CPEN 400Q Lecture 03
Measurement I (projective measurements)

Monday 15 January 2024

Announcements

open book! notes, code, etc.

- Quiz 1 today
- Assignment 0 due tonight at 23:59 - remember to cite sources in contribution statement
- Assignment 1 coming this week

Quiz solution

We saw how qubits can be represented in 3D space on the Bloch sphere, and how unitary operations rotate the Bloch vector.

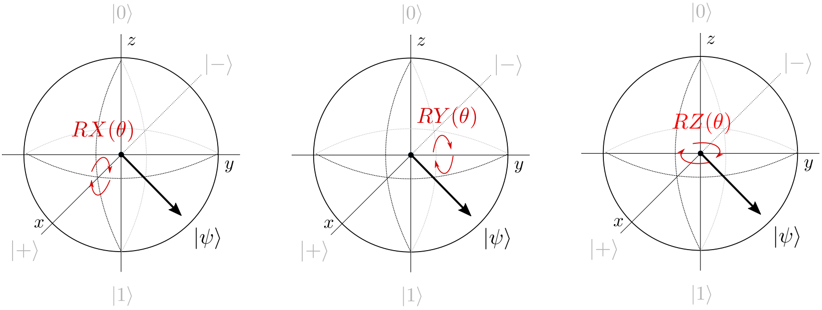
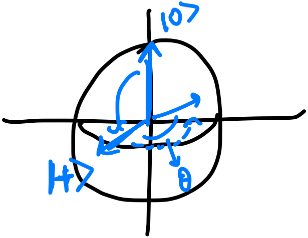


Image credit: Codebook node I.6

We learned about the three Pauli rotations

	Math	Matrix	Code	Special cases
RZ	$e^{-i\frac{\theta}{2}Z}$	$\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$	`qml.RZ`	$Z(\pi), S(\pi/2), T(\pi/4)$
RY	$e^{-i\frac{\theta}{2}Y}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RY`	$Y(\pi)$
RX	$e^{-i\frac{\theta}{2}X}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RX`	$X(\pi), SX(\pi/2)$

We saw a curious example where changing the state didn't change the measurement outcomes, even though it wasn't a global phase...

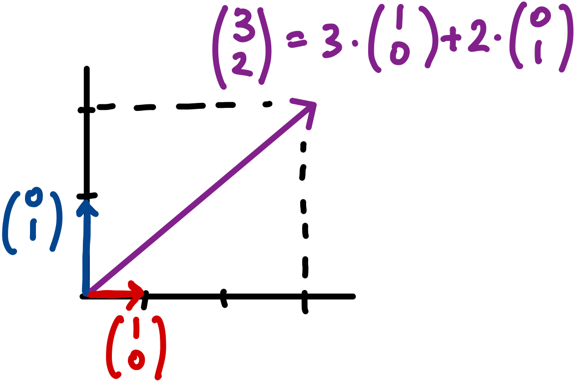


Learning outcomes

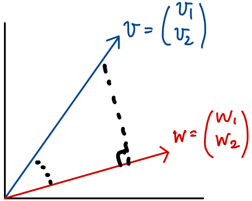
- compute the inner product between two quantum states
- perform a projective measurement
- distinguish between global phase and relative phase
- measure a qubit in different bases

We can now create every single single-qubit quantum state: how do we *compare* them?

Recall what things look like in a classical vector space.



We can define an **inner product** between two vectors that tells us how much overlap they have.



$$\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= v_1 w_1 + v_2 w_2$$

$$= \sum_{i=1}^{2} v_i w_i$$

$$= |\vec{v}||\vec{w}| \cos\theta$$

orthogonal: 0

Take just one of these representations:

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \vec{w}$$

The Hilbert space has complex valued vectors. The inner product looks *similar*, but slightly different. Let

$$|v\rangle = v_1 |0\rangle + v_2 |1\rangle \qquad |w\rangle = w_1 |0\rangle + w_2 |1\rangle$$

$$v_i, w_i \in \mathbb{C}$$

The inner product is defined as

$$\big\langle |v\rangle, |w\rangle \big\rangle = \left(|v\rangle^T\right)^* |w\rangle$$

$$= \left(|v\rangle\right)^\dagger |w\rangle$$

This notation is cumbersome, so let's complete our knowledge of Dirac notation by introducing the **bra**:

$$\langle v| = (|v\rangle)^\dagger = (v_1^* \quad v_2^*)$$

The inner product is defined as

$$\langle |v\rangle, |w\rangle \rangle = \langle v| \cdot |w\rangle = \langle v|w\rangle$$

Written another way,

$$\langle v|w\rangle = (\langle w|v\rangle)^* \qquad \langle v|w\rangle = v_1^* w_1 + v_2^* w_2$$

Pro tip:

np.vdot (v, w)

Inner products

$$\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$$

Exercise: compute the inner product of the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with itself.

$$\langle\psi|\psi\rangle = \left(\alpha^*\langle 0| + \beta^*\langle 1|\right)\left(\alpha|0\rangle + |1\rangle\right)$$

$$= \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

$$= \alpha^*\alpha + \beta^*\beta$$

$$= |\alpha|^2 + |\beta|^2$$

$$= \underline{1}$$

Exercise: compute the inner product between all possible combinations of $|0\rangle$ and $|1\rangle$.

