the channel by assigning it a distinct carrier frequency; at the receiver, a bank of filters is used to separate the different modulated signals and prepare them individually for demodulation.

- *Time-division multiplexing* (TDM), in which pulse modulation is used to position samples of the different message signals in nonoverlapping time slots.
- *Code-division multiplexing* (CDM), in which each message signal is identified by a distinctive code.

In FDM the message signals overlap with each other at the channel input; hence the system may suffer from *crosstalk* (i.e., interaction between message signals) if the channel is nonlinear. In TDM the message signals use the full passband of the channel, but on a time-shared basis. In CDM the message signals are permitted to overlap in both time and frequency across the channel.

Mention should also be made of *wavelength-division multiplexing* (*WDM*), which is special to optical fibers. In WDM, wavelength is used as a new degree of freedom by concurrently operating distinct portions of the wavelength spectrum (i.e., distinct *colors*) that are accessible within the optical fiber. However, recognizing the reciprocal relationship that exists between the wavelength and frequency of an electromagnetic wave, we may say that WDM is a form of FDM.

Analog and Digital Types of Communication

Typically, in the design of a communication system the information source, communication channel, and information sink (end user) are all specified. The challenge is to design the transmitter and the receiver with the following guidelines in mind:

- Encode/modulate the message signal generated by the source of information, transmit it over the channel, and produce an "estimate" of it at the receiver output that satisfies the requirements of the end user.
- Do all of this at an affordable cost.

We have the option of using a digital or analog communication system.

Consider first the case of a *digital communication* system represented by the block diagram of Figure 9, the rationale for which is rooted in information theory. The functional blocks of the transmitter and the receiver, starting from the far end of the channel, are paired as follows:

- *Source encoder-decoder.*
- *Channel encoder-decoder.*
- *Modulator-demodulator.*

The source encoder removes redundant information from the message signal and is responsible for the efficient use of the channel. The resulting sequence of symbols is called the *source code word*. The data stream is processed next by the channel encoder, which produces a new sequence of symbols called the *channel code word*. The channel code word is longer than the source code word by virtue of the *controlled* redundancy built into its construction. Finally, the modulator represents each symbol of the channel code word by a corresponding analog symbol, appropriately selected from a finite set of possible analog symbols. The sequence of analog symbols produced by the modulator is called a *waveform*, which is suitable for transmission over the channel. At the receiver, the channel output (received signal) is processed in reverse order to that in the transmitter, thereby recon-

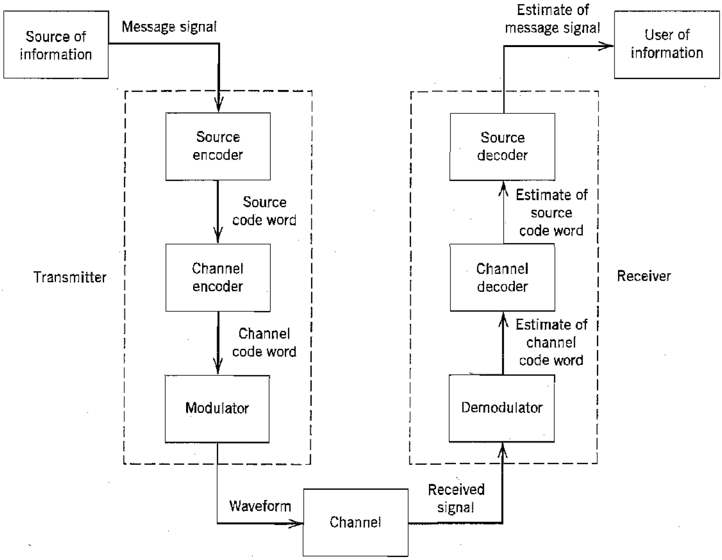


FIGURE 9 Block diagram of digital communication system.

structing a recognizable version of the original message signal. The reconstructed message signal is finally delivered to the user of information at the destination. From this description it is apparent that the design of a digital communication system is rather complex in conceptual terms but easy to build. Moreover, the system is *robust*, offering greater tolerance of physical effects (e.g., temperature variations, aging, mechanical vibrations) than its analog counterpart.

In contrast, the design of an *analog communication system* is simple in conceptual terms but difficult to build because of stringent requirements on linearity and system adjustment. For example, voice communication requires nonlinear distortion products at least 40 dB below the wanted message signal. In signal-processing terms, the transmitter consists of a modulator and the receiver consists of a demodulator, the details of which are determined by the type of CW modulation used.

The conceptual simplicity of analog communications is due to the fact that analog modulation techniques, exemplified by their wide use in radio and television, make relatively superficial changes to the message signal in order to prepare it for transmission over the channel. More specifically, there is no significant effort made by the system designer to tailor the waveform of the transmitted signal to suit the channel at any deeper level. On the other hand, digital communication theory endeavors to find a finite set of waveforms that are closely matched to the characteristics of the channel and which are therefore more tolerant of channel impairments. In so doing, reliable communication is established over the channel. In the selection of good waveforms for digital communication over a noisy channel, the design is influenced solely by the channel characteristics. However, once the appropriate set of waveforms for transmission over the channel has been selected, the source information can be encoded into the channel waveforms, and the efficient transmission of information from the source to the user is thereby ensured. In summary, the use of digital communications provides the capability for information transmission that is both *efficient* and *reliable.*

From this discussion, it is apparent that the use of digital communications requires a considerable amount of electronic circuitry, but nowadays electronics are inexpensive, due to the ever-increasing availability of very-large-scale integrated (VLSI) circuits in the form of silicon chips. Thus although cost considerations used to be a factor in selecting analog communications over digital communications in the past, that is no longer the case.

Despite the trend toward the ever-increasing use of digital communications, a strong case can be made for the study of analog communications for two important reasons:

1. As long as we hear and see analog communications around us via radio and television, we need to understand how these communications systems work. Moreover, the study of analog modulation motivates other digital modulation schemes.
2. Analog devices and circuits have a natural affinity for operating at very high speeds and they consume very little power compared to their digital counterparts. Accordingly, the implementation of very high-speed or very low-power communication systems dictates the use of an analog approach.

Shannon's Information Capacity Theorem

The goal of a communication system designer is to configure a system that transports a message signal from a source of interest across a noisy channel to a user at the other end of the channel with the following objective:

> The message signal is delivered to the user both efficiently and reliably, subject to certain design constraints: allowable transmit power, available channel bandwidth, and affordable cost of building the system.

In the case of a digital communication system, reliability is commonly expressed in terms of *bit error rate* (BER) or *probability of bit error* measured at the receiver output. Clearly, the smaller the BER, the more reliable the communication system is. A question that comes to mind in this context is whether it is possible to design a communication system that operates with zero BER even through the channel is noisy. In an *ideal* setting, the answer to this question is an emphatic *yes*. The answer is embodied in one of Shannon's celebrated theorems,[10] which is called the information capacity theorem.

Let B denote the channel bandwidth, and let SNR denote the received signal-to-noise ratio. The *information capacity theorem* states that ideally these two parameters are related as

$$C = B \log_2(1 + \mathrm{SNR}) \text{ b/s} \tag{1}$$

where C is the information capacity of the channel. The *information capacity* is defined as the maximum rate at which information can be transmitted across the channel without error; it is measured in *bits per second* (b/s). For a prescribed channel bandwidth B and received SNR, the information capacity theorem tells us that a message signal can be transmitted through the system without error even when the channel is noisy, provided that the *actual signaling rate* R in bits per second, at which data are transmitted through the channel, is less than the information capacity C.

steady regardless of the distance from the transmitter. Thus, the signal quality is continuously worsening along the length of the channel. Amplification of the received signal to make up for the attenuation is to no avail because the noise will be amplified by the same proportion, and the quality remains, at best, unchanged.* These are the key challenges that we must face in designing modern communication systems.

1.2 ANALOG AND DIGITAL MESSAGES

Messages are digital or analog. Digital messages are ordered combinations of finite symbols or codewords. For example, printed English consists of 26 letters, 10 numbers, a space, and several punctuation marks. Thus, a text document written in English is a digital message constructed from the ASCII keyboard of 128 symbols. Human speech is also a digital message, because it is made up from a finite vocabulary in a language.† Music notes are also digital, even though the music sound itself is analog. Similarly, a Morse-coded telegraph message is a digital message constructed from a set of only **two** symbols—dash and dot. It is therefore a **binary** message, implying only two symbols. A digital message constructed with M symbols is called an M**-ary** message.

Analog messages, on the other hand, are characterized by data whose values vary over a continuous range and are defined for a continuous range of time. For example, the temperature or the atmospheric pressure of a certain location over time can vary over a continuous range and can assume an (uncountable) infinite number of possible values. A piece of music recorded by a pianist is also an analog signal. Similarly, a particular speech waveform has amplitudes that vary over a continuous range. Over a given time interval, an infinite number of possible different speech waveforms exist, in contrast to only a finite number of possible digital messages.

1.2.1 Noise Immunity of Digital Signals

It is no secret to even a casual observer that every time one looks at the latest electronic communication products, newer and better "digital technology" is replacing the old analog technology. Within the past decade, cellular phones have completed their transformation from the first-generation analog AMPS to the current second-generation (e.g., GSM, CDMA) and third-generation (e.g., WCDMA) digital offspring. More visibly in every household, digital video technology (DVD) has made the analog VHS cassette systems almost obsolete. Digital television continues the digital assault on analog video technology by driving out the last analog holdout of color television. There is every reason to ask: Why are digital technologies better? The answer has to do with both economics and quality. The case for economics is made by noting the ease of adopting versatile, powerful, and inexpensive high-speed digital microprocessors. But more importantly at the quality level, one prominent feature of digital communications is the enhanced immunity of digital signals to noise and interferences.

Digital messages are transmitted as a finite set of electrical waveforms. In other words, a digital message is generated from a finite alphabet, while each character in the alphabet can be represented by one waveform or a sequential combination of such waveforms. For example, in sending messages via Morse code, a dash can be transmitted by an electrical pulse of amplitude $A/2$ and a dot can be transmitted by a pulse of negative amplitude

* Actually, amplification may further deteriorate the signal because of additional amplifier noise.

† Here we imply the information contained in the speech rather than its details such as the pronunciation of words and varying inflections, pitch, and emphasis. The speech signal from a microphone contains all these details and is therefore an analog signal, and its information content is more than a thousand times greater than the information accessible from the written text of the same speech.

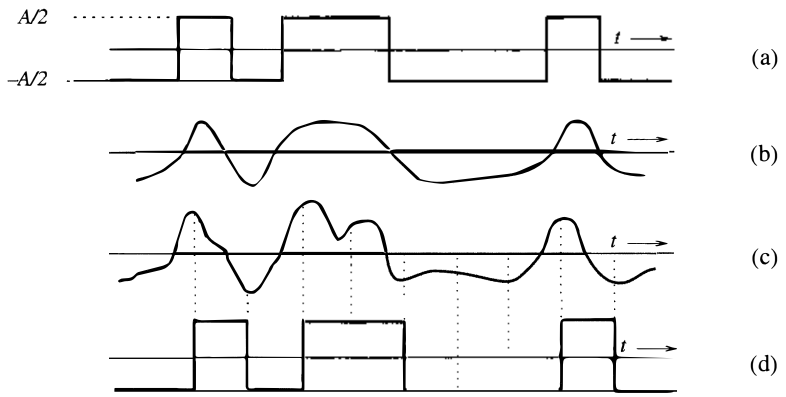


Figure 1.3
(a) Transmitted signal.
(b) Received distorted signal (without noise).
(c) Received distorted signal (with noise).
(d) Regenerated signal (delayed).

