CfNM

Root finding :-

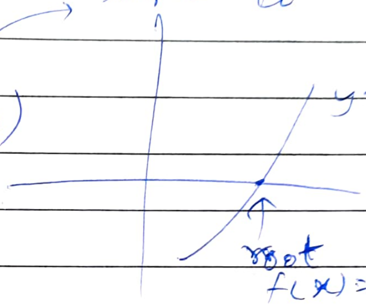
$$f(x) = 0$$

A number 'r' is called a root or zero of $f(x)$ if $f(r) = 0$.

simple eq^n

$$f(x) = 6x^2 - 7x + 2$$
$$= (2x-1)(3x-2)$$

$$x = \frac{1}{2}, \frac{2}{3}$$



Complex eq^n :-

eg $f(x) = 3.24x^8 - 2.42x^7 + 10.36x^6 \dots$

$g(x) = 2^{x^2} - \log x + 1$

→ We have to apply numerical methods to find roots

1) Bisection method
2) Newton method
3) Secant method
4) fixed point method

→ Iterative

CfNM

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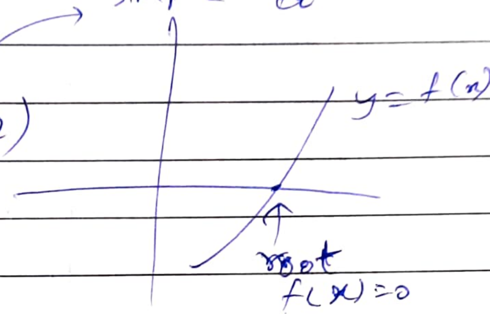

Complex eqn :-

eg $f(x) = 3.24x^8 - 2.42x^7 + 10.86x^6 \ldots$

$$g(x) = 2^{x^2} - \log x + 1$$

→ We have to apply numerical methods to find roots

1) Bisection method
2) Newton method
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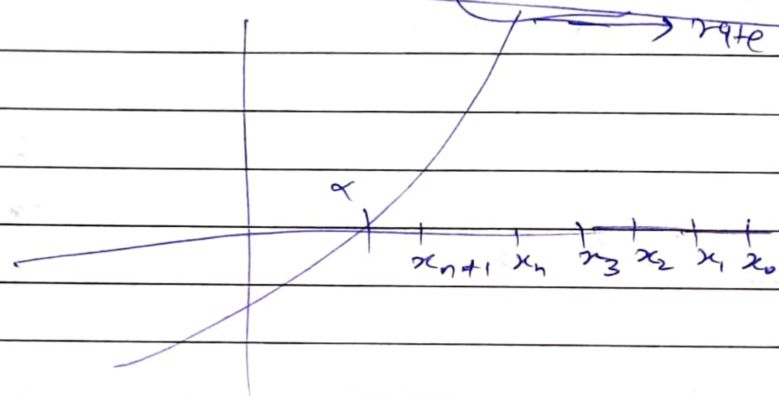
→ Iterative

1) Bisection method :-

Def^n A seq. of iterates $\{x_n \mid n \geq 0\}$ is said to converge with order $p \geq 1$ to a point α if

$$\boxed{|\alpha - x_{n+1}| \leq \underbrace{C}_{\text{rate of convergence}} |\alpha - x_n|^p}$$



(C<1) $p=1 \rightarrow$ Linear convergence

$p=2 \rightarrow$ Quadratic Convergence

$p=3 \rightarrow$ Cubic convergence

$$|\alpha - x_n| \leq C|\alpha - x_{n-1}| \leq C^2 |\alpha - x_{n-2}|$$

$$\ldots \leq C^n |\alpha - x_0|$$

$$0 \leq C < 1$$

$$\lim_{n \to \infty} C^n = 0 \longrightarrow \boxed{x_n \rightarrow \alpha}$$

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$f(x) \rightarrow$ real valued func

To find the root $f(x) = 0$

→ Intermediate value Thm :-

Let $f(x)$ is continuous on an interval $[a, b]$ & $f(a) \cdot f(b) < 0$ then $f(x) = 0$ has at least one or odd no. of roots real root on the interval $[a, b]$.

→ Suppose the interval $[a, b]$ is given error tolerance $\epsilon > 0$ given.

Algorithm :-

B1 → Define $c = \dfrac{a+b}{2}$

B2 → If $b - c \leq \epsilon$, then accept c as the root & stop.

