ECO311: Game Theory

Take-Home Quiz Solutions

Divyajeet Singh (2021529) **Anoushka Kumar (2021133)**

Solution 1.

Given a set of 3 indivisible objects $S = \{A, B, C\}$.

Part (a)

We first model the given scenario as an extensive-form game $\Gamma = (N, (V, E, r), \{V_i\}_{i \in N}, \{A(x)\}_{x \in V}, \{U_i\}_{i \in N})$. The game tree is given in Figure 1.
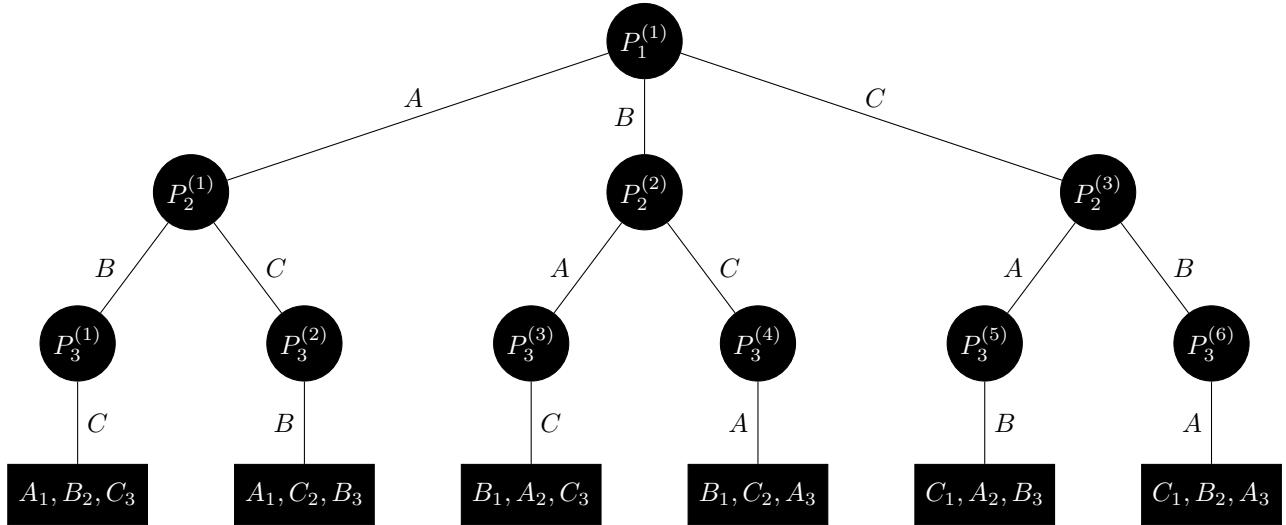

Figure 1: Game tree for the given scenario.

where s_i refers to the value of object $s \in S$ to player i. For example, B_3 refers to the value of object B to player 3. We have the set of players $N = \{1, 2, 3\}$. Vertices $P_i^{(j)}$ lie in the set V_i for each player $i \in N$. Let $v_i(s)$ denote the value of object $s \in S$ to player $i \in N$. Then, the utility U_i for player i is given by

$$U_i(a_1, a_2, a_3) = v_i(a_i) \tag{1}$$

where (a_1, a_2, a_3) is a leaf node in the game tree. From the game tree, we can find the set of strategies for each player. These are

$$
\begin{aligned}
S_1 &= \{A, B, C\} \\
S_2 &= \{BAA, BAB, BCA, BCB, CAA, CAB, CCA, CCB\} \\
S_3 &= \{CCB, CCA, CAB, CAA, BCB, BCA, BAB, BAA\}
\end{aligned}
\tag{2}
$$

We know that any extensive-form game can be represented as a strategic-form (matrix-form) game. The strategic form of the game is given by the set of strategies for each player and the payoff matrix. Using the above information, we can represent the game in strategic form by creating a $3 \times 8 \times 8$ payoff matrix. To represent the utilities, we can draw out 3 tables of 8×8 each, one for each possible choice of object by player 1.

However, we make a key observation - when player 1 and player 2 have made their choices, player 3 is forced to

select the last remaining object. This means that given any strategy of player 1 and player 2, the payoff of player 3 is fixed, i.e. the strategy of player 3 is a function of the actions taken by player 1 and player 2. So, this is effectively a 2-player game between player 1 and player 2, based on whose choices, payoffs are generated for 3 players. Thus, we can simplify the payoff matrix as shown in Table 1.

