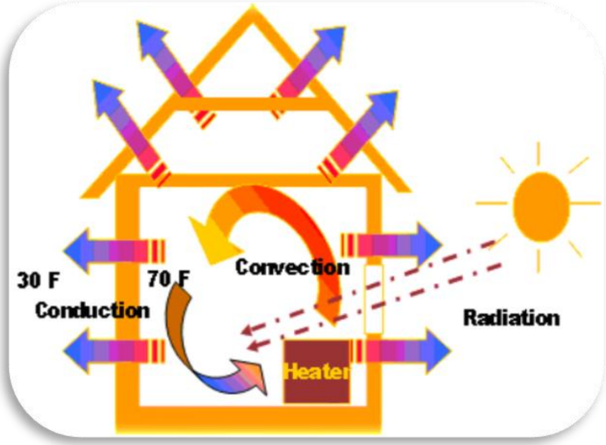
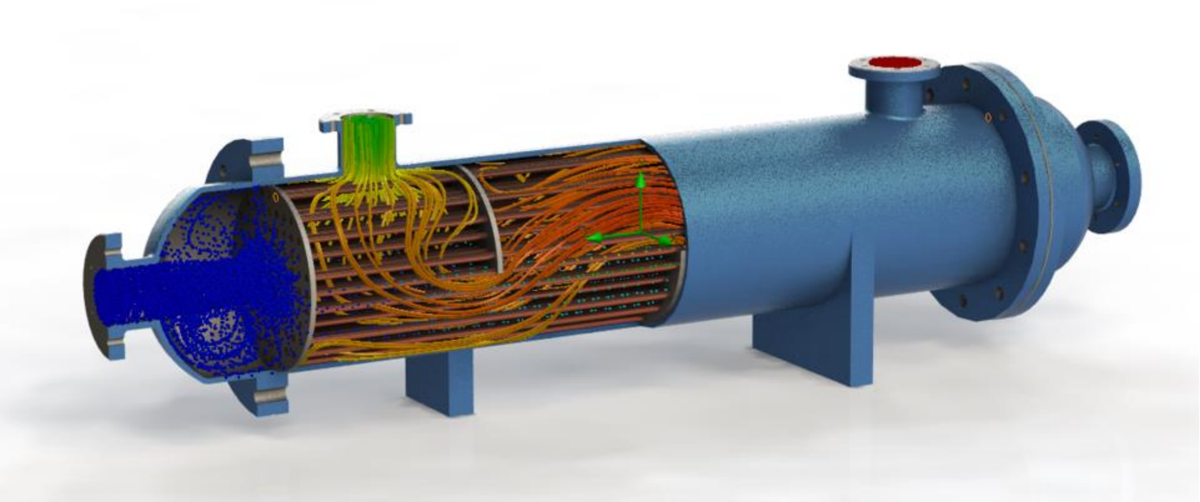
Heat Transfer Operations

Session-2. Heat Conduction





Dr. Jogender Singh (JS)

Summary of Session-1

Defined *energy*: product of a force over a distance which it is applied.

Temperature is the measure of the average kinetic energy of a given substance, or internal energy. Defined °F, °C, and K scales and their conversions.

Assuming pressure remains constant, an increase in temperature means the volume of air expands and its density decreases.

Heat is energy in the process of being transferred.

Conduction: heat transfer molecule by molecule

Convection: mass movement of a fluid or gas

Radiation: electromagnetic waves pass through the empty space

Latent heat is energy associated with phase changes from one state of matter to another.

Specific Objectives of Session-2

- ✓ Know and understand Fourier's law.
- ✓ Apply Fourier's law in steady state heat transfer problems using rectangular coordinates, unidirectional heat flow and constant thermal conductivity. Identify the normal area to the heat flow.
- ✓ Understand the analogy between electrical and thermal resistances. Apply the analogy with resistances in series.
- ✓ Apply Fourier's law in heat transfer problems through cylindrical walls.
- ✓ Understand the concept of logarithmic mean area in heat transfer through cylindrical walls.
- ✓ Solve problems of heat transfer through multiple cylindrical walls with different thermal conductivities.

Fourier's law of conduction

Jean Baptiste Joseph Fourier

Various flows and their driving forces

Flow	Driving force
Electricity flow	Electric potential gradient
Fluid flow	Pressure gradient
Heat flow	Temperature gradient

Thus, the heat flow per unit area per unit time (heat flux) can be represented as,

$$\dot{q}' \propto \frac{dT}{dx} \tag{2.1}$$

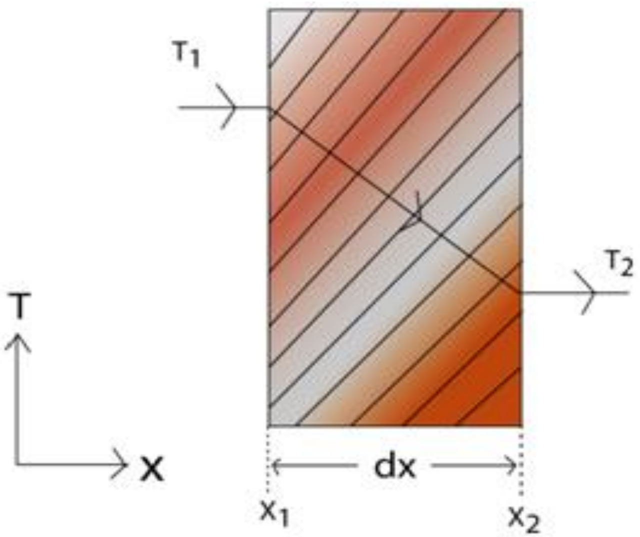
$$\dot{q}' = k\frac{dT}{dx} \tag{2.2}$$

where, proportionality constant *k* is the thermal conductivity of the material, *T* is the temperature and *x* is the distance in the direction of heat flow. This is known as Fourier's law of conduction.

Steady-state conduction is defined as the condition which prevails when temperatures at fixed points do not change with time.

One-dimensional is applied to a heat conduction problem when only one coordinate is required to describe the distribution of temperature within the body. Such a situation hardly exists in real engineering problems. However, the real problem is solved fairly up to the accuracy of practical engineering interest.

Steady-state conduction through constant area



- The flat wall of thickness d_x is separated by two regions, the one region is at high temperature (T_1) and the other one is at temperature T_2. The wall is very large in comparison of the thickness so that the heat losses from the edges are negligible.

- Consider there is no generation or accumulation of the heat in the wall and the external surfaces of the wall are at isothermal temperatures T_1 and T_2.

- The area of the surface through which the heat transfer takes place is A. Then the eq.2.2 can be written as,

$$\dot{q}' = -k\frac{dT}{dx} \tag{2.3}$$

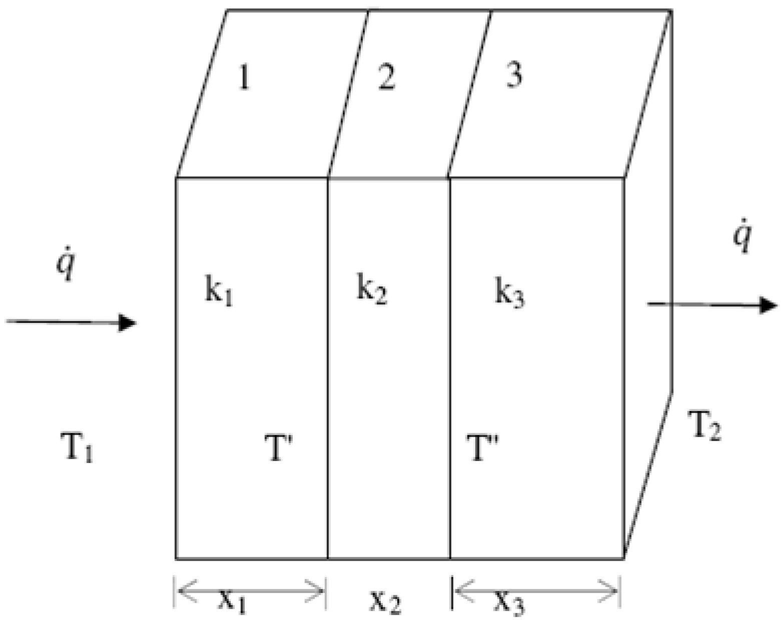
The negative sign (-) shows that the heat flux is from the higher temperature surface to the lower temperature surface.

$$\frac{\dot{q}}{A} = -k\frac{dT}{dx} \tag{2.4}$$

$$\frac{\dot{q}}{A} = -k\frac{T_2 - T_1}{x_2 - x_1} \tag{2.5}$$

Analogy between electrical and thermal resistances

Consider three different layers of materials having different thermal conductivities and thicknesses of the layers.



The rate of heat transfer through layer-1 to 2 will be,

$$\dot{q} = \frac{k_1 A\,(T_1 - T')}{x_1} \quad or \quad (T_1 - T') = \frac{\dot{q}}{1/(x_1/k_1 A)} \tag{2.6}$$

and rate of heat transfer through layer-2 to 3 will be,

$$\dot{q} = \frac{k_2 A\,(T' - T'')}{x_2} \quad or \quad (T' - T'') = \frac{\dot{q}}{1/(x_2/k_2 A)} \tag{2.7}$$

The rate of heat transfer through layer 3 to the other side of the wall

$$\dot{q} = \frac{k_3 A\,(T'' - T_2)}{x_3} \quad or \quad (T'' - T_2) = \frac{\dot{q}}{1/(x_3/k_3 A)} \tag{2.8}$$

On adding the above three equations (2.6 to 2.8)

$$\dot{q} = \frac{T_1 - T_2}{\frac{x_1}{k_1 A} + \frac{x_2}{k_2 A} + \frac{x_3}{k_3 A}} \tag{2.9}$$

$$\dot{q} = \frac{T_1 - T_2}{R_1 + R_2 + R_3} \tag{2.10}$$

Where, R represents the thermal resistance of the layers. The above relation can be written analogous to the electrical circuit as,

$$Rate\ of\ heat\ flow = \frac{Temperaure\ difference}{Resistance} \qquad (2.11)$$

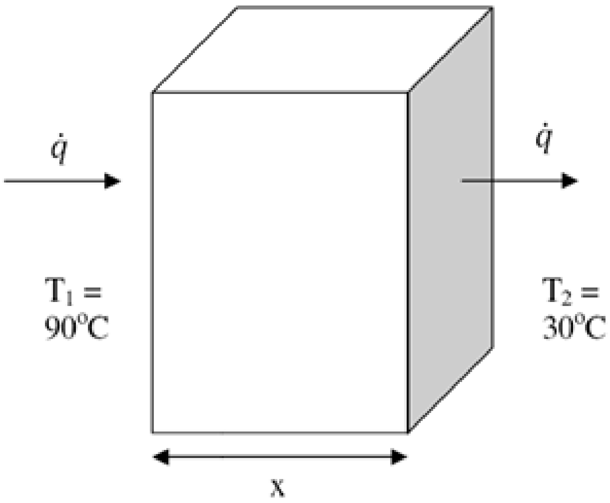
$$R_1 = \frac{x_1}{k_1 A} \qquad R_2 = \frac{x_2}{k_2 A} \qquad R_3 = \frac{x_3}{k_3 A}$$

The wall is composed of 3-different layers in series and thus the total thermal resistance was represented by R (= $R_1 + R_2 + R_3$). The discussed concept can be understood by the illustrations shown here.