$\langle 0\|0\rangle$	1
$\langle 0\|1\rangle$	0
$\langle 1\|0\rangle$	0
$\langle 1\|1\rangle$	1

$$(1 \;\; 0)\begin{pmatrix}0\\1\end{pmatrix} = 0$$

$$(0 \;\; 1)\begin{pmatrix}1\\0\end{pmatrix} = 0$$

$$0 \le |\langle\phi|\psi\rangle|^2 \le 1$$

Linearity

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

$$(\gamma^*\langle 0| + \delta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$$

$$= \gamma^*\alpha\langle 0|0\rangle^{1} + \cancel{\delta^*\alpha\langle 1|0\rangle} + \cancel{\gamma^*\beta\langle 0|1\rangle} + \delta^*\beta\langle 1|1\rangle^{1}$$

$$= \gamma^*\alpha + \delta^*\beta$$

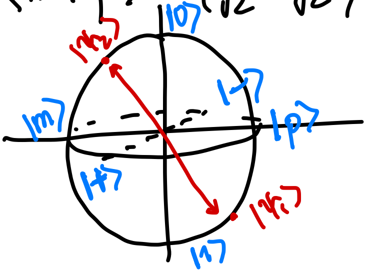
For a single qubit, a pair of states that are **normalized** and **orthogonal** constitute an **orthonormal basis** for the Hilbert space.

Exercise: do the states

"Y" basis
$$\begin{cases} |p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle \end{cases}$$
YES

form an orthonormal basis?

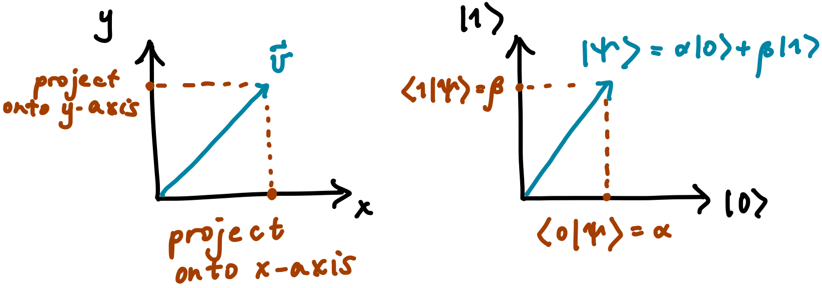
$$\langle m | p \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} + \frac{i^2}{2} = 0 = \langle p | m \rangle$$



"antipodal"

Projective measurements

Measurement is performed with respect to a basis; we perform **projections** to determine the overlap with a given basis state.



(Image for expository purposes only!)

Image credit: Xanadu Quantum Codebook I.9

$\{|\psi_1\rangle, |\psi_2\rangle\}$

When we measure state $|\varphi\rangle$ with respect to basis $\{|\psi_i\rangle\}$, the probability of obtaining outcome i is

$$\Pr(\text{outcome } \underset{|\psi_1\rangle, |\psi_2\rangle}{\underset{\downarrow}{i}}) = |\underbrace{\langle \psi_i | \varphi \rangle}_{\text{"how much"?}}|^2$$

If we observe outcome i, following the measurement the system will be left in state $|\psi_i\rangle$.

Measurement in computational basis

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Then if we measure $|\psi\rangle$ is the computational basis,

$$Pr(0) = |\langle 0|\psi\rangle|^2 = |\alpha \cdot \underbrace{\langle 0|0\rangle}_{1} + \beta \cdot \cancel{\langle 0|1\rangle}|^2 = |\alpha|^2$$

$$Pr(1) = |\langle 1|\psi\rangle|^2 = |\beta|^2$$

Exercise: what are the measurement outcome probabilities if we measure

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

$|+i\rangle$ $|-i\rangle$

in the computational basis?

$|p\rangle$: $Pr(0) = \frac{1}{2}$ $Pr(1) = \frac{1}{2}$

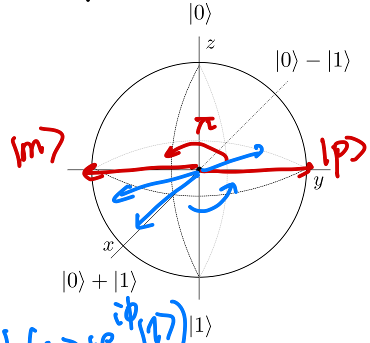
$|m\rangle$: $Pr(0) = \frac{1}{2}$ $Pr(1) = \frac{1}{2}$

indistinguishable in comp. basis.

