Linear Algebra Proofs Assignment

Problem taken from *Linear Algebra Done Right, 3rd edition*

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1 Theorems and Definitions

Theorem 1. *A subspace S of vector space V is closed under vector addition, i.e.:*

$$\forall x_1, x_2 \in S,\ x_1 + x_2 \in S \tag{1}$$

Theorem 2. *A subspace S of vector space V is closed under scalar multiplication, i.e.:*

$$\forall x \in S,\ c \in \mathbb{R},\ cx \in S \tag{2}$$

Corollary 1 (From **Theorem 1** and **Theorem 2**)**.** *A subspace S of vector space V is closed under the operation of taking linear combinations, i.e.:*

$$\forall x_1, x_2, \ldots, x_n \in S,\ c_1, c_2, \ldots, c_n \in \mathbb{R},\ c_1x_1 + c_2x_2 + \cdots + c_nx_n \in S \tag{3}$$

Definition 1 (Set-Minus)**.** *The Set-Minus operator is used to denote the difference of two sets.*

$$A \setminus B = \{x :\ x \in A,\ x \notin B\} \tag{4}$$

2 Problem Statement and Solution

Problem 1. *Prove that the union of two subspaces of a vector space is a subspace iff one of the two subspaces is contained in the other.*

Proof. Given: A and B are two subspaces of V.

($\Longleftarrow$)

Suppose $A \subseteq B$. Then $A \cup B = B$. B is already a subspace of V.

Similarly, $B \subseteq A \implies A \cup B = A$. A is already a subspace of V.

$\therefore$ In either case, $A \cup B$ is a subspace of V.

($\Longrightarrow$)

Assume: $A \nsubseteq B$ and $B \nsubseteq A$. Then,

(a) $\exists\, x_1 \in A \setminus B$, i.e., by (4), $x_1 \in A$ and $x_1 \notin B$.

(b) $\exists\, x_2 \in B \setminus A$, i.e., by (4), $x_2 \in B$ and $x_2 \notin A$.

Let $C = A \cup B$. Then, $x_1 \in C$ and $x_2 \in C$. Since C is a subspace of V, it is closed under vector addition by (1). Then:

$$x_1 + x_2 \in C, \text{ i.e., } x_1 + x_2 \in A \cup B \tag{5}$$

Then, either $x_1 + x_2 \in A$ or $x_1 + x_2 \in B$.

(a) Suppose $x_1 + x_2 \in A$. Then we write $x_2 = (x_1 + x_2) - x_1$, i.e. x_2 is a linear combination of elements of A. Hence, by (3), $x_2 \in A$. This contradicts our assumption that $x_2 \notin A$.

(b) Next, suppose $x_1 + x_2 \in B$. Then we write $x_1 = (x_1 + x_2) - x_2$, i.e. x_1 is a linear combination of elements of B. Hence, by (3), $x_1 \in B$. This contradicts our assumption that $x_1 \notin B$.

$\therefore$ In either case, we reach a contradiction, and thus, our assumption must be incorrect.

Therefore, it must be the case that either $A \subseteq B$ or $B \subseteq A$, i.e., one of the two subspaces must be contained in the other.

$\square$