

# Parallel Depth-First Search for Directed Acyclic Graphs

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Depth-First Search (DFS) is a common algorithm, often used as a building block for topological sort, connectivity and planarity testing, among many other applications. We will be implementing the work-efficient parallel algorithm for the DFS traversal of directed acyclic graphs (DAGs) as proposed in [2][3]. We will be reporting the speedup of this parallel algorithm (in comparison with the serial implementation).

Additional Key Words and Phrases: Depth-First Search, Parallel Programming, Directed Acyclic Graph, Work efficient Algorithm.

## 1 INTRODUCTION

Let a graph  $G = (V, E)$ , be defined by its vertex  $V = \{1, 2, \dots, n\}$  and edge  $E = \{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\}$  sets, with  $|V| = n$  and  $|E| = m$ . The sequential lexicographic Depth-First Search (DFS) algorithm was proposed in [4].

The (DFS) traversal problem requires us to compute : parent information , pre-order (start time) and post-order (end time) for every node in  $G$ , see 1.

## 2 LITERATURE SURVEY

The lexicographical DFS problem can be subdivided into two major sets of algorithms:

- (1) Finding the Pre-order (start time) and the Post-order (end time) for every node in the Graph  $G$ .
- (2) Conversion from a DAG to a Directed Tree (DT).

Before we proceed to discuss on how the paper [3] suggests to find the pre-order and post-order, we need to establish some math.

*Definition 1.* Let  $\varsigma_p$  and  $\zeta_p$  denote the number of nodes reachable under and including node  $p$ , where if a sub-graph is reachable from  $k$  multiple parents then its nodes are counted once and  $k$  times, respectively.

LEMMA 1. For a DT, where each node has a single parent,  $\varsigma_p = \zeta_p$  is simply the sub-graph size, while for a DAG we have  $\zeta_p \geq \varsigma_p$ .

For example, in Fig. 1 we have  $\zeta_a = 7$  and  $\varsigma_a = 6$ , because we double counted the node  $f$  in the former case. Notice that the aforementioned lemma follows directly from the definition of a DT.

Also, the following recursive relationship could be observed :

$$\zeta_p = 1 + \sum_{i \in C_p} \zeta_i$$

where  $C_p$  is ordered set of children of  $p$ .

It so seems that it's easy to compute  $\zeta_p$  (recursive relation), compared to  $\varsigma_p$  can be expensive to compute [3].

Let  $\tilde{\zeta}_l$  where  $l$  is an index of exclusive prefix sum list, of the list  $\zeta_i$ , where  $i \in C_p$ .

$$\tilde{\zeta}_l = \sum_{i < l, i \in C_p} \zeta_i$$

Having established that, the values of  $\zeta_p$  can be computed for a DT or DAG by traversing the graph bottom-up, from leafs to roots, as shown in Algorithm 1.

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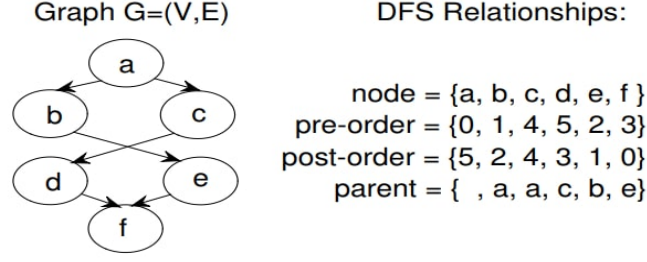


Fig. 1. Graph and associated DFS relationships

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**Algorithm 1** Sub-Graph Size (bottom-up traversal)

