# CPEN 400Q Lecture 05
# Entanglement and multi-qubit gates; superdense coding

Monday 20 January 2025

# Announcements

- Quiz 2 today
- Assignment 1 due 26 Jan at 23:59
- Tomorrow's tutorial: TA office hour / suggest topics on Piazza
- Midterm in class on Wed 29 Jan - details on PrairieLearn.

# Last time

We performed measurements in different bases by applying basis rotations with `qml.adjoint`:

```
def convert_to_other_basis():
    gate1()
    gate2()

@qml.qnode(dev)
def my_circuit():
    gates()
    qml.adjoint(convert_to_other_basis)()
    return qml.sample()
```

`qml.adjoint` is a special type of function called a **transform**.

# Last time

We began working with more than one qubit.

Hilbert spaces combine under the *tensor product*. If

then

# Last time

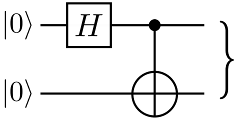
$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

If we measure in the computational basis, the outcome probabilities are:

- $|\alpha|^2 = |\langle 00|\psi\rangle|^2$ for $|00\rangle$
- $|\beta|^2 + |\delta|^2$ to observe the second qubit in state $|1\rangle$
- ...

We applied single-qubit operations to individual qubits:

We ended with a very interesting two-qubit operation.

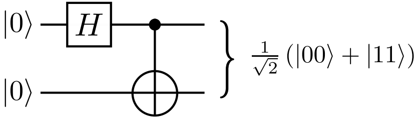



$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

# Learning outcomes

- Describe the action of common multi-qubit gates
- Define and give examples of entangled states
- Make any gate a controlled gate
- Measure a two-qubit state in the Bell basis
- Outline and implement the superdense coding algorithm

# Entanglement

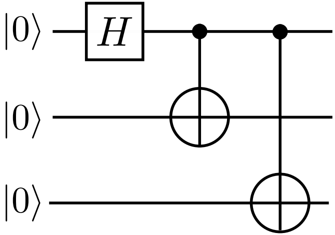



We *cannot express*

as a tensor product of two single-qubit states. Moreover, the measurement outcomes are correlated!

This state is **entangled**, and CNOT is an **entangling gate**!

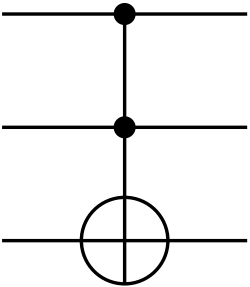
Entanglement generalizes to more than two qubits:




**Exercise**: Express the output state of this circuit in the computational basis.

## Toffoli

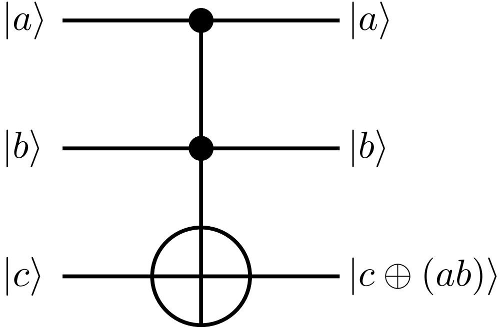
There are also gates on more than two qubits, like the Toffoli gate, which is a controlled-CNOT, or controlled-controlled-NOT.



$$TOF|000\rangle =$$
$$TOF|001\rangle =$$
$$TOF|010\rangle =$$
$$TOF|011\rangle =$$
$$TOF|100\rangle =$$
$$TOF|101\rangle =$$
$$TOF|110\rangle =$$
$$TOF|111\rangle =$$

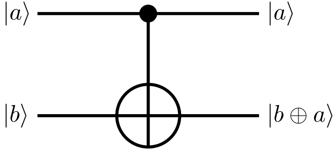
PennyLane: `qml.Toffoli`

The Toffoli implements a reversible AND gate.

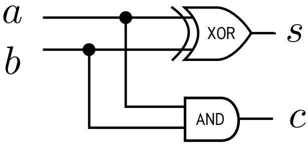


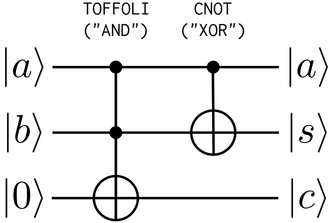

It is also universal for classical reversible computing.

CNOT also implements a reversible Boolean function.




X, CNOT, TOF can be used to create Boolean arithmetic circuits.

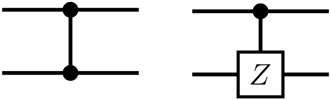



Fun for you: assignment problem 3 & 6.

# Example: controlled-$Z$ ($CZ$)

What does this operation do?




