CSE643: Artificial Intelligence
Assignment-3

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Theory

Problem-1

(i)

We identify the following random variables in the given sentences

1. T: a boolean RV indicating if the person travelled or not
2. D: an RV indicating the type of disease the person caught (CORONA, OTHER, NONE)
3. S: an RV indicating the severity of the disease (MILD, SEVERE, NONE)
4. L: a boolean RV indicating if the person died

Using probability notation, the given sentences can be written as follows

(a) $\mathbb{P}[T = \text{TRUE} \wedge D \neq \text{NONE}] = 0.825$

(b) $\mathbb{P}[D = \text{CORONA} \wedge S = \text{MILD} \mid T = \text{TRUE}] = 0.15$
$\mathbb{P}[D = \text{CORONA} \wedge S = \text{SEVERE} \mid T = \text{TRUE}] = 0.22$

(c) $\mathbb{P}[D = \text{OTHER} \mid T = \text{TRUE}] = 0.485$

(d) $\mathbb{P}[L = \text{TRUE} \wedge D = \text{OTHER} \mid T = \text{TRUE}] = 0.24$

(e) $\mathbb{P}[T = \text{FALSE} \wedge D = \text{CORONA} \wedge S = \text{SEVERE}] = 0.025$

(f) $\mathbb{P}[S = \text{SEVERE} \mid T = \text{FALSE}] = 0.457$

(g) $\mathbb{P}[L = \text{TRUE} \wedge D = \text{CORONA}] = 0.059$

(h) $\mathbb{P}[S \neq \text{NONE}] = 0.7$

(i) $\mathbb{P}[T = \text{TRUE} \mid S = \text{SEVERE}] = 0.8$

(j) $\mathbb{P}[D = \text{CORONA} \mid T = \text{TRUE}] = \mathbb{P}[D = \text{CORONA} \mid T = \text{FALSE}] = 0.5$

(ii)

Now, we verify that the above system satisfy probability axioms and from a valid distribution. We can infer from the above that

1. $\mathbb{P}[D = \text{NONE} \mid T = \text{TRUE}] = 1 - 0.5 - 0.485 = 0.015$
2. $\mathbb{P}[D = \text{CORONA} \wedge S = \text{NONE} \mid T = \text{TRUE}] = 0.5 - 0.15 - 0.22 = 0.13$

It is hence clear that the given problem forms a valid distribution. So, it follows the following conditional probability axioms

1. All conditional probabilities are non-negative and at most 1, i.e.
$$0 \leq \mathbb{P}[e \mid B] \leq 1 \quad \forall\, e \in B \subseteq S$$
2. The sum of all conditional probabilities given condition is 1, i.e.
$$\sum_{e \in B} \mathbb{P}[e \mid B] = 1 \quad \forall\, B \subseteq S$$
3. Bayes' Theorem, which states
$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}, \quad \forall A, B \subseteq S$$

(iii)

The joint probability distribution table is given in table 1.

T	D	S	L	$\mathbb{P}[T = t \wedge D = d \wedge S = s \wedge L = l]$

Table 1: Joint Probability Distribution Table

(iv)

It is clear that all variables are conditionally dependent on each other.

Problem-3

We know apriori that the key is behind one of the three doors. If we choose a door that does not have the key, we lose a life.

(a)

Apriori, with no other information, the probability that the key is behind any door is $\frac{1}{3}$. Let K be the random variable indicating the door behind which the key is. Then,

$$\mathbb{P}[K = i] = \frac{1}{3} \implies \mathbb{P}[K \neq i] = 1 - \frac{1}{3} = \frac{2}{3} \quad \forall\, i \in \{1, 2, 3\} \tag{1}$$

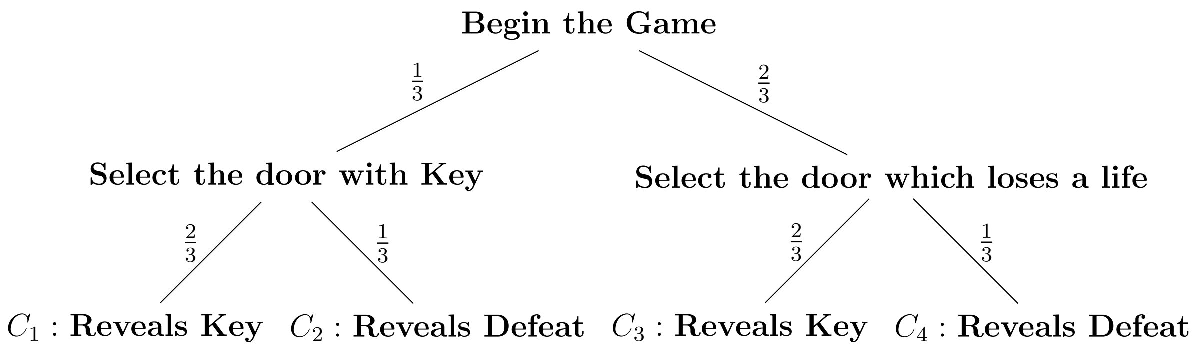
Without loss of generality, let us assume that we picked door 1, and the adversary who knows where the key is, opens door 2. There are clearly only three possible cases, which are listed in table 2. Clearly, the probability of winning the key is $\frac{1}{3}$ if we stick to our choice and $\frac{2}{3}$ if we switch. Hence, **the optimal strategy is to switch**.
Intuitively, switching is more likely to land on a door with the key because you are more likely to pick an empty door in the first place. So, when the adversary reveals another empty door, it is *more* likely that the key is behind the third door.

