

# **CPEN 400Q Lecture 13**

## **QFT and quantum phase estimation**

Wednesday 26 February 2025

# Announcements

- Quiz 6 on Monday
- Assignment 2 due tomorrow 23:59
- Assignment 3 coming next week - last “big” technical assignment
- Fill out peer review survey by Friday (link in Piazza)

## Last time

We introduced the quantum Fourier transform, and saw how it is the analog of the classical inverse discrete Fourier transform.

$$QFT|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle$$

$$QFT = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix}$$

where for  $n$  qubits,  $N = 2^n$ , and  $\omega = e^{2\pi i/N}$

## Last time

We saw the circuits for some special cases.

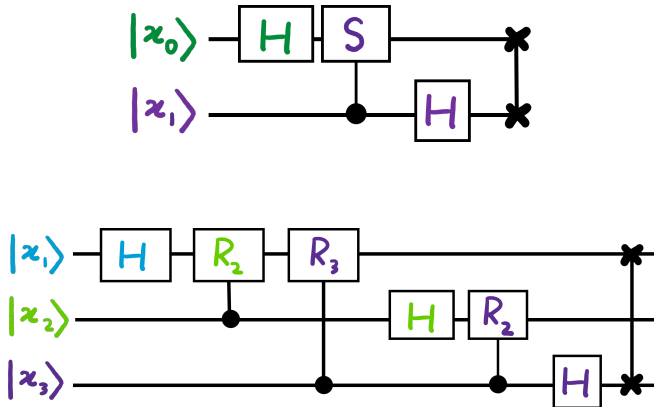


Image credit: Xanadu Quantum Codebook node F.2, F.3

## Last time

The general form of the circuit is:

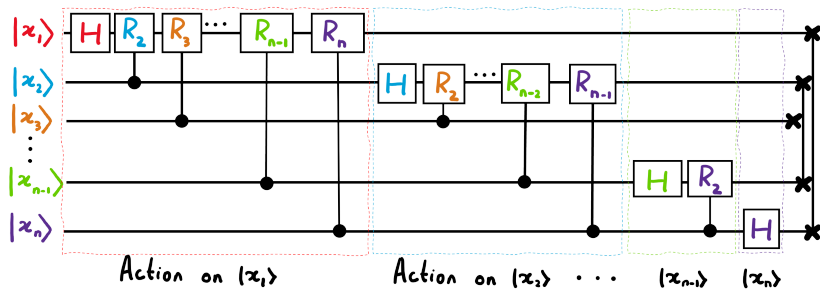


Image credit: Xanadu Quantum Codebook node F.3

- Redevelop the circuit for the quantum Fourier transform
- Outline the steps of the quantum phase estimation (QPE) subroutine
- Use the QFT to implement QPE

# A circuit for the QFT

Start with the one-qubit case:

$$\text{QFT} = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix}$$

$$0.01 \Rightarrow 0.25 = \frac{0}{2} + \frac{1}{4}$$

$$\omega = e^{\frac{2\pi i}{N}} = e^{\frac{2\pi i}{2}} = e^{\pi i}$$

$$\text{QFT} |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$\text{QFT} |1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + \omega |1\rangle)$$

$$\text{QFT} |x_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_1} |1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + \omega^{x_1} |1\rangle)$$

$x_1 \in \{0,1\}$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i \frac{x_1}{2}} |1\rangle)$$

$$\frac{x_1}{2} = 2^{-1} \cdot x_1 = 0 \cdot x_1$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i \cdot 0 \cdot x_1} |1\rangle)$$

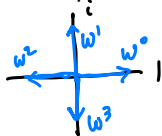
# A circuit for the QFT

Next, consider the two-qubit case:

$$QFT = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Re-express higher powers of  $\omega = e^{\frac{2\pi i}{4}}$ :

Powers of  $\omega$ : 0, 1, 2, 3



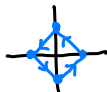
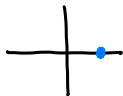
$$QFT = \frac{1}{2} \begin{pmatrix} 1 & \boxed{1} & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

