CPEN 400Q Lecture 06
Superdense coding and quantum teleportation; Measurement II (expectation values)

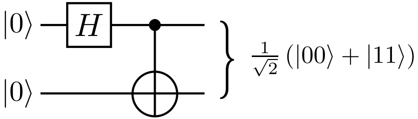
Wednesday 22 January 2025

Announcements

- Assignment 1 due Sun 26 1 at 23:59 (will be adjusted based on what we cover today)
- Midterm in class on Wed 29 Jan (see PrairieLearn for details)
- Quiz 3 on Monday

Last time

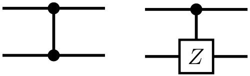
We defined entangled states and entangling gates:

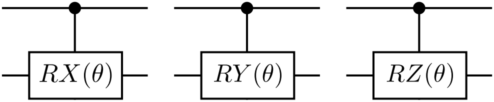


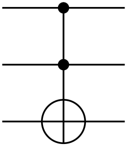
Entangled states *cannot be expressed* as a tensor products of all constituent single-qubit states.

An **entangling gate** sends some non-entangled (separable state) to an entangled state.

We saw more examples of two-qubit gates:







A1

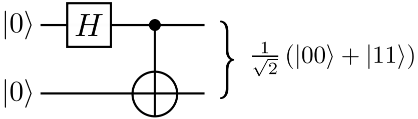
Image credit: Codebook node I.13, I.14

Last time

We saw how to make arbitrary controlled operations (or functions) in PennyLane using `qml.ctrl`.

```
@qml.qnode(dev)
def my_circuit():
    qml.CNOT(wires=[2, 3])
    qml.ctrl(qml.S, control=1)(wires=0)
    qml.Toffoli(wires=[0, 1, 2])
    return qml.sample()
```

In preparation for some algorithms, we created an orthonormal basis of entangled states called the Bell basis:



$$
\begin{aligned}
|\Psi_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Psi_{01}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi_{10}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Learning outcomes

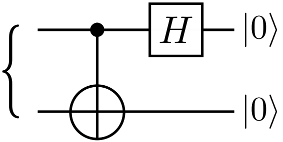
- leverage entanglement to implement superdense coding
- prove that arbitrary quantum states cannot be cloned
- teleport a qubit
- define observables and expectation values
- compute expectation values of an observable after running a circuit

The Bell basis

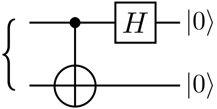
$$
\begin{aligned}
|\Psi_{00}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
|\Psi_{01}\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\
|\Psi_{10}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\
|\Psi_{11}\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)
\end{aligned}
$$

We can measure in this basis by applying the adjoint of the circuit:

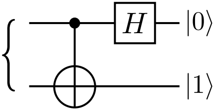
$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$



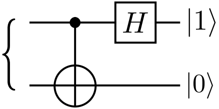
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



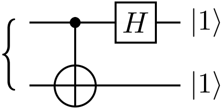
$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$



$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

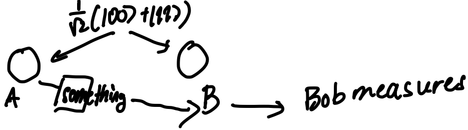


$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$



Two famous quantum algorithms, **superdense coding** and **teleportation** work by performing a measurement in the Bell basis.

Superdense coding



Suppose Alice wants to send Bob two classical bits of information, say ‘1’ and ‘0’.

Q1: How many classical bits must she send to Bob to do this?

2

Q2: How many *qubits* must she send to Bob to do this?

1 provided they share an entangled pair.

Superdense coding

Alice and Bob start the protocol with this shared entangled state:

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Next, depending on her bits, Alice performs one of the following operations on her qubit:

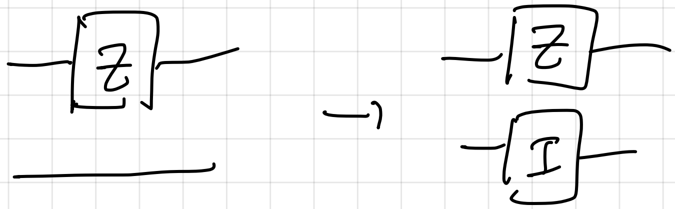
$$\begin{aligned} 00 &\rightarrow I \\ 01 &\rightarrow X \\ 10 &\rightarrow Z \\ 11 &\rightarrow ZX \end{aligned}$$

What happened to the entangled state?