$-A/2$ (Fig 1.3a). In an M-ary case, M distinct electrical pulses (or waveforms) are used; each of the M pulses represents one of the M possible symbols. Once transmitted, the receiver must extract the message from a distorted and noisy signal at the channel output. Message extraction is often easier from digital signals than from analog signals because the digital decision must belong to the finite-sized alphabet. Consider a binary case: two symbols are encoded as rectangular pulses of amplitudes $A/2$ and $-A/2$. The only decision at the receiver is to select between two possible pulses received; the fine details of the pulse shape are not an issue. A finite alphabet leads to noise and interference immunity. The receiver's decision can be made with reasonable certainty even if the pulses have suffered modest distortion and noise (Fig. 1.3). The digital message in Fig. 1.3a is distorted by the channel, as shown in Fig. 1.3b. Yet, if the distortion is not too large, we can recover the data without error because we need make only a simple binary decision: Is the received pulse positive or negative? Figure 1.3c shows the same data with channel distortion and noise. Here again, the data can be recovered correctly as long as the distortion and the noise are within limits. In contrast, the waveform shape itself in an analog message carries the needed information, and even a slight distortion or interference in the waveform will show up in the received signal. Clearly, a digital communication system is more rugged than an analog communication system in the sense that it can better withstand noise and distortion (as long as they are within a limit).

1.2.2 Viability of Distortionless Regenerative Repeaters

One main reason for the superior quality of digital systems over analog ones is the viability of **regenerative** repeaters and network nodes in the former. Repeater stations are placed along the communication path of a digital system at distances short enough to ensure that noise and distortion remain within a limit. This allows pulse detection with high accuracy. At each repeater station, or network node, the incoming pulses are detected such that new, "clean" pulses are retransmitted to the next repeater station or node. This process prevents the accumulation of noise and distortion along the path by cleaning the pulses at regular repeater intervals. We can thus transmit messages over longer distances with greater accuracy. There has been widespread application of distortionless regeneration by repeaters in long-haul communication systems and by nodes in a large (possibly heterogeneous) network.

For analog systems, signals and noise within the same bandwidth cannot be separated. Repeaters in analog systems are basically filters plus amplifiers and are not "regenerative."

CHAPTER 3

PULSE MODULATION

This chapter, representing the transition from analog to digital communications, covers the following topics:

- *Sampling, which is basic to all forms of pulse modulation.*
- *Pulse-amplitude modulation, which is the simplest form of pulse modulation.*
- *Quantization, which, when combined with sampling, permits the representation of an analog signal in discrete form in both amplitude and time.*
- *Pulse-code modulation, which is the standard method for the transmission of an analog message signal by digital means.*
- *Time-division multiplexing, which provides for the time sharing of a common channel by a plurality of users by means of pulse modulation.*
- *Digital multiplexers, which combine many slow bit streams into a single faster stream.*
- *Other forms of digital pulse modulation, namely, delta modulation and differential pulse-code modulation.*
- *Linear prediction, which is basic to the encoding of analog message signals at reduced bit rates as in differential pulse-code modulation.*
- *Adaptive forms of differential pulse-code modulation and delta modulation.*
- *The MPEG-1/audio coding standard, which is a transparent, perceptually lossless compression system.*

3.1 Introduction

In *continuous-wave (CW) modulation*, which we studied in Chapter 2, some parameter of a sinusoidal carrier wave is varied continuously in accordance with the message signal. This is in direct contrast to pulse modulation, which we study in the present chapter. In *pulse modulation*, some parameter of a pulse train is varied in accordance with the message signal. We may distinguish two families of pulse modulation: *analog pulse modulation* and *digital pulse modulation*. In analog pulse modulation, a periodic pulse train is used as the carrier wave, and some characteristic feature of each pulse (e.g., amplitude, duration, or position) is varied in a continuous manner in accordance with the corresponding *sample* value of the message signal. Thus in analog pulse modulation, information is transmitted basically in analog form, but the transmission takes place at discrete times. In digital pulse modulation, on the other hand, the message signal is represented in a form that is discrete in both time and amplitude, thereby permitting its transmission in digital form as a sequence of *coded pulses*; this form of signal transmission has *no* CW counterpart.

The use of coded pulses for the transmission of analog information-bearing signals represents a basic ingredient in the application of digital communications. This chapter may therefore be viewed as a transition from analog to digital communications in our study of the principles of communication systems. We begin the discussion by describing the sampling process, which is basic to all pulse modulation systems, whether they are analog or digital.

3.2 *Sampling Process*

The *sampling process* is usually described in the time domain. As such, it is an operation that is basic to digital signal processing and digital communications. Through use of the sampling process, an analog signal is converted into a corresponding sequence of samples that are usually spaced uniformly in time. Clearly, for such a procedure to have practical utility, it is necessary that we choose the sampling rate properly, so that the sequence of samples uniquely defines the original analog signal. This is the essence of the sampling theorem, which is derived in what follows.

Consider an arbitrary signal $g(t)$ of finite energy, which is specified for all time. A segment of the signal $g(t)$ is shown in Figure 3.1*a*. Suppose that we sample the signal $g(t)$ instantaneously and at a uniform rate, once every T_s seconds. Consequently, we obtain an infinite sequence of samples spaced T_s seconds apart and denoted by $\{g(nT_s)\}$, where n takes on all possible integer values. We refer to T_s as the *sampling period*, and to its reciprocal $f_s = 1/T_s$ as the *sampling rate*. This ideal form of sampling is called *instantaneous sampling.*

Let $g_\delta(t)$ denote the signal obtained by individually weighting the elements of a periodic sequence of delta functions spaced T_s seconds apart by the sequence of numbers $\{g(nT_s)\}$, as shown by (see Figure 3.1*b*)

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\, \delta(t - nT_s) \tag{3.1}$$

We refer to $g_\delta(t)$ as the *ideal sampled signal.* The term $\delta(t - nT_s)$ represents a delta function positioned at time $t = nT_s$. From the definition of the delta function, we recall that such an idealized function has unit area; see Appendix 2. We may therefore view the multiplying factor $g(nT_s)$ in Equation (3.1) as a "mass" assigned to the delta function $\delta(t - nT_s)$. A delta function weighted in this manner is closely approximated by a rectangular pulse of

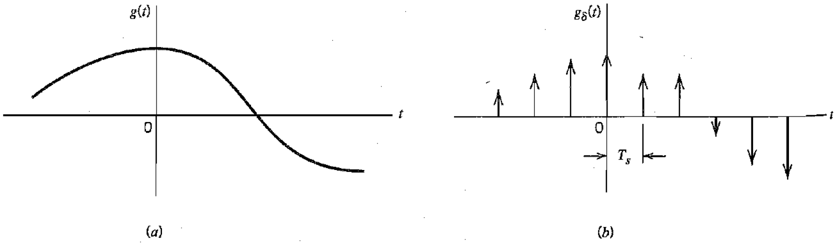


FIGURE 3.1 The sampling process. (*a*) Analog signal. (*b*) Instantaneously sampled version of the analog signal.

duration Δt and amplitude $g(nT_s)/\Delta t$; *the smaller we make* Δt *the better will be the approximation.*

Using the table of Fourier-transform pairs, we may write (see the last item of Table A6.3)

$$g_\delta(t) \rightleftharpoons f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \tag{3.2}$$

where $G(f)$ is the Fourier transform of the original signal $g(t)$, and f_s is the sampling rate. Equation (3.2) states that *the process of uniformly sampling a continuous-time signal of finite energy results in a periodic spectrum with a period equal to the sampling rate.*

Another useful expression for the Fourier transform of the ideal sampled signal $g_\delta(t)$ may be obtained by taking the Fourier transform of both sides of Equation (3.1) and noting that the Fourier transform of the delta function $\delta(t - nT_s)$ is equal to $\exp(-j2\pi nfT_s)$. Let $G_\delta(f)$ denote the Fourier transform of $g_\delta(t)$. We may therefore write

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\exp(-j2\pi nfT_s) \tag{3.3}$$

This relation is called the *discrete-time Fourier transform.* It may be viewed as a complex Fourier series representation of the periodic frequency function $G_\delta(f)$, with the sequence of samples $\{g(nT_s)\}$ defining the coefficients of the expansion.

The relations, as derived here, apply to any continuous-time signal $g(t)$ of finite energy and infinite duration. Suppose, however, that the signal $g(t)$ is *strictly band-limited,* with no frequency components higher than W Hertz. That is, the Fourier transform $G(f)$ of the signal $g(t)$ has the property that $G(f)$ is zero for $|f| \geq W$, as illustrated in Figure 3.2*a*; the shape of the spectrum shown in this figure is intended for the purpose of illustration only. Suppose also that we choose the sampling period $T_s = 1/2W$. Then the corresponding spectrum $G_\delta(f)$ of the sampled signal $g_\delta(t)$ is as shown in Figure 3.2*b*. Putting $T_s = 1/2W$ in Equation (3.3) yields

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right) \tag{3.4}$$

From Equation (3.2), we readily see that the Fourier transform of $g_\delta(t)$ may also be expressed as

$$G_\delta(f) = f_sG(f) + f_s \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} G(f - mf_s) \tag{3.5}$$

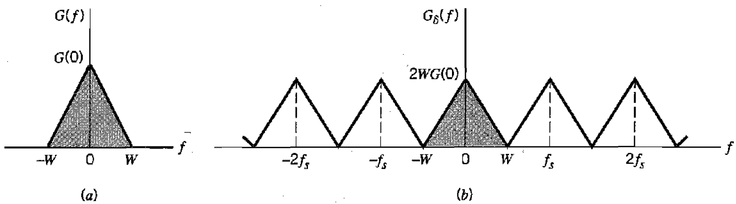


FIGURE 3.2 (*a*) Spectrum of a strictly band-limited signal $g(t)$. (*b*) Spectrum of the sampled version of $g(t)$ for a sampling period $T_s = 1/2W$.

Hence, under the following two conditions:

1. $G(f) = 0$ for $|f| \geq W$
2. $f_s = 2W$

we find from Equation (3.5) that

$$G(f) = \frac{1}{2W} G_\delta(f), \qquad -W < f < W \tag{3.6}$$

Substituting Equation (3.4) into (3.6), we may also write

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi n f}{W}\right), \qquad -W < f < W \tag{3.7}$$

Therefore, if the sample values $g(n/2W)$ of a signal $g(t)$ are specified for all n, then the Fourier transform $G(f)$ of the signal is uniquely determined by using the discrete-time Fourier transform of Equation (3.7). Because $g(t)$ is related to $G(f)$ by the inverse Fourier transform, it follows that the signal $g(t)$ is itself uniquely determined by the sample values $g(n/2W)$ for $-\infty < n < \infty$. In other words, the sequence $\{g(n/2W)\}$ has all the information contained in $g(t)$.

