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Mathematical Modelling in Industry **ED15B013**

Traffic Flow

The code is available in the zip file attached with the mail.

1 Implementation

The programming for this assignment is done in Python. A class **TrafficFlow** is created. The user needs to make an instance of **TrafficFlow** class, describing how the road is and the initial conditions of the density. This class is capable of using the Godunov's scheme for any general flow function. The user needs to provide the flow function, as well as its derivative function.

The variable named ***u*** is used in the Godunov's scheme, ***rho*** is used to define the density.

The user needs to provide function that are used to convert u to rho, and rho to u based on the flow function defined.

Functions named ***enable_signal()*** and ***enable_speed_breaker()*** are available to enable the traffic signal and speed breaker, respectivvely. The function ***update_gudonov()*** should be called to start the calculations. The function named
show_animation() can be called to density vs x animation as time progresses.

Documentation on how to use the functions is provided in the code file itself.

2 Greenshield's Model

Velocity:

$$v(\rho) = v_{max}(1 - \frac{\rho}{\rho_{max}}), \;\; 0 \leq \rho \leq \rho_{max}$$

The model:

$$\rho_t + (\rho v_{max}(1 - \frac{\rho}{\rho_{max}}))_x = 0, \;\; x \in \mathbf{R}, \; t \geq 0$$

Initial conditions:

$$\rho(x, 0) = \rho_0(x), \;\; x \in \mathbf{R}$$

2.1 Burger's Equation

Transforming the above into Burger's Equation

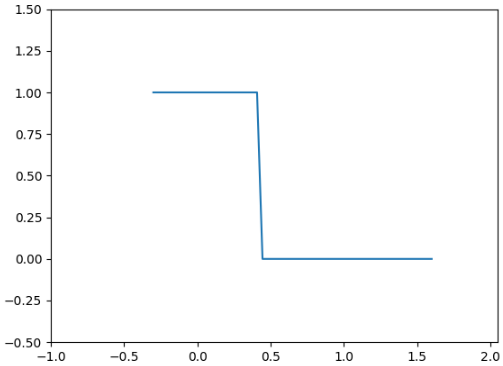
$$u = 1 - \frac{2\rho}{\rho_{max}}$$

$$u_t + (\frac{u^2}{2})_x = 0$$

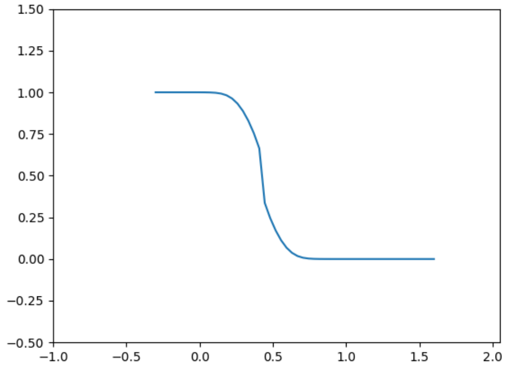
$$u(x, 0) = u_0(x)$$

Godunov's scheme is used to find the numerical solution for the above mentioned equations.

