CPEN 400Q Lecture 07
Measurement II (expectation values)

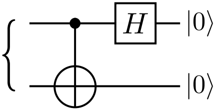
Monday 27 January 2025

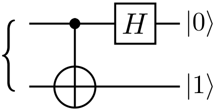
Announcements

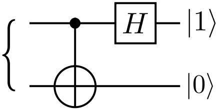
- Quiz 3 today
- Tutorial tomorrow: midterm practice
- Midterm in class on Wed (info on PrairieLearn) covers "the basics", i.e., lectures 01-06, A1, Q1-3 (may see today's material at high level, but learning outcomes not being tested)

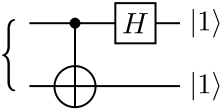
We implemented **superdense coding** and **teleportation**.

Both algorithms leverage **shared entanglement**, and perform measurements in the Bell basis.

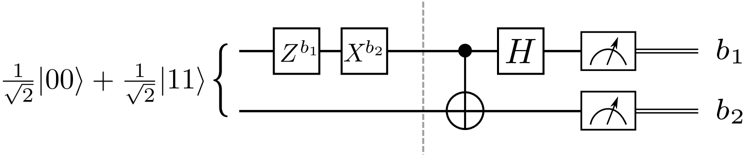
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ 

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ 

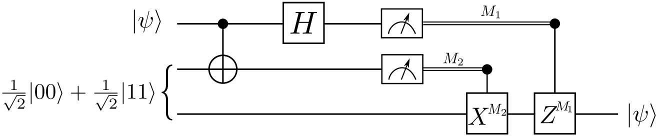
$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ 

$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ 

We "sent" two bits of information by transmitting only one qubit with the **superdense coding** protocol.



While we cannot clone arbitrary states, we *can* teleport them!



You asked lots of awesome questions - this stuff is pretty weird!

Learning outcomes

Core outcomes:

- define observables and expectation values
- compute the expectation value of an observable after performing a quantum computation

If there's time:

- distinguish between projective measurements and positive operator-valued measurements (POVMs)
- develop a POVM to help differentiate between two non-orthogonal quantum states

Projective measurements: recap

Our current view of measurements involves computing an inner product w.r.t. a basis state to determine the outcome probability:

In other contexts, we are interested in measuring real, physical quantities. In physics, these are called **observables**.

(The two kinds of measurements are related)

Observables are represented mathematically by Hermitian matrices. An operator (matrix) H is Hermitian if

$$B = B^\dagger$$

Why Hermitian?

- the possible measurement outcomes of an observable are its eigenvalues
- the eigenvalues of Hermitian operators are **real**.

Example:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Z is Hermitian:

Its eigensystem is

Example:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

X is Hermitian and its (normalized) eigensystem is

Example:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Y is Hermitian and its (normalized) eigensystem is

Analytically, the **expectation value** of measuring the observable B given the state $|\psi\rangle$ is

When we measure an observable (e.g., X, Y, or Z), for each shot we observe the system in one of its eigenstates, and associate the outcome to its eigenvalue.

The expectation value is what we expect to see *on average* over multiple shots.

Expectation values: analytical

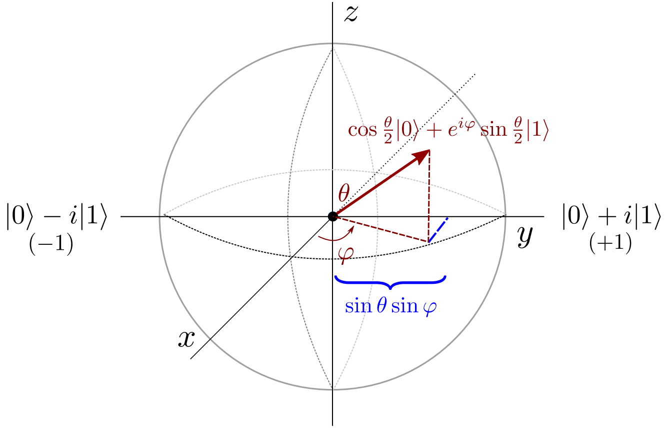
Exercise: consider the quantum state

$$|\psi\rangle = \frac{1}{2}|0\rangle - i\frac{\sqrt{3}}{2}|1\rangle.$$

Compute the expectation value of Y:

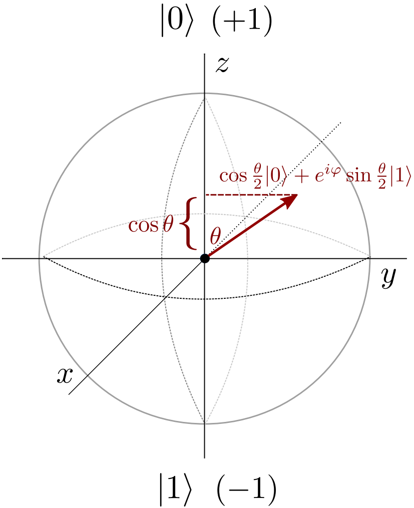
Expectation values and the Bloch sphere

The Bloch sphere offers us some more insight.



Exercise: derive the expression in blue by computing $\langle\psi| Y |\psi\rangle$.

Expectation values and the Bloch sphere



Why this works

More formally, the **spectral theorem** from linear algebra states

where λ_k, $|\varphi_k\rangle$ are the eigenvalues and eigenstates of B

Exercise: show that the spectral theorem holds for Pauli X.