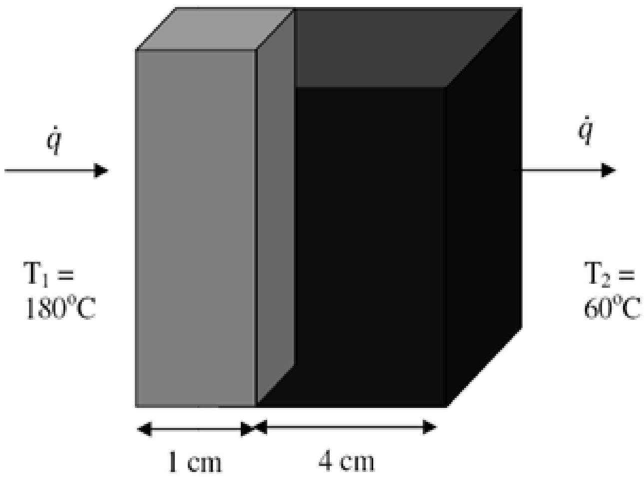
The unit of the various parameters used above is summarized as follows,

Parameter	Symbol used	Unit
Heat flux	$\dot{q}'$	W/m^2 or J/(s·m^2)
Heat flow rate	$\dot{q}$	W or J/s
Thermal conductivity	k	W/(m·°C)
Thermal resistance	R	°C/W

P. 2.1. *The two sides of a wall (2 mm thick, with a cross-sectional area of 0.2 m^2) are maintained at 30 ºC and 90 ºC. The thermal conductivity of the wall material is 1.28 W/(m. ºC). Find out the rate of heat transfer through the wall?*



P. 2.2. *One side of a 1 cm thick stainless steel wall (k_1 = 19 W/m.ºC) is maintained at 180 and the other side is insulated with a layer of 4 cm fiberglass (k_2 = 0.04 W/m.ºC). The outside of the fiberglass is maintained at 60 ºC and the heat loss through the wall is 300 W. Determine the area of the wall.*



Concept of equivalent resistance (Composite wall)

The wall is composed of seven different layers indicated by 1 to 7. The interface temperatures of the composite are T_1 to T_5 as shown in the fig. 1. The equivalent electrical circuit of the above composite is shown in the fig. 2. below

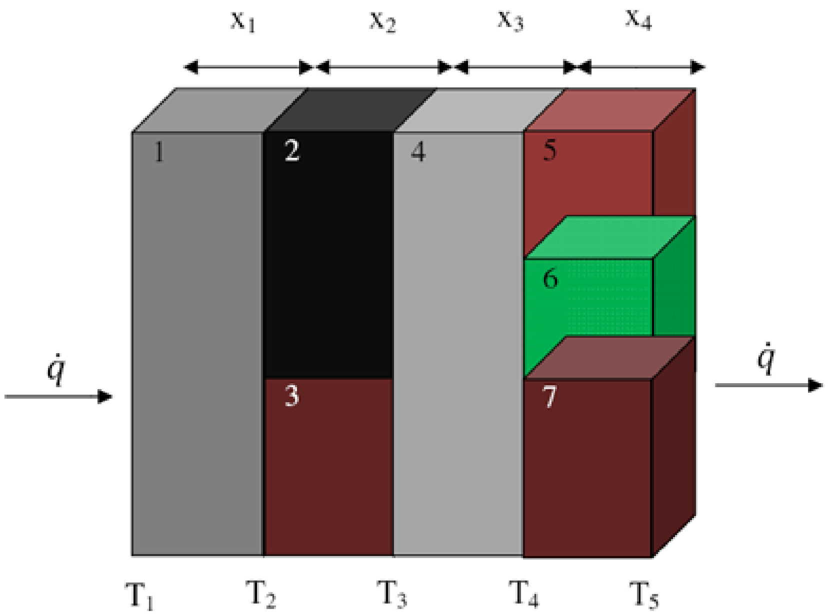


fig. 1.

The equivalent resistance of the wall will be,

$$R = R_1 + \frac{1}{{}^1/_{R_2} + {}^1/_{R_3}} + R_4 + \frac{1}{{}^1/_{R_5} + {}^1/_{R_6} + {}^1/_{R_7}} \qquad (2.12)$$

where, $R_i = \dfrac{x_i}{k_i A_i}$

Therefore,

$$\dot{q} = \frac{T_1 - T_5}{R} \qquad (2.13)$$

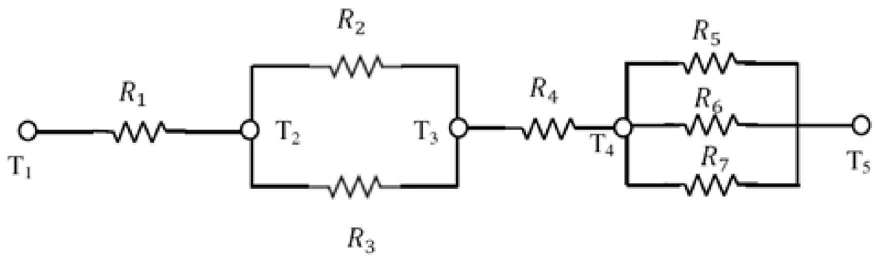
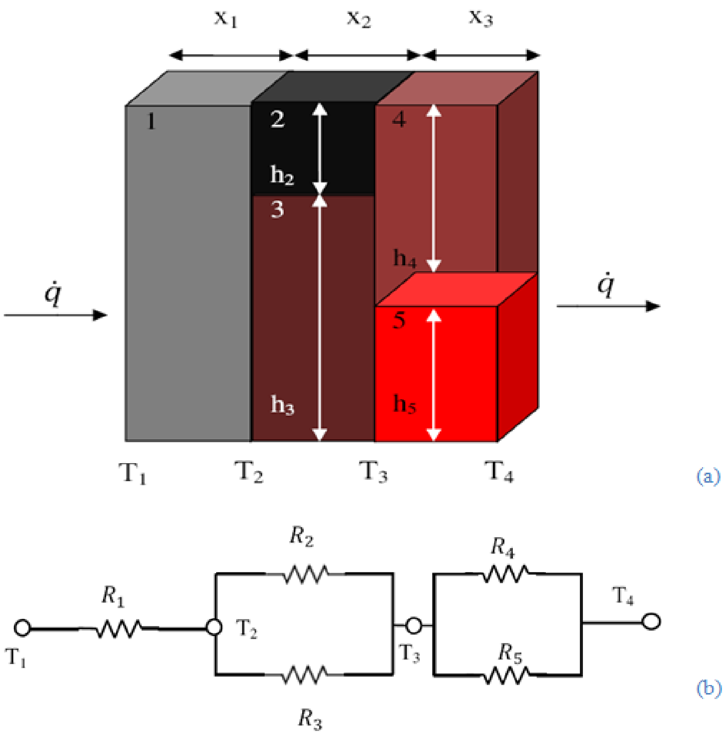


fig. 2.

P. 2.3. Consider a composite wall containing 5-different materials as shown in the fig. Calculate the rate of heat flow through the composite from the following data?



$x_1 = 0.1\ m$	$x_2 = 0.2\ m$	$x_3 = 0.15\ m$
$k_1 = 15\ W/m\cdot{}^oC$	$k_2 = 25\ W/m\cdot{}^oC$	$k_3 = 30\ W/m\cdot{}^oC$
$k_4 = 20\ W/m\cdot{}^oC$	$k_5 = 35\ W/m\cdot{}^oC$	
$h_2 = 1\ m$	$h_3 = 3\ m$	$h_4 = 2.5\ m$
$h_5 = 1.5\ m$	$T_A = 120^oC$	$T_B = 50^oC$

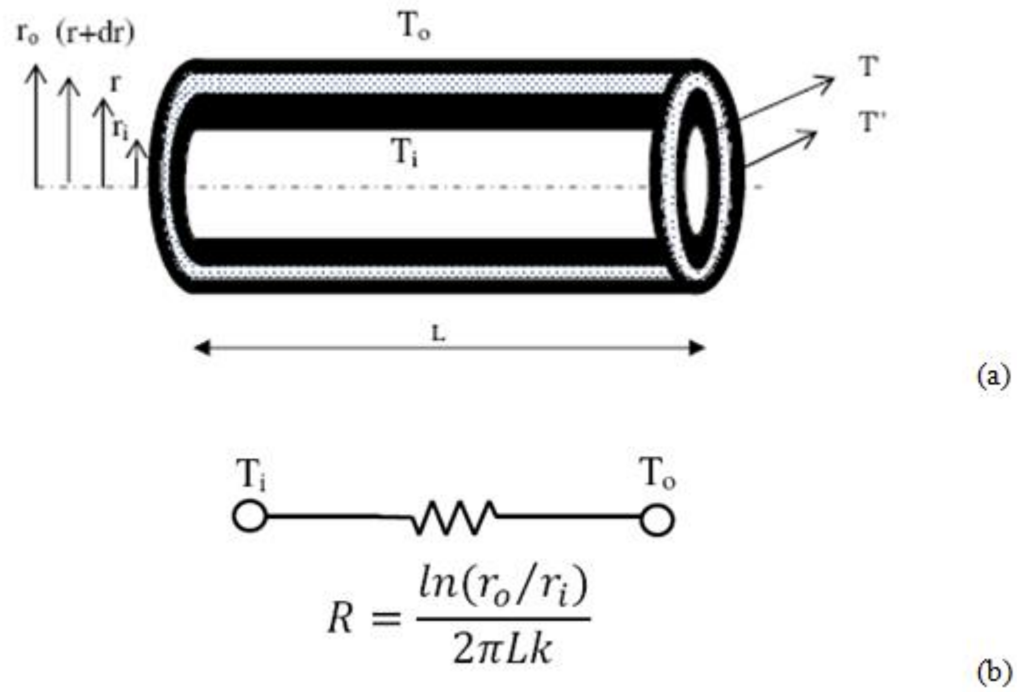
Assumptions:

1. Steady-state one-dimensional conduction.
2. Thermal conductivity is constant for the temperature range of interest.
3. The heat loss through the edge side surface is insignificant.
4. The layers are in perfect thermal contact.
5. Area in the direction of heat flow is 1 m^2.

Steady-state heat conduction through a variable area

Cylinder

Consider a hollow cylinder as shown in the fig. The inner and outer radius is represented by r_i and r_o, whereas T_i and T_o ($T_i > T_o$) represent the uniform temperature of the inner and outer wall, respectively.



$$R = \frac{ln(r_o/r_i)}{2\pi Lk}$$

(a) Hollow cylinder, (b) equivalent electrical circuit

$$\dot{q} = -k\frac{dT}{dr}\,2\pi rL \qquad (2.14)$$

On rearranging

$$\frac{dr}{r} = -\frac{2\pi Lk}{\dot{q}}\,dT \qquad (2.15)$$

Or

To get the heat flow through the thick wall cylinder, the above equation can be integrated between the limits,

$$\int_{r_i}^{r_o}\frac{dr}{r} = -\frac{2\pi Lk}{\dot{q}}\int_{T_i}^{T_o} dT \qquad (2.16)$$

On solving,

$$\dot{q} = k(2\pi L)\frac{(T_i - T_o)}{ln(r_o/r_i)} \qquad (2.17)$$

$$\dot{q} = kA_{LM}\frac{(T_i - T_o)}{r_o - r_i} \qquad (2.18)$$

Where

$$A_{LM} = \frac{2\pi L(r_o - r_i)}{ln(r_o/r_i)} = 2\pi L r_{LM} \qquad (2.19)$$

The careful analysis of the above equation shows that the expression is same as for heat flow through the plane wall of thickness (r_o–r_i) except the expression for the area. The A_{LM} is known as log mean area of the cylinder, whose length is L and radius is r_{LM}

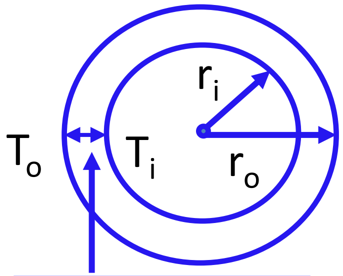
$$r_{LM} = \frac{r_{o-}r_i}{\ln(r_{o-}r_i)} \qquad (2.20)$$

Log mean area (A_{LM})

Assumption

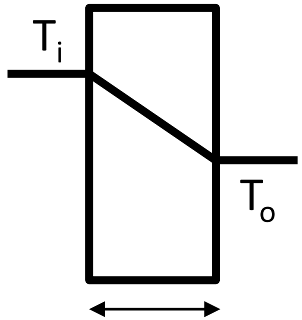
(i) $T_i > T_o$

(ii) $Equal\ thickness\ for\ heat\ transfer$ (dr)



$$\boxed{dr = (r_o - r_i)}$$

$$\dot{q_c} = \frac{(T_o - T_i)}{\ln\left({r_o}/{r_i}\right)\Big/_{2\pi LK}}$$



$$\boxed{dr = (r_o - r_i)}$$

$$\dot{q_w} = \frac{(T_o - T_i)}{(r_o - r_i)\Big/_{KA}}$$

Since thickness is equal the rate of heat transfer will also be equal.

