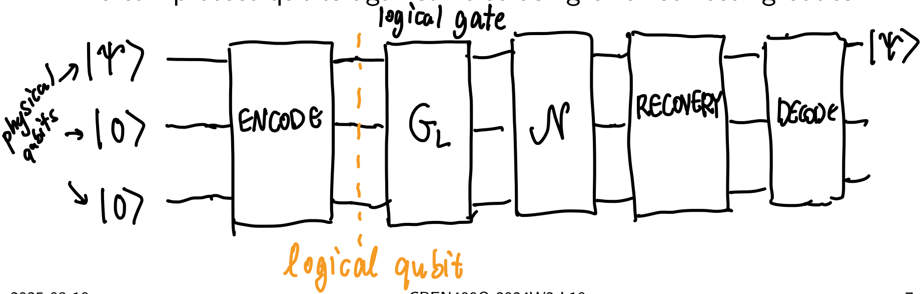
Quantum error correction

Our current picture of noise:



$$\rho = \mathcal{N}\left(G|\psi\rangle\langle\psi|G^{\dagger}\right)$$

We can protect qubits against noise using error correcting codes.



Repetition codes

Imagine sending a bit string through a classical channel that flips each bit (individually) with probability p.

$$\phi\,(0\ 0\ 0\ 1\ 1) \longrightarrow 0\ 0\ 1\ 1\ 1$$

Idea: add redundant information to enable *detection* and *correction* of bit flips.

Define *encoding* and *decoding* operations, $\mathcal{E}$ and $\mathcal{D}$,

$$\mathcal{E}(0) \longrightarrow 000 \quad \text{"logical 0"}$$
$$\mathcal{E}(1) \longrightarrow 111 \quad \text{"logical 1"}$$

$$\mathcal{D}(000) \longrightarrow 0$$
$$\mathcal{D}(111) \longrightarrow 1$$

$$0\ 0\ 0\ 1\ 1 \longrightarrow \underbrace{000}_{001}\ 000\ 000\ 111\ \underset{101}{111}$$

Repetition codes

Devise a procedure, $\mathcal{R}$, to recover from an error: majority voting.

$$R(b_0, b_1, b_2) = MAJ(b_0, b_1, b_2)$$

Operation	Outcome				
	0	0	1	0	1
$\mathcal{E}$	000	000	111	000	111
ϕ	010	000	110	000	111
$\mathcal{R}$	000	000	111	000	111
$\mathcal{D}$	0	0	1	0	1

000 → 101 → 111 → 1

Is this better? When will this fail? → 2 bit flips!

$$\text{prob error} = \underbrace{3p^2(1-p)}_{\text{(two flips)}} + \underbrace{p^3}_{\text{(all three flipped)}} \longrightarrow P_e < P_{\text{original}}$$

Quantum repetition code (bit flip code)

Idea: $\mathcal{E}(|\psi\rangle) \Rightarrow |\psi\rangle^{\otimes 3}$

Why won't this work?

- NO CLONING!!!
- Measuring to compare them destroys them
- More than one type of error can happen

Alternative: quantum-specific methods!

$$\mathcal{E}(|0\rangle) = |000\rangle = |0\rangle_L$$

$$\mathcal{E}(|1\rangle) = |111\rangle = |1\rangle_L$$

Bit flip code: encoding circuit

Exercise: (a) How does $\mathcal{E}$ affect a general state,

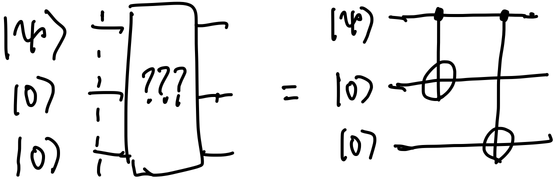
$$\mathcal{E}(\alpha|0\rangle + \beta|1\rangle)$$

(b) What does the corresponding circuit look like?

(a) $\mathcal{E}(\alpha|0\rangle+\beta|1\rangle) = \alpha|0\rangle_L + \beta|1\rangle_L = \alpha|000\rangle+\beta|111\rangle$

$(= \mathcal{E}(\alpha|0\rangle)+\mathcal{E}(\beta|1\rangle))$

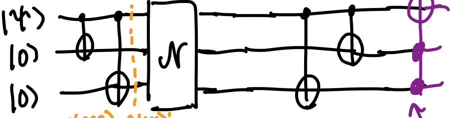
(b)



$(\alpha|0\rangle+\beta|1\rangle)|00\rangle = \alpha|000\rangle+\beta|100\rangle$

Bit flip code: recovery operation

Let's design a circuit to detect (a) *whether* an error occurs, and (b) *which* error occurs.



$|\psi\rangle$ ✓

$|00\rangle$ if no bit flip
$|11\rangle$ if X_0
$|10\rangle$ if X_1
$|01\rangle$ if X_2

syndrome

$\alpha|000\rangle + \beta|111\rangle$

No bit flip or one bit flip

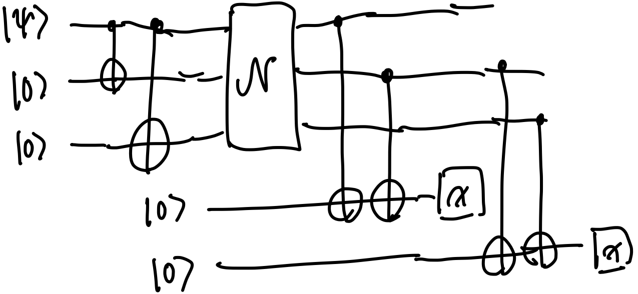
$I: \alpha|000\rangle + \beta|111\rangle$
$X_0: \alpha|100\rangle + \beta|011\rangle$
$X_1: \alpha|010\rangle + \beta|101\rangle$
$X_2: \alpha|001\rangle + \beta|110\rangle$