Consider next the problem of reconstructing the signal $g(t)$ from the sequence of sample values $\{g(n/2W)\}$. Substituting Equation (3.7) in the formula for the inverse Fourier transform defining $g(t)$ in terms of $G(f)$, we get

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t)\, df \\ &= \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi n f}{W}\right) \exp(j2\pi f t)\, df \end{aligned}$$

Interchanging the order of summation and integration:

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{1}{2W} \int_{-W}^{W} \exp\left[j2\pi f\left(t - \frac{n}{2W}\right)\right] df \tag{3.8}$$

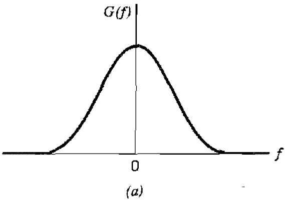
The integral term in Equation (3.8) is readily evaluated, yielding the final result

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{\sin(2\pi W t - n\pi)}{(2\pi W t - n\pi)} \\ &= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wt - n), \qquad -\infty < t < \infty \end{aligned} \tag{3.9}$$

Equation (3.9) provides an *interpolation formula* for reconstructing the original signal $g(t)$ from the sequence of sample values $\{g(n/2W)\}$, with the sinc function $\operatorname{sinc}(2Wt)$ playing the role of an *interpolation function*. Each sample is multiplied by a delayed version of the interpolation function, and all the resulting waveforms are added to obtain $g(t)$.

We may now state the *sampling theorem* for strictly band-limited signals of finite energy in two equivalent parts, which apply to the transmitter and receiver of a pulse-modulation system, respectively:

1. *A band-limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signal at instants of time separated by 1/2W seconds.*
2. *A band-limited signal of finite energy, which has no frequency components higher than W Hertz, may be completely recovered from a knowledge of its samples taken at the rate of 2W samples per second.*



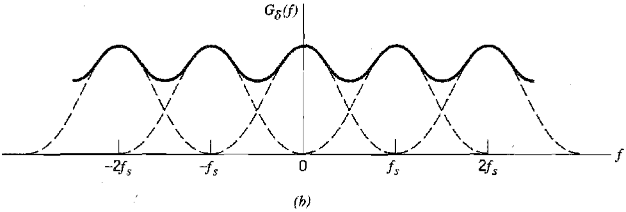


FIGURE 3.3 (*a*) Spectrum of a signal. (*b*) Spectrum of an undersampled version of the signal exhibiting the aliasing phenomenon.

The sampling rate of $2W$ samples per second, for a signal bandwidth of W Hertz, is called the *Nyquist rate*; its reciprocal $1/2W$ (measured in seconds) is called the *Nyquist interval.*

The derivation of the sampling theorem, as described herein, is based on the assumption that the signal $g(t)$ is strictly band limited. In practice, however, an information-bearing signal is *not* strictly band limited, with the result that some degree of undersampling is encountered. Consequently, some *aliasing* is produced by the sampling process. Aliasing refers to the phenomenon of a high-frequency component in the spectrum of the signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version, as illustrated in Figure 3.3. The aliased spectrum, shown by the solid curve in Figure 3.3*b*, pertains to an "undersampled" version of the message signal represented by the spectrum of Figure 3.3*a*.

To combat the effects of aliasing in practice, we may use two corrective measures, as described here:

1. Prior to sampling, a low-pass *anti-aliasing filter* is used to attenuate those high-frequency components of the signal that are not essential to the information being conveyed by the signal.
2. The filtered signal is sampled at a rate slightly higher than the Nyquist rate.

The use of a sampling rate higher than the Nyquist rate also has the beneficial effect of easing the design of the *reconstruction filter* used to recover the original signal from its sampled version. Consider the example of a message signal that has been anti-alias (low-pass) filtered, resulting in the spectrum shown in Figure 3.4*a*. The corresponding spectrum of the instantaneously sampled version of the signal is shown in Figure 3.4*b*, assuming a sampling rate higher than the Nyquist rate. According to Figure 3.4*b*, we readily see that the design of the reconstruction filter may be specified as follows (see Figure 3.4*c*):

- The reconstruction filter is low-pass with a passband extending from $-W$ to W, which is itself determined by the anti-aliasing filter.
- The filter has a transition band extending (for positive frequencies) from W to $f_s - W$, where f_s is the sampling rate.

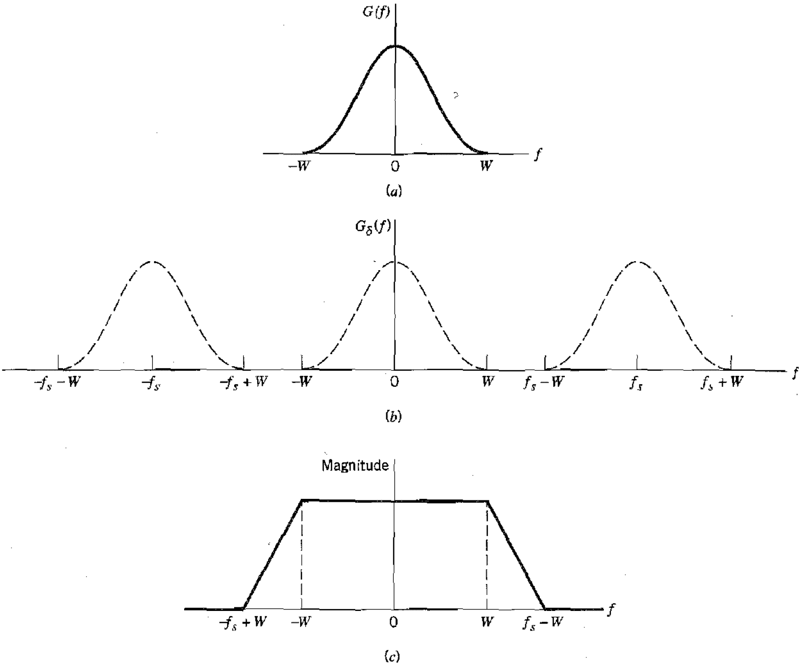


FIGURE 3.4 (*a*) Anti-alias filtered spectrum of an information-bearing signal. (*b*) Spectrum of instantaneously sampled version of the signal, assuming the use of a sampling rate greater than the Nyquist rate. (*c*) Magnitude response of reconstruction filter.

The fact that the reconstruction filter has a well-defined transition band means that it is physically realizable.

3.3 *Pulse-Amplitude Modulation*

Now that we understand the essence of the sampling process, we are ready to formally define pulse-amplitude modulation, which is the simplest and most basic form of analog pulse modulation. In *pulse-amplitude modulation (PAM), the amplitudes of regularly spaced pulses are varied in proportion to the corresponding sample values of a continuous message signal*; the pulses can be of a rectangular form or some other appropriate shape. Pulse-amplitude modulation as defined here is somewhat similar to natural sampling, where the message signal is multiplied by a periodic train of rectangular pulses. However, in natural sampling the top of each modulated rectangular pulse varies with the message signal, whereas in PAM it is maintained flat; natural sampling is explored further in Problem 3.2.

The waveform of a PAM signal is illustrated in Figure 3.5. The dashed curve in this figure depicts the waveform of a message signal $m(t)$, and the sequence of amplitude-

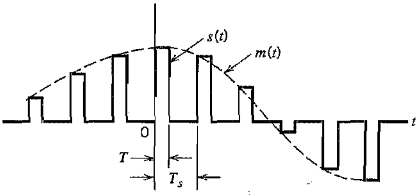


FIGURE 3.5 Flat-top samples, representing an analog signal.

modulated rectangular pulses shown as solid lines represents the corresponding PAM signal $s(t)$. There are two operations involved in the generation of the PAM signal:

1. *Instantaneous sampling* of the message signal $m(t)$ every T_s seconds, where the sampling rate $f_s = 1/T_s$ is chosen in accordance with the sampling theorem.
2. *Lengthening* the duration of each sample so obtained to some constant value T.

In digital circuit technology, these two operations are jointly referred to as "sample and hold." One important reason for intentionally lengthening the duration of each sample is to avoid the use of an excessive channel bandwidth, since bandwidth is inversely proportional to pulse duration. However, care has to be exercised in how long we make the sample duration T, as the following analysis reveals.

Let $s(t)$ denote the sequence of flat-top pulses generated in the manner described in Figure 3.5. We may express the PAM signal as

$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) \tag{3.10}$$

where T_s is the *sampling period* and $m(nT_s)$ is the sample value of $m(t)$ obtained at time $t = nT_s$. The $h(t)$ is a standard rectangular pulse of unit amplitude and duration T, defined as follows (see Figure 3.6*a*):

$$h(t) = \begin{cases} 1, & 0 < t < T \\ \frac{1}{2}, & t = 0,\ t = T \\ 0, & \text{otherwise} \end{cases} \tag{3.11}$$

By definition, the instantaneously sampled version of $m(t)$ is given by

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\,\delta(t - nT_s) \tag{3.12}$$

where $\delta(t - nT_s)$ is a time-shifted delta function. Therefore, convolving $m_\delta(t)$ with the pulse $h(t)$, we get

$$\begin{aligned} m_\delta(t) \star h(t) &= \int_{-\infty}^{\infty} m_\delta(\tau)h(t - \tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m(nT_s)\,\delta(\tau - nT_s)h(t - \tau)\,d\tau \\ &= \sum_{n=-\infty}^{\infty} m(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s)h(t - \tau)\,d\tau \end{aligned} \tag{3.13}$$

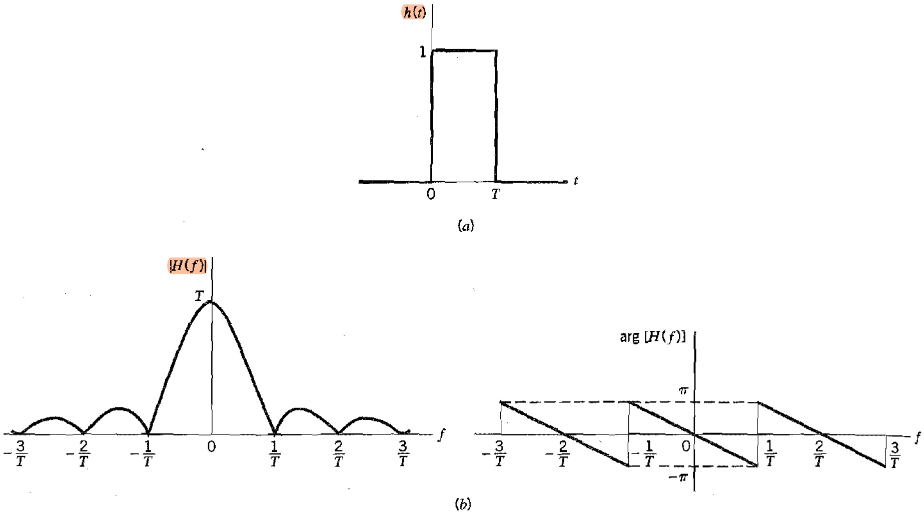


FIGURE 3.6 (*a*) Rectangular pulse $h(t)$. (*b*) Spectrum $H(f)$, made up of the magnitude $|H(f)|$, and phase $\arg[H(f)]$.