So $\dot{q_c} = \dot{q_w}$

$$\frac{(T_o - T_i)}{\ln\left({r_o}/{r_i}\right)\Big/_{2\pi LK}} = \frac{(T_o - T_i)}{(r_o - r_i)\Big/_{KA}}$$

Or

$$A_{LM} = \frac{2\pi L(r_o - r_i)}{\ln\left({r_o}/{r_i}\right)}$$

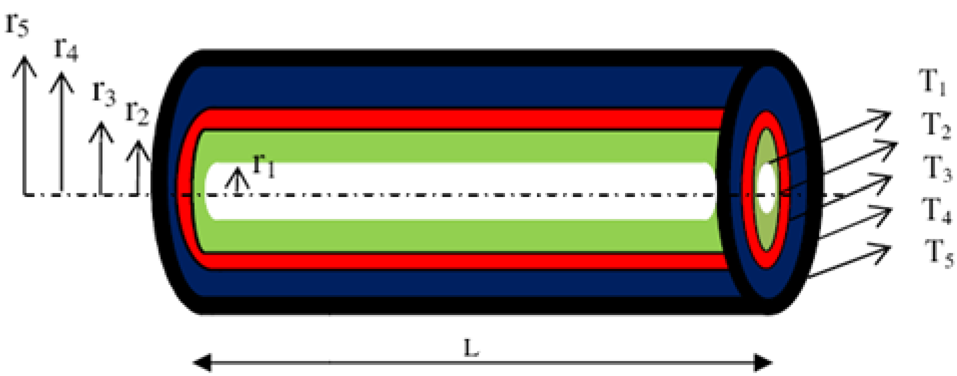
$$A_{LM} = \frac{2\pi L r_o - 2\pi L r_i}{\ln\left({2\pi L r_o}/{2\pi L r_i}\right)}$$

Or

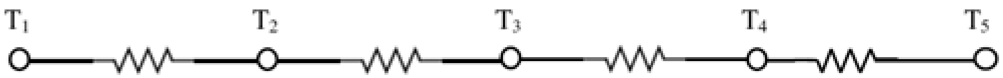
$$\boxed{A_{LM} = \frac{A_o - A_i}{\ln\left({A_o}/{A_i}\right)}}$$

4-layers of solid material of different inner and outer diameter as well as thermal conductivity.

Four layer composite hollow cylinder



Equivalent electrical circuit



$$R_1 = \frac{ln(r_2/r_1)}{2\pi L k_1} \quad R_2 = \frac{ln(r_3/r_2)}{2\pi L k_2} \quad R_3 = \frac{ln(r_4/r_3)}{2\pi L k_3} \quad R_4 = \frac{ln(r_5/r_4)}{2\pi L k_4}$$

The total heat transfer at steady-state will be,

$$\dot{q} = \frac{T_1 - T_5}{R_1 + R_2 + R_3 + R_4} \qquad (2.21)$$

where R_1 , R_2, R_3 , and R_4 are represented in the fig

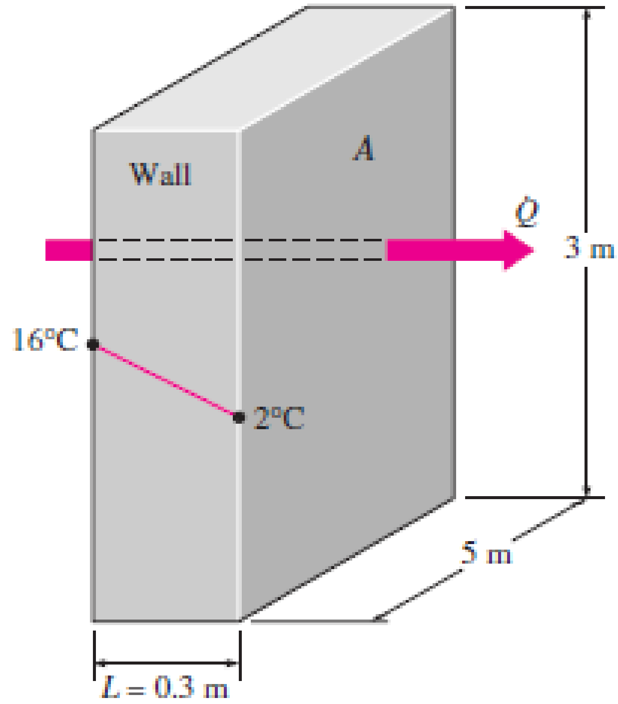
Sphere

The rate of heat transfer through a hollow sphere can be determined in a similar manner as for cylinder. The students are advised to derive the following expression shown below.

The final expression for the rate of heat flow is,

$$\dot{q} = 4\pi K \frac{(T_i - T_o)}{{}^{1}/_{r_i} - {}^{1}/_{r_o}} \qquad (2.22)$$

- **P. 2.4.** Consider a 3-m-high, 5-m-wide, and 0.3-m-thick wall whose thermal conductivity is k = 0.9 W/m ·°C. On a certain day, the temperatures of the inner and the outer surfaces of the wall are measured to be 16°C and 2°C, respectively. Determine the rate of heat loss through the wall on that day.



Insulation types and applications

Thermal **insulation**, materials used to reduce the rate of heat transfer.

	Type	Temperature range, °C	Thermal conductivity, mW/m · °C	Density, kg/m^3	Application
1	Linde evacuated superinsulation	−240–1100	0.0015–0.72	Variable	Many
2	Urethane foam	−180–150	16–20	25–48	Hot and cold pipes
3	Urethane foam	−170–110	16–20	32	Tanks
4	Cellular glass blocks	−200–200	29–108	110–150	Tanks and pipes
5	Fiberglass blanket for wrapping	−80–290	22–78	10–50	Pipe and pipe fittings
6	Fiberglass blankets	−170–230	25–86	10–50	Tanks and equipment
7	Fiberglass preformed shapes	−50–230	32–55	10–50	Piping
8	Elastomeric sheets	−40–100	36–39	70–100	Tanks
9	Fiberglass mats	60–370	30–55	10–50	Pipe and pipe fittings
10	Elastomeric preformed shapes	−40–100	36–39	70–100	Pipe and fittings
11	Fiberglass with vapor barrier blanket	−5–70	29–45	10–32	Refrigeration lines
12	Fiberglass without vapor barrier jacket	to 250	29–45	24–48	Hot piping
13	Fiberglass boards	20–450	33–52	25–100	Boilers, tanks, heat exchangers
14	Cellular glass blocks and boards	20–500	29–108	110–150	Hot piping
15	Urethane foam blocks and boards	100–150	16–20	25–65	Piping
16	Mineral fiber preformed shapes	to 650	35–91	125–160	Hot piping
17	Mineral fiber blankets	to 750	37–81	125	Hot piping
18	Mineral wool blocks	450–1000	52–130	175–290	Hot piping
19	Calcium silicate blocks, boards	230–1000	32–85	100–160	Hot piping, boilers, chimney linings
20	Mineral fiber blocks	to 1100	52–130	210	Boilers and tanks

- **P. 2.5.** A thick-walled tube of stainless steel [18% Cr, 8% Ni, k =19 W/m ▪ ◦C] with 2-cm inner diameter (ID) and 4-cm outer diameter (OD) is covered with a 3-cm layer of asbestos insulation [k =0.2 W/m ▪ ◦C]. If the inside and final temperature of the wall of pipe are maintained at 600◦C and 100 ◦C, respectively, calculate the heat loss per meter of length. Also calculate the tube–insulation interface temperature.

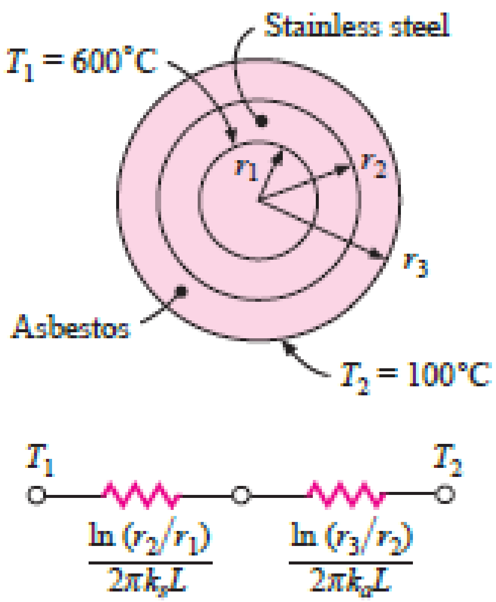


Figure shows the thermal network for this problem. The heat flow is given by

$$\frac{q}{L} = \frac{2\pi(T_1 - T_2)}{\ln(r_2/r_1)/k_s + \ln(r_3/r_2)/k_a} = \frac{2\pi(600 - 100)}{(\ln 2)/19 + (\ln \frac{5}{2})/0.2} = 680 \text{ W/m}$$

This heat flow may be used to calculate the interface temperature between the outside tube wall and the insulation. We have

$$\frac{q}{L} = \frac{T_a - T_2}{\ln(r_3/r_2)/2\pi k_a} = 680 \text{ W/m}$$

where Ta is the interface temperature, which may be obtained as

$$T_a = 595.8°\text{C}$$

The largest thermal resistance clearly results from the insulation, and thus the major portion of the temperature drop is through that material.

