CPEN 400Q Lecture 13
QFT and quantum phase estimation

Wednesday 26 February 2025

Announcements

- Quiz 6 on Monday
- Assignment 2 due tomorrow 23:59
- Assignment 3 coming next week - last "big" technical assignment
- Fill out peer review survey by Friday (link in Piazza)

Last time

We introduced the quantum Fourier transform, and saw how it is the analog of the classical inverse discrete Fourier transform.

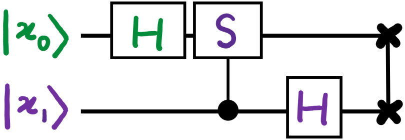
$$QFT|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle$$

$$QFT = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix}$$

where for n qubits, $N = 2^n$, and $\omega = e^{2\pi i/N}$

Last time

We saw the circuits for some special cases.



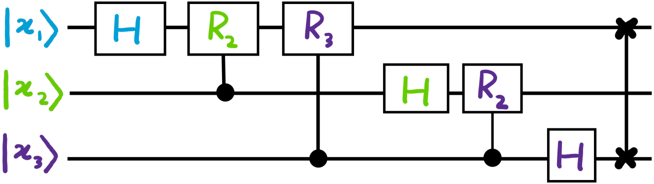


Image credit: Xanadu Quantum Codebook node F.2, F.3

The general form of the circuit is:

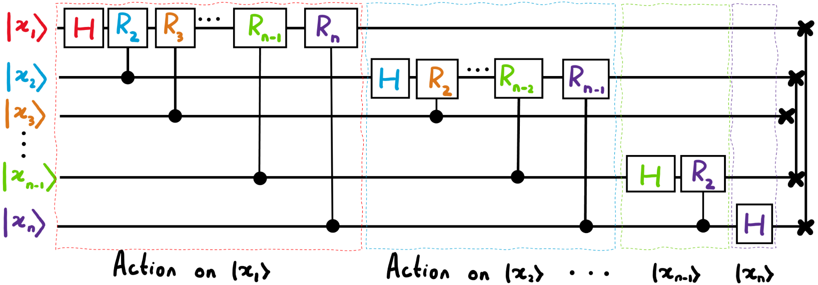


Image credit: Xanadu Quantum Codebook node F.3

Learning outcomes

- Redevelop the circuit for the quantum Fourier transform
- Outline the steps of the quantum phase estimation (QPE) subroutine
- Use the QFT to implement QPE

A circuit for the QFT

Start with the one-qubit case:

A circuit for the QFT

Next, consider the two-qubit case:

$$QFT = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Re-express higher powers of $\omega = e^{\frac{2\pi i}{4}}$:

A circuit for the QFT

$$QFT = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix}$$

How does this affect the computational basis states?

A circuit for the QFT

A circuit for the QFT

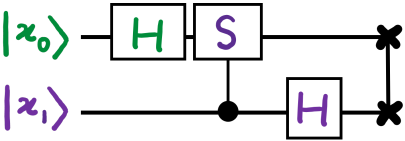


Image credit: Xanadu Quantum Codebook node F.3

A circuit for the QFT

Three-qubit case:

$$QFT = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \cdots \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \cdots \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \cdots \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \cdots \end{pmatrix}$$

$$QFT|x_1x_2x_3\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i 0.x_3}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.x_2x_3}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.x_1x_2x_3}|1\rangle)$$

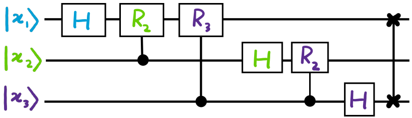


Image credit: Xanadu Quantum Codebook node F.3

A circuit for the QFT

General expression:

$$
\begin{aligned}
|x\rangle \quad &\rightarrow \quad \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle \\
&= \quad \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}
\end{aligned}
$$

Full mathematical derivation at end of slides.

