Real Analysis

Divya Bhagia

Boston College

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Overview

Overview

Set Notation

A **set** is any well-specified collection of elements.

- A set that contains no elements is called an **empty set** denoted by $\emptyset$.
- $A \subseteq B$ (A is a **subset** of B) if $x \in A \Rightarrow x \in B$.
- $A \subset B$ (A is a **proper subset** of B) if $A \subseteq B$ is true but $B \subseteq A$ is not true.
- The set of all subsets of a set X is called the **power set** of X denoted by $P(X)$ or 2^X.

Set Operations

- **Union** of two sets: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- **Intersection** of two sets: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- **Set Difference**: $A - B$ or $A \backslash B = \{x : x \in A \text{ and } x \notin B\}$
- **Complement**: A^c or $\bar{A} = U \backslash A$ where U is the universal set.
- **Cartesian product**: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

De Morgan's Law

Proposition

Let $\{A_i\}_{i\in I}$ be a collection of subsets of U, then

(1) $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \bar{A}_i$ (2) $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \bar{A}_i$

Proof: (1) $\subset$: Take $x \in \overline{\bigcup_{i\in I} A_i}$, this implies that $x \notin A_i$ for all $i \in I$. Then $x \in \bar{A}_i$ for all $i \in I$ and therefore $x \in \bigcap_{i\in I} \bar{A}_i$.

$\supset$: Take $x \in \bigcap_{i\in I} \bar{A}_i$, this implies that $x \in \bar{A}_i$ for all $i \in I$. Then $x \notin A_i$ for all $i \in I$ and therefore $x \notin \bigcup_{i\in I} A_i$ which implies that $x \in \overline{\bigcup_{i\in I} A_i}$.

(2) Follows from (1).

Relations

A binary **relation** R from A to B is a subset of $A \times B$. (e.g. $\geq$ on $X = \{1, 2, 3\}$)

- R is **reflexive** if for all $x \in X$, we have xRx.
- R is **transitive** if for all $x, y, z \in X$ s.t xRy and yRz, we have xRz.
- R is **anti-symmetric** if for all $x, y \in X$ s.t xRy and yRx, we have $x = y$.
- R is **complete** if for all $x, y \in X$, either xRy or yRx.
- R is **symmetric** if for all $x, y \in X$ s.t xRy, we have yRx.
- R^{-1} is **inverse** of R such that $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$

Partially Ordered Set

A relation $\leq$ on X is a **partial order** if $\leq$ is reflexive, transitive and anti-symmetric. In this case, we call $(X, \leq)$ a **partially ordered set**.

Upper/Lower Bound, Maximum/Minimum, and Supremum/Infimum

Definitions

Let $(X, \leq)$ be a partially ordered set, and let $A \subset X$.

- $u \in X$ is an **upper bound** of A if $u \geq a$, $\forall a \in A$.
- $l \in X$ is a **lower bound** of A if $l \leq a$, $\forall a \in A$.
- $x \in A$ is a **maximum** of A if x is an upper bound of A.
- $x \in A$ is a **minimum** of A if x is an lower bound of A.
- $x \in X$ is the **least upper bound** or **supremum** of A if x is an upper bound of A and $x \leq y$ for any upper bound y of A.
- $x \in X$ is the **greatest lower bound** or **infimum** of A if x is a lower bound of A and $x \geq y$ for any lower bound y of A.

Functions

Definition

A relation f from X to Y is a **function** if

1. $\forall x \in X, \exists\ y \in Y$ s.t. $(x, y) \in f$, and
2. $\forall x \in X$ and $y_1, y_2 \in Y$ if $(x, y_1) \in f$ and $(x, y_2) \in f$ then $y_1 = y_2$

Given a function $f : X \to Y$

- X is called the **domain** of f
- Y is called the **codomain** of f
- **Image** of a set $S \subset X$ under f is given by

$$f(S) = \{f(x) : x \in S\}$$

- f(X) is called the **range** of f

Functions

Consider a function $f : X \to Y$, then

- f is an **injective** function if $\forall x_1, x_2 \in X$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
- f is a **surjective** function if $f(X) = Y$
- f is a **bijective** function if f is both injective and surjective.

