CPEN 400Q Lecture 17
Quantum channels

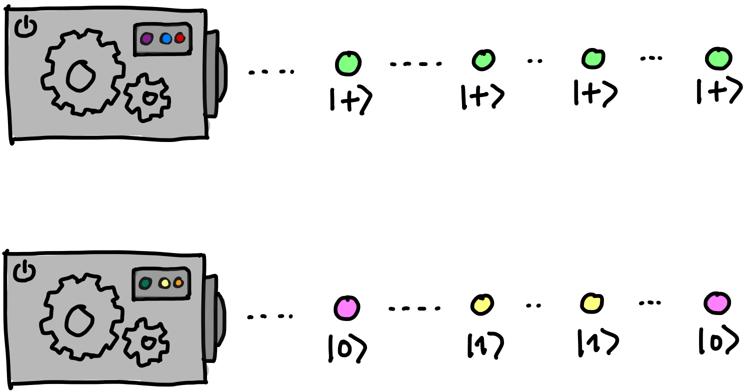
Wednesday 12 March 2025

Announcements

- Quiz 8 beginning of class Monday
- Signup link for MT checkpoint meetings on Piazza (**please use group number, not your name**)
- Signup link for final oral interviews distributed later this week
- Upcoming deadlines
  - MT checkpoint report: this Friday 12:00
  - TA3: this Friday 23:59
  - A3: Tuesday 25 March 23:59

Last time

We introduced *mixed states*.



Mixed states are probabilistic mixtures of pure states.

Last time

	Pure state	Pure state ρ	Mixed state ρ
States	$\|\psi\rangle$	$\rho = \|\psi\rangle\langle\psi\|$	$\rho = \sum_i p_i \|\psi_i\rangle\langle\psi_i\|$
Ops.	$\|\psi'\rangle = U\|\psi\rangle$	$\rho' = \|\psi'\rangle\langle\psi'\| = U\underbrace{\|\psi\rangle\langle\psi\|}_{\rho}U^\dagger$	$\rho' = \sum_i p_i U\|\psi_i\rangle\langle\psi_i\|U^\dagger = U\rho U^\dagger$
Meas.*	$\|\langle\varphi_i\|\psi\rangle\|^2$ $B \mapsto \langle\psi\|B\|\psi\rangle = \langle B\rangle$	$\|\varphi_i\rangle\langle\varphi_i\| \to \mathrm{Tr}(\|\varphi_i\rangle\langle\varphi_i\| \cdot \|\psi\rangle\langle\psi\|)$ $\mathrm{Tr}(B\rho)$	$\mathrm{Tr}(P_i\rho)$ $\mathrm{Tr}(B\rho)$

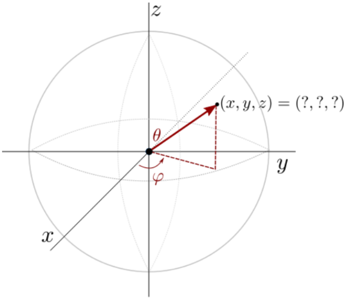
* where $\{\varphi_i\}$ form an orthonormal basis, and $\{P_i\}$ is a set of projectors $P_i = |\varphi_i\rangle\langle\varphi_i|$ or more generally a POVM ($\sum_i P_i = I$).

Learning outcomes

- Identify mixed states on the Bloch sphere
- Define fidelity and trace distance, and use them to compute the distance between two arbitrary quantum states
- Define and apply quantum channels to quantum states
- Express operations, measurements, and partial trace as quantum channels

Recall the following two problems from A2:

Write a function that uses quantum measurements to extract its Bloch vector, i.e., its coordinates in 3-dimensional space, as demonstrated by the following graphic:

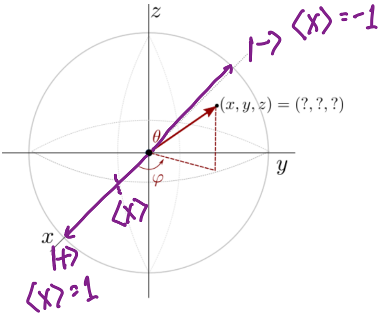


Suppose we wish to determine the expectation value of a general Hermitian observable, $\langle M \rangle$, for an arbitrary single-qubit state, but are restricted to measuring Pauli expectation values.

a. (1 point) Show that, for any single-qubit M, we can determine $\langle M \rangle$ by measuring only $\langle X \rangle$, $\langle Y \rangle$, and $\langle Z \rangle$. Upload a hand-written or typeset solution below. *Hint: write down a general single-qubit Hermitian operator, then leverage linearity of expectation values.*

Mixed states and the Bloch sphere

Write a function that uses quantum measurements to extract its Bloch vector, i.e., its coordinates in 3-dimensional space, as demonstrated by the following graphic:



$$\langle X \rangle = \langle \psi | X | \psi \rangle \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= 1 \cdot Pr(+) + (-1) \cdot Pr(-)$$

$$= Tr(|+\rangle\langle+| \cdot \rho) - Tr(|-\rangle\langle-| \rho)$$

$$= Tr\left[|+\rangle\langle+| \rho - |-\rangle\langle-| \rho\right]$$

$$= Tr\left[\underbrace{(|+\rangle\langle+| - |-\rangle\langle-|)}_{?} \rho\right]$$

$$= Tr(X\rho)$$

$$\langle X \rangle = Tr(X\rho)$$

$$\langle Y \rangle = Tr(Y\rho)$$

$$\langle Z \rangle = Tr(Z\rho)$$

Given $\langle X\rangle$, $\langle Y\rangle$, $\langle Z\rangle$, can we determine ρ?

