Mid-Semester Exam: Linear Algebra

Indraprastha Institute of Information Technology, Delhi

19th February, 2022

Duration: 60 minutes **Maximum Marks:** 50

Question 1.

(a) (5 marks) Let A and B be $m \times n$ matrices (where $m, n \in \mathbb{N}$) having columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{b}_1, \ldots, \mathbf{b}_n$, respectively. Suppose $\mathbf{b}_j = j^2 \mathbf{a}_{j-1}$ for $j = 2, \ldots, n$ and $\mathbf{b}_1 = \mathbf{a}_n$. Find an $n \times n$ matrix E such that $AE = B$.

(b) (5 marks) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \\ 0 & 9 & 0 \end{bmatrix}$. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$. Solve the equations $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ by row reducing exactly one matrix.

Question 2. (10 marks) All subparts of this question carry equal marks.

In each part of this question, V is a vector space and W is a subset of V. Decide whether W is a subspace of V. Justify your answers with a short proof or counterexample.

(a) $V = \mathbb{R}(t)$, the set of *all* polynomials in t which have real coefficients (please note that the degrees of the polynomials are not bounded).
$W = \{p(t) = a_0 + \cdots + a_n t^n \mid a_{2k} = 0, \text{ if } k \in \mathbb{N} \text{ and } 2k \in \{0, \ldots, n\}\}$

(b) $V = \mathbb{R}^n$
$W = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n \geq 0\}$

(c) $V = \mathbb{R}^n$
$W = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \geq 0\}$

(d) $V = \mathbb{R}^\infty$, the set of all sequences indexed by $\mathbb{N}$
Fix $k \in \mathbb{N}$.
$W = \{(a_n) \mid a_1 + \cdots + a_k = 0\}$

(e) $V = \mathbb{R}^{n \times n}$, the set of all $n \times n$ matrices having real entries.
$(Note : \mathbb{R}^{n \times n}$ is the same as $M_n(\mathbb{R}))$
$W = \{A \mid A \text{ is in reduced row echelon form}\}$

Question 3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \geq 0,\ a + c = b + d = 1$ and $A \neq I_2$. Let $P = \begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix}$.

(a) (7 marks) Show that P is invertible. Find P^{-1} and $P^{-1}AP$.

(b) (3 marks) Find a formula for A^n.

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) (4 marks) Show that: If $\exists\ c \in \mathbb{R} \setminus \{0\}$ such that $f(x) = cx,\ \forall x \in \mathbb{R}$, then the graph of f is a proper nontrivial subspace of $\mathbb{R}^2$.
(The graph of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as the set $\{(x, y) \mid y = f(x)\}$.)

(b) (1 mark) What is the converse of the statement that is too be proved in part (a)?

(c) (5 marks) Is the converse that you stated in part (b) true? Justify with a short proof or an appropriate counterexample.

($Note$: You may assume the following statement without proof -
Any proper nontrivial subspace of $\mathbb{R}^2$ is of the form Span$\{\vec{v}\}$ where $\vec{v}$ is a non-zero vector in $\mathbb{R}^2$.)

Question 5. (10 marks) Solve ONE of the following two problems (either part (a) or part (b)).

(a) Prove or disprove:
If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for a vector spave V, then so is $\{\alpha_1\vec{v}_1, \dots, \alpha_n\vec{v}_n\}$ where the α_i are non-zero scalars.

(b) Prove or disprove:
If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for a vector spave V, then so is $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \dots, \vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_n\}$.