Math Review Part II
Problem Set 2: Convexity

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1. In vector space V, let $\{S_\alpha\}_{\alpha\in A}$ be a family of convex sets. For the following statements, prove it if true or disprove it by providing a counterexample if false.

   1.1. $\cap_{\alpha\in A} S_\alpha$ is convex

   1.2. $\cup_{\alpha\in A} S_\alpha$ is convex

2. In vector space V, let $\{x_1, x_2, ..., x_n\}$ be a finite set of vectors. Then prove that

$$Co(\{x_1, x_2, ..., x_n\}) = \left\{\sum_{i=1}^{n} \lambda_i x_i : \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+ \text{ and } \sum_{i=1}^{n} \lambda_i = 1\right\}$$

3. Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V. Prove the following:

   3.1. f is (strictly) convex/quasiconvex *iff* $-f$ is (strictly) concave/quasiconcave.

   3.2. If f is (convex) concave and $\phi : \mathbb{R} \to \mathbb{R}$ is weakly increasing and (convex) concave, then $\phi \circ f$ is (convex) concave.

4. **(Jensen's Inequality)** Consider a function $f : S \to \mathbb{R}$, where S is a convex set in vector space V. Prove the following:

   4.1. f is convex *iff*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

   for any $x_1, x_2, ..., x_n \in S$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$.

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4.2. f is concave *iff*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i f(x_i)$$

for any $x_1, x_2, ..., x_n \in S$ and $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$.

5. Consider a function $f : S \to \mathbb{R}$, where S is a convex set in $\mathbb{R}^n$. For the following statements, prove it if true or disprove it by providing a counterexample if false.

   5.1. If f is concave, then it is quasiconcave

   5.2. If f is quasiconcave, then it is concave

   5.3. If f is convex, then it is quasiconvex

   5.4. If f is quasiconvex, then it is convex

6. Determine if possible, which of the following properties each of the following functions satisfies: convexity, strict convexity, concavity, strict concavity, quasiconcavity, strict quasiconcavity.

   6.1. $f(x, y) = x^2 y^2$ for $x \geq 0$ and $y \geq 0$.

   6.2. $f(x, y) = x - e^x - e^{x+y}$

7. Consider the Cobb-Douglas function $f(x, y) = cx^a y^b$ with $a, b, c > 0$ and $x, y > 0$. What are the conditions required for this function to be concave?

8. Show that a generalized Constant Elasticity of Substitution (CES) function given by

$$f(x) = \left(\sum_{i=1}^{n} a_i x_i^{\rho}\right)^{\frac{1}{\rho}}$$

   with $a_i > 0$ for all i and $\rho \in (0, 1)$ is quasiconcave.