Algorithms Under Uncertainty : Homework 3
Full Marks : 20
Due : 15/11/2023

Solutions must be done in teams of at most two. Clearly mention the name of your partner at the top. Also, both the partners are required to submit their individual solution copies on Google Classroom. All solutions to be written in latex. No other format is acceptable. You are free to consult online resources, but must mention them. If found hiding your sources, it will be considered an act of plagiarism.

Problem 1. (10 points) Consider the problem of online matching where one side L is fixed and vertices from the other side R arrive on-line. The bipartite graph will be generated randomly. Initialize a set S_1 to be the set of all vertices in L. When the first vertex v_1 in R arrives, it will have edges to all the vertices in S_1. Next, we randomly remove a vertex from S_1 to get a set S_2. The next vertex v_2 will have edges to all the vertices in S_2, and so on. Thus, if S_i is the current set (after arrival of $v_1, \ldots, v_i$), we randomly remove a vertex from S_i to get S_{i+1}. The next arriving vertex v_{i+1} will have edges to all the vertices in S_{i+1}. This process goes on for n steps.

a) Prove that the off-line optimum can match all the vertices.

b) Let $\mathcal{A}$ be any deterministic algorithm for online matching. Suppose it has the property that when a vertex v_i arrives, and at least one neighbour of v_i is unmatched, then it will match v_i to some vertex. Order the vertices in L as $w_1, w_2, \ldots, w_n$, where w_1 is the vertex in $S_1 \setminus S_2$, w_2 is the vertex in $S_2 \setminus S_3$ and so on (note that w_i a random vertex). Show that the probability that $\mathcal{A}$ matches w_i is at most

$$\frac{1}{n} + \frac{1}{n-1} + \ldots \frac{1}{n-i+1}.$$

c) Show that the expected competitive ratio of any online matching algorithm is at most $(1 - 1/e)$.

Problem 2. (10 points) Consider the problem of learning from experts where there are n experts, and at each time t, each expert i incurs a loss ‘ $\ell_t^i \in [0, 1]$. There is an additional complication now: each time the algorithm switches from one expert to another, it incurs an additional cost of 1. In other words, define a variable I_t which is 1 if the expert chosen by the algorithm at time t is different from the expert chosen by the algorithm at time $t + 1$; this variable is 0 if the two experts are the same. Let e_t be the expert chosen by the algorithm at time t. Then the total loss of the algorithm is

$$\sum_{t=1}^{T} \ell_{e_t}^t + \sum_{t=1}^{T} I_t$$

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and the regret is again defined as the total loss minus the loss incurred by the best expert. Design a randomized algorithm for which the expected regret grows sublinearly with T, i.e., if $R(T)$ is the expected regret of the algorithm, the the ratio $R(T)/T$ goes to 0 as T goes to infinity. (Hint: You may want to divide time into blocks of certain length such that in each block the algorithm does not change expert.)

Problem 3. (10 points) We consider Online Linear Regression as introduced in the lecture. Recall that

$$f_t(w_1, w_2) = (w_1 x^{(t)} + w_2 - y^{(t)})^2$$

Derive a regret bound for Follow-the-Regularized-Leader with Euclidean regularization under the assumption that $|x^{(t)}|, |y^{(t)}| \leq 1$ for all t and $S = \{\mathbf{w} \in \mathbb{R}^2 : ||\mathbf{w}||_2 \leq r\}$