Chapter 8

Hypothesis Testing with Two Samples

§ 8.1

Testing the Difference Between Means (Large Independent Samples)

Two Sample Hypothesis Testing

In a two-sample hypothesis test, two parameters from two populations are compared.

For a two-sample hypothesis test,
1. the **null hypothesis** H_0 is a statistical hypothesis that usually states there is no difference between the parameters of two populations. The null hypothesis always contains the symbol $\leq$, $=$, or $\geq$.
2. the **alternative hypothesis** H_a is a statistical hypothesis that is true when H_0 is false. The alternative hypothesis always contains the symbol $>$, $\neq$, or $<$.

Two Sample Hypothesis Testing

To write a null and alternative hypothesis for a two-sample hypothesis test, translate the claim made about the population parameters from a verbal statement to a mathematical statement.

$$\begin{cases} H_0: \mu_1 = \mu_2 \\ H_a: \mu_1 \neq \mu_2 \end{cases} \qquad \begin{cases} H_0: \mu_1 \leq \mu_2 \\ H_a: \mu_1 > \mu_2 \end{cases} \qquad \begin{cases} H_0: \mu_1 \geq \mu_2 \\ H_a: \mu_1 < \mu_2 \end{cases}$$

Regardless of which hypotheses used, $\mu_1 = \mu_2$ is always assumed to be true.

Two Sample *z*-Test

Three conditions are necessary to perform a *z*-test for the difference between two population means μ_1 and μ_2.

1. The samples must be randomly selected.

2. The samples must be independent. Two samples are **independent** if the sample selected from one population is not related to the sample selected from the second population.

3. Each sample size must be at least 30, or, if not, each population must have a normal distribution with a known standard deviation.

Two Sample *z*-Test

If these requirements are met, the sampling distribution for $\bar{x}_1 - \bar{x}_2$ (the difference of the sample means) is a normal distribution with mean and standard error of

$$\mu_{\bar{x}_1 - \bar{x}_2} = \mu_{\bar{x}_1} - \mu_{\bar{x}_2} = \mu_1 - \mu_2$$

and

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\sigma_{\bar{x}_1}^2 + \sigma_{\bar{x}_2}^2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}.$$

Sampling distribution
for $\bar{x}_1 - \bar{x}_2$

2
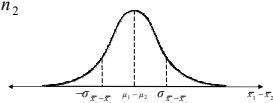
Two Sample z-Test

Two-Sample z-Test for the Difference Between Means

A two-sample z-test can be used to test the difference between two population means μ_1 and μ_2 when a large sample (at least 30) is randomly selected from each population and the samples are independent. The test statistic is   and the standardized test statistic is $\bar{x}_1 - \bar{x}_2$

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}} \quad \text{where} \quad \sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}.$$

When the samples are large, you can use s_1 and s_2 in place of σ_1 and σ_2. If the samples are not large, you can still use a two-sample z-test, provided the populations are normally distributed and the population standard deviations are known.

Two Sample z-Test for the Means

Using a Two-Sample z-Test for the Difference Between Means (Large Independent Samples)

In Words	In Symbols
1. State the claim mathematically. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Sketch the sampling distribution.	
4. Determine the critical value(s).	Use Table 4 in Appendix B.
5. Determine the rejection regions(s).	

Continued.

Two Sample z-Test for the Means

Using a Two-Sample z-Test for the Difference Between Means (Large Independent Samples)

| In Words	In Symbols
6. Find the standardized test statistic.	$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}}$
7. Make a decision to reject or fail to reject the null hypothesis.	
8. Interpret the decision in the context of the original claim.	If z is in the rejection region, reject H_0. Otherwise, fail to reject H_0.

Two Sample *z*-Test for the Means

Example:

A high school math teacher claims that students in her class will score higher on the math portion of the ACT then students in a colleague's math class. The mean ACT math score for 49 students in her class is 22.1 and the standard deviation is 4.8. The mean ACT math score for 44 of the colleague's students is 19.8 and the standard deviation is 5.4. At $\alpha = 0.10$, can the teacher's claim be supported?

H_0: $\mu_1 \le \mu_2$

H_a: $\mu_1 > \mu_2$ (Claim)
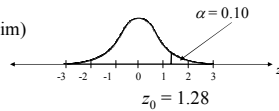
$\alpha = 0.10$

-3 -2 -1 0 1 2 3 z

$z_0 = 1.28$

Continued.

Two Sample *z*-Test for the Means

Example continued:

H_0: $\mu_1 \le \mu_2$

H_a: $\mu_1 > \mu_2$ (Claim)

$z_0 = 1.28$

-3 -2 -1 0 1 2 3 z

Reject H_0.
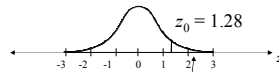
The standardized error is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{4.8^2}{49} + \frac{5.4^2}{44}} \approx 1.0644.$$

The standardized test statistic is

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}} = \frac{(22.1 - 19.8) - 0}{1.0644} \approx 2.161$$

There is enough evidence at the 10% level to support the teacher's claim that her students score better on the ACT.