Composite Functions

Let $f : X \to Y$ and $g : Y \to Z$. Then the **composite function** of f and g, denoted as $g \circ f$, is a function from X to Z s.t. for each $x \in X$, the value of the function $(g \circ f)(x) = g(f(x))$.

Monotonic Functions

When both the domain and the codomain are ordered sets, we can talk about monotonicity of a function. A monotonic function is simply a function that preserves, or inverses, the order.

Definition

Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be posets, and consider a function $f : X \to Y$

1. f is **weakly increasing** *iff* $x \leq_X x'$ implies $f(x) \leq_Y f(x')$
2. f is **weakly decreasing** *iff* $x \leq_X x'$ implies $f(x) \geq_Y f(x')$
3. f is **strictly increasing** *iff* $x <_X x'$ implies $f(x) <_Y f(x')$
4. f is **strictly decreasing** *iff* $x <_X x'$ implies $f(x) >_Y f(x')$

Overview

Metric Spaces

Definition

Let X be a set. A function $d : X^2 \to \mathbb{R}_+$ is a **distance function** or a **metric** on X if it satisfies the following.

1. $d(x, y) = 0$ if $x = y$
2. Symmetry: $d(x, y) = d(y, x)\ \forall x, y \in X$, and
3. Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$

If d is a metric on X, then the couple (X, d) is called a metric space.

Corollary

Given a metric space (X, d) and a subset $S \subset X$. Define a new distance function $d|_S : S^2 \to \mathbb{R}_+$ as $d|_S(x, y) = d(x, y)$ for any $x, y \in S$, then $(S, d|_S)$ is a valid metric space.

Examples of Metrics on $\mathbb{R}^k$

- Euclidean distance function d_2

$$d_2(x, y) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

- d_n metric

$$d_n(x, y) = \left(\sum_{i=1}^{k} |x_i - y_i|^n \right)^{\frac{1}{n}}$$

- Discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Overview

Open Balls and Bounded Sets

Let (X, d) be a metric space and S be a subset of X.

Definition

The **open ball** centered at $x \in X$ with radius $r > 0$ is defined as the set

$$B_r(x) = \{z \in X : d(z, x) < r\}$$

Definition

The set S is said to be **bounded** if there exists $x \in X$ and $r > 0$ s.t. $S \subset B_r(x)$.

Let (X, d) be a metric space and S be a subset of X.

Interior Point

A point $x \in X$ is an **interior point** of S if $\exists\ r > 0$ s.t. $B_r(x) \subset S$. The set of interior points of S is denoted as $int(S)$.

Open Set

The set S is an **open set** if $S \subset int(S)$, i.e. all points in S are interior points.

Union of open sets is open

Proposition

Let $\{E_\alpha\}_{\alpha \in A}$ be an arbitrary family of open sets (potentially uncountably many of them) in a metric space (X, d). Then their union $\bigcup\limits_{\alpha \in A} E_\alpha$ is also open.

Proof: Take any $x \in \bigcup\limits_{\alpha \in A} E_\alpha$, we need to find $r > 0$ s.t. $B_r(x) \subset \bigcup\limits_{\alpha \in A} E_\alpha$. By definition of union, there exists $\hat{\alpha}$ s.t. $x \in E_{\hat{\alpha}}$. Because $E_{\hat{\alpha}}$ is open, we can find $r > 0$ s.t. $B_r(x) \subset E_{\hat{\alpha}}$. This is the $r > 0$ we needed to find as $B_r(x) \subset E_{\hat{\alpha}} \subset \bigcup\limits_{\alpha \in A} E_\alpha$.

Finite intersection of open sets is open

Proposition

Let $\{E_i\}_{i=1}^{n}$ be a finite family of open sets in a metric space (X, d). Then their intersection $\bigcap_{i=1}^{n} E_i$ is also open.

Proof: Take any $x \in \bigcap_{i=1}^{n} E_i$, we need to find $r > 0$ s.t. $B_r(x) \subset \bigcap_{i=1}^{n} E_i$. By definition of intersection, $x \in E_i$ for all $i = 1, ..., n$. For each i, because E_i is open, $\exists\ r_i > 0$ s.t. $B_{r_i}(x) \subset E_i$. Let $r = min\{r_1, ..., r_n\}$. Now clearly, $r > 0$ and $B_r(x) \subset B_{r_i}(x) \subset E_i$.

