The Threshold Theorem of Epidemiology

1. A small group of people introduces an ~~...~~ infectious disease in a large population.
2. The disease has a short incubation period.
3. Recovered individuals gain permanent immunity.

(SIR) There are three classes of population. They are:
i) $x \to$ The infected class, ii) $y \to$ The susceptible class.
iii) $z \to$ The removed class (recovered class).

Rule 1: $\boxed{x(t) + y(t) + z(t) = N}$, where N is the fixed total number of population. (Conserved condition)

Rule 2: $\boxed{\dfrac{dy}{dt} \propto xy} \Rightarrow \boxed{\dfrac{dy}{dt} = -Axy}$ $A \to$ The infection rate.

Rule 3: $\boxed{\dfrac{dz}{dt} \propto x} \Rightarrow \boxed{\dfrac{dz}{dt} = Bx}$ $B \to$ The removal rate.
(A, B → constants)

Using Rule 1 we can write $\boxed{\dfrac{dx}{dt} = -\dfrac{dy}{dt} - \dfrac{dz}{dt}}$, which gives $\boxed{\dfrac{dx}{dt} = Axy - Bx}$. In all the three $\boxed{dx/dt}$, $\boxed{dy/dt}$ and $\boxed{dz/dt}$ equations, the right hand side does not depend on z. This a pseudo-third-order system.

I.) <u>The x-y equation</u> :

$$\boxed{\frac{dx/dt}{dy/dt} = \frac{dx}{dy} = \frac{Axy - Bx}{-Axy} = -1 + \frac{B}{Ay}}$$

$$\Rightarrow \boxed{\int dx = \int \left(\frac{B}{Ay} - 1\right) dy} \Rightarrow \boxed{x = \frac{B}{A}\ln y - y + c_1}$$

($c_1 \to$ Integral constant).

<u>Initial condition</u>: At $\boxed{t=0}$ (<u>initially</u>), $\boxed{x = x_0}$, $\boxed{y = y_0}$ and $\boxed{z = 0}$. $\Rightarrow$ $\boxed{x_0 + y_0 = N}$.

$$\therefore \boxed{c_1 = x_0 + y_0 - \frac{B}{A}\ln y_0 = N - \frac{B}{A}\ln y_0}.$$

$$\Rightarrow \boxed{x = (x_0 + y_0) - y + \frac{B}{A}\ln(y/y_0)} \quad \boxed{x \equiv x(y)}$$

<u>in closed form</u>.

II.) <u>The y-z equation</u> :

$$\boxed{\frac{dy/dt}{dz/dt} = \frac{dy}{dz} = \frac{-Axy}{Bx} = -\frac{Ay}{B}}$$

$$\Rightarrow \boxed{\int \frac{dy}{y} = -\int \frac{A}{B} dz} \Rightarrow \boxed{\ln y = -\frac{Az}{B} + c_2}$$

($c_2 \to$ <u>Integral constant</u>).

At $\boxed{t=0}$, $\boxed{y = y_0}$ and $\boxed{z = 0}$ $\Rightarrow$ $\boxed{c_2 = \ln y_0}$

$\Rightarrow \boxed{\ln\left(\frac{y}{y_0}\right) = -\frac{A}{B}z}$. For <u>$t > 0$</u>, $\boxed{y < y_0}$

$\Rightarrow \boxed{\ln\left(\frac{y}{y_0}\right) < 0}$ for <u>$t > 0$</u> $\Rightarrow \boxed{z = -\frac{B}{A}\ln\left(\frac{y}{y_0}\right) > 0}$

for <u>$t > 0$</u>.

And $\boxed{y = y_0 \exp\left(-\frac{A}{B}z\right)}$ $\boxed{y = y(z)}$ in <u>closed form</u>.

III.) The z-x equation :

$$\boxed{\frac{dx/dt}{dz/dt} = \frac{dx}{dz} = \frac{Axy - Bx}{Bx} = \frac{Ay}{B} - 1}.$$

Now $\boxed{y = y_0 e^{-\frac{Az}{B}}}$ $\Rightarrow$ $\boxed{\frac{dx}{dz} = \frac{A}{B} y_0 e^{-\frac{Az}{B}} - 1}$.

$$\Rightarrow x = \int \frac{A}{B} y_0 e^{-\frac{A}{B}z}\, dz - z + C_3$$

[$C_3 \rightarrow$ Integral Constant]

$$\Rightarrow \quad x = \frac{\cancel{A}}{\cancel{B}} y_0 \frac{e^{-\frac{Az}{B}}}{-\cancel{A}/\cancel{B}} - z + C_3 .$$

$$\Rightarrow \quad \boxed{x = -y_0 e^{-Az/B} - z + C_3}.$$

When (at $t=0$), $\boxed{x = x_0}$ and $\boxed{z = 0}$,

$\boxed{C_3 = x_0 + y_0}$ $\Rightarrow$ $\boxed{x = x_0 + y_0\left(1 - e^{-Az/B}\right) - z}$

OR $\boxed{x = (x_0 + y_0) - y_0 \exp\left(-\frac{Az}{B}\right) - z}$ But z is not written in a closed form for x.