00      01      10      11

column 1

column 2

c. 3





## A circuit for the QFT

$$QFT = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix}$$

How does this affect the computational basis states?

$$QFT |00\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$QFT |01\rangle = \frac{1}{2} (|00\rangle + \omega |01\rangle + \omega^2 |10\rangle + \omega^3 |11\rangle)$$

$$QFT |10\rangle = \frac{1}{2} (|00\rangle + \omega^2 |01\rangle + |10\rangle + \omega^2 |11\rangle)$$

$$QFT |11\rangle = \frac{1}{2} (|00\rangle + \omega^3 |01\rangle + \omega^2 |10\rangle + \omega |11\rangle)$$

$$|x_1 x_2\rangle$$

# A circuit for the QFT

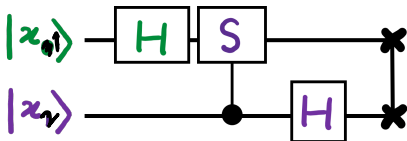
$$\left\{ \begin{array}{l} \text{QFT } |00\rangle = \frac{1}{2} \left[ (|0\rangle + |1\rangle) |0\rangle + (|0\rangle + |1\rangle) |1\rangle \right] \\ \text{QFT } |01\rangle = \frac{1}{2} \left[ (|0\rangle + \omega^2 |1\rangle) |0\rangle + (\omega |0\rangle + \omega^3 |1\rangle) |1\rangle \right] \\ \text{QFT } |10\rangle = \frac{1}{2} \left[ (|0\rangle + |1\rangle) |0\rangle + (\omega^2 |0\rangle + \omega^2 |1\rangle) |1\rangle \right] \\ \text{QFT } |11\rangle = \frac{1}{2} \left[ (|0\rangle + \omega^2 |1\rangle) |0\rangle + (\omega^3 |0\rangle + \omega |1\rangle) |1\rangle \right] \end{array} \right\}$$

$$\begin{aligned} \text{QFT } |00\rangle &= \frac{1}{2} \left[ (|0\rangle + |1\rangle) |0\rangle + (|0\rangle + |1\rangle) |1\rangle \right] = \frac{1}{2} (|0\rangle + |1\rangle) (|0\rangle + |1\rangle) \\ \text{QFT } |01\rangle &= \frac{1}{2} \left[ (|0\rangle + \omega^2 |1\rangle) |0\rangle + (\omega |0\rangle + \omega^3 |1\rangle) |1\rangle \right] = \frac{1}{2} (|0\rangle + \omega^2 |1\rangle) (|0\rangle + \omega |1\rangle) \\ \text{QFT } |10\rangle &= \frac{1}{2} \left[ (|0\rangle + |1\rangle) |0\rangle + (|0\rangle + |1\rangle) \omega^2 |1\rangle \right] = \frac{1}{2} (|0\rangle + |1\rangle) (|0\rangle + \omega^2 |1\rangle) \\ \text{QFT } |11\rangle &= \frac{1}{2} \left[ (|0\rangle + \omega^2 |1\rangle) |0\rangle + (|0\rangle + \omega^2 |1\rangle) \omega^3 |1\rangle \right] = \frac{1}{2} (\underbrace{|0\rangle + \omega^2 |1\rangle}_{x_2}) (\underbrace{|0\rangle + \omega^3 |1\rangle}_{2x_1 + x_2}) \end{aligned}$$

$$\text{QFT } |x_1 x_2\rangle = \frac{1}{2} (|0\rangle + \omega^{2x_2} |1\rangle) (|0\rangle + \omega^{2x_1 + x_2} |1\rangle)$$