The Heat Diffusion Equation

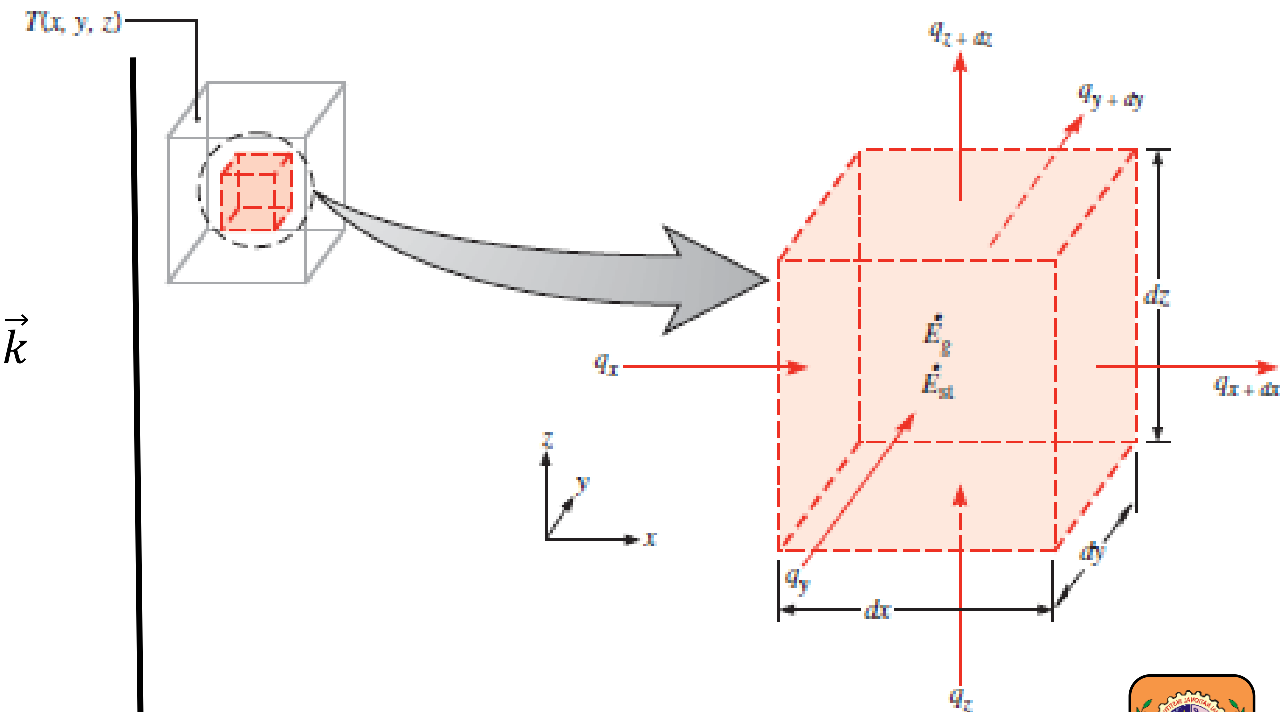
- The cases considered so far have been those in which the heat conducting solid is free of **internal heat generation**. However, the situations where the internal heat is generated are very common cases in chemical industries for example, the **exothermic reaction** in the solid pallet of a catalyst.
- Consider a homogeneous medium within which there is no bulk motion (advection) and the **temperature distribution *T*(*x*, *y*, *z*)** is expressed in Cartesian coordinates. The medium is assumed to be incompressible, that is, its density can be treated as constant.
- The dimensions of the **infinitesimal volume element** are d_x , d_y , and d_z in the respective direction as shown in the figure below.

Three dimensional form of Heat Conduction [T (x, y, z)]

$$\overrightarrow{q''} = -k\frac{\partial T}{\partial x}\vec{i} - k\frac{\partial T}{\partial y}\vec{j} - k\frac{\partial T}{\partial z}\vec{k}$$

Or

$$\overrightarrow{q''} = -k\overrightarrow{\nabla T}$$



- The heat is entering into the volume element from three different faces of the volume element and leaving from the opposite face of the control element.
- The heat source within the volume element generates the volumetric energy at the rate of $\dot{e}_g \left(\frac{w}{m^3}\right)$.
- According to Fourier's law of heat conduction, the heat flowing into the volume element from the left (in the x-direction) can be written as,

$$(heat\ transfer\ rate)_i = (heat\ flux)_i \times (surface\ area)$$

$$\dot{q}_x = -k\, dy\, dz \frac{\partial T}{\partial x} \qquad (2.23)$$

The heat flow out from the right surface (in the x-direction) of the volume element can be obtained by Taylor series expansion of the above equation. As the volume element is of infinitesimal volume, we may retain only first two element of the Taylor series expansion with a reasonable approximation (truncating the higher order terms). Thus,

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx \qquad (2.24)$$

or,

$$\dot{q}_x - \dot{q}_{x+dx} = -\frac{\partial \dot{q}_x}{\partial x} dx \qquad (2.25)$$

The left side of the above equation represent the net heat flow in the x-direction. If we put the value of $\dot{q_x}$ in the right side of the above equation,

$$\dot{q}_x - \dot{q}_{x+dx} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) dx\, dy\, dz \qquad (2.26)$$

In a similar way we can get the net heat flow in the *y* and *z*-directions,

$$\dot{q}_y - \dot{q}_{y+dy} = \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) dx\, dy\, dz \qquad (2.27)$$

and,

$$\dot{q}_z - \dot{q}_{z+dz} = \frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right) dx\, dy\, dz \qquad (2.28)$$

As we know some heat is entering, some heat is leaving and some ***heat in generating*** in the volume element and as we have **not considered any steady state assumption** till now, thus because of all these phenomena some of the heat will be absorbed by the element. Thus the rate of change of heat energy $(\dot{u})$ within the volume element can be written as,

$$\dot{u} = (\rho dx\, dy\, dz)c_p\left(\frac{\partial T}{\partial t}\right) \qquad (2.29)$$

where, c_p is the specific heat capacity at constant pressure (J/(kg·K)), ρ is the density (kg/m^3) of the material, and t is the time (s).

We know all the energy term related to the above problem, and with the help of energy conservation,

$$Rate\ of\ heat\ input + Rate\ of\ heat\ generation$$
$$= Rate\ of\ heat\ output$$
$$+\ Rate\ of\ change\ of\ heat\ energy\ within\ the\ body$$

(2.30)

On putting all the values in the above equation,

$$(\dot{q}_x + \dot{q}_y + \dot{q}_z) + (\dot{e}_g\, dx\, dy\, dz) = (\dot{q}_{x+dx} + \dot{q}_{y+dy} + \dot{q}_{z+dz}) + (\rho dx\, dy\, dz) c_p \left(\frac{\partial T}{\partial t}\right) \quad (2.31)$$

or,

$$(\dot{q}_x - \dot{q}_{x+dx}) + (\dot{q}_y - \dot{q}_{y+dy}) + (\dot{q}_z - \dot{q}_{z+dz}) + (\dot{e}_g dx\, dy\, dz) = (\rho dx\, dy\, dz) c_p \left(\frac{\partial T}{\partial t}\right) \quad (2.32)$$

or,

$$\left\{\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right)\right\}(dx\, dy\, dz) + (\dot{e}_g\, dx\, dy\, dz) = (\rho dx\, dy\, dz) c_p \left(\frac{\partial T}{\partial t}\right) \quad (2.33)$$

or,

$$\left\{\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\,\frac{\partial T}{\partial z}\right)\right\} + \dot{e}_g = \rho c_p \left(\frac{\partial T}{\partial t}\right) \quad (2.34)$$

As we have considered that the thermal conductivity of the solid is isotropic in nature, the above relation reduces to,

$$\frac{k}{\rho C_p}\left\{\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)\right\}+\frac{\dot{e_g}}{\rho c_p}=\left(\frac{\partial T}{\partial t}\right)$$

or,

$$\frac{k}{\rho C_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{\dot{e_g}}{\rho c_p}=\frac{\partial T}{\partial t}$$

or,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{e_g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

or,

$$\nabla^2 T+\frac{\dot{e_g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.35)$$

*Equation 2.35 is known as the **Fourier-Biot equation**, and it reduces to the different forms under specified conditions as shown on the next slide under **AA**….*

where $\alpha \left(= \frac{k}{\rho c_p}\right)$ is the thermal diffusivity of the material and its unit m²/s signifies the rate at which heat diffuses in to the medium during change in temperature with time. Thus, the higher value of the thermal diffusivity gives the idea of how fast the heat is conducting into the medium, whereas the low value of the thermal diffusivity shown that the heat is mostly absorbed by the material and comparatively less amount is transferred for the conduction. The ∇^2 called the Laplacian operator, and in Cartesian coordinate it is defined as $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Equation 2.35 is known as general heat conduction relation. When there is no heat generation term the eq. 2.35 will become,

$$\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \qquad (2.36)$$

and the equation is known as *Fourier Field Equation*.

(1) *Steady-state:*
(called the **Poisson equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0$$

(2) *Transient, no heat generation:*
(called the **diffusion equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

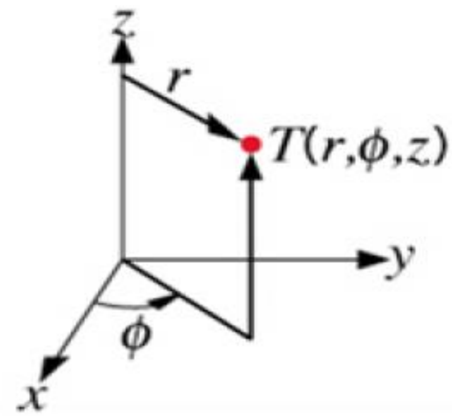
(3) *Steady-state, no heat generation:*
(called the **Laplace equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

AA

Cylindrical coordination systems:

For 3-dimensional problem in cylindrical coordination $(r, \emptyset, z)$ with temperature, $T(r, \emptyset, z)$, the conduction heat flux vector will be:

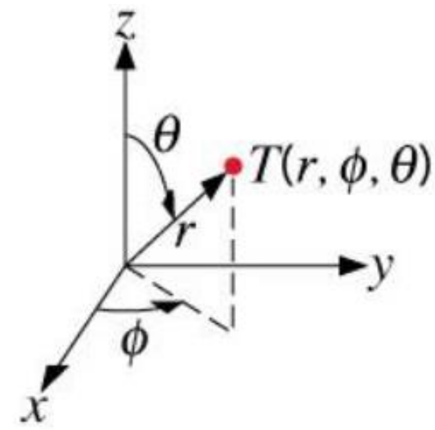


$$q'' = -k\frac{\partial T}{\partial r}i - k\frac{\partial T}{r\partial \emptyset}j - k\frac{\partial T}{\partial z}k$$

where: $q''_r = -k\frac{\partial T}{\partial r}$; $q''_\emptyset = -k\frac{\partial T}{r\partial \emptyset}$; $q''_z = -k\frac{\partial T}{\partial z}$

Spherical coordination systems:

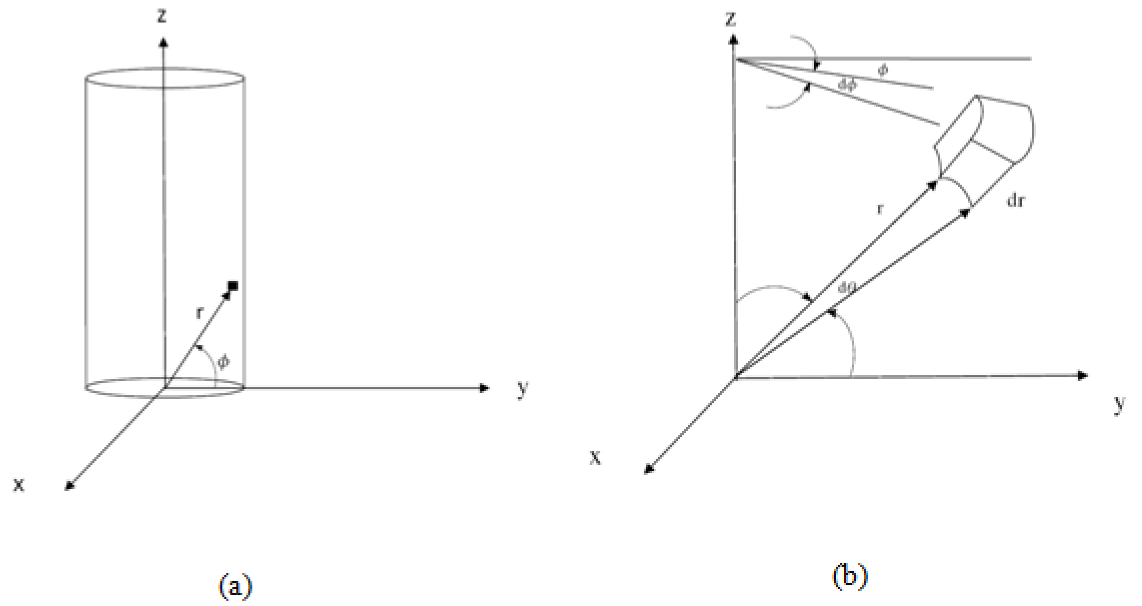
For 3-dimensional problem in spherical coordination $(r, \emptyset, \theta)$ with temperature, $T(r, \emptyset, \theta)$, the conduction heat flux vector will be:



$$q'' = -k\frac{\partial T}{\partial r}i - k\frac{\partial T}{r\partial \theta}j - k\frac{\partial T}{r\sin\theta\,\partial \emptyset}k$$

where: $q''_r = -k\frac{\partial T}{\partial r}$; $q''_\theta = -k\frac{\partial T}{r\partial \theta}$; $q''_\emptyset = -k\frac{\partial T}{r\sin\theta\,\partial \emptyset}$

