CPEN 400Q Lecture 21
Stabilizer codes

Wednesday 26 March 2025

Announcements

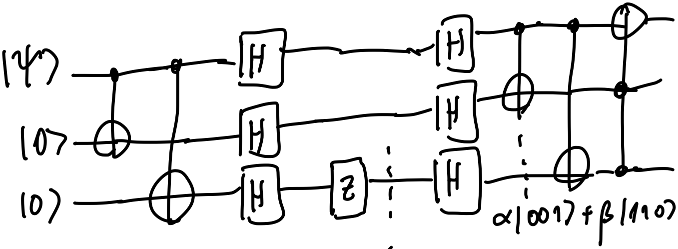
- Last content lecture on Monday; presentations for two classes + two tutorials after (attendance expected - come support your classmates!)
- Project rubric available on PrairieLearn
- Quiz 10 on *Tuesday* before presentation
- TA4 due Friday 23:59

Last time

We made a code to fix phase flip errors by making them look like bit flip errors in a different basis.

$\alpha|000\rangle + \beta|111\rangle$

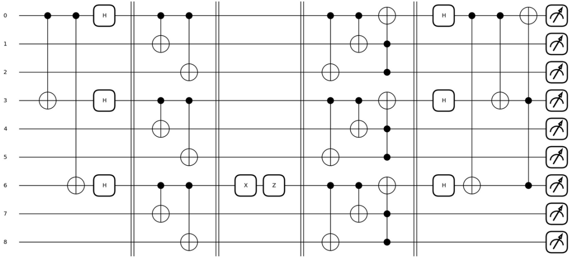
$$\mathcal{E}(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle_L + \beta|1\rangle_L = \alpha|+++\rangle + \beta|---\rangle$$



$\alpha|++-\rangle + \beta|--+\rangle$

Last time

We *concatenated* the bit and phase flip codes to create a 9-qubit code that corrected *any* single-qubit error.



This worked because if a code can correct a set of error operations $\{E_j\}$, it can also correct linear combinations of them.

Learning outcomes

Today:

1. Outline the conditions under which errors can be corrected
2. Define the stabilizers of a quantum error correcting code
3. Express the bit flip, phase flip, Shor code, and 5-qubit code in the stabilizer formalism

Conditions for quantum error correction

Formal definition of a quantum error correcting code is a *subspace*, C, called the **codespace**.

Example: bit flip code.

Define **projector** onto codespace,

$$P = |000\rangle\langle 000| + |111\rangle\langle 111|$$

$$P^2 = P$$

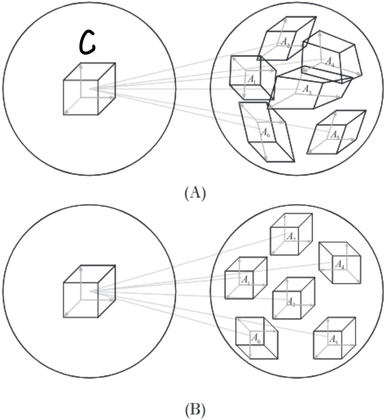


Image: Nielsen & Chuang, Fig. 10.5

Conditions for quantum error correction

$$PE_i^\dagger E_j P = \alpha_{ij} P$$

Theorem 10.1: (**Quantum error-correction conditions**) Let C be a quantum code, and let P be the projector onto C. Suppose $\mathcal{E}$ is a quantum operation with operation elements $\{E_i\}$. A necessary and sufficient condition for the existence of an error-correction operation $\mathcal{R}$ correcting $\mathcal{E}$ on C is that

$$\underbrace{P E_i^\dagger} \underbrace{E_j P} = \alpha_{ij} P \tag{10.16}$$

for some Hermitian matrix α of complex numbers.

If such an $\mathcal{R}$ exists, $\{E_i\}$ is called a *correctable set of errors*.

