CPEN 400Q Lecture 05
Entanglement and multi-qubit gates; superdense coding

Monday 20 January 2025

Announcements

- Quiz 2 today
- Assignment 1 due 26 Jan at 23:59
- Tomorrow's tutorial: TA office hour / suggest topics on Piazza
- Midterm in class on Wed 29 Jan - details on PrairieLearn.

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5}{4}}|1\rangle$$

$$|b_1\rangle = \frac{2i}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}e^{i\frac{2}{3}}|1\rangle$$

$$|b_2\rangle = -\frac{1}{\sqrt{5}}e^{-i\frac{2}{3}}|0\rangle - \frac{2i}{\sqrt{5}}|1\rangle$$

(1) $\langle b_2|\psi\rangle$

(2) $\mathrm{Prob}(|b_1\rangle)$

$$\langle b_2|\psi\rangle = \left(-\frac{1}{\sqrt{5}}e^{i\frac{2}{3}} \quad \frac{2i}{\sqrt{5}}\right)\begin{pmatrix}\frac{\sqrt{3}}{2} \\ -\frac{1}{2}e^{i5/4}\end{pmatrix}$$

$$= -\frac{1}{\sqrt{5}}\frac{\sqrt{3}}{2}e^{i\frac{2}{3}} - \frac{i}{\sqrt{5}}e^{i5/4}$$

$$\Rightarrow a+bi = z$$

$\mathrm{Prob}(|b_1\rangle) \Rightarrow \mathrm{Prob}(|b_2\rangle) = |\langle b_2|\psi\rangle|^2$

$\Downarrow$

$|\langle b_1|\psi\rangle|^2 = 1 - \mathrm{Prob}(|b_2\rangle)$

Last time

We performed measurements in different bases by applying basis rotations with `qml.adjoint`:

```
def convert_to_other_basis():
    gate1()
    gate2()

@qml.qnode(dev)
def my_circuit():
    gates()
    qml.adjoint(convert_to_other_basis)()
    return qml.sample()
```

`qml.adjoint` is a special type of function called a **transform**.

Last time

We began working with more than one qubit.

Hilbert spaces combine under the *tensor product*. If

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$$

then

$$|\psi\rangle \otimes |\varphi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \beta \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

$$|10\rangle = \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} 0 \\ 1 \\ 2 \leftarrow \\ 3 \end{array}$$

$$= \alpha\gamma \underset{0}{|00\rangle} + \alpha\delta \underset{1}{|01\rangle} + \beta\gamma \underset{2}{|10\rangle} + \beta\delta \underset{3}{|11\rangle}$$

$$\sum |\cdot|^2 = 1$$

Last time

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$|\gamma|^2 + |\delta|^2$

If we measure in the computational basis, the outcome probabilities are:

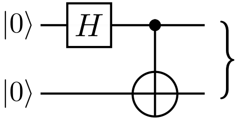
- $|\alpha|^2 = |\langle 00|\psi\rangle|^2$ for $|00\rangle$
- $|\beta|^2 + |\delta|^2$ to observe the second qubit in state $|1\rangle$
- ...

Last time

We applied single-qubit operations to individual qubits:

$$(U_1 \otimes U_2)(|\psi\rangle \otimes |\varphi\rangle) = U_1|\psi\rangle \otimes U_2|\varphi\rangle$$

We ended with a very interesting two-qubit operation.



CNOT

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ → Bell state

$= \cancel{|\psi\rangle \otimes |\varphi\rangle}$

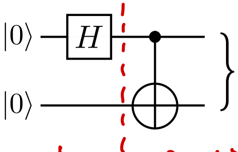
entangled state.

Learning outcomes

- Describe the action of common multi-qubit gates
- Define and give examples of entangled states
- Make any gate a controlled gate
- Measure a two-qubit state in the Bell basis
- Outline and implement the superdense coding algorithm

Entanglement

entangled



$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$ not entangled

We *cannot express*

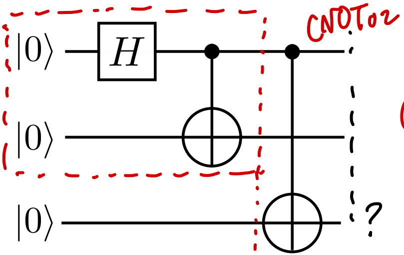
$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

as a tensor product of two single-qubit states. Moreover, the measurement outcomes are correlated!

