Modern Algorithm Design (Monsoon 2024)
Homework 3

Deadline: 23rd November, 2024, 11 pm (IST)

Release Date: 12th November, 2024

1. **Yet Another Randomized Quicksort.** Given an array A of n numbers (which we assume are distinct for simplicity), the algorithm picks a pivot x uniformly at random from A and computes the rank of x. If the rank of x is between $n/4$ and $3n/4$ (call such a pivot a good pivot), it behaves like the normal QuickSort in partitioning the array A and recursing on both sides. If the rank of x does not satisfy the desired property (the pivot picked is not good), the algorithm simply repeats the process of picking a pivot until it finds a good one. Note that in principle the algorithm may never terminate!

   (a) Prove that the expected runtime of the above algorithm is $\mathcal{O}(n \log n)$ in expectation.

   (b) Prove that the runtime of the above algorithm is $\mathcal{O}(n \log n)$ with probability at least $1 - 1/n$

2. **Nearly Orthonormal Vectors.** We call a set of unit vectors "near-orthonormal" if the inner product of any two of them is close to zero. In this problem we will show that while there are at most d orthonormal vectors in $\mathbb{R}^d$, there can be exponentially more near-orthonormal vectors. For vectors $x, y \in \mathbb{R}^d$, we use $\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i$ to denote the inner product.

   (a) Let $x = (x_1, x_2, \cdots, x_d)$ and $y = (y_1, y_2, \cdots, y_d)$ be two independently and uniformly chosen vectors in $\{-1, 1\}^d$. (I.e., each bit x_i and y_i in each vector is independently and uniformly chosen from $\{-1, 1\}$.) Show that

   $$\mathbb{P}r[|\langle x, y \rangle| \geq \varepsilon d] \leq 2 \exp(-\varepsilon^2 d/6)$$

   (b) Given any constant $\varepsilon > 0$, a set S of unit vectors is called ε-*orthonormal* if for all $\vec{x}, \vec{y} \in S$,

   $$|\langle \vec{x}, \vec{y} \rangle| \leq \varepsilon.$$

   Show that there exist constants $c, d_0 > 0$ (possibly depending on ε) such that for any $d \geq d_0$, if you sample $N := exp(c\varepsilon^2 d)$ random vectors independently and uniformly from the set $\left\{-\frac{1}{\sqrt{d}}, +\frac{1}{\sqrt{d}}\right\}^d$, this sampled set is ε-orthonormal with probability at least $1/2$.