CPEN 400Q Lecture 14
Quantum phase estimation and order finding

Monday 3 March 2025

Announcements

- Quiz 6 today
- Technical assignment 3 available later this week
- Project midterm checkpoint details available later this week
- Tomorrow's tutorial: intro to RSA

We implemented the quantum Fourier transform using a *polynomial* number of gates:

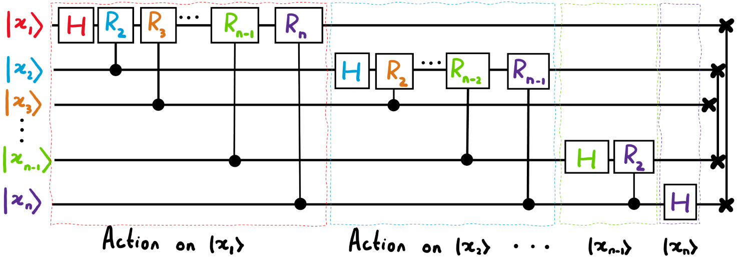


Image credit: Xanadu Quantum Codebook node F.3

We started learning about the quantum phase estimation subroutine which estimates the eigenvalues of unitary matrices.

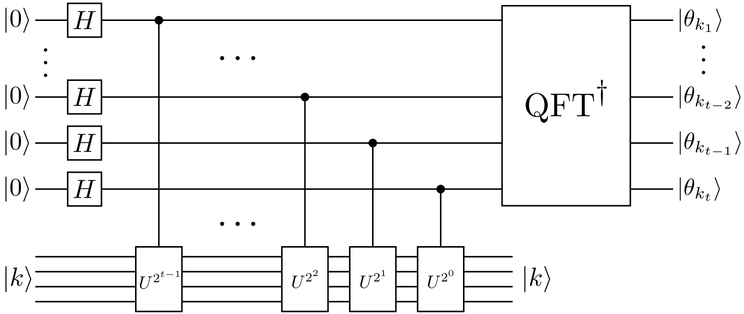
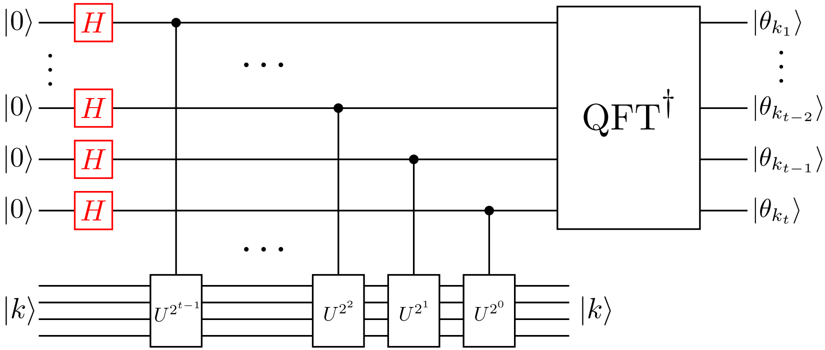


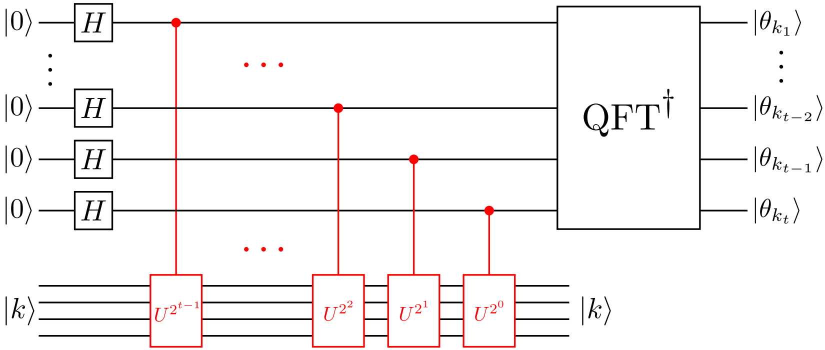
Image credit: Xanadu Quantum Codebook node P.2

Learning outcomes

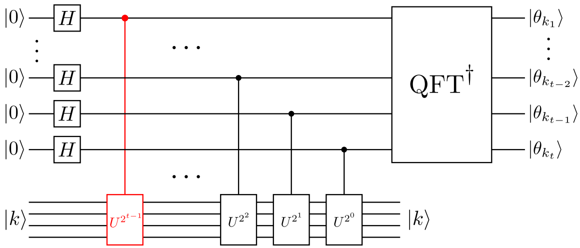
- Outline the steps of the quantum phase estimation (QPE) subroutine
- Use QPE to implement the order finding algorithm