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§ 8.2

Testing the Difference Between Means   (Small Independent Samples)

Two Sample *t*-Test

If samples of size less than 30 are taken from normally-distributed populations, a *t*-test may be used to test the difference between the population means μ_1 and μ_2.

Three conditions are necessary to use a *t*-test for small independent samples.

1. The samples must be randomly selected.
2. The samples must be independent. Two samples are **independent** if the sample selected from one population is not related to the sample selected from the second population.
3. Each population must have a normal distribution.

Two Sample *t*-Test

Two-Sample *t*-Test for the Difference Between Means

A **two-sample *t*-test** is used to test the difference between two population means μ_1 and μ_2 when a sample is randomly selected from each population. Performing this test requires each population to be normally distributed, and the samples should be independent. The standardized test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}}.$$

If the population variances are equal, then information from the two samples is combined to calculate a **pooled estimate of the standard deviation** $\hat{\sigma}$.

$$\hat{\sigma} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Continued.

Two Sample *t*-Test

Two-Sample *t*-Test (Continued)

The standard error for the sampling distribution of $\bar{x}_1 - \bar{x}_2$ is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \hat{\sigma} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \qquad \text{Variances equal}$$
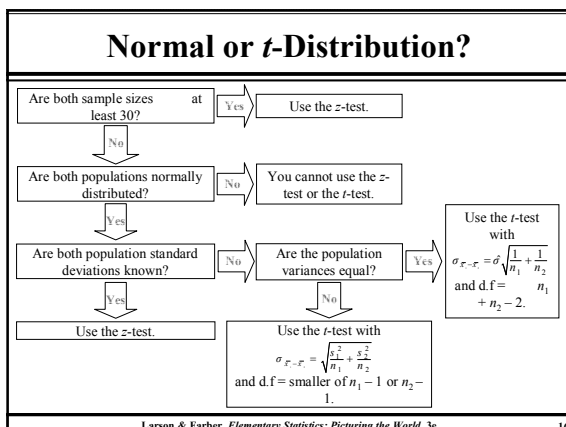
and d.f.$= n_1 + n_2 - 2$.

If the population variances are not equal, then the standard error is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \qquad \text{Variances not equal}$$

and d.f = smaller of $n_1 - 1$ or $n_2 - 1$.

Normal or *t*-Distribution?

Are both sample sizes at least 30? — **Yes** → Use the *z*-test.

No ↓

Are both populations normally distributed? — **No** → You cannot use the *z*-test or the *t*-test.

Yes ↓

Are both population standard deviations known? — **No** → Are the population variances equal? — **Yes** → Use the *t*-test with $\sigma_{\bar{x}_1-\bar{x}_2} = \hat{\sigma}\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}$ and d.f $= n_1 + n_2 - 2$.

Yes ↓   **No** ↓

Use the *z*-test.   Use the *t*-test with $\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\dfrac{s_1^2}{n_1}+\dfrac{s_2^2}{n_2}}$ and d.f = smaller of $n_1 - 1$ or $n_2 - 1$.

Two Sample *t*-Test for the Means

Using a Two-Sample *t*-Test for the Difference Between Means (Small Independent Samples)

In Words	*In Symbols*
1. State the claim mathematically. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Identify the degrees of freedom and sketch the sampling distribution.	d.f. $= n_1 + n_2 - 2$ or d.f. = smaller of $n_1 - 1$ or $n_2 - 1$.
4. Determine the critical value(s).	Use Table 5 in Appendix B.

Continued.

Two Sample *t*-Test for the Means

Using a Two-Sample *t*-Test for the Difference Between Means (Small Independent Samples)

In Words	*In Symbols*
5. Determine the rejection regions(s).	
6. Find the standardized test statistic.	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1-\bar{x}_2}}$
7. Make a decision to reject or fail to reject the null hypothesis.	If *t* is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
8. Interpret the decision in the context of the original claim.	

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Two Sample *t*-Test for the Means

Example:

A random sample of 17 police officers in Brownsville has a mean annual income of \$35,800 and a standard deviation of \$7,800. In Greensville, a random sample of 18 police officers has a mean annual income of \$35,100 and a standard deviation of \$7,375. Test the claim at $\alpha = 0.01$ that the mean annual incomes in the two cities are not the same. Assume the population variances are equal.
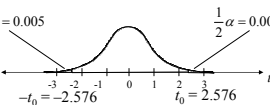
$H_0: \mu_1 = \mu_2$

$H_a: \mu_1 \neq \mu_2$ (Claim)

$\frac{1}{2}\alpha = 0.005$ $\frac{1}{2}\alpha = 0.005$

$-t_0 = -2.576$ $t_0 = 2.576$

d.f. $= n_1 + n_2 - 2$
$= 17 + 18 - 2 = 33$

Continued.

