Math Review Part II
Problem Set 1: Solutions

Divya Bhagia*

August 19, 2019

1. $\subset$: Take any $x \in f^{-}(T_1 \cup T_2)$ then $f(x) \in T_1 \cup T_2$ by definition of inverse image. Then either $f(x) \in T_1$ or $f(x) \in T_2$. Then again by definition of the inverse image, either $x \in f^{-}(T_1)$ or $x \in f^{-}(T_2)$. Therefore, $x \in f^{-}(T_1) \cup f^{-}(T_2)$.
   $\supset$: Take any $x \in f^{-}(T_1) \cup f^{-}(T_2)$. Then either $x \in f^{-}(T_1)$ or $x \in f^{-}(T_2)$. That is, either $f(x) \in T_1$ or $f(x) \in T_2$. Therefore, $f(x) \in T_1 \cup T_2$, which implies $x \in f^{-}(T_1 \cup T_2)$.

2. Take any open ball $B_r(x)$ in metric space, and take any point $z \in B_r(x)$. Let $\varepsilon = r - d(z, x)$. First, because $z \in B_r(x)$, we have $d(z, x) < r$ and thus $\varepsilon > 0$. Second, take any $y \in B_\varepsilon(z)$, we have

$$d(y, x) \leq d(y, z) + d(z, x) < \varepsilon + d(z, x) = r$$

   and therefore $y \in B_r(x)$.

3. Proof by contradiction. Suppose $x \neq x'$, then $d(x, x') > 0$ (see definition of a metric). Let $\varepsilon = d(x, x')/2$. Because $x_n \to x$, $\exists N$ s.t $d(x_n, x) < \varepsilon \ \forall n > N$. And because $x_n \to x'$, $\exists N'$ s.t $d(x_n, x') < \varepsilon \ \forall n > N'$. Let $\hat{n} = max\{N, N'\} + 1$. Since $\hat{n} > N$ and $\hat{n} > N'$ and since $d(x_{\hat{n}}, x) < \varepsilon$ and $d(x_{\hat{n}}, x') < \varepsilon$, we have

$$d(x_{\hat{n}}, x) + d(x_{\hat{n}}, x') < 2\varepsilon = d(x, x')$$

   which contradicts triangle inequality of d. Therefore we must have $x = x'$.

4. $\Rightarrow$: Take any $i \in \{1, 2, ..., k\}$ and take any $\varepsilon > 0$, we want to find N^i s.t. $d_2(x_n^i, x^i) < \varepsilon$ for any $n > N^i$. Because $x_n \to x$, there exists N s.t. $d_2(x_n, x) < \varepsilon$ for any $n > N$. Let

*PhD Candidate in Economics at Boston College (email: bhagia@bc.edu)

$N^i = N$ and this is the N^i we need to find. This is because for any $n > N^i = N$, we have

$$\begin{aligned} d_2(x_n^i, x^i) = |x_n^i - x^i| &= \sqrt{(x_n^i - x^i)^2} \\ &\leq \sqrt{\sum_{j=1}^{k}(x_n^j - x^j)^2} = d_2(x_n, x) < \varepsilon \end{aligned}$$

$\Leftarrow$: Take any $\varepsilon > 0$, we want to find N s.t. $d_2(x_n, x) < \varepsilon$ for any $n > N$. Because $x_n^i \to x^i$, there exists N^i s.t. $d_2(x_n^i, x^i) < \varepsilon/\sqrt{k}$ for any $n > N^i$. Let $N = max\{N_1, ..., N_k\}$ and this is the N we want to find. This is because for any $n > N$, we have $n > N^i$ and thus $d_2(x_n^i, x^i) < \varepsilon/\sqrt{k}$ for any i, and therefore

$$d_2(x_n, x) = \sqrt{\sum_{j=1}^{k}(x_n^j - x^j)^2} < \sqrt{k(\varepsilon/\sqrt{k})^2} = \varepsilon$$

5. Take any $\varepsilon > 0$, we want to find N s.t. $|x_n y_n - xy| < \varepsilon$ for any $n > N$. Because (y_n) is convergent, it is bounded, i.e. there exists an open ball $(z - r, z + r)$ that contains $\{y_1, y_2, ...\}$. Let $M = max\{|z - r|, |z + r|\}$, and by construction $|y_n| < M$ for any n. Because $x_n \to x$, there exists N_x s.t. $|x_n - x| < \varepsilon/2M$. Because $y_n \to y$, there exists N_y s.t. $|y_n - y| < \varepsilon/2(|x| + 1)$.
Let $N = max\{N_x, N_y\}$ and this is the N we need to find. This is because for any $n > N$, we have

