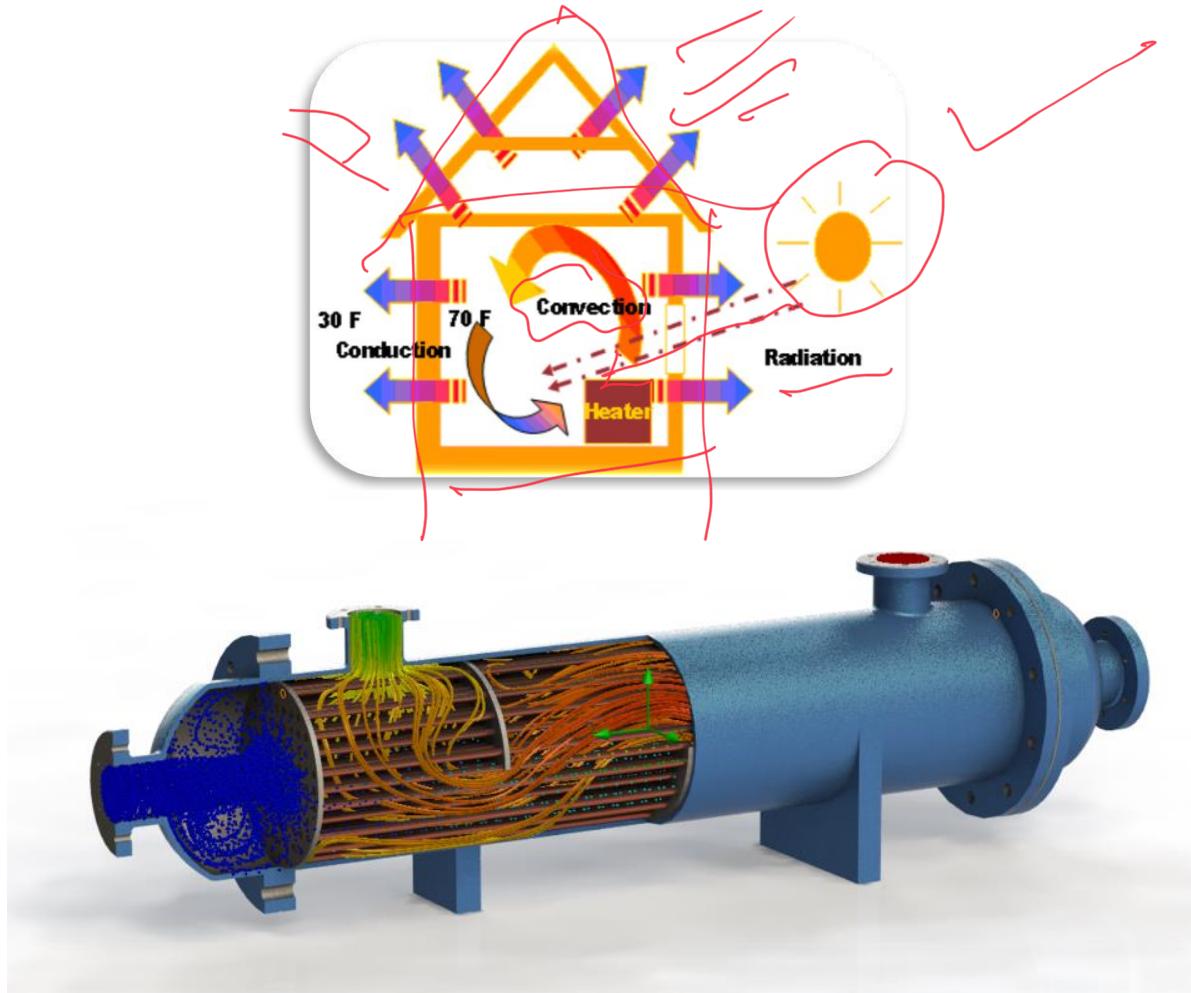


Heat Transfer Operations

Session-2. Heat Conduction



Dr. Jogender Singh (JS)

Summary of Session-1

Defined **energy**: product of a force over a distance which it is applied.

Temperature is the measure of the average kinetic energy of a given substance, or internal energy. Defined °F, °C, and K scales and their conversions.

Assuming pressure remains constant, an increase in temperature means the volume of air expands and its density decreases.

Heat is energy in the process of being transferred.

Conduction: heat transfer molecule by molecule

Convection: mass movement of a fluid or gas

Radiation: electromagnetic waves pass through the empty space

Latent heat is energy associated with phase changes from one state of matter to another.



Specific Objectives of Session-2

- ✓ Know and understand Fourier's law.
- ✓ Apply Fourier's law in steady state heat transfer problems using rectangular coordinates, unidirectional heat flow and constant thermal conductivity. Identify the normal area to the heat flow.
- ✓ Understand the analogy between electrical and thermal resistances.
Apply the analogy with resistances in series.
- ✓ Apply Fourier's law in heat transfer problems through **cylindrical walls**.
- ✓ Understand the concept of **logarithmic mean area** in heat transfer through cylindrical walls.
- ✓ Solve problems of heat transfer through **multiple cylindrical walls** with different thermal conductivities.



Fourier's law of conduction



Various flows and their driving forces

Flow	Driving force
Electricity flow	Electric potential gradient
Fluid flow	Pressure gradient
Heat flow	Temperature gradient

Thus, the heat flow per unit area per unit time (heat flux) can be represented as,

$$\dot{q}' \propto \frac{dT}{dx} \quad (2.1)$$

$$\dot{q}' = k \frac{dT}{dx} \quad (2.2)$$

where, proportionality constant k is the thermal conductivity of the material, T is the temperature and x is the distance in the direction of heat flow. This is known as Fourier's law of conduction.

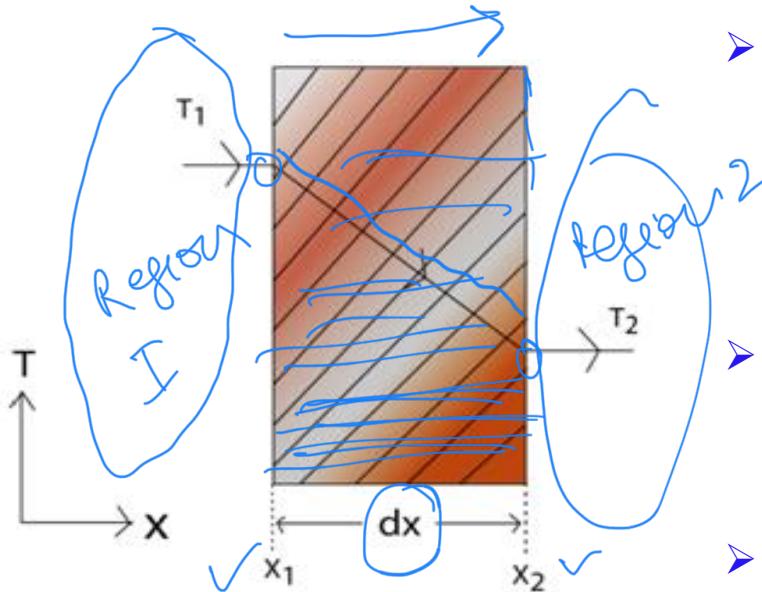
Jean Baptiste Joseph Fourier

Steady-state conduction is defined as the condition which prevails when temperatures at fixed points do not change with time.

One-dimensional is applied to a heat conduction problem when only one coordinate is required to describe the distribution of temperature within the body. Such a situation hardly exists in real engineering problems. However, the real problem is solved fairly up to the accuracy of practical engineering interest.