Using the sifting property of the delta function (see Appendix 2), we thus obtain

$$m_\delta(t) \star h(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) \tag{3.14}$$

From Equations (3.10) and (3.14) it follows that the PAM signal $s(t)$ is mathematically equivalent to the convolution of $m_\delta(t)$, the instantaneously sampled version of $m(t)$, and the pulse $h(t)$, as shown by

$$s(t) = m_\delta(t) \star h(t) \tag{3.15}$$

Taking the Fourier transform of both sides of Equation (3.15) and recognizing that the convolution of two time functions is transformed into the multiplication of their respective Fourier transforms, we get

$$S(f) = M_\delta(f)H(f) \tag{3.16}$$

where $S(f) = F[s(t)]$, $M_\delta(f) = F[m_\delta(t)]$, and $H(f) = F[h(t)]$. Adapting Equation (3.2) to the problem at hand, we note that the Fourier transform $M_\delta(f)$ is related to the Fourier transform $M(f)$ of the original message signal $m(t)$ as follows:

$$M_\delta(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) \tag{3.17}$$

where f_s is the sampling rate. Therefore, substitution of Equation (3.17) into (3.16) yields

$$S(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s)H(f) \tag{3.18}$$

Given a PAM signal $s(t)$ whose Fourier transform $S(f)$ is as defined in Equation (3.18), how do we recover the original message signal $m(t)$? As a first step in this recon-

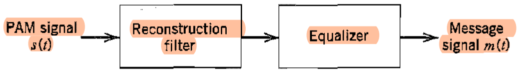


FIGURE 3.7 System for recovering message signal $m(t)$ from PAM signal $s(t)$.

struction, we may pass $s(t)$ through a low-pass filter whose frequency response is defined in Figure 3.4*c*; here it is assumed that the message is limited to bandwidth W and the sampling rate f_s is larger than the Nyquist rate $2W$. Then, from Equation (3.18) we find that the spectrum of the resulting filter output is equal to $M(f)H(f)$. This output is equivalent to passing the original message signal $m(t)$ through another low-pass filter of frequency response $H(f)$.

From Equation (3.11) we note that the Fourier transform of the rectangular pulse $h(t)$ is given by

$$H(f) = T\,\mathrm{sinc}(fT)\exp(-j\pi fT) \tag{3.19}$$

which is plotted in Figure 3.6*b*. We see therefore that by using flat-top samples to generate a PAM signal, we have introduced *amplitude distortion* as well as a *delay* of $T/2$. This effect is rather similar to the variation in transmission with frequency that is caused by the finite size of the scanning aperture in television. Accordingly, the distortion caused by the use of pulse-amplitude modulation to transmit an analog information-bearing signal is referred to as the *aperture effect.*

This distortion may be corrected by connecting an *equalizer* in cascade with the low-pass reconstruction filter, as shown in Figure 3.7. The equalizer has the effect of decreasing the in-band loss of the reconstruction filter as the frequency increases in such a manner as to compensate for the aperture effect. Ideally, the magnitude response of the equalizer is given by

$$\frac{1}{|H(f)|} = \frac{1}{T\,\mathrm{sinc}(fT)} = \frac{\pi f}{\sin(\pi fT)} \tag{3.20}$$

The amount of equalization needed in practice is usually small. Indeed, for a duty cycle $T/T_s \leq 0.1$, the amplitude distortion is less than 0.5 percent, in which case the need for equalization may be omitted altogether.

The transmission of a PAM signal imposes rather stringent requirements on the magnitude and phase responses of the channel, because of the relatively short duration of the transmitted pulses. Furthermore, the noise performance of a PAM system can never be better than baseband-signal transmission. Accordingly, we find that for transmission over long distances, PAM would be used only as a means of message processing for time-division multiplexing, from which conversion to some other form of pulse modulation is subsequently made; time-division multiplexing is discussed in Section 3.9.

3.4 *Other Forms of Pulse Modulation*

In a pulse modulation system we may use the increased bandwidth consumed by the pulses to improve the noise performance of the system. This can be achieved by representing the sample values of the message signals by some property of the pulse other than amplitude:

- *Pulse-duration modulation* (PDM), also referred to as *pulse-width modulation*, where samples of the message signal are used to vary the duration of the individual pulses in the carrier.

- *Pulse-position modulation* (PPM), where the position of a pulse relative to its unmodulated time of occurrence is varied in accordance with the message signal.

These two other forms of pulse modulation are illustrated in Figure 3.8 for the case of a sinusoidal modulating wave.

In PDM, long pulses expend considerable power while bearing no additional information. If this unused power is subtracted from PDM so that only time transitions are preserved, we obtain PPM. Accordingly, PPM is a more efficient form of pulse modulation than PDM.

Since in a PPM system the transmitted information is contained in the relative positions of the modulated pulses, the presence of additive noise affects the performance of such a system by falsifying the time at which the modulated pulses are judged to occur. Immunity to noise can be established by making the pulse build up so rapidly that the time interval during which noise can exert any perturbation is very short. Indeed, additive noise would have no effect on the pulse positions if the received pulses were perfectly rectangular, because the presence of noise introduces only vertical perturbations. However, the reception of perfectly rectangular pulses would require an infinite channel bandwidth, which is of course impractical. Thus with a finite channel bandwidth in practice, we find that the received pulses have a finite rise time, so the performance of the PPM receiver is affected by noise, which is to be expected.

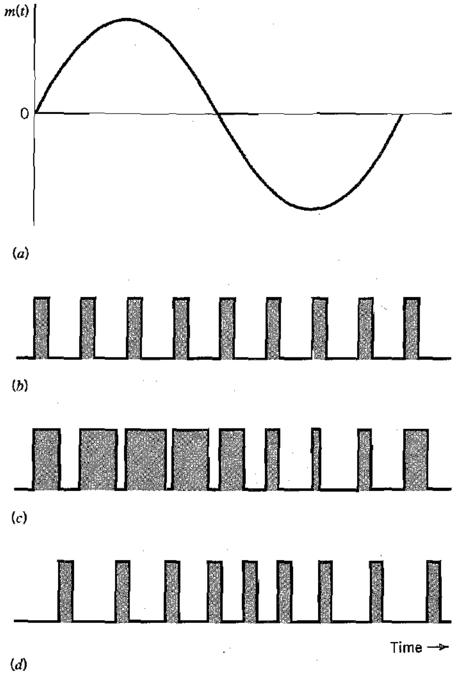


FIGURE 3.8 Illustrating two different forms of pulse-time modulation for the case of a sinusoidal modulating wave. (*a*) Modulating wave. (*b*) Pulse carrier. (*c*) PDM wave. (*d*) PPM wave.

As in a CW modulation system, the noise performance of a PPM system may be described in terms of the output signal-to-noise ratio (SNR). Also, to find the noise improvement produced by PPM over baseband transmission of a message signal, we may use the figure of merit defined as the output signal-to-noise ratio of the PPM system divided by the channel signal-to-noise ratio; see Section 2.10. Assuming that the average power of the channel noise is small compared to the peak pulse power, the figure of merit of the PPM system is proportional to the square of the transmission bandwidth B_T normalized with respect to the message bandwidth W. When, however, the input signal-to-noise ratio drops below a critical value, the system suffers a loss of the wanted message signal at the receiver output. That is, a PPM system suffers from a threshold effect of its own.

on increasing B noise Inc.

3.5 *Bandwidth–Noise Trade-Off*

In the context of noise performance, a PPM system is the optimum form of analog pulse modulation. The noise analysis of a PPM system reveals that pulse-position modulation (PPM) and frequency modulation (FM) systems exhibit a similar noise performance, as summarized here.[1]

1. Both systems have a figure of merit proportional to the square of the transmission bandwidth normalized with respect to the message bandwidth.
2. Both systems exhibit a threshold effect as the signal-to-noise ratio is reduced.

The practical implication of point 1 is that, in terms of a trade-off of increased transmission bandwidth for improved noise performance, the best that we can do with continuous-wave (CW) modulation and analog pulse modulation systems is to follow a *square law*. A question that arises at this point in the discussion is: Can we produce a trade-off better than a square law? The answer is an emphatic yes, and *digital pulse modulation* is the way to do it. The use of such a method is a radical departure from CW modulation.

Specifically, in a basic form of digital pulse modulation known as *pulse-code modulation* (PCM),[2] a message signal is represented in discrete form in both time and amplitude. This form of signal representation permits the transmission of the message signal as a sequence of *coded binary pulses*. Given such a sequence, the effect of channel noise at the receiver output can be reduced to a negligible level simply by making the average power of the transmitted binary PCM wave large enough compared to the average power of the noise.

Two fundamental processes are involved in the generation of a binary PCM wave: *sampling* and *quantization*. The sampling process takes care of the discrete-time representation of the message signal; for its proper application, we have to follow the sampling theorem described in Section 3.2. The quantization process takes care of the discrete-amplitude representation of the message signal; quantization is a new process, the details of which are described in the next section. For now it suffices to say that the combined use of sampling and quantization permits the transmission of a message signal in coded form. This, in turn, makes it possible to realize an *exponential law* for the bandwidth-noise trade-off, which is also demonstrated in the next section.

3.6 *Quantization Process*[3]

A continuous signal, such as voice, has a continuous range of amplitudes and therefore its samples have a continuous amplitude range. In other words, within the finite amplitude

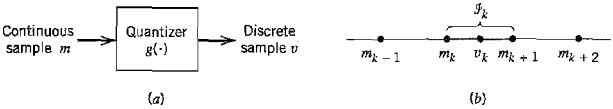


FIGURE 3.9 Description of a memoryless quantizer.

range of the signal, we find an infinite number of amplitude levels. It is not necessary in fact to transmit the exact amplitudes of the samples. Any human sense (the ear or the eye), as ultimate receiver, can detect only finite intensity differences. This means that the original continuous signal may be *approximated* by a signal constructed of discrete amplitudes selected on a minimum error basis from an available set. The existence of a finite number of discrete amplitude levels is a basic condition of pulse-code modulation. Clearly, if we assign the discrete amplitude levels with sufficiently close spacing, we may make the approximated signal practically indistinguishable from the original continuous signal.

Amplitude *quantization* is defined as *the process of transforming the sample amplitude $m(nT_s)$ of a message signal $m(t)$ at time $t = nT_s$ into a discrete amplitude $v(nT_s)$ taken from a finite set of possible amplitudes.* We assume that the quantization process is *memoryless* and *instantaneous*, which means that the transformation at time $t = nT_s$ is not affected by earlier or later samples of the message signal. This simple form of scalar quantization, though not optimum, is commonly used in practice.

