CPEN 400Q Lecture 18
Noise and error channels

Monday 17 March 2025

Announcements

- Quiz 8 today
- MT checkpoints this week
- Sign up for final oral interview (Canvas calendar)
- A3 due Tuesday 25 March 23:59
- New slide format - feedback welcome

I will post sign ups for group presentations

Last time

We expressed density matrices (for pure or mixed states) as

$$\rho = \frac{1}{2} I + \frac{\langle X \rangle}{2} X + \frac{\langle Y \rangle}{2} Y + \frac{\langle Z \rangle}{2} Z$$

$$= \frac{1}{2} \begin{pmatrix} 1 + \langle Z \rangle & \langle X \rangle - i\langle Y \rangle \\ \langle X \rangle + i\langle Y \rangle & 1 - \langle Z \rangle \end{pmatrix}$$

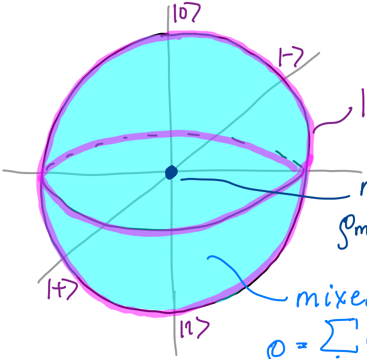
Since ρ is positive semidefinite,

$$\det(\rho) = \frac{1}{2}\left(1 - \langle X \rangle^2 - \langle Y \rangle^2 - \langle Z \rangle^2\right) \geq 0$$

$$\Rightarrow \quad \langle X \rangle^2 + \langle Y \rangle^2 + \langle Z \rangle^2 \leq 1$$

Last time

The set $(\langle X\rangle, \langle Y\rangle, \langle Z\rangle)$ form the Bloch vector, $\langle P\rangle = \mathrm{Tr}(P\rho)$.



pure states

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

maximally mixed state

$\rho_{max} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{1}$

mixed states

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Last time

Unitary operations preserve length of Bloch vector

$$\rho \rightarrow \rho' = V \rho V^\dagger \qquad \det(\rho') = \det(V \rho V^\dagger) = \det(\rho)$$

To take pure states to mixed states, apply a *quantum channel*:

$$\rho \longrightarrow \rho' = \Phi(\rho)$$

Quantum channels are CPTP linear maps.

- **T**race-**P**reserving: $\mathrm{Tr}(\rho') = \mathrm{Tr}(\rho)$
- Positive: ρ eig vals $\geq 0 \Rightarrow \rho'$ eig vals ≥ 0
- **C**ompletely **P**ositive: $I_n \otimes \Phi \otimes I_m \longrightarrow$ also has to be positive for all n, m

Last time

Quantum channels are characterized by **Kraus operators** $\{K_i\}$,

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger \qquad \sum_i K_i K_i^\dagger = I$$

Example: unitary channel, $\mathcal{U}$, $\{K_i\} = \{U\}$.

$$\mathcal{U}(\rho) = U \rho U^\dagger, \qquad UU^\dagger = I$$

Example: a projective measurement $\mathcal{M}$, $\{K_i\} = \{\Pi_i\}$

$$\mathcal{M}(\rho) = \sum_i (\text{Prob outcome } i) \cdot (\text{State if observe outcome } i)$$

$$= \sum_i \Pi_i \rho \Pi_i^\dagger \qquad \sum_i \Pi_i \Pi_i^\dagger = I$$

(before you actually observe it)

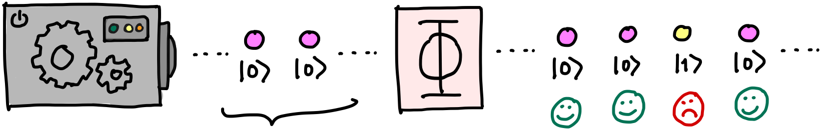
Learning outcomes

1. Express noise in quantum systems using quantum channels
2. Compare density matrices with fidelity and trace distance
3. Apply noise to quantum circuits in PennyLane

Next time: how to correct errors

The bit flip channel

Suppose a "bit flip" (Pauli X) error occurs with probability p.



pure state

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

How do we write this as a channel? What are the Kraus operators?

$$\rho \longmapsto \boxed{\Phi(\rho) = p\cdot X\rho X^\dagger + (1-p)\rho}$$

$$\Phi(|0\rangle\langle 0|) = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0| = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}$$

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$$

$$\boxed{K_0 = \sqrt{p}\,X \qquad K_1 = \sqrt{1-p}\,I}$$

$$K_0K_0^\dagger + K_1K_1^\dagger = pI + (1-p)I = I$$

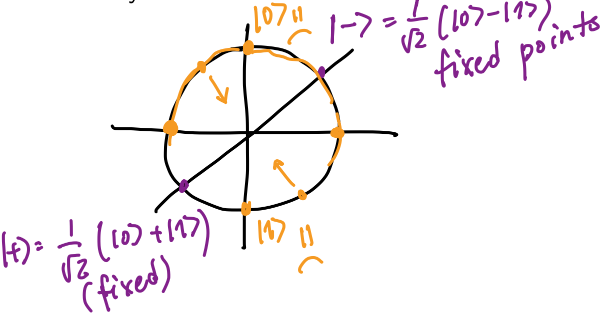
The bit flip channel

We can visualize the effects of the channel by observing how it deforms the Bloch sphere.