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|\overset{A}{0}0\rangle + |11\rangle) \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

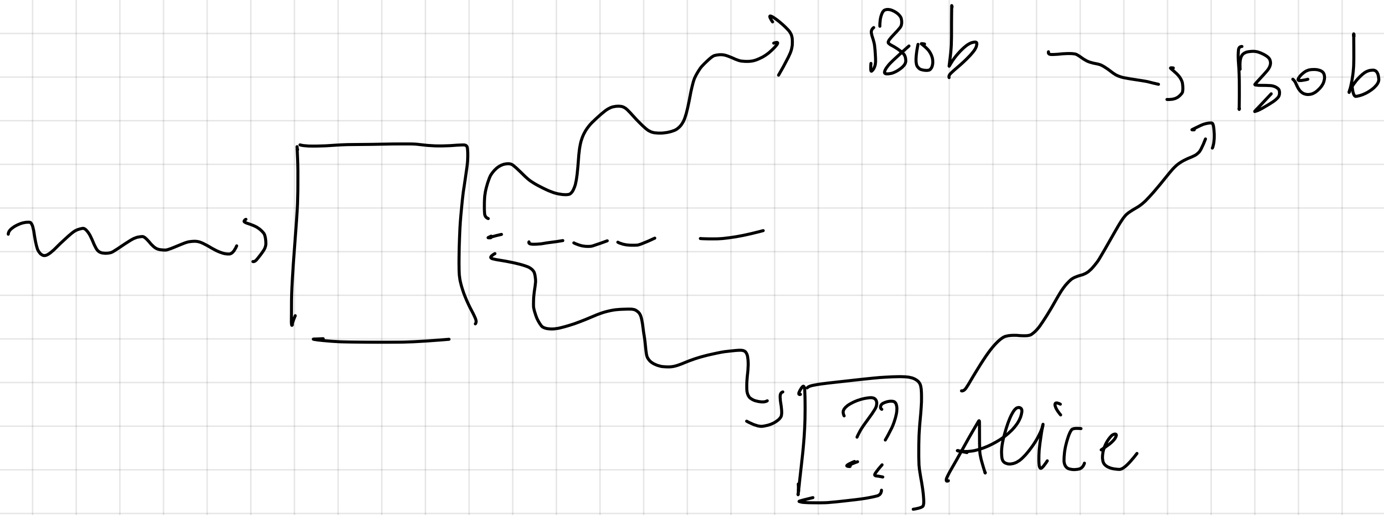
It will transform to:

$$\left.\begin{aligned}
00 &\quad \rightarrow_I \quad \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
01 &\quad \rightarrow_X \quad \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\
10 &\quad \rightarrow_Z \quad \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
11 &\quad \rightarrow_{ZX} \quad \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle)
\end{aligned}\right\} \text{Bell basis states}$$



$$Z \otimes I = \begin{pmatrix} 1 & & & \\ & 1 & \multicolumn{2}{c}{0} \\ \multicolumn{2}{c}{0} & -1 & \\ & & & -1 \end{pmatrix}$$

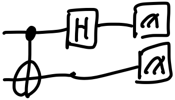
$$(Z \otimes I) \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) = \frac{1}{\sqrt{2}} \left(|00\rangle - |11\rangle \right)$$



Now, Bob can either

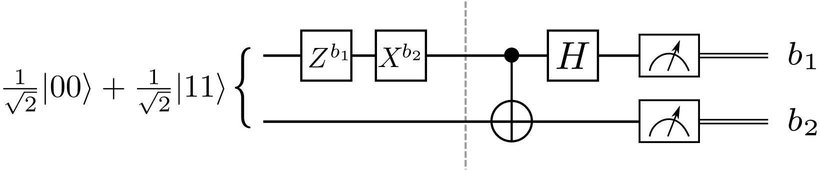
- perform a measurement directly in the Bell basis
- perform a basis transformation from the Bell basis back to the computational basis

to determine his state with certainty, and thus, the bits Alice sent.



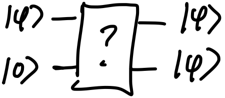
$$
\begin{aligned}
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) &= |00\rangle \\
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) &= |01\rangle \\
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) &= |10\rangle \\
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) &= |11\rangle
\end{aligned}
$$

Let's go implement it!

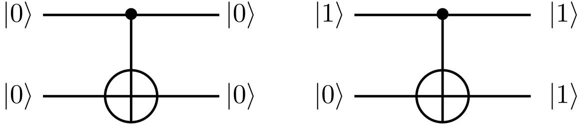


Copying quantum states

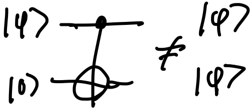
Suppose you found a really cool quantum state, and you want to send a copy to a friend. Can you?



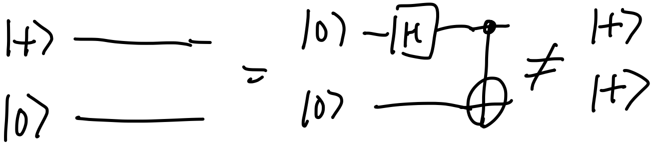
Idea: CNOT sends $|00\rangle$ to $|00\rangle$, and $|10\rangle$ to $|11\rangle$, thus copying the first qubit's state to the second.



Everything is linear, so will this work in general?



Very easy to find a state for which this fails:



$$|+\rangle \otimes |+\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

(Not) copying quantum states

The no-cloning theorem

It is impossible to create a copying circuit that works for arbitrary quantum states.