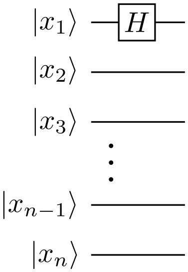
A circuit for the QFT

$$|x\rangle \rightarrow \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

Starting with the state

$$|x\rangle = |x_1 \cdots x_n\rangle,$$

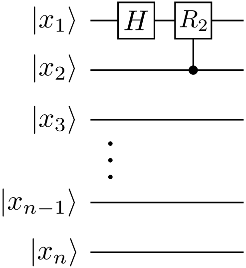
apply a Hadamard to qubit 1:



A circuit for the QFT

$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

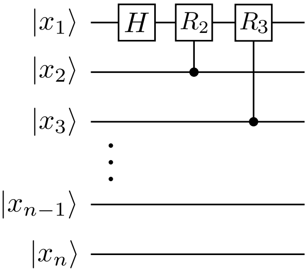
Apply controlled R_2 from qubit $2 \to 1$



A circuit for the QFT

$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

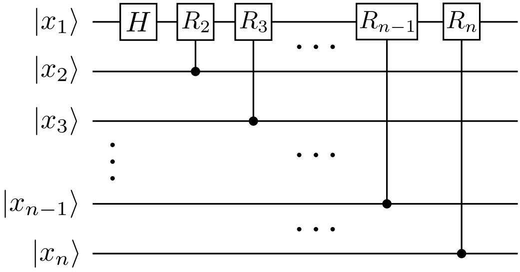
Apply controlled R_3 from qubit $3 \to 1$



A circuit for the QFT

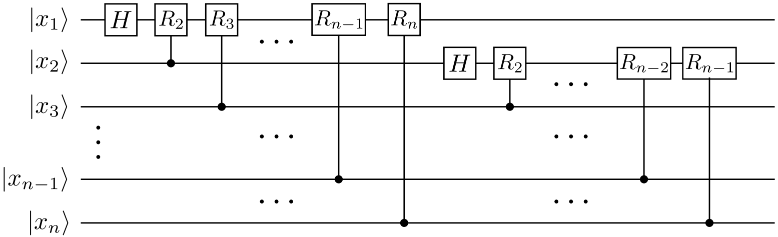
$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

Apply a controlled R_4 from $4 \to 1$, etc. up to the n-th qubit to get



A circuit for the QFT

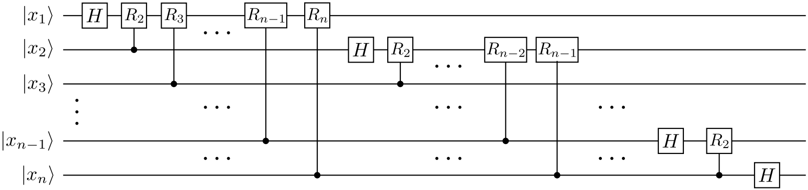
Similar thing with second qubit: apply H then controlled rotations from every qubit from 3 to n to get



A circuit for the QFT

Do this for all qubits to get that big ugly state from earlier:

$$|x\rangle \rightarrow \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$



Qubit order is backwards - easily fixed with SWAP gates.

Exercise: What are the gate counts and depth of this circuit?

-
-
-
-

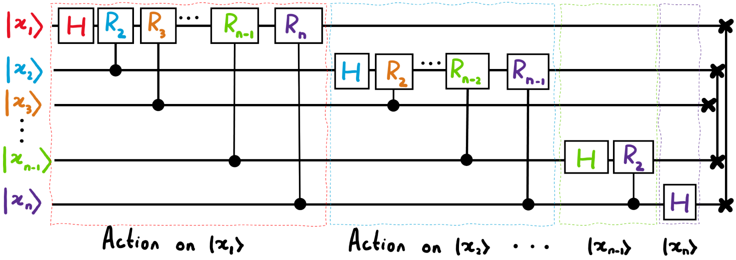


Image credit: Xanadu Quantum Codebook node F.3

Reminder: where are we going?

