CPEN 400Q Lecture 03
Measurement I (projective measurements)

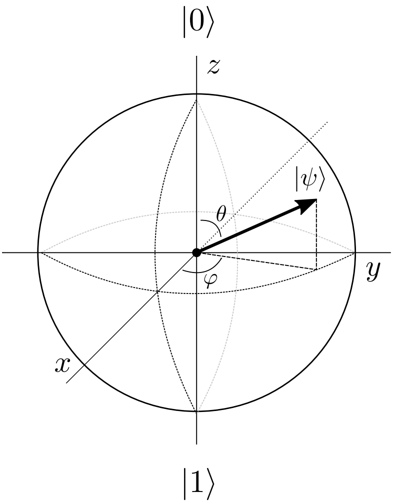
Monday 13 January 2024

Announcements

- Quiz 1 today
- Tomorrow's tutorial: first hands-on assignment (for submission). **New room: CEME 1215**
- Assignment 1 due Sunday 26 Jan at 23:59

Last time

We saw the most general single-qubit state parametrization, and how it can be represented in 3D space on the Bloch sphere write out params



We learned about the three Pauli rotations

	Math	Matrix	Code	Special cases (θ)
RZ	$e^{-i\frac{\theta}{2}Z}$	$\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$	`qml.RZ`	$Z(\pi), S(\pi/2), T(\pi/4)$
RY	$e^{-i\frac{\theta}{2}Y}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RY`	$Y(\pi)$
RX	$e^{-i\frac{\theta}{2}X}$	$\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	`qml.RX`	$X(\pi), SX(\pi/2)$

Last time

We interpreted unitary operations as rotations of the state vector on the Bloch sphere.

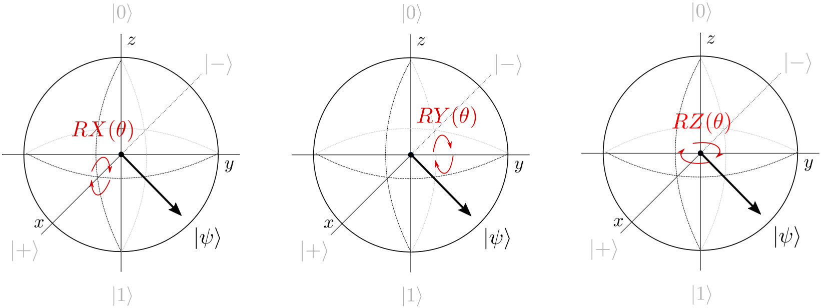


Image credit: Codebook node I.6

Last time

We left off with a few exercises.

Exercise 1: design a quantum circuit that prepares

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5}{4}}|1\rangle$$

Exercise 2: In PennyLane, implement the circuit below

Run your circuit with two different values of θ and take 1000 shots. How does θ affect the measurement outcome probabilities?

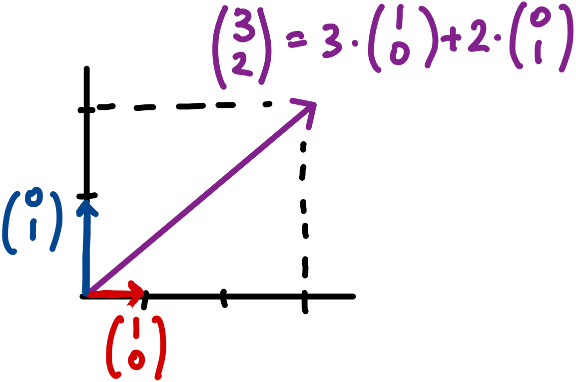
Learning outcomes

- compute the inner product between two quantum states
- perform a projective measurement
- distinguish between global phase and relative phase
- measure a qubit in different bases

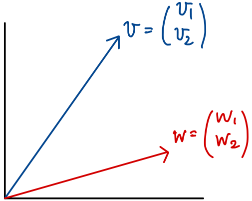
Inner products

We can now create any single-qubit quantum state: how do we *compare* them?

For intuition, consider a classical vector space.



We can define an **inner product** between two vectors to quantify much overlap they have.



Inner products

Take just one of these representations:

The inner product in Hilbert space is defined as

To avoid cumbersome notation define the **bra** (*braket* notation):

Exercise: compute the inner product of the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with itself.

Inner products

Exercise: compute the inner product between all possible combinations of $|0\rangle$ and $|1\rangle$.

$\langle 0\|0\rangle$	
$\langle 0\|1\rangle$	
$\langle 1\|0\rangle$	
$\langle 1\|1\rangle$	

For a single qubit, a pair of states that are **normalized** and **orthogonal** constitute an **orthonormal basis** for the Hilbert space.