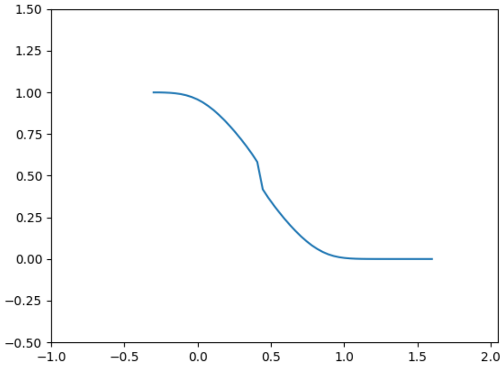
2.2 Rarefaction Wave



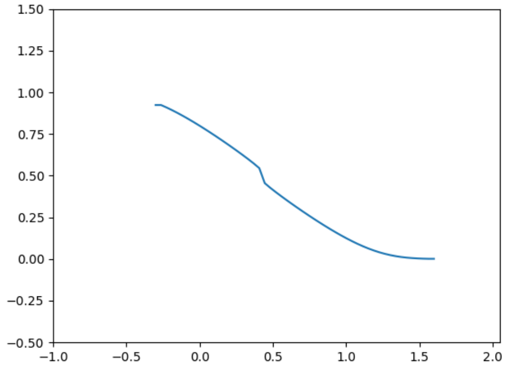
(a) t = 0.0



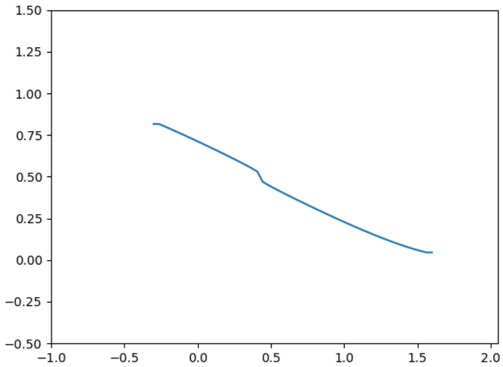
(b) t = 0.2



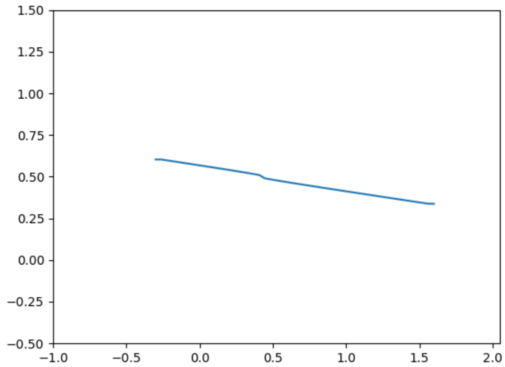
(c) t = 0.5



(d) t = 1.0



(e) t = 1.5

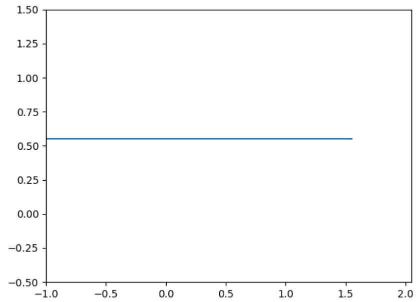


(f) t = 5.0

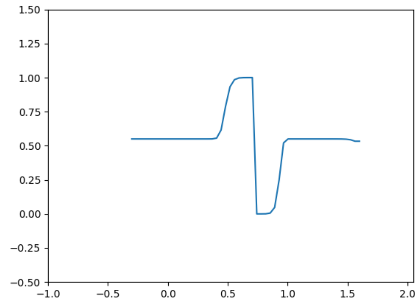
Figure 1: Rarefaction Wave

It can be seen in from the plots, that the density at the left side x = 0.4 is 1 and on the right side is 0. As time progresses, the density distributes over the road. At t = 5, the density starts to settle down to rho = 0.5

