CPEN 400Q Lecture 07
Measurement II (expectation values)

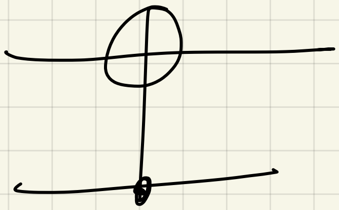
Monday 27 January 2025

Announcements

- Quiz 3 today
- Tutorial tomorrow: midterm practice
- Midterm in class on Wed (info on PrairieLearn) covers "the basics", i.e., lectures 01-06, A1, Q1-3 (may see today's material at high level, but learning outcomes not being tested)

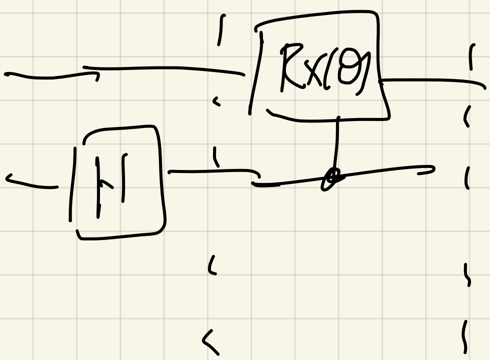
2024 midterm: 1.6 and 2.2 not applicable
(or Hermitian mats in 1.1)

CNOT_{10}



$$\begin{aligned} 00 &\to 00 \\ 01 &\to 11 \\ 10 &\to 10 \\ 11 &\to 01 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{U} \\ 0 & 0 & & \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{00} & 0 & U_{01} \\ 0 & 0 & 1 & 0 \\ 0 & U_{10} & 0 & U_{11} \end{pmatrix}$$



$$|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

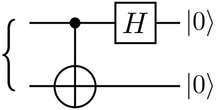
$$|\phi\rangle = CRX(\theta)(\equiv) = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}\left[\cos\theta|0\rangle - i\sin\theta|1\rangle\right]|1\rangle$$

$$|\langle\Psi_{00}|\phi\rangle|^2 = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{\sqrt{2}}(\cos\theta|01\rangle - i\sin\theta|11\rangle)$$

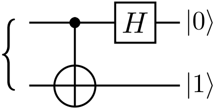
We implemented **superdense coding** and **teleportation**.

Both algorithms leverage **shared entanglement**, and perform measurements in the Bell basis.

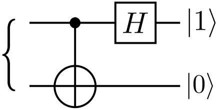
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



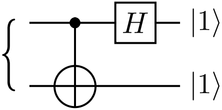
$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$



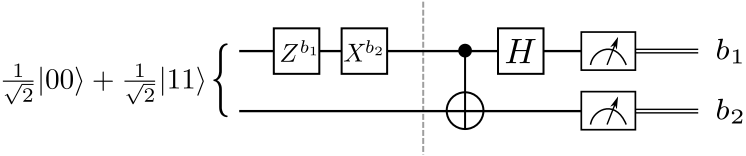
$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$



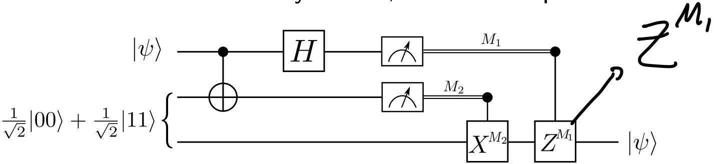
$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$



We "sent" two bits of information by transmitting only one qubit with the **superdense coding** protocol.



While we cannot clone arbitrary states, we *can* teleport them!



You asked lots of awesome questions - this stuff is pretty weird!

Learning outcomes

Core outcomes:

- define observables and expectation values
- compute the expectation value of an observable after performing a quantum computation

If there's time:

- distinguish between projective measurements and positive operator-valued measurements (POVMs)
- develop a POVM to help differentiate between two non-orthogonal quantum states

Projective measurements: recap

Our current view of measurements involves computing an inner product w.r.t. a basis state to determine the outcome probability:

$$\{|\phi_i\rangle\} \qquad\qquad |\langle\phi_i|\psi\rangle|^2 = \text{Prob}(\text{outcome } i)$$

In other contexts, we are interested in measuring real, physical quantities. In physics, these are called **observables**.

(The two kinds of measurements are related) unitary: $UU^\dagger = U^\dagger U = I$

Observables are represented mathematically by Hermitian matrices. An operator (matrix) H is Hermitian if

$$B = B^\dagger$$

Why Hermitian?

- the possible measurement outcomes of an observable are its eigenvalues
- the eigenvalues of Hermitian operators are **real**.