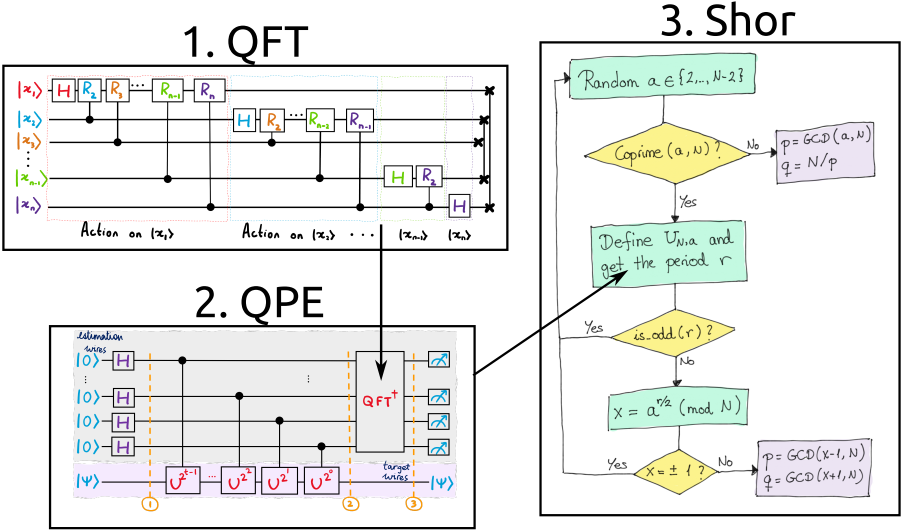


Image credit: Xanadu Quantum Codebook nodes F.3, P.3, S.4

Eigenvalues of unitary matrices are complex numbers with magnitude 1, i.e.,

where θ_k is a phase, $|\theta_k| \leq 1$. (See proof at end of slides)

What if we want to *estimate* an unknown θ_k?

Eigenvalues of unitary matrices

Idea: apply U to the relevant eigenvector, because that's "what makes the phase come out".

...but this is an unobservable *global* phase!

We have to do something different: eigenvalue estimation, or quantum phase estimation (QPE).

Quantum phase estimation

Given unitary U and eigenvector $|k\rangle$, estimate θ_k such that

$$U|k\rangle = e^{2\pi i \theta_k}|k\rangle$$

Must determine:

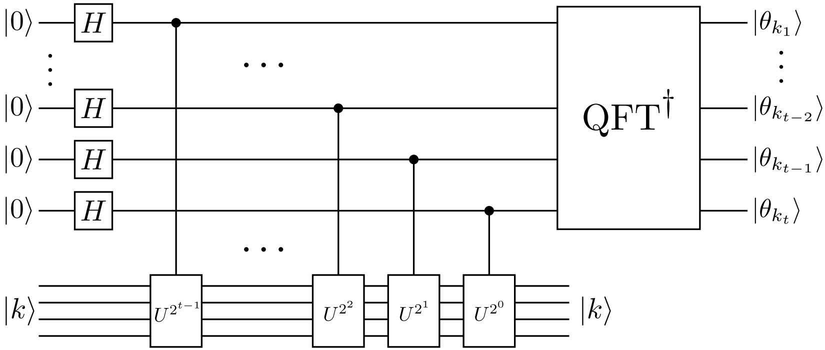
- How to design a circuit that extracts θ_k
- What precision can we estimate it to
- What to do if we don't know a $|k\rangle$ in advance

(*You will explore the last two in your homework!*)