When dealing with a memoryless quantizer, we may simplify the notation by dropping the time index. We may thus use the symbol m in place of $m(nT_s)$, as indicated in the block diagram of a quantizer shown in Figure 3.9*a*. Then, as shown in Figure. 3.9*b*, the signal amplitude m is specified by the index k if it lies inside the *partition cell*

$$\mathcal{J}_k: \{m_k < m \leq m_{k+1}\}, \qquad k = 1, 2, \ldots, L \tag{3.21}$$

where L is the total number of amplitude levels used in the quantizer. The discrete amplitudes m_k, $k = 1, 2, \ldots, L$, at the quantizer input are called *decision levels* or *decision thresholds*. At the quantizer output, the index k is transformed into an amplitude v_k that represents all amplitudes of the cell $\mathcal{J}_k$; the discrete amplitudes v_k, $k = 1, 2, \ldots, L$, are called *representation levels* or *reconstruction levels*, and the spacing between two adjacent representation levels is called a *quantum* or *step-size*. Thus, the quantizer output v equals v_k if the input signal sample m belongs to the interval $\mathcal{J}_k$. The mapping (see Figure 3.9*a*)

$$v = g(m) \tag{3.22}$$

is the *quantizer characteristic*, which is a staircase function by definition.

Quantizers can be of a *uniform* or *nonuniform* type. In a uniform quantizer, the representation levels are uniformly spaced; otherwise, the quantizer is nonuniform. In this section, we consider only uniform quantizers; nonuniform quantizers are considered in Section 3.7. The quantizer characteristic can also be of *midtread* or *midrise type*. Figure 3.10*a* shows the input–output characteristic of a uniform quantizer of the midtread type, which is so called because the origin lies in the middle of a tread of the staircaselike graph. Figure 3.10*b* shows the corresponding input–output characteristic of a uniform quantizer of the midrise type, in which the origin lies in the middle of a rising part of the staircaselike graph. Note that both the midtread and midrise types of uniform quantizers illustrated in Figure 3.10 are *symmetric* about the origin.

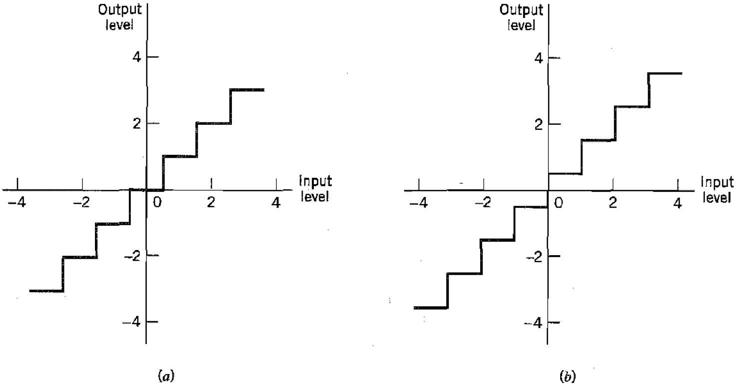


FIGURE 3.10 Two types of quantization: (*a*) midtread and (*b*) midrise.

QUANTIZATION NOISE

The use of quantization introduces an error defined as the difference between the input signal m and the output signal v. The error is called *quantization noise*. Figure 3.11 illustrates a typical variation of the quantization noise as a function of time, assuming the use of a uniform quantizer of the midtread type.

Let the quantizer input m be the sample value of a zero-mean random variable M. (If the input has a nonzero mean, we can always remove it by subtracting the mean from the input and then adding it back after quantization.) A quantizer $g(\cdot)$ maps the input

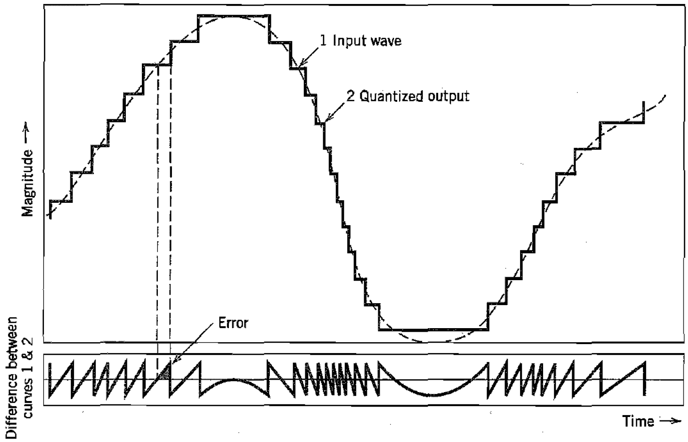


FIGURE 3.11 Illustration of the quantization process. (Adapted from Bennett, 1948, with permission of AT&T.)

random variable M of continuous amplitude into a discrete random variable V; their respective sample values m and v are related by Equation (3.22). Let the quantization error be denoted by the random variable Q of sample value q. We may thus write

$$q = m - v \tag{3.23}$$

or, correspondingly,

$$Q = M - V \tag{3.24}$$

With the input M having zero mean, and the quantizer assumed to be symmetric as in Figure 3.10, it follows that the quantizer output V and therefore the quantization error Q, will also have zero mean. Thus for a partial statistical characterization of the quantizer in terms of output signal-to-(quantization) noise ratio, we need only find the mean-square value of the quantization error Q.

Consider then an input m of continuous amplitude in the range $(-m_{\max}, m_{\max})$. Assuming a uniform quantizer of the midrise type illustrated in Figure 3.10*b*, we find that the step-size of the quantizer is given by

$$\Delta = \frac{2m_{\max}}{L} \tag{3.25}$$

where L is the total number of representation levels. For a uniform quantizer, the quantization error Q will have its sample values bounded by $-\Delta/2 \leq q \leq \Delta/2$. If the step-size Δ is sufficiently small (i.e., the number of representation levels L is sufficiently large), it is reasonable to assume that the quantization error Q is a *uniformly distributed* random variable, and the interfering effect of the quantization noise on the quantizer input is similar to that of thermal noise. We may thus express the probability density function of the quantization error Q as follows:

$$f_Q(q) = \begin{cases} \dfrac{1}{\Delta}, & -\dfrac{\Delta}{2} < q \leq \dfrac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases} \tag{3.26}$$

For this to be true, however, we must ensure that the incoming signal does *not* overload the quantizer. Then, with the mean of the quantization error being zero, its variance σ_Q^2 is the same as the mean-square value:

$$\begin{aligned} \sigma_Q^2 &= E[Q^2] \\ &= \int_{-\Delta/2}^{\Delta/2} q^2 f_Q(q)\, dq \end{aligned} \tag{3.27}$$

Substituting Equation (3.26) into (3.27), we get

$$\begin{aligned} \sigma_Q^2 &= \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} q^2\, dq \\ &= \frac{\Delta^2}{12} \end{aligned} \tag{3.28}$$

Typically, the L-ary number k, denoting the kth representation level of the quantizer, is transmitted to the receiver in binary form. Let R denote the number of *bits per sample* used in the construction of the binary code. We may then write

$$L = 2^R \tag{3.29}$$

or, equivalently,

$$R = \log_2 L \tag{3.30}$$

Hence, substituting Equation (3.29) into (3.25), we get the step size

$$\Delta = \frac{2m_{\max}}{2^R} \tag{3.31}$$

Thus the use of Equation (3.31) in (3.28) yields

$$\sigma_Q^2 = \tfrac{1}{3} m_{\max}^2 2^{-2R} \tag{3.32}$$

Let P denote the average power of the message signal $m(t)$. We may then express the *output signal-to-noise ratio* of a uniform quantizer as

$$\begin{aligned}(\mathrm{SNR})_O &= \frac{P}{\sigma_Q^2} \\ &= \left(\frac{3P}{m_{\max}^2}\right) 2^{2R}\end{aligned} \tag{3.33}$$

Equation (3.33) shows that the output signal-to-noise ratio of the quantizer increases *exponentially* with increasing number of bits per sample, R. Recognizing that an increase in R requires a proportionate increase in the channel (transmission) bandwidth B_T, we thus see that the use of a binary code for the representation of a message signal (as in pulse-code modulation) provides a more efficient method than either frequency modulation (FM) or pulse-position modulation (PPM) for the trade-off of increased channel bandwidth for improved noise performance. In making this statement, we presume that the FM and PPM systems are limited by receiver noise, whereas the binary-coded modulation system is limited by quantization noise. We have more to say on the latter issue in Section 3.8.

▶ Example 3.1 Sinusoidal Modulating Signal

Consider the special case of a full-load sinusoidal modulating signal of amplitude A_m, which utilizes all the representation levels provided. The average signal power is (assuming a load of 1 ohm)

$$P = \frac{A_m^2}{2}$$

The total range of the quantizer input is $2A_m$, because the modulating signal swings between $-A_m$ and A_m. We may therefore set $m_{\max} = A_m$, in which case the use of Equation (3.32) yields the average power (variance) of the quantization noise as

$$\sigma_Q^2 = \tfrac{1}{3} A_m^2 2^{-2R}$$

Thus the output signal-to-noise ratio of a uniform quantizer, for a full-load test tone, is

$$(\mathrm{SNR})_O = \frac{A_m^2/2}{A_m^2 2^{-2R}/3} = \frac{3}{2}(2^{2R}) \tag{3.34}$$

Expressing the signal-to-noise ratio in decibels, we get

$$10 \log_{10}(\mathrm{SNR})_O = 1.8 + 6R \tag{3.35}$$

TABLE 3.1 *Signal-to-(quantization) noise ratio for varying number of representation levels for sinusoidal modulation*

Number of Representation Levels, L	*Number of Bits per Sample,* R	*Signal-to-Noise Ratio (dB)*
32	5	31.8
64	6	37.8
128	7	43.8
256	8	49.8

For various values of L and R, the corresponding values of signal-to-noise ratio are as given in Table 3.1. From Table 3.1 we can make a quick estimate of the number of bits per sample required for a desired output signal-to-noise ratio, assuming sinusoidal modulation. ◀

Thus far in this section we have focused on how to characterize memoryless scalar quantizers and assess their performance. In so doing, however, we avoided the optimum design of quantizers, that is, the issue of selecting the representation levels and partition cells so as to minimize the average quantization power for a prescribed number of representation levels. Unfortunately, this optimization problem does not lend itself to a closed-form solution because of the highly *nonlinear* nature of the quantization process. Rather, we have effective algorithms for finding the optimum design in an iterative manner. A well-known algorithm that deserves to be mentioned in this context is the Lloyd-Max quantizer, which is discussed next.

■ CONDITIONS FOR OPTIMALITY OF SCALAR QUANTIZERS

In designing a scalar quantizer the challenge is how to select the representation levels and surrounding partition cells so as to minimize the average quantization power for a fixed number of representation levels.

To state the problem in mathematical terms, consider a message signal $m(t)$ drawn from a stationary process $M(t)$. Let $-A \leq m \leq A$ denote the dynamic range of $m(t)$, which is partitioned into a set of L cells, as depicted in Figure 3.12. The boundaries of the partition cells are defined by a set of real numbers $m_1, m_2, \ldots, m_{L+1}$ that satisfy the following three conditions:

$$
\begin{aligned}
m_1 &= -A \\
m_{L+1} &= A \\
m_k &\leq m_{k+1} \text{ for } k = 1, 2, \ldots, L
\end{aligned}
$$

The kth partition cell is defined by

$$\mathcal{J}_k\colon m_k < m \leq m_{k+1} \text{ for } k = 1, 2, \ldots, L \tag{3.36}$$

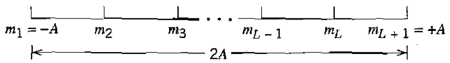


FIGURE 3.12 Illustrating the partitioning of the dynamic range $-A \leq m \leq A$ of a message signal $m(t)$ into a set of L cells.