## A circuit for the QFT

$$\begin{aligned}
 \text{QFT} |x_1 x_2\rangle &= \frac{1}{2} \left( |0\rangle + \omega^{2x_2} |1\rangle \right) \left( |0\rangle + \omega^{2x_1 + x_2} |1\rangle \right) \\
 &\quad \begin{array}{l} \downarrow \\ e^{2\pi i \cdot \frac{2x_2}{4}} = e^{2\pi i \cdot 0 \cdot x_2} \end{array} \quad \begin{array}{l} \downarrow \\ e^{2\pi i \cdot \frac{2x_1 + x_2}{4}} = e^{2\pi i \left( \frac{x_1}{2} + \frac{x_2}{4} \right)} \\ \phantom{e^{2\pi i \cdot \frac{2x_1 + x_2}{4}}} = e^{2\pi i \cdot 0 \cdot x_1 x_2} \end{array} \\
 &= \underbrace{\frac{1}{\sqrt{2}} \left( |0\rangle + e^{2\pi i \cdot 0 \cdot x_2} |1\rangle \right)}_{H|x_2\rangle} \otimes \underbrace{\frac{1}{\sqrt{2}} \left( |0\rangle + e^{2\pi i \cdot 0 \cdot x_1 x_2} |1\rangle \right)}_{\left( \text{Phase Controlled on } x_2 \right) \cdot H|x_1\rangle}
 \end{aligned}$$



# A circuit for the QFT

Three-qubit case:

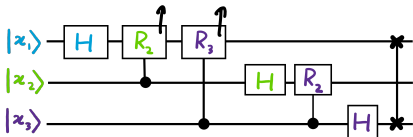
$$QFT = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \dots \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \dots \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \dots \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \dots \end{pmatrix}$$

$$QFT|x_1x_2x_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.x_3}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.x_2x_3}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.x_1x_2x_3}|1\rangle)$$

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

$$R_2 = S$$

$$R_3 = T$$



# A circuit for the QFT

General expression:

$$\begin{aligned} |x\rangle &\rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle \\ &= \frac{(|0\rangle + e^{2\pi i 0.x_n} |1\rangle) (|0\rangle + e^{2\pi i 0.x_{n-1}x_n} |1\rangle) \cdots (|0\rangle + e^{2\pi i 0.x_1 \cdots x_n} |1\rangle)}{\sqrt{N}} \end{aligned}$$

Full mathematical derivation at end of slides.

# A circuit for the QFT

$$|x\rangle \rightarrow \frac{(|0\rangle + e^{2\pi i 0 \cdot x_n} |1\rangle) (|0\rangle + e^{2\pi i 0 \cdot x_{n-1} x_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0 \cdot x_1 \dots x_n} |1\rangle)}{\sqrt{N}}$$

Starting with the state

$$|x\rangle = |x_1 \dots x_n\rangle,$$

apply a Hadamard to qubit 1:

$$\frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 0 \cdot x_1} |1\rangle) |x_2 \dots x_n\rangle$$

$$|x_1\rangle \text{ --- } \boxed{H} \text{ ---}$$

$$|x_2\rangle \text{ ---}$$

$$|x_3\rangle \text{ ---}$$

$\vdots$

$$|x_{n-1}\rangle \text{ ---}$$

$$|x_n\rangle \text{ ---}$$

# A circuit for the QFT

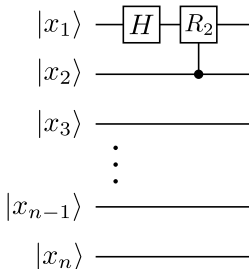
$$|x\rangle \rightarrow \frac{(|0\rangle + e^{2\pi i 0 \cdot x_n} |1\rangle) (|0\rangle + e^{2\pi i 0 \cdot x_{n-1} x_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0 \cdot x_1 \dots x_n} |1\rangle)}{\sqrt{N}}$$

Apply controlled  $R_2$  from qubit

$2 \rightarrow 1$

$$\frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 0 \cdot x_1} e^{2\pi i / 2^2 \cdot x_2} |1\rangle) |x_2 \dots x_n\rangle$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 0 \cdot x_1 x_2} |1\rangle) |x_2 \dots x_n\rangle$$



