Set 4: Constrained Growth beyond the logistic modelling

Divya Patel (202001420)* and Aryan Shah (202001430)†

Dhirubhai Ambani Institute of Information & Communication Technology,
Gandhinagar, Gujarat 382007, India
CS302, Modeling and Simulation

In this lab, we numerically and analytically analyze constrained growth beyond the logistic model using the Gompertz equation and the Allee effect for different initial values.

I. GOMPERTZ EQUATION

The Gompertz equation is a mathematical model that describes the growth or decline of a population over time. It was developed by Benjamin Gompertz, a British mathematician and actuary, in 1825. The equation is often used to model the growth or decline of biological populations, such as tumor growth or bacterial growth, and has been applied in fields such as demography, epidemiology, and economics. We have for modeling the tumour growth, the Gompertz Equation as:

$$\dot{x} = -axln(bx) \tag{1}$$

On rescaling $X = x/b^{-1}$ and $T = at$ we obtain a parameter-free differential equation,

$$\dot{X} \equiv dX/dT = -XlnX \tag{2}$$

1. Plots of $\dot{X} versus X$

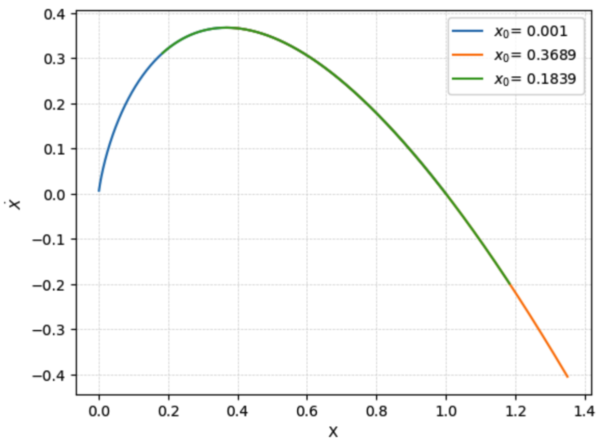

FIG. 1: Plot of $\dot{X}$ vs X for the initial values (i) 0.001, (ii) 0.3689, (iii) 0.1839

2. Plot of $X(T)$ using Euler Method for 3 different values of (X_o)

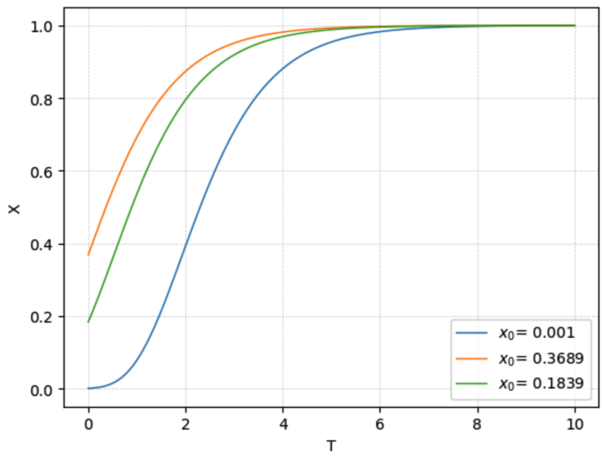

FIG. 2: Plot of X vs T for $\Delta T = 0.001$ using Euler Method

3. Euler's and analytical intergration for for 3 different values of (X_o)

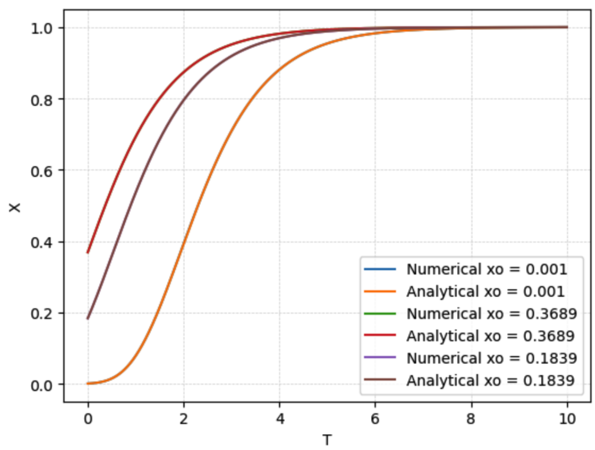

FIG. 3: Plot of $\dot{X} vs X$. $\Delta T = 0.001 for Euler$

4. Relative error between Euler's and analytical intergration

*Electronic address: 202001420@daiict.ac.in
†Electronic address: 202001430@daiict.ac.in