Let the representation levels (i.e., quantization values) be denoted by v_k, $k = 1, 2, \ldots, L$. Then, assuming that $d(m, v_k)$ denotes a *distortion measure* for using v_k to represent all those values of the input m that lie inside the partition cell $\mathcal{J}_k$, the goal is to find the two sets, $\{v_k\}_{k=1}^{L}$ and $\{\mathcal{J}_k\}_{k=1}^{L}$, that minimize the *average distortion*

$$D = \sum_{k=1}^{L} \int_{m \in \mathcal{J}_k} d(m, v_k) f_M(m)\, dm \tag{3.37}$$

where $f_M(m)$ is the probability density function of the random variable M with sample value m.

A commonly used distortion measure is

$$d(m, v_k) = (m - v_k)^2 \tag{3.38}$$

in which case we speak of the *mean-square distortion*. In any event, the optimization problem stated herein is nonlinear, defying an explicit, closed-form solution. To get around this difficulty, we resort to an algorithmic approach for solving the problem in an iterative manner.

Structurally speaking, the quantizer consists of two components with interrelated design parameters:

- An encoder characterized by the set of partition cells $\{\mathcal{J}_k\}_{k=1}^{L}$; it is located in the transmitter.
- A decoder characterized by the set of representation levels $\{v_k\}_{k=1}^{L}$; it is located in the receiver.

Accordingly, we may identify two critically important conditions that provide the mathematical basis for all algorithmic solutions to the optimum quantization problem. One condition assumes that we are given a decoder and the problem is to find the optimum encoder in the transmitter. The other condition assumes that we are given an encoder and the problem is to find the optimum decoder in the receiver. Henceforth, these two conditions are referred to as condition I and condition II, respectively.

Condition I. Optimality of the Encoder for a Given Decoder

The availability of a decoder means that we have a certain *codebook* in mind. Let the codebook be defined by

$$\mathcal{C}: \{v_k\}_{k=1}^{L} \tag{3.39}$$

Given the codebook $\mathcal{C}$, the problem is to find the set of partition cells $\{\mathcal{J}_k\}_{k=1}^{L}$ that minimizes the average distortion D. That is, we wish to find the encoder defined by the nonlinear mapping

$$g(m) = v_k, \qquad k = 1, 2, \ldots, L \tag{3.40}$$

such that we have

$$D = \int_{-A}^{A} d(m, g(m)) f_M(m)\, dM \geq \sum_{k=1}^{L} \int_{m \in \mathcal{J}_k} [\min_{v_k \in \mathcal{C}} d(m, v_k)] f_M(m)\, dm \tag{3.41}$$

For the lower bound specified in Equation (3.41) to be attained, we require that the nonlinear mapping of Equation (3.40) be satisfied only if the condition

$$d(m, v_k) \leq d(m, v_j) \qquad \text{holds for all } j \neq k \tag{3.42}$$

The necessary condition described in Equation (3.42) for optimality of the encoder for a specified codebook $\mathscr{C}$ is recognized as the *nearest neighbor condition.* In words, the nearest neighbor condition requires that the partition cell $\mathscr{I}_k$ should embody all those values of the input m that are closer to v_k than any other element of the codebook C. This optimality condition is indeed intuitively satisfying.

Condition II. Optimality of the Decoder for a Given Encoder

Consider next the reverse situation to that described under condition I, which may be stated as follows: Optimize the codebook $\mathscr{C} = \{v_k\}_{k=1}^{L}$ for the decoder, given that the set of partition cells $\{\mathscr{I}_k\}_{k=1}^{L}$ characterizing the encoder is fixed. The criterion for optimization is the average (mean-square) distortion:

$$D = \sum_{k=1}^{L} \int_{m \in \mathscr{I}_k} (m - v_k)^2 f_M(m)\, dm \tag{3.43}$$

The probability density function $f_M(m)$ is clearly independent of the codebook $\mathscr{C}$. Hence, differentiating D with respect to the representation level v_k, we readily obtain

$$\frac{\partial D}{\partial v_k} = -2 \sum_{k=1}^{L} \int_{m \in \mathscr{I}_k} (m - v_k) f_M(m)\, dm \tag{3.44}$$

Setting $\partial D/\partial v_k$ equal to zero and then solving for v_k, we obtain the optimum value

$$v_{k,\text{opt}} = \frac{\displaystyle\int_{m \in \mathscr{I}_k} m f_M(m)\, dm}{\displaystyle\int_{m \in \mathscr{I}_k} f_M(m)\, dm} \tag{3.45}$$

The denominator in Equation (3.45) is just the probability, p_k, that the random variable M with sample value m lies in the partition cell $\mathscr{I}_k$, as shown by

$$\begin{aligned} p_k &= P(m_k < M \le m_k + 1) \\ &= \int_{m \in \mathscr{I}_k} f_M(m)\, dm \end{aligned} \tag{3.46}$$

Accordingly, we may interpret the optimality condition of Equation (3.45) as choosing the representation level v_k to equal the *conditional mean* of the random variable M, given that M lies in the partition cell $\mathscr{I}_k$. We can thus formally state the condition for optimality of the decoder for a given encoder as follows:

$$v_{k,\text{opt}} = E[M \mid m_k < M \le m_{k+1}] \tag{3.47}$$

where E is the expectation operator. Equation (3.47) is also intuitively satisfying.

Note that the nearest neighbor condition (condition I) for optimality of the encoder for a given decoder was proved for a generic average distortion. However, the conditional mean requirement (condition II) for optimality of the decoder for a given encoder was proved for the special case of a mean-square distortion. In any event, these two conditions are necessary for optimality of a scalar quantizer. Basically, the algorithm for designing the quantizer consists of alternately optimizing the encoder in accordance with condition I, then optimizing the decoder in accordance with condition II, and continuing in this manner until the average distortion D reaches a minimum. An optimum quantizer designed in this manner is called a *Lloyd-Max quantizer*.[4]

3.7 *Pulse-Code Modulation*

With the sampling and quantization processes at our disposal, we are now ready to describe pulse-code modulation, which, as mentioned previously, is the most basic form of digital pulse modulation. In *pulse-code modulation (PCM), a message signal is represented by a sequence of coded pulses, which is accomplished by representing the signal in discrete form in both time and amplitude.* The basic operations performed in the transmitter of a PCM system are *sampling, quantizing,* and *encoding*, as shown in Figure 3.13*a*; the low-pass filter prior to sampling is included to prevent aliasing of the message signal. The quantizing and encoding operations are usually performed in the same circuit, which is called an *analog-to-digital converter*. The basic operations in the receiver are *regeneration* of impaired signals, *decoding*, and *reconstruction* of the train of quantized samples, as shown in Figure 3.13*c*. Regeneration also occurs at intermediate points along the transmission path as necessary, as indicated in Figure 3.13*b*. When time-division multiplexing is used, it becomes necessary to synchronize the receiver to the transmitter for the overall system to operate satisfactorily, as discussed in Section 3.9. In what follows, we describe the various operations that constitute a basic PCM system.

Sampling

The incoming message signal is sampled with a train of narrow rectangular pulses so as to closely approximate the instantaneous sampling process. To ensure perfect reconstruction of the message signal at the receiver, the sampling rate must be greater than twice the highest frequency component W of the message signal in accordance with the sampling theorem. In practice, a low-pass anti-aliasing filter is used at the front end of the sampler to exclude frequencies greater than W before sampling. Thus the application of sampling

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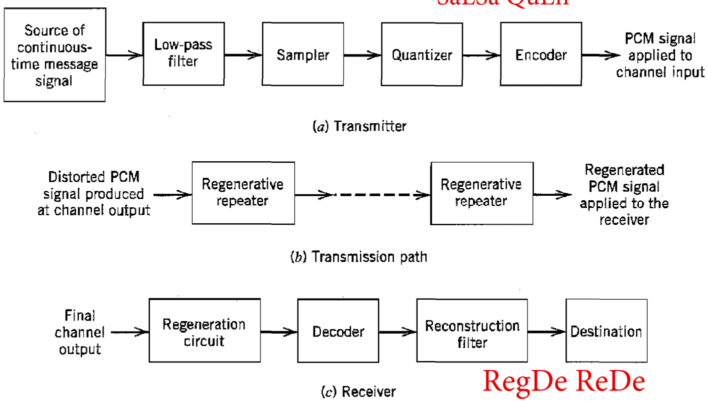

RegDe ReDe

Figure 3.13 The basic elements of a PCM system.

permits the reduction of the continuously varying message signal (of some finite duration) to a limited number of discrete values per second.

■ QUANTIZATION

The sampled version of the message signal is then quantized, thereby providing a new representation of the signal that is discrete in both time and amplitude. The quantization process may follow a uniform law as described in Section 3.6. In telephonic communication, however, it is preferable to use a variable separation between the representation levels. For example, the range of voltages covered by voice signals, from the peaks of loud talk to the weak passages of weak talk, is on the order of 1000 to 1. By using a *nonuniform quantizer* with the feature that the step-size increases as the separation from the origin of the input–output amplitude characteristic is increased, the large end steps of the quantizer can take care of possible excursions of the voice signal into the large amplitude ranges that occur relatively infrequently. In other words, the weak passages, which need more protection, are favored at the expense of the loud passages. In this way, a nearly uniform percentage precision is achieved throughout the greater part of the amplitude range of the input signal, with the result that fewer steps are needed than would be the case if a uniform quantizer were used.

The use of a nonuniform quantizer is equivalent to passing the baseband signal through a *compressor* and then applying the compressed signal to a uniform quantizer. A particular form of compression law that is used in practice is the so-called *μ-law*,[5] which is defined by

$$|v| = \frac{\log(1 + \mu|m|)}{\log(1 + \mu)} \tag{3.48}$$

where m and v are the normalized input and output voltages, and μ is a positive constant. In Figure 3.14*a*, we have plotted the μ-law for three different values of μ. The case of uniform quantization corresponds to $\mu = 0$. For a given value of μ, the reciprocal slope

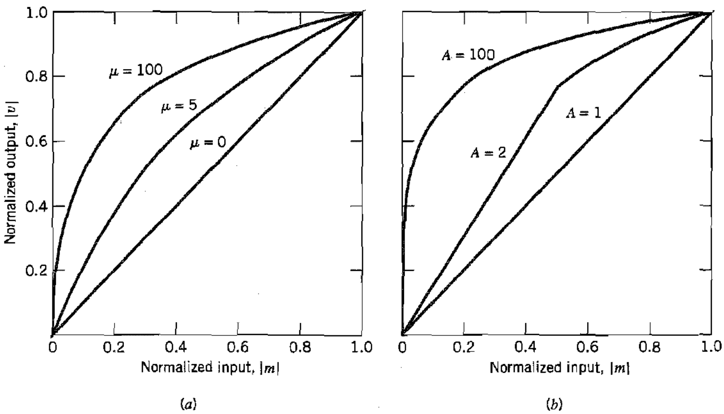

FIGURE 3.14 Compression laws. (*a*) μ-law. (*b*) A-law.

of the compression curve, which defines the quantum steps, is given by the derivative of $|m|$ with respect to $|v|$; that is,

$$\frac{d|m|}{d|v|} = \frac{\log(1+\mu)}{\mu}(1 + \mu|m|) \tag{3.49}$$

We see therefore that the μ-law is neither strictly linear not strictly logarithmic, but it is approximately linear at low input levels corresponding to $\mu|m| \ll 1$, and approximately logarithmic at high input levels corresponding to $\mu|m| \gg 1$.

