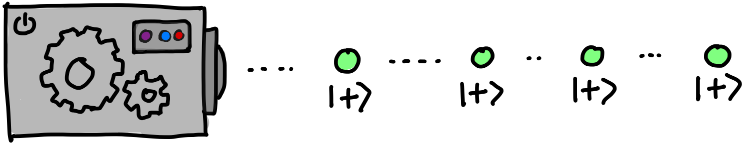
CPEN 400Q Lecture 17
Quantum channels

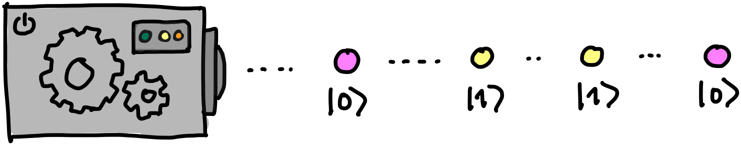
Wednesday 12 March 2025

- Quiz 8 beginning of class Monday
- Signup link for MT checkpoint meetings on Piazza (**please use group number, not your name**)
- Signup link for final oral interviews distributed later this week
- Upcoming deadlines
  - MT checkpoint report: this Friday 12:00
  - TA3: this Friday 23:59
  - A3: Tuesday 25 March 23:59

Last time

We introduced *mixed states*.





Mixed states are probabilistic mixtures of pure states.

Last time

	Pure state	Pure state ρ	Mixed state ρ
States			
Ops.			
Meas.*			

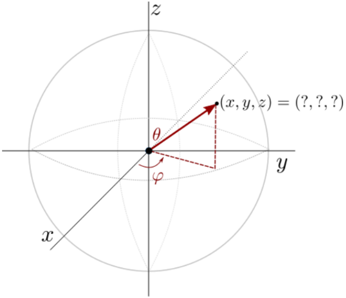
* where $\{\varphi_i\}$ form an orthonormal basis, and $\{P_i\}$ is a set of projectors $P_i = |\varphi_i\rangle \langle\varphi_i|$ or more generally a POVM ($\sum_i P_i = I$).

Learning outcomes

- Identify mixed states on the Bloch sphere
- Define fidelity and trace distance, and use them to compute the distance between two arbitrary quantum states
- Define and apply quantum channels to quantum states
- Express operations, measurements, and partial trace as quantum channels

Recall the following two problems from A2:

Write a function that uses quantum measurements to extract its Bloch vector, i.e., its coordinates in 3-dimensional space, as demonstrated by the following graphic:

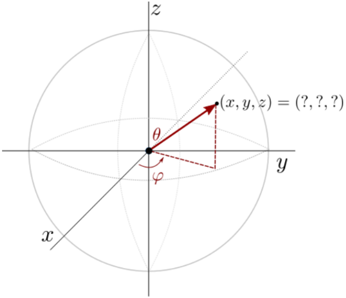


Suppose we wish to determine the expectation value of a general Hermitian observable, $\langle M \rangle$, for an arbitrary single-qubit state, but are restricted to measuring Pauli expectation values.

a. (1 point) Show that, for any single-qubit M, we can determine $\langle M \rangle$ by measuring only $\langle X \rangle$, $\langle Y \rangle$, and $\langle Z \rangle$. Upload a hand-written or typeset solution below. *Hint: write down a general single-qubit Hermitian operator, then leverage linearity of expectation values.*

Mixed states and the Bloch sphere

Write a function that uses quantum measurements to extract its Bloch vector, i.e., its coordinates in 3-dimensional space, as demonstrated by the following graphic:



Given $\langle X \rangle$, $\langle X \rangle$, $\langle X \rangle$, can we determine ρ?

Recall ρ is Hermitian; Paulis are a basis.

ρ must have trace 1:

Trace out another Pauli:

More formally, we can write any ρ as

where $a_P = \mathrm{Tr}(P\rho) = \langle P \rangle$.

