Math Review Part II
Problem Set 4: Solutions

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1. Substitute $xz = 3$ into the objective function and set up the Lagrangian

$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = 3 + yz - \lambda(y^2 + z^2 - 1)$$

First Order Conditions:

$$\frac{d\mathcal{L}}{dy} = z - 2\lambda y = 0$$
$$\frac{d\mathcal{L}}{dz} = y - 2\lambda z = 0$$
$$\frac{d\mathcal{L}}{d\lambda} = y^2 + z^2 - 1 = 0$$

The solutions are four (y, z) pairs such that $y = \pm\frac{1}{\sqrt{2}}$ and $z = \pm\frac{1}{\sqrt{2}}$. For a maximum, y and z must have the same sign.

2. *Simon and Blume, Example 18.5 and 19.7*

3. Let's rewrite the problem as

$$\begin{aligned} \max \quad & -(x-a)^2 - (y-b)^2 \\ \text{s.t} \quad & x \leq 1 \\ & y \leq 2 \end{aligned}$$

Setting up the Lagrangian

$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = -(x-a)^2 - (y-b)^2 + \lambda_1(1-x) + \lambda_2(2-y)$$

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First Order Conditions:

$$\frac{d\mathcal{L}}{dx} = -2(x-a) - \lambda_1 = 0$$
$$\frac{d\mathcal{L}}{dy} = -2(y-b) - \lambda_2 = 0$$

Complementary Slackness Conditions:

$$\lambda_1(1-x) = 0 \quad \text{where} \quad \lambda_1 \geq 0, 1-x \geq 0$$
$$\lambda_2(2-y) = 0 \quad \text{where} \quad \lambda_2 \geq 0, 2-y \geq 0$$

Notice that

$$D^2 f(x,y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

which implies that f is concave and so the Kuhn-Tucker theorem will give us the global max.

- **Case A**: $a \leq 1,\ b \leq 2$
  The solution is (a, b).
- **Case B**: $a \leq 1, b > 2$:
  Then the second constraint must bind, while the first is slack.
  $\lambda_1 = 0, y = 2, x = a, \lambda_2 = 2(b-a) \geq 0$. Therefore $(a, 2)$ is the solution.
- **Case C**: $a > 1, b \leq 2$:
  Then the first constraint must bind, while the second is slack.
  $\lambda_2 = 0, x = 1, y = b, \lambda_1 = 2(a-1) \geq 0$. Therefore $(1, b)$ is the solution.
- **Case D**: $a > 1, b > 2$:
  Then both constraints bind.
  $x = a, y = b, \lambda_1 = \lambda_2 = 0$. Therefore (a, b) is the solution.

4. 4.1. Setting up the Lagrangian:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = ln(x+1) + ln(y+1) + \lambda_1(c - x - 2y) + \lambda_2(2 - x - y)$$

First Order Conditions:

$$\frac{d\mathcal{L}}{dx} = \frac{1}{x+1} - \lambda_1 - \lambda_2 = 0$$
$$\frac{d\mathcal{L}}{dy} = \frac{1}{y+1} - 2\lambda_1 - \lambda_2 = 0$$

Complementary Slackness Conditions:

$$\lambda_1(c - x - 2y) = 0 \quad \text{where} \quad \lambda_1 \geq 0, c - x - 2y \geq 0$$
$$\lambda_2(2 - x - y) = 0 \quad \text{where} \quad \lambda_2 \geq 0, 2 - x - y \geq 0$$

4.2. There are four case

- *Case 1*: Both constraints are slack
  $\lambda_1 = \lambda_2 = 0$. This contradicts the FOCs.
- *Case 2*: Only the first constraint binds.
  $\lambda_2 = 0$, then $x + 2y = \frac{5}{2}$. Combining the first two FOCs we get $x + 1 = 2y + 2$. So $x = \frac{7}{4}$, $y = \frac{3}{8}$. But then $x + y = \frac{17}{8} > 2$, contradicting the second constraint.
- *Case 3*: Only the second constraint binds
  $\lambda_1 = 0$, $x + y = 2$. Combining first two FOCs, we get $x = y = 1$. But then $x + 2y = 3 > \frac{5}{2}$, violating the second constraint.
- *Case 4*: Both bind
  $x + 2y = \frac{5}{2}$ and $x + y = 2$, so $x = \frac{3}{2}$ and $y = \frac{1}{2}$. Combing the first two FOCs we get $\lambda_1 = \frac{4}{15} \geq 0$, $\lambda_2 = \frac{2}{15} \geq 0$ and so $(x, y) = (\frac{3}{2}, \frac{1}{2})$ is the solution.

4.3. By the Envelope Theorem $V'(\frac{5}{2}) = \frac{d}{dc}\mathcal{L}(x^*, y^*, \lambda_1^*, \lambda_2^*, c) = \lambda_1^* = \frac{4}{15}$