CPEN 400Q Lecture 04
Basis rotations; entanglement and multi-qubit systems

Wednesday 15 January 2025

Announcements

- Assignment 1 due Sunday 26 Jan at 23:59
- Tutorial hands-on due Friday 17 Jan at 23:59
  - Bonus TA office hour Friday 2-3:30, KAIS 4037
- Quiz 2 on Monday (covers L3, L4)
- Will make Piazza thread for tutorial topic suggestions

We introduced the "bra" part of the "bra-ket notation"

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \langle v| = \left(|v\rangle\right)^\dagger = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix}$$

The inner product between two states is defined as

$$\langle v|w\rangle = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1^* w_1 + v_2^* w_2$$

Inner product tells about the *overlap* (similarity) between states.

We introduced the concept of *orthonormal bases* for qubit states:

$$\{ |\psi_1\rangle, |\psi_2\rangle \} \longrightarrow \langle \psi_i | \psi_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Examples:

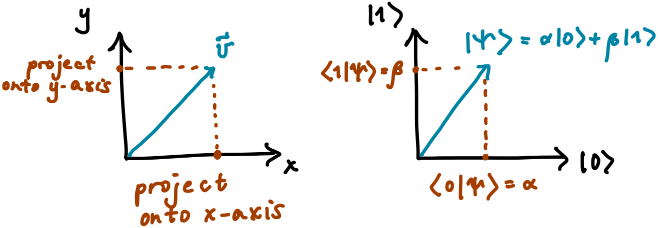
$$|0\rangle \text{ and } |1\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

Last time

We discussed *projective measurement* with respect to a basis.



When we measure state $|\varphi\rangle$ with respect to basis $\{|\psi_i\rangle\}$, the probability of obtaining outcome i is

$$\Pr(\text{outcome } i) = |\langle \varphi | \psi_i \rangle|^2$$

Image credit: Xanadu Quantum Codebook I.9

Projective measurements can be performed with respect to any orthonormal basis. For example, $\{|+\rangle, |-\rangle\}$:

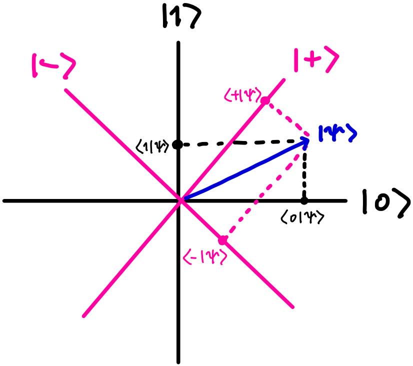


Image credit: Codebook node I.9

Learning outcomes

- Measure a qubit in different bases
- Mathematically describe a system of multiple qubits
- Describe the action of common multi-qubit gates
- Make any gate a controlled gate
- Perform measurements on multiple qubits

Basis rotations

So far we've seen 3 ways of extracting information out of a QNode:

1. `qml.state()`
2. `qml.probs(wires=x)`
3. `qml.sample()`

But these return results of measurements in the computational basis; and most hardware only allows for this.

How can we measure with respect to *different bases*?

Basis rotations

Use a basis rotation to “trick” the quantum computer.

Suppose we want to measure in the “Y” basis:

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right).$$

Recall: unitary operations preserve length *and* angles between normalized quantum state vectors (prove on A1!)

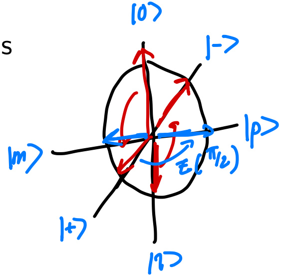
There exists a unitary operation that will convert between this basis and the computational basis.

Basis rotations

Exercise: determine a quantum circuit that sends

$$|0\rangle \quad \to \quad |p\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|1\rangle \quad \to \quad |m\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$



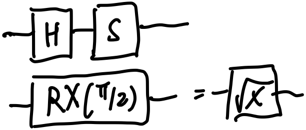
$$H|0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$Z\left(\frac{\pi}{2}\right): S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \sim \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$SH|0\rangle = |p\rangle$$

$$SH|1\rangle = |m\rangle$$



Basis rotations

At the end of our circuit, apply the reverse (adjoint) of this transformation to rotate *back* to the computational basis.

$UU^\dagger = I$

$|p\rangle \longrightarrow |0\rangle$

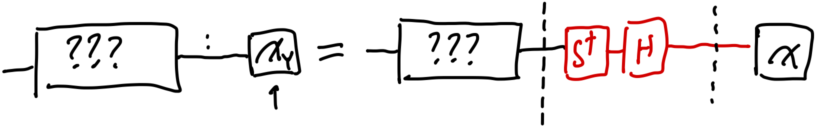
$|m\rangle \longrightarrow |1\rangle$

$-\boxed{S^\dagger}-\boxed{H}-$

$Z(-\frac{\pi}{2})$

$-\boxed{RX(-\pi/2)}- = -\boxed{\sqrt{X}^\dagger}-$

If we measure and observe $|0\rangle$, we know the qubit was previously $|p\rangle$ in the Y basis (analogous result for $|m\rangle$).