B3 → If $\text{sign}[f(b)] \cdot \text{sign}[f(c)] < 0$ then $a = c$, otherwise set $b = c$.

B4 → Return to step 1.

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eg

$f(x) = x^6 - x - 1 = 0$

$\epsilon = 0.001$

$f(1) = -1$

$f(2) = 61$

1 — 2
interval

$f(1)\,f(2) < 0$

n	a	b	c	b − c	f(c)
1	1.000	2.000	1.5	0.5	8.89 (>0, b=c)
2	1.000	1.500	1.25	0.2500	1.5647
3	1.000	1.25	1.125	0.125	−0.0967
9	1.1328	1.1367	1.1348	0.002	0.0004
10	1.1328	1.1348	1.1338	0.00098	−0.0090

$|b-c| \le 0.001$

root

Error bounds :-

Let a_n, b_n, c_n denote the n^{th} computed values of a, b, c.

Then $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \qquad n \geq 1$$

Since α lies in either $[a_n, c_n]$ or $[c_n, b_n]$

$$|\alpha - c_n| \leq c_n - a_n = b_n - c_n = \frac{1}{2}(b_n - a_n)$$

$$\leq \frac{1}{2} \cdot \frac{1}{2}(b_{n-1} - a_{n-1})$$

$$|\alpha - c_n| \leq \frac{1}{2^n}(b-a)$$

$$\lim_{n \to \infty} \frac{1}{2^n} = 0$$

$\hookrightarrow c_n \to \alpha$

We want $|\alpha - c_n| \leq \epsilon$

if $\frac{1}{2^n}|b-a| \leq \epsilon$, taking logarithms,

$$\boxed{n \geq \frac{\log\left(\frac{b-a}{\epsilon}\right)}{\log 2}}$$

Fixed Point Iteration

$$f(x) = 0$$

→ A number α is a fixed point of a func $g(x)$ if $g(\alpha) = \alpha$

→ If $f(x) = 0$ is written as $x = g(x)$

$$f(x) = x - g(x)$$

$$f(\alpha) = \alpha - g(\alpha) = 0$$

→ If α is a fixed point of $g(x)$, then α is also a root of $f(x)$.

→ Exp $x^2 - a = 0$ for $a > 0$

(1) $x = x^2 + x - a$

(2) $x = a/x$

(3) $x = \frac{1}{2}\left(x + \frac{a}{x}\right)$

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→ method :-

→ Start with an initial guess x_0 of the root.

→ define by $x_{n+1} = g(x_n)$

→ If $\lim_{n \to \infty} x_n \Rightarrow \alpha$

$\lim_{n \to \infty} x_{n+1} = g(\lim_{n \to \infty} x_n)$

$\alpha = g(\alpha) \Rightarrow$ a fixed point

→ α a root of $f(x) = 0$

$a = 3$

$x_0 = 2$

$\alpha = \sqrt{3} = 1.732051$

Ex

n	(I) x_n	(II) x_n	(III) x_n
0	2.0	2.0	2.0
1	3	1.5	1.75
2	9.0	2.0	1.732143
3	87.0	1.5	1.732051

→ When the iteration $x_{n+1} = g(x_n)$ will converge. conditions for a fixed point of $g(x)$.

Lemma Let $g(x)$ be a continuous on the interval $a \le x \le b$ & assume that $a \le g(x) \le b$ i.e.

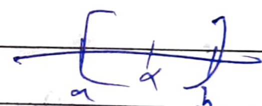
$$g([a,b]) \subset [a,b].$$

→ Then $x = g(x)$ has atleast one fixed point α in $[a,b]$.

Proof Define $f(x) = x - g(x)$

→ $f(x)$ is continuous $a \le x \le b$

$f(a) \le 0$ $f(b) \ge 0$

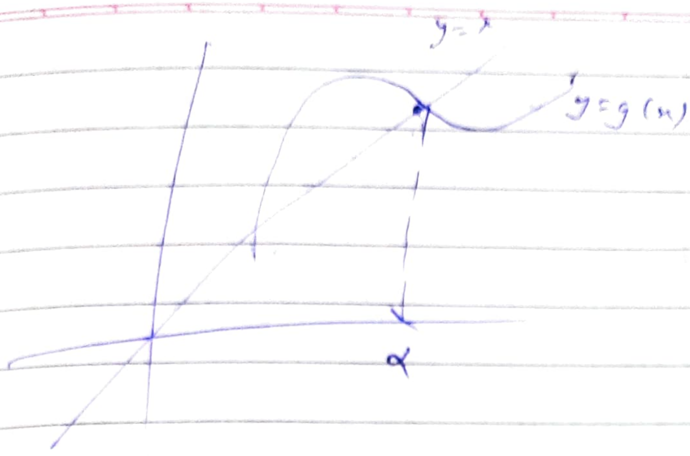


$f(a) \le 0$ $f(b) \ge 0$

→ By IVT, there exists at least one point $x \in (a,b)$ s.t. $f(x) = 0$,

$x = g(x)$.

