CPEN 400Q Lecture 03
Measurement I (projective measurements)

Monday 13 January 202~~4~~5

Announcements

- Quiz 1 today
- Tomorrow's tutorial: first hands-on assignment (for submission). **New room: CEME 1215**
- Assignment 1 due Sunday 26 Jan at 23:59

A mysterious quantum black box prepares a system in state $\frac{1}{\sqrt{4}}e^{2.0i}|0\rangle + \frac{\sqrt{3}}{\sqrt{4}}e^{-0.5i}|1\rangle$. Is this a legal quantum state?

$a \to aa^* = |a|^2$

$\frac{1}{\sqrt{4}} e^{2i} \qquad \frac{\sqrt{3}}{\sqrt{4}} e^{-0.5i}$

$|\,|^2 = \frac{1}{4} \qquad |\,|^2 = \frac{3}{4} \Rightarrow$ ✓

Consider the circuit below.

What is the probability of measuring and observing the qubit in state $|0\rangle$ if the gates in the circuit are $A = H$ and $B = H$?

Prob($\lvert 0\rangle$)	number (rtol=0.01, atol=0.0001) 1	?

Next, suppose a Hadamard gate is applied to different qubit in state $\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$. What is the probability of measuring and observing that qubit in state $|1\rangle$?

Prob($\lvert 1\rangle$)	number (rtol=0.01, atol=0.0001)	?

$H\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right) \to$ Prob $|1\rangle$?

$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1/\sqrt{3}\\ \sqrt{2}/\sqrt{3}\end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$

Method 1 matrices

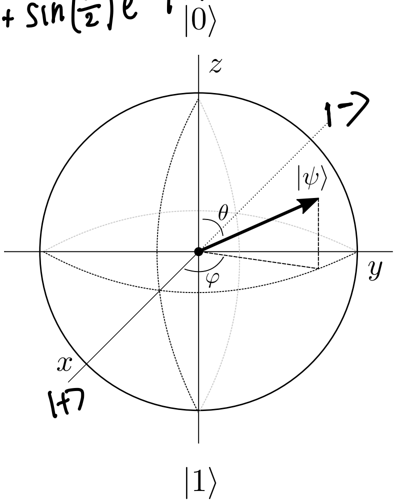
$H|\psi\rangle = \frac{1}{\sqrt{3}}H|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}H|1\rangle = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) + \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right.$

...

Last time

We saw the most general single-qubit state parametrization, and how it can be represented in 3D space on the Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle$$



Last time

We learned about the three Pauli rotations

	Math	Matrix	Code	Special cases (θ)
RZ	$e^{-i\frac{\theta}{2}Z}$	$\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$	`qml.RZ`	$Z(\pi), S(\pi/2), T(\pi/4)$
RY	$e^{-i\frac{\theta}{2}Y}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RY`	$Y(\pi)$
RX	$e^{-i\frac{\theta}{2}X}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RX`	$X(\pi), SX(\pi/2)$

Last time

We interpreted unitary operations as rotations of the state vector on the Bloch sphere.

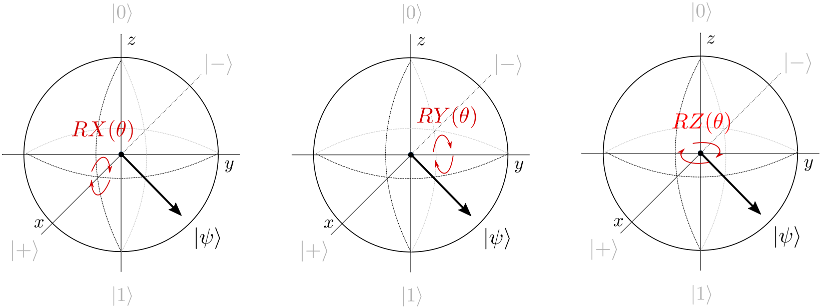


Image credit: Codebook node I.6

$|\psi\rangle = e^{i\phi}(\alpha|0\rangle + \beta|1\rangle)$

We left off with a few exercises.