$IIX \quad XXI$

$$(|000\rangle\langle 000| + |111\rangle\langle 111|)\, \underbrace{IIX \cdot XXI}_{XXX}\, (|000\rangle\langle 000| + |111\rangle\langle 111|)$$

$$= (|000\rangle\langle 000| + |111\rangle\langle 111|)(|111\rangle\langle 000| + |000\rangle\langle 111|)$$

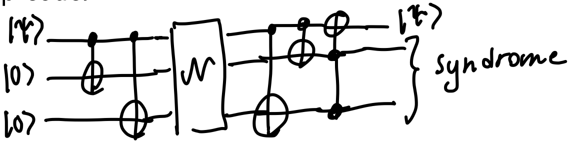
$$= |111\rangle\langle 000| + |000\rangle\langle 111|$$

$$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$$

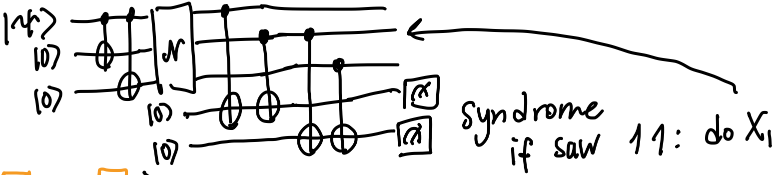
Bit flip code: recovery revisited

Last week / last class, we considered two recovery circuits for the bit flip code.

v1



v2



$|100\rangle + |011\rangle$

$|010\rangle + |101\rangle$

Stabilizers

We can consider a more general invariant than parity: eigenvalues w.r.t. special subsets of the Pauli group.

ZZZ:

$\|000\rangle$	$+1$
$\|001\rangle$	-1
$\|010\rangle$	-1
$\|011\rangle$	$+1$
$\|100\rangle$	-1
$\vdots$	$\vdots$

The n-qubit Pauli group is

$$\mathcal{P}_n = \{I, X, Y, Z\}^{\otimes n} \times \{1, -1, i, -1\}$$

Example: Which two-qubit Paulis is this Bell state a $+1$ eigenstate of?

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$P|\psi\rangle = (+1)|\psi\rangle$

(II) $\quad ZZ \quad XX$

bit flip: $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

ZZ: -1 eig-state

XX: $+1$ eig-state

Bit flip code: stabilizers

Consider our logical states:

$$
|0\rangle_L = |000\rangle \\
|1\rangle_L = |111\rangle
$$

Which three-qubit Paulis are these states $+1$ eigenstates of?

ZZI ZIZ IZZ

Stabilizers

Let S be a subgroup of $\mathcal{P}_n$.

Let V_S be a set of states that are $+1$ eigenstates for all $P \in S$.

Then, S is the **stabilizer** of V_S, and V_S is stabilized by S.

Facts about S:

- $-I$ is never in S
- all items of S commute
- choosing S uniquely defines the fixed subspace, V_S

Bit flip code: stabilizers generators

Let's determine a *minimal* representation of the group in terms of its **stabilizer generators**.

$$ZZI \quad \underbrace{ZIZ}_{=(ZZI)(IZZ)} \quad IZZ$$

generating set : $\underbrace{\langle ZZI, \quad IZZ \rangle}_{\text{+1 eig. state of generators is +1 eig. state of whole stabilizer group}}$

Bit flip code: stabilizers and logical operations

$\langle ZZI, IZZ \rangle$

$|\psi\rangle$

Logical Z: need a Pauli that

- stabilizes the logical state but is *not* in S
- *commutes* with everything in S
- has the right action on the subspace

$Z_L = ZZZ$

see N&C 10.5.5

note: need to redefine logical states to include Z_L as a stabilizer:

$|0\rangle_L$ stab. by $\langle ZZI, IZZ, ZZZ \rangle$

$|1\rangle_L$ stab. by $\langle ZZI, IZZ, -ZZZ \rangle$

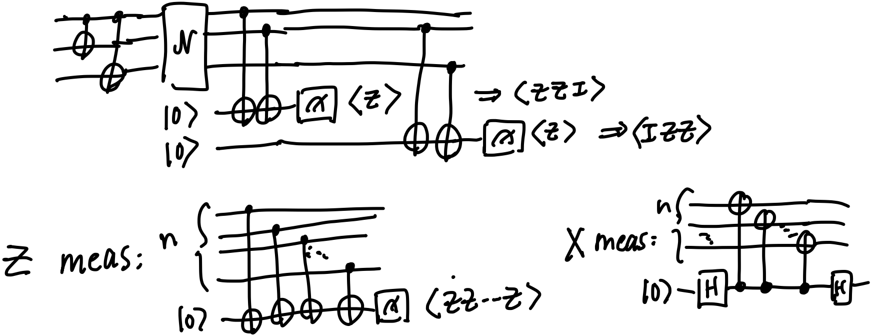
Logical X: need a Pauli that

- stabilizes the logical state but is *not* in S
- *commutes* with everything in S
- has the right action on the subspace
- anticommutes with logical Z