→ Denote this x as α.



Lemma: Let $g(x)$ be continuous on $[a, b]$ & $g([a,b]) \subset [a,b]$. Further, assume there is a constant $0 < \lambda < 1$ with

$$|g(x) - g(y)| \leq \lambda |x - y| \quad \text{for all } x, y \in [a,b]$$

Then $x = g(x)$ has a unique soln α in $[a,b]$.

Also the iterates $x_n = g(x_{n-1})$ $n \geq 1$ will converge to α for any choice $x_0 \in [a,b]$ &

$$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda} |x_1 - x_0| \quad \text{(error bound)}$$

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Pf Suppose, $x = g(x)$ has two sol^n α & β in $[a, b]$

$$|\alpha - \beta| = |g(\alpha) - g(\beta)| \leq \lambda |\alpha - \beta|$$

$\underbrace{(1-\lambda)}_{+ve} \underbrace{|\alpha - \beta|}_{+ve} \leq 0$, so it has to be equal to 0.

$|\alpha - \beta| = 0$, $\underbrace{\boxed{\alpha = \beta}}_{\text{unique}}$

→ To prove $x_n \to \alpha$, if $x_n \in [a, b]$ then $x_{n+1} = g(x_n)$ also lies in $[a, b]$.

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)| \leq \lambda |\alpha - x_n|$$

By induction,

$$|\alpha - x_n| \leq \lambda |\alpha - x_{n-1}| \leq \lambda^2 |\alpha - x_{n-2}|$$

$$\dots \leq \lambda^n |\alpha - x_0|$$

$$|\alpha - x_n| \leq \lambda^n |\alpha - x_0|$$

as $0 < \lambda < 1$, $\lambda^n \to 0$, $\alpha - x_n \to 0$

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$$|\alpha - x_0| \le |\alpha - x_1| + |x_1 - x_0| \le \lambda |\alpha - x_0| + |x_1 - x_0|$$

$$(1-\lambda)|\alpha - x_0| \le |x_1 - x_0|$$

$$|\alpha - x_0| \le \frac{1}{1-\lambda} |x_1 - x_0|$$

$$\therefore \boxed{|\alpha - x_n| \le \lambda^n |\alpha - x_0| \le \frac{\lambda^n}{1-\lambda} |x_1 - x_0|}$$

$$\boxed{|\alpha - x_{n+1}| \le \frac{\lambda}{1-\lambda} |x_{n+1} - x_n|}$$

→ If $g(x)$ is differentiable on $[a,b]$, then

$g(x) - g(y) = g'(r)(x-y)$ where r is b/w x & y.

Define

(5374

Assume that $g(x)$ & $g'(x)$ are continuous for $a \le x \le b$ s.t.

$$g([a,b]) \subset [a,b]$$ Further assume that

$$\lambda = \max_{a \le x \le b} |g'(x)| < 1$$

Then (I) $x = g(x)$ has a unique sol^n $\alpha \in (a,b]$

(II) for any choice of $x_0 \in [a,b]$ with $x_{n+1} = g(x_n)$, $n \ge 0$

(III) $$|\alpha - x_n| \le \lambda^n |\alpha - x_0| \le \frac{\lambda^n}{1-\lambda}(x_1 - x_0)$$

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$

Then for x_n close to α, $\alpha - x_{n+1} \approx g'(\alpha)(\alpha - x_n)$

Pf: $g([a,b]) \subset [a,b] \Rightarrow \exists$ at least one sol^n in $[a,b]$. By MVT, for any 2 points w & $z \in [a,b]$.

$$g(w) - g(z) = g'(c)(w - z)$$

for some c b/w w & z.