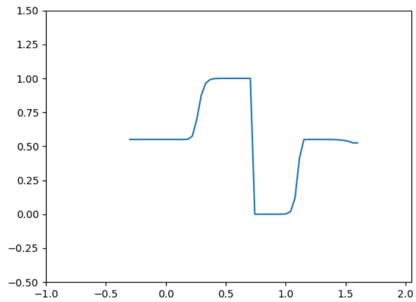
2.3 Traffic Signal



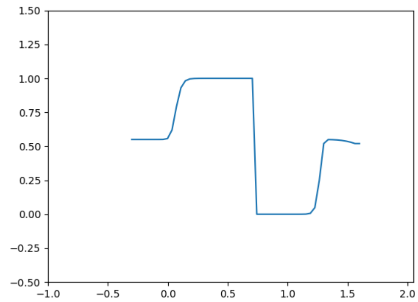
(a) t = 0.0



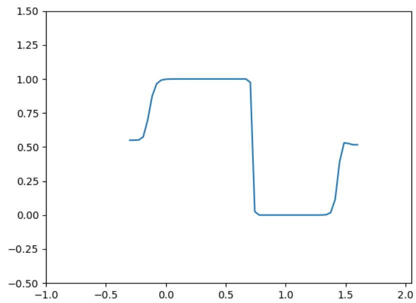
(b) t = 0.5



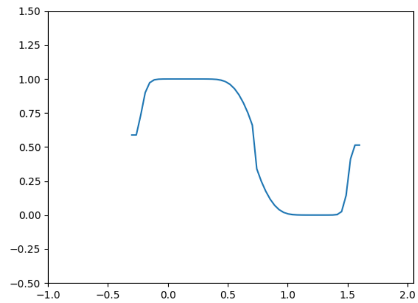
(c) t = 1.0



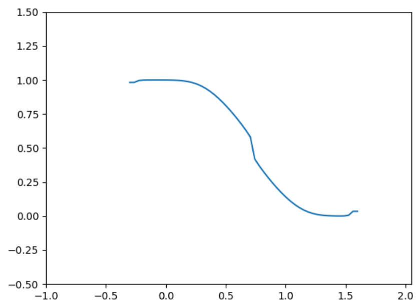
(d) t = 1.5



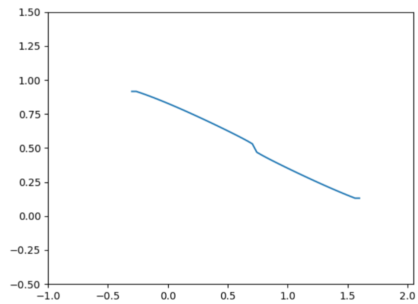
(e) t = 2.0



(f) t = 2.2



(g) t = 2.5



(h) t = 3.5

Figure 2: Traffic Signal

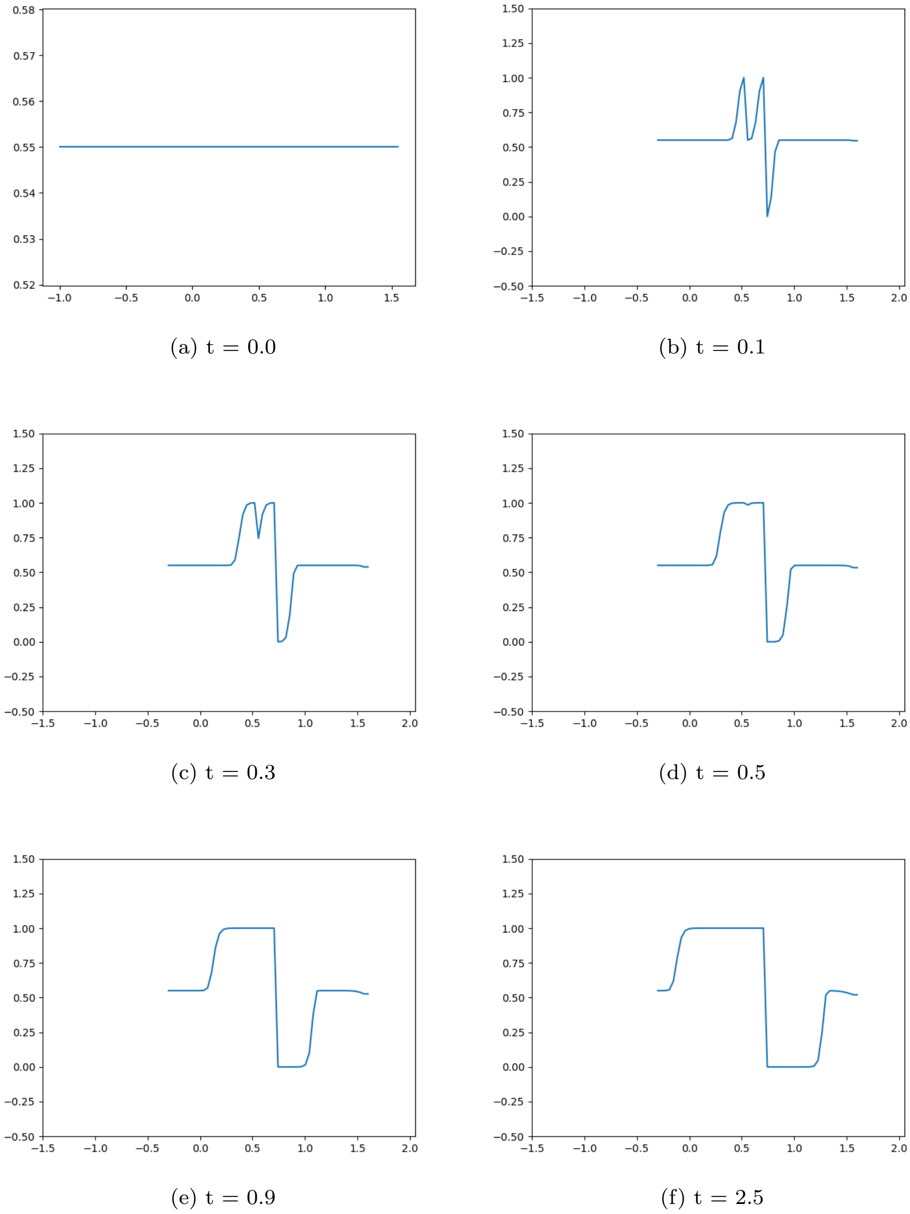
The traffic signal is present at x = 0.7

Initially the density is evenly distributed as rho = 0.55 throughout the entire road. Then the red light is turned on. Two things start to happen, vehicles begin to collect on the left side of the signal, and since vehicles on the right side of the signal are free to move, density starts decreasing on the right side.

At t = 2, the signal turns green again, and therefore the vehicle density starts getting distributed.