Steady-state conduction through constant area



- The flat wall of thickness d_x is separated by two regions, the one region is at high temperature (T_1) and the other one is at temperature T_2 . The wall is very large in comparison of the thickness so that the heat losses from the edges are negligible.
- Consider there is no generation or accumulation of the heat in the wall and the external surfaces of the wall are at isothermal temperatures T_1 and T_2 .
- The area of the surface through which the heat transfer takes place is A . Then the eq.2.2 can be written as,

$$\dot{q}' = -k \frac{dT}{dx} \quad (2.3)$$

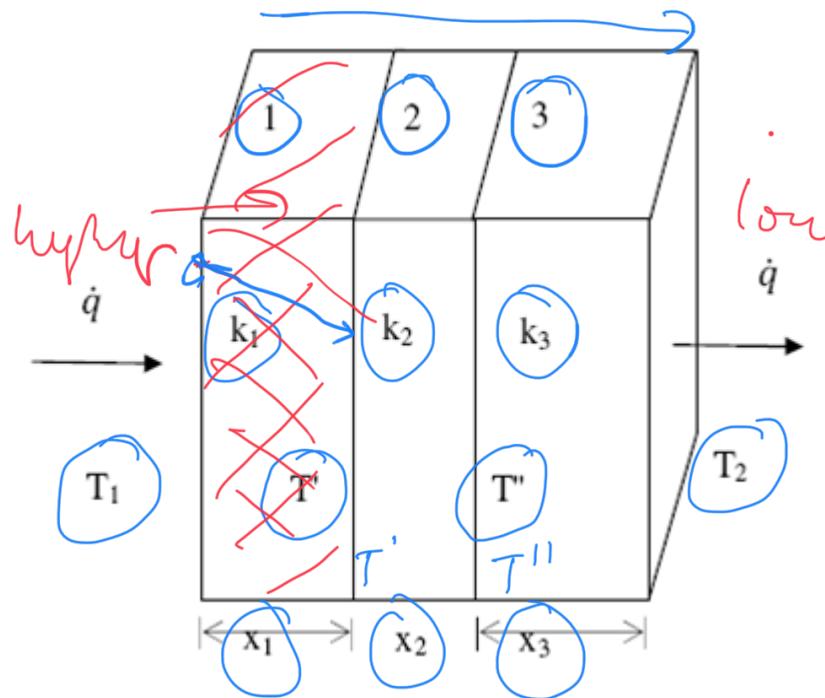
$$\dot{q} = -k \frac{dT}{dx} \quad (2.4)$$

$$\frac{\dot{q}}{A} = -k \frac{T_2 - T_1}{x_2 - x_1} \quad (2.5)$$

The negative sign (-) shows that the heat flux is from the higher temperature surface to the lower temperature surface.

Analogy between electrical and thermal resistances

Consider three different layers of materials having different thermal conductivities and thicknesses of the layers.



$$\dot{q}_r = -kA \frac{dT}{dx}$$

The rate of heat transfer through layer-1 to 2 will be,

$$\dot{q} = +k_1 A \frac{(T_1 - T')}{x_1} \quad \text{or} \quad (T_1 - T') = \frac{\dot{q}}{1/(x_1/k_1 A)} \quad (2.6)$$

and rate of heat transfer through layer-2 to 3 will be,

$$\checkmark \dot{q} = \frac{k_2 A (T' - T'')}{x_2} \quad \text{or} \quad (T' - T'') = \frac{\dot{q}}{1/(x_2/k_2 A)} \quad (2.7)$$

The rate of heat transfer through layer 3 to the other side of the wall

$$\checkmark \dot{q} = \frac{k_3 A (T'' - T_2)}{x_3} \quad \text{or} \quad (T'' - T_2) = \frac{\dot{q}}{1/(x_3/k_3 A)} \quad (2.8)$$

On adding the above three equations (2.6 to 2.8)

$$\dot{q} = \frac{T_1 - T_2}{\frac{x_1}{k_1 A} + \frac{x_2}{k_2 A} + \frac{x_3}{k_3 A}}$$

$$\dot{q} = \frac{T_1 - T_2}{R_1 + R_2 + R_3}$$

$$R_1 = \frac{x_1}{k_1 A} \quad \checkmark$$

$$R_2 = \frac{x_2}{k_2 A} \quad \checkmark$$

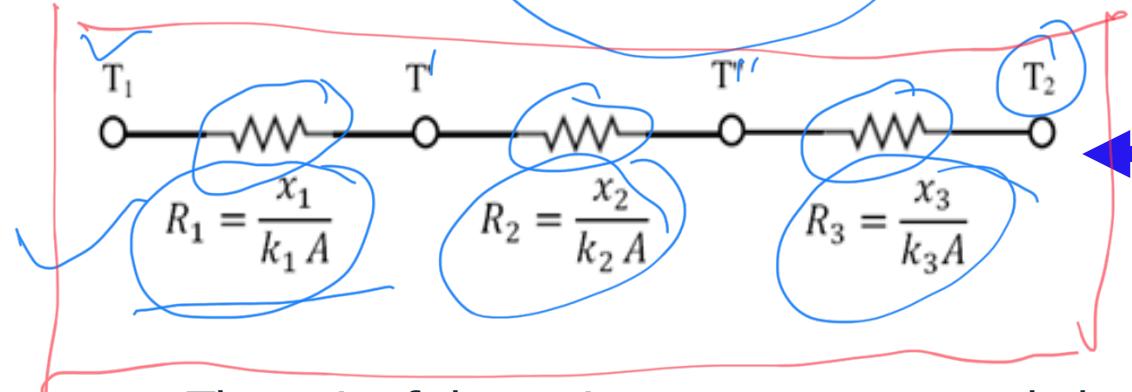
$$R_3 = \frac{x_3}{k_3 A}$$

(2.10)

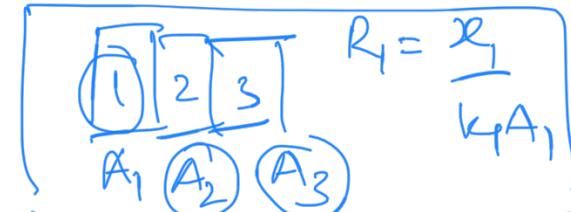


Where, R represents the thermal resistance of the layers. The above relation can be written analogous to the electrical circuit as,

$$\text{Rate of heat flow} = \frac{\text{Temperature difference}}{\text{Resistance}}$$



(2.11)



The wall is composed of 3-different layers in series and thus the total thermal resistance was represented by $R (= R_1 + R_2 + R_3)$. The discussed concept can be understood by the illustrations shown here.

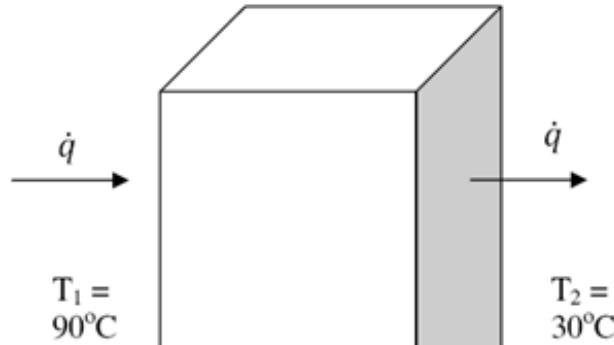
The unit of the various parameters used above is summarized as follows,

Parameter	Symbol used	Unit
Heat flux	\dot{q}'	W/m^2 or $\text{J/(s}\cdot\text{m}^2)$
Heat flow rate	\dot{q}	W or J/s
Thermal conductivity	k	$\text{W/(m}\cdot\text{C)}$
Thermal resistance	R	$^{\circ}\text{C/W}$



$$0.0028 \quad | \quad 0.1378$$

P. 2.1. The two sides of a wall (2 mm thick, with a cross-sectional area of 0.2 m^2) are maintained at 30°C and 90°C . The thermal conductivity of the wall material is $1.28 \text{ W}/(\text{m. }^\circ\text{C})$. Find out the rate of heat transfer through the wall?



