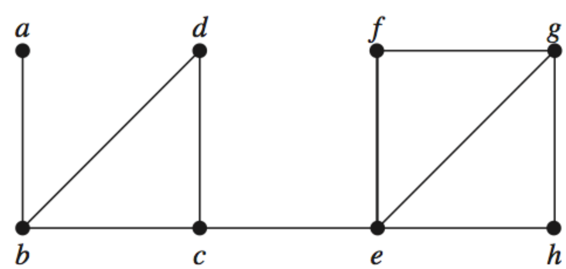
How Connected is a Graph?

Suppose that a graph represents a computer network. Knowing that this graph is connected tells us that any two computers on the network can communicate. However, we would also like to understand how reliable this network is. For instance, will it still be possible for all computers to communicate after a router or a communications link fails? To answer this and similar questions, we now develop some new concepts.

Sometimes the removal from a graph of a vertex and all incident edges produces a subgraph with more connected components. Such vertices are called **cut vertices** (or **articulation points**). The removal of a cut vertex from a connected graph produces a subgraph that is not connected. Analogously, an edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge** or **bridge**. Note that in a graph representing a computer network, a cut vertex and a cut edge represent an essential router and an essential link that cannot fail for all computers to be able to communicate.

EXAMPLE

Find the cut vertices and cut edges in the graph G_1



G_1

The cut vertices of G_1 are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects G_1.

VERTEX CONNECTIVITY

Not all graphs have cut vertices. For example, the complete graph K_n, where $n \geq 3$, has no cut vertices. When you remove a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph. Connected graphs without cut vertices are called **nonseparable graphs**, and can be thought of as more connected than those with a cut vertex. We can extend this notion by defining a more granulated measure of graph connectivity based on the minimum number of vertices that can be removed to disconnect a graph.

DEFINITION

A subset V' of the vertex set V of $G = (V, E)$ is a **vertex cut**, or **separating set**, if $G - V'$ is disconnected. For instance, in the graph Figure 1, the set $\{b, c, e\}$ is a vertex cut with three vertices. The **vertex connectivity** of a non-complete graph G, denoted by $\kappa(G)$, as the minimum number of vertices in a vertex cut.

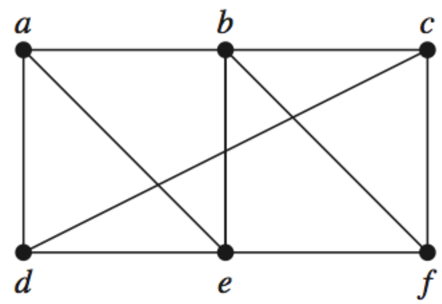


FIGURE 1 **A Simple Graph.**

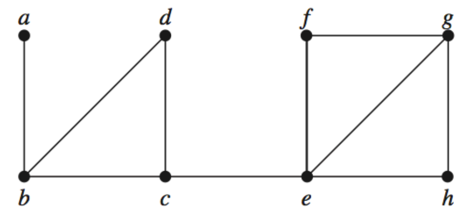
Note that the vertex set is not unique. The set $\{a, c, e\}$ is also a vertex cut. But the set $\{a, d, e\}$ is not a vertex cut.

Consequently, for every graph G, $\kappa(G)$ is minimum number of vertices that can be removed from G to either disconnect G or produce a graph with a single vertex. We have $0 \leq \kappa(G) \leq n - 1$ if G has n vertices, $\kappa(G) = 0$ if and only if G is disconnected or $G = K_1$, and $\kappa(G) = n - 1$ if and only if G is complete.