2.4 Speed Breaker

2.4.1 Before the Traffic Signal



(a) t = 0.0

(b) t = 0.1

(c) t = 0.3

(d) t = 0.5

(e) t = 0.9

(f) t = 2.5

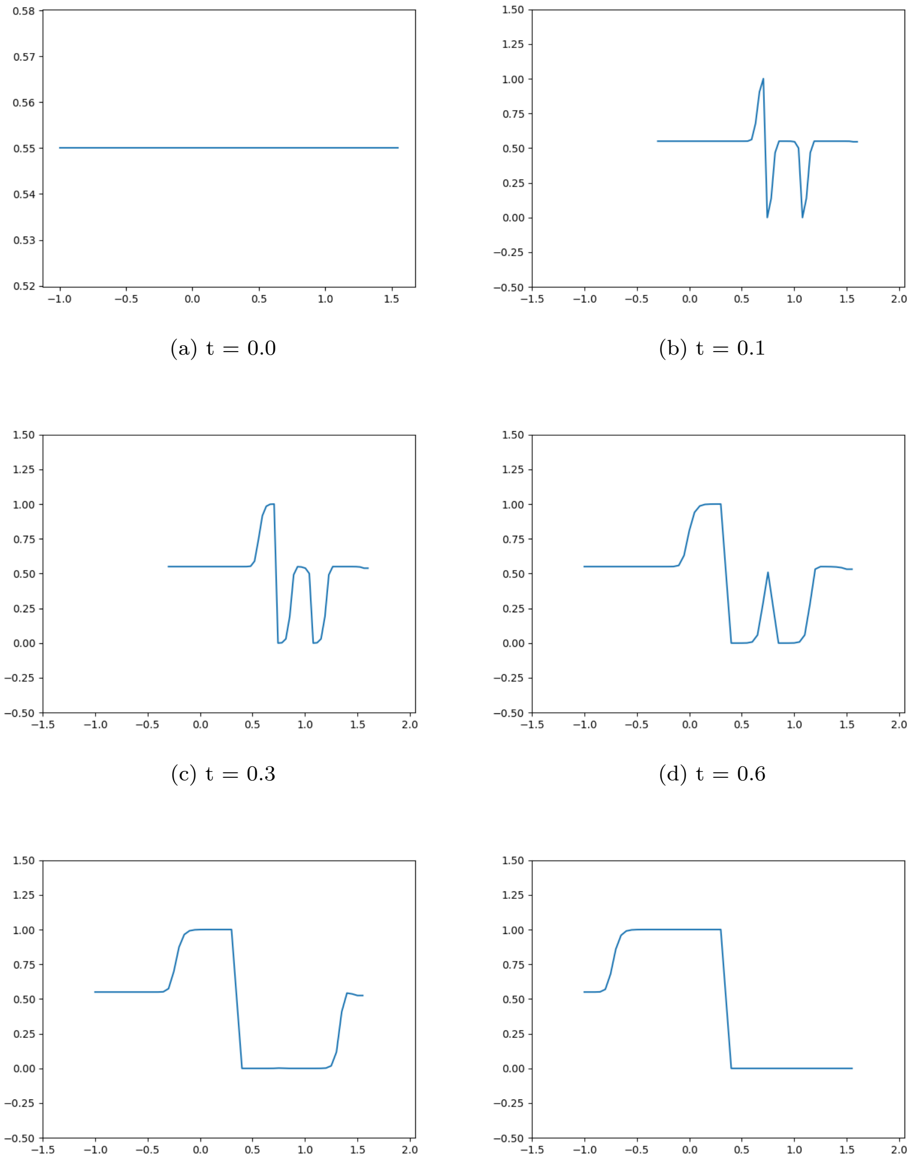
Figure 3: Speed breaker before the traffic signal

The traffic signal is present at x = 0.7, and speed breaker is at x = 0.5

Initially the density is evenly distributed throughout the entire road, rho = 0.55

At t = 0.1, it can be seen that density on the right of the traffic signal starts to decrease as vehicles begin to move out. On the left side, two peaks can be seen. Since, speed breaker decreases the vehicle speed, vehicles begin to pile up just behind the speed breaker, while another group of vehicles begins to pile behind the traffic signal. As the vehicles cross the speed breaker, the density to its right begins to increase, but there is still some gap between the two peaks, as vehicles slow down before crossing the speed breaker. At t = 0.5, it can be seen that this gap on the left and right side of the speed breaker is almost full. And later, vehicles begin to pile up normally.

2.4.2 After the Traffic Signal



(a) t = 0.0

(b) t = 0.1

(c) t = 0.3

(d) t = 0.6

(e) t = 1.0

(f) t = 1.9

Figure 4: Speed breaker before the traffic signal

The traffic signal is present at x = 0.7, and speed breaker is at x = 1.1

Initially the density is evenly distributed throughout the entire road, rho = 0.55

At t = 0.1, it can be seen that the vehicles begin to pile up behind the traffic signal. On the right of the signal, two things are happening, vehicles on the right side of the speed breaker begin to move out. But, since vehicles have to slow down before the speed breaker, we can see an area between the traffic signal and the speed breaker where the dinsity is high. This density decreases as the vehicle cross the speed breaker. At t = 1.9, no vehicle are present on the right side of the traffic signal, as all of them have crossed the speed breaker, and vehicles have piled to the right of the signal.

3 Speed as Hyperbolic

The velocity:

$$v(\rho) = \begin{cases} v_{max}, & \rho \leq \rho_{critical} \\ v_{max}(\frac{1}{\rho} - \frac{1}{\rho_{jam}}), & \rho_{critical} \leq \rho \leq \rho_{jam} \\ 0, & \rho \geq \rho_{jam} \end{cases}$$