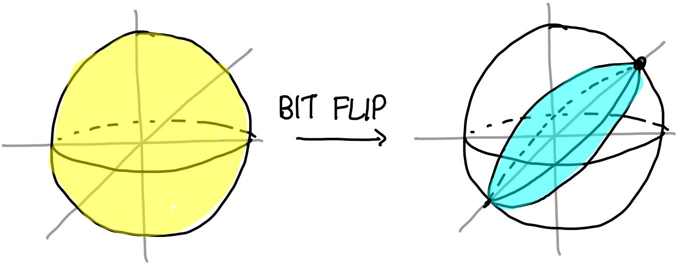
Consider:

- Which states are not affected by a bit flip channel?
- Which states are *most* affected, and where do they go?

Then look at the density matrices of these states.



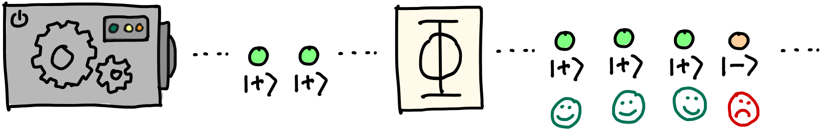
The bit flip channel



x : same
yz : contracts
(factor 1-2p)

The phase flip channel

Suppose a “phase flip” (Pauli Z) error occurs with probability p.



How do we write this as a channel? What are the Kraus operators?

$$\Phi(\rho) = p \cdot Z \rho \underbrace{Z^\dagger}_{= Z} + (1-p)\rho$$

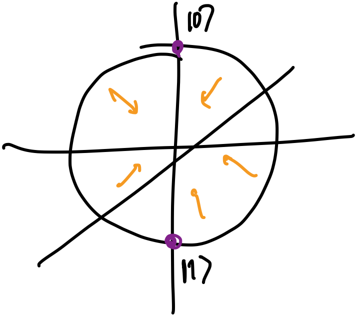
$$K_0 = \sqrt{p}\, Z \qquad\qquad K_1 = \sqrt{1-p}\, I$$

The phase flip channel

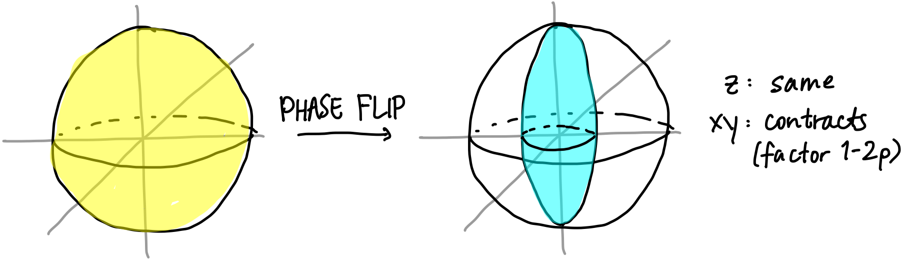
Same questions:

- Which states are not affected by a phase flip channel?
- Which states are *most* affected, and where do they go?

Then look at the density matrices of these states.

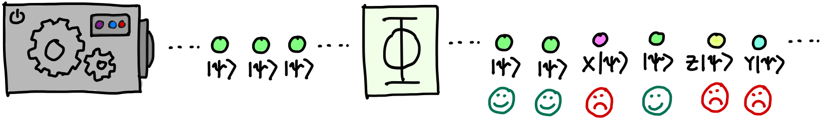


The phase flip channel



The depolarizing channel

Suppose each Pauli error occurs with probability $p/3$.



How do we write this as a channel? What are the Kraus operators?

$$\Phi(\rho) = (1-p)\rho + \frac{p}{3} X\rho X + \frac{p}{3} Y\rho Y + \frac{p}{3} Z\rho Z$$

$$K_0 = \sqrt{1-p}\, I \qquad K_1 = \sqrt{\frac{p}{3}}\, X \qquad K_2 = \sqrt{\frac{p}{3}}\, Y \qquad K_3 = \sqrt{\frac{p}{3}}\, Z$$

Q: What will happen to the Bloch sphere?