This state is **entangled**, and CNOT is an **entangling gate**!

$$CNOT_{02}\left(\frac{1}{\sqrt{2}}(|000\rangle+|110\rangle)\right)=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$$

Entanglement generalizes to more than two qubits:



CNOT$_{ij}$

$$\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\otimes|0\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|110\rangle)$$

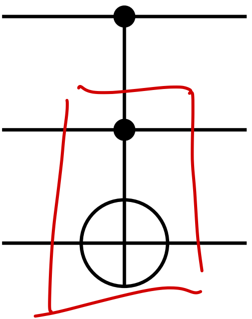
$$\Rightarrow \frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$$

Exercise: Express the output state of this circuit in the computational basis.

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|111\rangle+|000\rangle)$$

$\underbrace{\text{GHZ state}}$

There are also gates on more than two qubits, like the Toffoli gate, which is a controlled-CNOT, or controlled-controlled-NOT.



no change

$$TOF|000\rangle = |000\rangle$$
$$TOF|001\rangle = |001\rangle$$
$$TOF|010\rangle = |010\rangle$$
$$TOF|011\rangle = |011\rangle$$

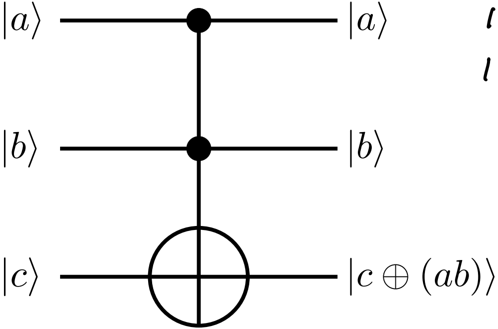
$CNOT_{12}$

$$TOF|100\rangle = |100\rangle$$
$$TOF|101\rangle = |101\rangle$$
$$TOF|110\rangle = |111\rangle$$
$$TOF|111\rangle = |110\rangle$$

PennyLane: `qml.Toffoli`

The Toffoli implements a reversible AND gate.

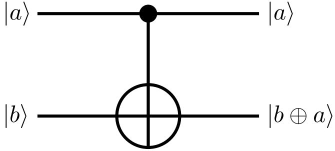
a	b	AND(a,b)
0	0	0
0	1	0
1	0	0
1	1	1



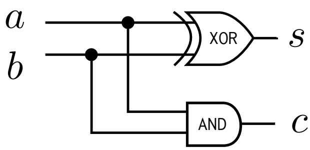
It is also universal for classical reversible computing.

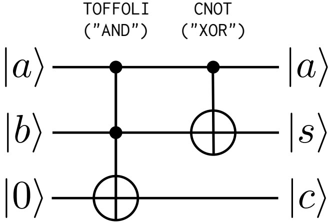
CNOT, Toffoli, and classical reversible circuits

CNOT also implements a reversible Boolean function.



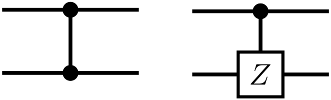
X, CNOT, TOF can be used to create Boolean arithmetic circuits.





Fun for you: assignment problem 3 & 6.

What does this operation do?