Closed Sets

Let (X, d) be a metric space and S be a subset of X.

Limit Point

A point $x \in X$ is a **limit point** of S if $(B_r(x) \backslash \{x\}) \cap S \neq \emptyset$, $\forall r > 0$.
Denote the set of limit points of S as S'.

Closed Set

The set S is a **closed set** if $S' \subset S$, i.e. S contains all of it's limit points.

Topological definition of closed sets

Definition

The set S is a **closed set** if S^c is open.

Proposition

Both definitions are equivalent.

Proof: $\Leftarrow$: Suppose S is closed. Then choose $x \in S^c$, then $x \notin S$, and so x is not a limit point of S. Hence there exists $r > 0$ such that $S \cap B_r(x) = \phi$, such that $B_r(x) \subset S^c$. Thus x is an interior point of S^c and S^c is open.
$\Rightarrow$: Suppose S^c is open. Let x be a limit point of S, then every open ball around x contains a point of S, so that x is not an interior point of S^c. Since S^c is open, this means that $x \in S$, Therefore S is closed.

Union and intersection of closed sets

Proposition

In metric space (X, d):

1. Let $\{F_\alpha\}_{\alpha \in A}$ be an arbitrary family of closed sets (potentially uncountably many of them). Then their intersection $\bigcap_{\alpha \in A} F_\alpha$ is also closed.
2. Let $\{F_i\}_{i=1}^n$ be a finite family of closed sets. Then their union $\bigcup_{i=1}^n F_i$ is also closed.

Overview

Sequences and Convergence

Definitions

- Let X be a set. The function $x : \mathbb{N} \to X$ is called a **sequence** in X.
- Given a sequence (x_n), a **subsequence** of (x_n) is a sequence (x_{nk}) indexed by $k \in \mathbb{N}$ where (n_k) is a strictly increasing sequence in $\mathbb{N}$.

Definition

Let (X, d) be a metric space. A sequence (x_n) in X is said to **converge** to a point $x \in X$, *iff* $\forall \epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n > N$. When the sequence (x_n) converges to x, the point x is called a **limit** of the sequence (x_n), and we use the notation $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Closed Set: Alternative Definition

Proposition

Let (X, d) be a metric space, and S a subset of X. Then the following statements are equivalent.

1. S is a closed set.
2. For any sequence (x_n) in S convergent to some point $x \in X$, we have $x \in S$.

Proof: (1) $\Rightarrow$ (2): Suppose S is closed and $x_n \to x$. We will prove the claim by contradiction, so assume $x \notin S$. But because $x_n \to x$, for all $r > 0$, $\exists\ n$ such that $x_n \in B_r(x) \backslash \{x\}$. Therefore $x_n \in B_r(x) \backslash \{x\} \cap S$ which implies $B_r(x) \backslash \{x\} \cap S \neq \phi$. So we have shown that $x \in S'$.
(2) $\Rightarrow$ (1): We will prove the claim by contrapositive. Suppose S is an open set. Then S^c is closed and $\exists\ x \in S^c$ such that $\forall r > 0$, $B_r(x) \backslash \{x\} \cap S \neq \phi$. For every $n \in \mathbb{N}$, let $x_n \in B_{\frac{1}{n}}(x) \cap S$, then $x_n \to x$ but $x \notin S$.

Proposition

Let (X, d) be a metric space. If (x_n) is a convergent sequence in X, then x_n must be bounded.

Proof: Let the limit of (x_n) be x. Let $\epsilon = 1$, and by definition of convergence, there exists N s.t. $d(x_n, x) < 1$ for any $n > N$. Then let

$$r = max\{d(x_1, x), d(x_2, x), ..., d(x_N, x)\} + 1$$

Then clearly we have $B_r(x) \supset \{x_1, x_2, ..\}$.