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Recall ρ is Hermitian; Paulis are a basis.

$$\rho = \frac{a_I}{2} I + \frac{a_X}{2} X + \frac{a_Y}{2} Y + \frac{a_Z}{2} Z$$

ρ must have trace 1:

$$\mathrm{Tr}(\rho) = 1 = \mathrm{Tr}\left[\underset{\checkmark}{\frac{a_I}{2} I} + \underset{\times}{\frac{a_X}{2} X} + \underset{\times}{\frac{a_Y}{2} Y} + \underset{\times}{\frac{a_Z}{2} Z} \right]$$

$$= \frac{a_I}{2} \cdot \mathrm{Tr}(I)$$

$$1 = a_I$$

Trace out another Pauli:

$$\mathrm{Tr}(X\rho) = \mathrm{Tr}\left(\cancel{\frac{1}{2} X} + \frac{a_X}{2} I + \cancel{\frac{a_Y}{2} iZ} + \cancel{\frac{a_Z}{2}(-iY)} \right)$$

$$= a_X = \langle X \rangle$$

More formally, we can write any ρ as

$$\rho = \frac{1}{2} I + \frac{a_X}{2} X + \underbrace{\frac{a_Y}{2}}_{} Y + \frac{a_Z}{2} Z$$

$$a_Y \to \langle Y \rangle \qquad a_Z \to \langle Z \rangle$$

where $a_P = \mathrm{Tr}(P\rho) = \langle P \rangle$.

(Note that all of this generalizes to multiple qubits as well)

Re-express:

$$\rho = \frac{1}{2} I + \frac{\langle X \rangle}{2} X + \frac{\langle Y \rangle}{2} Y + \frac{\langle Z \rangle}{2} Z$$

$$= \frac{1}{2} \begin{pmatrix} 1 + \langle Z \rangle & \langle X \rangle - i \langle Y \rangle \\ \langle X \rangle + i \langle Y \rangle & 1 - \langle Z \rangle \end{pmatrix}$$

$(\langle X \rangle, \langle Y \rangle, \langle Z \rangle)$

Exercise: As ρ is positive semidefinite, its eigvals are ≥ 0. What constraint does this put on $\langle X \rangle, \langle Y \rangle, \langle Z \rangle$? (Hint: look at $\det(\rho)$)

$$\det(\rho) = \frac{1}{2} \Big((1 + \langle Z \rangle)(1 - \langle Z \rangle) - (\langle X \rangle - i \langle Y \rangle)(\langle X \rangle + i \langle Y \rangle) \Big)$$

$$= \frac{1}{2} \left(1 - \langle X \rangle^2 - \langle Y \rangle^2 - \langle Z \rangle^2 \right)$$

$$\geq 0$$

$$\Rightarrow \langle X \rangle^2 + \langle Y \rangle^2 + \langle Z \rangle^2 \leq 1$$

eigvecs

eigvals

$\rho = A D A^\dagger$

$\det(\rho) = \det(A D A^\dagger)$

$= \det(D)$

Mixed states and the Bloch sphere

Exercise: How many eigenvalues does ρ have if it is pure? What constraint does this put on $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$?

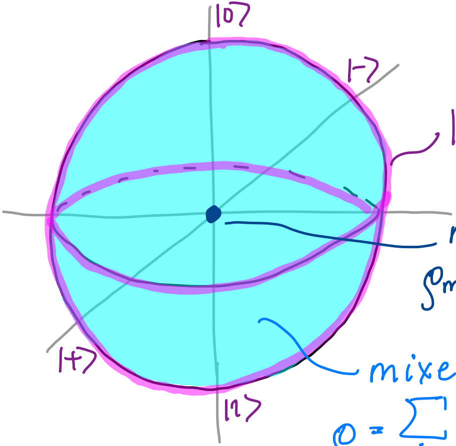
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|0\rangle\langle 0| \qquad |1\rangle\langle 1| \qquad |+\rangle\langle +|$$

$$\rho \text{ pure} \longrightarrow \det(\rho) = 0$$

$$\Rightarrow \langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 = 1$$

Mixed states live *in* the Bloch sphere!