$|0\rangle$
H
$|0\rangle$
H
$|0\rangle$
H
$|0\rangle$
H
$|k\rangle$
$U^{2^{t-1}}$
U^{2^2}
U^{2^1}
U^{2^0}
$|k\rangle$
$\mathrm{QFT}^\dagger$
$|\theta_{k_1}\rangle$
$|\theta_{k_{t-2}}\rangle$
$|\theta_{k_{t-1}}\rangle$
$|\theta_{k_t}\rangle$

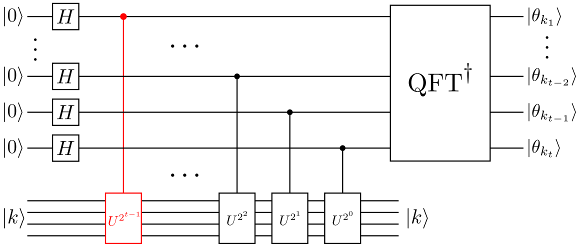
Quantum phase estimation: step 2



Quantum phase estimation: step 2

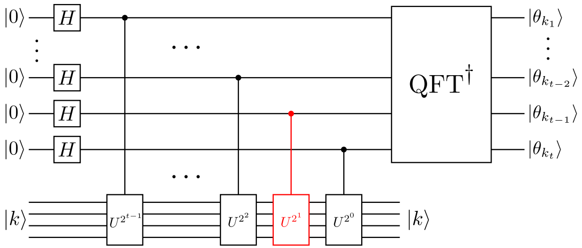


Quantum phase estimation: step 2



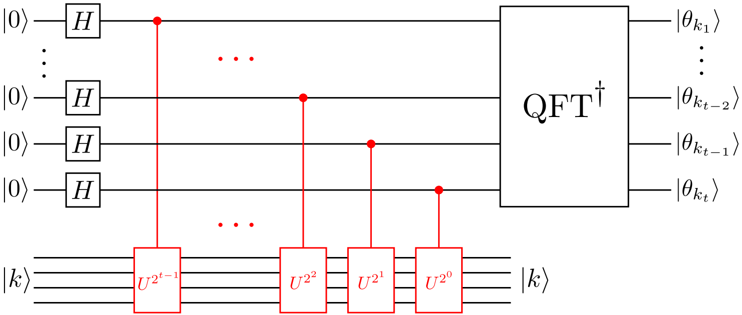
Use phase kickback

Quantum phase estimation: step 2



Check second-last qubit (ignore the others)

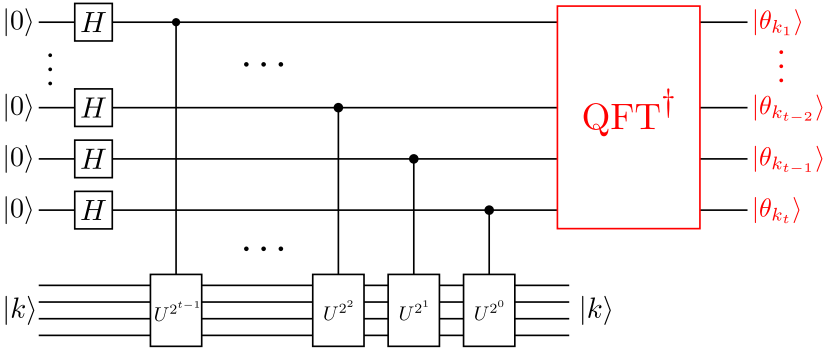
Quantum phase estimation: step 2



After step 2, we have the state

$$\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_t}}|1\rangle) \cdots \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}|1\rangle) \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_1}\cdots\theta_{k_t}}|1\rangle)|k\rangle$$

Measure to learn the bits of θ_k.



Example: QPE for the T gate

Let's apply QPE to estimate the phase of an eigenstate of T: $|1\rangle$.