Two Sample *t*-Test for the Means

Example continued:

$H_0: \mu_1 = \mu_2$

$H_a: \mu_1 \neq \mu_2$ (Claim)

$-t_0 = -2.576$ $t_0 = 2.576$
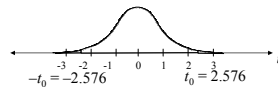
The standardized error is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \hat{\sigma}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= \sqrt{\frac{(17 - 1)7800^2 + (18 - 1)7375^2}{17 + 18 - 2}} \cdot \sqrt{\frac{1}{17} + \frac{1}{18}}$$

$$\approx 7584.0355(0.3382)$$

$$\approx 2564.92$$

Continued.

Two Sample *t*-Test for the Means

Example continued:

$H_0: \mu_1 = \mu_2$

$H_a: \mu_1 \neq \mu_2$ (Claim)

$-t_0 = -2.576$ $t_0 = 2.576$
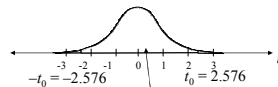
The standardized test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}} = \frac{(35800 - 35100) - 0}{2564.92} \approx 0.273$$

Fail to reject H_0.

There is not enough evidence at the 1% level to support the claim that the mean annual incomes differ.

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§ 8.3

Testing the Difference Between Means (Dependent Samples)

Independent and Dependent Samples

Two samples are **independent** if the sample selected from one population is not related to the sample selected from the second population. Two samples are **dependent** if each member of one sample corresponds to a member of the other sample. Dependent samples are also called **paired samples** or **matched samples**.
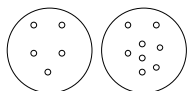
Independent Samples   Dependent Samples

Independent and Dependent Samples

Example:
Classify each pair of samples as independent or dependent.

Sample 1: The weight of 24 students in a first-grade class
Sample 2: The height of the same 24 students

These samples are dependent because the weight and height can be paired with respect to each student.

Sample 1: The average price of 15 new trucks
Sample 2: The average price of 20 used sedans

These samples are independent because it is not possible to pair the new trucks with the used sedans. The data represents prices for different vehicles.

t-Test for the Difference Between Means

To perform a two-sample hypothesis test with dependent samples, the difference between each data pair is first found:

$$d = x_1 - x_2 \qquad \text{Difference between entries for a data pair.}$$

The test statistic is the mean $\bar{d}$ of these differences.

$$\bar{d} = \frac{\sum d}{n}. \qquad \begin{array}{l}\text{Mean of the differences between paired data} \\ \text{entries in the dependent samples.}\end{array}$$

Three conditions are required to conduct the test.

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t-Test for the Difference Between Means

1. The samples must be randomly selected.

2. The samples must be dependent (paired).

3. Both populations must be normally distributed.

If these conditions are met, then the sampling distribution for $\bar{d}$ is approximated by a *t*-distribution with $n - 1$ degrees of freedom, where n is the number of data pairs.
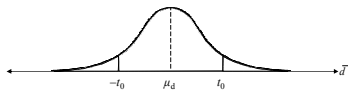
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t-Test for the Difference Between Means

The following symbols are used for the *t*-test for μ_d.

Symbol	Description
n	The number of pairs of data
d	The difference between entries for a data pair, $d = x_1 - x_2$
μ_d	The hypothesized mean of the differences of paired data in the population
$\bar{d}$	The mean of the differences between the paired data entries in the dependent samples $\bar{d} = \frac{\sum d}{n}$
s_d	The standard deviation of the differences between the paired data entries in the dependent samples $s_d = \sqrt{\frac{n(\sum d^2) - (\sum d)^2}{n(n-1)}}$

t-Test for the Difference Between Means

_t_-Test for the Difference Between Means

A *t*-test can be used to test the difference of two population means when a sample is randomly selected from each population. The requirements for performing the test are that each population must be normal and each member of the first sample must be paired with a member of the second sample.

The **test statistic** is

$$\bar{d} = \frac{\sum d}{n}$$

and the **standardized test statistic** is

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}.$$

The degrees of freedom are

d.f. = $n - 1$.

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t-Test for the Difference Between Means

Using the _t_-Test for the Difference Between Means (Dependent Samples)

In Words	*In Symbols*
1. State the claim mathematically. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Identify the degrees of freedom and sketch the sampling distribution.	d.f. = $n - 1$
4. Determine the critical value(s).	Use Table 5 in Appendix B.

Continued.