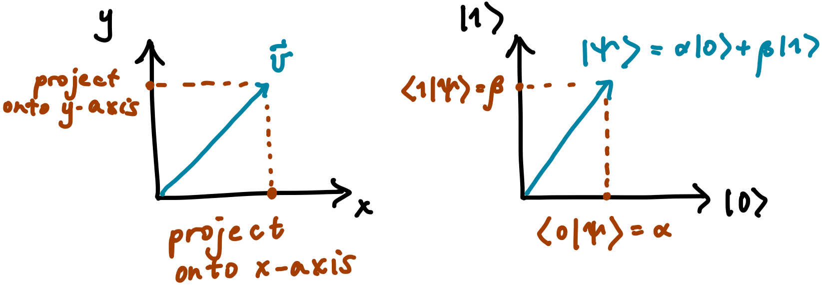
Exercise: do the states

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

form an orthonormal basis?

Projective measurements

Measurement is performed with respect to a basis; we perform **projections** to determine the overlap with a given basis state.



(Image for expository purposes only!)

Image credit: Xanadu Quantum Codebook I.9

Projective measurements

When we measure state $|\varphi\rangle$ with respect to basis $\{|\psi_i\rangle\}$, the probability of obtaining outcome i is

If we observe outcome i, the system will be left in state $|\psi_i\rangle$ after the measurement.

Measurement in the computational basis

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Then if we measure $|\psi\rangle$ is the computational basis,

Exercise: what are the measurement outcome probabilities if we measure

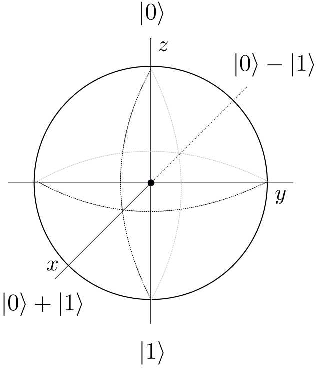
$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

in the computational basis?

Measurement in the computational basis

These are *not* the same state: there is a relative phase.

$$|p\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \quad |m\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$



How to tell them apart?

Basis rotations

Projective measurements can be performed with respect to any orthonormal basis. For example, $\{|+\rangle, |-\rangle\}$:

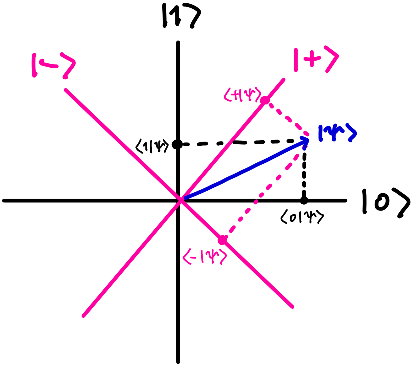


Image credit: Codebook node I.9

Basis rotations

So far we've seen 3 ways of extracting information out of a QNode:

1. `qml.state()`
2. `qml.probs(wires=x)`
3. `qml.sample()`

But these return results of measurements in the computational basis; and most hardware only allows for this.

How can we measure with respect to *different bases*?

Use a basis rotation to "trick" the quantum computer.

Suppose we want to measure in the "Y" basis:

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right).$$

Recall: unitary operations preserve length *and* angles between normalized quantum state vectors (prove on A1!)

There exists a unitary operation that will convert between this basis and the computational basis.

Exercise: determine a quantum circuit that sends

$$
\begin{aligned}
|0\rangle \quad &\rightarrow \quad |p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \\
|1\rangle \quad &\rightarrow \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)
\end{aligned}
$$

Basis rotations

At the end of our circuit, apply the reverse (adjoint) of this transformation to rotate *back* to the computational basis.

If we measure and observe $|0\rangle$, we know the qubit was previously $|p\rangle$ in the Y basis (analogous result for $|m\rangle$).

Adjoints

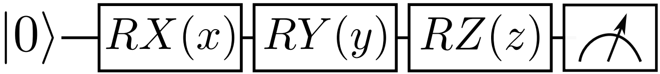
In PennyLane, we can compute adjoints of operations *and* entire quantum functions using `qml.adjoint`:

```
def some_function(x):
    qml.RZ(Z, wires=0)

def apply_adjoint(x):
    qml.adjoint(qml.S)(wires=0)
    qml.adjoint(some_function)(x)
```

`qml.adjoint` is a special type of function called a **transform**. We will cover transforms in more detail later in the course.

Let's run the following circuit, and measure in the Y basis



Hands-on time...

Next time

Content:

- Mathematical representation of multi-qubit systems
- Multi-qubit gates
- Entanglement

Action items:

1. Work on Assignment 1 (can do questions 1 & 5, 2ai,ii now)

Recommended reading:

- For today: Codebook modules IQC, SQ; Nielsen & Chuang 1.3.3, 2.2.3-2.2.5
- For next class: Codebook module MQ; Nielsen & Chuang 4.3