$RZ(\theta) \to e^{-i\theta/2}|0\rangle + e^{i\theta/2}|1\rangle$

Exercise 1: design a quantum circuit that prepares

qml. State Preparation

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5}{4}}|1\rangle$$

$\to |\psi\rangle = e^{i\frac{5}{8}}\left(\frac{\sqrt{3}}{2}e^{-i\frac{5}{8}}|0\rangle - \frac{1}{2}e^{i\frac{5}{8}}|1\rangle\right)$

$-\boxed{RY(\pi/3)}-\boxed{RZ(5/4)}-$

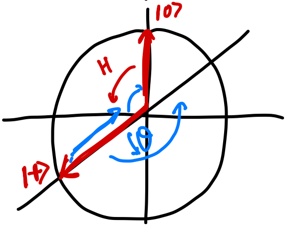
$-\boxed{RY(-\pi/3)}-\boxed{RZ(5/4)}-$

Exercise 2: In PennyLane, implement the circuit below

Run your circuit with two different values of θ and take 1000 shots. How does θ affect the measurement outcome probabilities?

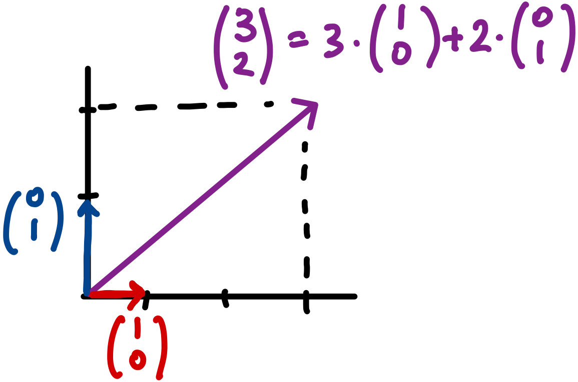
Learning outcomes

- compute the inner product between two quantum states
- perform a projective measurement
- distinguish between global phase and relative phase
- measure a qubit in different bases

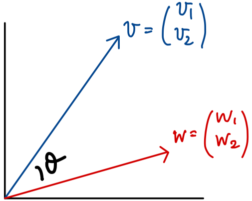


We can now create any single-qubit quantum state: how do we *compare* them?

For intuition, consider a classical vector space.



We can define an **inner product** between two vectors to quantify much overlap they have.



$$\begin{aligned} \vec{v} \cdot \vec{w} &= |\vec{v}| \cdot |\vec{w}| \cos\theta \\ &= \sum_{i=1}^{2} v_i w_i \\ &= \vec{v}^T \cdot \vec{w} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

Take just one of these representations:

$$|v\rangle = v_1|0\rangle + v_2|1\rangle \qquad |w\rangle = w_1|0\rangle + w_2|1\rangle \qquad v_i, w_i \in \mathbb{C}$$

$$= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The inner product in Hilbert space is defined as

$$\langle |v\rangle, |w\rangle \rangle = \underbrace{\begin{pmatrix} v_1^* & v_2^* \end{pmatrix}}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

To avoid cumbersome notation define the **bra** (*braket* notation):

$$\langle v| = \left(|v\rangle\right)^\dagger \quad \text{conj transpose.}$$

$$\langle |v\rangle, |w\rangle \rangle = \boxed{\langle v|w\rangle}$$

np.vdot

Exercise: compute the inner product of the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with itself.

$$\langle\psi|\psi\rangle = \alpha^*\alpha + \beta^*\beta = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

$$= 1$$

Exercise: compute the inner product between all possible combinations of $|0\rangle$ and $|1\rangle$.

$$\underbrace{|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{computational basis}}$$

$\langle 0\|0\rangle$	1
$\langle 0\|1\rangle$	0
$\langle 1\|0\rangle$	0
$\langle 1\|1\rangle$	1

For a single qubit, a pair of states that are **normalized** and **orthogonal** constitute an **orthonormal basis** for the Hilbert space.

Exercise: do the states

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

form an orthonormal basis?