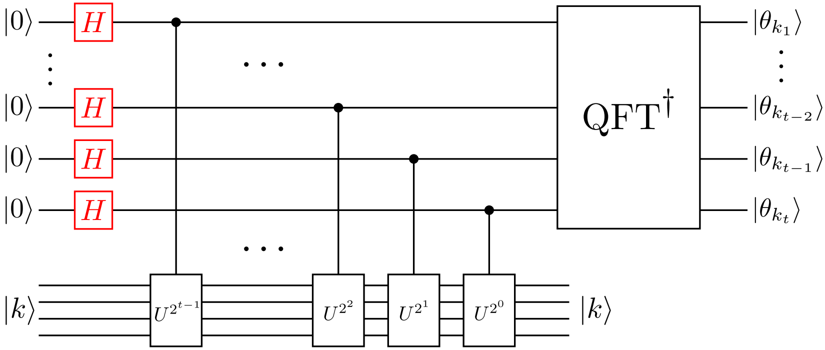
Let U be an n-qubit unitary; $|k\rangle$ is an n-qubit eigenstate.

Assume θ_k can be represented *exactly* using t bits:

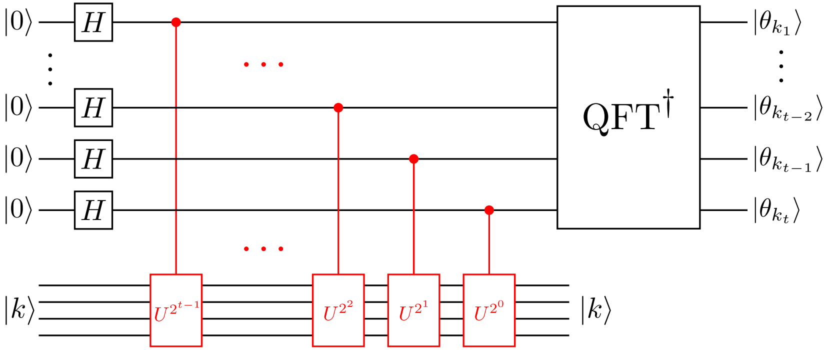
$$\theta_k = 0.\theta_{k_1} \cdots \theta_{k_t}$$