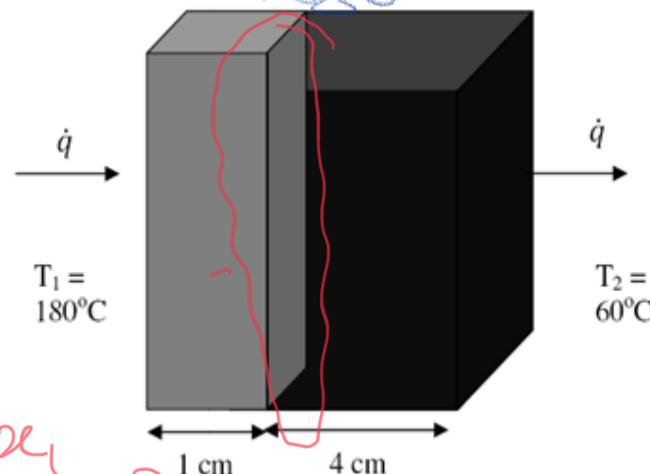
$$\left| \frac{dQ}{dT} = AK \frac{dT}{L} \right|$$

$$Q = -K A \frac{\Delta T}{\Delta x}$$

$$Q = -1.28 \times 0.2 \times \frac{90 - 30}{0.02}$$

$$Q = 7680 \text{ W}$$

P. 2.2. One side of a 1 cm thick stainless steel wall ($k_1 = 19 \text{ W}/(\text{m. }^\circ\text{C})$) is maintained at 180°C and the other side is insulated with a layer of 4 cm fiberglass ($k_2 = 0.04 \text{ W}/(\text{m. }^\circ\text{C})$). The outside of the fiberglass is maintained at 60°C and the heat loss through the wall is 300 W. Determine the area of the wall.



$$q_i = \frac{\Delta T}{R} = \frac{T_1 - T_2}{R} = \frac{180 - 60}{R} = \frac{120}{R}$$

$$R = \frac{L}{kA}$$

$$R = R_1 + R_2$$

$$R = \frac{x_1}{k_1 A} + \frac{x_2}{k_2 A}$$

$$R = \frac{0.01}{19 A} + \frac{0.04}{0.04 A}$$

$$R = \frac{0.01}{19 A} + \frac{1}{A}$$

$$R = \frac{0.01 + 19}{19 A} = \frac{19.01}{19 A} = \frac{1.01}{A}$$

$$300 = \frac{120}{1.01/A}$$

$$A = \frac{120}{300} = 0.4 \text{ m}^2$$

Concept of equivalent resistance (Composite wall)

The wall is composed of seven different layers indicated by 1 to 7. The interface temperatures of the composite are T_1 to T_5 as shown in the fig. 1. The equivalent electrical circuit of the above composite is shown in the fig. 2. below

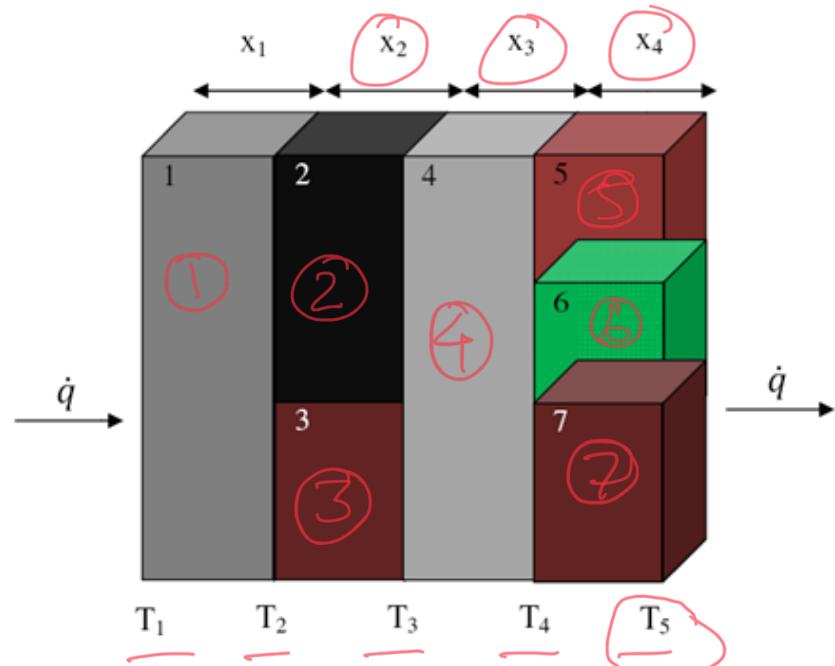


fig. 1.

The equivalent resistance of the wall will be,

$$R = R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}} + R_4 + \frac{1}{\frac{1}{R_5} + \frac{1}{R_6} + \frac{1}{R_7}} \quad (2.12)$$

$$\text{where, } R_i = \frac{x_i}{k_i A_i}$$

Therefore,

$$\dot{q} = \frac{T_1 - T_5}{R} \quad (2.13)$$

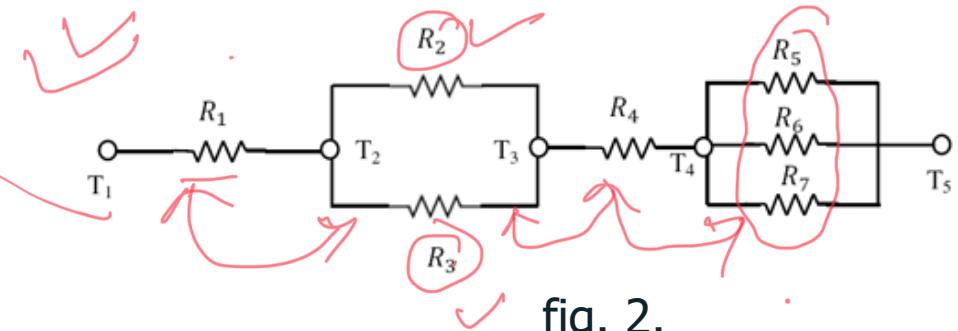


fig. 2.

P. 2.3. Consider a composite wall containing 5-different materials as shown in the fig. Calculate the rate of heat flow through the composite from the following data?

$$x_1 = 0.1 \text{ m}$$

$$k_1 = 15 \text{ W/m}^\circ\text{C}$$

$$k_4 = 20 \text{ W/m}^\circ\text{C}$$

$$h_2 = 1 \text{ m}$$

$$h_5 = 1.5 \text{ m}$$

$$x_2 = 0.2 \text{ m}$$

$$k_2 = 25 \text{ W/m}^\circ\text{C}$$

$$k_3 = 35 \text{ W/m}^\circ\text{C}$$

$$h_3 = 3 \text{ m}$$

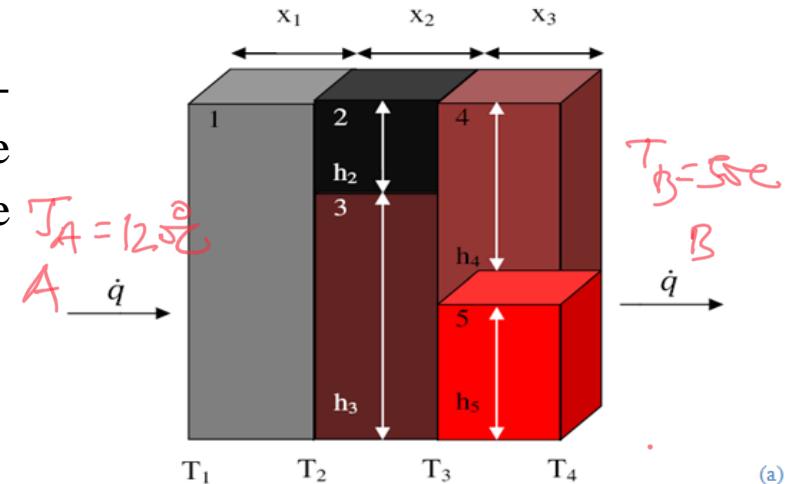
$$T_A = 120^\circ\text{C}$$

$$x_3 = 0.15 \text{ m}$$

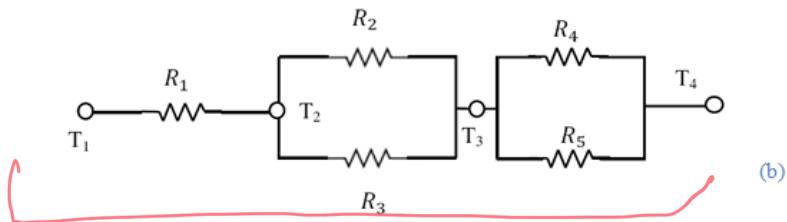
$$k_3 = 30 \text{ W/m}^\circ\text{C}$$

$$h_4 = 2.5 \text{ m}$$

$$T_B = 50^\circ\text{C}$$



(a)



(b)

Assumptions:

1. Steady-state one-dimensional conduction.
2. Thermal conductivity is constant for the temperature range of interest.
3. The heat loss through the edge side surface is insignificant.
4. The layers are in perfect thermal contact.
5. Area in the direction of heat flow is 1 m².