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1: Initialize all sub-graph sizes to 0.
2: Find leafs and insert them into queue  $Q$ .
3: while  $Q \neq \{\phi\}$  do
4:   for node  $i \in Q$  do in parallel
5:     Let  $P_i$  be set of parents of  $i$  and queue  $C = \{\phi\}$ 
6:     for node  $p \in P_i$  do in parallel
7:       Mark  $p$  outgoing edge  $(p, i)$  as visited
8:       Insert  $p$  into  $C$  if all outgoing edges are visited
9:     end for
10:  end for
11:  for node  $p \in C$  do in parallel
12:    Let  $C_p$  be an ordered set of children of node  $p$ 
13:    Compute a prefix-sum on  $C_p$ , obtaining  $\zeta_p$  (use lexicographic ordering of elements in  $C_p$ )
14:  end for
15:  Set  $Q = C$  for the next iteration
16: end while

```

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Having computed  $\zeta_p$  and  $\tilde{\zeta}_p$  for each node  $p$  in the graph the paper now expresses the pre and post-order times as their functions.

**THEOREM 1.** Let  $\zeta_i$  be the sub-graph size for node  $i$  in a DT and  $\tilde{\zeta}_i$  be the corresponding prefix-sum value. Then,

$$\text{preorder}(p) = k + \tau_p \quad (1)$$

$$\text{postorder}(p) = (\zeta_p - 1) + \tau_p \quad (2)$$

where  $\mathfrak{P}_{r,p} = \{r, i_1, i_2, \dots, i_{k-1}, p\}$  and,  $\tau_p = \sum_{l \in \mathfrak{P}_{r,p}} \tilde{\zeta}_l$

**COROLLARY 1.** Let a path from root  $r$  to node  $p$  be an ordered set of nodes  $\mathfrak{P}_{i,p} = \{r, i_1, \dots, i_{k-1}, p\}$ . Then,  $\tau_p$  can be obtained recursively

$$\tau_p = \tau_{i_{k-1}} + \tilde{\zeta}_p \quad (3)$$

The above algorithm works well when the input is a DT. So, we now discuss how to convert any DAG to a DT.

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**Algorithm 2** Pre- and Post-Order (top-down traversal)

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1: Initialize pre and post-order of every node to 0.
2: Find roots and insert them into queue  $Q$ .
3: while  $Q \neq \{\phi\}$  do
4:   for node  $p \in Q$  do in parallel
5:     Let  $\text{pre} = \text{pre-order}(p)$  and  $\text{post} = \text{post-order}(p)$ 
6:     Let  $C_p$  be a set of children of  $p$  and queue  $P = \{\phi\}$ 
7:     for node  $i \in C_p$  do in parallel
8:       Set  $\text{pre-order}(i) = \text{pre} + \tilde{\zeta}_i$ 
9:       Set  $\text{post-order}(i) = \text{pre} + \tilde{\zeta}_i$ 
10:      Mark  $i$  incoming edge  $(p, i)$  as visited
11:      Insert  $i$  into  $P$  if all incoming edges are visited
12:   end for
13:   Set  $\text{pre-order}(p) = \text{pre} + \text{depth}(p)$ 
14:   Set  $\text{post-order}(p) = \text{post} + (\zeta_p - 1)$ 
15: end for
16: Set  $Q = P$  for the next iteration
17: end while
```

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*Definition 2.* Let  $\mathfrak{P}_{r,p} = \{r, i_1, i_2, \dots, i_{k-1}, p\}$  and  $\mathfrak{Q}_{r,p} = \{r, j_1, j_2, \dots, j_{k-1}, p\}$  be two paths of potentially different length to node  $p$ . We say that path  $\mathfrak{P}$  has the first lexicographically smallest node and denote it by

$$\mathfrak{P}_{r,p} < \mathfrak{Q}_{r,p} \quad (4)$$

when during the pair-wise comparison of the elements in the two paths going from left-to-right the path  $\mathfrak{P}_{r,p}$  has the lexicographically smallest element in the first mismatch.

For example, in Fig. 1 the two paths to node  $f$  are

$$\mathfrak{P}_{r,p} = [a, b, e, f]$$

$$\mathfrak{Q}_{r,p} = [a, c, d, f]$$

Clearly  $\mathfrak{P}_{r,p} = [a, b, e, f] \leq \mathfrak{Q}_{r,p} = [a, c, d, f]$  as  $b \leq c$  lexicographically.

**THEOREM 2.** Let  $\mathfrak{P}_{r,p} = \{r, i_1, i_2, \dots, i_{k-1}, p\}$  and  $\mathfrak{Q}_{r,p} = \{r, j_1, j_2, \dots, j_{k-1}, p\}$  be two paths of potentially different length to node  $p$ . If  $\mathfrak{P}_{r,p} < \mathfrak{Q}_{r,p}$  then  $\mathfrak{P}_{r,p}$  is the path taken by DFS traversal.

Putting it all together, the parallel DFS algorithm can be written as Algorithm 4.

### 3 ANALYSIS

#### 3.1 Complexity Analysis

The following definitions, lemmas and theorems establish the fact that the aforementioned algorithm is a work-efficient one.