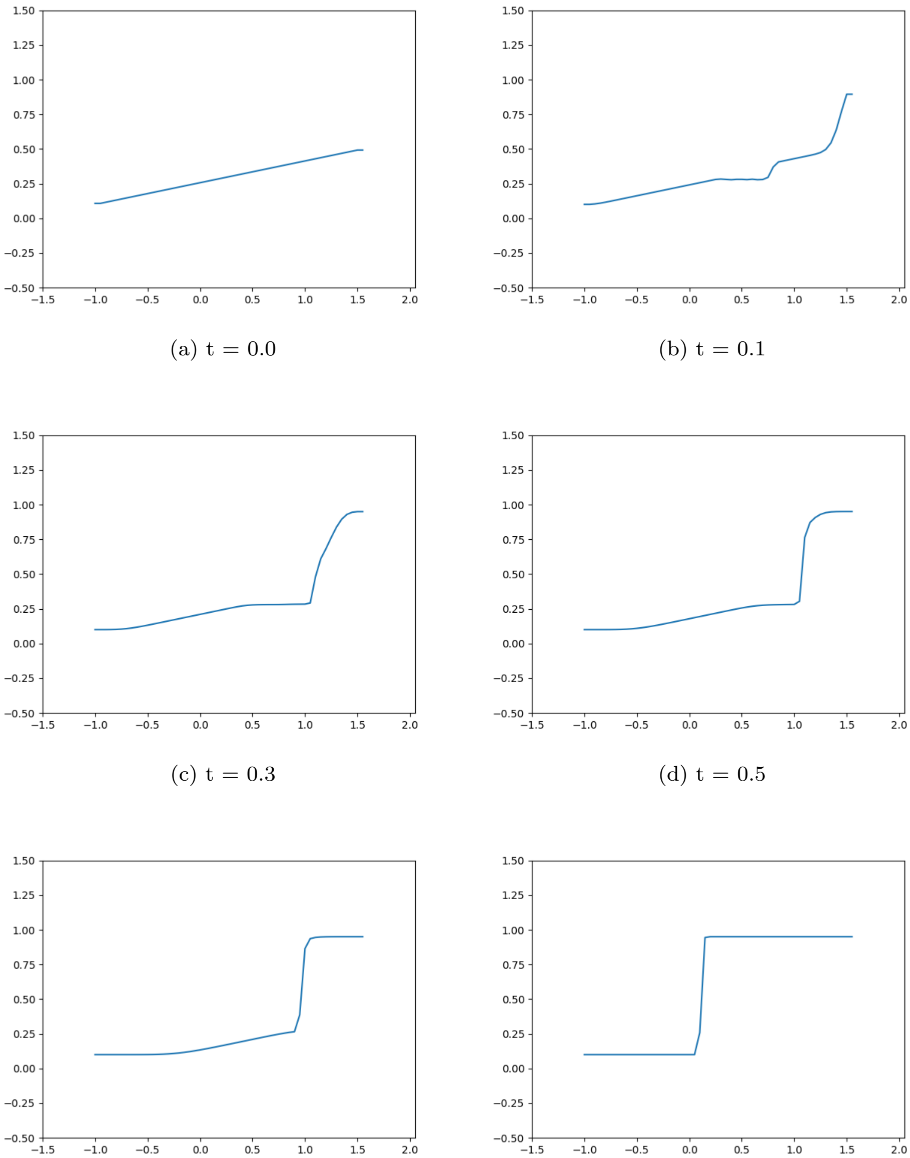
Corresponding flow function:

$$f(\rho) = \begin{cases} \rho v_{max}, & \rho \leq \rho_{critical} \\ v_{max}(1 - \frac{\rho}{\rho_{jam}}), & \rho_{critical} \leq \rho \leq \rho_{jam} \\ 0, & \rho \geq \rho_{jam} \end{cases}$$

Initial conditions:

$$\rho(x, 0) = \rho_0(x)$$

3.1 Traffic jam occurs



(a) t = 0.0

(b) t = 0.1

(c) t = 0.3

(d) t = 0.5

(e) t = 0.8

(f) t = 7.0

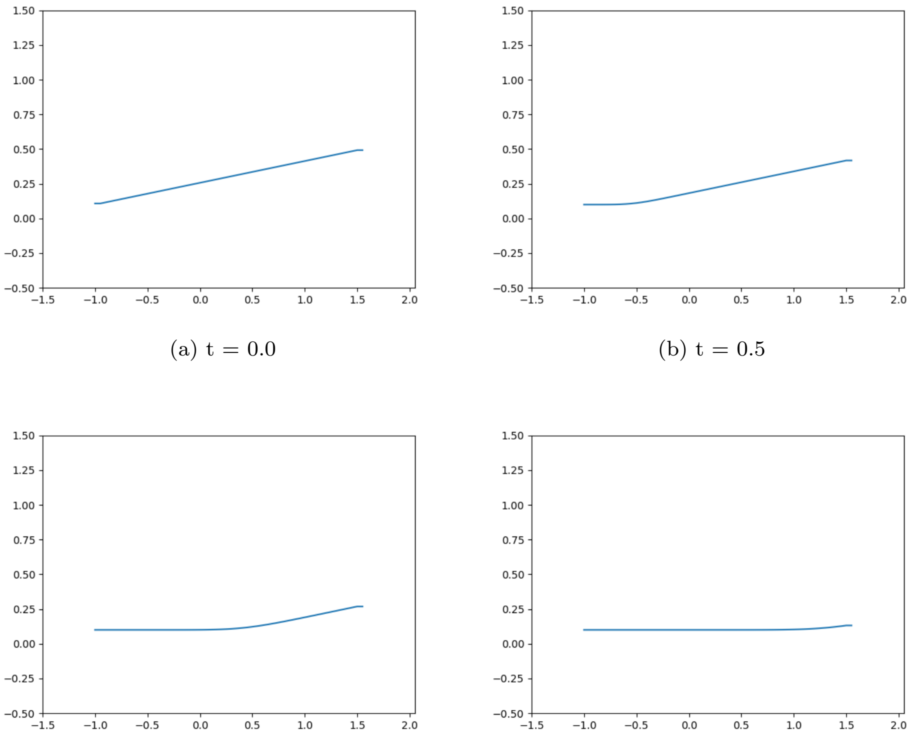
Figure 5: Speed as hyperbolic - Traffic jam