# A circuit for the QFT

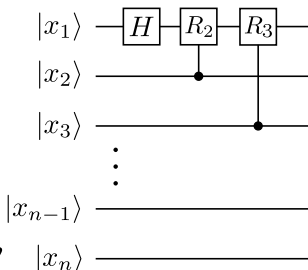
$$|x\rangle \rightarrow \frac{(|0\rangle + e^{2\pi i 0 \cdot x_n} |1\rangle) (|0\rangle + e^{2\pi i 0 \cdot x_{n-1} x_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0 \cdot x_1 \dots x_n} |1\rangle)}{\sqrt{N}}$$

Apply controlled  $R_3$  from qubit

$3 \rightarrow 1$

$$\frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 0 \cdot x_1 x_2} e^{\frac{2\pi i}{8} x_3} |1\rangle) |x_2 \dots x_n\rangle$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 0 \cdot x_1 x_2 x_3} |1\rangle) |x_2 \dots x_n\rangle$$



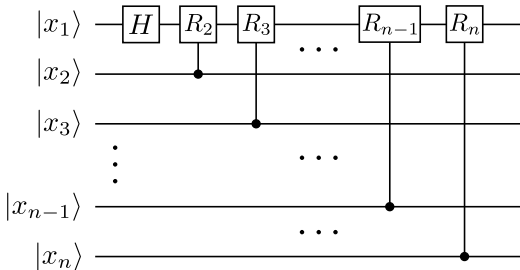


# A circuit for the QFT

$$|x\rangle \rightarrow \frac{(|0\rangle + e^{2\pi i 0 \cdot x_n} |1\rangle) (|0\rangle + e^{2\pi i 0 \cdot x_{n-1} x_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0 \cdot x_1 \dots x_n} |1\rangle)}{\sqrt{N}}$$

Apply a controlled  $R_4$  from  $4 \rightarrow 1$ , etc. up to the  $n$ -th qubit to get

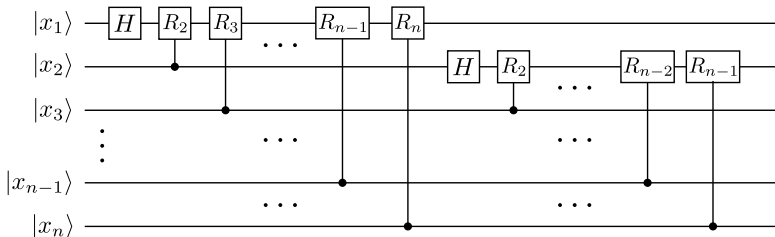
$$\frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 0 \cdot x_1 \dots x_n} |1\rangle) |x_1 \dots x_n\rangle$$



# A circuit for the QFT

Similar thing with second qubit: apply  $H$  then controlled rotations from every qubit from 3 to  $n$  to get

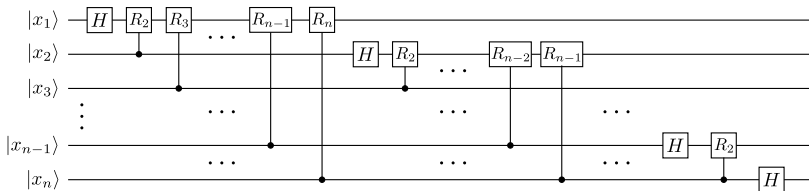
$$\frac{1}{\sqrt{2}} \left( |0\rangle + e^{2\pi i 0 \cdot x_1 \dots x_n} |1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left( |0\rangle + e^{2\pi i 0 \cdot x_2 \dots x_n} |1\rangle \right) |x_3 \dots x_n\rangle$$



# A circuit for the QFT

Do this for all qubits to get that big ugly state from earlier:

$$|x\rangle \rightarrow \frac{(|0\rangle + e^{2\pi i 0.x_n} |1\rangle) (|0\rangle + e^{2\pi i 0.x_{n-1}x_n} |1\rangle) \dots (|0\rangle + e^{2\pi i 0.x_1 \dots x_n} |1\rangle)}{\sqrt{N}}$$