Example:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Z is Hermitian:

$$Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

Its eigensystem is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{associate}} \lambda_0 = 1$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \lambda_1 = -1$$

Example:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X^\dagger$$

X is Hermitian and its (normalized) eigensystem is

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \lambda_+ = 1$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad \lambda_- = -1$$

Example:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Y is Hermitian and its (normalized) eigensystem is

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \qquad \lambda_p = 1$$

$$|m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) \qquad \lambda_m = -1$$

Expectation values

Analytically, the **expectation value** of measuring the observable B given the state $|\psi\rangle$ is

$$\langle B \rangle = \langle \psi | B | \psi \rangle$$

When we measure an observable (e.g., X, Y, or Z), for each shot we observe the system in one of its eigenstates, and associate the outcome to its eigenvalue.

The expectation value is what we expect to see *on average* over multiple shots.

Expectation values: analytical

Exercise: consider the quantum state

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{2}|0\rangle - i\frac{\sqrt{3}}{2}|1\rangle.$$

Compute the expectation value of Y: $\langle Y \rangle = \langle \psi | Y | \psi \rangle$

$$Y|\psi\rangle = \frac{1}{2} Y|0\rangle - i\frac{\sqrt{3}}{2} Y|1\rangle = \frac{1}{2}\left(i|1\rangle\right) - i\frac{\sqrt{3}}{2}\left(-i|0\rangle\right) = \frac{i}{2}|1\rangle - \frac{\sqrt{3}}{2}|0\rangle$$

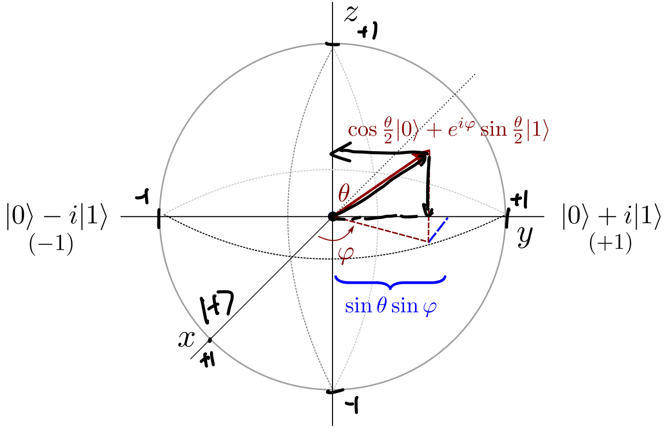
$$\langle\psi| \overbrace{Y|\psi\rangle} = \left(\frac{1}{2}\langle 0| + i\frac{\sqrt{3}}{2}\langle 1|\right)\left(-\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right) = \begin{pmatrix}\frac{1}{2} & \frac{i\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}-\frac{\sqrt{3}}{2} \\ \frac{i}{2}\end{pmatrix}$$

$$= -\frac{\sqrt{3}}{4}\underbrace{\langle 0|0\rangle}_{=1} - \cancel{\frac{3}{4}i\langle 1|0\rangle}^{\,0} + \cancel{\frac{i}{4}\langle 0|1\rangle}^{\,0} - \frac{\sqrt{3}}{4}\underbrace{\langle 1|1\rangle}_{=1}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

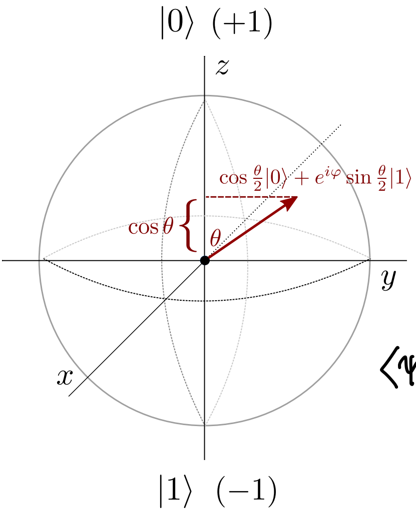
$$= -\sqrt{3}/2$$

The Bloch sphere offers us some more insight.



Exercise: derive the expression in blue by computing $\langle\psi|\, Y |\psi\rangle$.

Expectation values and the Bloch sphere



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$\langle Z\rangle = \langle\psi|Z|\psi\rangle$$

$$Z|\psi\rangle = \cos\frac{\theta}{2}|0\rangle - e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$\langle\psi|Z|\psi\rangle = \left(\cos\frac{\theta}{2}\langle 0| + e^{-i\varphi}\sin\frac{\theta}{2}\langle 1|\right)\cdot \left(\cos\frac{\theta}{2}|0\rangle - e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= \cos\theta$$

More formally, the **spectral theorem** from linear algebra states

$$B = \sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k|$$

where λ_k, $|\varphi_k\rangle$ are the eigenvalues and eigenstates of B

$$|+\rangle\langle+| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

Exercise: show that the spectral theorem holds for Pauli X.

$$X = (+1\ |+\rangle\langle+|) + (-1\cdot |-\rangle\langle-|)$$

$\langle\varphi_k|\varphi_k\rangle \rightarrow$ inner product $\in \mathbb{C}$

$|\varphi_k\rangle\langle\varphi_k| \rightarrow$ outer product

$$|\psi\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \Rightarrow |\psi\rangle\langle\psi| = \begin{pmatrix}\alpha\\\beta\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}\alpha\alpha^* & \alpha\beta^*\\\beta\alpha^* & \beta\beta^*\end{pmatrix}$$

