Bounce Cosmology from a Compact Internal Degree of Freedom

A standalone derivation for the X–θ framework

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Abstract

We show how a nonsingular cosmological bounce arises once a compact internal angle $\theta \in S^1$—the hallmark of the X–θ framework—and its mixed-curvature backreaction are included in FRW dynamics. The θ-sector by itself acts as stiff matter ($w = 1$), which is not enough to bounce flat FRW under standard GR. However, the cross-coupling encoded by the mixed curvature $F_{i\theta}$ produces an *effective negative* a^{-6} term in the Friedmann constraint, acting as a centrifugal barrier in the scale-factor dynamics. We derive the bounce conditions, obtain an explicit expression for the minimum scale factor $a_{\min}$, and give a simple numerical recipe that reproduces the simulated bounce reported elsewhere.

1 Setup: FRW + a compact rotor

We take a homogeneous, isotropic FRW metric (spatial curvature $k \in \{0, \pm 1\}$):

$$ds^2 = -dt^2 + a(t)^2 \, \gamma_{ij} \, dx^i dx^j, \qquad \gamma_{ij} \, dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2, \tag{1.1}$$

with scale factor $a(t)$ and Hubble parameter $H \equiv \dot{a}/a$. In the X–θ picture, every particle carries an internal compact angle $\theta \in S^1$. At the coarse-grained level, the θ-sector behaves as a homogeneous "rotor field" with Lagrangian density

$$\mathcal{L}_\theta = \frac{I}{2} g^{\mu\nu} \, \partial_\mu \theta \, \partial_\nu \theta, \qquad I > 0, \tag{1.2}$$

so for a homogeneous configuration $\theta(t)$,

$$\mathcal{L}_\theta = \frac{I}{2} \, \dot{\theta}^2. \tag{1.3}$$

The conjugate momentum and the effective fluid variables are

$$\Pi_\theta \equiv \frac{\partial (a^3 \mathcal{L}_\theta)}{\partial \dot{\theta}} = I a^3 \dot{\theta}, \qquad \rho_\theta = \frac{I}{2} \dot{\theta}^2 = \frac{\Pi_\theta^2}{2 I a^6}, \qquad p_\theta = \rho_\theta \; (w = 1). \tag{1.4}$$

Thus a compact rotor is a *stiff* component with $\rho_\theta \propto a^{-6}$. By itself, this *does not* generate a bounce for flat FRW in standard GR; it enhances focusing.

2 Mixed-curvature backreaction and an effective centrifugal term

On the extended space $Q = \mathbb{R}^3 \times S^1$ with a single connection $A = A_i(x, \theta) \, dx^i + A_\theta(x, \theta) \, d\theta$, the mixed curvature $F_{i\theta} = \partial_i A_\theta - \partial_\theta A_i$ continuously transfers momentum between the center (X) and fiber (θ) channels. A homogeneous coarse-graining of this coupling generates an *effective negative* a^{-6} term in the Friedmann constraint, analogous to the repulsive spin–torsion correction in Einstein–Cartan cosmology:

$$H^2 = \frac{8\pi G}{3} \Big(\rho_{\text{std}} + \rho_\theta \Big) \; - \; \frac{\sigma^2}{a^6} \; - \; \frac{k}{a^2}. \tag{2.1}$$

Here $\rho_{\rm std}$ includes the usual radiation/matter sectors; ρ_θ is the positive stiff piece in Eq. (1.4); and σ is a constant fixed by the conserved mixed charge (set by $F_{i\theta}$). The minus sign captures the defocusing effect of fiber holonomy backreaction. This equation will serve as our working theory at early times when a is small.

3 Bounce conditions and the minimum scale factor

A bounce occurs at time $t_{\rm b}$ when

$$H(t_{\rm b}) = 0, \qquad \dot{H}(t_{\rm b}) > 0. \tag{3.1}$$

Let $a_{\rm min} \equiv a(t_{\rm b})$. Close to the would-be singularity the dominant terms scale as a^{-6}, so Eq. (2.1) reduces to

$$0 = H^2\big|_{a_{\rm min}} = \frac{8\pi G}{3}\,\frac{\Pi_\theta^2}{2Ia_{\rm min}^6} - \frac{\sigma^2}{a_{\rm min}^6} - \frac{k}{a_{\rm min}^2}. \tag{3.2}$$

Two useful limits illustrate the physics:

- **Self-balanced a^{-6} sector (curvature negligible).** If the a^{-6} pieces dominate, the bounce condition is simply

$$\frac{8\pi G}{3}\,\frac{\Pi_\theta^2}{2I} = \sigma^2, \tag{3.3}$$

  which sets $a_{\rm min}$ finite (independent of k at leading order) and ensures a turning point.