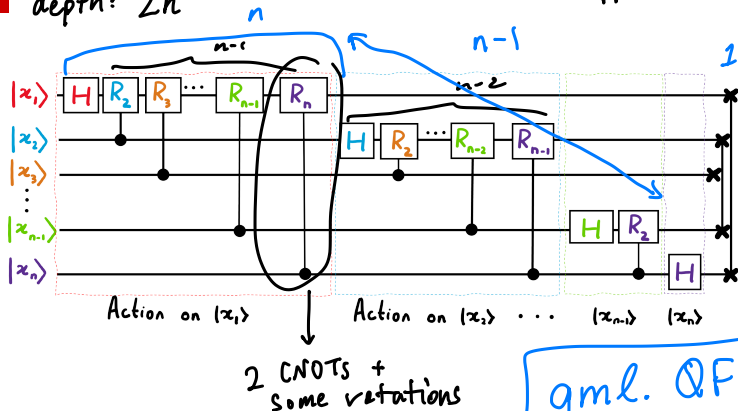
Qubit order is backwards - easily fixed with SWAP gates.

# Quantum Fourier transform

**Exercise:** What are the gate counts and depth of this circuit?

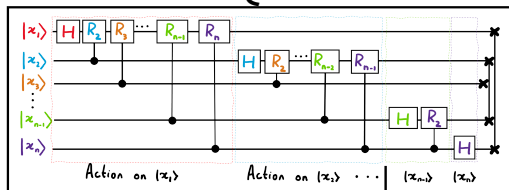
- $H$ :  $n$
- rotations:  $\frac{(n-1)n}{2}$
- SWAPs:  $\lfloor n/2 \rfloor$
- depth:  $2n$

$\Rightarrow$  Polynomial in  $n$  <sup>quantum resources</sup>  
= efficient!

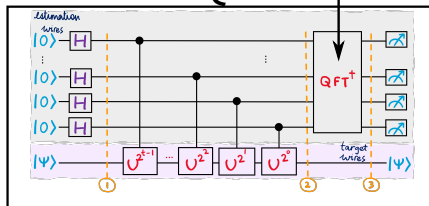


# Reminder: where are we going?

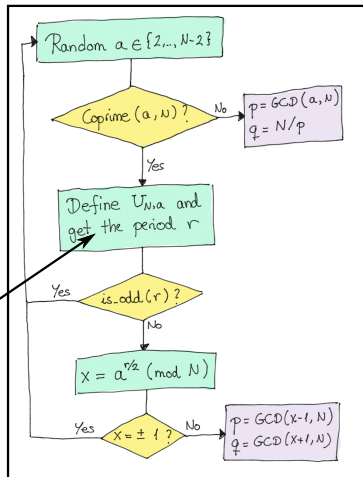
## 1. QFT



## 2. QPE



## 3. Shor



# Eigenvalues of unitary matrices

Eigenvalues of unitary matrices are complex numbers with magnitude 1, i.e.,

$$U|k\rangle = \lambda_k |k\rangle \Rightarrow \lambda_k = e^{2\pi i \theta_k}$$

where  $\theta_k$  is a phase,  $|\theta_k| \leq 1$ . (See proof at end of slides)

What if we want to *estimate* an unknown  $\theta_k$ ?

# Eigenvalues of unitary matrices

Idea: apply  $U$  to the relevant eigenvector, because that's "what makes the phase come out".

$$U|k\rangle = e^{2\pi i \theta_k} |k\rangle$$

...but this is an unobservable *global* phase!

We have to do something different: eigenvalue estimation, or **quantum phase estimation** (QPE).