Here, $\rho_{critical} = 0.4$ and $\rho_{jam} = 0.95$

Initially, the density increases linearly from rho = 0.1 to rho = 0.5.

The vehicles at density less than $\rho_{critical}$ can move at full velocity, while vehicles at density between $\rho_{critical}$ and ρ_{jam} have to move slower. This leads to increase in density in the center of the road, which leads to piling up of vehicles towards the end. As a result a traffic jam occurs at the end, since density over there becomes equal to ρ_{jam}, and no vehicle can move forward now. Therefore, vehicles start collecting behind the jam.

3.2 Traffic jam does not occurs



(a) t = 0.0

(b) t = 0.5

(c) t = 1.5

(d) t = 2.5

Figure 6: Speed as hyperbolic - No traffic jam

Here, $\rho_{critical} = 0.5$ and $\rho_{jam} = 0.95$

Initially, the density increases linearly from rho = 0.1 to rho = 0.5.

In this case, since the critical density is more, all the cars are able to move at maximum velocity and no car slows down due to increase in density. Therefore, The density slowly settles down to a constant value, as all the old cars move out of the road.

4 Characteristics

Model under consideration:

$$\rho_t + (\frac{\rho^2}{2})_x = 0$$

Here, the velocity is defined as:

$$v = \frac{\rho}{2}$$

Rankine-Hugoniot Condition:

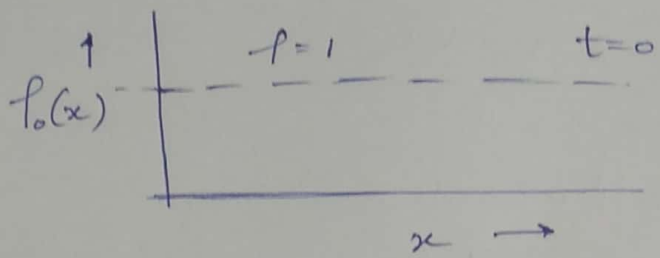
$$s(t) = \frac{dx}{dt} = \frac{f(\rho_{left}) - f(\rho_{right})}{\rho_{left} - \rho_{right}}$$

Entropy Condition:

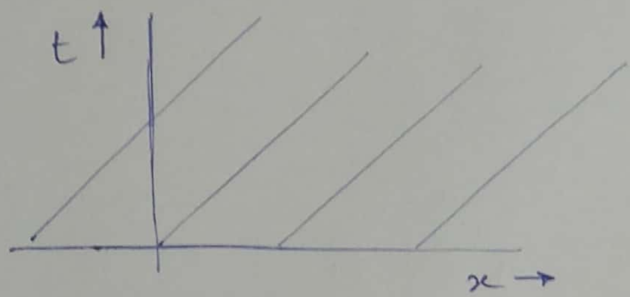
$$\frac{df(\rho_{left})}{d\rho} > \frac{dx}{dt} = s(t) > \frac{df(\rho_{right})}{d\rho}$$

Case 1 :

$$f(x,0) = 1 \quad \forall x \in \mathbb{R}$$



Characteristics

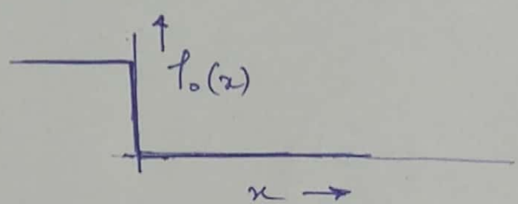


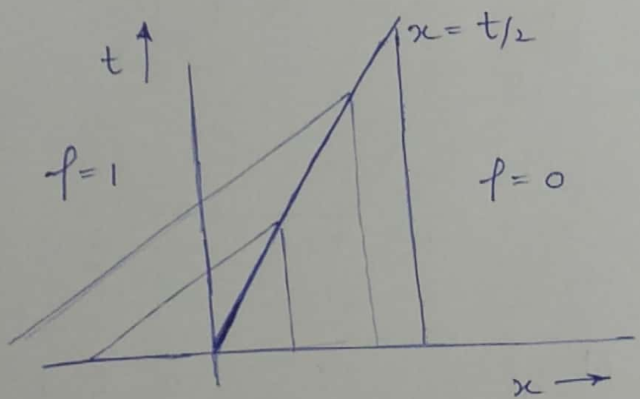
$$f(x,t) = 1, \quad x \in \mathbb{R}, \ t > 0$$

