2019CS50429

Q.1. $f(x) = y$ $\quad f'(x) = \frac{\partial y}{\partial y}$ $\quad f: X \rightarrow Y$

$g(y) = z$ $\quad g'(y) = \frac{\partial z}{\partial y}$ $\quad g: Y \rightarrow Z$

(a) $h(x) = z$ $\quad h'(z) = \frac{\partial z}{\partial z}$ $\quad h: X \rightarrow Z$

$$k = \frac{\| J(x) \|}{\| f(x) \| / \| x \|}$$

$$k_f = \frac{f'(x) \cdot x}{f(x)}$$

$$k_g = \frac{g'(y) \cdot y}{g(y)}$$

$$k_h = \frac{h'(x) \cdot x}{h(x)} = \frac{f'(x) \cdot x}{f(x)} \cdot \frac{g'(y) \cdot y}{g(y)} = k_f * k_g$$

The tighest possible upper bound is $k_f * k_g$.

OR

$$k_{h(x)} = \frac{x (g \circ f)'(x)}{(g \circ f)(x)} = \frac{x f'(x) g'(f(x))}{g(f(x))} = \frac{x f'(x)}{f(x)} \cdot \frac{f(x) g'(f(x))}{g(f(x))}$$

$$= k_f * k_g.$$

(b) Unable to find an example.

For $k_f \geq 10$, $k_g \geq 10$ I get $k_h \geq 100$.

2019CS50429

Q.2. $f(x) = x(1-x)$

$\hat{f}(x) = \hat{x} \otimes (1 \ominus \hat{x})$

$\hat{x} = fl(x) = x(1+\epsilon_1)$

$$\therefore \hat{f}(x) = x(1+\varepsilon_1) \otimes (1 \ominus x(1+\varepsilon_1))$$

$$= x(1+\varepsilon_1) \otimes \left[\left(1 - x(1+\varepsilon_1)\right)(1+\varepsilon_2)\right]$$

$$= x(1+\varepsilon_1) \otimes \left[(1+\varepsilon_2 - x(1+\varepsilon_3))\right]$$

$$= x(1+\varepsilon_1)\left[(1+\varepsilon_2 - x(1+\varepsilon_3))\right](1+\varepsilon_4)$$

$$= x(1+\varepsilon_5)\left[(1+\varepsilon_2) - x(1+\varepsilon_3)\right]$$

$$= x(1+\varepsilon_6) - x^2(1+\varepsilon_7)$$

$$= (x-x^2)(1+\varepsilon_8)$$

for some $\varepsilon_i < |\epsilon_m|$ for $i=1,\dots 8$.

$\epsilon_m = \epsilon_{machine}$

(a) $\Delta f(x) = \hat{f}(x) - f(x)$

$$= (x-x^2)(1+\varepsilon) - (x-x^2)$$

$$= \varepsilon(x-x^2)$$

Max value of $x-x^2$ is $\frac{1}{4}$ in real plane.

$$\therefore \Delta f(x) = \varepsilon(x-x^2) \le \varepsilon\left(\frac{1}{4}\right) < \boxed{\frac{\epsilon_m}{4}}$$

Required upper bound.

(b) The result fails to be accurate when $f(x)=0$ i.e. for $x=0,1$.

It fails to be backward stable for $x=0$ since $\ominus$ will introduce absolute errors of size $O(\epsilon_m)$.

2019CS50429

Q.3. To solve $UX + XL = Y$ where $U, L, Y \in \mathbb{C}^{m\times m}$ are known $U \rightarrow$ Upper triangular
$L \rightarrow$ Lower triangular

To find $X \in \mathbb{C}^{m\times m}$

Since, it is already given that we have to find an $O(m^3)$ algorithm and backward substitution takes $O(m^2)$, we can roughly say that back substitution is taking place in the order of m times.

From the hint, we proceed by trying to find columns of X at a time. Let x_i denote the i^{th} column of X. similarly, y_i for Y.

Now, the contribution of the first term UX in the column form will be just Ux_i

Meaning,
$$Ux_i + \underbrace{b_i}_{\{\text{contribution from } XL;\ \text{Unknown for now}\}} = y_i$$
for some column i.

Observing XL:

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ x_{m1} & \cdots & \cdots & x_{mm} \end{bmatrix} \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ l_{m1} & \cdots & \cdots & l_{mm} \end{bmatrix} = \underbrace{B}_{\text{assume}}$$
(some matrix)

Upon few calculations, we can observe that B will be of the form:

$$B = \begin{bmatrix} x_{11} l_{11} + \sum_{j=1+1}^{m} x_{1j} l_{j1} & \cdots & x_{1m} l_{mm} + \sum_{j=m+1}^{m} x_{mj} l_{jm} \\ \vdots & \ddots & \vdots \\ x_{mm} l_{mm} + \sum_{j=1+1}^{m} x_{mj} l_{jm} & \cdots & x_{mm} l_{mm} + \sum_{j=m+1}^{m} x_{mj} l_{jm} \end{bmatrix}$$