Steady-state heat conduction through a variable area

Cylinder

Consider a hollow cylinder as shown in the fig. The inner and outer radius is represented by r_i and r_o , whereas T_i and T_o ($T_i > T_o$) represent the uniform temperature of the inner and outer wall, respectively.

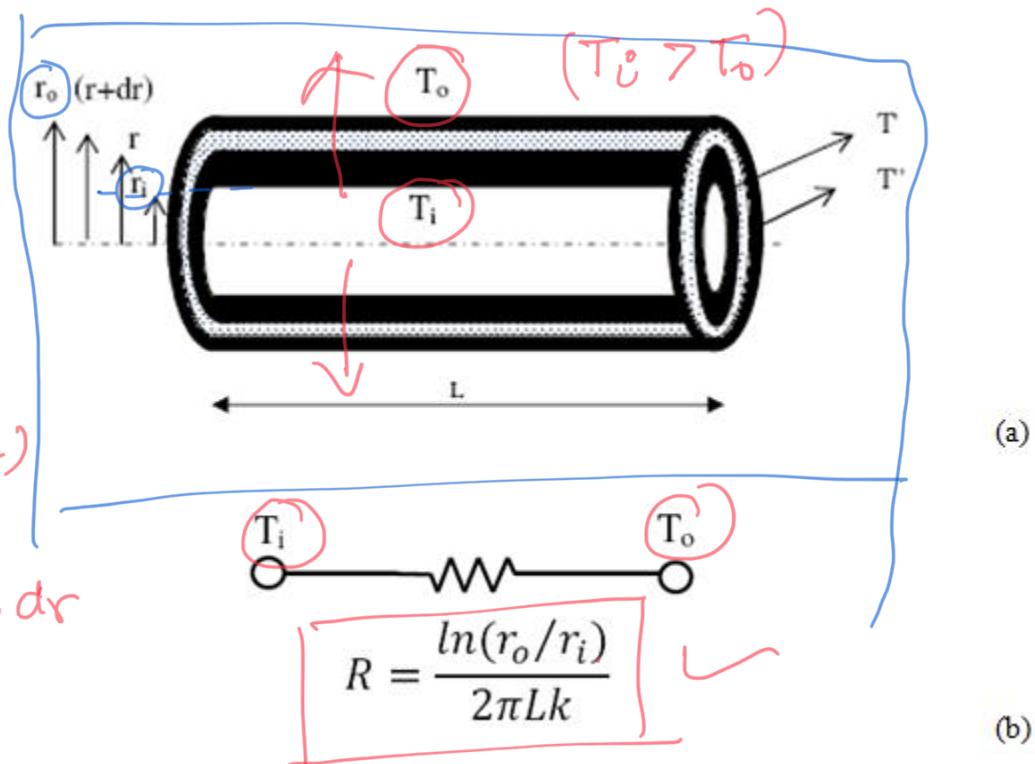
$$\checkmark \dot{q} = -k \frac{dT}{dr} (2\pi r L) \quad \dot{q} = -k A \frac{dT}{dr} \xrightarrow{(A \propto r^2)} \quad (2.14)$$

On rearranging

$$\frac{dr}{r} = -\frac{2\pi L k}{\dot{q}} dT \quad (2.15)$$

Or

To get the heat flow through the thick wall cylinder, the above equation can be integrated between the limits,



$$\int_{r_i}^{r_o} \frac{dr}{r} = -\frac{2\pi L k}{\dot{q}} \int_{T_i}^{T_o} dT \quad (2.16)$$



On solving,

$$\dot{q} = k(2\pi L) \frac{(T_i - T_o)}{\ln(r_o/r_i)} \quad (2.17)$$

$$\dot{q} = kA_{LM} \frac{(T_i - T_o)}{r_o - r_i} \quad (2.18)$$

Where

$$A_{LM} = \frac{2\pi L(r_o - r_i)}{\ln(r_o/r_i)} = 2\pi L r_{LM} \quad (2.19)$$

The careful analysis of the above equation shows that the expression is same as for heat flow through the plane wall of thickness ($r_o - r_i$) except the expression for the area. The A_{LM} is known as log mean area of the cylinder, whose length is L and radius is r_{LM}

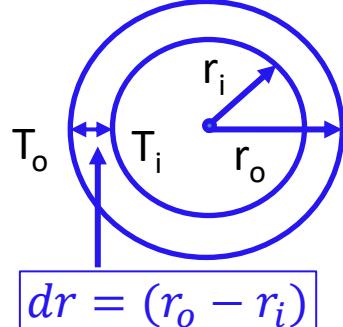
$$r_{LM} = \frac{r_o - r_i}{\ln(r_o - r_i)} \quad (2.20)$$



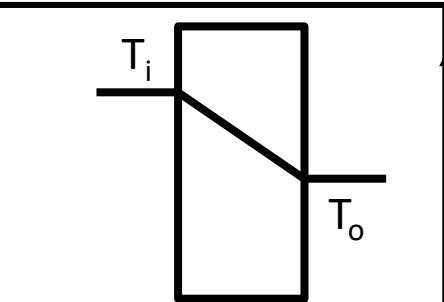
Log mean area (A_{LM})

Assumption

- (i) $T_i > T_o$
- (ii) Equal thickness for heat transfer (dr)



$$\dot{q}_c = \frac{(T_o - T_i)}{\ln(r_o/r_i)/2\pi LK}$$



$$\dot{q}_w = \frac{(T_o - T_i)}{(r_o - r_i)/KA}$$

Since thickness is equal the rate of heat transfer will also be equal.

So $\dot{q}_c = \dot{q}_w$

$$\frac{(T_o - T_i)}{\ln(r_o/r_i)/2\pi LK} = \frac{(T_o - T_i)}{(r_o - r_i)/KA}$$

Or

$$A_{LM} = \frac{2\pi L(r_o - r_i)}{\ln(r_o/r_i)}$$

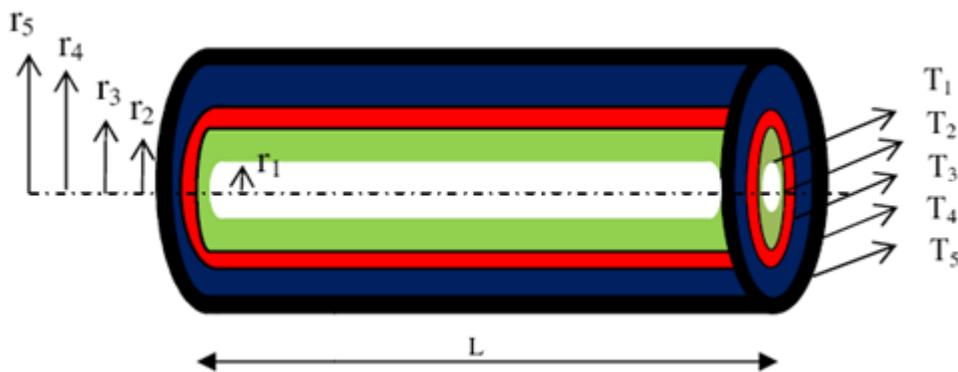
$$A_{LM} = \frac{2\pi Lr_o - 2\pi Lr_i}{\ln(2\pi Lr_o/2\pi Lr_i)}$$

Or

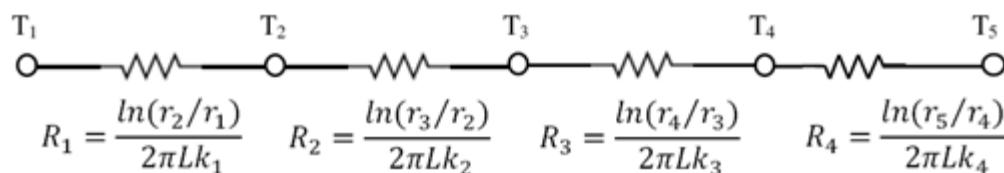
$$A_{LM} = \frac{A_o - A_i}{\ln(A_o/A_i)}$$

4-layers of solid material of different inner and outer diameter as well as thermal conductivity.