General heat conduction relation in cylindrical coordinate system



Cylindrical coordinate system (a) and an element of the cylinder

The energy conservation for the system is written as,

I + II = III + IV 2.37

where,

I : Rate of heat energy conducted in
II : Rate of heat energy generated within the volume element
III : Rate of heat energy conducted out
IV : Rate of energy accumulated

and the above terms are defines as,

I : $\dot{q}_r + \dot{q}_\emptyset + \dot{q}_z$

II : $\dot{e}_g \, dr \, dz \, r \, d\emptyset$

III : $\dot{q}_{r+dr} + \dot{q}_{\emptyset+d\emptyset} + \dot{q}_{z+dz}$

IV : $\rho \, dr \, dz \, r d\emptyset \, c_p \frac{\partial T}{\partial t}$ (T is a function of space and time)

Thus,

(I) - (III) $\quad : (-\frac{\partial \dot{q}_r}{\partial r}\, dr) + (-\frac{\partial \dot{q}_\emptyset}{\partial \emptyset}\, d\emptyset) + (-\frac{\partial \dot{q}_z}{\partial z}\, dz)$

Using Fourier's law

$$\dot{q}_r = -k(r\, d\emptyset\, dz)\frac{\partial T}{\partial r}$$

$$\dot{q}_\emptyset = -k(dr\, dz)\frac{\partial T}{r\partial \emptyset}$$

$$\dot{q}_z = -k(r\, d\emptyset\, dr)\frac{\partial T}{\partial z}$$

(I) - (III) $\quad : \frac{\partial}{\partial r}(k\, r\, d\emptyset\, dz\frac{\partial T}{\partial r})dr + \frac{\partial}{\partial \emptyset}(k\, dr\, dz\frac{\partial T}{r\partial \emptyset})d\emptyset + \frac{\partial}{\partial z}(k\, r\, d\emptyset\, dr\frac{\partial T}{\partial z})dz$

If k is isotropic,

$$= k\, d\emptyset\, dz\, dr\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + k\, d\emptyset\, dz\, dr\frac{\partial}{\partial \emptyset}(\frac{\partial T}{r\partial \emptyset}) + r\, k\, d\emptyset\, dr\, dz\frac{\partial}{\partial z}(\frac{\partial T}{\partial z})$$

$$= k\, r\, d\emptyset\, dz\, dr\, [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{1}{r^2}\frac{\partial}{\partial \emptyset}(\frac{\partial T}{\partial \emptyset}) + \frac{\partial}{\partial z}(\frac{\partial T}{\partial z})]$$

$$= k\, r\, d\emptyset\, dz\, dr\, [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{\partial^2 T}{\partial z^2}]$$

On putting the values in equation 2.37

$$k\, r\, d\emptyset\, dz\, dr \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{\partial^2 T}{\partial z^2}\right] + \dot{e}_g\, dr\, dz\, r\, d\emptyset = \rho\, dr\, dz\, r d\emptyset\, C_p \frac{\partial T}{\partial t}$$

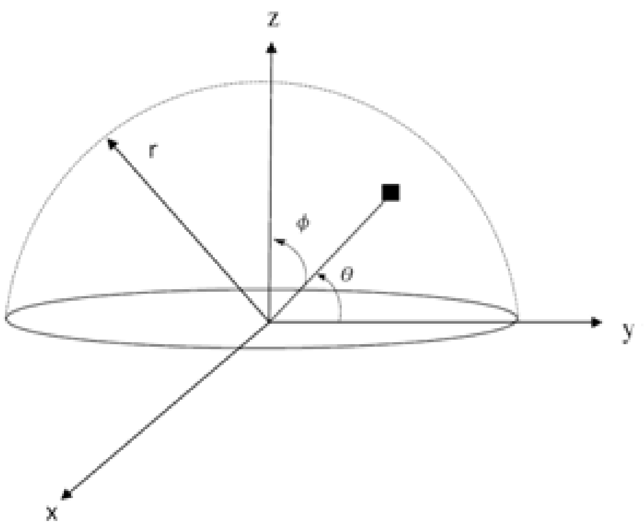
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_g}{k} = \frac{\rho C_p}{k}\ \frac{\partial T}{\partial t}$$

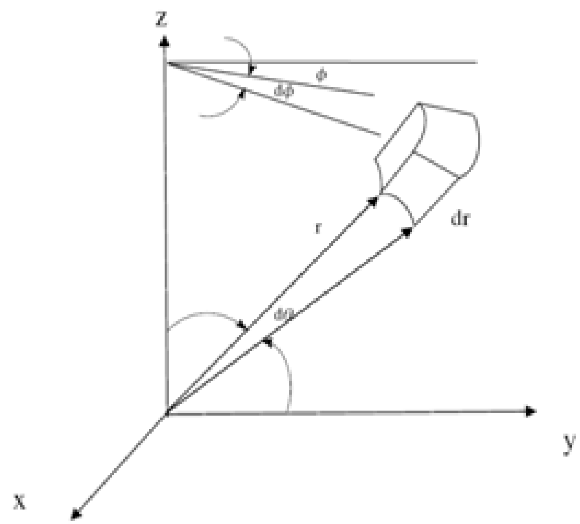
$$\nabla^2 T + \frac{\dot{e}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \left(\text{where } \alpha = \frac{k}{\rho C_p}\right) \qquad 2.38$$

Thus the Laplacian operator is,

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \emptyset^2} + \frac{\partial^2}{\partial z^2} \qquad 2.39$$

Spherical coordinate system





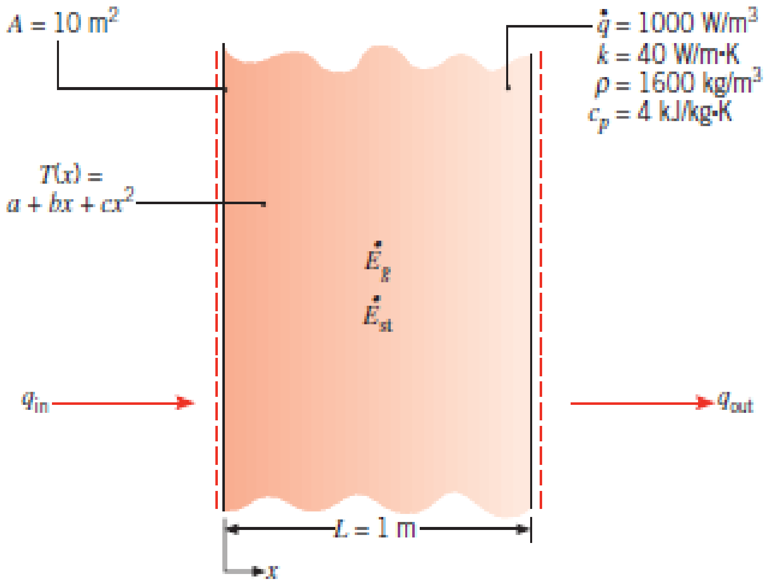
Spherical coordinate system and an element of the sphere

In a similar way the general expression for the conduction heat transfer in spherical body with heat source can also be found out as per the previous discussion. The Laplacian operator for the spherical coordinate system (fig.2.13) is given below and the students are encouraged to derive the expression themselves.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin \emptyset}\frac{\partial}{\partial \emptyset}\left(sin\emptyset\ \frac{\partial}{\partial \emptyset}\right) + \frac{1}{r^2 \sin^2 \emptyset}\frac{\partial^2}{\partial \theta^2} \qquad 2.40$$

P. 2.6. The temperature distribution across a wall 1 m thick at a certain instant of time is given as $T(x) = a + bx + cx^2$, where T is in degrees Celsius and x is in meters, while a = 900°C, b = −300°C/m, and c = −50°C/m2. A uniform heat generation, q = 1000 W/m^3 is present in the wall of area 10 m^2 having the properties ρ = 1600 kg/m^3, k = 40 W/m · K, and c_p = 4 kJ/kg · K.

1. Determine the rate of heat transfer entering the wall (x = 0) and leaving the wall (x = 1 m).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at x = 0, 0.25, and 0.5 m.



Known: Temperature distribution $T(x)$ at an instant of time t in a one-dimensional wall with uniform heat generation.

Find:

1. Heat rates entering, q_{in} ($x = 0$), and leaving, q_{out} ($x = 1$ m), the wall.
2. Rate of change of energy storage in the wall, E_{st}.
3. Time rate of temperature change at $x = 0$, 0.25, and 0.5 m.

Assumptions:

1. One-dimensional conduction in the x-direction.
2. Incompressible, isotropic medium with constant properties.
3. Uniform internal heat generation, q (W/m^3).

Recall that once the temperature distribution is known for a medium, it is a simple matter to determine the conduction heat transfer rate at any point in the medium or at its surfaces by using Fourier's law. Hence the desired heat rates may be determined by using the prescribed temperature distribution with Equation 2.1. Accordingly,

$$q_{\text{in}} = q_x(0) = -kA\frac{\partial T}{\partial x}\bigg|_{x=0} = -kA(b + 2cx)_{x=0}$$

$$q_{\text{in}} = -bkA = 300°\text{C/m} \times 40 \text{ W/m} \cdot \text{K} \times 10 \text{ m}^2 = 120 \text{ kW} \qquad \triangleleft$$

Similarly,

$$q_{\text{out}} = q_x(L) = -kA\frac{\partial T}{\partial x}\bigg|_{x=L} = -kA(b + 2cx)_{x=L}$$

$$q_{\text{out}} = -(b + 2cL)kA = -[-300°\text{C/m}$$

$$+ 2(-50°\text{C/m}^2) \times 1 \text{ m}] \times 40 \text{ W/m} \cdot \text{K} \times 10 \text{ m}^2 = 160 \text{ kW} \qquad \triangleleft$$

The rate of change of energy storage in the wall $\dot{E}_{\text{st}}$ may be determined by applying an overall energy balance to the wall. Using Equation 1.12c for a control volume about the wall,

$$\dot{E}_{\text{in}} + \dot{E}_g - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

$\boxed{\dot{E}_g = \dot{q}AL}$

where $\dot{E}_g = \dot{q}AL$, it follows that

$$\dot{E}_{\text{st}} = \dot{E}_{\text{in}} + \dot{E}_g - \dot{E}_{\text{out}} = q_{\text{in}} + \dot{q}AL - q_{\text{out}}$$

$$\dot{E}_{\text{st}} = 120 \text{ kW} + 1000 \text{ W/m}^3 \times 10 \text{ m}^2 \times 1 \text{ m} - 160 \text{ kW}$$

$$\dot{E}_{\text{st}} = -30 \text{ kW} \qquad \triangleleft$$

The time rate of change of the temperature at any point in the medium may be determined from the heat equation,

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

From the prescribed temperature distribution, it follows that

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(b + 2cx) = 2c = 2(-50°\text{C/m}^2) = -100°\text{C/m}^2$$

Note that this derivative is independent of position in the medium. Hence the time rate of temperature change is also independent of position and is given by

$$\frac{\partial T}{\partial t} = \frac{40\ \text{W/m}\cdot\text{K}}{1600\ \text{kg/m}^3 \times 4\ \text{kJ/kg}\cdot\text{K}} \times (-100°\text{C/m}^2)$$

$$+ \frac{1000\ \text{W/m}^3}{1600\ \text{kg/m}^3 \times 4\ \text{kJ/kg}\cdot\text{K}}$$

$$\frac{\partial T}{\partial t} = -6.25 \times 10^{-4°}\text{C/s} + 1.56 \times 10^{-4°}\text{C/s}$$

$$= -4.69 \times 10^{-4°}\text{C/s} \qquad \triangleleft$$

Comments:

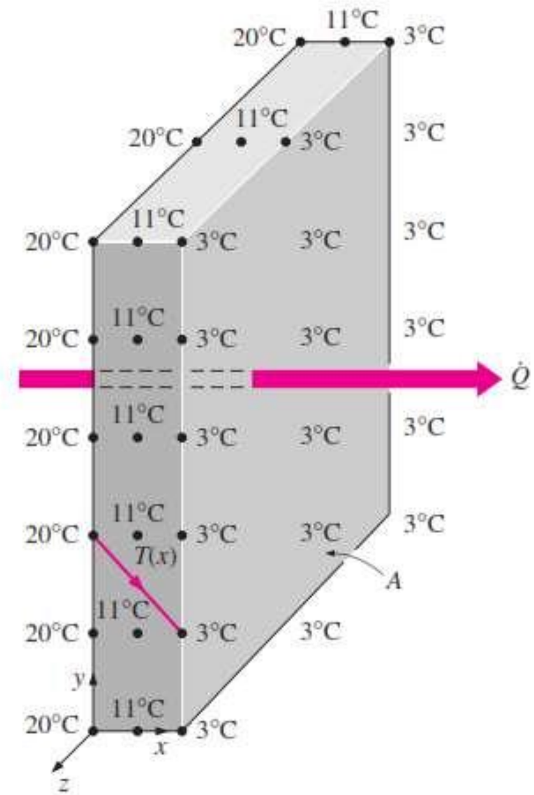
1. From this result, it is evident that the temperature at every point within the wall is decreasing with time.