Every term of B is of the form

$$B_{ba} = x_{ba} l_{aa} + \sum_{j=a+1}^{m} x_{bj} l_{ja}$$

$a \rightarrow$ column number
$b \rightarrow$ row number

We can observe that in the $x_{ba} l_{aa}$ term the column is same. We can write both the terms in column form

$$b_i = (l_{ii} I) x_i + \sum_{j=i+1}^{m} l_{ji} x_j \quad \text{—— (1)}$$

(As earlier, i denotes column number)

$\therefore \quad U x_i + b_i = y_i$ (As stated earlier)

From (1),

$$U x_i + \left[(l_{ii} I) x_i + \sum_{j=i+1}^{m} l_{ji} x_j \right] = y_i$$

$$\Rightarrow [U + l_{ii} I] x_i + \underbrace{\sum_{j=i+1}^{m} l_{ji} x_j}_{} = y_i \quad \text{——} (2)$$

In this term we need colums of x from $(i+1)$ to (m) to calculate i^{th} column of x in previous term.

Hence, we will start calculating x_m first the x_{m-1}, etc to calculate any column x_i for $i \leq m$. In each step, we will need to use backsubstitution of $O(m^2)$ time.

We have total 'm' columns.

Therefore, our algorithm will be to calculate $x_m, x_{m-1}, \ldots x_0$ using the formula in (2) and backsubstitution at each step. ∴ It will take $O(\underbrace{m^2}_{\text{Backsubstitution}} * \underbrace{m}_{\text{Number of columns of } X})$ time.

2019CS50429

Q.4. $A \in \mathbb{C}^{m \times n}$

$\|Ax\|_2 = \|x\|_2 \quad \forall \; x \in \mathbb{C}^n$

$*$ → denotes transpose

We can write $A = uv^*$

where u is an m-vector.

v is an n-vector. since size of A is $m \times n$.

Now,

$\|Ax\|_2 = \|uv^*x\|_2$

$= \|u\|_2 |v^*x| \leq \|u\|_2 \|v\|_2 \|x\|_2$ — (i)

We can observe that equality holds for $v = x$. Here, setting $v = x$ doesn't affect any further calculations since we can have many values of u & v hence x.

∴ (1) can be re-written as

$\|Ax\|_2 = \|u\|_2 \|x\|_2^2$

∵ $\|Ax\|_2 = \|x\|_2$

$\Rightarrow 1 = \|u\|_2 \|x\|_2$

∵ $v = x \Rightarrow \|v\|_2 = \|x\|_2$

$\Rightarrow 1 = \|u\|_2 \|v\|_2$

$\Rightarrow \|u\|_2 \|v\|_2 = 1$

$\Rightarrow (u^*u)(v^*v) = I$

$\Rightarrow u^*(uv^*)v = I$

$u^*(A)v = I$

$\Rightarrow u^*(A)v v^* = u^*$

$\Rightarrow u^*(A)(A^*) = u^*$

$\Rightarrow (A)(A^*) = I$

Hence, A should an orthogonal matrix.

Q.5. The question asks us to find a positive real number δ st $\forall |\lambda| < \delta \Rightarrow M(\lambda)$ is non singular.

We can infer from this that there are several such δ for which this possible. (If γ is the largest possible value of δ then our statement is always true for $|\lambda| < \delta \leq \gamma$)

Here, I will NOT find γ (or largest possible value of δ). I will assume δ to be $\frac{1}{2\|A\|}$ and prove that $M(\lambda)$ is nonsingular whenever $|\lambda| < \frac{1}{2\|A\|}$.

$M(\lambda) = ~~\text{...}~~ \; I + \lambda A \qquad \lambda \in \mathbb{C}$ and $A \in \mathbb{C}^{m \times m}$

Considering action of $M(\lambda)$ on vector x & taking its norm gives us : $\|(I+\lambda A)x\|$

Now,

$$\|M(\lambda)x\| = \|(I+\lambda A)x\| = \|x + \lambda A x\|$$
$$\geq \|x\| - \|\lambda A x\|$$
$$\geq \|x\| - |\lambda| \, \|Ax\| \qquad \text{—— (1)}$$

We have,
$$\|Ax\| \leq \|A\| \, \|x\|$$
and $\quad |\lambda| < \delta$
$$\Rightarrow \|Ax\| \, |\lambda| < \delta \|A\| \, \|x\| \qquad \text{—— (2)}$$
$$\Rightarrow - \|Ax\| \, |\lambda| > -\delta \|A\| \, \|x\|$$
$\Rightarrow$

From (1) and (2), we have
$$\|(I+\lambda A)x\| > \|x\| (1 - \delta\|A\|) \qquad \text{—— (3)}$$

Now, substituting our assumption $\delta = \frac{1}{2\|A\|}$

in (3):

$$\|(I+\lambda A)x\| > \|x\| \left(1 - \left(\frac{1}{2\|A\|}\right)\|A\|\right)$$

$$\Rightarrow \|(I+\lambda A)x\| > \|x\| \left(1-\frac{1}{2}\right) = \frac{\|x\|}{2}$$