This means that the traffic density will remain constant throughout the road $\forall \, t > 0$

Case 2:

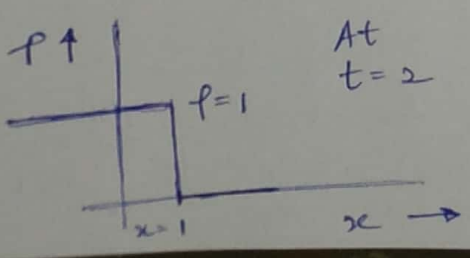
$$f(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$$



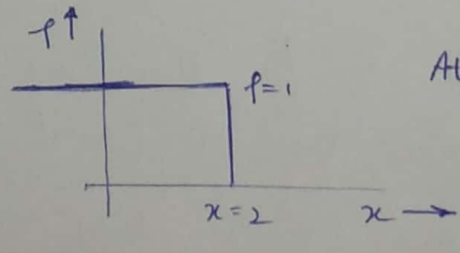


$$f(x,t) = \begin{cases} 1 & x < \frac{t}{2} \\ 0 & x \geq \frac{t}{2} \end{cases}$$

According to the characteristics, the discontinuity will move in x as time progresses

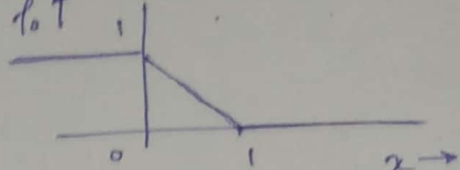


At t = 2

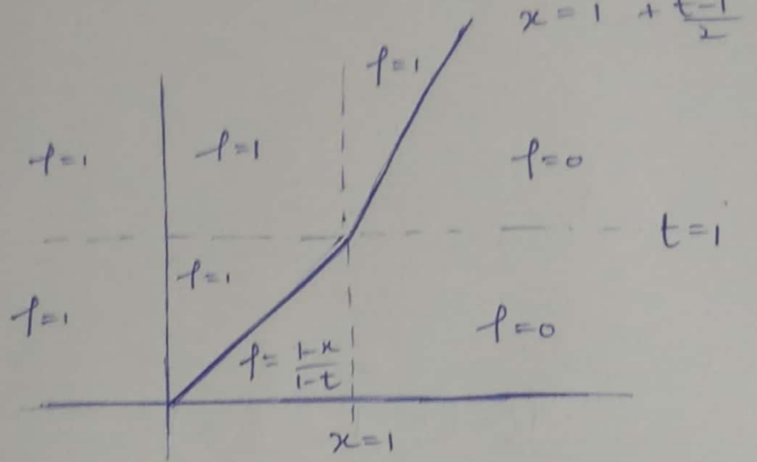


At t = 4

3. $$f(x,0) = \begin{cases} 1 & x < 0 \\ 1-x & 0 \leq x < 1 \\ 0 & x \leq 1 \end{cases}$$



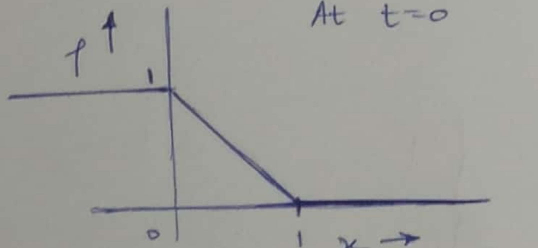
Characteristics

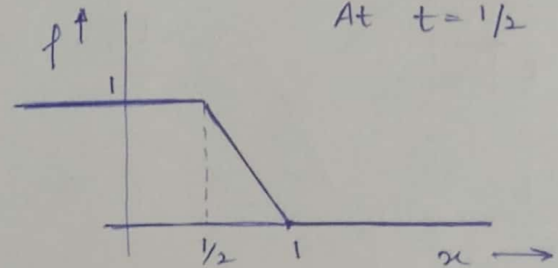


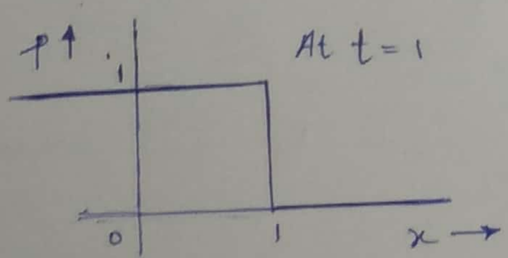
For $t < 1$ $$f(x,t) = \begin{cases} 1 & x < t \\ \frac{1-x}{1-t} & t \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