Monotone Convergence Theorem

Theorem

Every monotone and bounded real sequence (x_n) is convergent in $(\mathbb{R}, d_2)$.

Proof: (1) Take any increasing and bounded from above real sequence (x_n). Because the range of the sequence $\{x_1, x_2, ..\}$ is bounded from above, by l.u.b. property of $\mathbb{R}$, it has a least upper bound. Let $x = sup\{x_1, x_2, ...\}$, then we want to show that $x_n \to x$.
Take any $\epsilon > 0$. We want to find N s.t. $|x_n - x| < \epsilon$ for any $n > N$.
Because x is the least upper bound of $\{x_1, x_2, ..\}$, $x - \epsilon$ is not an upper bound and therefore $\exists$ N s.t. $x_N > x - \epsilon$. Therefore, for any $n > N$, we have

$$x \geq x_n \geq x_N > x - \epsilon$$

and therefore $|x_n - x| < \epsilon$.
(2) Take any decreasing and bounded from below real sequence (x_n). Then $(-x_n)$ is increasing and bounded from above. By (1) we have $(-x_n)$ is convergent and thus (x_n) is also convergent.

Sequences and Convergence

Lemma

Every sequence in $\mathbb{R}$ has a monotone subsequence.

Proof: Take any sequence (x_n) in $\mathbb{R}$. Call the term x_n a dominant term if $x_n \geq x_m$ for any $m \geq n$.
Case 1: (x_n) has infinitely many dominant terms. Then these dominant terms constitute a decreasing subsequence.
Case 2: (x_n) has finitely many dominant terms. The let x_N be the last dominant term and let $n_1 = N + 1$. By definition, $\exists\ n_2 > n_1$ s.t. $x_{n_2} > x_{n_1}$. Similarly, $\exists\ n_3 > n_2$ s.t. $x_{n_3} > x_{n_2}$ and so on. Therefore we obtain a strictly increasing subsequence.
Case 3: (x_n) has no dominant term. The let $n_1 = 1$ and construct strictly increasing subsequence as in *Case 2*.

Bolzano-Weierstrass Theorem

Every bounded sequence in $(\mathbb{R}^k, d_2)$ has a convergent subsequence.

Overview

Compactness: Two Equivalent Definitions

Compactness

Let (X, d) be a metric space, and S be a subset of X.

- A family of open sets $\{E_a\}_{a \in A}$ is an **open cover** of S if $S \subset \bigcup_{a \in A} E_a$.
- The set S is **compact** if $\forall$ open cover $\{E_a\}_{a \in A}$ of S, $\exists$ a finite $B \subset A$ s.t. $\{E_a\}_{a \in B}$ is also an open cover of S.

Sequential Compactness

Let (X, d) be a metric space, and S a subset of X. The set S is **sequentially compact** if any sequence (x_n) in S has a subsequence convergent to some $x \in S$.

Compactness

Lemma 1

Any closed interval $[a, b]$ is compact in $(\mathbb{R}, d_2)$.

Lemma 2

Every k-cell $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_k, b_k]$ is compact in $(\mathbb{R}^k, d_2)$.

Heine-Borel Theorem

In $(\mathbb{R}^k, d_2)$, a set S is compact *iff* it is closed and bounded.

Sequential Compactness

Theorem

Let (X, d) be a metric space, and S a subset of X. The set S is compact *iff* it is sequentially compact.

Proof: The theorem is true in general but let us just prove it for $(\mathbb{R}^k, d_2)$.
$\Rightarrow$: If S in $(\mathbb{R}^k, d_2)$ is compact, then it is bounded. Then any sequence (x_n) in S must be bounded. Then by Bolzano-Weierstrass theorem, (x_n) has a subsequence convergent to some $x \in \mathbb{R}^k$. By sequential definition of closed sets, we have $x \in S$. Therefore S is sequentially compact.
$\Leftarrow$: If S in $(\mathbb{R}^k, d_2)$ is sequentially compact then it must be closed; otherwise we can find a sequence (x_n) in S convergent to some x^* outside S, and any subsequence of (x_n) must also converge to $x^* \notin S$, so it does not have a subsequence convergent to some point in S. Also, the set S must be bounded. Otherwise we can construct a sequence (x_n) s.t. $d_2(x_n, 0) > n$, and so (x_n) does not even have a convergent subsequence.

