CPEN 400Q Lecture 03
Measurement I (projective measurements)

Monday 15 January 2024

Announcements

- Quiz 1 today
- Assignment 0 due tonight at 23:59 - remember to cite sources in contribution statement
- Assignment 1 coming this week

Quiz solution

Last time

We saw how qubits can be represented in 3D space on the Bloch sphere, and how unitary operations rotate the Bloch vector.

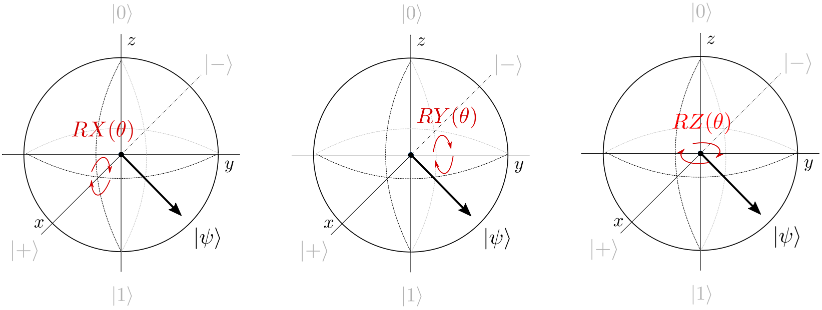


Image credit: Codebook node I.6

We learned about the three Pauli rotations

	Math	Matrix	Code	Special cases
RZ	$e^{-i\frac{\theta}{2}Z}$	$\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$	`qml.RZ`	$Z(\pi), S(\pi/2), T(\pi/4)$
RY	$e^{-i\frac{\theta}{2}Y}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RY`	$Y(\pi)$
RX	$e^{-i\frac{\theta}{2}X}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RX`	$X(\pi), SX(\pi/2)$

Last time

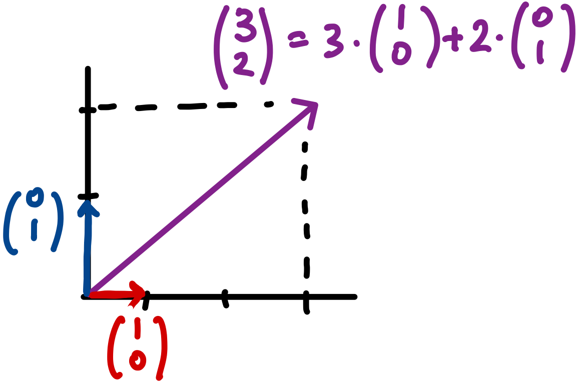
We saw a curious example where changing the state didn't change the measurement outcomes, even though it wasn't a global phase...

Learning outcomes

- compute the inner product between two quantum states
- perform a projective measurement
- distinguish between global phase and relative phase
- measure a qubit in different bases

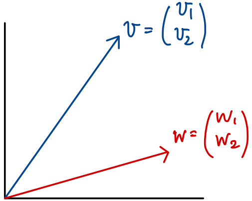
We can now create every single single-qubit quantum state: how do we *compare* them?

Recall what things look like in a classical vector space.



Inner products

We can define an **inner product** between two vectors that tells us how much overlap they have.



Inner products

Take just one of these representations:

The Hilbert space has complex valued vectors. The inner product looks *similar*, but slightly different. Let

The inner product is defined as

This notation is cumbersome, so let's complete our knowledge of Dirac notation by introducing the **bra**:

The inner product is defined as

Written another way,

Pro tip:

Exercise: compute the inner product of the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with itself.

Exercise: compute the inner product between all possible combinations of $|0\rangle$ and $|1\rangle$.

$\langle 0\|0\rangle$	
$\langle 0\|1\rangle$	
$\langle 1\|0\rangle$	
$\langle 1\|1\rangle$	

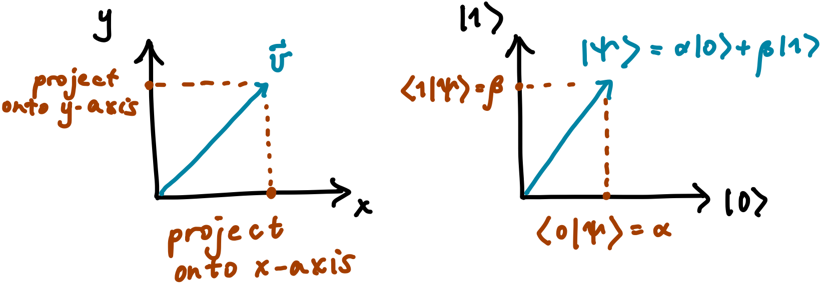
For a single qubit, a pair of states that are **normalized** and **orthogonal** constitute an **orthonormal basis** for the Hilbert space.

