Correspondences

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Overview

Overview

Correspondences

Definition

A **correspondence** F from X to Y is a set-valued function that associates every element in X to a subset of Y

$$F : X \rightrightarrows Y \qquad \text{s.t.} \qquad x \mapsto F(x) \subset Y$$

The set X is called the **domain** of the correspondence F, and Y is called the **codomain** of F. $F(x)$ is called the **image** of point $x \in X$.

Correspondences

Definitions

For each x in $\{$

1. non-empty
2. single (singleton)
3. open
4. closed
5. compact
6. convex

A correspondence $F : X \rightrightarrows Y$ is said to be x-valued at $x_0 \in X$ if $F(x_0)$ is a x set. If F is x-valued for all $x_0 \in X$, we say F is x-valued.
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Overview

Upper Hemi-Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be **upper hemi-continuous (uhc) at** $x_0 \in X$ if $\forall$ open sets U in (Y, d_Y) with $F(x_0) \subset U$, $\exists\ \delta > 0$ s.t. $F(B_\delta(x_0)) \subset U$.

The correspondence $F : X \rightrightarrows Y$ is said to be **upper hemi-continuous (uhc)** if it is upper hemi-continuous at x_0 for all $x_0 \in X$.

Upper Hemi-Continuity: Examples

$F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ is **not** upper hemi-continuous

$$F_1(x) = \begin{cases} \{0\} & x \leq 0 \\ [-1, 1] & x > 0 \end{cases}$$

$F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ is upper hemi-continuous

$$F_2(x) = \begin{cases} \{0\} & x < 0 \\ [-1, 1] & \geq 0 \end{cases}$$

Upper Hemi-Continuity: Another Example

Let's formally show that $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_3(x) = [x, x+1]$$

is uhc for any $x \in \mathbb{R}$.

Take any $x_0 \in \mathbb{R}$ and take any open set $U \supset [x_0, x_0+1]$. We want to show that $\exists \delta > 0$ s.t. $F(x) \subset U$ for any $x \in (x_0 - \delta, x_0 + \delta)$.
Since U is an open set and $x_0, x_0 + 1 \in U$ then x_0 and $x_0 + 1$ are interior points of U, so $\exists\ \delta > 0$ s.t.

$$(x_0 - \delta, x_0 + \delta) \subset U$$

$$(x_0 + 1 - \delta, x_0 + 1 + \delta) \subset U$$

Therefore $(x_0 - \delta, x_0 + 1 + \delta) \subset U$ and hence for any $x \in (x_0 - \delta, x_0 + \delta)$, we have

$$F(x) = [x, x+1] \subset (x_0 - \delta, x_0 + 1 + \delta) \subset U$$

What about $F_4 : \mathbb{R} \rightrightarrows \mathbb{R}$ with $F_4(x) = (x, x+1)$?

Upper Hemi-Continuity

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a correspondence $F : X \rightrightarrows Y$, and let $x_0 \in X$. Then the following two statements are equivalent

- F is compact-valued at x_0, and F is uhc at x_0
- For any sequence (x_n) in X convergent to x_0, any sequence (y_n) s.t. $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, there exists a subsequence (y_{n_k}) convergent to some $y_0 \in F(x_0)$.

Proof: (1) $\Rightarrow$ (2): Take any sequence (x_n) in X convergent to x_0, any sequence (y_n) s.t. $y_n \in F(x_n)$ for each $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, consider the set

$$U_k := \bigcup_{y \in F(x_0)} B_{1/k}(y)$$

By construction, U_k is an open set and $F(x_0) \subset U_k$. Since F is uhc at x_0, there exists $\delta_k > 0$ s.t. $F(B_{\delta_k}(x_0)) \subset U_k$.

Upper Hemi-Continuity

Proof Cont.

Because $x_n \to x_0$, there exists N_k s.t. $x_n \in B_{\delta_k}(x_0)$, and thus $y_n \in U_k$ for any $n > N_k$. Therefore, we can find a subsequence (y_{n_k}) s.t. $y_{n_k} \in U_k$ for each $k \in \mathbb{N}$. By construction of U_k, for each k, there exists $z_k \in F(x_0)$ s.t. $d_Y(y_{n_k}, z_k) < 1/k$. Because F is compact-valued at x_0, we know that $F(x_0)$ is compact in (Y, d_Y). So there exists a subsequence (z_{k_l}) convergent to some $y_0 \in F(x_0)$. So we have $d_Y(z_{k_l}, y_0) \to 0$, and

$$\begin{aligned} 0 \leq d_Y(y_{n_{k_l}}, y_0) &\leq d_Y(y_{n_{k_l}}, z_{k_l}) + d_Y(z_{k_l}, y_0) \\ &< 1/k_l + d_Y(z_{k_l}, y_0) \to 0 + 0 = 0 \end{aligned}$$

Therefore, we have $d_Y(y_{n_{k_l}}, y_0) \to 0$, which means $y_{n_{k_l}} \to y_0$.