*Definition 3.* Let  $k_i$  be the number of elements inserted into the queue at iteration  $i$  and  $k = \max_i k_i$  in Algorithms 1-3.

*Definition 4.* Let  $d_i$  denote the degree of node  $i$  and  $d = \max_i d_i$  denote the maximum degree in DAG.

*Definition 5.* Let  $\delta_i$  denote the minimum depth of node  $i$  and  $\delta = \max_i \delta_i$  denote the diameter in a DAG.

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**Algorithm 3** Compute DFS-Parent using Path (top-down traversal)

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```
1: Initialize path to  $\{\phi\}$  and parent to -1 for every node.
2: Find roots and insert them into queue  $Q$ .
3: while  $Q \neq \{\phi\}$  do
4:   for node  $p \in Q$  do in parallel
5:     Let  $C_p$  be a set of children of  $p$  and queue  $P = \{\phi\}$ 
6:     for node  $i \in C_p$  do in parallel
7:       Let  $\mathfrak{Q}_{r,i}$  be existing and  $\mathfrak{P}_{r,i}$  be a new path. ( $\mathfrak{P}_{r,i}$  is a concatenation of path to  $p$  and node  $i$ )
8:       if then  $\mathfrak{P}_{r,i} \leq \mathfrak{Q}_{r,i}$ 
9:         Set  $\mathfrak{Q}_{r,i} = \mathfrak{P}_{r,i}$ 
10:        Set parent( $i$ ) =  $p$ 
11:       end if
12:       Mark  $i$  incoming edge  $(p, i)$  as visited
13:       Insert  $i$  into  $P$  if all incoming edges are visited
14:     end for
15:   end for
16:   Set  $Q = P$  for the next iteration
17: end while
```

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**Algorithm 4** Parallel DFS (Path)

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```
1: Let graph  $G = (V, E)$  and its adjacency matrix  $A$ 
2: Run Algorithm 3
3: Run Algorithm 1
4: Run Algorithm 2
5: Resulting in parent, pre- and post-order for every node in  $G$ 
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*Definition 6.* Let  $\eta_i$  denote the maximum depth of node  $i$  and  $\eta = \max_i \eta_i$  denote the length of the longest path in a DAG.

LEMMA 2. *The parallel prefixsum of  $n$  numbers, can be computed in  $O(\log n)$  steps. Also, the algorithm performs  $O(n)$  work*

LEMMA 3. *Let  $n = \min(n_1, n_2)$ , then identifying the first left-to-right pair of digits in two sequences of  $n_1$  and  $n_2$  numbers can be performed in  $O(\log n)$  steps, by doing  $O(n)$  work.*

LEMMA 4. *The queue can be implemented such that parallel insertion and extraction of  $n$  numbers, can be performed in  $O(\log n)$  and  $O(1)$  steps, respectively. Also, the algorithm performs  $O(n)$  work.*