Norm of a vector is zero iff the vector is zero.

$$\therefore \frac{\|x\|}{2} > 0$$

$$\Rightarrow \|(I+\lambda A)x\| > \frac{\|x\|}{2} > 0$$

$$\therefore \|(I+\lambda A)x\| = \|\underbrace{M(\lambda)x}_{\text{A vector}}\| > 0$$

Lemma 1: If A is a singular matrix $\exists\, x \neq 0$ st $Ax = 0$

Lemma 2: $\exists\, x$ st $\|Ax\| = 0 \Rightarrow A$ is singular

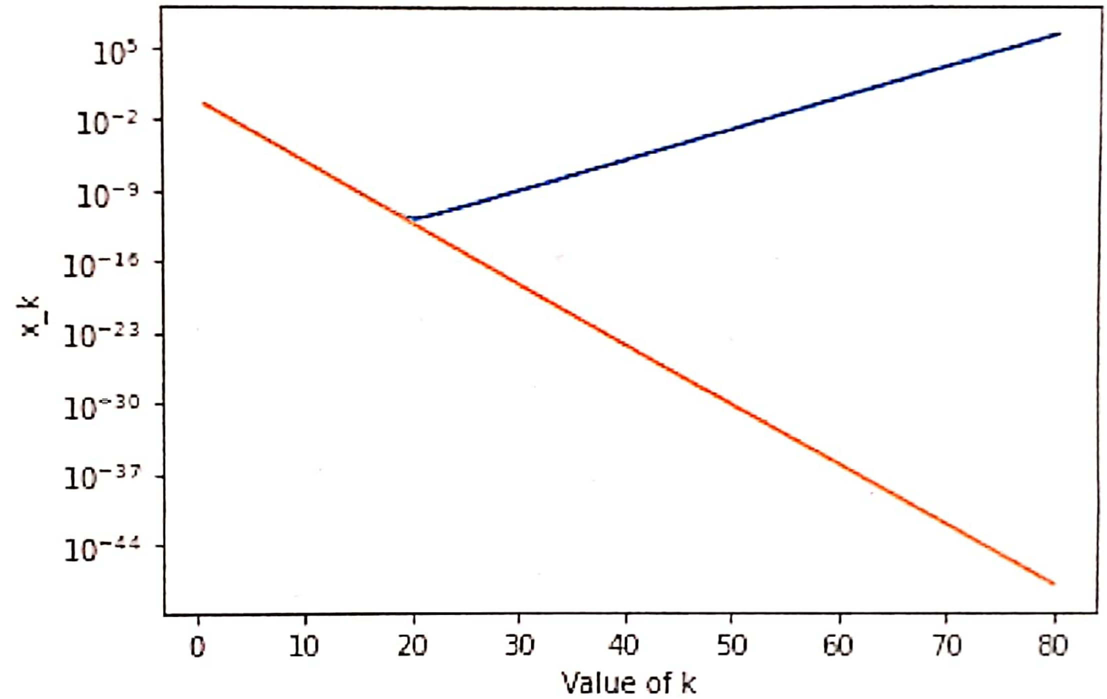
$\because \|M(\lambda)x\| > 0$ & $M(\lambda)x$ is a vector & $x \neq 0$

We can conclude that $M(\lambda)$ will always be non-singular. for $\delta = \frac{1}{2\|A\|}$.

Q.6.

(b) The semilog plot of the computed terms and that of $x_k = 4^{-k/3}$ is :

Blue:- computed terms
orange:- $x_k = 4^{-k/3}$



(c) The exact solution decreases monotonically as k increases. This is also seen in the graph plotted above. However, my computed solution does not have the same behaviour. At first, I thought that this happened due to an error in choosing the right type which must have caused some error in my program. But soon realized that there was an error but on a lower level.

To examine this behaviour, I decided to derive the exact solution. This will help me identify if at any point the problem is ill-conditioned. and/or the solution is unstable.

The difference recurrance relation given to us is : $x_{k+1} = 2.25x_k - 0.5x_{k-1}$

w/ $x_0 = 1/3$ $x_1 = 1/12$

Now; to solve any recurrance relation of the form $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$, we must solve the characteristic polynomial $x^2 + \alpha x + \beta$.

In our case, the solutions are 2 and $\frac{1}{4}$

$\therefore$ The solution to recurrance is : $x_k = a2^k + b\left(\frac{1}{4}\right)^k$

we have

$$\frac{1}{3} = a2^0 + b\left(\frac{1}{4}\right)^0$$

$$\frac{1}{12} = a\cdot 2^1 + b\left(\frac{1}{4}\right)^1$$

Solving these, we get $a = 0$, $b = 1/3$

Now, $a = 0$ reminds me of problem 2b where our function was not backward stable. We know that a and b are both represented in the form of $a(1+\epsilon_a)$ and $b(1+\epsilon_b)$ for some $\epsilon_m > \epsilon_a, \epsilon_b > 0$. It is possible that ϵ_a here causes accumulation of rounding error which builds up until we notice it at $k \sim 20$. (from the graph).