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$$|g(w)-g(z)| = |g'(c)|\,|w-z| \le \lambda\,|w-z|$$

(I) Let α, β are two sol^n of $x = g(x)$.

$$\alpha = g(\alpha) \quad , \quad \beta = g(\beta)$$

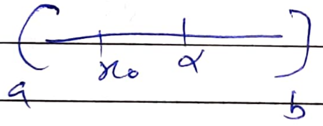
$$\alpha - \beta = g(\alpha) - g(\beta)$$

$$|\alpha-\beta| = |g(\alpha)-g(\beta)| \le \lambda\,|\alpha-\beta|$$

$$(1-\lambda)\,|\alpha-\beta| \le 0 \qquad \lambda < 1$$

$$|\alpha-\beta| = 0 \implies \alpha = \beta$$

(II)



$$x_n \in [a,b] \implies g(x_n) \in [a,b]$$

$$\implies x_{n+1} \in [a,b]$$

$$x_{n+1} = g(x_n) \quad , \quad \alpha = g(\alpha)$$

$$\alpha - x_{n+1} = g(\alpha) - g(x_n) = g'(c)\,(\alpha - x_n)$$

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$$|\alpha - x_{n+1}| = |g'(c)|\,|\alpha - x_n| \le \lambda\,|\alpha - x_n|$$

$\rightarrow$ linearly convergent

$$|\alpha - x_n| \le \lambda\,|\alpha - x_{n-1}| \le \lambda^2\,|\alpha - x_{n-2}|$$

$$\le \lambda^n\,|\alpha - x_0|$$

$x_n \rightarrow \alpha$ & $\lim_{n\to\infty} \lambda^n = 0$

(iii) $$|\alpha - x_0| \le |\alpha - x_1| + |x_1 - x_0|$$

$$\le \lambda\,|\alpha - x_0| + |x_1 - x_0|$$

$$(1-\lambda)\,|\alpha - x_0| \le |x_1 - x_0|$$

$$|\alpha - x_0| \le \frac{1}{1-\lambda}\,|x_1 - x_0|$$

$$|\alpha - x_n| \le \frac{\lambda^n}{1-\lambda}\,|x_1 - x_0|$$

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n\to\infty} g'(c_n)$$ c_n is b/w α & x_n.

$$= g'(\alpha)$$

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$\Rightarrow \quad \alpha - x_{n+1} \approx g'(\alpha)\,(\alpha - x_n)$

for x_n very close to α.

Corollary Assume that $g(x)$ & $g'(x)$ are combines for some intend $c < x < d$ with the fixed point α contained in the interval. moreover assume that $|g'(\alpha)| < 1$. Then there is an interval $[a, b]$ around α for which the hypothesis & hence also the conclusion of above th^m are true. And if to the contrary $|g'(\alpha)| > 1$ then the iteration method $x_{n+1} = g(x_n)$ will not converge to α. When $|g'(\alpha)| = 1$, no conclusion can be drawn.

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Exp: $x^2 - 5 = 0$ $(\alpha = \sqrt{5})$

1) $x = 5 + x - x^2$ $\rightarrow |g'(\alpha)| > 3$

2) $x = 5/x$ $\rightarrow g'(\alpha) = 1$

3) $x = 1 + x - \frac{x^2}{5}$ $\rightarrow |g'(\alpha)| < 1$ $\rightarrow$ converge

4) $x = \frac{1}{2}\left(x + 5/x\right)$ $\rightarrow |g'(\alpha)| < 1$ $\rightarrow$ converge

CNM (CS374)

→ **Interpolation Theory:-**

Thm Given $n+1$ distinct points $x_0, x_1, \ldots, x_n$ and $n+1$ ~~ordinates~~ ordinates $y_0, y_1, \ldots, y_n$ there is a polynomial $p(x)$ of degree $\leq n$ that interpolates y_i at x_i, $i = 0, 1, 2, \ldots, n$. This polynomial $p(x)$ is unique among the set of all polynomial of degree at most n.

→ Consider a special interpolation problem in which $y_i = 1$ & $y_j = 0$ for $j \neq i$, for some i, $0 \leq i \leq n$.

→ we construct a poly of deg $\leq n$ with n zeros x_j, $j \neq i$

$$p(x) = c\,(x-x_0)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)$$

for some constant c.