$|\psi\rangle = \alpha_Y|p\rangle + \beta_Y|m\rangle$
$= \alpha|0\rangle + \beta|1\rangle$

$|\psi\rangle = \alpha_Y|0\rangle + \beta_Y|1\rangle$

Adjoints

In PennyLane, we can compute adjoints of operations *and* entire quantum functions using `qml.adjoint`:

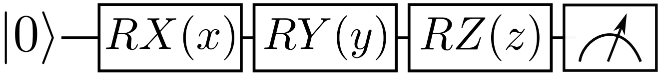
```
def some_function(x):
    qml.RZ(Z, wires=0)

def apply_adjoint(x):
    qml.adjoint(qml.S)(wires=0)
    qml.adjoint(some_function)(x)
```

`qml.adjoint` is a special type of function called a **transform**. We will cover transforms in more detail later in the course.

Basis rotations: hands-on

Let's run the following circuit, and measure in the Y basis



Hands-on time...

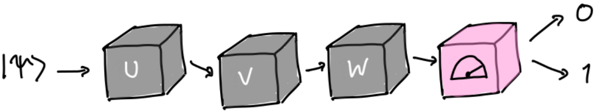
Multi-qubit systems

Recall this slide from lecture 1...

Quantum computing

Quantum computing is the act of manipulating the state of qubits in a way that represents solution of a computational problem:

1. **Prepare** qubits in a **superposition**
2. Apply **operations** that **entangle** the qubits and manipulate the amplitudes
3. **Measure** qubits to extract an answer
4. Profit



Let's simulate this using NumPy.

How do we express the mathematical space of multiple qubits?

Hilbert spaces compose under the *tensor product*.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \;\; B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

The tensor product of A and B, $A \otimes B$ is

$$A \otimes B = \begin{pmatrix} a \begin{pmatrix} e & f \\ g & h \end{pmatrix} & b \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ c \begin{pmatrix} e & f \\ g & h \end{pmatrix} & d \begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

Multi-qubit systems

$$n \text{ qubits} \longrightarrow 2^n \times 2^n \text{ matrices}$$
$$2^n \text{ vectors}$$

Qubit state vectors also combine under the *tensor product*:

$$|0\rangle|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\ 0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = |01\rangle$$
$$= |0\rangle\otimes|1\rangle$$

$$\underbrace{|100\rangle}_{4} = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{array}{c} \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} \end{array}\begin{array}{l}0\\1\\2\\3\\4..\\ \\ \\ \\ \end{array}$$

$$|000\rangle = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}$$

The states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are the computational basis vectors for 2 qubits:

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We can create arbitrary linear combinations of them (coefficients must still be normalized).

Same pattern for 3 qubits: $|000\rangle, |001\rangle, \ldots, |111\rangle$.

$|\psi\rangle_1 = \alpha|0\rangle + \beta|1\rangle$ $\quad |\psi_2\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$

$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$

Multi-qubit systems

The tensor product is linear and distributive. Given

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\varphi\rangle = \gamma|0\rangle + \delta|1\rangle,$$

$$|\psi\rangle \otimes |\varphi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \dots$$

Unitary operations also compose under tensor product.

For example, apply U_1 to qubit $|\psi\rangle$ and U_2 to qubit $|\varphi\rangle$:

$$(U_1 \otimes U_2)(|\psi\rangle \otimes |\varphi\rangle) = (U_1|\psi\rangle) \otimes (U_2|\varphi\rangle)$$

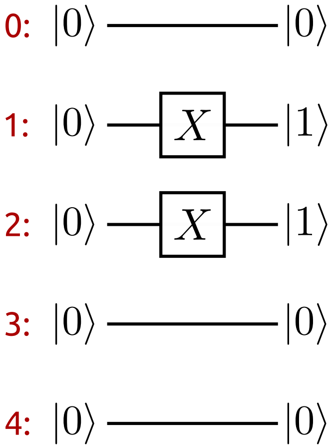
$$\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

Qubit ordering (very important!)

In PennyLane:

$$\underset{\color{darkred}{0\ 1\ 2\ 3\ 4}}{|01100\rangle} \longrightarrow$$

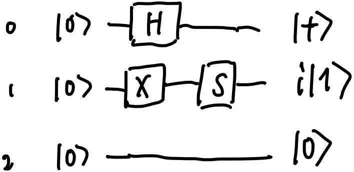
$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\cdots$$



(This is different in other frameworks!)