# Quantum phase estimation

Given unitary  $U$  and eigenvector  $|k\rangle$ , estimate  $\theta_k$  such that

$$U|k\rangle = e^{2\pi i\theta_k}|k\rangle$$

Must determine:

- How to design a circuit that extracts  $\theta_k$
- What precision can we estimate it to
- What to do if we don't know a  $|k\rangle$  in advance

*(You will explore the last two in your homework!)*



# Quantum phase estimation

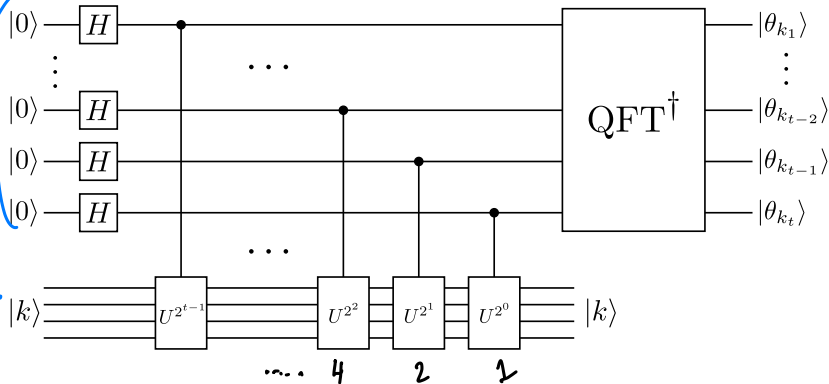
Let  $U$  be an  $n$ -qubit unitary;  $|k\rangle$  is an  $n$ -qubit eigenstate.

Assume  $\theta_k$  can be represented *exactly* using  $t$  bits:

