How to do proofs?

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In mathematics, a **statement** is a sentence that is either true or false. A **proof** is a sound argument for the truth of a particular statement expressed in mathematical language. A **proposition** is a true statement of interest to be proved – the proof would accept the truth of some number of statements (the **premises**) and logically and cogently argue for the truth of the proposition. A **theorem** is a proposition that is subjectively considered to be of great import or value. Sometimes, because of the length of an argument for a theorem, the proof is broken into stages, with each linking proposition being proved as a **lemma**. Lastly, **corollaries** are propositions that follow almost immediately from a theorem; the proof of such a statement is usually trivial, but the subjective value of the knowledge of its truth is not. An **axiom** is a statement whose truth value is accepted without formal proof.

Definitions:

Let A and B be two statements

- We say "A implies B" and denote $A \Rightarrow B$ if B is true when A is true.
- The operator *and* denoted $\wedge$ is a binary operator such that $A \wedge B$ is true if and only if A is true and B is true.
- The operator *or* denoted $\vee$ is a binary operator such that $A \vee B$ is true if and only if at least one of A and B is true.
- The operator *not* denoted $\neg$ is a unary operator such that $\neg A$ is true if and only if A is false.
- The **converse** of $A \Rightarrow B$ is $B \Rightarrow A$.
- The **inverse** of $A \Rightarrow B$ is $\neg A \Rightarrow \neg B$.
- The **contrapositive** of $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$.
- If the proposition $A \Rightarrow B$ and its converse $B \Rightarrow A$ are both true, we say that A holds **if and only if** (short: iff) B holds or that A is **equivalent** to B.

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Different Ways of Proving:

Suppose we have to prove that the implication $A \Rightarrow B$ is true.

1. **Direct proof**: Assume A holds true and produce a chain of implications showing that B is true.

   Example 1. *Prove that the sum of two even integers is even.*

   Proof. a is even whenever there exists an integer t such that $a = 2t$. Now suppose a and b are even. Then there exists integers t and s such that $a = 2t$ and $b = 2s$. Then $k = a + b = 2s + 2t = 2(s + t)$, since integers are closed under addition $s + t$ is also an integer. □

   Alternatively we could attempt an *indirect proof* by contraposition or contradiction.

2. **Proof by Contraposition**: $\neg B \Rightarrow \neg A$, which is equivalent to $A \Rightarrow B$.

   Example 2. *Prove that if x^2 is odd then x is odd.*

   Proof. If x is not odd then x is even. Then there exists y such that $x = 2y$ and $x^2 = 2y.2y = 2(2y^2)$. Since integers are closed under multiplication, x^2 is even and is not odd. □

3. **Proof by Contradiction**: Assume that $A \Rightarrow B$ is false, and then show that this assumption leads to a contradiction of a previously proved (or assumed) statement.

   Example 3. *There is no greatest even integer.*

   Proof. Suppose there is greatest even integer N. Then for every even integer x, $N \geq x$. Now suppose $M = N + 2$. Then, M is an even integer. Also, $M > N$. Therefore, M is an integer that is greater than the greatest integer. This contradicts the supposition that $N \geq x$ for every even integer x. □

4. **Proof by Induction**: Suppose that we are considering a sequence of statements indexed by the natural numbers, so that the first statement is $P(1)$, the second statement is P(2), and the n-th statement is $P(n)$. Suppose that we can verify two facts about this sequence of statements:

   (a) **Base case**: Statement $P(1)$ is true.

   (b) **Inductive step**: Whenever any statement $P(k)$ is true for some k, then $P(k+1)$ is also true.

   Then we can conclude that *all* of the statements in the sequence are true.

   Example 4. *Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.*

   Proof. For $n = 1$ the statement is trivially true. Now assume that for $n = k$, $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. Then for $n = k + 1$, $\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$. □

Common Types of Proofs in Economics

- Existence Proofs: When proving that there exists at least one $x \in X$ that satisfies some properties, we often do so by construction i.e. we explicitly construct a specific x, and then try to show that the x we have constructed satisfies those required properties.

- Uniqueness Proofs: When proving that there exists at most one $x \in X$ that satisfies some properties, we start by taking any x_1 and x_2 that both satisfy those properties, and then try to get to the conclusion that they must be the same element.

References

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