Connectivity

- Problem of determining whether a message can be sent between two computers using intermediate links
- Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on

Paths

A **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

DEFINITION 1

Let n be a nonnegative integer and G an undirected graph. A *path* of *length n* from u to v in G is a sequence of n edges $e_1,...,e_n$ of G for which there exists a sequence $x_0 = u, x_1,..., x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1,...,n$, the end points x_{i-1} and x_i . When the graph is simple, we denote this path by its vertex sequence $x_0 , x_1 , . . . , x_n$ (because listing these vertices uniquely determines the path). The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to *pass through* the vertices x_1, $x_2, . . . , x_{n-1}$ or *traverse* the edges $e_1, e_2, . . . , e_n$. A path or circuit is *simple* if it does not contain the same edge more than once.

- When it is not necessary to distinguish between multiple edges, we will denote a path $e_1, e_2,...,e_n$, where e_i is associated with $\{x_{i-1}, x_i\}$ for $i = 1, 2,..., n$ by its vertex sequence $x_0, x_1, . . . , x_n$.
- The notation identifies a path only as far as which vertices it passes through
- It does not specify a unique path when there is more than one path that passes through this sequence of vertices, which will happen if and only if there are multiple edges between some successive vertices in the list.
- A path of length zero consists of a single vertex.

Remark: There is considerable variation of terminology concerning the concepts defined in Definition 1. For instance, in some books, the term **walk** is used instead of *path*, where a walk is defined to be an alternating sequence of vertices and edges of a graph, $v_0, e_1, v_1, e_2,...,v_{n-1}, e_n, v_n$, where v_{i-1} and v_i are the endpoints of e_i for $i = 1, 2,..., n$. When this terminology is used, **closed walk** is used instead of *circuit* to indicate a walk that begins and ends at the same vertex, and **trail** is used to denote a walk that has no repeated edge (replacing the term *simple path*). When this terminology is used, the terminology **path** is often used for a trail with no repeated vertices, conflicting with the terminology in Definition 1.

EXAMPLE

In the simple graph shown in Figure 1, a, d, c, f , e is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, d, e, c, a is not a path, because $\{e, c\}$ is not an edge. Note that b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at b. The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice.

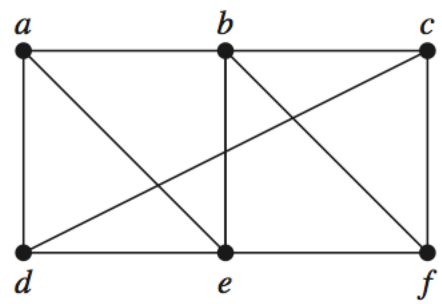


FIGURE 1 **A Simple Graph.**

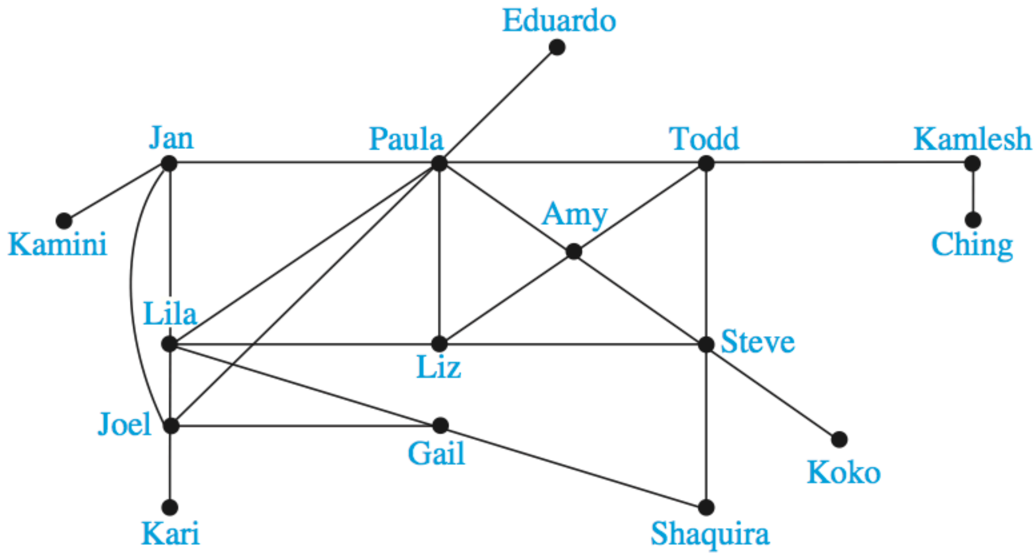
Paths and circuits in directed graphs

DEFINITION 2

Let n be a nonnegative integer and G a directed graph. A *path* of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a *circuit* or *cycle*. A path or circuit is called *simple* if it does not contain the same edge more than once.

Paths in graph models

Paths in Acquaintanceship Graphs In an acquaintanceship graph there is a path between two people if there is a chain of people linking these people, where two people adjacent in the chain know one another. For example, in acquaintanceship graph given below, there is a chain of six people linking Kamini and Ching.