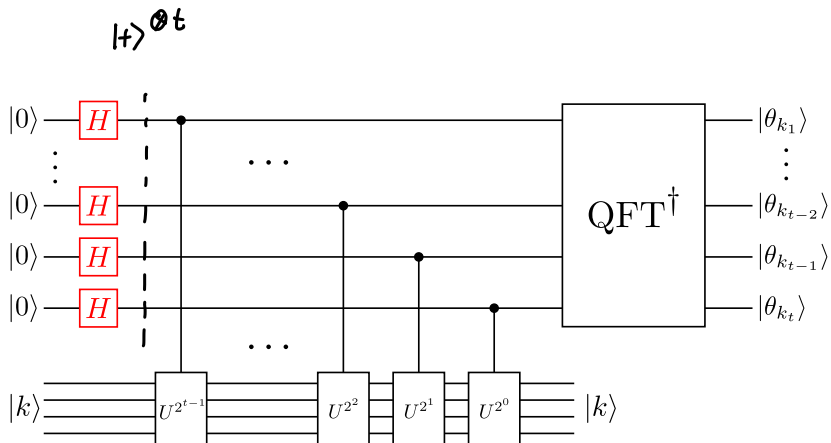
*q.m.l. QPE*

$$\theta_k = 0.\theta_{k_1} \cdots \theta_{k_t}$$

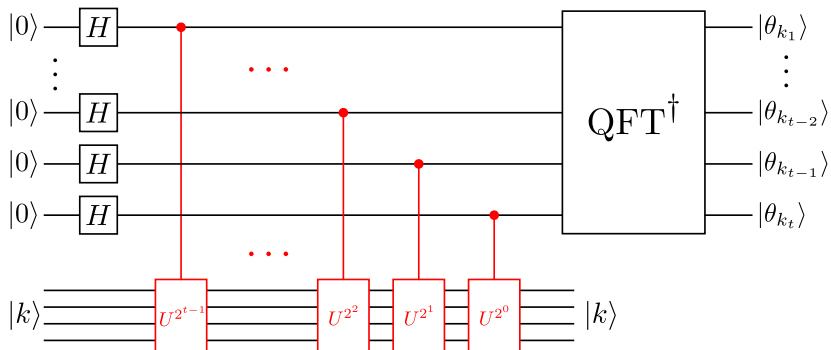
*estimation wires*  
*target wires*



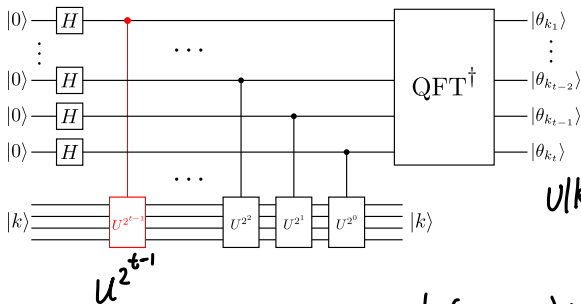
# Quantum phase estimation: step 1



## Quantum phase estimation: step 2



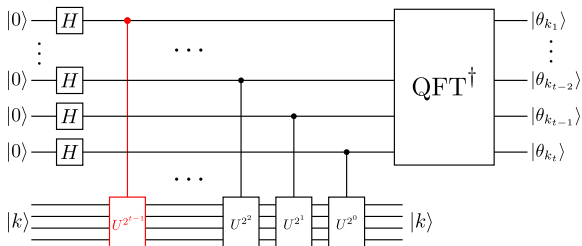
# Quantum phase estimation: step 2



$$U|k\rangle = e^{2\pi i \theta_k} |k\rangle$$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \text{other qubits} \otimes |k\rangle \Rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |k\rangle$   
 Apply controlled  $U^{2^{t-1}}$ :  $\frac{1}{\sqrt{2}}|0\rangle|k\rangle + \frac{1}{\sqrt{2}}|1\rangle|k\rangle \xrightarrow{CU^{2^{t-1}}} \frac{1}{\sqrt{2}}|0\rangle|k\rangle + \frac{1}{\sqrt{2}}|1\rangle(U^{2^{t-1}}|k\rangle)$   
 $= \frac{1}{\sqrt{2}}|0\rangle|k\rangle + \frac{1}{\sqrt{2}}|1\rangle(e^{2\pi i \theta_k})^{2^{t-1}}|k\rangle$   
 $= \frac{1}{\sqrt{2}}|0\rangle|k\rangle + \frac{1}{\sqrt{2}}|1\rangle(e^{2\pi i \theta_k \cdot 2^{t-1}})|k\rangle$   
 $= \frac{1}{\sqrt{2}}|0\rangle|k\rangle + \frac{1}{\sqrt{2}}e^{2\pi i \theta_k \cdot 2^{t-1}}|1\rangle|k\rangle$

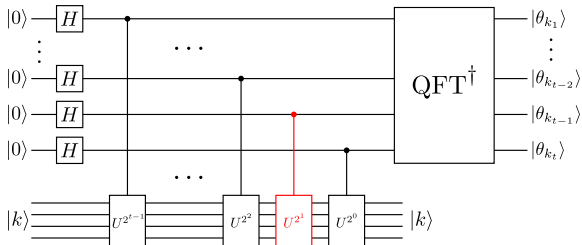
## Quantum phase estimation: step 2



Use phase kickback

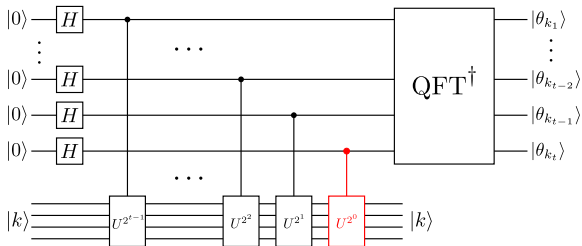
$$= \frac{1}{\sqrt{2}} \left( |0\rangle + e^{\underbrace{2\pi i \theta_k \cdot 2^{t-1}}_{\theta_k = 0. \theta_{k_1} \theta_{k_2} \dots}} |1\rangle \right) |k\rangle$$

## Quantum phase estimation: step 2



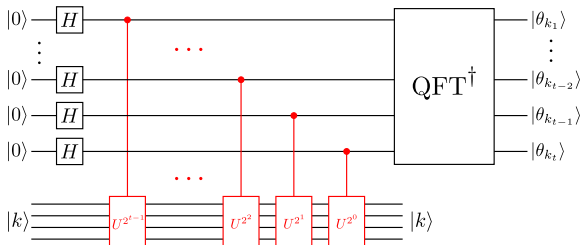
Check second-last qubit (ignore the others)