Why this works

$$\text{Basis } \{|\varphi_k\rangle\} \to \Pr(\text{outcome } k) = |\underbrace{\langle\varphi_k|\psi\rangle}_{\alpha}|^2$$

The spectral theorem shows how this relates to measurement outcome probabilities of projective measurements:

$$|\alpha|^2 = \alpha\alpha^*$$

$$\langle B\rangle = \langle\psi|B|\psi\rangle = \langle\psi|\left(\sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k|\right)|\psi\rangle$$

$$= \sum_k \lambda_k \underbrace{\langle\psi|\varphi_k\rangle}_{\alpha^*}\underbrace{\langle\varphi_k|\psi\rangle}_{\alpha}$$

$$= \sum_k \lambda_k \cdot |\langle\varphi_k|\psi\rangle|^2$$

Expectation values: from measurement data

Let's compute the expectation value of Z for the following circuit using 10 samples:

```
dev = qml.device('default.qubit', wires=1, shots=10)

@qml.qnode(dev)
def circuit():
    qml.RX(2*np.pi/3, wires=0)
    return qml.sample()
```

Results might look something like this:

$$[1,\ 1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 1,\ 1]$$

Expectation values: from measurement data

The expectation value pertains to the measured eigenvalue; recall Z eigenstates are

$$\lambda_1 = +1, \qquad |\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = -1, \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So when we observe $|0\rangle$, this is eigenvalue $+1$ (and if $|1\rangle$, -1). Our samples shift from

```
[1, 1, 1, 0, 1, 1, 1, 0, 1, 1]
```

to

```
[-1, -1, -1, 1, -1, -1, -1, 1, -1, -1]
```

Expectation values: from measurement data

The expectation value is the weighted average, where the weights are the eigenvalues:

$$\langle Z \rangle = \frac{1 \cdot n_1 + (-1) \cdot n_{-1}}{N}$$

where

- n_1 is the number of $+1$ eigenvalues
- n_{-1} is the number of -1 eigenvalues
- N is the total number of shots

For our example, $\langle Z \rangle = -0.6$

Expectation values

Let's do this in PennyLane instead:

```
dev = qml.device('default.qubit', wires=1)

@qml.qnode(dev)
def measure_z():
    qml.RX(2*np.pi/3, wires=0)
    return qml.expval(qml.PauliZ(0))
```

Multi-qubit expectation values

Example: operator $Z \otimes Z$.

Eigenvalues are computational basis states:

$$\begin{aligned}
(Z \otimes Z)|00\rangle &= |00\rangle \\
(Z \otimes Z)|01\rangle &= -|01\rangle \\
(Z \otimes Z)|10\rangle &= -|10\rangle \\
(Z \otimes Z)|11\rangle &= |11\rangle
\end{aligned}$$

To compute an expectation value from data:

$$\langle Z \otimes Z \rangle = \frac{1 \cdot n_1 + (-1) \cdot n_{-1}}{N}$$

Example: operator $X \otimes I$.

Eigenvalues of X are the $|+\rangle$ and $|-\rangle$ states:

$$(X \otimes I)|+0\rangle = |+0\rangle$$
$$(X \otimes I)|+1\rangle = |+1\rangle$$
$$(X \otimes I)|-0\rangle = -|-0\rangle$$
$$(X \otimes I)|-1\rangle = -|-1\rangle$$

Fun fact: All Pauli operators have an equal number of $+1$ *and* -1 *eigenvalues!*

Multi-qubit expectation values

How to compute expectation value of X from data, when we can only measure in the computational basis?

Basis rotation: apply H to first qubit

$$(H \otimes I)(X \otimes I)|+0\rangle = |00\rangle$$
$$(H \otimes I)(X \otimes I)|+1\rangle = |01\rangle$$
$$(H \otimes I)(X \otimes I)|-0\rangle = -|10\rangle$$
$$(H \otimes I)(X \otimes I)|-1\rangle = -|11\rangle$$

When we measure and obtain $|10\rangle$ or $|11\rangle$, we know those correspond to the -1 eigenstates of $X \otimes I$.

Hands-on: multi-qubit expectation values

Multi-qubit expectation values can be created using the `@` symbol:

```
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0) @ qml.PauliZ(1))
```

Can return *multiple* expectation values (if no shared qubits)

```
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0)), qml.expval(qml.PauliZ(
                                      1))
```

Next time

Content:

- Oracle-based algorithms (Deutsch and Grover)

Action items:

1. Study for midterm

Recommended reading (lecture 01-07, and midterm)

- Codebook nodes IQC, SQ, MQ
- Nielsen & Chuang 1.2, 1.3.1-1.3.7, 2.2.3-2.2.5, 2.3, 4.3

Read ahead for next time:

- Codebook modules BA (basic quantum algorithms), GA (Grover's algorithm)
- Nielsen and Chuang 1.4, 6.1