(2) $\Rightarrow$ (1): Take any sequence $(y_n) \in F(x_0)$. Let $x_n = x_0$ for all $n \in \mathbb{N}$. Then $x_n \to x_0$ and $y_n \in F(x_n)$ for each $n \in \mathbb{N}$. Then by (2) there exists a subsequence (y_{n_k}) convergent to some $y_0 \in F(x_0)$. So F is compact-valued at x_0.

Upper Hemi-Continuity

Proof Cont.

Now assume F is not uhc at x_0. Then $\exists U$ open in (x, d_x) s.t. $F(x_0) \subset U$, but $\forall \delta > 0$ we have $F(B_\delta(x_0)) \not\subset U$. Then for $n \in \mathbb{N}$, we have $F(B_{1/n}(x_0)) \not\subset U$ i.e. there exists $x_n \in B_{1/n}(x_0)$ and $y_n \in F(x_n)$ s.t. $y_n \notin U$. Because $x_n \to x_0$, by assumption there exists a subsequence (y_{n_k}) convergent to some $y_0 \in F(x_0)$. Because (y_{n_k}) is in $Y \backslash U$, which is closed in (Y, d_Y), we have $y_0 \in Y \backslash U$ and so $y_0 \in F(x_0)$. Contradiction.

Note: Without compactness, uhc alone does not imply (2). For example consider $F_5 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $F_5(x) = (0, 1)$ for any $x \in \mathbb{R}$.

Lower Hemi-Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be **lower hemi-continuous (lhc) at** $x_0 \in X$ if $\forall$ open sets U in (Y, d_Y) s.t. $F(x_0) \cap U \neq \emptyset$, $\exists \delta > 0$ s.t. $F(x) \cap U \neq \emptyset$ for any $x \in B_\delta(x_0)$.

The correspondence $F : X \rightrightarrows Y$ is said to be **lower hemi-continuous (lhc)** if it is lower hemi-continuous at x_0 for all $x_0 \in X$.

Lower Hemi-Continuity: Examples

$F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ is **not** lower hemi-continuous

$$F_2(x) = \begin{cases} \{0\} & x < 0 \\ [-1, 1] & x \geq 0 \end{cases}$$

$F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ is upper lower-continuous

$$F_1(x) = \begin{cases} \{0\} & x \leq 0 \\ [-1, 1] & x > 0 \end{cases}$$

Lower Hemi-Continuity: Another Example

Let's formally show that $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_3(x) = [x, x+1]$$

is lhc for any $x \in \mathbb{R}$.

Take any $x_0 \in \mathbb{R}$ and take any open set U s.t. $U \cap [x_0, x_0 + 1] \neq \emptyset$. We want to show that $\exists \delta > 0$ s.t. $U \cap [x, x+1] \neq \emptyset$ for any $x \in (x_0 - \delta, x_0 + \delta)$. Let $\hat{x} \in [x_0, x_0 + 1] \cap U$. Because U is open, $\exists \delta > 0$ s.t. $(\hat{x} - \delta, \hat{x} + \delta) \subset U$. Take any $x \in (x_0 - \delta, x_0 + \delta)$. By construction, we have $x - x_0 \in (-\delta, \delta)$, and so

$$\hat{x} + (x - x_0) \in (\hat{x} - \delta, \hat{x} + \delta) \subset U$$

Because $\hat{x} \in [x_0, x_0 + 1]$, we have

$$\hat{x} + (x - x_0) \in [x_0 + (x - x_0), x_0 + (x - x_0) + 1] = [x, x+1]$$

Therefore $\hat{x} + (x - x_0) \in [x, x+1] \cap U \neq \emptyset$
What about $F_4 : \mathbb{R} \rightrightarrows \mathbb{R}$ with $F_4(x) = (x, x+1)$?

Lower Hemi-Continuity

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces. A correspondence $F : X \rightrightarrows Y$ is lhc at $x_0 \in X$, *iff* for any $y_0 \in F(x_0)$ and sequence (x_n) in X convergent to x_0, there exists $N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ s.t. the sequence $(y_n)_{n>N}$ converges to y_0.

Proof: $\Rightarrow$: Take any $y_0 \in F(x_0)$ and sequence $(x_n) \in X$ convergent to x_0. We want to show that $\exists N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ s.t. the sequence $(y_n)_{n>N}$ converges to y_0.
For each $k \in \mathbb{N}$, we have $y \in F(x_0) \cap B_{1/k}(y_0)$, and so $F(x_0) \cap B_{1/k}(y_0) \neq \emptyset$. By lhc, $\exists \delta_k > 0$ s.t. for any $x \in B_{\delta_k}(x_0)$, we have $F(x) \cap B_{1/k}(y_0) \neq \emptyset$.
Because $x_n \to x$, $\exists N \in \mathbb{N}$ s.t. $x_n \in B_{\delta_1}(x_0)$ for any $n > N$. For each $n > N$, arbitrarily take

$$y_n \in \bigcap_{k \in \{k' \in \mathbb{N} : x_n \in B_{\delta_{k'}}(x_0)\}} [F(x_n) \cap B_{1/k}(y_0)]$$

This is possible because $F(x_n) \cap B_{1/k}(y_0) \neq \emptyset$ whenever $x_n \in B_{\delta_k}(x_0)$.