Measurement in computational basis

These are *not* the same state: there is a relative phase.

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$



Exercise: are $|p\rangle / |m\rangle$ dist. in $|+\rangle / |-\rangle$ basis?

$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$-\boxed{H}-\boxed{RZ(\theta)}- \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$

But they are indistinguishable when we measure in the computational basis. How to tell them apart?

Projective measurements can be performed with respect to any orthonormal basis. For example, $\{|+\rangle, |-\rangle\}$:

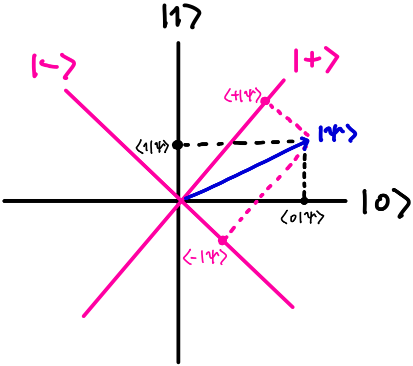


Image credit: Codebook node I.9

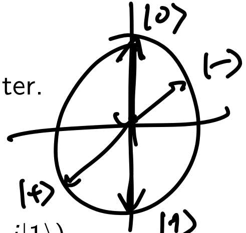
Basis rotations

So far we've seen 3 ways of extracting information out of a QNode:

1. `qml.state()`
2. `qml.probs(wires=x)`
3. `qml.sample()`

But these return results of measurements in the computational basis; and most hardware only allows for this.

How can we measure with respect to *different bases*?



Use a basis rotation to "trick" the quantum computer.

Suppose we want to measure in the "Y" basis:

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right).$$

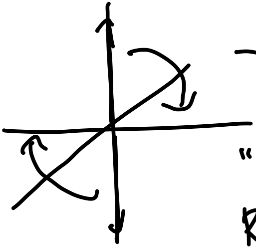
Unitary operations preserve length *and* angles between normalized quantum state vectors. (**Exercise:** prove it!)

There exists some unitary transformation that will convert between this basis and the computational basis.

Exercise: determine a quantum circuit that sends

$$|0\rangle \quad \to \quad |p\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|1\rangle \quad \to \quad |m\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

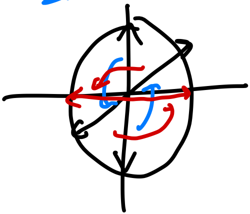


$-\boxed{\sqrt{X}}-$

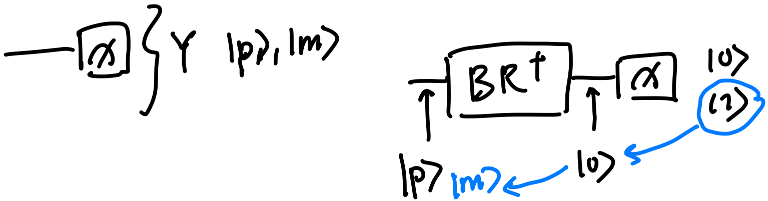
"sqrt X"

$Rx(\pi/2)$

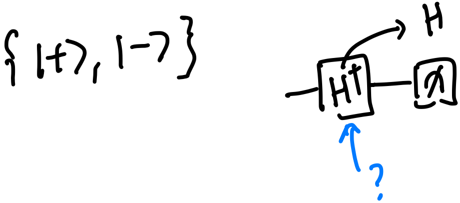
$-\boxed{H}-\boxed{RZ(\pi/2)}-$



At the end of our circuit, we can then apply the reverse (adjoint) of this transformation rotate *back* to the computational basis.



That way, if we measure and observe $|0\rangle$, we know that this was previously $|p\rangle$ in the Y basis (and similarly for $|m\rangle$).



We will start here on Wednesday

In PennyLane, we can compute adjoints of operations *and* entire quantum functions using `qml.adjoint`:

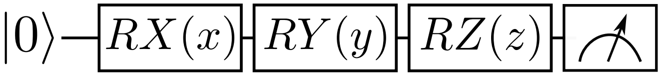
```
def some_function(x):
    qml.RZ(Z, wires=0)

def apply_adjoint(x):
    qml.adjoint(qml.S)(wires=0)
    qml.adjoint(some_function)(x)
```

`qml.adjoint` is a special type of function called a **transform**. We will cover transforms in more detail later in the course.

Basis rotations: hands-on

Let's run the following circuit, and measure in the Y basis



Hands-on time...

Recap

- compute the inner product between two quantum states
- perform a projective measurement
- distinguish between global phase and relative phase
- measure a qubit in different bases

Next time

Content:

- Mathematical representation of multi-qubit systems
- Multi-qubit gates
- Entanglement

Action items:

1. Finish assignment 0
2. Keep an eye out for A1

Recommended reading:

- From today: Codebook nodes I.9
- For next time: Codebook nodes I.11-I.14
- Nielsen & Chuang 4.3