pure states

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

maximally mixed state

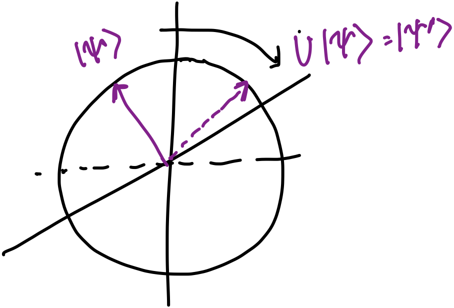
$\rho_{max} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{1}$

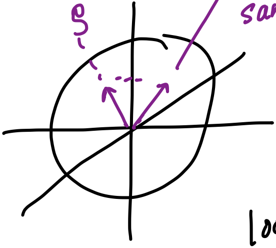
mixed states

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Mixed states and the Bloch sphere

Exercise: What happens to a mixed state on the Bloch sphere if we apply a unitary U?





still mixed, same "length"

Look at determinant of ρ vs $U\rho U^\dagger$

To “get inside” the Bloch sphere, we have to apply a different kind of operation: a quantum channel.

A quantum channel Φ *maps* density matrices to density matrices.

$$\rho \longmapsto \rho' = \Phi(\rho)$$

Quantum channels are completely positive, trace-preserving (CPTP) linear maps.

- Trace-preserving: $\mathrm{Tr}(\rho') = \mathrm{Tr}(\rho) = 1$
- Positive: ρ positive semidefinite $\longrightarrow$ ρ' pos. semidef.
- Completely positive: channel $I_n \otimes \Phi$ must be positive for all n

Quantum channels are characterized by a set of **Kraus operators** $\{K_i\}$,

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$$

where

$$\sum_i K_i^\dagger K_i = I$$

A channel's Kraus operators represent, loosely, a set of things that can happen to a system, including *measurement*, and *errors*.

1 Kraus operator: $\Phi(\rho) = K \rho K^\dagger \quad K^\dagger K = I$

↳ K is unitary!

Example: unitary channel

A channel with a single Kraus operator is a unitary operation ("unitary channel"): $\mathcal{U}$.

$$\mathcal{U}(\rho) = U \rho U^{\dagger}$$

Concrete example with a projective measurement $\{M_m\} = \{\Pi_+, \Pi_-\}$:

$$\sum_m M_m^\dagger M_m = I$$

Consider mixed state

$$\rho = \sum_i p_i \rho_i = \frac{1}{5}|0\rangle\langle 0| + \frac{4}{5}|-\rangle\langle -|$$

$$\frac{1}{2}|+\rangle \quad \frac{1}{2}|-\rangle \qquad 0\cdot|+\rangle \quad 1\cdot|-\rangle$$

Intuitively, after measurement expect

$$\rho' = \frac{1}{5}\left(\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|\right) + \frac{4}{5}|-\rangle\langle -| = \frac{1}{10}|+\rangle\langle +| + \frac{9}{10}|-\rangle\langle -|$$

$$= \sum_m M_m \rho M_m^\dagger$$

Example: measurement

* We didn't cover this in detail, including for completion

Recall last class we discussed projective measurements and the more general POVM.

Let $\{M_m\}$ be a set of measurement operators, where the M_m are not necessarily projectors, and $E_m = M_m^\dagger M_m$, where

$$\sum_m E_m = I \qquad \text{Prob}(m) = \text{Tr}(E_m \rho)$$

The set of operators

$$\{E_m\}$$

constitutes a POVM, and can be used to make a measurement on a quantum system.

Example: measurement

The $\{M_m\}$ can be viewed as Kraus operators:

$$\sum_m M_m^\dagger M_m = I$$

Consider arbitrary mixed state

$$\rho = \sum_i p_i \rho_i = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

What is the probability of measuring and obtaining outcome m?

$$p(m) = \sum_i p(m|i)\, p_i$$

$$= \sum_i p_i \langle\psi_i| M_m^\dagger M_m |\psi_i\rangle = \sum_i p_i \operatorname{Tr}\left[M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|\right]$$

$$= \sum_i p_i \operatorname{Tr}(M_m^\dagger M_m \rho_i)$$

$$= \operatorname{Tr}(M_m^\dagger M_m \rho) = \operatorname{Tr}(E_m \rho)$$

(Normalized) state after measurement and obtaining outcome m:

$$\rho \mapsto \frac{M_m \rho M_m^\dagger}{p(m)} \leftarrow \text{for normalization}$$

Overall state transforms as

$$\rho \mapsto \rho' = \sum_m \text{prob } m \cdot \text{state if obtain } m$$

$$= \sum_m p(m) \frac{M_m \rho M_m^\dagger}{p(m)}$$

$$= \sum_m M_m \rho M_m^\dagger \rightarrow \text{channel w/ Kraus ops } \{M_m\}!$$

Next time

Last class:

- Error channels
- Noise in quantum systems

Action items:

1. MT checkpoint reports
2. TA3 and A3

Recommended reading:

- From this class: Codebook NT, DM; N&C 2.2.6, 2.4
- For next class: Codebook EC; N& C 8.2-8.3,