Exercise: determine the state of a 3-qubit system if H is applied to qubit 0, X and then S are applied to qubit 1.

$$|\underset{0}{0}\underset{1}{0}\underset{2}{0}\rangle$$



$$i|+10\rangle$$

$$\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)\otimes\left(i|1\rangle\right)\otimes|0\rangle$$

$$=\frac{1}{\sqrt{2}}i\left[(|0\rangle+|1\rangle)\otimes|1\rangle\otimes|0\rangle\right]$$

$$=\frac{1}{\sqrt{2}}i\left[(|0\rangle+|1\rangle)\otimes|10\rangle\right]$$

$$=\frac{1}{\sqrt{2}}i\left[|010\rangle+|110\rangle\right]$$

Multi-qubit measurement outcome probabilities

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

If we measure in the computational basis, the outcome probabilities are:

- $|\alpha|^2 = |\langle 00|\psi\rangle|^2$ for $|00\rangle$
- $|\beta|^2 = |\langle 01|\psi\rangle|^2$ for $|01\rangle$
- ...

Exercise: what is the probability of the qubits being in state $|110\rangle$ after measuring $(H \otimes SX \otimes I)|000\rangle$ in the computational basis?

Multi-qubit measurement outcome probabilities

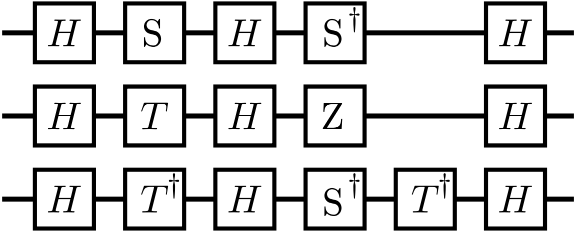
$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

We can also measure *just one qubit*:

- The probability of the first qubit being in state $|0\rangle$ is $|\alpha|^2 + |\beta|^2$
- The probability of the second qubit being in state $|1\rangle$ is $|\beta|^2 + |\delta|^2$

Exercise: what is the probability of the second qubit being in state $|0\rangle$ after measuring $(H \otimes SX \otimes I)|000\rangle$ in the computational basis?

The circuits we've seen so far only involve single-qubit gates:



Surely this isn't all we can do...

Image credit: Xanadu Quantum Codebook I.11

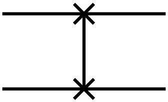
SWAP

We can swap the state of two qubits using the SWAP operation.
First define what it does to the basis states...

$$\mathrm{SWAP}\,|00\rangle = |00\rangle$$
$$\mathrm{SWAP}\,|01\rangle = |10\rangle$$
$$\mathrm{SWAP}\,|10\rangle = |01\rangle$$
$$\mathrm{SWAP}\,|11\rangle = |11\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Circuit element:



PennyLane: `qml.SWAP`

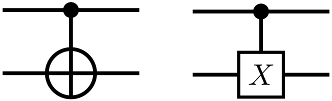
CNOT

CNOT = "controlled-NOT". A NOT (X) is applied to second qubit only if first qubit is in state $|1\rangle$.

$$\mathrm{CNOT}\,|00\rangle = |00\rangle$$
$$\mathrm{CNOT}\,|01\rangle = |01\rangle$$
$$\mathrm{CNOT}\,|10\rangle = |11\rangle$$
$$\mathrm{CNOT}\,|11\rangle = |10\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Circuit elements:



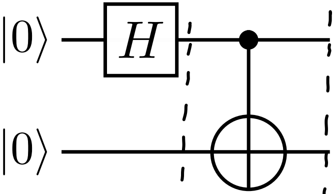
PennyLane: `qml.CNOT`

Image credit: Xanadu Quantum Codebook I.12

CNOT hands-on

Exercise: What does CNOT do to qubits in a superposition? Determine the output state of this circuit, and the measurement outcome probabilities of the computational basis states.

$(\alpha|0\rangle+\beta|1\rangle)\otimes(\gamma|0\rangle+\delta|1\rangle)$



???

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)$

$\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$

Next time

Content:

Entanglement, more 2-qubit gates and controlled operations.

- Superdense coding
- The no-cloning theorem
- ~~Quantum teleportation~~

Action items:

1. Work on Assignment 1 (can do 1 & 5, 2ab, 3 & 6 now)
2. Quiz 2 Monday about contents from this week

Recommended reading:

- For this week: Codebook IQC, SQ, MQ; Nielsen & Chuang 1.2, 1.3.1-1.3.5, 2.2.3-2.2.5, 4.3
- For next week: Codebook SQ (what did you expect?), MQ (all tied up); Nielsen & Chuang 1.3.6, 1.3.7, 2.3