1. What answer do we expect?
2. How many estimation bits?
3. What does the circuit look like?

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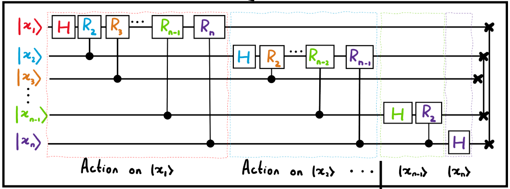
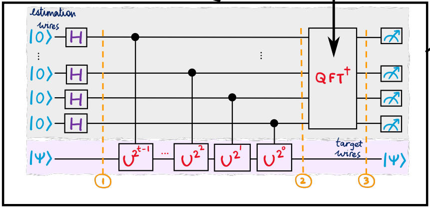
1. QFT



2. QPE



3. Shor

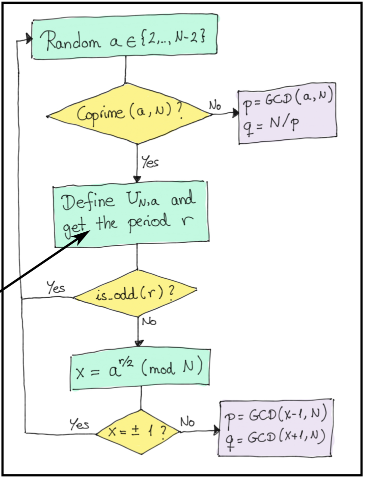


Image credit: Xanadu Quantum Codebook nodes F.3, P.3, S.4

Order finding on a quantum computer

Suppose we have a function

over the integers modulo N.

If there exists $r \in \mathbb{Z}$ s.t.

$f(x)$ is periodic with period r.

Suppose

The *order* of a is the smallest m such that

Note that this is also the period:

Order finding on a quantum computer

Exercise: find the order of $a = 5$ for $N = 7$.

Order finding on a quantum computer

More formally, define

Define a unitary operation that performs

If m is the order of a, and we apply $U_{N,a}$ m times,

So m is also the order of $U_{N,a}$! We can find it efficiently using a quantum computer.

Let U be an operator, and consider a state $|\phi\rangle$. How do we find the minimum r such that

QPE does the trick if we set things up in a clever way:

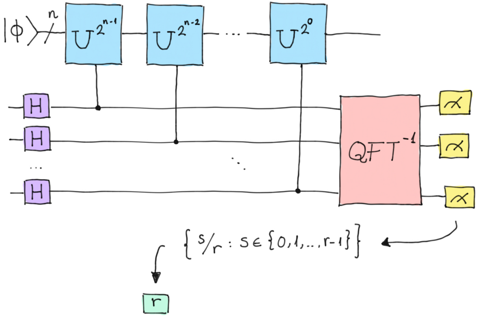


Image credit: Xanadu Quantum Codebook node S.3

Order finding on a quantum computer

Consider the state

If we apply U to this:

Order finding on a quantum computer

Now consider the state

If we apply U to this:

This generalizes to $|\Psi_s\rangle$

It has eigenvalue

Idea: if we can create *any* one of these $|\Psi_s\rangle$, we could run QPE and get an estimate for s/r, and then recover r.

Problem: to construct any $|\Psi_s\rangle$, we would need to know r in advance!

Solution: construct the uniform superposition of all of them.

But what does this equal?

The superposition of all $|\Psi_s\rangle$ is just our original state $|\phi\rangle$!

$$
\begin{aligned}
|\Psi\rangle = \frac{1}{\sqrt{r}} \cdot \Big(|\Psi_0\rangle + |\Psi_1\rangle + \cdots + |\Psi_{r-1}\rangle \Big)
= \frac{1}{\sqrt{r}} \cdot \Big[& \tfrac{1}{\sqrt{r}} \left(|\phi\rangle + 1 \cdot U|\phi\rangle + \cdots + 1 \cdot U^{r-1}|\phi\rangle \right) \\
+ & \tfrac{1}{\sqrt{r}} \left(|\phi\rangle + e^{\frac{-2\pi i}{r}} U|\phi\rangle + \cdots + e^{\frac{-2\pi i(r-1)}{r}} U^{r-1}|\phi\rangle \right) \\
& \vdots \\
+ & \tfrac{1}{\sqrt{r}} \left(|\phi\rangle + e^{\frac{-2\pi i(r-1)}{r}} U|\phi\rangle + \cdots + e^{\frac{-2\pi i(r-1)^2}{r}} U^{r-1}|\phi\rangle \right) \Big]
\end{aligned}
$$

$$
\underbrace{r \qquad 0 \quad \cdots \quad 0 \quad \cdots \quad 0}_{= \frac{1}{\sqrt{r}} \cdot \frac{1}{\sqrt{r}} \cdot r |\phi\rangle = \boxed{|\phi\rangle}}
$$

If we run QPE, the output will be s/r for one of these states.

Image credit: Xanadu Quantum Codebook node S.3

Next time

Content:

- Hands-on about RSA
- Shor's algorithm

Action items:

1. A3 when available
2. Work on project

Recommended reading:

- Codebook modules QFT, QPE, SH
- Nielsen & Chuang 5.3, Appendix A.5