2. Fourier's law can always be used to compute the conduction heat rate from knowledge of the temperature distribution, even for unsteady conditions with internal heat generation.

Steady Heat Conduction in Plane Walls

- ✓ For one-dimensional conduction in a plane wall, temperature is a function of the *x-coordinate* only and heat is transferred exclusively in this direction.

- ✓ There will be no heat transfer in a direction in which there is no change in temperature. (**isothermal**).



Heat transfer through a plane wall:

Temperature distribution and its equivalent thermal circuit.

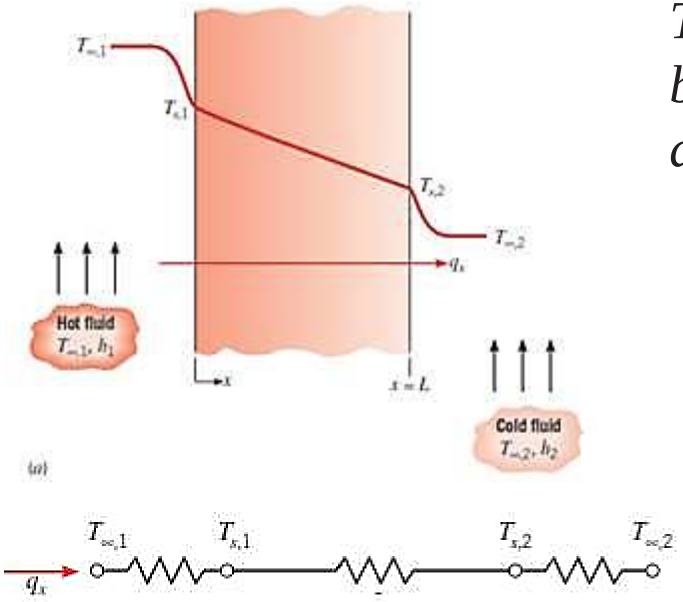
- ✓ For steady-state conditions with no distributed source or sink of energy within the wall, the appropriate form of the heat equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \ldots \ldots \ldots (2.41)$$

- ✓ *No heat generation*, *the heat flux is a constant*, *independent of x. and k is constant* , the equation may be integrated twice to obtain the *general solution* .

$$T(x) = C_1 x + C_2 \ldots \ldots \ldots (2.42)$$

Steady Heat Conduction in Plane Walls



To obtain the constants of integration, C_1 and C_2, boundary conditions must be introduced. x = 0 and x = L, in which case $T(x) = C_1x + C_2$.

$$T(0) = T_{s,1} \;\&\; T(L) = T_{s,2}$$

$$At\ x = 0,\ T_{s,1} = C_2,$$

$$At\ x = L,\ T_{s,2} = C_1L + C_2$$

$$Or\ T_{s,2} = C_1L + T_{s,1}$$

$$Thus,\quad \frac{T_{s,2} - T_{s,1}}{L} = C_1 \ldots\ldots\ldots (2.43)$$

$$Substituting\ into\ eq.(2)\ T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1} \ldots\ldots (2.44),$$

- ✓ *Therefore eq. (2) states that the temperature varies linearly with x.*

$$Heat\ rate\ (Fourier's\ Law),\qquad q_x = -kA\frac{dT}{dx} = \frac{KA}{L}(T_{s,1} - T_{s,2}) \ldots\ldots\ldots (2.45)$$

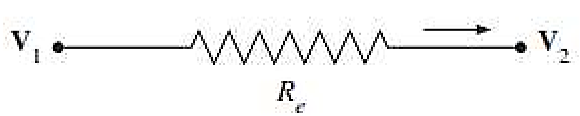
$$For\ Heat\ Flux,\qquad q_x'' = \frac{q_x}{A} = \frac{K}{L}(T_{s,1} - T_{s,2}) \ldots\ldots\ldots (2.46)$$

- ✓ Heat Rate and Heat Flux are constant, independent of x

.

Thermal Resistance

- ✓ *Just as an electrical resistance is associated with the conduction of electricity, a thermal resistance may be associated with the conduction of heat.*
- ✓ *Defining resistance as the ratio of a driving potential to the corresponding transfer rate,*
- ✓ *For electrical conduction in the same system, Ohm's law provides an electrical resistance of the form*



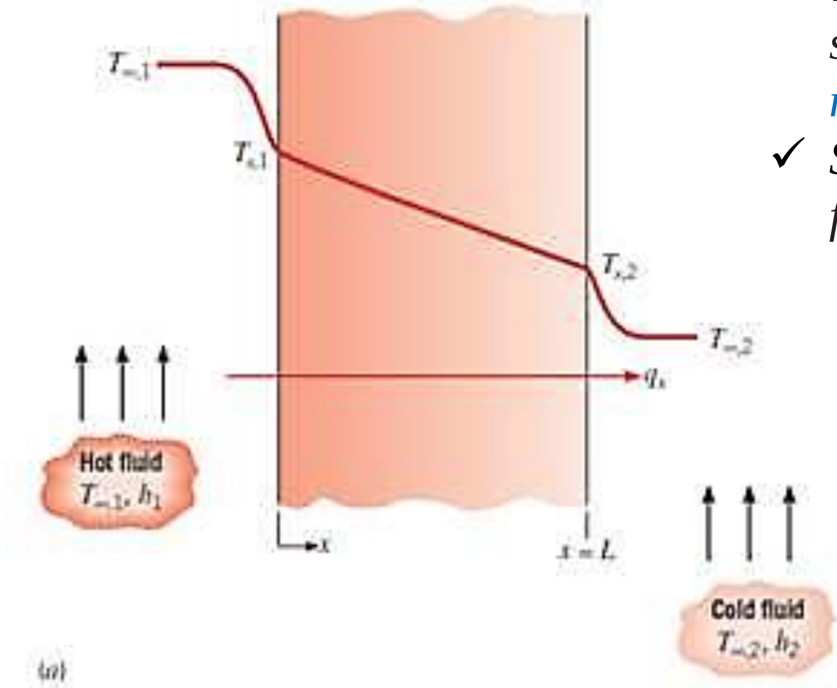
$$R_e = \frac{V_1 - V_2}{I} = \frac{E_{s,1} - E_{s,2}}{I} = \frac{L}{\sigma A}$$

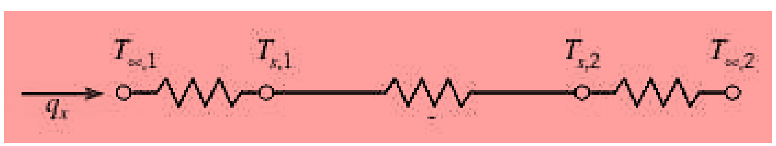
- ✓ *thermal resistance for conduction in a plane wall is* $R_t = \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}$

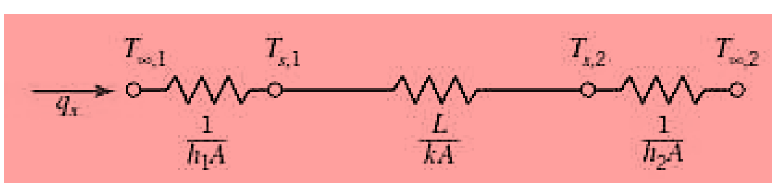
- ✓ Thermal resistance for convection heat transfer at a surface. (Newton's cooling system) $q = hA(T_s - T_\infty)$

$$R_{t,Con} = \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

The equivalent thermal circuit for the plane wall with convection surface conditions







- ✓ *The heat transfer rate may be determined from separate consideration of each element in the network.*
- ✓ *Since q_x is constant throughout the network, it follows that*

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

- ✓ In terms of the *overall temperature difference*, $T_{\infty,1} - T_{\infty,2}$, and the *total thermal resistance*, R_{tot} the heat transfer rate

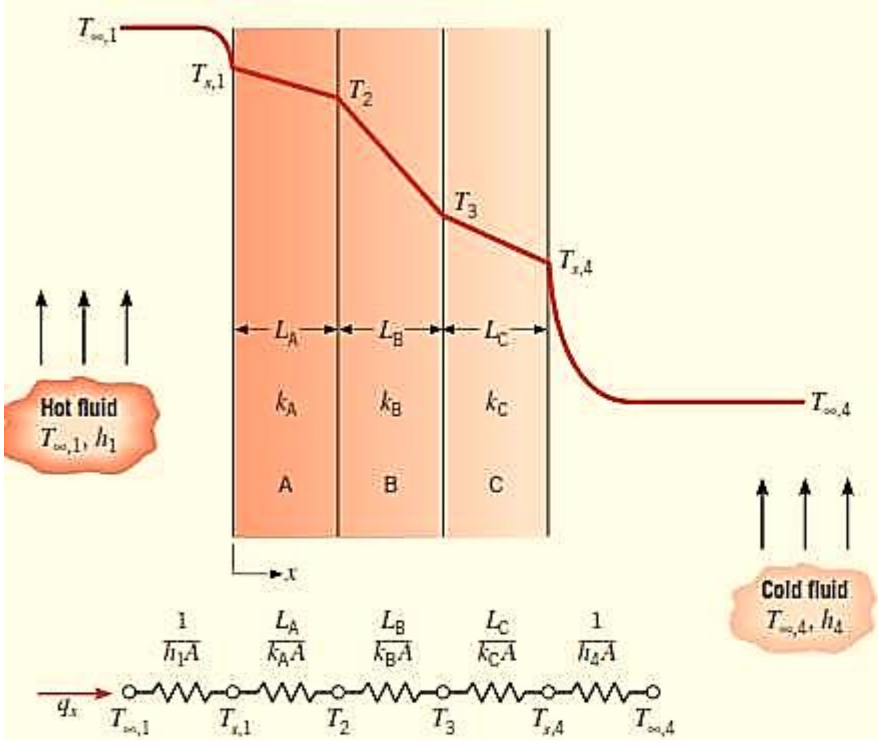
$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$

- ✓ *Thermal resistance for radiation*

$$R_{t,rad} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A}$$

The Composite Wall (Multilayer Plane)



The composite walls that involve any number of series and parallel thermal resistances due to layers of different materials.