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - y_n)y_n + (y_n - y)x| \\ &\leq |x_n - x|.|y_n| + |y_n - y|.|x| \\ &< \frac{\varepsilon}{2M}.M + \frac{\varepsilon}{2(|x| + 1)}.|x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

6. We will show that S^c is open. Take any $x \in S^c$, we want to find $r > 0$ s.t. $B_r(x) \subset S^c$. Take any $y \in S$, let $r_y = d(y, x)/2$. Then clearly $B_{r_y}(x)$ and $B_{r_y}(y)$ are disjoint. Also note that $\{B_{r_y}(y)\}_{y \in S}$ is an open cover S. By compactness of S, there exists $\{y_1, y_2, ..., y_n\}$ s.t. $\{B_{r_{y_i}}(y_i)\}_{i=1}^n$ is also an open cover of S. Now let $r = min\{r_{y_1}, r_{y_2}, ..., r_{y_n}\}$. We want to show that $B_r(x)$ is disjoint with $B_{r_{y_i}}(y_i)$ for any i. So $B_r(x)$ is disjoint with the union of $B_{r_{y_i}}(y_i)$s as well and thus $B_r(x)$ is disjoint with S, which implies $B_r(x) \subset S^c$.

7. Take any open cover $\{E_\alpha\}_{\alpha \in A}$ of S. We want to find a finite family chosen from $\{E_\alpha\}_{\alpha \in A}$ that also covers S. Clearly, $\{E_\alpha\}_{\alpha \in A} \cup S^c$ covers the whole space, and thus covers Y. Because Y is compact, there exists a finite family chosen from $\{E_\alpha\}_{\alpha \in A} \cup S^c$ that covers Y. Because $S \subset Y$, the finite family also covers S. If the finite family contains S^c,

then we can remove it from the family, then the family still covers S, since S^c has no contribution to covering S. So we have obtained a finite family chosen from $\{E_\alpha\}_{\alpha \in A}$ that covers S.

8. Take any closed interval $[a, b]$, and suppose that it is not compact. Then there exists an open cover $\{E_\alpha\}_{\alpha \in A}$ of $[a, b]$ without a finite subcover.
   Let $a_0 = a$ and $b_0 = b$.
   Cut the interval $[a_0, b_0]$ in half: $[a_0, (a_0 + b_0)/2]$ and $[(a_0 + b_0)/2, b_0]$. At least one of them cannot be finitely covered (otherwise the interval $[a_0, b_0]$ can be finitely covered). Take the one that cannot be finitely covered and label it as $[a_1, b_1]$.
   Cut the interval $[a_1, b_1]$ in half: $[a_1, (a_1 + b_1)/2]$ and $[(a_1 + b_1)/2, b_1]$. At least one of them cannot be finitely covered . Take the one that cannot be finitely covered and label it as $[a_2, b_2]$.
   Repeat this process, and we get a shrinking sequence of intervals $[a_0, b_0] \supset [a_1, b_1] \supset ...$, and each of them cannot be finitely covered using the open cover $\{E_\alpha\}_{\alpha \in A}$.
   Because (a_n) is increasing and bounded from above by b_0, (a_n) converges to some limit a^*. Symmetrically, (b_n) converges to some limit b^*. Because $b_n - a_n = (1/2)^n(b - a) \to 0$, we know that

$$b_n = a_n + (b_n - a_n) \to a^* + 0 = a^*$$

   and therefor $b^* = a^*$. That is, the sequence of intervals $[a_0, b_0] \supset [a_1, b_1] \supset ...$ shrinks to one point a^*. Because $a^* \in [a, b]$, it is covered by some open set E_{α^*} in the open cover.
   Therefore, there exists $B_r(a^*) \subset E_{\alpha^*}$. Because (a_n) and (b_n) both converge to a^*, there exists $\hat{n}$ s.t. $a_{\hat{n}}, b_{\hat{n}} \in B_r(a^*)$, and therefore $[a_{\hat{n}}, b_{\hat{n}}] \subset B_r(a^*) \subset E_{\alpha^*}$. So $[a_{\hat{n}}, b_{\hat{n}}]$ can be finitely covered using open cover $\{E_\alpha\}_{\alpha \in A}$, which contradicts the construction of the sequence $([a_n, b_n])$.