Math Review Part II
Problem Set 2: Solutions

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1. 1.1. Pick any $x, x' \in \cap_{\alpha \in A} S_\alpha$, then by definition of intersection, $x, x' \in S_\alpha$ for all $\alpha \in A$. Given that each S_α is convex, for any $\lambda \in [0,1]$, $\lambda x + (1-\lambda)x' \in S_\alpha$ for all $\alpha \in A$. Therefore, $\lambda x + (1-\lambda)x' \in \cap_{\alpha \in A} S_\alpha$.

   1.2. Intervals $[1,2]$ and $[3,4]$ are both convex but their union is not.

2. $\subset$: Let's call the set on the left hand side A. It is sufficient to show that A covers $\{x_1, ..., x_n\}$ and is convex. Clearly A covers $\{x_1, ..., x_n\}$. To show convexity, take any two vectors $\sum_{i=1}^n \lambda_i x_i$ and $\sum_{i=1}^n \mu_i x_i$ in A and consider any $\lambda \in [0,1]$, we have

$$\lambda \sum_{i=1}^n \lambda_i x_i + (1-\lambda) \sum_{i=1}^n \mu_i x_i = \sum_{i=1}^n [\lambda \lambda_i + (1-\lambda)\mu_i] x_i$$

Clearly, each coefficient $\lambda \lambda_i + (1-\lambda)\mu_i \geq 0$, and their sum

$$\begin{aligned} \sum_{i=1}^n [\lambda \lambda_i + (1-\lambda)\mu_i] &= \sum_{i=1}^n \lambda \lambda_i + \sum_{i=1}^n (1-\lambda)\mu_i \\ &= \lambda \sum_{i=1}^n \lambda_i + (1-\lambda) \sum_{i=1}^n \mu_i \\ &= \lambda + (1-\lambda) = 1 \end{aligned}$$

and so $\lambda \sum_{i=1}^n \lambda_i x_i + (1-\lambda) \sum_{i=1}^n \mu_i x_i \in A$.
$\supset$: Take any $\sum_{i=1}^n \lambda_i x_i \in A$. $Co(\{x_1, ..., x_n\})$ is the intersection all $X \subset V$ convex sets that contain $\{x_1, ..., x_n\}$. Therefore $\sum_{i=1}^n \lambda_i x_i \in X$ for all such Xs and hence $\sum_{i=1}^n \lambda_i x_i \in Co(\{x_1, ..., x_n\})$.

3. 3.1. Follows directly from definitions.

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3.2. Take any $x_1, x_2 \in S$, $\lambda \in [0,1]$. Then since f is convex, $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$. In addition if ϕ is weakly increasing then $(\phi \circ f)(\lambda x_1 + (1-\lambda)x_2) \leq \phi(\lambda f(x_1) + (1-\lambda)f(x_2))$. Also ϕ is convex so $\phi(\lambda f(x_1) + (1-\lambda)f(x_2)) \leq \lambda(\phi \circ f)(x_1) + (1-\lambda)(\phi \circ f)(x_2)$. From the last two inequalities, it follows that

$$(\phi \circ f)(\lambda x_1 + (1-\lambda)x_2) \leq \lambda(\phi \circ f)(x_1) + (1-\lambda)(\phi \circ f)(x_2)$$

4. 4.1. Proof by induction.

For $n = 1$, the statement is trivial. If $n = 2$, then $f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$ by definition of convex functions.

Suppose inequality holds for $n = k$.

Consider $n = k+1$. For any $x_1, x_2, ..., x_{k+1} \in S$ and $\lambda_i > 0$, $\sum_{i=1}^{k+1} \lambda_i = 1$

$$\begin{aligned}
f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) &= f\left(\sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1}\right) \\
&= f\left(\sum_{i=1}^{k} \lambda_i \times \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i} x_i + \lambda_{k+1} x_{k+1}\right) \\
&\leq \left(\sum_{i=1}^{k} \lambda_i\right) f\left(\sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i} x_i\right) + \lambda_{k+1} f(x_{k+1}) \\
&\leq \left(\sum_{i=1}^{k} \lambda_i\right) \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i} f(x_i) + \lambda_{k+1} f(x_{k+1}) \\
&= \sum_{i=1}^{k+1} \lambda_{k+1} f(x_{k+1})
\end{aligned}$$

4.2. Can be proved analogously.

5. 5.1. For any α consider points $x_1, x_2 \in C^\alpha = \{x \in S : f(x) \geq a\}$. Then $f(x_1) \geq a$ and $f(x_2) \geq a$ which implies that $\lambda f(x_1) + (1-\lambda)f(x_2) \geq a$. In addition if f is concave then $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$. The last two inequalities imply that $f(\lambda x_1 + (1-\lambda)x_2) \geq a$ and hence $\lambda x_1 + (1-\lambda)x_2 \in C^\alpha$ implying that C^α is convex and f is quasiconcave.

5.2. Consider the function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that $f(x_1, x_2) = x_1 x_2$. This function is quasiconcave but not concave.

5.3. Proof analogous to (5.1).

5.4. Consider $-f$ from (5.2).

6. 6.1. $f(x,y) = x^2 y^2$ for $x \geq 0$ and $y \geq 0$. The Hessian of the function is

$$\begin{bmatrix} 2y^2 & 4xy \\ 4xy & 2x^2 \end{bmatrix}$$

After checking the principle minors of the above matrix we can conclude that the function is neither concave nor convex.

To test for quasiconcavity, let us look at the bordered Hessian:

$$\begin{bmatrix} 0 & 2xy^2 & 2x^2y \\ 2xy^2 & 2y^2 & 4xy \\ 2x^2y & 4xy & 2x^2 \end{bmatrix}$$

The determinant of the above 3×3 matrix is $16x^4y^4$ and so the function is quasi-concave but we cannot say anything about strict-quasiconcavity.

6.2. The Hessian of the function is

$$\begin{bmatrix} -e^x - e^{x+y} & -e^{x+y} \\ -e^{x+y} & -e^{x+y} \end{bmatrix}$$

$|M_1| = -e^x - e^{x+y} < 0$ and $|M_2| = e^{2x+1} > 0$ for all x and y. So the function is strictly concave, hence strictly quasiconcave.

7. Hessian of f is

$$\begin{bmatrix} a(a-1)cx^{a-2}y^b & abcx^{a-1}y^{b-1} \\ abcx^{a-1}y^{b-1} & b(b-1)cx^a y^{b-2} \end{bmatrix}$$

The principal minors of order 1 of this matrix are

$$M_1 = a(a-1)cx^{a-2}y^b \qquad M_1' = b(b-1)cx^a y^{b-2}$$

and only principal minor of order 2 is

$$M_2 = abcx^{2a-2}y^{2b-2}(1-a-b)$$

For f to be concave the Hessian must be negative semidefinite i.e. when $M_1, M_1' \leq 0$ and $M_2 \geq 0$. So f will be concave if $a + b \leq 1$ as it also implies that $a \leq 1$ and $b \leq 1$.

8. $\sum_{i=1}^n a_i x_i^\rho$ is a concave function because it is a linear combination of concave functions with positive coefficients.[1] Moreover, $q(z) = z^{\frac{1}{\rho}}$ is a monotonic transformation. In (5.1) we proved that if a function is concave then it is also quasiconcave and in the class we already proved that a monotonic transformation of a quasiconcave function is quasiconcave.

[1] You should be able to see why this is true.