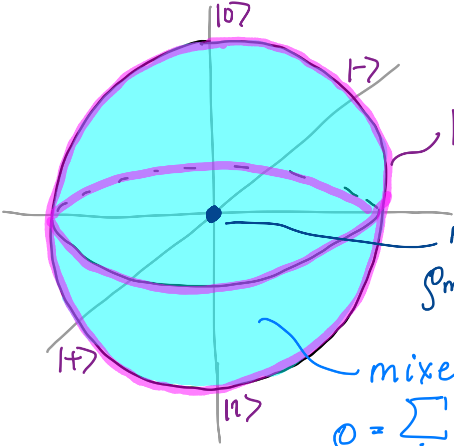
(Note that all of this generalizes to multiple qubits as well)

Re-express:

Exercise: As ρ is positive semidefinite, its eigvals are ≥ 0. What constraint does this put on $\langle X \rangle, \langle Y \rangle, \langle Z \rangle$? (Hint: look at $\det(\rho)$)

Exercise: How many eigenvalues does ρ have if it is pure? What constraint does this put on $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$?

Mixed states live *in* the Bloch sphere!



pure states

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

maximally mixed state

$\rho_{max} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{1}$

mixed states

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Comparing density matrices

The inner product tells us how close two pure states are:

How close are two mixed states σ, ρ?

One common metric is the **trace distance**:

Bounded by $0 \leq T(\rho, \sigma) \leq 1$; *lower* trace distance is better.

Another is the **fidelity**:

Bounded by $0 \leq F(\rho, \sigma) \leq 1$, and *higher* fidelity is better.

Exercise: Suppose both σ and ρ are pure states. What does the expression for fidelity reduce to?

Exercise: Suppose ρ is pure but σ is not. What does the expression for fidelity reduce to?

Exercise: What is the fidelity of any pure ρ with the maximally mixed state, $\sigma = \frac{1}{2}\boldsymbol{I}$?

Exercise: What happens to a mixed state on the Bloch sphere if we apply a unitary U?

Quantum channels

To "get inside" the Bloch sphere, we have to apply a different kind of operation: a quantum channel.

A quantum channel Φ *maps* density matrices to density matrices.

Quantum channels are completely positive, trace-preserving (CPTP) linear maps.

- Trace-preserving:
- Positive:
- Completely positive:

Quantum channels are characterized by a set of **Kraus operators** $\{K_i\}$,

where

A channel's Kraus operators represent, loosely, a set of things that can happen to a system, including *measurement*, and *errors*.

Example: unitary channel

A channel with a single Kraus operator is a unitary operation ("unitary channel"): $\mathcal{U}$.

Example: measurement

Recall last class we discussed projective measurements and the more general POVM.

Let $\{M_m\}$ be a set of measurement operators, where the M_m are not necessarily projectors, and $E_m = M_m^\dagger M_m$, where

The set of operators

constitutes a POVM, and can be used to make a measurement on a quantum system.

Example: measurement

The $\{M_m\}$ can be viewed as Kraus operators:

Consider arbitrary mixed state

What is the probability of measuring and obtaining outcome m?

Example: measurement

(Normalized) state after measurement and obtaining outcome m:

Overall state transforms as

Concrete example with a projective measurement $\{M_m\} = \{\Pi_+, \Pi_-\}$:

Consider mixed state

Intuitively, after measurement expect

Example: partial trace

What happens to a system if we only measure part of a state?

Defined as

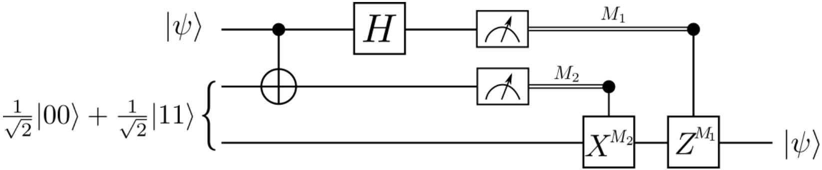
Example: two pure states

Example: partial trace

Example: tensor product of two mixed states

Example: Bell state

Consider what happens when we trace out Alice's system.



Next time

Last class:

- Error channels
- Noise in quantum systems

Action items:

1. MT checkpoint reports
2. TA3 and A3

Recommended reading:

- From this class: Codebook NT, DM; N&C 2.2.6, 2.4
- For next class: Codebook EC; N& C 8.2-8.3,