Four layer composite hollow cylinder



Equivalent electrical circuit



The total heat transfer at steady-state will be,

$$\dot{q} = \frac{T_1 - T_5}{R_1 + R_2 + R_3 + R_4} \quad (2.21)$$

where R_1, R_2, R_3 , and R_4 are represented in the fig

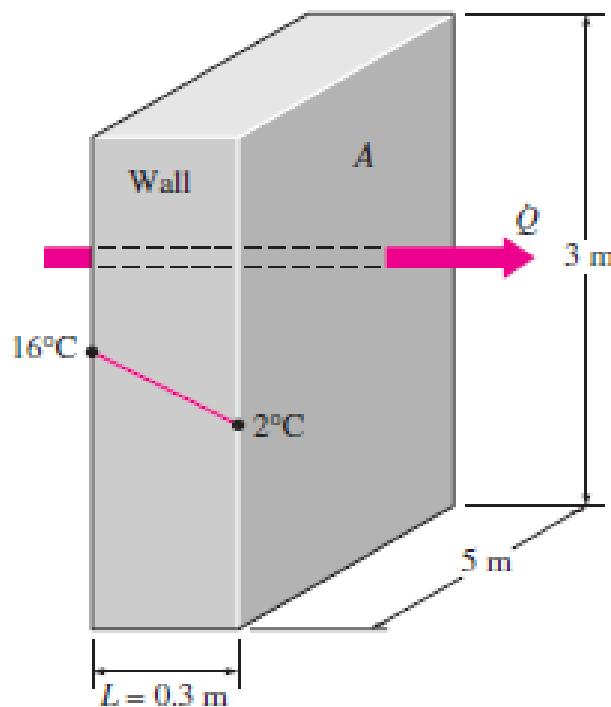
Sphere

The rate of heat transfer through a hollow sphere can be determined in a similar manner as for cylinder. The students are advised to derive the following expression shown below.

The final expression for the rate of heat flow is,

$$\dot{q} = 4\pi K \frac{(T_i - T_o)}{\frac{1}{r_i} - \frac{1}{r_o}} \quad (2.22)$$

- **P. 2.4.** Consider a 3-m-high, 5-m-wide, and 0.3-m-thick wall whose thermal conductivity is $k = 0.9 \text{ W/m} \cdot ^\circ\text{C}$. On a certain day, the temperatures of the inner and the outer surfaces of the wall are measured to be 16°C and 2°C , respectively. Determine the rate of heat loss through the wall on that day.



Thermal insulation, materials used to reduce the rate of heat transfer.

Insulation types and applications

Type	Temperature range, °C	Thermal conductivity, mW/m · °C	Density, kg/m ³	Application
1 Linde evacuated superinsulation	-240–1100	0.0015–0.72	Variable	Many
2 Urethane foam	-180–150	16–20	25–48	Hot and cold pipes
3 Urethane foam	-170–110	16–20	32	Tanks
4 Cellular glass blocks	-200–200	29–108	110–150	Tanks and pipes
5 Fiberglass blanket for wrapping	-80–290	22–78	10–50	Pipe and pipe fittings
6 Fiberglass blankets	-170–230	25–86	10–50	Tanks and equipment
7 Fiberglass preformed shapes	-50–230	32–55	10–50	Piping
8 Elastomeric sheets	-40–100	36–39	70–100	Tanks
9 Fiberglass mats	60–370	30–55	10–50	Pipe and pipe fittings
10 Elastomeric preformed shapes	-40–100	36–39	70–100	Pipe and fittings
11 Fiberglass with vapor barrier blanket	-5–70	29–45	10–32	Refrigeration lines
12 Fiberglass without vapor barrier jacket	to 250	29–45	24–48	Hot piping
13 Fiberglass boards	20–450	33–52	25–100	Boilers, tanks, heat exchangers
14 Cellular glass blocks and boards	20–500	29–108	110–150	Hot piping
15 Urethane foam blocks and boards	100–150	16–20	25–65	Piping
16 Mineral fiber preformed shapes	to 650	35–91	125–160	Hot piping
17 Mineral fiber blankets	to 750	37–81	125	Hot piping
18 Mineral wool blocks	450–1000	52–130	175–290	Hot piping
19 Calcium silicate blocks, boards	230–1000	32–85	100–160	Hot piping, boilers, chimney linings
20 Mineral fiber blocks	to 1100	52–130	210	Boilers and tanks



- **P. 2.5.** A thick-walled tube of stainless steel [18% Cr, 8% Ni, $k = 19 \text{ W/m} \cdot \text{°C}$] with 2-cm inner diameter (ID) and 4-cm outer diameter (OD) is covered with a 3-cm layer of asbestos insulation [$k = 0.2 \text{ W/m} \cdot \text{°C}$]. If the inside and final temperature of the wall of pipe are maintained at 600°C and 100°C , respectively, calculate the heat loss per meter of length. Also calculate the tube–insulation interface temperature.

Figure shows the thermal network for this problem. The heat flow is given by

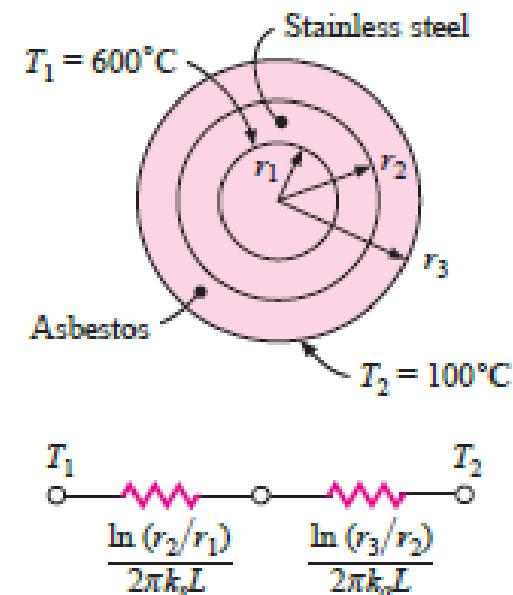
$$\frac{q}{L} = \frac{2\pi (T_1 - T_2)}{\ln(r_2/r_1)/k_s + \ln(r_3/r_2)/k_a} = \frac{2\pi (600 - 100)}{(\ln 2)/19 + (\ln \frac{5}{2})/0.2} = 680 \text{ W/m}$$

This heat flow may be used to calculate the interface temperature between the outside tube wall and the insulation. We have

$$\frac{q}{L} = \frac{T_a - T_2}{\ln(r_3/r_2)/2\pi k_a} = 680 \text{ W/m}$$

where T_a is the interface temperature, which may be obtained as

$$T_a = 595.8^\circ\text{C}$$



The largest thermal resistance clearly results from the insulation, and thus the major portion of the temperature drop is through that material.