	BAA	BAB	BCA	BCB	CAA	CAB	CCA	CCB
A	A_1, B_2, C_3	A_1, B_2, C_3	A_1, B_2, C_3	A_1, B_2, C_3	A_1, C_2, B_3	A_1, C_2, B_3	A_1, C_2, B_3	A_1, C_2, B_3
B	B_1, A_2, C_3	B_1, A_2, C_3	B_1, C_2, A_3	B_1, C_2, A_3	B_1, A_2, C_3	B_1, A_2, C_3	B_1, C_2, A_3	B_1, C_2, A_3
C	C_1, A_2, B_3	C_1, B_2, A_3	C_1, A_2, B_3	C_1, B_2, A_3	C_1, A_2, B_3	C_1, B_2, A_3	C_1, A_2, B_3	C_1, B_2, A_3

Table 1: Strategies of player 1 vs strategies of player 2 (payoff of player 3 gets fixed).

where each entry is of the form (X_1, Y_2, Z_3), denoting the payoff of player 1, player 2, and player 3 respectively on the given strategy profile. Thus, the above payoff matrix is the strategic-form of the game.

Part (b)

Yes, the game has a pure-strategy Nash equilibrium. Define s_1^*, s_2^*, and s_3^* as follows.

$$s_1^* = \operatorname*{argmax}_{s \in S} v_1(s) \qquad s_2^* = \operatorname*{argmax}_{s \in S \setminus \{s_1^*\}} v_2(s) \qquad s_3^* \in S \setminus \{s_1^*, s_2^*\} \tag{3}$$

Then, the strategy profile (s_1^*, s_2^*, s_3^*) is a Nash equilibrium. This is because player 1 gets the object of their highest preference (i.e. cannot obtain a higher payoff). Player 2 gets the object of their highest preference among the remaining objects. Player 3 gets the last object, which is the only one available. Since the payoff of each object is strictly positive, player 3 still receives a positive payoff. Thus, no player can benefit by unilateral deviation.

Deriving the equilibrium strategy profile using backward induction

We can also derive the same strategy profile as the Nash equilibrium using backward induction as follows. Nodes P_3^j effectively make no choice and their corresponding payoffs move up the tree. So, the action taken by player 3 can be written as

$$s_3^* \in S \setminus \{s_1^*, s_2^*\} = \begin{cases} C & \text{if } \{s_1^*, s_2^*\} = \{A, B\} \\ B & \text{if } \{s_1^*, s_2^*\} = \{A, C\} \\ A & \text{if } \{s_1^*, s_2^*\} = \{B, C\} \end{cases} \tag{4}$$

where s_1^* and s_2^* are the actions of player 1 and player 2 respectively. $P_2^{(1)}$ chooses between the payoffs (A_1, B_2, C_3) and (A_1, C_2, B_3), and chooses an action based on whether $B_2 > C_2$ or vice-versa. Similarly, the choices at nodes $P_2^{(2)}$ and $P_2^{(3)}$ is governed by whether $A_2 > C_2$ and $A_2 > B_2$ respectively. Mathematically, this choice can be written as

$$s_2^* = \operatorname*{argmax}_{s \in S \setminus \{s_1^*\}} v_2(s) = \begin{cases} \begin{cases} B & \text{if } v_2(B) > v_2(C) \\ C & \text{otherwise} \end{cases} & \text{if } s_1^* = A \\ \begin{cases} A & \text{if } v_2(A) > v_2(C) \\ C & \text{otherwise} \end{cases} & \text{if } s_1^* = B \\ \begin{cases} A & \text{if } v_2(A) > v_2(B) \\ B & \text{otherwise} \end{cases} & \text{otherwise} \end{cases} \tag{5}$$

Finally, player 1 can choose the object of their highest preference, since $P_2^{(1)}$, $P_2^{(2)}$, and $P_2^{(3)}$ forward a payoff of the form (A_1, s_2, s_3), (B_1, s_2, s_3), and (C_1, s_2, s_3) to $P_1^{(1)}$ respectively. Thus, the action taken by player 1 is

$$s_1^* = \operatorname*{argmax}_{s \in S} v_1(s) = \begin{cases} A & \text{if } v_1(A) > v_1(B) \text{ and } v_1(A) > v_1(C) \\ B & \text{if } v_1(B) > v_1(A) \text{ and } v_1(B) > v_1(C) \\ C & \text{otherwise} \end{cases} \tag{6}$$

Thus, the Nash equilibrium strategy profile obtained by backward induction is the same, i.e. (s_1^*, s_2^*, s_3^*).

Part (c)

Now, after the game, each player is asked to state whether they are happy with their allocation or want to exchange it with another player. If any player says that they want to exchange, then every player receives 0 payoff.
We make a few standard assumptions:

- **Principle of Individual Rationality:** The players are perfectly rational, and aim to maximize their payoff.