$\langle p\|p\rangle$	1
$\langle p\|m\rangle$	0
$\langle m\|p\rangle$	0
$\langle m\|m\rangle$	1

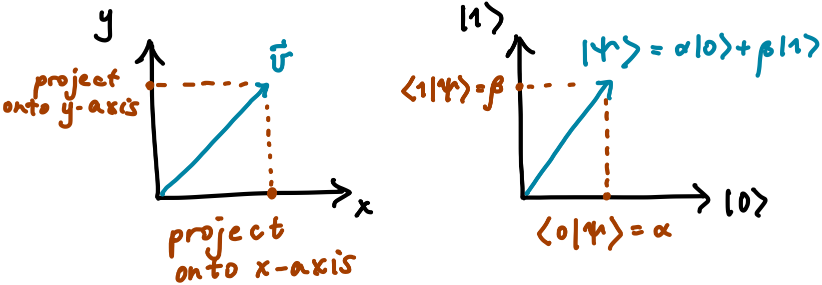
$$\langle p|p\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}\left(1+(-i)i\right) = 1$$

$$\langle p|m\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{2}\left(1+i^2\right) = 0$$

$$\langle m|p\rangle = \left(\langle p|m\rangle\right)^*$$

try proving

Measurement is performed with respect to a basis; we perform **projections** to determine the overlap with a given basis state.



(Image for expository purposes only!)

Image credit: Xanadu Quantum Codebook I.9

When we measure state $|\varphi\rangle$ with respect to basis $\{|\psi_i\rangle\}$, the probability of obtaining outcome i is

$$\Pr(\text{outcome } i) = |\langle \psi_i | \varphi \rangle|^2$$

If we observe outcome i, the system will be left in state $|\psi_i\rangle$ after the measurement.

Measurement in the computational basis

$$\Pr(\text{outcome } i) = |\langle \psi_i | \varphi \rangle|^2$$

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Then if we measure $|\psi\rangle$ is the computational basis, $\{|0\rangle, |1\rangle\}$

$$\Pr(0) = |\underbrace{\langle 0|\psi\rangle}|^2$$

$$\langle 0|\psi\rangle = \alpha \cdot \underbrace{\langle 0|0\rangle}_{1} + \underbrace{\cancel{\beta\langle 0|1\rangle}}_{0} = \alpha$$

$$\Pr(1) = |\langle 1|\psi\rangle|^2 = |\beta|^2$$

Measurement in the computational basis

Exercise: what are the measurement outcome probabilities if we measure

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

in the computational basis?

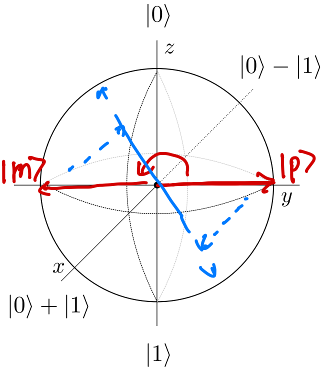
$|p\rangle$: $\quad \text{Prob}(0) = \frac{1}{2} \qquad \text{Prob}(1) = \frac{1}{2}$

$|m\rangle$: $\quad \text{Prob}(0) = \frac{1}{2} \qquad \text{Prob}(1) = \frac{1}{2}$

Measurement in the computational basis

These are *not* the same state: there is a relative phase.

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$



How to tell them apart?

Basis rotations

Projective measurements can be performed with respect to any orthonormal basis. For example, $\{|+\rangle, |-\rangle\}$:

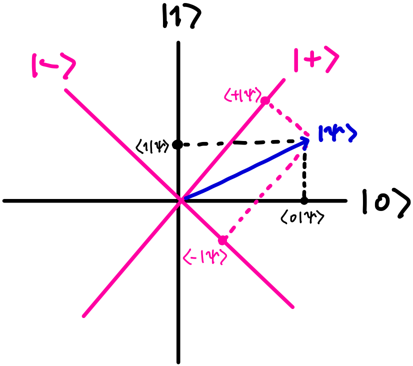


Image credit: Codebook node I.9

Next time

Content: basis rotations

- Mathematical representation of multi-qubit systems
- Multi-qubit gates
- Entanglement

Action items:

1. Work on Assignment 1 (can do questions 1 & 5, 2ai,ii now)

Recommended reading:

- For today: Codebook modules IQC, SQ; Nielsen & Chuang 1.3.3, 2.2.3-2.2.5
- For next class: Codebook module MQ; Nielsen & Chuang 4.3