→ $p(x_i) = 1$

$$1 = c\,(x_i-x_0)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)$$

$$c = [(x_i-x_0)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)]^{-1}$$

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Substitute value of c,

$$l_i(x) = p(x) = \frac{(x-x_0) \cdots (x-x_{i-1})(x-x_{i+1}) \cdots (x-x_n)}{(x_i-x_0) \cdots (x_i-x_{i-1})(x_i-x_{i+1}) \cdots (x_i-x_n)}$$

$$= \prod_{j \neq i} \left(\frac{x-x_j}{x_i-x_j} \right) \quad j = 0, 1, \ldots, n$$

$$l_i(x_j) = 0 \quad \forall\, j \neq i$$

$$l_i(x_i) = 1 \quad j = i$$

Write:-

$$p(x) = y_0 l_0(x) + y_1 l_1(x) \cdots + y_n l_n(x)$$

This will satisfy the interpolation problem

$$\downarrow$$

$$p(x_i) = y_i \quad \forall\, i = 0, 1, \ldots, n$$

$\rightarrow$ $\deg(p(x)) \leq n$

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→ let $q(x)$ be another poly of deg $\le n$ that satisfies $q(x_i) = y_i$

let $r(x) = q(x) - p(x)$

$\deg(r(x)) \le n$

$r(x_i) = q(x_i) - p(x_i) = y_i - y_i = 0$

for $i = 0, 1, \dots n$

$\Rightarrow r(x)$ has $n+1$ zeros but $\deg(r(x)) \le n$ from fundamental thm. of algebra, this is not possible. So $r(x) = 0 \quad \forall x$.

$q(x) = p(x)$

→ Lagrang's interpolating polynomial :-

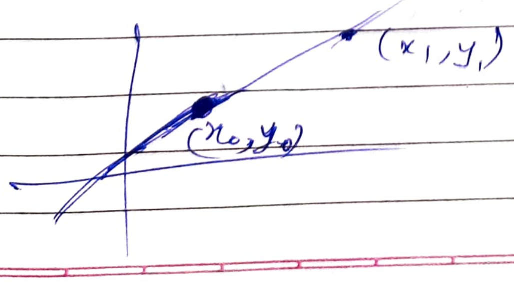
Linear interpolation :- (LI)

$(x_0, y_0) \qquad (x_1, y_1)$

$$P_1(x) = y_0 l_0(x) + y_1 l_1(x) = y_0 \frac{(x - x_1)}{(x_0 - x_1)} + y_1 \frac{(x - x_0)}{(x_1 - x_0)}$$

↓

represents a straight line

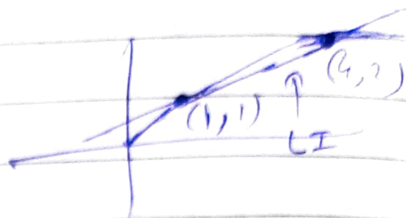


$\boxed{\dfrac{P_n(x)}{\text{deg} = n}}$

for LI, 2 pts. (points)

for QI, 3 pts.

→ Example :-



$y = \sqrt{x}$

$(1,1)$, $(4,2)$

By LI,

→ $P_1(x)$

$$P_1(x) = y_0\left(\frac{x - x_1}{x_0 - x_1}\right) + y_1\left(\frac{x - x_0}{x_1 - x_0}\right)$$

deg

$$P_1(x) = \frac{x-2}{3}$$

→ **Quadratic Interpolation :-** (QI)

(x_0, y_0), (x_1, y_1), (x_2, y_2) → 3 points

$$P_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$$

deg

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

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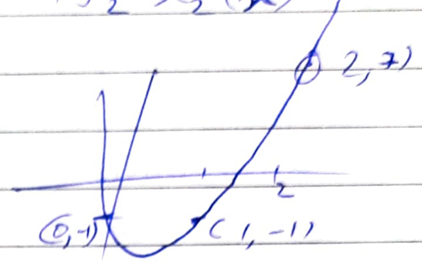
Exp :- $(0,-1)$ $(1,-1)$ $(2,7)$

$p_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$



→ Higher Degree polynomial :-

Exp Estimate $e^{0.826}$ using the func $f(x) = e^x$ values,

$e^{0.82} = 0.270500$

$e^{0.83} = 2.293319$

→ Find $p_1(x)$ & then find $p_1(0.826)$.

↳ LI

answer

$p_1(0.826) = 2.2841914$

$e^{0.826} = 2.2841638$

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→ Error in polynomial Interpolation :-

Thm Let $x_0, x_1, \ldots x_n$ be distinct real numbers in $[a,b]$, & let f be a given real valued function with $n+1$ continuous derivative on $[a,b]$. Then there exists $\eta \in [a,b]$ with

$$f(t) - P_n(t) = \frac{(t-x_0)(t-x_1)\ldots(t-x_n)}{(n+1)!} f^{(n+1)}(\eta)$$

for $a \le t \le b$ & $\min(x_0 \ldots x_n) < \eta < \max(x_0 \ldots x_n)$

Pf If t is a node point the result is trivially true.