Overview

Continuity

Definition

Let (X, d_X) and (Y, d_X) be metric spaces. Let S be a subset of X, function $f : S \to Y$, and $x_0 \in S$. Then the function f is said to be **continuous at** x_0 if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$f(B_\delta(x_0) \cap S) \subset B_\epsilon(f(x_0))$$

The function f is said to be a **continuous function** if f is continuous at x_0 for all $x_0 \in S$.

Continuity

Theorem

Let (X, d_X) and (Y, d_X) be metric spaces. Let S be a subset of X, function $f : S \to Y$, and $x_0 \in S$. Then f is continuous at x_0 *iff* $f(x_n) \to f(x_0)$ for any sequence (x_n) in S convergent to x_0.

Continuous image of a compact set is also compact

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, and function $f : X \to Y$ is continuous. Then $f(K)$ is compact in (Y, d_Y) for any K compact in (X, d_X).

Proof: Take K compact in (X, d_X) and take any open cover $\{E_\alpha\}_{\alpha \in A}$ of $f(K)$. We want to find a finite $B \subset A$ s.t. $\{E_\alpha\}_{\alpha \in B}$ is an open cover of $f(K)$.

First we claim that $\{f^{-1}(E_\alpha)\}_{\alpha \in A}$ is an open cover of K. Because each E_α is open in (Y, d_Y), the set $f^{-1}(E_\alpha)$ is open in (X, d_X). Take any $x \in K$, we have $f(x) \in f(K)$. Because $\{E_\alpha\}_{\alpha \in A}$ covers $f(K)$, $\exists \hat{\alpha} \in A$ s.t. $f(x) \in E_{\hat{\alpha}}$. So $x \in \{f^{-1}(E_\alpha)\}_{\alpha \in A}$. Therefore, $\{f^{-1}(E_\alpha)\}_{\alpha \in A}$ is an open cover of K.

Take any $y \in f(K)$. There exists $\hat{x} \in K$ s.t. $f(\hat{x}) = y$. Because $\{f^{-1}(E_\alpha)\}_{\alpha \in A}$ covers K, $\exists \hat{\alpha} \in B$ s.t. $\hat{x} \in f^{-1}(E_{\hat{\alpha}})$. Therefore, $y = f(\hat{x}) \in E_{\hat{\alpha}}$. Therefore $\{E_\alpha\}_{\alpha \in B}$ is an open cover of $f(K)$.

Claim

Let K be a compact set in $(\mathbb{R}, d_2)$. Then there exists $x^* \in K$ s.t. $x^* \geq x$ for any $x \in K$, and there exists $x_* \in K$ s.t. $x_* \leq x$ for any $x \in K$.

Proof: Because K is compact in $(\mathbb{R}, d_2)$, we know that K is bounded, i.e. there exists some $r > 0$ s.t. $K \subset B_r(x)$. Then $x + r$ is an upper bound of K. By l.u.b. property of $\mathbb{R}$, there exists sup K. Now we need to show that sup $K \in K$. If not, then sup K is a limit point of K because for any $\epsilon > 0$, there exists $x \in K$ s.t. $x >$ sup $k - \epsilon$. Since K is closed it must contain all it's limit points.

Weierstrass Theorem

Theorem

Let (X, d_X) be a metric space, and function $f : X \to \mathbb{R}$ is continuous. Let S be a compact set in (X, d_X). There exists $x^* \in S$ s.t. $f(x^*) \geq f(x)$ for any $x \in S$, and there exists $x_* \in S$ s.t. $f(x_*) \leq f(x)$ for any $x \in S$.

Proof: By the previous theorem we know that $f(S)$ is compact in $(\mathbb{R}, d_2)$. Therefore, there exists $y^* \in f(S)$ s.t. $y^* \geq f(x)$ for any $x \in S$. By definition of the image $f(S)$, there exists $x^* \in S$ s.t. $f(x^*) = y^*$, and therefore $f(x^*) \geq f(x)$ for any $x \in S$. Symmetrically, we can find x_*.