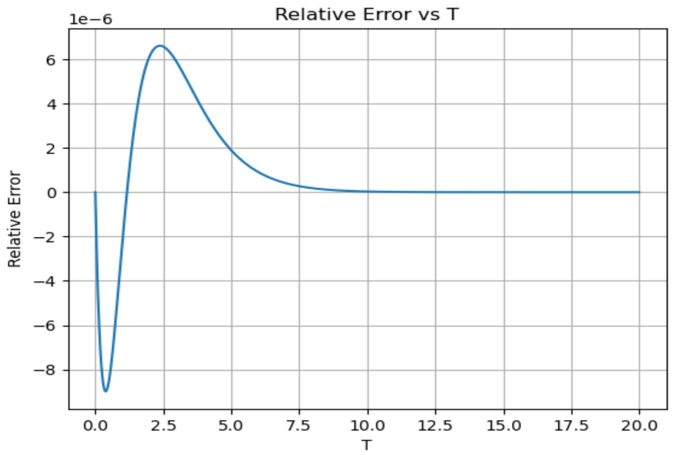

FIG. 4: Plot of relative error between Euler and analytical integration

II. ALLEE EFFECT

The Allee effect is a phenomenon observed in some populations where the fitness of individuals decreases as the population size becomes smaller. Specifically, it refers to situations where the reproductive success of individuals decreases as population density falls below a certain threshold. The equation is,

$$\dot{x} = x[r - a(x-b)^2] \tag{3}$$

1. Plots of $\dot{X} versus X$

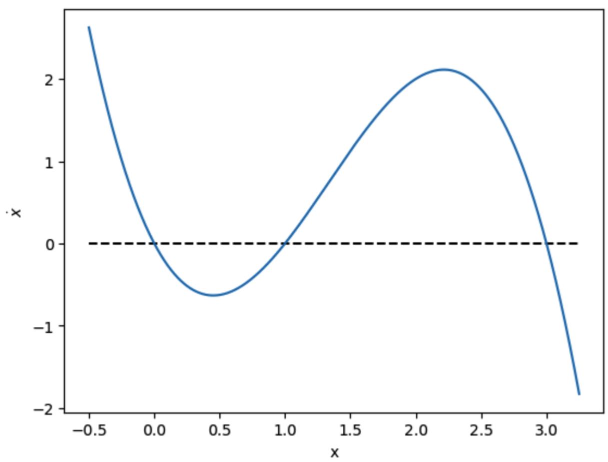

FIG. 5: Plot of $\dot{X} vs X$ for a = r = 1 and b = 2

2. Euler's Integration

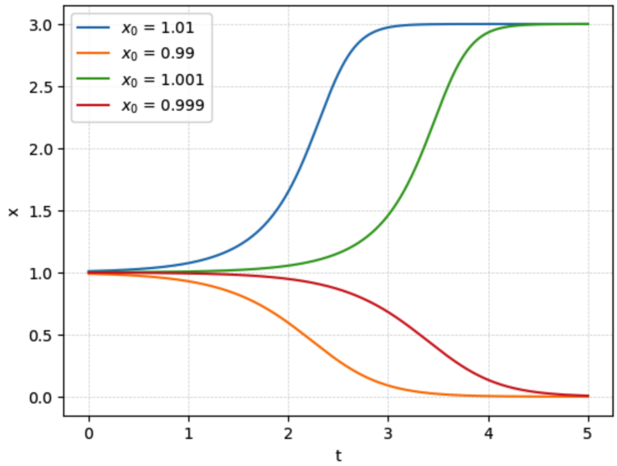

FIG. 6: Plot of Euler's integration for initial values $x_0 = 0.99, x_0 = 1.01, x_0 = 0.999$ and $x_0 = 1.001$

As $t \to \infty$,
For $x_o = 0.99$ and $x_o = 0.999$, Limiting value of $x(t) = 0$
For $x_o = 1.01$ and $x_o = 1.001$, Limiting value of $x(t) = 3$

III. CONCLUSIONS

1. Gompertz equation is a sigmoid function that models growth processes such as tumor growth. One of its key features is that it has an upper asymptote, meaning that the growth will eventually saturate and approach a maximum value regardless of the initial value.

2. Allee effect is a mathematical model that describes the positive relationship between population growth rate and population size when the initial population size has an intermediate value. The limiting values of x when $t \to \infty$ depend on the initial value of x_0. If x_0 is less than or greater than a certain threshold value, then the population will approach different limiting values and exhibit different growth behaviors.