THEOREM 3. *Algorithm 1 takes  $O(\eta(\log d + \log k))$  steps and performs  $O(m + n)$  total work to traverse a DAG. The number of processors  $t \leq m + n$  actively doing work varies at each step of the algorithm. Here  $\eta$  is the length of longest path in DAG,  $d$  is maximum degree in DAG and  $k$  is the maximum number of elements inserted into a queue.*

THEOREM 4. *Algorithm 2 takes  $O(\eta \log k)$  steps and performs  $O(n)$  total work to traverse a DAG. The number of processors  $t \leq n$  actively doing work varies at each step of the algorithm. Here  $\eta$  is the length of longest path in DAG and  $k$  is the maximum number of elements inserted into a queue.*

**THEOREM 5.** *Algorithm 3 takes  $O(\eta(\log \eta + \log k))$  steps and performs  $O(\eta m + n)$  total work to traverse a DAG. The number of processors  $t \leq \eta d + n$  actively doing work varies at each step of the algorithm. Here  $\eta$  is the length of longest path in DAG,  $d$  is maximum degree in DAG and  $k$  is the maximum number of elements inserted into a queue.*

**COROLLARY 2.** *Path based DFS takes  $O(\eta(\log d + \log k + \log \eta))$  steps and performs  $O(m + n + \eta m)$  total work to traverse a DAG. The number of processors  $t \leq m + n + \eta d$  actively doing work varies at each step of the algorithm. Here  $\eta$  is the length of longest path in DAG,  $k$  is the maximum number of elements inserted into a queue and  $d$  is maximum degree in DAG.*

Notice that in practice  $\eta m \rightarrow m$  because the data structure for storing the path detects the same "parent" block during comparisons, which implicitly eliminates additional work. Also, recall that  $d, k, \eta \leq n$  in a DAG and therefore the Path-based algorithm takes no more than  $O(\eta \log n)$  steps and performs  $O(m + n)$  total work.

### 3.2 Parallel Analysis

The Algorithms 1, 2 and 3 spawn multiple threads wherever **do in parallel** is mentioned. Apart from this the prefix sum calculation in line 13 of Algorithm 1 will also run in parallel.

**3.2.1 Task Dependency Graphs.** We subdivide the dependency graph of the parallel DFS traversal algorithm into three meta-tasks : the Algorithms 1, 2 and 3, and attach their dependency graphs separately.

#### (1) Algorithm 1 - Figure 2

- (a) Level A - In this level, the sole task is to *initialize all sub-graph sizes to 0 and finding the leaf nodes and initializing the queue  $Q$  with them.*