$X_L = XXX$

Bit flip code: stabilizer measurement and error detection

We can use this formalism to construct circuits for error detection and recovery: simply *measure the stabilizer generators*.



Bit flip code: stabilizer measurement and error detection

Correctible errors all anticommute with at least one of the generators, so we can detect their presence in the syndrome measurement.

stab. gen \ errors	XII	IXI	IIX
ZZI	-1	-1	+1
IZZ	+1	-1	-1

Bit flip code: stabilizers and logical errors

Danger: error that *commutes* with all elements of S, but isn't in S.

XXX — commutes w/ ZZI, IZZ

Phase flip code: stabilizer formalism

$$\alpha |{+}{+}{+}\rangle + \beta |{-}{-}{-}\rangle$$

$$S: \langle XXI, IXX \rangle$$

$$Z_L = XXX$$
$$X_L = ZZZ$$

S \ err	ZII	IZI	IIZ
XXI			
IXX			

fill out yourself!

Shor code: stabilizer formalism

Name	Operator
g_1	$Z\,Z\,I\,I\,I\,I\,I\,I\,I$
g_2	$I\,Z\,Z\,I\,I\,I\,I\,I\,I$
g_3	$I\,I\,I\,Z\,Z\,I\,I\,I\,I$
g_4	$I\,I\,I\,I\,Z\,Z\,I\,I\,I$
g_5	$I\,I\,I\,I\,I\,I\,Z\,Z\,I$
g_6	$I\,I\,I\,I\,I\,I\,I\,Z\,Z$
g_7	$X\,X\,X\,X\,X\,X\,I\,I\,I$
g_8	$I\,I\,I\,X\,X\,X\,X\,X\,X$
$\bar{Z}$	$X\,X\,X\,X\,X\,X\,X\,X\,X$
$\bar{X}$	$Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z$

Screenshot: Nielsen & Chuang, chapter 10.5.6.

The Hamming bound

What is the smallest number of physical qubits that we can use to make a logical qubit, and correct any single-qubit error?

General bound:

- n physical qubits
- k logical qubits
- up to t errors

$\binom{n}{j}$ $\qquad$ $k=1$

$$\sum_{j=0}^{t} \binom{n}{j} 3^j \cdot 2^k \leq 2^n$$

$$(3^0 + 3n)2 = (1 + 3n)2 \leq 2^n \Rightarrow n \geq 5$$

The smallest code: [[5, 1, 3]]

$$
\begin{aligned}
|0_L\rangle = \frac{1}{4}\Big[&|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\
&+|01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\
&- |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\
&- |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\Big] \\
|1_L\rangle = \frac{1}{4}\Big[&|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle \\
&+|10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\
&- |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle \\
&- |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\Big]
\end{aligned}
$$

Name	Operator
g_1	$XZZXI$
g_2	$IXZZX$
g_3	$XIXZZ$
g_4	$ZXIXZ$
$\bar{Z}$	$ZZZZZ$
$\bar{X}$	$XXXXX$

Screenshot: Nielsen & Chuang, chapter 10.5.6.

Properties of stabilizer codes

Stabilizer code usually described by notation $[[n, k, d]]$:

- code has 2^{n-k} stabilizer generators
- $d =$ distance (minimum weight of Paulis that commute with everything in S but aren't in S)
- a distance d code can correct $(d-1)/2$ errors

Next time

Next class (last class):

- More on stabilizer codes; fault-tolerant quantum computing
- Quiz on Tuesday (+ presentations)

Action items:

1. TA4 due Friday at 23:59
2. Work on project

Recommended reading:

- From this class: Codebook EC; N&C 10.1-10.5
- For next class: Codebook EC; N&C 10.6