Another compression law that is used in practice is the so-called *A-law* defined by

$$|v| = \begin{cases} \dfrac{A|m|}{1 + \log A}, & 0 \le |m| \le \dfrac{1}{A} \\[2ex] \dfrac{1 + \log(A|m|)}{1 + \log A}, & \dfrac{1}{A} \le |m| \le 1 \end{cases} \tag{3.50}$$

which is plotted in Figure 3.14*b* for varying A. The case of uniform quantization corresponds to $A = 1$. The reciprocal slope of this second compression curve is given by the derivative of $|m|$ with respect to $|v|$, as shown by (depending on the value assigned to the normalized input $|m|$)

$$\frac{d|m|}{d|v|} = \begin{cases} \dfrac{1 + \log A}{A}, & 0 \le |m| \le \dfrac{1}{A} \\[2ex] (1 + A)|m|, & \dfrac{1}{A} \le |m| \le 1 \end{cases} \tag{3.51}$$

To restore the signal samples to their correct relative level, we must, of course, use a device in the receiver with a characteristic complementary to the compressor. Such a device is called an *expander*. Ideally, the compression and expansion laws are exactly inverse so that, except for the effect of quantization, the expander output is equal to the compressor input. The combination of a *comp*ressor and an exp*ander* is called a *compander*.

For both the μ-law and A-law, the dynamic range capability of the compander improves with increasing μ and A, respectively. The SNR for low-level signals increases at the expense of the SNR for high-level signals. To accommodate these two conflicting requirements (i.e., a reasonable SNR for both low- and high-level signals), a compromise is usually made in choosing the value of paramenter μ for the μ-law and parameter A for the A-law. The typical values used in practice are: $\mu = 255$ and $A = 87.6$.

It is also of interest to note that in actual PCM systems, the companding circuitry does not produce an exact replica of the nonlinear compression curves shown in Figure 3.14. Rather, it provides a *piecewise linear* approximation to the desired curve. By using a large enough number of linear segments, the approximation can approach the true compression curve very closely. This form of approximation is illustrated in Example 3.2.

Encoding

In combining the processes of sampling and quantization, the specification of a continuous message (baseband) signal becomes limited to a discrete set of values, but not in the form best suited to transmission over a telephone line or radio path. To exploit the advantages of sampling and quantizing for the purpose of making the transmitted signal more robust to noise, interference and other channel impairments, we require the use of an *encoding*

TABLE 3.2 ***Binary number system for R = 4 bits/sample***

Ordinal Number of Representation Level	*Level Number Expressed as Sum of Powers of 2*	*Binary Number*
0		0000
1	2^0	0001
2	2^1	0010
3	$2^1 + 2^0$	0011
4	2^2	0100
5	$2^2 + 2^0$	0101
6	$2^2 + 2^1$	0110
7	$2^2 + 2^1 + 2^0$	0111
8	2^3	1000
9	$2^3 + 2^0$	1001
10	$2^3 + 2^1$	1010
11	$2^3 + 2^1 + 2^0$	1011
12	$2^3 + 2^2$	1100
13	$2^3 + 2^2 + 2^0$	1101
14	$2^3 + 2^2 + 2^1$	1110
15	$2^3 + 2^2 + 2^1 + 2^0$	1111

process to translate the discrete set of sample values to a more appropriate form of signal. Any plan for representing each of this discrete set of values as a particular arrangement of discrete events is called a *code*. One of the discrete events in a code is called a *code element* or *symbol*. For example, the presence or absence of a pulse is a symbol. A particular arrangement of symbols used in a code to represent a single value of the discrete set is called a *code word* or *character*.

In a *binary code*, each symbol may be either of two distinct values or kinds, such as the presence or absence of a pulse. The two symbols of a binary code are customarily denoted as 0 and 1. In a *ternary code*, each symbol may be one of three distinct values or kinds, and so on for other codes. However, *the maximum advantage over the effects of noise in a transmission medium* is *obtained by using a binary code, because a binary symbol withstands a relatively high level of noise and is easy to regenerate*. Suppose that, in a binary code, each code word consists of R bits: *bit* is an acronym for *binary digit*; thus R denotes the number of *bits per sample*. Then, using such a code, we may represent a total of 2^R distinct numbers. For example, a sample quantized into one of 256 levels may be represented by an 8-bit code word.

There are several ways of establishing a one-to-one correspondence between representation levels and code words. A convenient method is to express the ordinal number of the representation level as a binary number. In the binary number system, each digit has a place-value that is a power of 2, as illustrated in Table 3.2 for the case of four bits per sample (i.e., $R = 4$).

Line Codes

Any of several line codes can be used for the electrical representation of a binary data stream. Figure 3.15 displays the waveforms of five important line codes for the example data stream 01101001. Figure 3.16 displays their individual power spectra (for

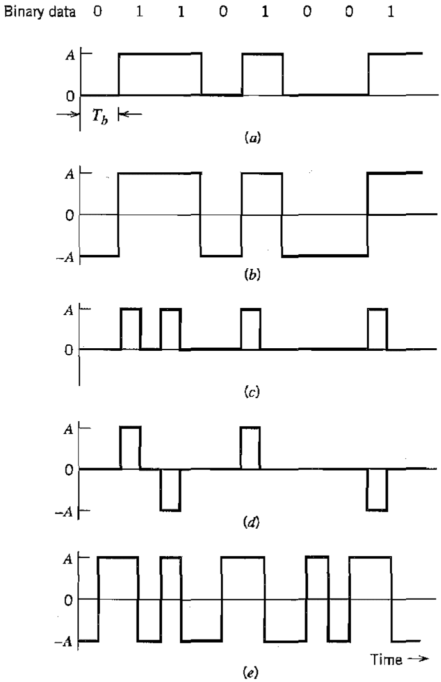

FIGURE 3.15 Line codes for the electrical representations of binary data. (*a*) Unipolar NRZ signaling. (*b*) Polar NRZ signaling. (*c*) Unipolar RZ signaling. (*d*) Bipolar RZ signaling. (*e*) Split-phase or Manchester code.

positive frequencies) for randomly generated binary data, assuming that (1) symbols 0 and 1 are equiprobable, (2) the average power is normalized to unity, and (3) the frequency f is normalized with respect to the bit rate $1/T_b$. (For the formulas used to plot the power spectra of Figure 3.16, the reader is referred to Problem 3.11.) The five line codes illustrated in Figure 3.15 are described here:

1. *Unipolar nonreturn-to-zero (NRZ) signaling*
In this line code, symbol 1 is represented by transmitting a pulse of amplitude A for the duration of the symbol, and symbol 0 is represented by switching off the pulse, as in Figure 3.15*a*. This line code is also referred to as *on-off signaling*. Disadvantages of on-off signaling are the waste of power due to the transmitted DC level and the fact that the power spectrum of the transmitted signal does not approach zero at zero frequency.

2. *Polar nonreturn-to-zero (NRZ) signaling*
In this second line code, symbols 1 and 0 are represented by transmitting pulses of amplitudes $+A$ and $-A$, respectively, as illustrated in Figure 3.15*b*. This line code is relatively easy to generate but its disadvantage is that the power spectrum of the signal is large near zero frequency.

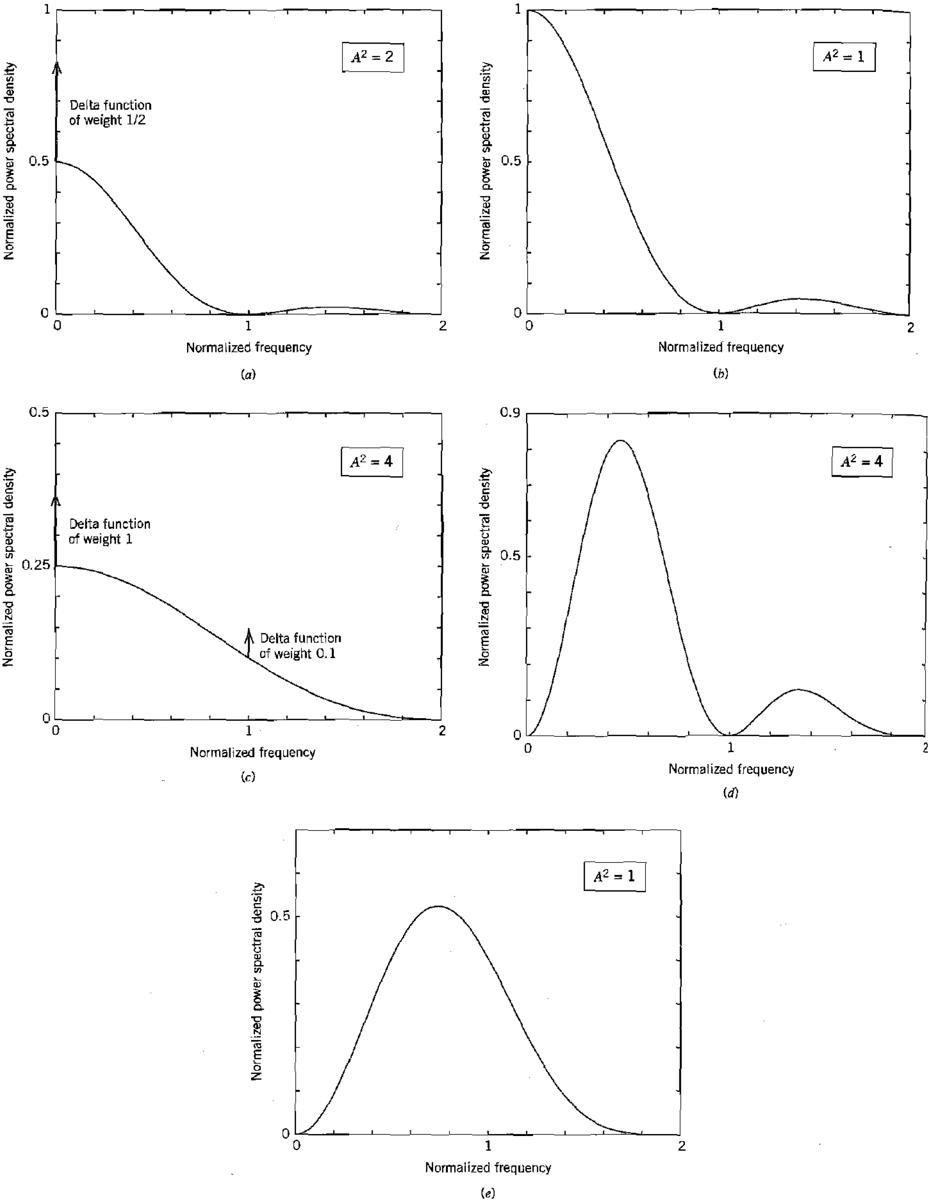

FIGURE 3.16 Power spectra of line codes: (*a*) Unipolar NRZ signal. (*b*) Polar NRZ signal. (*c*) Unipolar RZ signal. (*d*) Bipolar RZ signal. (*e*) Manchester-encoded signal. The frequency is normalized with respect to the bit rate $1/T_b$, and the average power is normalized to unity.

