Assignment 2

Randomized Algorithms Winter' 24

Due: Wed 10th April, by 11:55pm

Assignments are to be done individually. Submit a single pdf file to the classroom.

Problem 1 (2 points).
In the Morris counter, suppose we modify the step that increments the counter by $1/2^X$ instead by incrementing X with probability $1/(1+c)^X$ for some constant $c > 1$. This will change the estimator from $2^X - 1$ to something else. How small should c be so that our estimate $\tilde{n}$ satisfies $|\tilde{n} - n| \leq \epsilon n$ with at least 0.9 probability when we return the output of a single estimator instead of averaging the value of many estimators? Also derive a bound for $S(n)$, the space required by the algorithm, so that the algorithm uses $S(n)$ bits with at least 0.9 probability.

Problem 2 (2 points).
Consider a collection $X_1, ..., X_n$ of n independent integers chosen uniformly from the set $\{0, 1, 2\}$. Let $X = \sum_{i=1}^{n} X_i$, and $0 < \delta < 1$. Derive a Chernoff bound for $\mathbb{P}[X \geq (1+\delta)n]$ and $\mathbb{P}[X \leq (1-\delta)n]$.

Problem 3 (2 points).
Let A be a set of unknown size. Let h be a hash function chosen from a pairwise independent family of hash functions to a hash table of size m. For a parameter $t < m$, suppose we sample n elements from A uniformly at random. What is the expected number of elements in the sample hashing to a value at most t? Can we use this to estimate $|A|$? Using this technique, given two sets A and B, determine how we can estimate $|A \cup B|, |A \cap B|, |A \setminus B| \cup |B \setminus A|$.

Problem 4 (2 points).
Give an example of a hash function that is universal, but not strongly universal.

Problem 5. We plan to conduct an opinion poll to find out the percentage of people in a community who want its president impeached. Assume that every person answers either yes or no. If the actual fraction of people who want the president impeached is p, we want to find an estimate X of p such that $\mathbb{P}[|X - p| \leq \epsilon p] > 1 - \delta$ for given $0 < \epsilon, \delta < 1$.