Lanchester's Combat Models (For battlefield tactics)

(Frederick William Lanchester, 1916)

An "x-force" and a "y-force" are engaged in combat. Strength is the number of combatants.

$x \equiv x(t) \rightarrow$ Number of combatants in x.

$y \equiv y(t) \rightarrow$ Number of combatants in y.

$t \rightarrow$ Measured in days from the start.

$\boxed{\frac{dx}{dt}}$ = reinforcement rate − operational loss rate − combat loss rate.

Same principle applies for $\boxed{dy/dt}$.

Operational loss: Due to disease, accidents, desertions etc.

Operational loss rate $\propto$ strength (Lanchester)

We assume zero operational loss.

I. Conventional- Conventional Combat:

In modern combat, where x is a conventional force, all of x is within the kill range of the enemy y.

i.) $\boxed{\frac{dx}{dt}} \propto$ fraction of x exposed to $y \equiv 1$ (in modern combat)

ii.) $\boxed{\frac{dx}{dt} \propto y}$ (strength of the enemy)

(P.T.O.)

The fraction of x exposed to y is 1 in modern conventional combat, since enemy fire is concentrated on all of x.

Hence, $\boxed{\frac{dx}{dt} = -Ay}$ $A \to$ Combat effectiveness of y $(A>0)$

Similarly $\boxed{\frac{dy}{dt} = -Bx}$ $B \to$ Combat effectiveness of x. $(B>0)$

If the reinforcement rate of x is $f(t)$ and for y it is $g(t)$, we get.

$\boxed{\frac{dx}{dt} = f(t) - Ay}$ and $\boxed{\frac{dy}{dt} = g(t) - Bx}$.

II/. Conventional - Guerilla Combat:

x is the guerilla force. Not all of it is exposed to the enemy fire of y, which is a conventional force.

$\therefore$ fraction of x exposed to y < 1. We write this fraction $\propto x$. $(C, D > 0)$

Hence, $\boxed{\frac{dx}{dt} \propto x}$ and $\boxed{\frac{dx}{dt} \propto y}$.

Jointly, $\boxed{\frac{dx}{dt} = -Cxy}$ and $\boxed{\frac{dy}{dt} = -Dx}$

With reinforcements. $\boxed{\frac{dx}{dt} = f(t) - Cxy}$ and

$\boxed{\frac{dy}{dt} = g(t) - Dx}$ C and D $\to$ Combat effectiveness.

Isolated Combat (Special Case):

No reinforcement in isolated battle formation

In this case $\boxed{f(t) = g(t) = 0}$.

1/. Conventional- Conventional Combat:

We have $\boxed{\frac{dx}{dt} = -Ay}$ and $\boxed{\frac{dy}{dt} = -Bx}$.

$$\therefore \boxed{\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{+Bx}{+Ay}} \Rightarrow \boxed{\int Ay\,dy = \int Bx\,dx}$$

Integration gives $\boxed{A\frac{y^2}{2} - B\frac{x^2}{2} = \text{Constant}}$.

$\Rightarrow \boxed{Ay^2 - Bx^2 = K}$. Initially $\boxed{x = x_0, y = y_0}$.

$\therefore \boxed{K = Ay_0^2 - Bx_0^2}$ (Lanchester's Square Law).

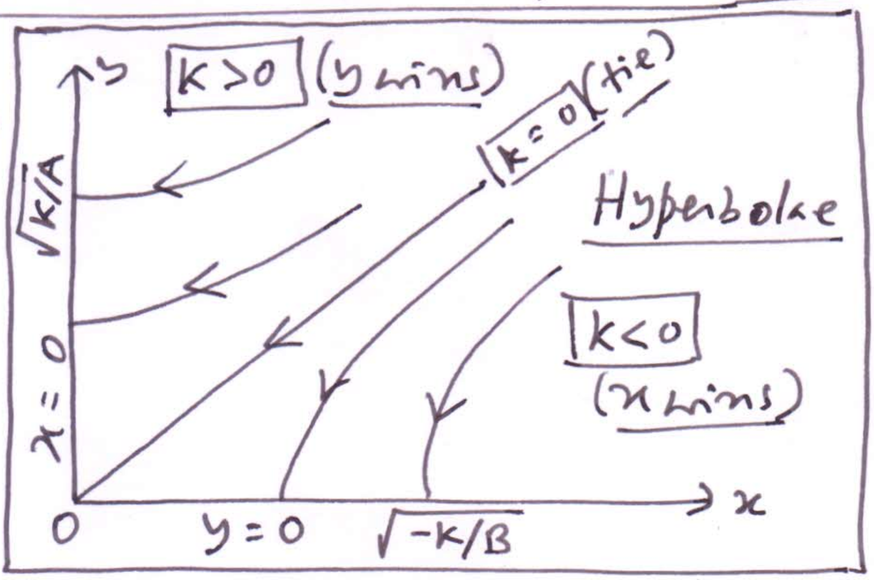
Victory criterion: One force wins the battle if the other force vanishes completely.

i) If $\boxed{x = 0}$, y wins.

