CPEN 400Q Lecture 20
Conditions for quantum error correction; intro to stabilizers

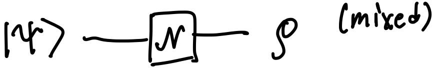
Monday 24 March 2025

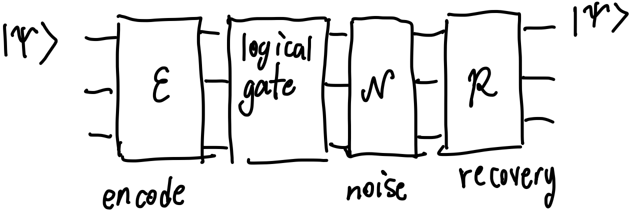
Announcements

- Quiz 9 today
- Last tutorial assignment tomorrow
- Sign up for project presentations, and final oral interview (Canvas calendar)
- Project rubric will be available in next few days
- A3 due tomorrow 23:59

Last time

We discussed the motivation for quantum error correction, and developed a mathematical (/graphical) model for noise.



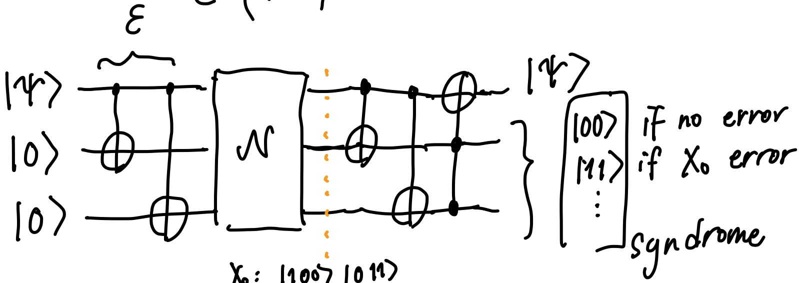


Last time

We saw our first quantum error correcting code: the quantum repetition code, or bit flip code.

$$\mathcal{E}(|0\rangle) = |000\rangle = |0\rangle_L$$

$$\mathcal{E}(|1\rangle) = |111\rangle = |1\rangle_L$$



X_0: $|100\rangle$ $|011\rangle$

X_1: $|010\rangle$ $|101\rangle$

X_2: $|001\rangle$ $|110\rangle$

Learning outcomes

Today:

1. Correct bit *and* phase flip errors with the 9-qubit code
2. Outline the conditions under which errors can be corrected
3. Define the stabilizers of a quantum error correcting code

Phase flip errors

With our encoding

$$\mathcal{E}(\alpha|0\rangle+\beta|1\rangle) = \alpha|0\rangle_L + \beta|1\rangle_L$$
$$= \alpha|000\rangle + \beta|111\rangle$$

and appropriate circuitry, we can correct *bit flip errors* but not phase flip errors.

Pauli Z error

$$Z_2 : \alpha|0\rangle_L - \beta|1\rangle_L$$
$$Z_1 : \alpha|0\rangle_L - \beta|1\rangle_L$$
$$Z_0 : \alpha|0\rangle_L - \beta|1\rangle_L$$

Phase flip code: encoding circuit

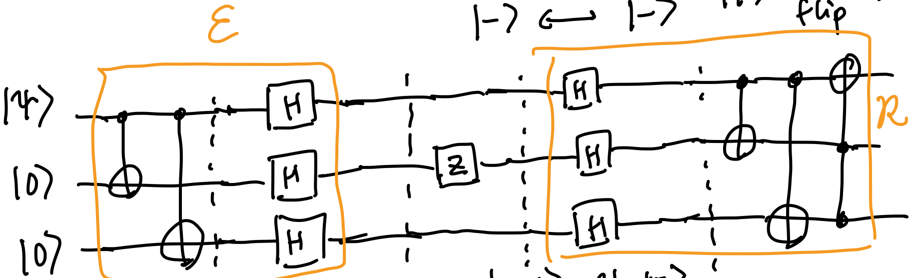
Main idea: make phase flip errors “look like” bit flip errors.

$|+\rangle / |-\rangle$

$|+\rangle \xleftrightarrow{\text{bit flip}} |+\rangle$

$|-\rangle \longleftrightarrow |-\rangle$

$|+\rangle \xleftrightarrow[\text{flip}]{\text{phase}} |-\rangle$



$\alpha|+-+\rangle + \beta|-+-\rangle$

$\alpha|000\rangle + \beta|111\rangle$

$\alpha|010\rangle + \beta|101\rangle$

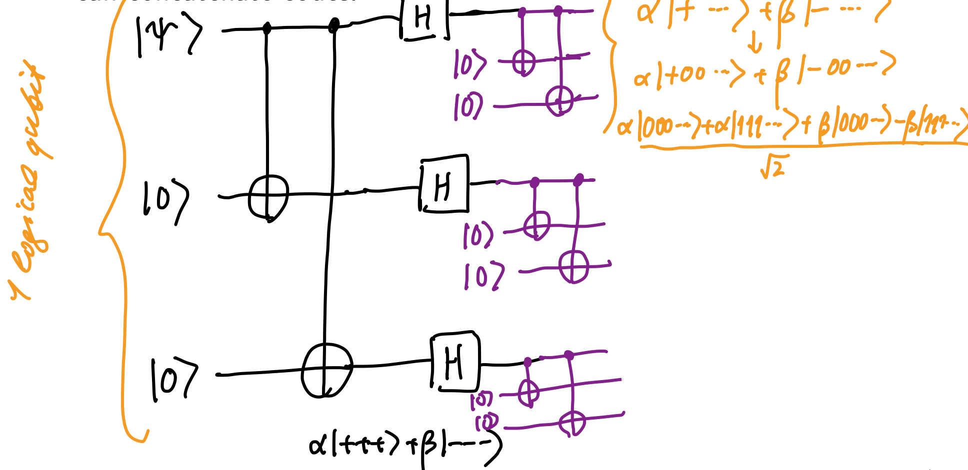
$\alpha|+++\rangle + \beta|---\rangle$

$|0\rangle_L = |+++\rangle$

$|1\rangle_L = |---\rangle$

Shor code

To correct a combination of one bit flip and/or phase flip error, we can *concatenate* codes.