Random-variable	Stick	Switch
$K = 1$	`Win the key`	`Lose a life`
$K = 2$	`Lose a life`	`Win the key`
$K = 3$	`Lose a life`	`Win the key`

Table 2: Possible cases in the escape room puzzle

(b)

Now, we are given that the adversary can sometimes makes a mistake. So, we get the following sequence of events.



where "Defeat" means the loss of a life. We can now calculate the probability of winning the key if we stick to our choice and if we switch. For this, we assume that no matter what, the adversary will always follow the given probabilities to select a door to open (i.e., the man might open the door we pick in case it has the key). We now look at each case

1. **Case-1**: We picked the door with the key and the adversary reveals the key. In this case, we win the key if we stick to our choice.
2. **Case-2**: We picked the door with the key and the adversary reveals defeat. In this case, we should ideally stick to our choice.
3. **Case-3**: We picked the door without the key and the adversary reveals the key. In this case, we lose anyway. So, let's assume switching is better.
4. **Case-4**: We picked the door without the key and the adversary reveals defeat. In this case, we should switch, since this is equivalent to the original problem.

So, we get the following table of probabilities.

Case	Probability	Stick	Switch
C_1	$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$	`Win the key`	`Lose a life`
C_2	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$	`Win the key`	`Lose a life`
C_3	$\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$	`Lose a life`	`Lost the game`
C_4	$\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$	`Lose a life`	`Win the key`

Table 3: Possible cases in the escape room puzzle with mistakes

So, the probability of winning the key if we stick to our choice is $\frac{3}{9}$, and the probability of winning the key if we switch[1] is $\frac{2}{9}$. Hence, **the optimal strategy is to stick**.

(c)

Here, we choose to swtich. We need to find the conditional probability of winning the key given that the man/adversary has revealed the door that shows the loss of a life. Let the random variable W indicate if we selected the winning door or not. Then, we are interested in the probability

$$\begin{aligned}
\mathbb{P}[W = \text{TRUE} \mid C_2 \vee C_4] &= \frac{\mathbb{P}[W = \text{TRUE} \wedge (C_2 \vee C_4)]}{\mathbb{P}[C_2 \vee C_4]} && (2)\\
&= \frac{\mathbb{P}[W = \text{TRUE} \wedge C_4]}{\mathbb{P}[C_2] + \mathbb{P}[C_4]} && (3)\\
&= \frac{\mathbb{P}[W = \text{TRUE}] \cdot \mathbb{P}[C_4]}{\mathbb{P}[C_2] + \mathbb{P}[C_4]} && (4)\\
&= \frac{\frac{1}{3} \cdot \left(\frac{2}{3} \cdot \frac{1}{3}\right)}{\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3}} = \frac{2}{9} && (5)
\end{aligned}$$

Where the second equality follows because if we win by switching, then C_2 is impossible, and the third equality follows by the independence of the player picking a door and the person picking the door to reveal.

(d)

Let us assume that the reward R for winning the key is 1 and losing a life is -1. Then, the expected reward is

$$\begin{aligned}
\mathbb{E}[R \mid \texttt{Switch}] &= \mathbb{P}[W = \text{TRUE} \mid C_2 \vee C_4] \cdot 1 + \mathbb{P}[W = \text{FALSE} \mid C_2 \vee C_4] \cdot (-1) && (6)\\
&= \frac{2}{9} \cdot 1 + \frac{7}{9} \cdot (-1) = -\frac{5}{9} \approx -0.555 && (7)
\end{aligned}$$

We now find the probability of winning if we choose to stick.

$$\begin{aligned}
\mathbb{P}[W \mid \texttt{Stick}] &= \mathbb{P}[W = \text{TRUE} \wedge (C_1 \vee C_2)] && (8)\\
&= \mathbb{P}[W = \text{TRUE} \wedge C_1] + \mathbb{P}[W = \text{TRUE} \wedge C_2] && (9)\\
&= \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3} && (10)
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathbb{E}[R \mid \texttt{Stick}] &= \mathbb{P}[W = \text{TRUE} \mid C_1 \vee C_2] \cdot 1 + \mathbb{P}[W = \text{FALSE} \mid C_1 \vee C_2] \cdot (-1) && (11)\\
&= \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (-1) = -\frac{1}{3} \approx -0.333 && (12)
\end{aligned}$$

So, the expected reward for switching is -0.555, and for sticking is -0.333. Hence, **the optimal strategy is to stick**.

[1] Here, we assume that we cannot switch to the open door, which contains the key. Hence, on switching, the chances of losing increase. Case C_3 happens with probability $\frac{4}{9}$, which is a destined loss.

Computational

The solution to the computational part of the assignment is given in the code in `main.ipynb` in the submission. The code uses the `bnlearn` package in Python to implement Bayesian Networks. The constructed Bayesian Network is used in prediction of the target variable in the Wine Quality Dataset provided on the UCI-ML Repo.

References

1. BNLearn Official Documentation