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\Sigma R_t}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1 A) + (L_A/k_A A) + (L_B/k_B A) + (L_C/k_C A) + (1/h_4 A)]}$$

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1 A)} = \frac{T_{s,1} - T_2}{(L_A/k_A A)} = \frac{T_2 - T_3}{(L_B/k_B A)} = \cdots$$

Overall heat transfer coefficient (U)

With composite systems, it is often convenient to work with an overall heat transfer coefficient U, which is defined by an expression analogous to Newton's law of cooling. Accordingly, $q_x \equiv UA\Delta T$ ... (2.47), *where* ΔT is overall temperature difference.

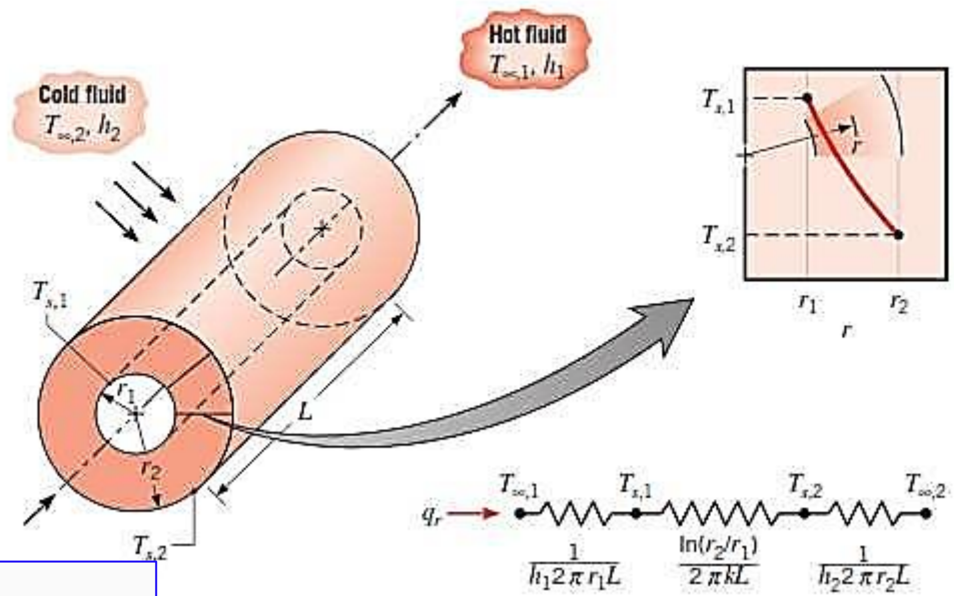
$$U = \frac{1}{R_{tot} A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]}$$

Or

$$R_{tot} = \sum R_t = \frac{\Delta T}{q} = \frac{1}{UA}$$

Steady Heat Conduction in Cylinder

- ✓ A common example is the hollow cylinder whose inner and outer surfaces are exposed to fluids at different temperatures
- ✓ For steady-state conditions with no heat generation, the appropriate form of the heat equation,



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(kr\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{g}=\rho C\frac{\partial T}{\partial t}$$

Or $\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0$ ……………..(2.48)

For Fourier's law $q_r=-kA\frac{dT}{dr}=-k(2\pi rL)\frac{dT}{dr}$ …(2.49)

where $A = 2\pi rL$ is the area normal to the direction of heat transfer.

From eq. **2.48,** the quantity $(kr(^{dT}/_{dr})=0$ is independent of r, then the conduction heat transfer rate q_r **eq. 2.49** is constant in the radial direction. But, for the heat flux q_r'' is dependent on radial direction.

Steady Heat Conduction in Cylinder

Double integration for eq. **2.48** by assuming the value of *k to be constant*

$$T(r) = C_1 \ln r + C_2$$

To obtain the constants of integration **C1** and **C2**, we introduce the following boundary conditions:

$$T(r_1) = T_{s,1} \quad \text{and} \quad T(r_2) = T_{s,2}$$

Applying these conditions to the general solution, we then obtain

$$T_{s,1} = C_1 \ln r_1 + C_2 \quad \text{and} \quad T_{s,2} = C_1 \ln r_2 + C_2$$

Solving for **C1** and **C2** and substituting into the general solution, we then obtain

$$q_r = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)} \quad \text{……………..(2.50)}$$

By substituting eq. **2.50** into eq. **2.49**

$$q_r = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)} \quad \text{……………..(2.51)}$$

The thermal resistance for cylinder:

$$R_{t,\mathrm{cond}} = \frac{\ln(r_2/r_1)}{2\pi Lk} \quad \text{……………..(2.52)}$$

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln(r_2/r_1)}{2\pi k_A L} + \dfrac{\ln(r_3/r_2)}{2\pi k_B L} + \dfrac{\ln(r_4/r_3)}{2\pi k_C L} + \dfrac{1}{2\pi r_4 L h_4}}$$

……………..(2.53)

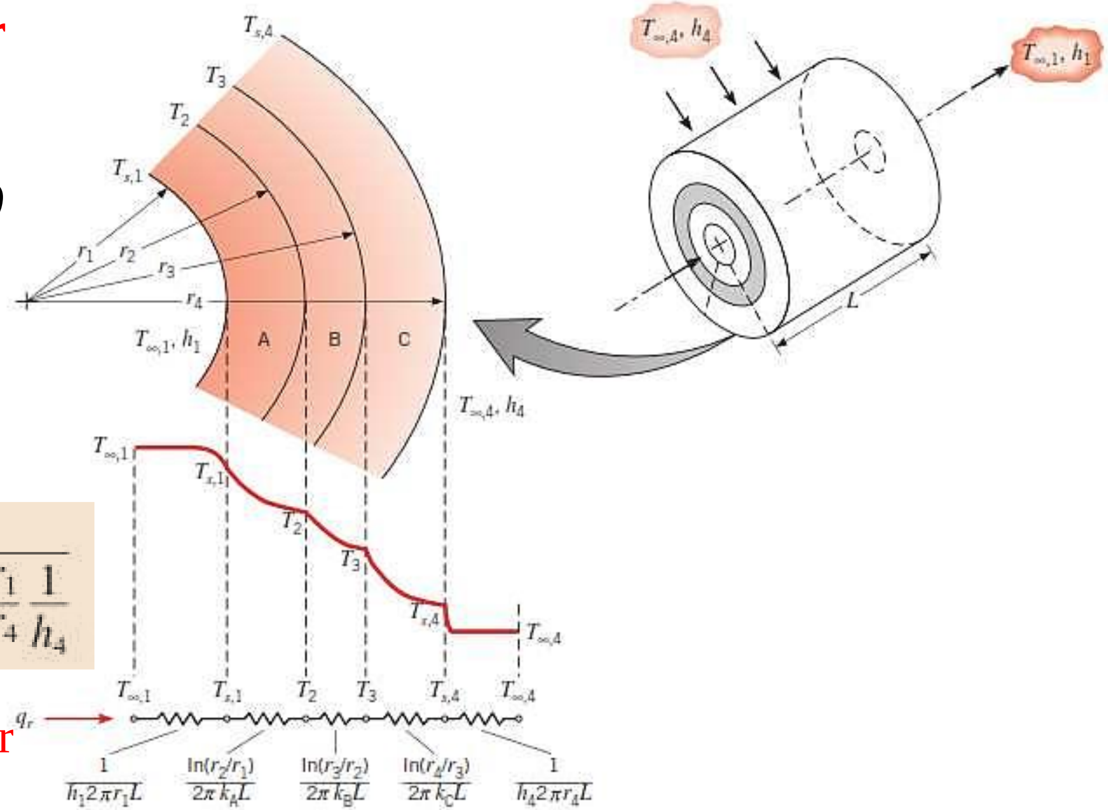
Steady Heat Conduction in Cylinder

For an **overall heat transfer** coefficient

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA(T_{\infty,1} - T_{\infty,4}) \quad \text{...(2.54)}$$

*If U is defined in terms of the inside area, A = 2π**r**₁L, Eq(s) 2.53 & 2.54 can be yield*

$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\frac{r_2}{r_1} + \frac{r_1}{k_B}\ln\frac{r_3}{r_2} + \frac{r_1}{k_C}\ln\frac{r_4}{r_3} + \frac{r_1}{r_4}\frac{1}{h_4}}$$

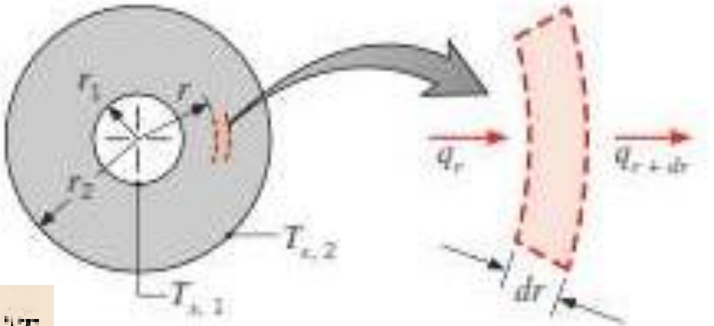


For arbitrary overall heat transfer coefficient

$$U_1A_1 = U_2A_2 = U_3A_3 = U_4A_4 = (\Sigma R_t)^{-1} \quad \text{...(2.55)}$$

Steady Heat Conduction in Sphere

By energy conservation on the differential control volume of the figure, $q_r = q_{r+dr}$



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$$

For steady-state, one-dimensional conditions with no heat generation. The appropriate form of Fourier's law is

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr} \quad ...(2.56)$$

Where $A = 4\pi r^2$ is the area normal to the direction of heat transfer. q_r is constant, independent of r. eq. **(2.56) may be expressed integral form**

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)\,dT$$

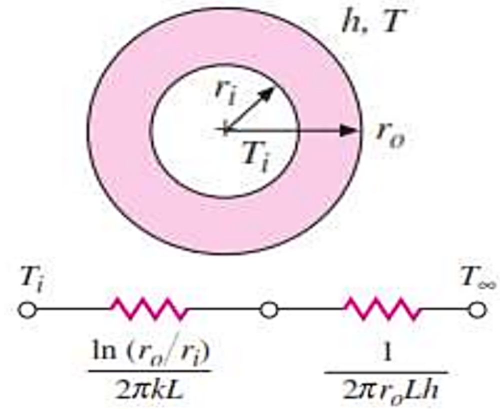
For constant *k*,

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)} \quad ...(2.57)$$

Thermal resistance for sphere

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad ...(2.58)$$

Critical Thickness of Insulation : sphere and cylinder



Let us consider a layer of insulation which might be installed around a circular pipe, (**cylinder**) as shown in Figure.

The inner temperature of the insulation is fixed at T_i, and the outer surface exposed to a convection environment T_∞.