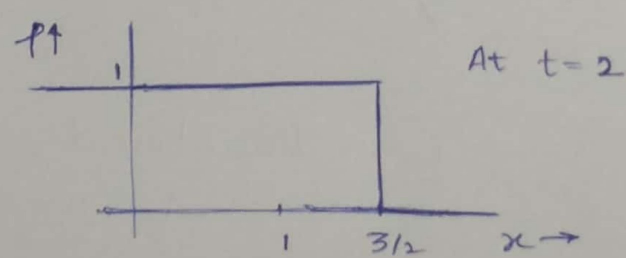
For $t \geq 1$ $$f(x,t) = \begin{cases} 1 & x < \frac{1}{2}(t-1) + 1 \\ 0 & x > 1 + \frac{1}{2}(t-1) \end{cases}$$

The density vs x plots for different time





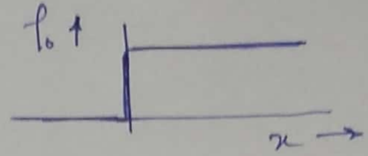




In this model $V = \frac{f}{2}$, therefore, where the density is 0, $V=0$
Hence the wave does not travel beyond $x=1$ for $t<1$. After $t \geq 1$, the wave reaches $x=1$ and this is now similar to case 1
The wave travels along with the discontinuity

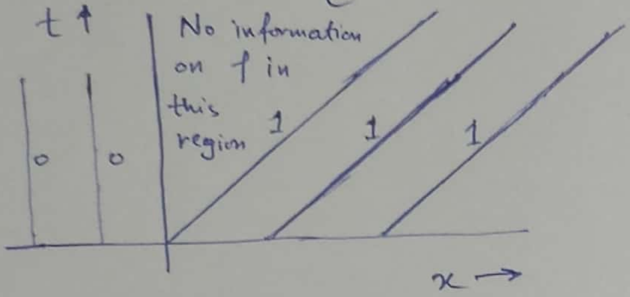
Case 4:

$$f(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Solution 1 :

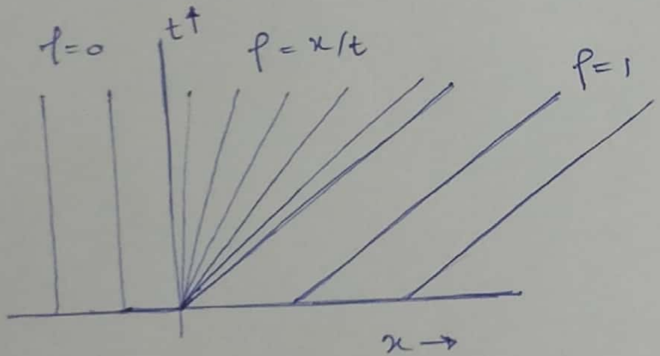
$$f(x,t) = \begin{cases} 0 & x < t/2 \\ 1 & x \geq t/2 \end{cases}$$



This is an unrealistic Solution because there exists a region in this case where density information is not available

Solution 2:

$$f(x,t) = \begin{cases} 0 & x \leq 0 \\ x/t & 0 \leq x < t \\ 1 & x \geq t \end{cases}$$



This solution is physically realistic

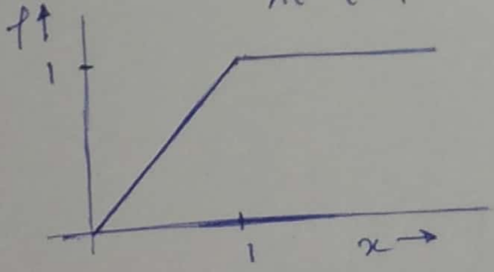
It satisfies entropy condition

$$f'(f_L) > \frac{dx}{dt} = s(t) > f'(f_R)$$

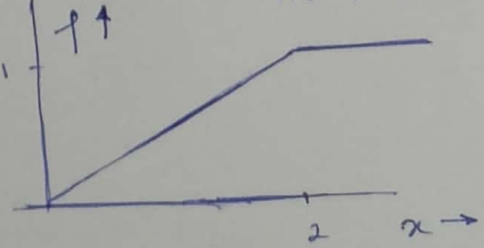
Which guarantees uniqueness

$f(x,t)$ For different time

At $t=1$



At $t=2$



This says that taller waves move faster than shorter waves. In case 1 and 2, the wave was taller on the left at t = 0, and, thus moving faster than the wave on the right. Therefore, it was expected that the wae on the left would overtake the wave on the right, and hence this resulted in discontinuity is the solution.

In case 4, initially the wave is higher to the right. Therefore the wave on the right should move faster. We choose the solution that is more realistic, physically.

5 References

- MA5710 Class Notes
- Traffic Flow Model - Sushmita Rose John
- Partial Differential Equations of Applied Mathematics (Stanford) - Class Notes
- Generalized Rankine-Hugonoit condition and shock solutions for quasilinear hyperbolic systems - Xiao-Biao Lin
- Burgers Equation - Mikel Landajuela
- https://www.uni-muenster.de/imperia/md/content/physik_tp/lectures/ws2016-2017/num_methods_i/burgers.pdf