$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

X

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |10\rangle$

$|11\rangle \rightarrow -|11\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Z

PennyLane: `qml.CZ`

Fun: write CZ using CNOT and H

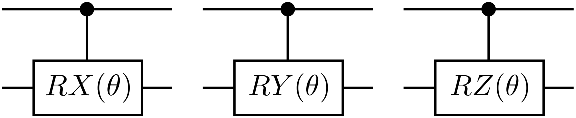
Image credit: Codebook node I.13

Example: controlled rotations (RX, RY, RZ)

Or this one?

$$CRY(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

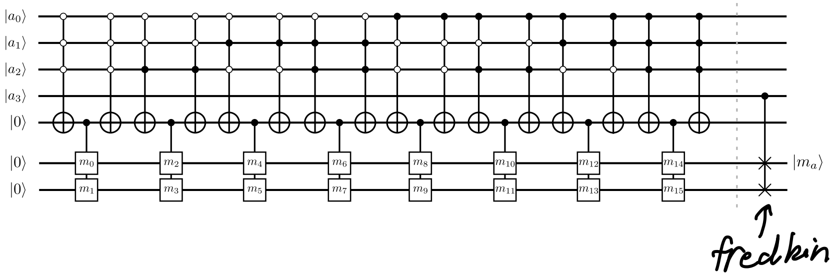
Circuit elements:



PennyLane: `qml.CRX`, `qml.CRY`, `qml.CRZ`

Image credit: Codebook node I.14

Any unitary operation can be turned into a controlled operation, controlled on any state.



Most common controls are controlled-on-$|1\rangle$ (filled circle), and controlled-on-$|0\rangle$ (empty circle).

Hands-on: qml.ctrl

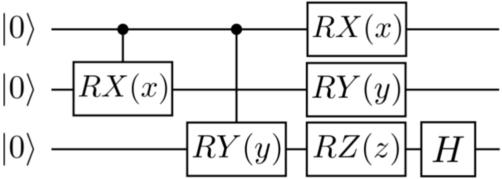
Remember qml.adjoint:

```
@qml.qnode(dev)
def my_circuit():
    qml.S(wires=0)
    qml.adjoint(qml.S)(wires=0)
    return qml.sample()
```

There is a similar *transform* that allows us to perform arbitrary controlled operations (or entire quantum functions)!

```
@qml.qnode(dev)
def my_circuit():
    qml.S(wires=0)
    qml.ctrl(qml.S, control=1)(wires=0)
    return qml.sample()
```

Let's go implement this circuit:

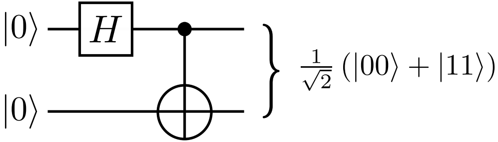


Bell states

Remember how we created

$$|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right),$$

from the $|00\rangle$ state:



Exercise: Apply the same circuit to the other 3 computational basis states? What is special about these four states?

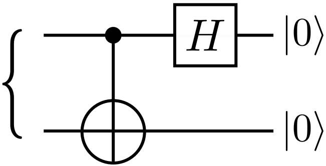
$$
\left.
\begin{array}{lll}
|00\rangle & \longrightarrow & |\psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
|01\rangle & \longrightarrow & |\psi_{01}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\
|10\rangle & \longrightarrow & |\psi_{10}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\
|11\rangle & \longrightarrow & |\psi_{11}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)
\end{array}
\right\}
\text{ Bell basis.}
$$

$$\langle \psi_{ab} | \psi_{cd} \rangle = 0 \text{ if } ab \neq cd$$

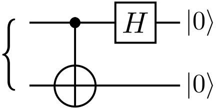
The Bell basis

We can measure in this basis by applying the adjoint of the circuit:

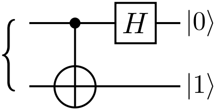
$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$



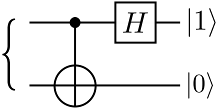
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



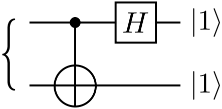
$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$



$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$



$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$



Two famous quantum algorithms, **superdense coding** and **teleportation** work by performing a measurement in the Bell basis.

Next time

Content:

- Quantum teleportation
- Measurement part 2: expectation values

Action items:

1. Assignment 1
2. Preparing for midterm

Recommended reading:

- Codebook nodes IQC, SQ, MQ; Nielsen & Chuang 1.2, 1.3.1-1.3.7, 2.2.3-2.2.5, 2.3, 4.3