  - (b) Level B - All tasks in this level execute parallelly for all the elements of  $Q$ , where in each task  $B_i$  involves initialization of the queue  $C$  accessing the parent list of the node ( $P_i$ ). Notice that, there will be  $|Q| = q$  tasks in this level.
  - (c) Level C - All tasks in this level execute parallelly for all the parents of the  $i$ th node initialized in  $B_i$  where in each task  $C_i$  involves marking the edges between the parent and it's child as visited; then, enqueueing the parents which have all their outgoing edges marked as visited into a queue  $C$ . Notice that, the number of tasks in this level will be  $|P_i| * |Q|$ .
  - (d) Level D - Wait for all the tasks in level C to finish updating the list  $C$ , for further computation.
  - (e) Level E - Wait for all the tasks in level B to finish their execution.
  - (f) Level F - All tasks in this level execute parallelly for all the elements  $p$  of the list  $C$  accessing it's children ( $C_p$ ). Each meta-task in this level  $F_i$  represents the dependency graph of the exclusive parallel-prefix sum (as discussed in the class). Notice that, the number of tasks in this level will be  $|C|$ .
  - (g) Level G - Wait for all the tasks in level E to finish their execution. Then, reinitialize  $Q$  as  $C$ .
- This block is repeated for every iteration of the outer while loop of the Algorithm 1 which is executed maximum  $\eta$  (defined 6) number of times.

#### (2) Algorithm 2 - Figure 3

- (a) Level A - In this level, the sole task is to *initialize all sub-graph sizes to 0 and finding the root nodes and initializing the queue  $Q$  with them.*
- (b) Level B - All tasks in this level execute parallelly for all the elements of  $Q$ , where in each task  $B_i$  involves initialization of preorder, postorder and accessing children ( $C_p$ ) of the node  $p$ . Notice that, there will be  $|Q| = q$  tasks in this level.
- (c) Level C - All tasks in this level execute parallelly for all the children of the  $i$ th node initialized in  $B_i$  where in each task  $C_i$  involves calculation of preorder and postorder, along with, marking the edges between

the child and it's parent as visited; then, enqueueing those children whose all incoming edges are marked visited into queue  $P$ . Notice that, the number of tasks in this level will be  $|C_p| * |Q|$ .