The Heat Diffusion Equation

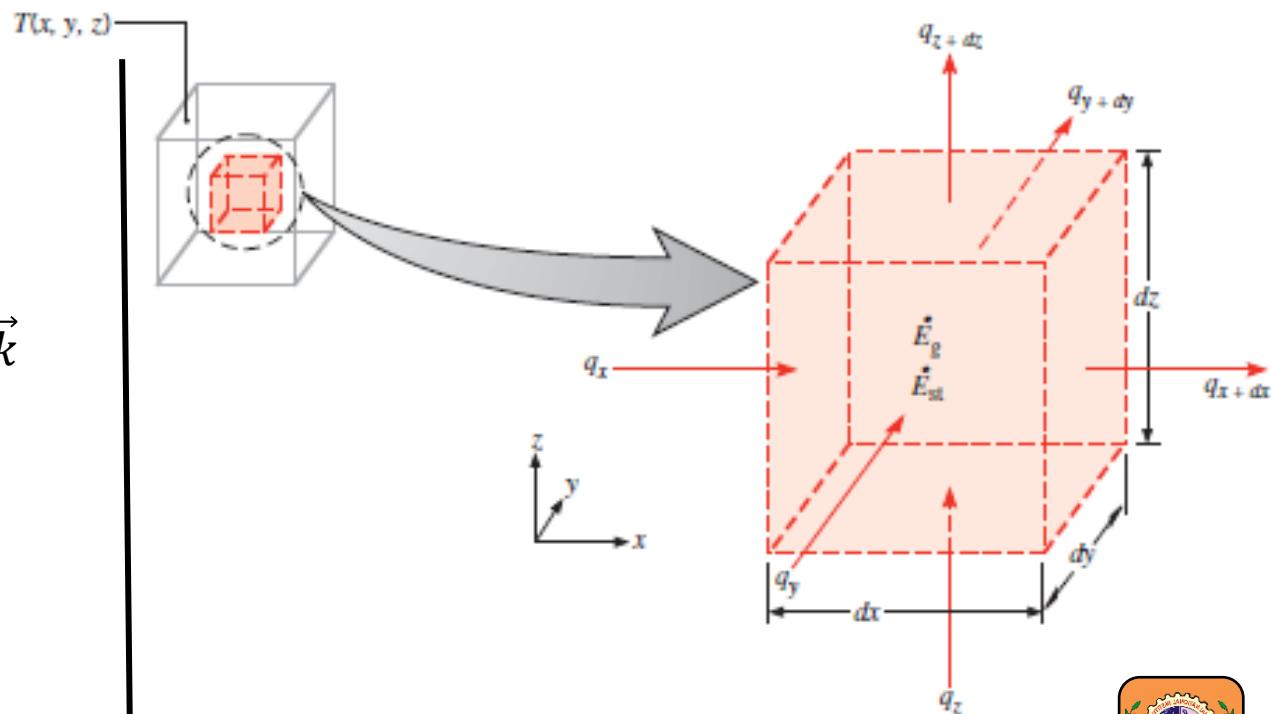
- The cases considered so far have been those in which the heat conducting solid is free of **internal heat generation**. However, the situations where the internal heat is generated are very common cases in chemical industries for example, the **exothermic reaction** in the solid pellet of a catalyst.
- Consider a homogeneous medium within which there is no bulk motion (advection) and the **temperature distribution $T(x, y, z)$** is expressed in Cartesian coordinates. The medium is assumed to be incompressible, that is, its density can be treated as constant.
- The dimensions of the **infinitesimal volume element** are d_x , d_y , and d_z in the respective direction as shown in the figure below.

Three dimensional form of Heat Conduction [$T(x, y, z)$]

$$\vec{q} = -k \frac{\partial T}{\partial x} \vec{i} - k \frac{\partial T}{\partial y} \vec{j} - k \frac{\partial T}{\partial z} \vec{k}$$

Or

$$\vec{q} = -k \nabla T$$



- The heat is entering into the volume element from three different faces of the volume element and leaving from the opposite face of the control element.
- The heat source within the volume element generates the volumetric energy at the rate of $\dot{e}_g \left(\frac{w}{m^3} \right)$.
- According to Fourier's law of heat conduction, the heat flowing into the volume element from the left (in the x-direction) can be written as,

$$(heat\ transfer\ rate)_i = (heat\ flux)_i \times (surface\ area)$$

$$\dot{q}_x = -k dy dz \frac{\partial T}{\partial x} \quad (2.23)$$

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx \quad (2.24)$$

or,

$$\dot{q}_x - \dot{q}_{x+dx} = - \frac{\partial \dot{q}_x}{\partial x} dx \quad (2.25)$$

$$\dot{q}_x - \dot{q}_{x+dx} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) dx dy dz \quad (2.26)$$

The heat flow out from the right surface (in the x-direction) of the volume element can be obtained by Taylor series expansion of the above equation. As the volume element is of infinitesimal volume, we may retain only first two elements of the Taylor series expansion with a reasonable approximation (truncating the higher order terms). Thus,

The left side of the above equation represents the net heat flow in the x-direction. If we put the value of \dot{q}_x in the right side of the above equation,



In a similar way we can get the net heat flow in the y and z -directions,

$$\dot{q}_y - \dot{q}_{y+dy} = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) dx dy dz \quad (2.27)$$

and,

$$\dot{q}_z - \dot{q}_{z+dz} = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) dx dy dz \quad (2.28)$$

$$\dot{u} = (\rho dx dy dz) c_p \left(\frac{\partial T}{\partial t} \right) \quad (2.29)$$

where, c_p is the specific heat capacity at constant pressure ($\text{J}/(\text{kg}\cdot\text{K})$), ρ is the density (kg/m^3) of the material, and t is the time (s).

As we know some heat is entering, some heat is leaving and some **heat in generating** in the volume element and as we have **not considered any steady state assumption** till now, thus because of all these phenomena some of the heat will be absorbed by the element. Thus the rate of change of heat energy (\dot{u}) within the volume element can be written as,

Rate of heat input + Rate of heat generation

= Rate of heat output

+ Rate of change of heat energy within the body

(2.30)

We know all the energy term related to the above problem, and with the help of energy conservation,



On putting all the values in the above equation,

$$(\dot{q}_x + \dot{q}_y + \dot{q}_z) + (\dot{e}_g dx dy dz) = (\dot{q}_{x+dx} + \dot{q}_{y+dy} + \dot{q}_{z+dz}) + \\ (\rho dx dy dz) C_p \left(\frac{\partial T}{\partial t} \right)$$

or,

$$(\dot{q}_x - \dot{q}_{x+dx}) + (\dot{q}_y - \dot{q}_{y+dy}) + (\dot{q}_z - \dot{q}_{z+dz}) + (\dot{e}_g dx dy dz) = \\ (\rho dx dy dz) C_p \left(\frac{\partial T}{\partial t} \right)$$

or,

$$\left\{ \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \right\} (dx dy dz) + (\dot{e}_g dx dy dz) = \\ (\rho dx dy dz) C_p \left(\frac{\partial T}{\partial t} \right)$$

or,

$$\left\{ \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \right\} + \dot{e}_g = \rho C_p \left(\frac{\partial T}{\partial t} \right)$$



As we have considered that the thermal conductivity of the solid is isotropic in nature, the above relation reduces to,

$$\frac{k}{\rho C_p} \left\{ \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial T}{\partial z} \right) \right\} + \frac{\dot{e}_g}{\rho c_p} = \left(\frac{\partial T}{\partial t} \right)$$

or,

$$\frac{k}{\rho C_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{\dot{e}_g}{\rho c_p} = \frac{\partial T}{\partial t}$$

or,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

or,

$$\nabla^2 T + \frac{\dot{e}_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2.35)$$



$$\alpha (= \frac{k}{\rho c_p})$$

where α is the thermal diffusivity of the material and its unit m^2/s signifies the rate at which heat diffuses in to the medium during change in temperature with time. Thus, the higher value of the thermal diffusivity gives the idea of how fast the heat is conducting into the medium, whereas the low value of the thermal diffusivity shows that the heat is mostly absorbed by the material and comparatively less amount is transferred for the conduction. The ∇^2 called the Laplacian operator,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and in Cartesian coordinate it is defined as

Equation 2.35 is known as general heat conduction relation. When there is no heat generation term the eq. 2.35 will become,

$$\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2.36)$$

and the equation is known as *Fourier Field Equation*.



