The Predator - Prey Model | Vito Volterra

(To solve the question of why the predator shark population rose ~~arose~~ disproportionately when fishing declined in the Mediterranean Sea.)

$\boxed{x(t)} \rightarrow$ Population of prey (food fish).
$\boxed{y(t)} \rightarrow$ Population of predators (sharks).

In the absence of fishing : $(A, B, C, D > 0)$

i.) Food supply for the prey fish is abundant.
ii.) Their population is not very dense.
iii.) Competition among the prey fish is, therefore, not very intense, and their population grows by the Malthusian law,

$$\boxed{\frac{dx}{dt} = Ax}$$. (The growth is exponential).

iv.) In contact with the predators, however, the population of prey declines. This interaction is modelled as $\boxed{-Bxy}$. This gives

$$\boxed{\frac{dx}{dt} = Ax - Bxy}$$ for the prey fish population.

v.) For the predators, a large population is unsustainable. Hence, with growing numbers, their growth rate decreases, as per $\boxed{dy/dt = -Cy}$.

(P.T.O.)

vi.) The predator population increases with their contact with the prey, an interaction that is modelled as $\boxed{Dxy}$.

Hence, $\boxed{\frac{dy}{dt} = -C\cancel{y} + Dxy}$ for the predator population.

Equilibrium conditions are $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$.

$\Rightarrow$ $\boxed{Ax_c - Bx_cy_c = 0}$ and $\boxed{-Cy_c + Dx_cy_c = 0}$.

We have solutions $\boxed{x_c = 0}$, $\boxed{y_c = A/B}$, $\boxed{y_c = 0}$ and $\boxed{x_c = C/D}$. An optimal equilibrium solution is both $\boxed{x_c \neq 0}$ and $\boxed{y_c \neq 0}$ to maintain Nature's balance (both species live)

In the presence of fishing: $(A, B, C, D > 0)$

Both prey and predators are affected in the same way (to the same extent).

Hence, $\boxed{\frac{dx}{dt} = Ax - Bxy - \epsilon x}$ and $\epsilon > 0$

$\boxed{\frac{dy}{dt} = -Cy + Dxy - \epsilon y}$. Fishing decreases both populations.

Equilibrium solutions for $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$ are

$\boxed{\frac{dx}{dt} = (A-\epsilon)x_c - Bx_cy_c = 0}$

and $\boxed{\frac{dy}{dt} = -(C+\epsilon)y_c + Dx_cy_c = 0}$. Two trivial solutions are $\boxed{x_c = 0}$ (Trivial implies uninteresting) and $\boxed{y_c = 0}$. ...

(P.T.O.)

(continued)

But two other solutions are $\boxed{x_c = \frac{C+E}{D}}$ and $\boxed{y_c = \frac{A-E}{B}}$. This implies that fishing boosts the population of the prey fish.

(y_c decreases) (x_c increases)

A more general model is ($E, F > 0$)

$$\boxed{\frac{dx}{dt} = Ax - Bxy - Ex^2 = x(A - By - Ex)}$$

and

$$\boxed{\frac{dy}{dt} = -Cy + Dxy - Fy^2 = y(-C + Dx - Fy)}.$$

Equilibrium solutions are $\boxed{x_c = y_c = 0}$ ($x = y = 0$). Also $\boxed{x_c \neq 0}$ and $\boxed{y_c \neq 0}$. The two latter solutions are optimal, because non-zero populations of both prey and predator maintain Nature's balance (both species live).

Comparison with the Equilibrium Solutions in the Principle of Competitive Exclusion.

$$\boxed{\frac{dx}{dt} = x\left(a_x - \frac{a_x}{K_x}x - \frac{\alpha a_x y}{K_x}\right)}$$ when $\boxed{\frac{dx}{dt} = 0}$ $\boxed{x_c = 0}$.

and $$\boxed{\frac{dy}{dt} = y\left(a_y - \frac{a_y}{K_y}y - \frac{\beta a_y x}{K_y}\right)}$$ When $\boxed{\frac{dy}{dt} = 0}$ $\boxed{y_c = 0}$.

Two other solutions are $\boxed{x_c \neq 0}$ and $\boxed{y_c \neq 0}$.

The ~~[illegible]~~ optimal combinations are either $\boxed{x_c = 0}$ and $\boxed{y_c \neq 0}$ or $\boxed{x_c \neq 0}$ and $\boxed{y_c = 0}$.