- (d) Level D - Wait for all the tasks in level  $C$  to finish updating the list  $C$ . Then, compute the final preorder and postorder value of the the nodes in queue  $Q$ . Notice that number of nodes in this level will be  $|Q|$  itself.
  - (e) Level E - Wait for all the tasks in level  $D$  to finish their execution. Then, reinitialize  $Q$  as  $P$ .
- This block is repeated for every iteration of the outer while loop of the Algorithm 2 which is executed maximum  $\eta$  (defined 6) number of times.

### (3) Algorithm 3 - Figure 4

- (a) Level A - In this level, the sole task is to *initialize all paths to  $\{\phi\}$ , parents to -1 and finding the root nodes and initializing the queue  $Q$  with them.*
  - (b) Level B - All tasks in this level execute parallelly for all the elements of  $Q$ , where in each task  $B_i$  involves accessing children ( $C_p$ ) of the node  $p$ . Notice that, there will be  $|Q| = q$  tasks in this level.
  - (c) Level C - All tasks in this level execute parallelly for all the children of the  $i$ th node initialized in  $B_i$  where in each task  $C_i$  involves marking the edges between the parent and it's child as visited; then, enqueueing the children which have all their incoming edges marked as visited into a queue  $P$ . Each meta-task in this level  $C_i$  represents the dependency graph of lexicographic path comparison function, which is same as the dependency graph for parallel-prefix sum - as mentioned in Lemma 3. Notice that, the number of tasks in this level will be  $|C_p| * |Q|$ .
  - (d) Level D - Wait for all the tasks in level  $C$  to finish updating the list  $P$ , for further computation.
  - (e) Level E - Wait for all the tasks in level  $D$  to finish their execution. Then, reinitialize  $Q$  as  $P$ .
- This block is repeated for every iteration of the outer while loop of the Algorithm 3 which is executed maximum  $\eta$  (defined 6) number of times.

- (4) **Algorithm 4 - Figure 5** This is the consolidated task-dependency graph of the parallel DFS traversal, where each meta-task represents the dependency graphs of the Algorithms mentioned within them.

## 4 IMPLEMENTATION PLAN OF ALGORITHM

We will use the standard CSC(Compressed Sparse Column) format to store the graph, which simply concatenates all non-zero entries of the matrix in column major order and records the starting position for the entries of each column. This format allows us to easily traverse and access information associated with outgoing edges.