3. *Unipolar return-to-zero (RZ) signaling*
In this other line code, symbol 1 is represented by a rectangular pulse of amplitude A and half-symbol width, and symbol 0 is represented by transmitting *no* pulse, as illustrated in Figure 3.15*c*. An attractive feature of this line code is the presence of delta functions at $f = 0, \pm 1/T_b$ in the power spectrum of the transmitted signal, which can be used for bit-timing recovery at the receiver. However, its disadvantage is that it requires 3 dB more power than polar return-to-zero signaling for the same probability of symbol error; this issue is addressed in Chapter 4 under Problem 4.10.

4. *Bipolar return-to-zero (BRZ) signaling*
This line code uses three amplitude levels as indicated in Figure 3.15*d*. Specifically, positive and negative pulses of equal amplitude (i.e., $+A$ and $-A$) are used alternately for symbol 1, with each pulse having a half-symbol width; no pulse is always used for symbol 0. A useful property of the BRZ signaling is that the power spectrum of the transmitted signal has no DC component and relatively insignificant low-frequency components for the case when symbols 1 and 0 occur with equal probability. This line code is also called *alternate mark inversion* (AMI) signaling.

5. *Split-phase (Manchester code)*
In this method of signaling, illustrated in Figure 3.15*e*, symbol 1 is represented by a positive pulse of amplitude A followed by a negative pulse of amplitude $-A$, with both pulses being half-symbol wide. For symbol 0, the polarities of these two pulses are reversed. The Manchester code suppresses the DC component and has relatively insignificant low-frequency components, regardless of the signal statistics. This property is essential in some applications.

Differential Encoding

This method is used to encode information in terms of *signal transitions*. In particular, a transition is used to designate symbol 0 in the incoming binary data stream, while no transition is used to designate symbol 1, as illustrated in Figure 3.17. In Figure 3.17*b* we show the differentially encoded data stream for the example data specified in Figure 3.17*a*. The original binary data stream used here is the same as that used in Figure 3.15. The waveform of the differentially encoded data is shown in Figure 3.17*c*, assuming the use of unipolar nonreturn-to-zero signaling. From Figure 3.17 it is apparent that a differentially encoded signal may be inverted without affecting its interpretation. The original binary information is recovered simply by comparing the polarity of adjacent binary symbols to establish whether or not a transition has occurred. Note that differential encoding requires the use of a *reference bit* before initiating the encoding process. In Figure 3.17, symbol 1 is used as the reference bit.

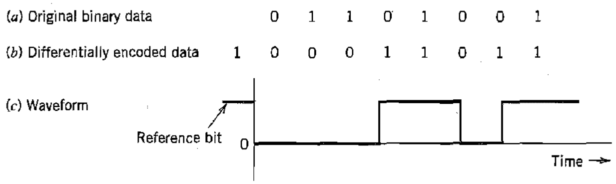

FIGURE 3.17 (*a*) Original binary data. (*b*) Differentially encoded data, assuming reference bit 1. (*c*) Waveform of differentially encoded data using unipolar NRZ signaling.

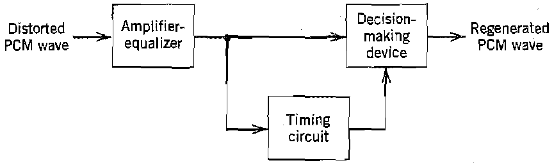

FIGURE 3.18 Block diagram of regenerative repeater.

REGENERATION

The most important feature of PCM systems lies in the ability to control the effects of distortion and noise produced by transmitting a PCM signal through a channel. This capability is accomplished by reconstructing the PCM signal by means of a chain of *regenerative repeaters* located at sufficiently close spacing along the transmission route. As illustrated in Figure 3.18, three basic functions are performed by a regenerative repeater: *equalization*, *timing*, and *decision making*. The equalizer shapes the received pulses so as to compensate for the effects of amplitude and phase distortions produced by the nonideal transmission characteristics of the channel. The timing circuitry provides a periodic pulse train, derived from the received pulses, for sampling the equalized pulses at the instants of time where the signal-to-noise ratio is a maximum. Each sample so extracted is compared to a predetermined *threshold* in the decision-making device. In each bit interval, a decision is then made whether the received symbol is a 1 or a 0 on the basis of whether the threshold is exceeded or not. If the threshold is exceeded, a clean new pulse representing symbol 1 is transmitted to the next repeater. Otherwise, another clean new pulse representing symbol 0 is transmitted. In this way, the accumulation of distortion and noise in a repeater span is completely removed, provided that the disturbance is not too large to cause an error in the decision-making process. Ideally, except for delay, the regenerated signal is exactly the same as the signal originally transmitted. In practice, however, the regenerated signal departs from the original signal for two main reasons:

1. The unavoidable presence of channel noise and interference causes the repeater to make wrong decisions occasionally, thereby introducing *bit errors* into the regenerated signal.
2. If the spacing between received pulses deviates from its assigned value, a *jitter* is introduced into the regenerated pulse position, thereby causing distortion.

DECODING

The first operation in the receiver is to regenerate (i.e., reshape and clean up) the received pulses one last time. These clean pulses are then regrouped into code words and decoded (i.e., mapped back) into a quantized PAM signal. The *decoding* process involves generating a pulse the amplitude of which is the linear sum of all the pulses in the code word, with each pulse being weighted by its place value ($2^0, 2^1, 2^2, \ldots, 2^{R-1}$) in the code, where R is the number of bits per sample.

FILTERING

The final operation in the receiver is to recover the message signal by passing the decoder output through a low-pass reconstruction filter whose cutoff frequency is equal to the message bandwidth W. Assuming that the transmission path is error free, the recovered signal includes no noise with the exception of the initial distortion introduced by the quantization process.

3.8 *Noise Considerations in PCM Systems*

The performance of a PCM system is influenced by two major sources of noise:

1. *Channel noise*, which is introduced anywhere between the transmitter output and the receiver input. Channel noise is always present, once the equipment is switched on.
2. *Quantization noise*, which is introduced in the transmitter and is carried all the way along to the receiver output. Unlike channel noise, quantization noise is *signal-dependent* in the sense that it disappears when the message signal is switched off.

Naturally, these two sources of noise appear simultaneously once the PCM system is in operation. However, the traditional practice is to consider them separately, so that we may develop insight into their individual effects on the system performance.

The main effect of channel noise is to introduce *bit errors* into the received signal. In the case of a binary PCM system, the presence of a bit error causes symbol 1 to be mistaken for symbol 0, or vice versa. Clearly, the more frequently bit errors occur, the more dissimilar the receiver output becomes compared to the original message signal. The fidelity of information transmission by PCM in the presence of channel noise may be measured in terms of the *average probability of symbol error*, which is defined as the probability that the reconstructed symbol at the receiver output differs from the transmitted binary symbol, on the average. The average probability of symbol error, also referred to as the *bit error rate* (BER), assumes that all the bits in the original binary wave are of equal importance. When, however, there is more interest in reconstructing the analog waveform of the original message signal, different symbol errors may need to be *weighted* differently; for example, an error in the most significant bit in a code word (representing a quantized sample of the message signal) is more harmful than an error in the least significant bit.

To optimize system performance in the presence of channel noise, we need to minimize the average probability of symbol error. For this evaluation, it is customary to model the channel noise as additive, white, and Gaussian. The effect of channel noise can be made practically negligible by ensuring the use of an adequate signal energy-to-noise density ratio through the provision of short-enough spacing between the regenerative repeaters in the PCM system. In such a situation, the performance of the PCM system is essentially limited by quantization noise acting alone.

From the discussion of quantization noise presented in Section 3.6, we recognize that quantization noise is essentially under the designer's control. It can be made negligibly small through the use of an adequate number of representation levels in the quantizer and the selection of a companding strategy matched to the characteristics of the type of message signal being transmitted. We thus find that the use of PCM offers the possibility of building a communication system that is *rugged* with respect to channel noise on a scale that is beyond the capability of any CW modulation or analog pulse modulation system.

Error Threshold

The underlying theory of bit error rate calculation in a PCM system is deferred until Chapter 4. For the present, it suffices to say that the average probability of symbol error in a binary encoded PCM receiver due to additive white Gaussian noise depends solely on

E_b/N_0, which is defined as *the ratio of the transmitted signal energy per bit, E_b, to the noise spectral density, N_0*. Note that the ratio E_b/N_0 is dimensionless even though the quantities E_b and N_0 have different physical meaning. In Table 3.3 we present a summary of this dependence for the case of a binary PCM system using polar nonreturn-to-zero signaling. The results presented in the last column of the table assume a bit rate of 10^5 b/s.

From Table 3.3 it is clear that there is an *error threshold* (at about 11 dB). For E_b/N_0 below the error threshold the receiver performance involves significant numbers of errors, and above it the effect of channel noise is practically negligible. In other words, provided that the ratio E_b/N_0 exceeds the error threshold, channel noise has virtually no effect on the receiver performance, which is precisely the goal of PCM. When, however, E_b/N_0 drops below the error threshold, there is a sharp increase in the rate at which errors occur in the receiver. Because decision errors result in the construction of incorrect code words, we find that when the errors are frequent, the reconstructed message at the receiver output bears little resemblance to the original message.

Comparing the figure of 11 dB for the error threshold in a PCM system using polar NRZ signaling with the 60–70 dB required for high-quality transmission of speech using amplitude modulation, we see that PCM requires much less power, even though the average noise power in the PCM system is increased by the R-fold increase in bandwidth, where R is the number of bits in a code word (i.e., bits per sample).

In most transmission systems, the effects of noise and distortion from the individual links accumulate. For a given quality of overall transmission, the longer the physical separation between the transmitter and the receiver, the more severe are the requirements on each link in the system. In a PCM system, however, because the signal can be regenerated as often as necessary, the effects of amplitude, phase, and nonlinear distortions in one link (if not too severe) have practically no effect on the regenerated input signal to the next link. We have also seen that the effect of channel noise can be made practically negligible by using a ratio E_b/N_0 above threshold. For all practical purposes, then, the transmission requirements for a PCM link are almost independent of the physical length of the communication channel.

Another important characteristic of a PCM system is its *ruggedness to interference*, caused by stray impulses or cross-talk. The combined presence of channel noise and interference causes the error threshold necessary for satisfactory operation of the PCM system to increase. If an adequate margin over the error threshold is provided in the first place, however, the system can withstand the presence of relatively large amounts of interference. In other words, a PCM system is *robust* to channel noise and interference.

TABLE 3.3 ***Influence of E_b/N_0 on the probability of error***

E_b/N_0	*Probability of Error P_e*	*For a Bit Rate of 10^5 b/s, This Is About One Error Every*
4.3 dB	10^{-2}	10^{-3} second
8.4	10^{-4}	10^{-1} second
10.6	10^{-6}	10 seconds
12.0	10^{-8}	20 minutes
13.0	10^{-10}	1 day
14.0	10^{-12}	3 months