$\xrightarrow{\text{detect}}$

$\alpha|000\rangle + \beta|100\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle$ ✓
$\alpha|111\rangle + \beta|011\rangle = \boxed{(\alpha|1\rangle + \beta|0\rangle)|11\rangle}$ × Apply X
$\alpha|010\rangle + \beta|110\rangle = (\alpha|0\rangle + \beta|1\rangle)|10\rangle$ ✓
$\alpha|001\rangle + \beta|101\rangle = (\alpha|0\rangle + \beta|1\rangle)|01\rangle$ ✓

Bit flip code: recovery operation

Let's design a circuit to detect (a) *whether* an error occurs, and (b) *which* error occurs.



m_0, m_1 } apply X based on these

odd

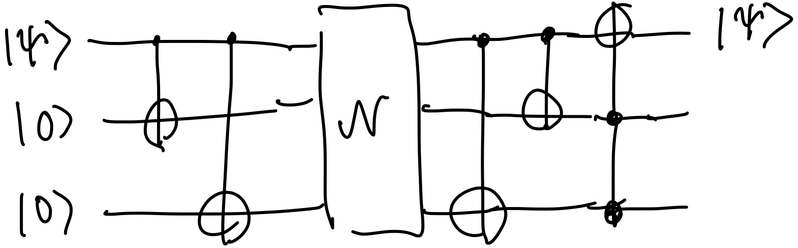
$\alpha|100\rangle + \beta|011\rangle$

even

Bit flip code: recovery operation

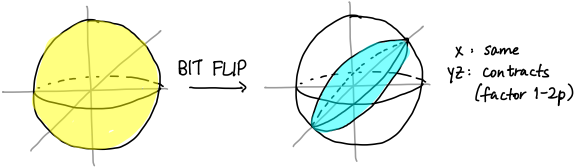
Design a circuit to *recover* the state of the first qubit

Bit flip code: full circuit for bit flip code



Bit flip code: analysis

Imagine any possible state that can go through the channel:



Exercise (from last time): Determine $F(|\psi\rangle, \Phi(|\psi\rangle\langle\psi|))$ where Φ is the *bit flip channel* with parameter p.

$$F(|\psi\rangle, \rho) = \langle\psi|\rho|\psi\rangle$$
$$= \langle\psi|\big((1-p)|\psi\rangle\langle\psi| + pX|\psi\rangle\langle\psi|X\big)|\psi\rangle$$
$$= (1-p)\langle\psi|\psi\rangle\langle\psi|\psi\rangle + p\,\underbrace{\langle\psi|X|\psi\rangle}_{\langle X\rangle}\langle\psi|X|\psi\rangle$$
$$= 1 - p + p\langle X\rangle^2$$
$$\geq 1-p$$

Bit flip code: analysis

Compare overall fidelity of the state *after the error*, vs. fidelity *after recovery*.

After the error

$$F \geq 1 - p$$

no error

After recovery, have the state

Mixed state : $(1-p)^3 |\psi\rangle\langle\psi| + 3p(1-p)^2 |\psi\rangle\langle\psi|$ + other stuff

$$F\left(|\psi\rangle, \quad \right) \geq (1-p)^3 + 3p(1-p)^2$$

$p < \frac{1}{2} \Rightarrow$ better to do correction

Designing logical operations

Logical operations are specific to codes.

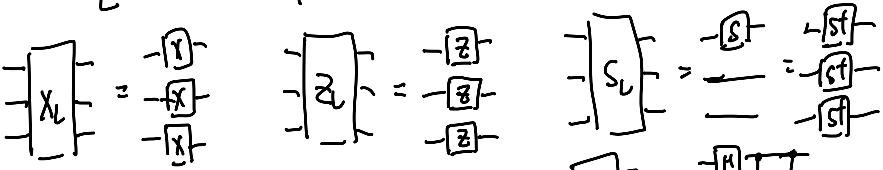
They should act on the logical states $|0\rangle_L$, $|1\rangle_L$ the same way the physical operations act.

Exercise: design circuits for logical X, Z, H, S, and $CNOT$.

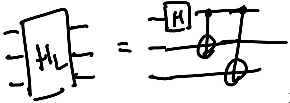
$X: \quad \alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|1\rangle + \beta|0\rangle$

$X_L: \quad \alpha|0\rangle_L + \beta|1\rangle_L \longrightarrow \alpha|1\rangle_L + \beta|0\rangle_L$

$S^3 = S^\dagger$



$|0\rangle_L \rightarrow \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L)$



Phase flip errors

With our encoding

☆ Start here on Monday.

and appropriate circuitry, we can correct *bit flip errors* but not phase flip errors.

Phase flip code: encoding circuit

Main idea: make phase flip errors "look like" bit flip errors.

Shor code

To correct a combination of one bit flip and/or phase flip error, we can *concatenate* codes: use logical qubit of a phase flip code as the “physical” qubits in a bit flip code.

Next time

Next class:

- Properties of errors and error correcting codes
- Stabilizers and stabilizer codes

Action items:

1. A3 (due 25 March 23:59)
2. Work on project

Recommended reading:

- From this class: Codebook EC; N&C 10.1-10.2
- For next class: Codebook EC; 10.3, 10.5