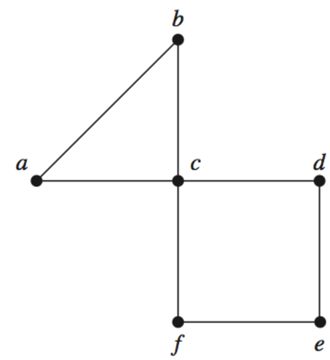
The larger $\kappa(G)$ is, the more connected we consider G to be. Disconnected graphs and K_1 have $\kappa(G) = 0$, connected graphs with cut vertices and K_2 have $\kappa(G) = 1$, graphs without cut vertices that can be disconnected by removing two vertices and K_3 have $\kappa(G) = 2$, and so on. We say that a graph is k-**connected** (or k-**vertex-connected**), if $\kappa(G) \geq k$. A graph G is 1- connected if it is connected and not a graph containing a single vertex; a graph is 2-connected, or **bi-connected**, if it is non-separable and has at least three vertices. Note that if G is a k-connected graph, then G is a j-connected graph for all j with $0 \leq j \leq k$.

EXAMPLE

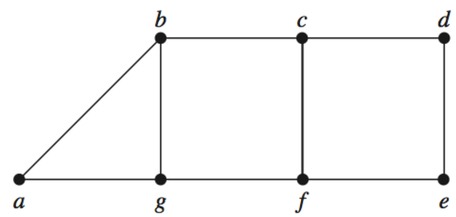
Find the vertex connectivity for each of the following graphs:



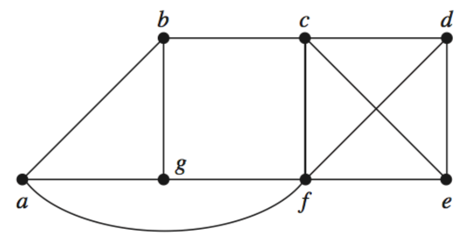
G_1



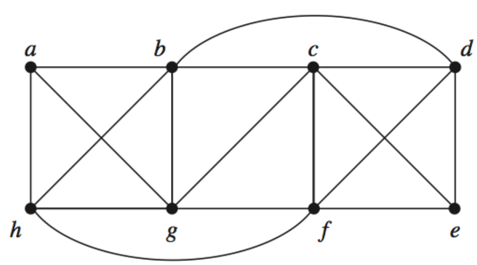
G_2



G_3



G_4



G_5

G_1 is a connected graph with a cut vertex b (c or e). Therefore, $\kappa(G_1) = 1$. Similarly, $\kappa(G_2) = 1$, because c is a cut vertex of G_2. $\kappa(G_3) = 2$ because we need two vertices (verify what all these two vertices can be) to disconnect the graph. Find $\kappa(G_4)$ and $\kappa(G_5)$

EDGE CONNECTIVITY

We can also measure the connectivity of a connected graph $G = (V, E)$ in terms of the minimum number of edges that we can remove to disconnect it. If a graph has a cut edge, then we need only remove it to disconnect G. If G does not have a cut edge, we look for the smallest set of edges that can be removed to disconnect it. A set of edges E' is called an **edge cut** of G if the subgraph $G - E'$ is disconnected. The **edge connectivity** of a graph G, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of G. This defines $\lambda(G)$ for all connected graphs with more than one vertex because it is always possible to disconnect such a graph by removing all edges incident to one of its vertices. Note that $\lambda(G) = 0$ if G is not connected. We also specify that $\lambda(G) = 0$ if G is a graph consisting of a single vertex. It follows that if G is a graph with n vertices, then $0 \leq$

$\lambda(G) \le n - 1$.

EXAMPLE

Find the edge connectivity of each of the graphs in the preceding example.

Verify that $\lambda(G_1) = 1$; $\lambda(G_2) = \lambda(G_3) = 2$; $\lambda(G_4) = \lambda(G_5) = 3$

EXERCISES

Q1) Show that you cannot have a set with 1 or two elements that will disconnect the graph in Figure 1.

Q2) Show that every connected graph, except a complete graph, has a vertex cut. Consequently, a complete graph has no vertex cut.

Q3) Show that $\lambda(G) = n - 1$ where G is a graph with n vertices if and only if $G = K_n$, which is equivalent to the statement that $\lambda(\mathrm{G}) \le n - 2$ when G is not a complete graph.

Q4) Show that a simple graph G with n vertices is connected if it has more than

$(n - 1)(n - 2)/2$ edges.