From the thermal network the heat transfer is

$$q = \frac{2\pi L\,(T_i - T_\infty)}{\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}}$$

Now let us manipulate this expression to determine the outer radius of insulation r_0, which will maximize the heat transfer. The maximization condition is

$$\frac{dq}{dr_o} = 0 = \frac{-2\pi L\,(T_i - T_\infty)\left(\dfrac{1}{kr_o} - \dfrac{1}{hr_o^2}\right)}{\left[\dfrac{\ln(r_o/r_i)}{k} + \dfrac{1}{r_o h}\right]^2}$$

which gives the result $\quad r_o = \dfrac{k}{h} \qquad$ ...(2.59)

For sphere $\quad r_{cr} = 2k/h \qquad$ ...(2.60)

One-dimensional, steady-state solutions to the heat equation with no generation (no thermal energy generation)

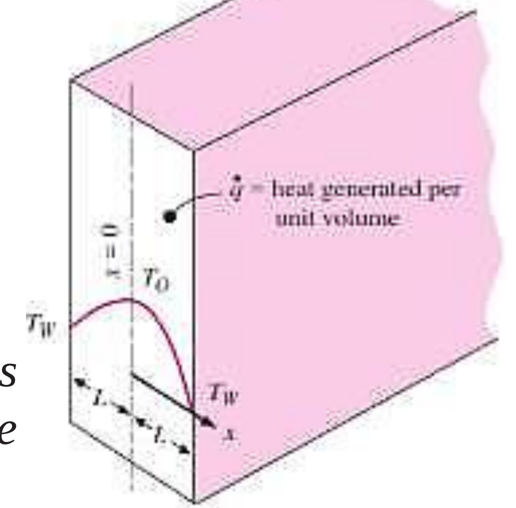
	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

Heat Conduction with Thermal Energy Generation

Plane wall with thermal energy Generation

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction at}\\ x, y, \text{and } z\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x,\\ y+\Delta y, \text{and } z+\Delta z\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$



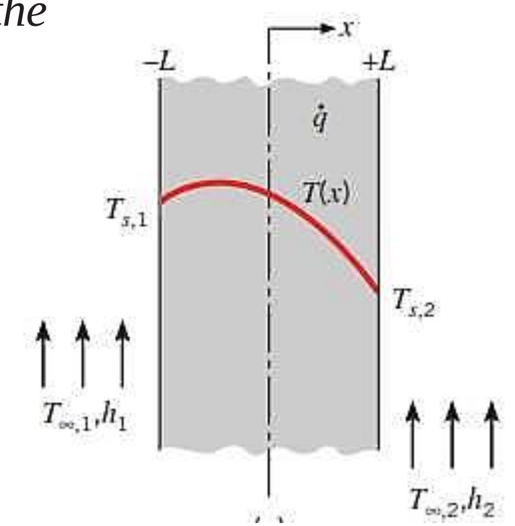
- ✓ Consider the Asymmetric plane wall of **Figure *a***, *in which there is uniform energy generation per unit volume (q is constant) the surface is maintained at* $T_{s,1}$ *and* $T_{s,2}$.
- ✓ For constant thermal conductivity ***k***, *the appropriate form of the heat equation,*

$$\frac{d^2T}{dx^2} + \frac{q}{k} = 0$$

General solution is

$$T = -\frac{q}{2k}x^2 + C_1x + C_2$$

where C_1 and C_2 are the constants of integration. For the prescribed boundary conditions,



$$T(-L) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

Heat Conduction with Thermal Energy Generation

The constants may be evaluated and are of the form

$$C_1 = \frac{T_{s,2} - T_{s,1}}{2L} \quad \text{and} \quad C_2 = \frac{q}{2k}L^2 + \frac{T_{s,1} + T_{s,2}}{2}$$

in which case the temperature distribution is

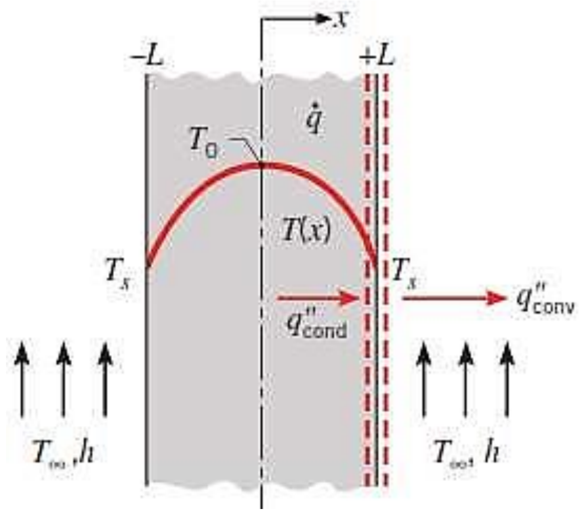
$$T(x) = \frac{qL^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2} \qquad ...(2.61)$$

The heat flux at any point in the wall may, of course, be determined by using eq. **2.61** with Fourier's law. Note, however, that *with generation the heat flux is no longer independent of x*.

For symmetric plane wall shown in the figure, both surfaces are maintained at a common temperature, $T_{s,1} \equiv T_{s,2} \equiv T_s$

The temperature distribution is then *symmetrical about the* mid plane, using above eq. **2.61.**

$$T(x) = \frac{qL^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s \qquad ...(2.62)$$



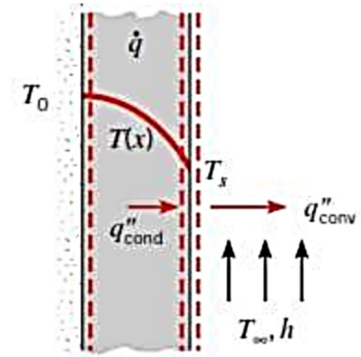
In which plane that the maximum temperature exists?

Heat Conduction with Thermal Energy Generation

The maximum temperature exists at the mid plane $T(0) \equiv T_0 = \dfrac{qL^2}{2k} + T_s$

in which case the temperature distribution, Equation 2.62 , may be expressed as

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2 \qquad \text{….(2.63)}$$



- ***Fig is represented*** *no heat transfer across the surface* (***adiabatic* surface at mid plane**)
- *Using* eq. 2.62 , the plane walls that are perfectly insulated on one side (***x=0***) *and maintained at a fixed temperature* ***T****s.* on the other side (***x=L***).
- Neglecting radiation and substituting the appropriate rate equations, the energy balance given by Equation $-k\left.\dfrac{dT}{dx}\right|_{x=L} = h(T_s - T_\infty)$

Substituting from Equation 2.62 to obtain the temperature gradient at $x = L$, *it follows that*

$$T_s = T_\infty + \frac{qL}{h} \qquad (2.64)$$

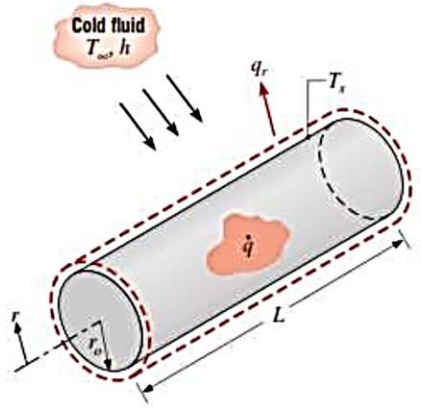
Hence ***T****s* may be computed from knowledge of T_∞. , *L, and h.*

Radial system with thermal energy Generation

Heat generation may occur in a variety of radial geometries. Consider the long, solid cylinder of figure 2a, which could represent a **current-carrying wire** or a **fuel element in a nuclear reactor**.

For constant thermal conductivity ***k***,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q}{k}=0$$



Separating variables and assuming uniform generation, this expression may be integrated to obtain

$$r\frac{dT}{dr}=-\frac{q}{2k}r^2+C_1 \qquad (2.65)$$

Repeating the procedure, the general solution for the temperature distribution becomes;

$$T(r)=-\frac{q}{4k}r^2+C_1\ln r+C_2 \qquad (2.66)$$

To obtain the constants of integration C_1 and C_2, we apply boundary conditions

$$\left.\frac{dT}{dr}\right|_{r=0}=0 \quad \text{and} \quad T(r_0)=T_s$$

Radial system with thermal energy Generation

For solid cylinder the centerline is a line of symmetry (**symmetry cond$\underline{n}$**)

- temperature distribution and the temperature gradient must be zero.
- symmetrical boundary conditions (Figure *b)*.
- $r = 0$ and Eq (2.65) , it is evident that $\mathbf{C_1 = 0}$

Using the surface boundary condition at $\mathbf{r = r_0}$ with equation (2.66) then obtain

$$C_2 = T_s + \frac{q}{4k} r_o^2$$

The temperature distribution is therefore $T(r) = \frac{q r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$

∴ the heat rate at any radius in the cylinder may, of course, be evaluated by using Eq (2.67) with **Fourier's law.**

- Evaluating Eq (2.67) at the centerline and dividing the result into Equation (2.67) we obtain the temperature distribution in non dimensional form,

$$\frac{T(r) - T_s}{T_o - T_s} = 1 - \left(\frac{r}{r_o}\right)^2 \qquad (2.68)$$

Where $\mathbf{T_0}$ is the centerline temperature.

Radial system with thermal energy Generation

To relate the surface temperature, T_s, to the temperature of the cold fluid T_∞, there is two methods

i. Surface energy balance

ii. An overall energy balance.

Choosing the second approach, we obtain

$$q(\pi r_o^2 L) = h(2\pi r_o L)(T_s - T_\infty) \quad \text{or} \quad T_s = T_\infty + \frac{q r_o}{2h} \qquad (2.69)$$

C. Sphere system with thermal energy Generation

E at conduction on sphere (polar) one dimension and steady state with thermal generation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{q}{k} = 0 \qquad (2.70)$$

Or,

$$T(r) = T_s + \frac{q r_2^2}{6k}\left(1 - \frac{r^2}{r_2^2}\right) \qquad (2.71)$$

One-Dimensional, Steady-State Solutions to the *Heat Equation for Plane, cylindrical, and Spherical Walls with Uniform Generation and Asymmetrical Surface Conditions.*

Plane Wall	$T(x) = \frac{qL^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$
Cylindrical Wall	$T(r) = T_{s,2} + \frac{qr_2^2}{4k}\left(1 - \frac{r^2}{r_2^2}\right) - \left[\frac{qr_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]\frac{\ln(r_2/r)}{\ln(r_2/r_1)}$
Spherical Wall	$T(r) = T_{s,2} + \frac{qr_2^2}{6k}\left(1 - \frac{r^2}{r_2^2}\right) - \left[\frac{qr_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]\frac{(1/r) - (1/r_2)}{(1/r_1) - (1/r_2)}$

Temperature Distribution

One-Dimensional, Steady-State Solutions to the *Heat Equation for Plane, cylindrical, and Spherical Walls with Uniform Generation and Asymmetrical Surface Conditions.*

Plane Wall	$q''(x) = \dot{q}x - \frac{k}{2L}(T_{s,2} - T_{s,1})$
Cylindrical Wall	$q''(r) = \frac{\dot{q}r}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r\ln(r_2/r_1)}$
Spherical Wall	$q''(r) = \frac{\dot{q}r}{3} - \frac{k\left[\frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r^2[(1/r_1) - (1/r_2)]}$

Heat Flux

***One-Dimensional, Steady-State Solutions to the** Heat Equation for Plane, cylindrical, and Spherical Walls with Uniform Generation and Asymmetrical Surface Conditions.*

Plane Wall	$q(x) = \left[\dot{q}x - \frac{k}{2L}(T_{s,2} - T_{s,1})\right]A_x$
Cylindrical Wall	$q(r) = \dot{q}\pi L r^2 - \frac{2\pi L k}{\ln(r_2/r_1)} \cdot \left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]$
Spherical Wall	$q(r) = \frac{\dot{q}4\pi r^3}{3} - \frac{4\pi k\left[\frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{(1/r_1) - (1/r_2)}$

Heat Rate

P. 2.7 *A pipe (k = 180 W/m °C) having inner and outer diameters of 80 mm and 100 mm, respectively is located in a space at 25 C. Hot gases at temperature160 C flows through the pipe. Neglecting surface heat transfer coefficients, calculate:*

a) *The heat loss through the pipe per unit length,*
b) *The temperature at a point halfway between the inner and outer surface, and.*
c) *The surface area normal to the direction of the heat flow so that the heat transfer through the pipe can be determine y considering material of pipe as a plane of the same thickness.*

Take away from todays session

- ✓ To understand the Fourier's law and its applications in steady state heat transfer problems using rectangular coordinates, unidirectional heat flow and constant thermal conductivity. Identify the normal area to the heat flow.
- ✓ Understand the analogy between electrical and thermal resistances. Apply the analogy with resistances in series.
- ✓ Apply Fourier's law in heat transfer problems through cylindrical walls.
- ✓ Understand the concept of logarithmic mean area in heat transfer through cylindrical walls.
- ✓ Solve problems of heat transfer through multiple cylindrical walls with different thermal conductivities.

Next Session…. Transient Heat Conduction