Assuming that storing a node requires  $\beta = 4$  (or 8) bytes. Then, the sequential DFS algorithm uses approximately  $(m + 5n)\beta$  bytes to store the input DAG, queue, parent, pre- and post-order output arrays.

The parallel DFS algorithm only requires an additional  $(m + kn)\beta$  bytes for its data structures. Here, the first term in the sum corresponds to  $\zeta$ , while the last term corresponds to the path data structure storage, where  $k$  is the block size.

## 5 EVALUATION

Since, there are no existing codes available for the parallel DFS traversal algorithms, we would be reporting the speedup in comparison with the sequential DFS algorithm.

First, to check for correctness of our implementation, we would run it on some small toy-testcases. Then, we would move on to bigger datasets (Table 1)[1] and report the running time (of both sequential and serial implementations) and the speedup.

Table 1. Sample DIMACS graphs/adjacency matrices

Datasets	n	m	d	$\eta$
<b>hugebubbles-00</b>	18318144	30144175	3	6724
<b>fl2010</b>	484482	1270757	121	105
<b>ca2010</b>	710146	1880571	119	125
<b>great-britain_os</b>	7733823	8523976	7	7076
<b>germany_osm</b>	11548846	12793527	12	8890

When necessary, we would create DAGs based on these general graphs by dropping the back edges, in other words, only considering the lower triangular part of the adjacency matrix these graphs.

## 6 DELIVERABLES

### 6.1 First

- Sequential Algorithm
- Conversion from a DAG to a DT

### 6.2 Second

- Pre-order and Post-order computation.

### 6.3 Third

- Optimizations:
  - (1) Path Pruning
  - (2) Path Compression
  - (3) Path Data Structure (as given the the paper [3])
  - (4) SSSP Based DFS (*If time permits*).

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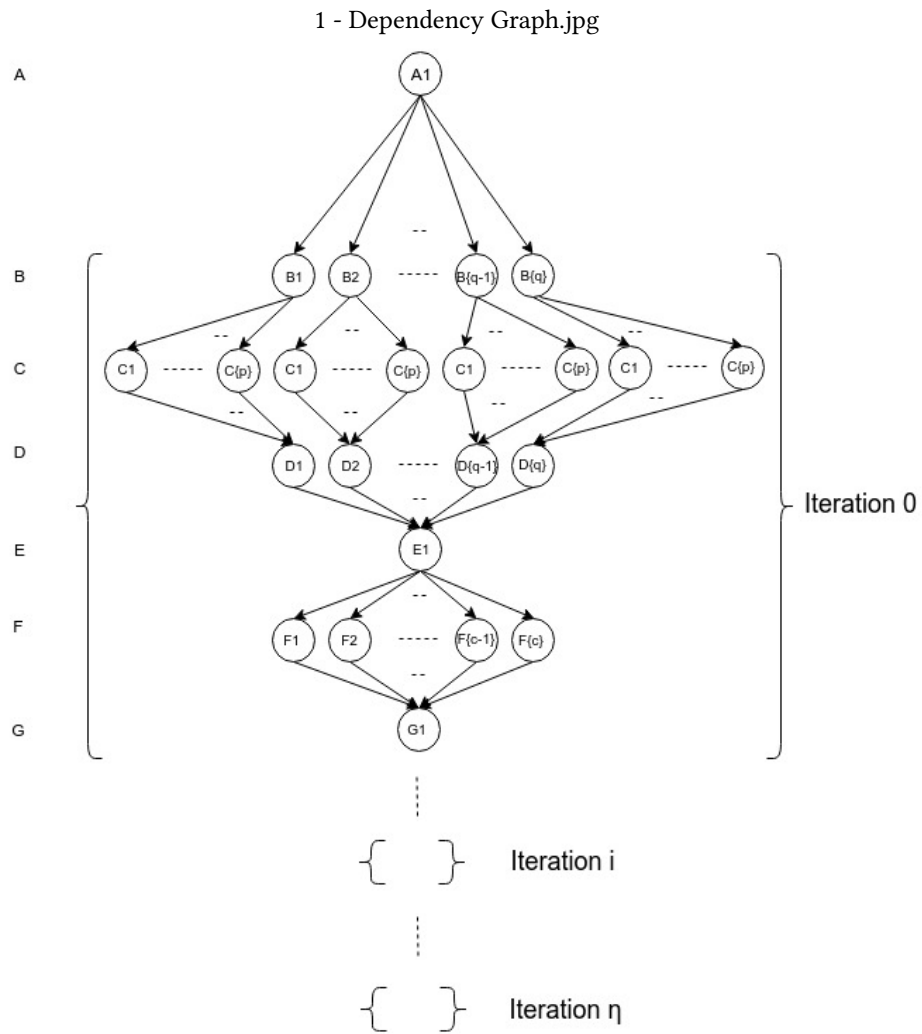


Fig. 2. Task dependency graph for Algorithm 1 with explanation in Section 3.2.1



2 - Dependency Graph.jpg

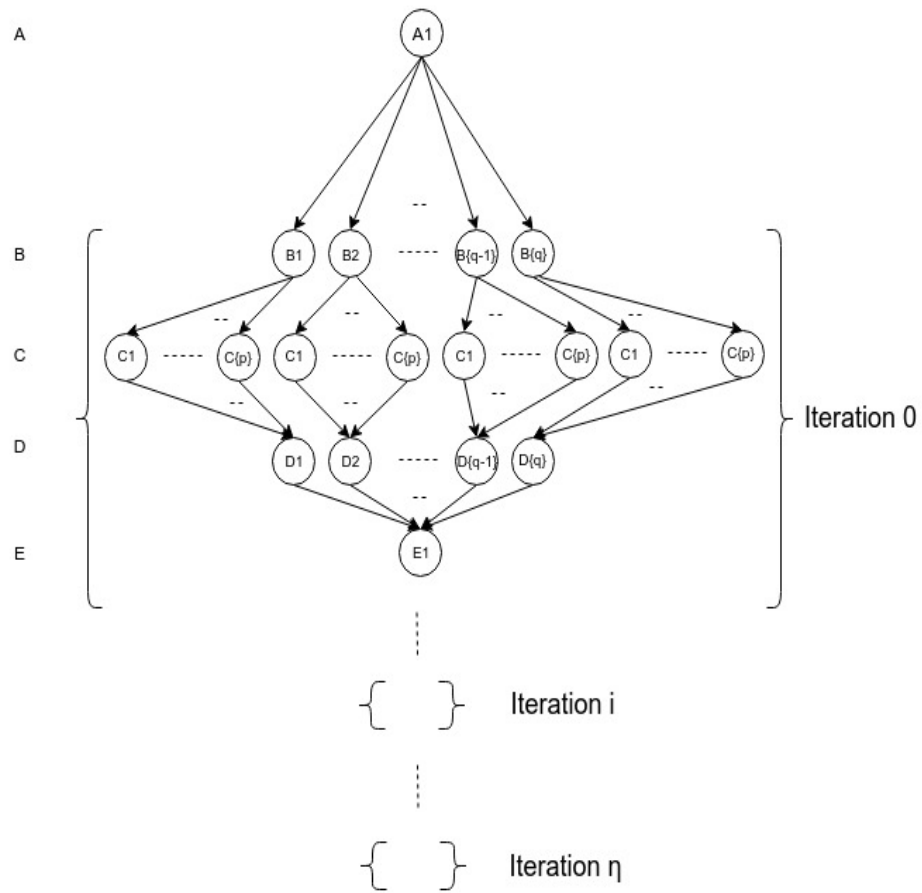


Fig. 3. Task dependency graph for Algorithm 2 with explanation in Section 3.2.1

3 - Dependency Graph.jpg

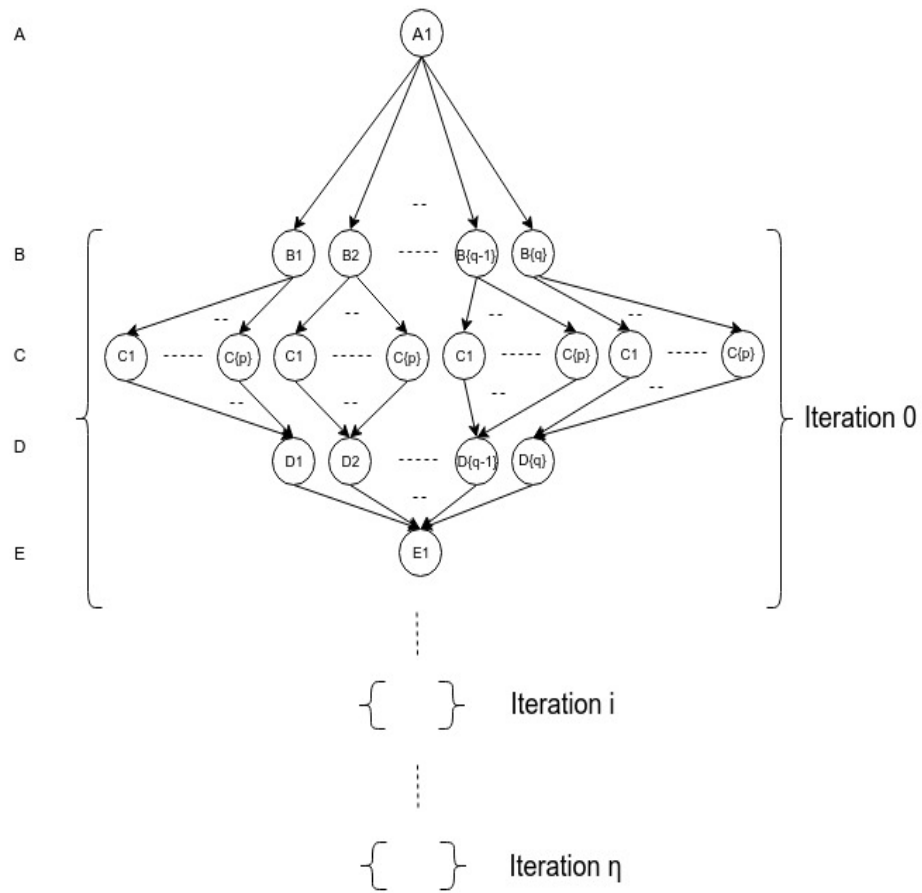
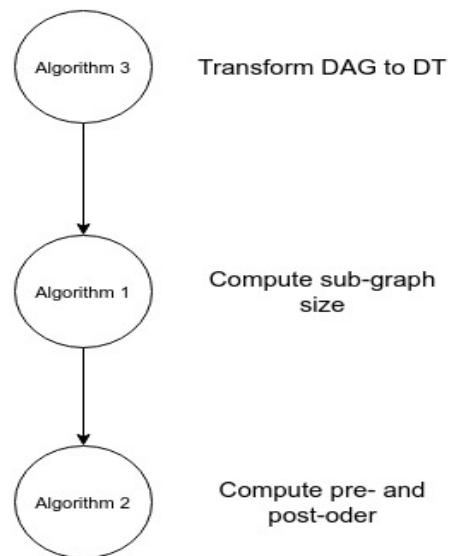


Fig. 4. Task dependency graph for Algorithm 3 with explanation in Section 3.2.1



4 - Dependency Graph.jpg

Fig. 5. Task dependency graph for Algorithm 4 with explanation in Section 3.2.1