Cylindrical coordination systems:

For 3-dimensional problem in cylindrical coordination (r, ϕ, z) with temperature, $T(r, \phi, z)$, the conduction heat flux vector will be:

$$\mathbf{q}'' = -k \frac{\partial T}{\partial r} \mathbf{i} - k \frac{\partial T}{r \partial \phi} \mathbf{j} - k \frac{\partial T}{\partial z} \mathbf{k}$$

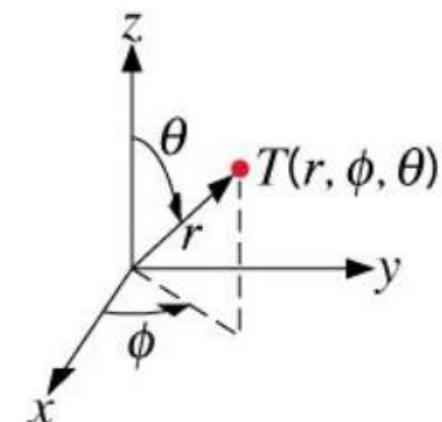
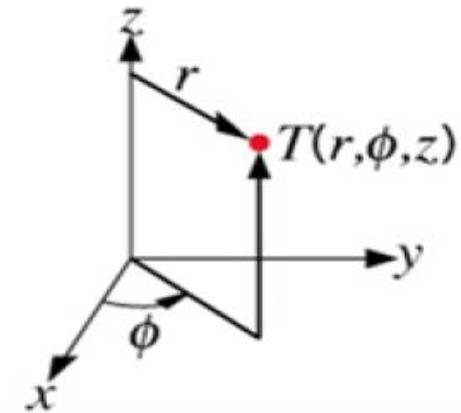
where: $q_r'' = -k \frac{\partial T}{\partial r}$; $q_\phi'' = -k \frac{\partial T}{r \partial \phi}$; $q_z'' = -k \frac{\partial T}{\partial z}$

Spherical coordination systems:

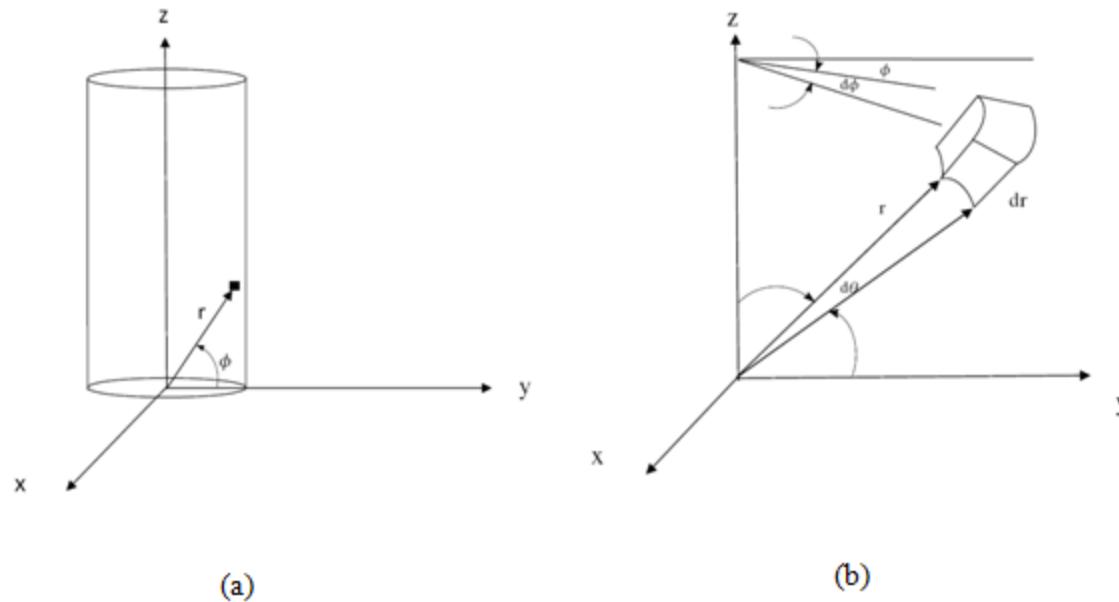
For 3-dimensional problem in spherical coordination (r, ϕ, θ) with temperature, $T(r, \phi, \theta)$, the conduction heat flux vector will be:

$$\mathbf{q}'' = -k \frac{\partial T}{\partial r} \mathbf{i} - k \frac{\partial T}{r \partial \theta} \mathbf{j} - k \frac{\partial T}{r \sin \theta \partial \phi} \mathbf{k}$$

where: $q_r'' = -k \frac{\partial T}{\partial r}$; $q_\theta'' = -k \frac{\partial T}{r \partial \theta}$; $q_\phi'' = -k \frac{\partial T}{r \sin \theta \partial \phi}$



General heat conduction relation in cylindrical coordinate system



Cylindrical coordinate system (a) and an element of the cylinder

The energy conservation for the system is written as,

$$\text{I} + \text{II} = \text{III} + \text{IV}$$

2.37

where,

I : Rate of heat energy conducted in

II : Rate of heat energy generated within the volume element

III : Rate of heat energy conducted out

IV : Rate of energy accumulated

and the above terms are defined as,

$$\text{I} : \dot{q}_r + \dot{q}_\theta + \dot{q}_z$$

$$\text{II} : \dot{e}_{\text{g}} dr dz r d\theta$$

$$\text{III} : \dot{q}_{r+dr} + \dot{q}_{\theta+d\theta} + \dot{q}_{z+dz}$$

$$\text{IV} : \rho dr dz r d\theta c_p \frac{\partial T}{\partial t} \quad (\text{T is a function of space and time})$$

Thus,



$$(I) - (III) : (-\frac{\partial q_r}{\partial r} dr) + (-\frac{\partial q_\theta}{\partial \theta} d\theta) + (-\frac{\partial q_z}{\partial z} dz)$$

Using Fourier's law

$$q_r = -k(r d\theta dz) \frac{\partial T}{\partial r}$$

$$q_\theta = -k(dr dz) \frac{\partial T}{r \partial \theta}$$

$$q_z = -k(r d\theta dr) \frac{\partial T}{\partial z}$$

$$(I) - (III) : \frac{\partial}{\partial r} (k r d\theta dz \frac{\partial T}{\partial r}) dr + \frac{\partial}{\partial \theta} (k dr dz \frac{\partial T}{r \partial \theta}) d\theta + \frac{\partial}{\partial z} (k r d\theta dr \frac{\partial T}{\partial z}) dz$$

If k is isotropic,

$$= k d\theta dz dr \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) + k d\theta dz dr \frac{\partial}{\partial \theta} (\frac{\partial T}{r \partial \theta}) + r k d\theta dr dz \frac{\partial}{\partial z} (\frac{\partial T}{\partial z})$$

$$= k r d\theta dz dr [\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) + \frac{1}{r^2} \frac{\partial}{\partial \theta} (\frac{\partial T}{\partial \theta}) + \frac{\partial}{\partial z} (\frac{\partial T}{\partial z})]$$

$$= k r d\theta dz dr [\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}]$$



On putting the values in equation 2.37

$$k r d\theta dz dr \left[\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \dot{e}_g dr dz r d\theta = \rho dr dz r d\theta C_p \frac{\partial T}{\partial t}$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_g}{k} = \frac{\rho C_p}{k} \frac{\partial T}{\partial t}$$

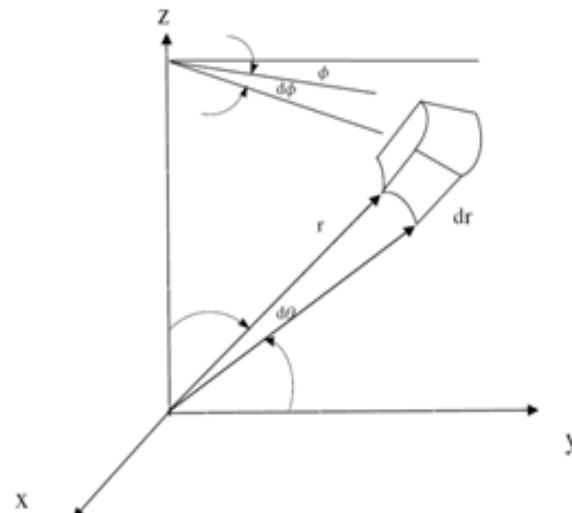
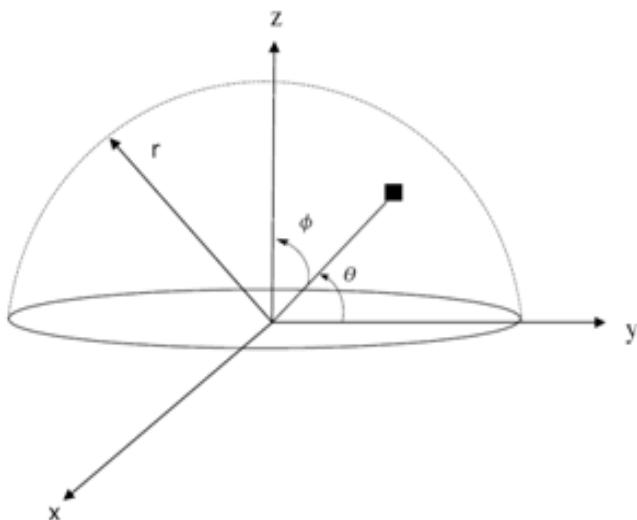
$$\nabla^2 T + \frac{\dot{e}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (\text{where } \alpha = \frac{k}{\rho C_p}) \quad 2.38$$

Thus the Laplacian operator is,

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r}) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad 2.39$$



Spherical coordinate system



Spherical coordinate system and an element of the sphere

In a similar way the general expression for the conduction heat transfer in spherical body with heat source can also be found out as per the previous discussion. The Laplacian operator for the spherical coordinate system (fig.2.13) is given below and the students are encouraged to derive the expression themselves.