In other words, there is no circuit that sends

$$|\psi\rangle \otimes |s\rangle \to |\psi\rangle \otimes |\psi\rangle$$

for any arbitrary $|\psi\rangle$.

Proof of the no-cloning theorem

Suppose we want to clone a state $|\psi\rangle$. We want a unitary operation that sends

$$U\left(|\psi\rangle \otimes |s\rangle\right) = |\psi\rangle \otimes |\psi\rangle$$

where $|s\rangle$ is some arbitrary state.

Suppose we find one. If our cloning machine is to be universal, we must also be able to clone some other state, $|\varphi\rangle$.

$$U\left(|\varphi\rangle \otimes |s\rangle\right) = |\varphi\rangle \otimes |\varphi\rangle$$

Proof of the no-cloning theorem

We purportedly have:

$\langle \phi | \psi \rangle$

$$
\begin{aligned}
U(|\psi\rangle \otimes |s\rangle) &= |\psi\rangle \otimes |\psi\rangle \\
U(|\varphi\rangle \otimes |s\rangle) &= |\varphi\rangle \otimes |\varphi\rangle
\end{aligned}
$$

Take the inner product of the LHS of both equations:

$$\left(\langle\varphi| \otimes \langle s|\, \underbrace{U^\dagger\right)\left(U}_{=I} |\psi\rangle \otimes |s\rangle\right) = \langle\varphi|\psi\rangle \cdot \underbrace{\langle s|s\rangle}_{=1} = \langle\varphi|\psi\rangle$$

Now take the inner product of the RHS of both equations:

$$\left(\langle\varphi| \otimes \langle\varphi|\right)\left(|\psi\rangle \otimes |\psi\rangle\right) = \langle\varphi|\psi\rangle \cdot \langle\varphi|\psi\rangle = \left(\langle\varphi|\psi\rangle\right)^2$$

complex z s.t. $z = z^2$ $\Rightarrow$ $z = 0$ or $z = 1$

$z = 0$ ↓ orthonormal

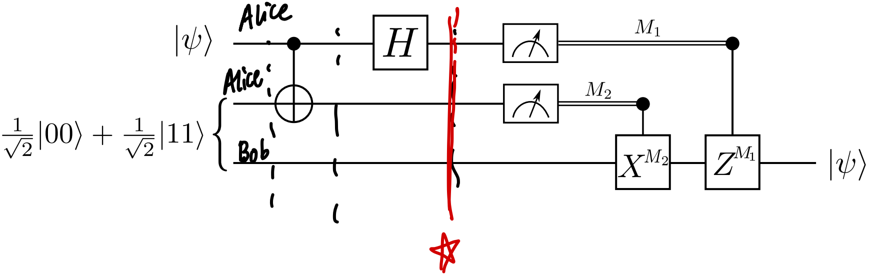
$z = 1$ ↓ same state

$\square$

Teleportation

We cannot clone arbitrary qubit states, but we *can* teleport them!

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



Homework: work through this circuit and determine the state after each gate (it is worth doing this once!).

Quantum teleportation: the details

Before measurements, the combined state of the system is

$$\frac{1}{2}\Big[\,|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle)+|10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]$$

What do you notice about this state?

Alice measure probs:

00 : 1/4

01 : 1/4 → Bob $\alpha|1\rangle+\beta|0\rangle$ → apply X → $\alpha|0\rangle+\beta|1\rangle$

10 : 1/4 → Bob applies Z

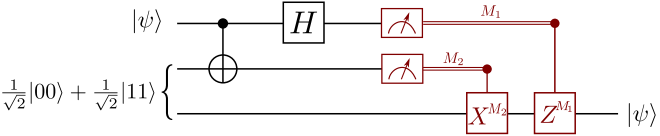
11 : 1/4

Quantum teleportation: the details

Alice measures in the computational basis and sends her results to Bob, who can adjust his state as needed.

$$
\begin{aligned}
00 &: I(\alpha|0\rangle + \beta|1\rangle) = (\alpha|0\rangle + \beta|1\rangle)\\
01 &: X(\alpha|1\rangle + \beta|0\rangle) = (\alpha|0\rangle + \beta|1\rangle)\\
10 &: Z(\alpha|0\rangle - \beta|1\rangle) = (\alpha|0\rangle + \beta|1\rangle)\\
11 &: ZX(\alpha|1\rangle - \beta|0\rangle) = (\alpha|0\rangle + \beta|1\rangle)
\end{aligned}
$$

Let's implement it!



Next time

Content: Expectation values

- ~~Measurement part 3: generalized measurements and state discrimination~~

Action items:

1. Assignment 1 due Sunday 23:59
2. Quiz 3 on Monday
3. Study for midterm

Recommended reading:

- Codebook nodes IQC, SQ, MQ; Nielsen & Chuang 1.2, 1.3.1-1.3.7, 2.2.3-2.2.5, 2.3, 4.3