## Quantum phase estimation: step 2



Can show in the same way for the last qubit (ignore others)

## Quantum phase estimation: step 2



After step 2, we have the state

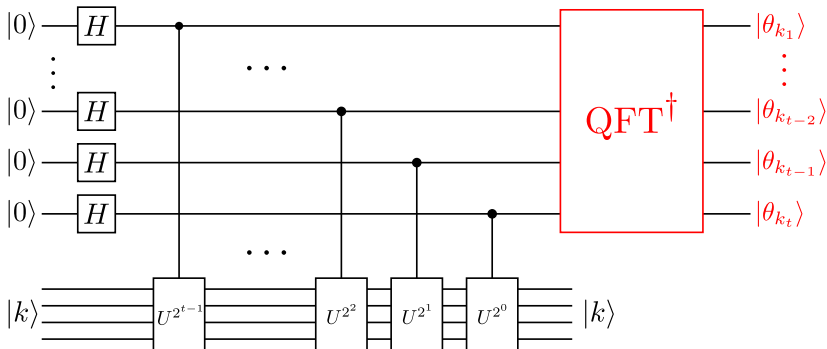
$$\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0 \cdot \theta_{k_t}} |1\rangle) \cdots \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0 \cdot \theta_{k_2} \cdots \theta_{k_t}} |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0 \cdot \theta_{k_1} \cdots \theta_{k_t}} |1\rangle) |k\rangle$$

Should look familiar!



## Quantum phase estimation: step 3

Measure to learn the bits of  $\theta_k$ .



Let's implement it.

## Next time

### Content:

- Quiz 6 on Monday
- Order finding

### Action items:

1. Work on project (do peer assessment)
2. Finish assignment 2

### Recommended reading:

- From today: Codebook QFT, QPE, Nielsen & Chuang 5.1-2
- For next class: Codebook SH, Nielsen & Chuang 5.3.1

## Derivation of output state of QFT

We will reexpress  $k/N$  in fractional binary notation, then reshuffle and *factor* the output state to uncover the circuit structure.

$$\begin{aligned} |x\rangle &\rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x (k/N)} |k\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{k_1=0}^1 \cdots \sum_{k_n=0}^1 e^{2\pi i x (\sum_{\ell=1}^n k_\ell 2^{-\ell})} |k_1 \cdots k_n\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{k_1=0}^1 \cdots \sum_{k_n=0}^1 \bigotimes_{\ell=1}^n e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \end{aligned}$$

# Derivation of output state of QFT

$$\begin{aligned}
 &= \frac{1}{\sqrt{N}} \sum_{k_1=0}^1 \cdots \sum_{k_n=0}^1 \bigotimes_{\ell=1}^n e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \\
 &= \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^n \left( \sum_{k_\ell=0}^1 e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \right) \\
 &= \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^n \left( |0\rangle + e^{2\pi i x 2^{-\ell}} |1\rangle \right) \\
 &= \frac{(|0\rangle + e^{2\pi i 0 \cdot x_n} |1\rangle) (|0\rangle + e^{2\pi i 0 \cdot x_{n-1} x_n} |1\rangle) \cdots (|0\rangle + e^{2\pi i 0 \cdot x_1 \cdots x_n} |1\rangle)}{\sqrt{N}}
 \end{aligned}$$

# Eigenvalues of unitary matrices

Fun fact: eigenvalues of unitaries are complex numbers with magnitude 1.

Proof: Let  $\lambda_k$  be the eigenvalue associated with eigenvector  $|k\rangle$  of a unitary  $U$ :

$$U|k\rangle = \lambda_k|k\rangle$$

Take the conjugate transpose:

$$\langle k| U^\dagger = \langle k| \lambda_k^*$$

Multiply the two sides together:

$$\langle k| U^\dagger U|k\rangle = \langle k| \lambda_k^* \lambda_k |k\rangle$$

$$\langle k|k\rangle = |\lambda_k|^2 \langle k|k\rangle$$

$$1 = |\lambda_k|^2$$