Paths in Collaboration Graphs

In a collaboration graph, two people *a* and *b* are connected by a path when there is a sequence of people starting with *a* and ending with *b* such that the endpoints of each edge in the path are people who have collaborated. The academic collaboration graph of people who have written papers in mathematics, the **Erdo˝s number** of a person *m*, is the length of the shortest path between *m* and the extremely prolific mathematician Paul Erdo˝s (who died in 1996). That is, the Erdo˝s number of a mathematician is the length of the shortest chain of mathematicians that begins with Paul Erdo˝s and ends with this mathematician, where each adjacent pair of mathematicians have written a joint paper. The number of mathematicians with each Erdo˝s number as of early 2006, according to the Erdo˝s Number Project, is shown in Table 1.

TABLE 1 The Number of Mathematicians with a Given Erdős Number (as of early 2006).	
Erdős Number	*Number of People*
0	1
1	504
2	6,593
3	33,605
4	83,642
5	87,760
6	40,014
7	11,591
8	3,146
9	819
10	244
11	68
12	23
13	5

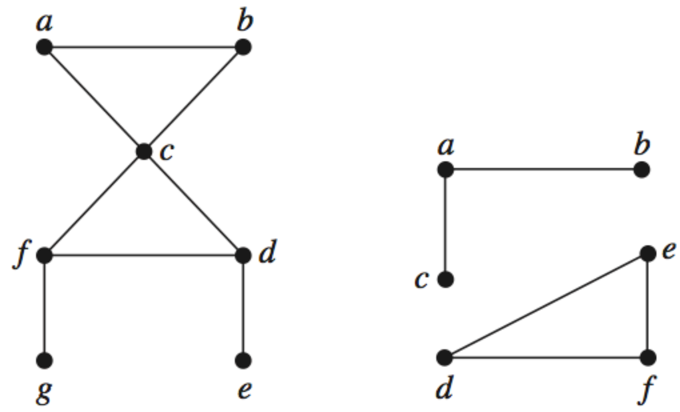
Connectedness in Undirected Graphs

DEFINITION 3

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not *connected* is called *disconnected*. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

EXAMPLE

The graph G_1 in Figure is connected, because for every pair of distinct vertices there is a path between them. However, the graph G_2 in Figure is not connected. For instance, there is no path in G_2 between vertices a and d.



G_1 G_2

THEOREM

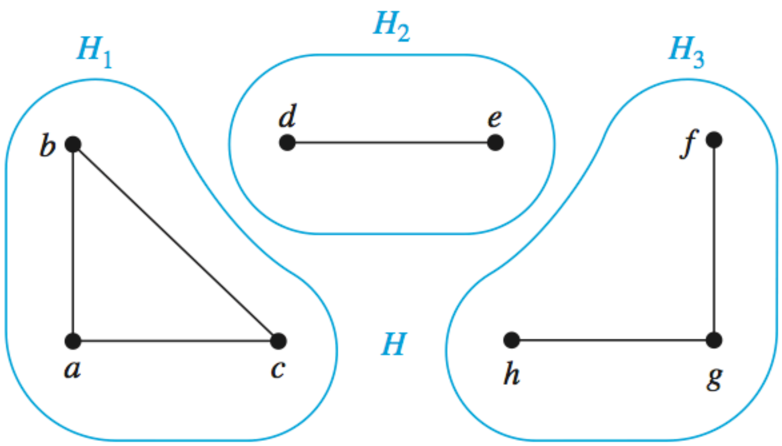
There is a simple path between every pair of distinct vertices of a connected undirected graph.

(The proof is left as an exercise for the students)

CONNECTED COMPONENTS A **connected component** of a graph G is a connected sub- graph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

EXAMPLE

The graph H is the union of three disjoint connected subgraphs H_1, H_2, and H_3, shown in Figure. These three subgraphs are the connected components of H.



Questions:

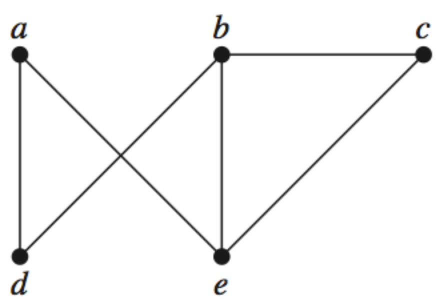
Q1) Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) *a, e ,b, c, b*

b) *a, e, a, d, b, c, a*

c) *e, b, a, d, b, e*

d) *c, b, d, a, e, c*



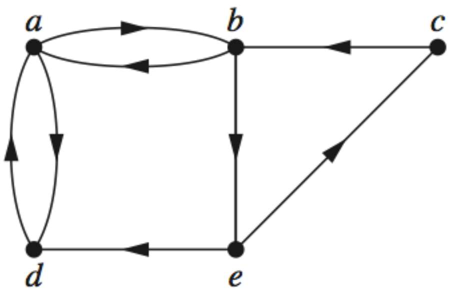
Q2) Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) *a, b, e, c, b*

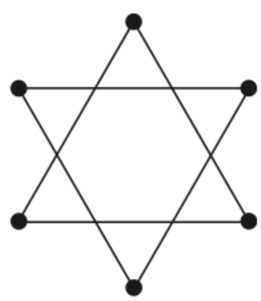
b) *a, d, a, d, a*

c) *a, d, b, e, a*

d) *a, b, e, c, b, d, a*



Q3) Determine whether the given graph is connected. If not, how many connected components does the graph have? Find each of its connected components.



Q4) What do the connected components of acquaintanceship graph represent?