$\boxed{y = \sqrt{K/A}}$.

ii) If $\boxed{y = 0}$, x wins

$\boxed{x = \sqrt{-K/B}}$.



a) For y to win,

$\boxed{K > 0} \Rightarrow \boxed{Ay_0^2 - Bx_0^2 > 0} \Rightarrow$ y wins. $(y > 0)$

b) For x to win, $\boxed{K < 0} \Rightarrow \boxed{Ay_0^2 - Bx_0^2 < 0}$

$\Rightarrow$ x wins. $(x > 0)$

(P.T.O.)

c.) When $\boxed{K=0}$, the battle is tied.

$\Rightarrow \boxed{Ay_0^2 - Bx_0^2 = 0} \Longrightarrow$ A tie.

Lanchester's Square Law: $\boxed{K = Ay_0^2 - Bx_0^2}$

A, B $\rightarrow$ Controlled by weaponry and equipment.

$x_0, y_0 \rightarrow$ Initial troops concentration.

The Decisiveness of well-trained and Concentrated troops is quadratic.

Decisiveness of weaponry is linear.

IV. Conventional- Guerilla Combat:

We have $\boxed{\frac{dx}{dt} = -Cxy}$ and $\boxed{\frac{dy}{dt} = -Dx}$.

$\therefore \boxed{\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{+Dx}{+Cxy}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{D}{Cy}}$

$\Rightarrow \boxed{\int Cy\,dy = \int D\,dx} \Rightarrow \boxed{Cy^2 - 2Dx = M}$ Parabola

when $\boxed{x = x_0,\ y = y_0}$, $\boxed{M = Cy_0^2 - 2Dx_0}$

i.) If $\boxed{x=0}$, y wins.