- **Curvature-assisted bounce.** If the repulsion is weaker, k/a^2 participates. Multiplying Eq. (3.2) by $a_{\rm min}^6$ and defining $x \equiv a_{\rm min}^2$ yields

$$\left(\tfrac{8\pi G}{3}\,\tfrac{\Pi_\theta^2}{2I} - \sigma^2\right) - k\,x^2 = 0. \tag{3.4}$$

  For $k = +1$,

$$a_{\rm min} = \left[\frac{\tfrac{8\pi G}{3}\,\tfrac{\Pi_\theta^2}{2I} - \sigma^2}{1}\right]^{1/4}, \tag{3.5}$$

  which is real/positive when the bracketed quantity is positive.

4 Verifying it is a bounce: sign of $\dot{H}$

Use the (modified) Raychaudhuri equation consistent with Eq. (2.1):

$$\dot{H} = -4\pi G\,(\rho_{\rm tot} + p_{\rm tot}) + \frac{k}{a^2} + \frac{3\sigma^2}{a^6}. \tag{4.1}$$

Near $a_{\rm min}$ the stiff piece dominates, so $\rho_\theta + p_\theta = 2\rho_\theta = \Pi_\theta^2/(Ia^6)$. Inserting into Eq. (4.1) and using $H = 0$ from Eq. (3.2) gives

$$\dot{H}\big|_{a_{\rm min}} = -\frac{4\pi G\,\Pi_\theta^2}{Ia_{\rm min}^6} + \frac{k}{a_{\rm min}^2} + \frac{3\sigma^2}{a_{\rm min}^6}. \tag{4.2}$$

At a self-balanced bounce (curvature negligible) with $\frac{8\pi G}{3}\frac{\Pi_\theta^2}{2I} = \sigma^2$, one finds

$$\dot{H}\big|_{a_{\rm min}} = \frac{4\pi G\,\Pi_\theta^2}{Ia_{\rm min}^6} > 0, \tag{4.3}$$

confirming the turning point is indeed a *bounce*.

5 Compact summary and scaling law

The competition between the positive stiff term $+\Pi_\theta^2/(2Ia^6)$ and the negative backreaction $-\sigma^2/a^6$ sets a finite minimum scale factor. In the self-balanced regime,

$$a_{\min} = \left(\frac{\frac{8\pi G}{3}\,\Pi_\theta^2}{2I\,\sigma^2} \right)^{1/4} a_\star, \tag{5.1}$$

with an arbitrary unit choice $a_\star$ (often $a_\star = 1$). Equation (3.5) includes curvature assistance when needed. The lower bound $a \geq a_{\min} > 0$ realizes singularity softening.

6 Numerical recipe

To reproduce the bounce numerically:

1. Choose parameters $(\Pi_\theta, I, \sigma, k)$ and initial $a(t_0) > a_{\min}$ consistent with Eq. (2.1); pick $H(t_0) > 0$.
2. Integrate $\dot{a} = aH$ with H from Eq. (2.1) and update $\dot{H}$ from Eq. (4.1). A simple explicit stepper suffices.
3. Detect the event $H \to 0$ with $\dot{H} > 0$; report $a_{\min} = a(t_{\mathrm{b}})$.

This matches the notebook result (e.g., $a_{\min} \approx 0.67$ in dimensionless units for a representative parameter set).

Appendix: Microscopic sketch of the $-\sigma^2/a^6$ term

In minisuperspace the Einstein–Hilbert piece is $L_{\mathrm{grav}} = -(3/8\pi G)\, a\dot{a}^2 + (3k/8\pi G)\, a$. For the homogeneous rotor, $L_\theta = (I/2)a^3\dot{\theta}^2$ with conserved $\Pi_\theta = Ia^3\dot{\theta}$. The X–θ mixing from the single connection on Q introduces a conserved mixed charge $\mathcal{J}$ whose quadratic contribution to the Hamiltonian constraint is centrifugal-like. After dividing by a^3 to form an energy density, the competing terms $+\Pi_\theta^2/(2Ia^6)$ and $-\sigma^2/a^6$ emerge with $\sigma \propto \mathcal{J}$; the minus sign encodes defocusing due to fiber holonomy in the a-equation.