Why this works

The spectral theorem shows how this relates to measurement outcome probabilities of projective measurements:

Expectation values: from measurement data

Let's compute the expectation value of Z for the following circuit using 10 samples:

```
dev = qml.device('default.qubit', wires=1, shots=10)

@qml.qnode(dev)
def circuit():
    qml.RX(2*np.pi/3, wires=0)
    return qml.sample()
```

Results might look something like this:

$$[1, 1, 1, 0, 1, 1, 1, 0, 1, 1]$$

Expectation values: from measurement data

The expectation value pertains to the measured eigenvalue; recall Z eigenstates are

$$\lambda_1 = +1, \qquad |\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1, \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So when we observe $|0\rangle$, this is eigenvalue $+1$ (and if $|1\rangle$, -1). Our samples shift from

```
[1, 1, 1, 0, 1, 1, 1, 0, 1, 1]
```

to

```
[-1, -1, -1, 1, -1, -1, -1, 1, -1, -1]
```

Expectation values: from measurement data

The expectation value is the weighted average, where the weights are the eigenvalues:

where

- n_1 is the number of $+1$ eigenvalues
- n_{-1} is the number of -1 eigenvalues
- N is the total number of shots

For our example,

Let's do this in PennyLane instead:

```
dev = qml.device('default.qubit', wires=1)

@qml.qnode(dev)
def measure_z():
    qml.RX(2*np.pi/3, wires=0)
    return qml.expval(qml.PauliZ(0))
```

Multi-qubit expectation values

Example: operator $Z \otimes Z$.

Eigenvalues are computational basis states:

$$
\begin{aligned}
(Z \otimes Z)|00\rangle &= |00\rangle \\
(Z \otimes Z)|01\rangle &= -|01\rangle \\
(Z \otimes Z)|10\rangle &= -|10\rangle \\
(Z \otimes Z)|11\rangle &= |11\rangle
\end{aligned}
$$

To compute an expectation value from data:

$$\langle Z \otimes Z \rangle = \frac{1 \cdot n_1 + (-1) \cdot n_{-1}}{N}$$

Example: operator $X \otimes I$.

Eigenvalues of X are the $|+\rangle$ and $|-\rangle$ states:

$$
\begin{aligned}
(X \otimes I)|+0\rangle &= |+0\rangle \\
(X \otimes I)|+1\rangle &= |+1\rangle \\
(X \otimes I)|-0\rangle &= -|-0\rangle \\
(X \otimes I)|-1\rangle &= -|-1\rangle
\end{aligned}
$$

Fun fact: All Pauli operators have an equal number of $+1$ *and* -1 *eigenvalues!*

How to compute expectation value of X from data, when we can only measure in the computational basis?

Basis rotation: apply H to first qubit

$$(H \otimes I)(X \otimes I)|+0\rangle = |00\rangle$$
$$(H \otimes I)(X \otimes I)|+1\rangle = |01\rangle$$
$$(H \otimes I)(X \otimes I)|-0\rangle = -|10\rangle$$
$$(H \otimes I)(X \otimes I)|-1\rangle = -|11\rangle$$

When we measure and obtain $|10\rangle$ or $|11\rangle$, we know those correspond to the -1 eigenstates of $X \otimes I$.

Hands-on: multi-qubit expectation values

Multi-qubit expectation values can be created using the `@` symbol:

```
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0) @ qml.PauliZ(1))
```

Can return *multiple* expectation values (if no shared qubits)

```
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0)), qml.expval(qml.PauliZ(
                                        1))
```

We've learned a couple ways to measure a qubit state, $|\psi\rangle$:

Option 1: make a **projective measurement** with respect to an orthonormal basis, $\{|\varphi_i\rangle\}$,

Option 2: measure the **expectation value** of a Hermitian observable, B

Let's play a game

Suppose I send you a mystery qubit, guaranteed to be either in

You must correctly determine which state I sent, but *you are only allowed to make one measurement*.

What measurement strategy maximizes your odds of being correct?

A more general framework for measurements

More generally,

The Π_k are called **projectors**. Similar terminology with "projective measurement" is no accident...

Example: Pauli Z

In a projective measurement, the $\{\Pi_k\}$ "project" a state down to their consituent eigenstates.

Example: Apply projector Π_0 to $|+\rangle$:

Apply projector again:

Key feature of any projector, $\Pi_\psi = |\psi\rangle \langle\psi|$:

A more general framework for measurements

A set of orthogonal projectors (eigenbasis of Hermitian operator) form a measurement, but it's not the only kind...

From Nielsen and Chuang 2.2.3, 3rd postulate of QM:

> *"Quantum measurements are described by a collection $\{M_m\}$ of measurement operators (...) If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is (...)*
>
> *and the state of the system after the measurement is*
>
> *The measurement operators satisfy the completeness equation,*

Check this with Pauli X:

Positive operator-valued measures (POVMs)

Let $\{M_m\}$ be a set of measurement operators, where the M_m are not necessarily projectors, and

The set of operators

constitutes a positive operator-valued measure (POVM), and can be used to make a measurement on a quantum system.

Suppose I send you a mystery qubit, guaranteed to be either in

You must correctly determine which state I sent, but *you are only allowed to make one measurement*.

What measurement strategy maximizes your odds of being correct?

Revisit our game

Choose the measurement to be a set of three POVM elements:

	$\Pr(E_1)$	$\Pr(E_2)$	$\Pr(E_3)$
$\lvert\psi_1\rangle = \lvert 0\rangle$			
$\lvert\psi_2\rangle = \lvert +\rangle$			

Next time

Content:

- Oracle-based algorithms (Deutsch and Grover)

Action items:

1. Study for midterm

Recommended reading (lecture 01-07, and midterm)

- Codebook nodes IQC, SQ, MQ
- Nielsen & Chuang 1.2, 1.3.1-1.3.7, 2.2.3-2.2.5, 2.3, 4.3

Read ahead for next time:

- Codebook modules BA (basic quantum algorithms), GA (Grover's algorithm)
- Nielsen and Chuang 1.4, 6.1