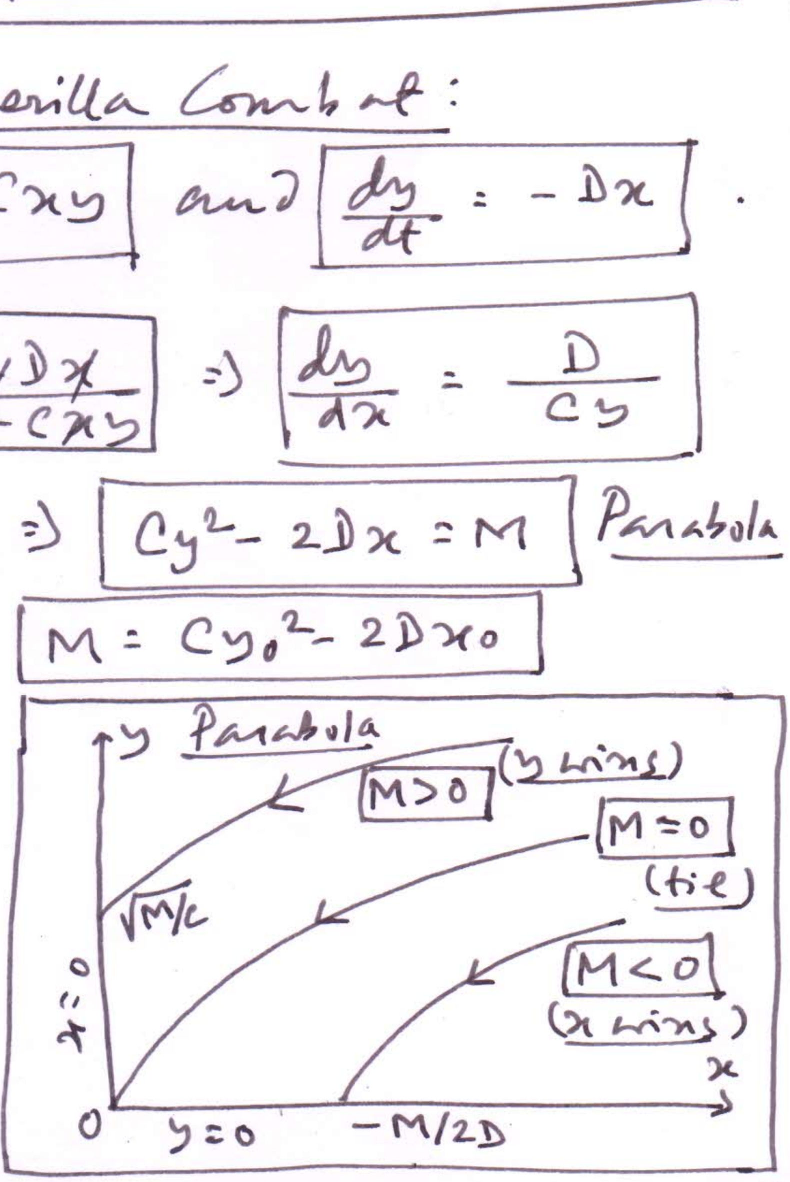
$\Rightarrow \boxed{y = \sqrt{M/c}}$

ii.) If $\boxed{y=0}$, x wins

$\Rightarrow \boxed{x = -M/2D}$

iii.) When $\boxed{x=y=0}$, $\boxed{M=0}$.

Condition for a tie.



Lanchester's Linear Law:

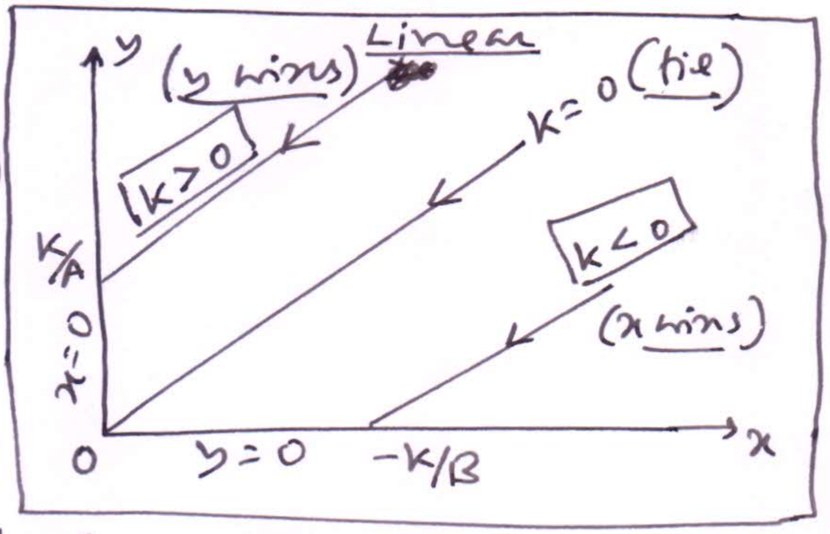
Appropriate for ancient combat

$$\boxed{\frac{dx}{dt} = -Axy} \text{ and } \boxed{\frac{dy}{dt} = -Bxy}$$

$$\therefore \boxed{\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{-Bxy}{-Axy}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{B}{A}}.$$

$$\Rightarrow \boxed{Ay - Bx = k}$$

(The Linear Equation)



i.) When $\boxed{x = 0}$, $\boxed{y = \frac{k}{A}}$ y wins.

ii.) When $\boxed{y = 0}$, $\boxed{x = -k/B}$ x wins.

iii.) $\boxed{k = 0}$, when $\boxed{x = y = 0}$. (Condition for a tie)

Initially $\boxed{x = x_0, y = y_0} \Rightarrow \boxed{k = Ay_0 - Bx_0}$.

Bracken's Generalisation of Lanchester's laws

$$\boxed{\frac{dx}{dt} = -Ax^p y^q} \text{ and } \boxed{\frac{dy}{dt} = -Bx^q y^p}.$$

$$\boxed{\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{-Bx^q y^p}{-Ax^p y^q} = \frac{B}{A}\frac{x^{q-p}}{y^{q-p}}}.$$

Integration gives $\boxed{Ay^\alpha - Bx^\alpha = \text{Constant}}$ in which $\boxed{\alpha = q - p + 1}$. In the context of modern combat a power law of $\boxed{1.5}$ is used (between 1 and 2)

Models are used for battles of Iwo Jima, Kursk and Ardennes (all from World War II).