Lower Hemi-Continuity

Proof Cont.

Now to show $(y_n)_{n>N}$ converges to y_0, take any $\epsilon > 0$, $\exists K$ s.t. $1/k < \epsilon$ for any $k > K$. Because $x_n \to x_0$, $\exists \hat{N} > N$ s.t. $x_n \in B_{\delta_k}(x_0)$ for any $n > \hat{N}$. Therefore for any $n > \hat{N}$, we have $x_n \in B_{\delta_k}(x_0)$, which implies $y_n \in B_{1/K}(y_0)$, which in turn implies $y_n \in B_\epsilon(y_0)$.

$\Leftarrow$: Suppose by contradiction, $\exists$ open set U in (Y, d_Y) s.t. $F(x_0) \cap U \neq \emptyset$ but $\forall \delta > 0$, $\exists x \in B_\delta(x_0)$ s.t. $F(x) \cap U = \emptyset$. This implies that for any $n \in \mathbb{N}$, $\exists x_n \in B_{1/n}(x_0)$ s.t. $F(x_n) \cap U \neq \emptyset$, i.e. $F(x_n) \subset Y \backslash U$.
By construction, we have $x_n \to x_0$. Take any $y_0 \in F(x_0) \cap U$, then by assumption $\exists N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ s.t. the sequence $(y_n)_{n>N}$ converges to y_0. Because $y_n \in F(x_n) \subset Y \backslash U$ for any $n > N$, and $Y \backslash U$ is closed in (Y, d_Y) since U is open, we have $y_0 \in Y \backslash U$. This contradicts construction of y_0.

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be **continuous** at $x_0 \in X$ if F is both uhc and lhc at x_0.

The correspondence $F : X \rightrightarrows Y$ is said to be **continuous** if it is continuous at x_0 for all $x_0 \in X$.

Overview

Closed Graph Property

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to have **closed graph property (cgp)** at $x_0 \in X$ if $\forall$ sequence (x_n) in X convergent to x_0, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$, we have $y_0 \in F(x_0)$.

The correspondence $F : X \rightrightarrows Y$ is said to have **closed graph property (cgp)** if it has closed graph property at x_0 for all $x_0 \in X$.

Closed Graph Property

Proposition 1

Let (X, d_X) and (Y, d_Y) be metric spaces. If a correspondence $F : X \rightrightarrows Y$ is uhc at $x_0 \in X$, and is closed-valued at x_0, then F has cgp at x_0.

Definition

A correspondence $F : X \rightrightarrows Y$, where (X, d_X) and (Y, d_Y) are metric spaces, is said to be **locally bounded** at x_0 if $\exists \delta > 0$ and a compact set K in (Y, d_Y) s.t. $F(B_\delta(x_0)) \subset K$.

Proposition 2

Let (X, d_X) and (Y, d_Y) be metric spaces. If a correspondence $F : X \rightrightarrows Y$ has cgp at $x_0 \in X$, and F is locally bounded at x_0, then F is uhc at x_0.

Overview

Brouwer's Fixed Point Theorem

Theorem

Let X be a nonempty, compact and convex set in $\mathbb{R}^N$ and consider a continuous function $f : X \to X$. Then there exists $x^* \in X$ s.t. $f(x^*) = x^*$.

Kakutani's Fixed Point Theorem

Definition

Consider a correspondence $F : X \rightrightarrows X$, a point $x^* \in X$ is called a **fixed point** of F if $x^* \in F(x^*)$.

Theorem

Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$. If the correspondence $F : X \rightrightarrows X$ is non-empty valued, compact valued, convex valued, and uhc, then there exists a fixed point $x^* \in X$ of F.

Note: since the codomain X of F is compact in the theorem, compact-valuedness is equivalent to closed-valuedness.

Overview

Berge's Theorem of Maximum

Theorem

Let (X, d_X) and (A, d_A) be metric spaces. Let $f : X \times A \to \mathbb{R}$ be a continuous function w.r.t. the metric $d_{X\times A}$. Let $\alpha_0 \in A$, and suppose that the correspondence $D : A \rightrightarrows X$ is nonempty-valued, compact-valued and continuous at α_0.
Define a correspondence $X^* : A \rightrightarrows X$ as

$$X^*(\alpha) = \arg\max_{x\in X}\{f(x, \alpha) : x \in D(\alpha)\}$$

for any $\alpha \in A$.
Let $\hat{A} = \{\alpha \in A : X^*(\alpha) \neq \emptyset\}$ and define the function $f^* : \hat{A} \to \mathbb{R}$ as

$$f^*(\alpha) = \max_{x\in X}\{f(x, \alpha) : x \in D(\alpha)\}$$

Then X^* is nonempty-valued, compact valued and uhc at α_0, and f^* is continuous at α_0.