PennyLane: `qml.CZ`

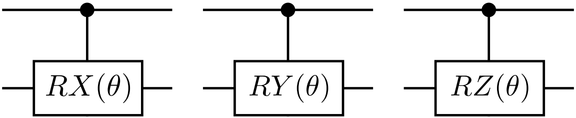
Image credit: Codebook node I.13

# Example: controlled rotations ($RX$, $RY$, $RZ$)

Or this one?

$$CRY(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

Circuit elements:

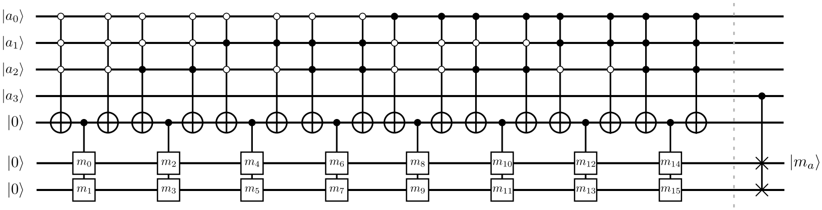



PennyLane: `qml.CRX`, `qml.CRY`, `qml.CRZ`

Image credit: Codebook node I.14

# Controlled unitary operations

Any unitary operation can be turned into a controlled operation, controlled on any state.




Most common controls are controlled-on-$|1\rangle$ (filled circle), and controlled-on-$|0\rangle$ (empty circle).

# Hands-on: `qml.ctrl`

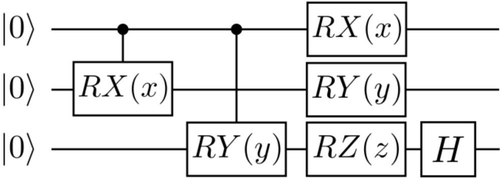
Remember `qml.adjoint`:

```
@qml.qnode(dev)
def my_circuit():
    qml.S(wires=0)
    qml.adjoint(qml.S)(wires=0)
    return qml.sample()
```

There is a similar *transform* that allows us to perform arbitrary controlled operations (or entire quantum functions)!

```
@qml.qnode(dev)
def my_circuit():
    qml.S(wires=0)
    qml.ctrl(qml.S, control=1)(wires=0)
    return qml.sample()
```

Let's go implement this circuit:

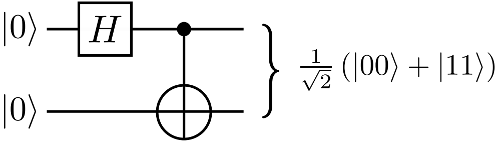


# Bell states

Remember how we created

$$|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right),$$

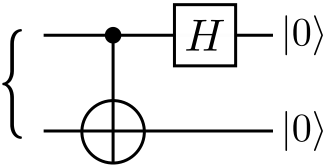
from the $|00\rangle$ state:



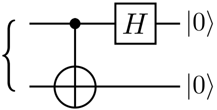
**Exercise**: Apply the same circuit to the other 3 computational basis states? What is special about these four states?

We can measure in this basis by applying the adjoint of the circuit:

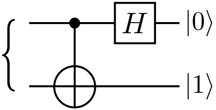
$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



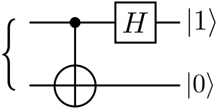
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



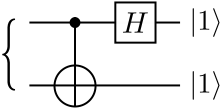

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$




$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$




$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$




Two famous quantum algorithms, **superdense coding** and **teleportation** work by performing a measurement in the Bell basis.

Suppose Alice wants to send Bob two classical bits of information, say ‘1’ and ‘0’.

Q1: How many classical bits must she send to Bob to do this?

Q2: How many *qubits* must she send to Bob to do this?

# Superdense coding

Alice and Bob start the protocol with this shared entangled state:

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Next, depending on her bits, Alice performs one of the following operations on her qubit:

$$\begin{aligned} 00 &\rightarrow I \\ 01 &\rightarrow X \\ 10 &\rightarrow Z \\ 11 &\rightarrow ZX \end{aligned}$$

What happened to the entangled state?

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

It will transform to:

$$\begin{aligned} 00 &\rightarrow_I \\ 01 &\rightarrow_X \\ 10 &\rightarrow_Z \\ 11 &\rightarrow_{ZX} \end{aligned}$$

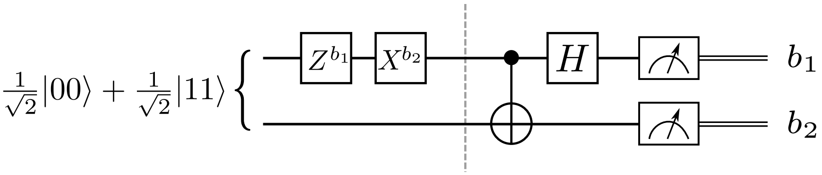
Now, Bob can either

- perform a measurement directly in the Bell basis
- perform a basis transformation from the Bell basis back to the computational basis

to determine his state with certainty, and thus, the bits Alice sent.

$$
\begin{aligned}
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) &= |00\rangle \\
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) &= |01\rangle \\
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) &= |10\rangle \\
(H \otimes I)\mathrm{CNOT} \cdot \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) &= |11\rangle
\end{aligned}
$$

Let's go implement it!



# Next time

Content:

- Quantum teleportation
- Measurement part 2: expectation values

Action items:

1. Assignment 1
2. Preparing for midterm

Recommended reading:

- Codebook nodes IQC, SQ, MQ; Nielsen & Chuang 1.2, 1.3.1-1.3.7, 2.2.3-2.2.5, 2.3, 4.3