Exercise: do the states

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

form an orthonormal basis?

Measurement is performed with respect to a basis; we perform **projections** to determine the overlap with a given basis state.



(Image for expository purposes only!)

Image credit: Xanadu Quantum Codebook I.9

Projective measurements

When we measure state $|\varphi\rangle$ with respect to basis $\{|\psi_i\rangle\}$, the probability of obtaining outcome i is

If we observe outcome i, following the measurement the system will be left in state $|\psi_i\rangle$.

Measurement in computational basis

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Then if we measure $|\psi\rangle$ is the computational basis,

Exercise: what are the measurement outcome probabilities if we measure

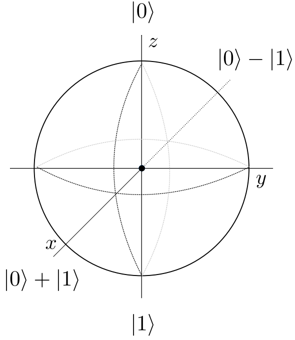
$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

in the computational basis?

Measurement in computational basis

These are *not* the same state: there is a relative phase.

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$



But they are indistinguishable when we measure in the computational basis. How to tell them apart?

Projective measurements can be performed with respect to any orthonormal basis. For example, $\{|+\rangle, |-\rangle\}$:

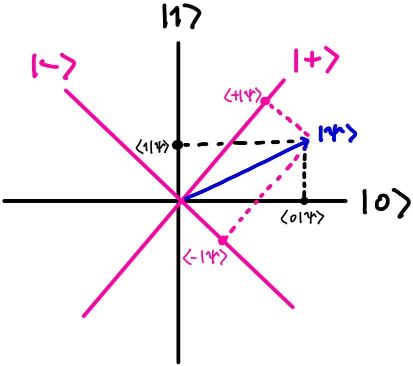


Image credit: Codebook node I.9

Basis rotations

So far we've seen 3 ways of extracting information out of a QNode:

1. `qml.state()`
2. `qml.probs(wires=x)`
3. `qml.sample()`

But these return results of measurements in the computational basis; and most hardware only allows for this.

How can we measure with respect to *different bases*?

Use a basis rotation to "trick" the quantum computer.

Suppose we want to measure in the "Y" basis:

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right).$$

Unitary operations preserve length *and* angles between normalized quantum state vectors. (**Exercise:** prove it!)

There exists some unitary transformation that will convert between this basis and the computational basis.

Exercise: determine a quantum circuit that sends

$$
\begin{aligned}
|0\rangle \quad &\rightarrow \quad |p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \\
|1\rangle \quad &\rightarrow \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)
\end{aligned}
$$

At the end of our circuit, we can then apply the reverse (adjoint) of this transformation rotate *back* to the computational basis.

That way, if we measure and observe $|0\rangle$, we know that this was previously $|p\rangle$ in the Y basis (and similarly for $|m\rangle$).

Adjoints

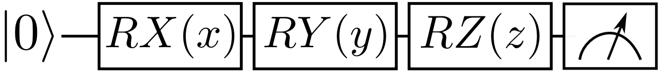
In PennyLane, we can compute adjoints of operations *and* entire quantum functions using `qml.adjoint`:

```
def some_function(x):
    qml.RZ(Z, wires=0)

def apply_adjoint(x):
    qml.adjoint(qml.S)(wires=0)
    qml.adjoint(some_function)(x)
```

`qml.adjoint` is a special type of function called a **transform**. We will cover transforms in more detail later in the course.

Let's run the following circuit, and measure in the Y basis



Hands-on time...

Recap

- compute the inner product between two quantum states
- perform a projective measurement
- distinguish between global phase and relative phase
- measure a qubit in different bases

Next time

Content:

- Mathematical representation of multi-qubit systems
- Multi-qubit gates
- Entanglement

Action items:

1. Finish assignment 0
2. Keep an eye out for A1

Recommended reading:

- From today: Codebook nodes I.9
- For next time: Codebook nodes I.11-I.14
- Nielsen & Chuang 4.3