Shor code

The Shor code can correct one *arbitrary* error *on a single qubit*.

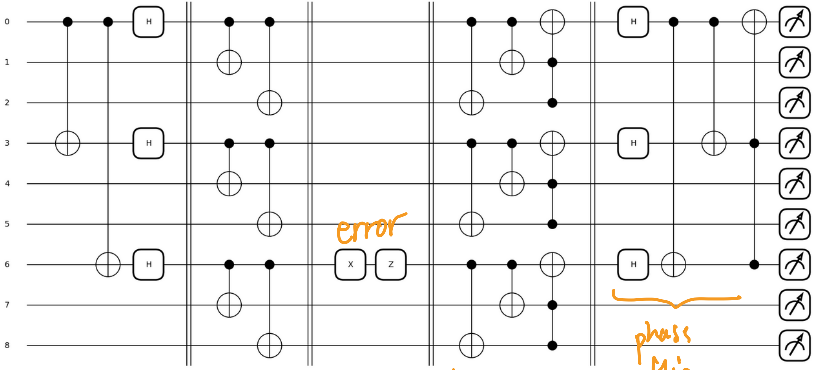
$$|0\rangle_L = \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle_L = \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Imagine a bit + phase flip on qubit 6:

$$|0\rangle_L \longrightarrow \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(-|100\rangle + |011\rangle)}{2\sqrt{2}}$$

Shor code


0
1
2
3
4
5
6
7
8
H
H
H
error
X
Z
H
H
H
bit flip recovery
phase flip recovery

Correcting arbitrary errors

The 9-qubit code can correct *arbitrary* errors on a *single qubit*.

Represent possible errors as $\{E_i\}$, full error channel as

$$\mathcal{N}(|\psi\rangle\langle\psi|) = \sum_i E_i |\psi\rangle\langle\psi| E_i^\dagger$$

We can correct $\{E_i\} = \{I, X_j, Z_j, X_j Z_j\}_j$. If we can write an arbitrary error E_k on qubit j as

$$E_k = e_{k0} I + e_{k1} X + e_{k2} Z + e_{k3} XZ$$

then we can correct it too.

$$E_k |\psi\rangle\langle\psi| E_k^\dagger = (\qquad) |\psi\rangle\langle\psi| (\qquad)$$

Conditions for quantum error correction

How do we know if a particular error is correctable by a QECC?

Formal definition of a quantum error correcting code is a *subspace*, C, called the **codespace**.

Example: bit flip code.

- Codewords: $|000\rangle, \; |111\rangle$
- Codespace: contents have the form $\alpha|000\rangle + \beta|111\rangle$

Define a **projector** onto the codespace,

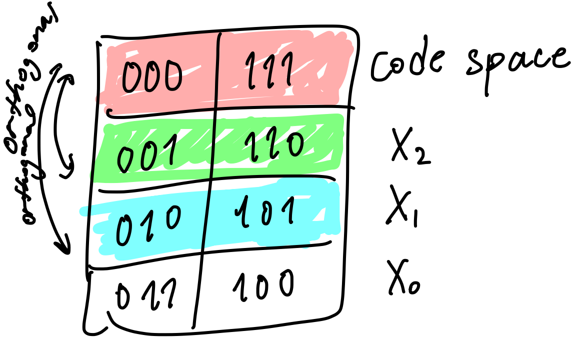
$$\Pi = |000\rangle\langle 000| + |111\rangle\langle 111|$$

$$\Pi\left(\alpha|000\rangle + \beta|111\rangle\right) = \alpha|000\rangle\langle 000|000\rangle + \alpha\cancel{|111\rangle\langle 111|000\rangle} + \ldots$$

$$= \alpha|000\rangle + \beta|111\rangle$$

Conditions for quantum error correction

The errors this code corrects are all mapped to different **orthogonal** subspaces of the full Hilbert space.



Conditions for quantum error correction

Theorem 10.1: **(Quantum error-correction conditions)** Let C be a quantum code, and let P be the projector onto C. Suppose $\mathcal{E}$ is a quantum operation with operation elements $\{E_i\}$. A necessary and sufficient condition for the existence of an error-correction operation $\mathcal{R}$ correcting $\mathcal{E}$ on C is that

$$P E_i^\dagger E_j P = \alpha_{ij} P \tag{10.16}$$

for some Hermitian matrix α of complex numbers.

If such an $\mathcal{R}$ exists, $\{E_i\}$ is called a *correctable set of errors*.

Next time

Next class:

- More stabilizer codes: phase flip, $[[5, 1, 3]]$ code, Steane code

Action items:

1. A3 (due 25 March 23:59)
2. Work on project

Recommended reading:

- From this class: Codebook EC; 10.3, 10.5
- For next class: Codebook EC; 10.1-10.5