Plot of x-y: $\boxed{x = (x_0 + y_0) - y + \frac{B}{A} \ln\left(\frac{y}{y_0}\right)}$.

i.) When $\boxed{y \rightarrow 0}$, $\boxed{x \longrightarrow -\infty}$.

ii.) $\boxed{\frac{dx}{dt} = x(Ay - B)}$ $\Rightarrow$ $\boxed{\frac{dx}{dt} = 0}$, when either $\boxed{x = 0}$ OR $\boxed{y = B/A}$.

iii.) $\boxed{\frac{dx}{dy} = \frac{B}{Ay} - 1}$ $\Rightarrow$ $\boxed{\frac{dx}{dy} = 0}$, when $\boxed{y = \frac{B}{A} = \rho \text{ (say)}}$

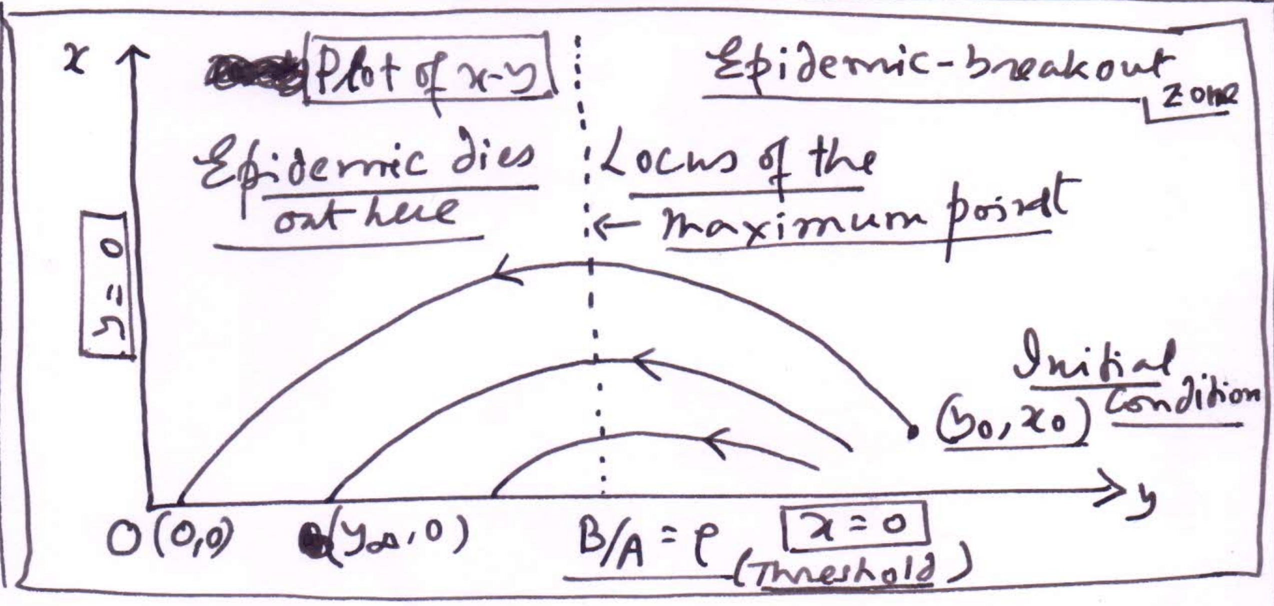
iv.) $\boxed{\frac{d^2x}{dy^2} = -\frac{B}{Ay^2}}$ At $\boxed{y = \frac{B}{A}}$, $\boxed{\frac{d^2x}{dy^2} = -\frac{A}{B} < 0}$. Hence, $y = B/A$ is a maximum.

(P.T.O.)

v.) When $\boxed{x = 0}$, write $\boxed{y = y_\infty}$ (say).

With $\boxed{x=0}$, the point becomes an equilibrium point, since both $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$ (in the phase plot)

Note: x is plotted along the vertical axis, and y along the horizontal.



Conclusion 1: An epidemic will breakout, if $(y_0 > \rho)$, i.e. the initial number of susceptibles are above the threshold ρ. This is the Threshold Theorem of Epidemiology (Kermack & McKendrick)

Conclusion 2: The spread of the disease stops $(x=0)$ because the infective population is reduced to zero, even though there may be some susceptibles left (at y_∞).

Practically speaking epidemics breakout due to overcrowding of a susceptible population in an unhygienic environment.