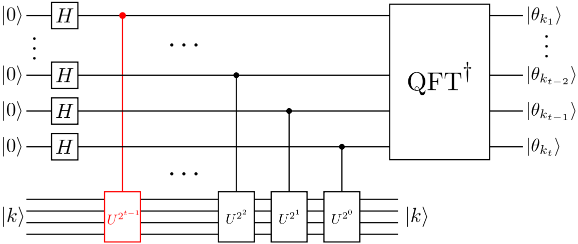
Quantum phase estimation: step 1



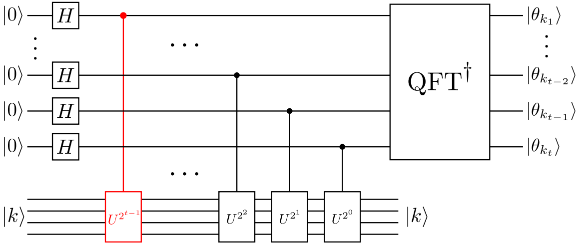
Quantum phase estimation: step 2



Quantum phase estimation: step 2

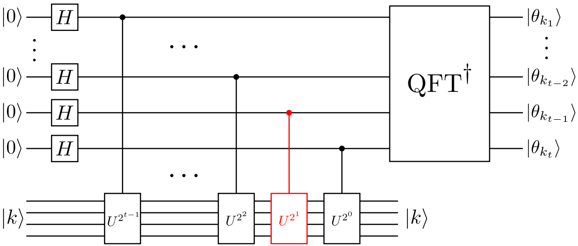


Quantum phase estimation: step 2



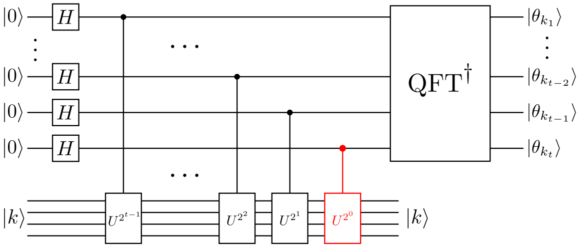
Use phase kickback

Quantum phase estimation: step 2



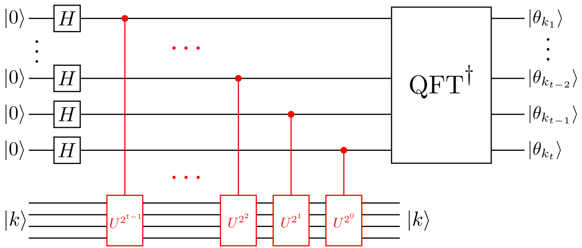
Check second-last qubit (ignore the others)

Quantum phase estimation: step 2



Can show in the same way for the last qubit (ignore others)

Quantum phase estimation: step 2

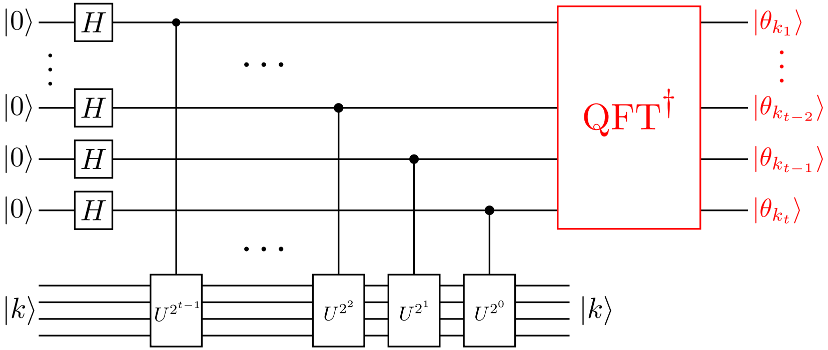


After step 2, we have the state

$$\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i 0.\theta_{k_t}}|1\rangle)\cdots\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i 0.\theta_{k_1}\cdots\theta_{k_t}}|1\rangle)|k\rangle$$

Should look familiar!

Measure to learn the bits of θ_k.



Let's implement it.

Next time

Content:

- Quiz 6 on Monday
- Order finding

Action items:

1. Work on project (do peer assessment)
2. Finish assignment 2

Recommended reading:

- From today: Codebook QFT, QPE, Nielsen & Chuang 5.1-2
- For next class: Codebook SH, Nielsen & Chuang 5.3.1

We will reexpress k/N in fractional binary notation, then reshuffle and *factor* the output state to uncover the circuit structure.

$$
\begin{aligned}
|x\rangle &\rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x(k/N)} |k\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i x\left(\sum_{\ell=1}^{n} k_\ell 2^{-\ell}\right)} |k_1 \cdots k_n\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{\ell=1}^{n} e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle
\end{aligned}
$$

Derivation of output state of QFT

$$
\begin{aligned}
&= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{\ell=1}^{n} e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \\
&= \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(\sum_{k_\ell=0}^{1} e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \right) \\
&= \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(|0\rangle + e^{2\pi i x 2^{-\ell}} |1\rangle \right) \\
&= \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}
\end{aligned}
$$

Fun fact: eigenvalues of unitaries are complex numbers with magnitude 1.

Proof: Let λ_k be the eigenvalue associated with eigenvector $|k\rangle$ of a unitary U:

$$U|k\rangle = \lambda_k|k\rangle$$

Take the conjugate transpose:

$$\langle k|\, U^\dagger = \langle k|\, \lambda_k^*$$

Multiply the two sides together:

$$\begin{aligned}\langle k|\, U^\dagger U|k\rangle &= \langle k|\, \lambda_k^* \lambda_k|k\rangle \\ \langle k|k\rangle &= |\lambda_k|^2\langle k|k\rangle \\ 1 &= |\lambda_k|^2\end{aligned}$$