The depolarizing channel

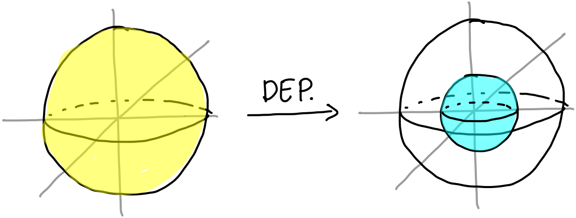
The depolarizing channel

$$\Phi(\rho) = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$$

can also be written as

$$\Phi(\rho) = (1-\lambda)\rho + \lambda\frac{I}{2}$$

$$\rho = \frac{I}{2} + \frac{\langle X\rangle}{2}X + \ldots$$



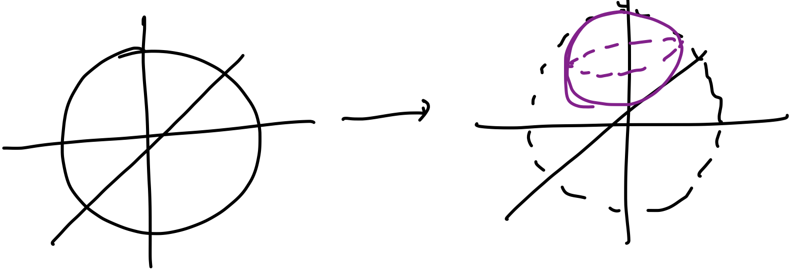
Amplitude damping channel

Suppose $|1\rangle$ relaxes to $|0\rangle$ with probability p.

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} = \sqrt{p}\, |0\rangle\langle 1|$$

$|0\rangle\langle 1|$

How do you think this affects the Bloch sphere?



Comparing density matrices

The inner product tells us how close two pure states are:

$$\langle \psi | \phi \rangle$$

How close are two mixed states ρ and σ?

One common metric is the **trace distance**:

$$Tr(\rho, \sigma) = \frac{1}{2} \| \rho - \sigma \|_1 = \frac{1}{2} Tr \sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)}$$

Bounded by $0 \leq T(\rho, \sigma) \leq 1$; *lower* trace distance is better.

Comparing density matrices

Can also compare using **fidelity**,

$$F(\rho, \sigma) = \left(\mathrm{Tr} \sqrt{\sqrt{\rho}\, \sigma \sqrt{\rho}} \right)^2$$ qml.fidelity

Bounded by $0 \leq F(\rho, \sigma) \leq 1$. *Higher* fidelity is better.

Example: Determine $F(\rho, \sigma)$ if both ρ, σ are pure.

$\Rightarrow$ try it - show it reduces to inner product, $|\langle \psi | \phi \rangle|^2$, $\rho = |\psi\rangle\langle\psi|$, $\sigma = |\phi\rangle\langle\phi|$

Comparing density matrices

Exercise: Determine $F(\rho, \sigma)$ if ρ is pure but σ is mixed.

☆ try yourself!

Comparing density matrices

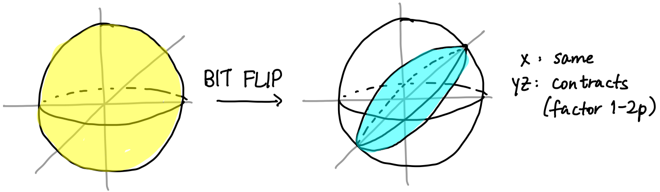
Exercise: What is the fidelity of any pure ρ with the maximally mixed state, $\sigma = \frac{1}{2}I$?

Try yourself!

Comparing density matrices

Exercise: Determine $F(\rho, \sigma)$ if ρ is pure and and $\sigma = \Phi(\rho)$, where Φ is the *bit flip channel* with parameter p.

☆ Try yourself; see answer in the code demo from today



Simulating noisy systems

Exercise: Suppose we prepare a system in the state

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

If a depolarizing channel with $p = 0.02$ is applied, what's the probability of observing $|0\rangle$?

Solution 1: solve by hand. Tedious, but can evaluate

Simulating noisy systems

Solution 2: use PennyLane's ``default.mixed'' device!

Resources:

- https://docs.pennylane.ai/en/stable/introduction/operations.html#noisy-channels
- https://docs.pennylane.ai/en/stable/code/qml_noise.html

Next time

Next class:

- Intro to quantum error correction

Action items:

1. MT checkpoint meetings
2. A3 (due 25 March 23:59)
3. Work on project

Recommended reading:

- From this class: Codebook NT, DM; N&C 8.2-8.3, 9.1-9.2
- For next class: Codebook EC; N&C 10.1-10.2