- **Principle of Common Knowledge:** The players know that all other players are rational.

- Players are aware that asking for an exchange results in a 0 payoff for them (and all other players).

Then, we can assume that being '*happy*' simply means whether a player has a non-zero payoff. Each object yields a strictly positive payoff to each player. So, even if a player receives an object not of their highest preference, they are better off 'lying' that they are *happy* with the allocation, since asking for an exchange would result in a 0 payoff. Thus, all (rational) players say that they are *happy* with the allocation (even if they are not!). Consequently, the Nash equilibrium (s_1^*, s_2^*, s_3^*) remains unchanged.

Solution 2.

Let us denote the set of players by $N = \{B, S\}$, where B is the buyer and S is the seller. B believes that the car is of high quality (H) with probability 0.4 and of low quality (L) with probability 0.6. Let p be the price at which the car is sold. Table 2 and Table **??** show the payoffs of the buyer and seller for the low-quality and high-quality car respectively.

	$p = 0$	$p = 1$
B	0	1
S	1	0

Table 2: Payoff matrix for the low-quality car.

Solution 3.

Given that in a zero-sum game, (σ_1, σ_2) and (σ_1', σ_2') are two mixed-strategy nash equilibria. Let A_1 and A_2 be the action sets of players P_1 and P_2 respectively. Then, let $U_1(\alpha, \beta) = U(\alpha, \beta)$ denote the expected utility of P_1 when P_1 plays mixed strategy $\alpha \in \Delta A_1$ and P_2 plays mixed strategy $\beta \in \Delta A_2$. Consequently, $U_2(\alpha, \beta) = -U(\alpha, \beta)$ is the expected utility of P_2 on the same strategy profile, since the game is zero-sum.
Let (ρ_1, ρ_2) be a mixed-strategy nash equilibrium. Then,

$$U_1(\rho_1, \rho_2) \geq U_1(\alpha, \rho_2) \implies U(\alpha, \rho_2) \leq U(\rho_1, \rho_2) \tag{7}$$

holds for any $\alpha \in \Delta A_1$. Similarly, for any $\beta \in \Delta A_2$,

$$U_2(\rho_1, \rho_2) \geq U_2(\rho_1, \beta) \implies -U(\rho_1, \rho_2) \geq -U(\rho_1, \beta) \implies U(\rho_1, \rho_2) \leq U(\rho_1, \beta) \tag{8}$$

Therefore, for any MSNE (ρ_1, ρ_2), we have, for every $\alpha \in \Delta A_1$ and $\beta \in \Delta A_2$,

$$U(\alpha, \rho_2) \leq U(\rho_1, \rho_2) \leq U(\rho_1, \beta) \tag{9}$$

Since (σ_1, σ_2) and (σ_1', σ_2') are two mixed-strategy nash equilibria, we have, for every $\alpha \in \Delta A_1$ and $\beta \in \Delta A_2$,

$$U(\alpha, \sigma_2) \leq U(\sigma_1, \sigma_2) \leq U(\sigma_1, \beta) \tag{10}$$
$$U(\alpha, \sigma_2') \leq U(\sigma_1', \sigma_2') \leq U(\sigma_1', \beta) \tag{11}$$

Substitute $\alpha = \sigma_1'$ and $\beta = \sigma_2'$ in (10) and $\alpha = \sigma_1$ and $\beta = \sigma_2$ in (11). This yields

$$U(\sigma_1', \sigma_2) \leq U(\sigma_1, \sigma_2) \leq U(\sigma_1, \sigma_2') \leq U(\sigma_1', \sigma_2') \leq U(\sigma_1', \sigma_2) \tag{12}$$

Since the left-hand and right-hand side of the inequality are equal, the inequalities appear as strict equalities. Thus, we have

$$U(\sigma_1, \sigma_2) = U(\sigma_1, \sigma_2') = U(\sigma_1', \sigma_2) = U(\sigma_1', \sigma_2') \tag{13}$$

Since all MSNE have the same expected utility in a zero-sum game, and we have shown the expected utility of the strategy profiles (σ_1, σ_2') and (σ_1', σ_2) to be equal to the expected utility of the MSNE, we are done. In other words,

$$U(\alpha, \sigma_2') \leq U(\sigma_1', \sigma_2') = U(\sigma_1, \sigma_2') \leq U(\sigma_1, \beta) \tag{14}$$
$$U(\alpha, \sigma_2) \leq U(\sigma_1, \sigma_2) = U(\sigma_1', \sigma_2) \leq U(\sigma_1', \beta) \tag{15}$$

holds for every $\alpha \in \Delta A_1$ and $\beta \in \Delta A_2$, which are the conditions for MSNE.