Assume t is not a node point.

Define $E(x) = f(x) - P_n(x)$

where $P_n(x)$ is Lagrange's Interpolation

Let $G(x) = E(x) - \frac{\psi(x)}{\psi(t)} E(t)$ for $x \in [a,b]$

with $\psi(x) = (x-x_0) \ldots (x-x_n)$

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→ $G(x)$ is $(n+1)$ times continuously differentiable.

$$G(x_i) = E(x_i) - \frac{\varphi(x_i)}{\varphi(t)} E(t) = 0$$

$i = 0, 1 \dots n$

$$G(t) = E(t) - E(t) = 0$$

→ So $G(x)$ has $(n+2)$ distinct roots. → ($i = 0 \dots n$ & t)

→ Using Roll's thm, G' has $(n+1)$ distinct zeros.

→ Let ξ be a zero of $G^{(n+1)}(x)$.

$$G^{n+1}(\xi) = 0$$

$$E^{n+1}(x) = f^{(n+1)}(x)$$

$$\varphi^{n+1}(x) = (n+1)!$$

$$G^{n+1}(x) = f^{n+1}(x) - \frac{(n+1)!}{\varphi(t)} E(t)$$

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Substitute $x = \xi$ & solve for $E(t)$

$$E(t) = \frac{\psi(t)}{(n+1)!} f^{n+1}(\xi)$$

Ex

$f(x) = e^{x}$ on $[0, 1]$

Interpolate $f(x)$ at x_0, x_1 s.t. $0 \le x_0 \le x_1 \le 1$

error in linear interpolation

$$e^{x} - P_1(x) = \underbrace{\frac{(x - x_0)(x - x_1)}{2} e^{\zeta_x}}_{E(t)}$$

for $\min\{x_0 \ldots x_n\} \le \zeta_x \le \max\{x_0, \ldots x_n\}$

Assume $x_0 < x < x_1$

$$E(x) = -\frac{(x_1 - x)(x - x_0)}{2} e^{\zeta_x}$$

Since $x_0 \le \zeta_x \le x_1$

$$\frac{(x_1 - x)(x - x_0)}{2} e^{x_0} \le |e^{x} - P_1(x)| \le \frac{(x_1 - x)(x - x_0)}{2} e^{x_1}$$

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$e^{x_1} \le e$ since $0 \le x_0 \le x_1 \le 1$

So we find max value,

$$\frac{(x_1 - x)(x - x_0)}{2} e^{x_1}$$

max of $(x_1 - x)(x - x_0)$ occurs at $\frac{x_0 + x_1}{2}$.

$$\frac{\left(x_1 - \left(\frac{x_1 + x_0}{2}\right)\right)\left(\frac{x_1 + x_0}{2} - x_0\right)}{2}$$

$$= \frac{\frac{x_1 - x_0}{2} \cdot \frac{x_1 - x_0}{2}}{2} = \frac{(x_1 - x_0)^2}{8}$$

step size $= x_1 - x_0 = h$

$$= \frac{h^2}{8}$$

$$\boxed{\left|e^{x} - P_1(x)\right| \le \frac{h^2}{8} e}$$

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Example $f(x) = e^x$ in $[0,1]$

$$e^x - P_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{6} e^{\xi_x}$$

$\min \le \xi_x \le \max$

Assume $0 \le x_0 \le x \le x_2 \le 1$

$h = x_1 - x_0 = x_2 - x_1$

$$|e^x - P_2(x)| \le \left|\frac{(x-x_0)(x-x_1)(x-x_2)}{6} e^{\xi}\right| \quad \xi_x \le 1$$

$\max(e^{\xi_x}) = e^1$

Let $x - x_1 = y$

$x - x_0 = x - (x_1 - h)$

$= x - x_1 + h$

$= y + h$

$x - x_2 = y - h$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{6} = \frac{(y+h)(y)(y-h)}{6}$$

max value comes at $y = \pm \frac{h}{\sqrt{3}}$, max value $\left(\pm \frac{h}{\sqrt{3}}\right) = \frac{h^3}{9\sqrt{2}}$

$$\left|e^x - P_2(x)\right| \leq \frac{h^3 e}{9\sqrt{2}}$$

0