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad 2.40$$

P. 2.6. The temperature distribution across a wall 1 m thick at a certain instant of time is given as $T(x) = a + bx + cx^2$, where T is in degrees Celsius and x is in meters, while $a = 900^\circ\text{C}$, $b = -300^\circ\text{C/m}$, and $c = -50^\circ\text{C/m}^2$. A uniform heat generation, $q = 1000 \text{ W/m}^3$ is present in the wall of area 10 m^2 having the properties $\rho = 1600 \text{ kg/m}^3$, $k = 40 \text{ W/m}\cdot\text{K}$, and $c_p = 4 \text{ kJ/kg}\cdot\text{K}$.

1. Determine the rate of heat transfer entering the wall ($x = 0$) and leaving the wall ($x = 1 \text{ m}$).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at $x = 0, 0.25$, and 0.5 m .

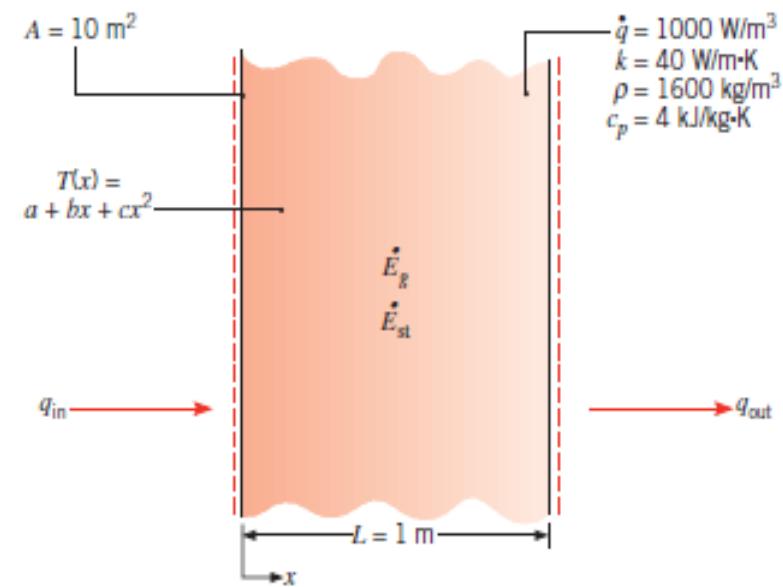
Known: Temperature distribution $T(x)$ at an instant of time t in a one-dimensional wall with uniform heat generation.

Find:

1. Heat rates entering, $q_{in}(x = 0)$, and leaving, $q_{out}(x = 1 \text{ m})$, the wall.
2. Rate of change of energy storage in the wall, E_{st} .
3. Time rate of temperature change at $x = 0, 0.25$, and 0.5 m .

Assumptions:

1. One-dimensional conduction in the x -direction.
2. Incompressible, isotropic medium with constant properties.
3. Uniform internal heat generation, $q (\text{W/m}^3)$.



Recall that once the temperature distribution is known for a medium, it is a simple matter to determine the conduction heat transfer rate at any point in the medium or at its surfaces by using Fourier's law. Hence the desired heat rates may be determined by using the prescribed temperature distribution with Equation 2.1. Accordingly,

$$q_{\text{in}} = q_x(0) = -kA \frac{\partial T}{\partial x} \Big|_{x=0} = -kA(b + 2cx)_{x=0}$$

$$q_{\text{in}} = -bkA = 300^{\circ}\text{C}/\text{m} \times 40 \text{ W/m} \cdot \text{K} \times 10 \text{ m}^2 = 120 \text{ kW}$$



Similarly,

$$q_{\text{out}} = q_x(L) = -kA \frac{\partial T}{\partial x} \Big|_{x=L} = -kA(b + 2cx)_{x=L}$$

$$q_{\text{out}} = -(b + 2cL)kA = -[-300^{\circ}\text{C}/\text{m}]$$

$$+ 2(-50^{\circ}\text{C}/\text{m}^2) \times 1 \text{ m}] \times 40 \text{ W/m} \cdot \text{K} \times 10 \text{ m}^2 = 160 \text{ kW}$$



The rate of change of energy storage in the wall \dot{E}_{st} may be determined by applying an overall energy balance to the wall. Using Equation 1.12c for a control volume about the wall,

$$\dot{E}_{\text{in}} + \dot{E}_g - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

$$\boxed{\dot{E}_g = \dot{q}AL}$$

where $\dot{E}_g = \dot{q}AL$, it follows that

$$\dot{E}_{\text{st}} = \dot{E}_{\text{in}} + \dot{E}_g - \dot{E}_{\text{out}} = q_{\text{in}} + \dot{q}AL - q_{\text{out}}$$

$$\dot{E}_{\text{st}} = 120 \text{ kW} + 1000 \text{ W/m}^3 \times 10 \text{ m}^2 \times 1 \text{ m} - 160 \text{ kW}$$

$$\dot{E}_{\text{st}} = -30 \text{ kW}$$



The time rate of change of the temperature at any point in the medium may be determined from the heat equation,

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

From the prescribed temperature distribution, it follows that

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) \\ &= \frac{\partial}{\partial x} (b + 2cx) = 2c = 2(-50^\circ\text{C}/\text{m}^2) = -100^\circ\text{C}/\text{m}^2\end{aligned}$$

Note that this derivative is independent of position in the medium. Hence the time rate of temperature change is also independent of position and is given by

$$\frac{\partial T}{\partial t} = \frac{40 \text{ W/m} \cdot \text{K}}{1600 \text{ kg/m}^3 \times 4 \text{ kJ/kg} \cdot \text{K}} \times (-100^\circ\text{C}/\text{m}^2)$$

$$+ \frac{1000 \text{ W/m}^3}{1600 \text{ kg/m}^3 \times 4 \text{ kJ/kg} \cdot \text{K}}$$

$$\begin{aligned}\frac{\partial T}{\partial t} &= -6.25 \times 10^{-4}^\circ\text{C/s} + 1.56 \times 10^{-4}^\circ\text{C/s} \\ &= -4.69 \times 10^{-4}^\circ\text{C/s}\end{aligned}$$



Comments:

1. From this result, it is evident that the temperature at every point within the wall is decreasing with time.
2. Fourier's law can always be used to compute the conduction heat rate from knowledge of the temperature distribution, even for unsteady conditions with internal heat generation.



Take away from todays session

- ✓ To understand the Fourier's law and its applications in steady state heat transfer problems using rectangular coordinates, unidirectional heat flow and constant thermal conductivity. Identify the normal area to the heat flow.
- ✓ Understand the analogy between electrical and thermal resistances. Apply the analogy with resistances in series.
- ✓ Apply Fourier's law in heat transfer problems through cylindrical walls.
- ✓ Understand the concept of logarithmic mean area in heat transfer through cylindrical walls.
- ✓ Solve problems of heat transfer through multiple cylindrical walls with different thermal conductivities.

Next Session.... Heat Transfer by Convection