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t-Test for the Difference Between Means

Using a Two-Sample _t_-Test for the Difference Between Means (Small Independent Samples)

In Words	*In Symbols*
5. Determine the rejection region(s).	
6. Calculate $\bar{d}$ and s_d. Use a table.	$\bar{d} = \frac{\sum d}{n}$ $s_d = \sqrt{\frac{n(\sum d^2) - (\sum d)^2}{n(n-1)}}$
7. Find the standardized test statistic.	$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$

t-Test for the Difference Between Means

Using a Two-Sample *t*-Test for the Difference Between Means (Small Independent Samples)

In Words	*In Symbols*
8. Make a decision to reject or fail to reject the null hypothesis.	If t is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
9. Interpret the decision in the context of the original claim.	

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t-Test for the Difference Between Means

Example:

A reading center claims that students will perform better on a standardized reading test after going through the reading course offered by their center. The table shows the reading scores of 6 students before and after the course. At $\alpha = 0.05$, is there enough evidence to conclude that the students' scores after the course are better than the scores before the course?

Student	1	2	3	4	5	6
Score (before)	85	96	70	76	81	78
Score (after)	88	85	89	86	92	89

H_0: $\mu_d \leq 0$

H_a: $\mu_d > 0$ (Claim)

Continued.

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t-Test for the Difference Between Means

Example continued:

H_0: $\mu_d \leq 0$

H_a: $\mu_d > 0$ (Claim)

d.f. = 6 – 1 = 5

$\alpha = 0.05$

$t_0 = 2.015$

d = (score before) – (score after)
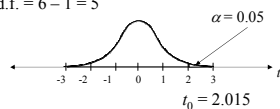
Student	1	2	3	4	5	6	
Score (before)	85	96	70	76	81	78	
Score (after)	88	85	89	86	92	89	
d	–3	11	–19	–10	–11	–11	$\Sigma d = -43$
d^2	9	121	361	100	121	121	$\Sigma d^2 = 833$

$$\bar{d} = \frac{\Sigma d}{n} = \frac{-43}{6} \approx -7.167$$

$$s_d = \sqrt{\frac{n(\Sigma d^2) - (\Sigma d)^2}{n(n-1)}} = \sqrt{\frac{6(833) - 1849}{6(5)}} \approx \sqrt{104.967} \approx 10.245$$
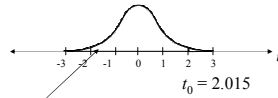
Continued.

t-Test for the Difference Between Means

Example continued:

H_0: $\mu_d \leq 0$

H_a: $\mu_d > 0$ (Claim)

$t_0 = 2.015$

The standardized test statistic is

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} = \frac{-7.167 - 0}{10.245 / \sqrt{6}} \approx -1.714.$$

Fail to reject H_0.

There is not enough evidence at the 5% level to support the claim that the students' scores after the course are better than the scores before the course.

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§ 8.4

Testing the Difference Between Proportions

Two Sample *z*-Test for Proportions

A z-test is used to test the difference between two population proportions, p_1 and p_2.

Three conditions are required to conduct the test.

1. The samples must be randomly selected.
2. The samples must be independent.
3. The samples must be large enough to use a normal sampling distribution. That is,

$$n_1 p_1 \geq 5, \ n_1 q_1 \geq 5,$$
$$n_2 p_2 \geq 5, \ \text{and} \ n_2 q_2 \geq 5.$$

Two Sample *z*-Test for Proportions

If these conditions are met, then the sampling distribution for $\hat{p}_1 - \hat{p}_2$ is a normal distribution with mean

$$\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2$$

and standard error

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, \quad \text{where } \bar{q} = 1 - \bar{p}.$$

A weighted estimate of p_1 and p_2 can be found by using

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2}, \quad \text{where } x_1 = n_1\hat{p}_1 \text{ and } x_2 = n_2\hat{p}_2.$$

Two Sample *z*-Test for Proportions

Two Sample *z*-Test for the Difference Between Proportions

A two sample *z*-test is used to test the difference between two population proportions p_1 and p_2 when a sample is randomly selected from each population.

The **test statistic** is

$$\hat{p}_1 - \hat{p}_1$$

and the **standardized test statistic** is

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} \text{ and } \bar{q} = 1 - \bar{p}.$$

Note:
$n_1\bar{p}$, $n_1\bar{q}$, $n_2\bar{p}$, and $n_2\bar{q}$ must be at least 5.

Two Sample *z*-Test for Proportions

Using a Two-Sample *z*-Test for the Difference Between Proportions

In Words	*In Symbols*
1. State the claim. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Determine the critical value(s).	Use Table 4 in Appendix B.
4. Determine the rejection region(s).	
5. Find the weighted estimate of p_1 and p_2.	$\bar{p} = \dfrac{x_1 + x_2}{n_1 + n_2}$

Continued.

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Two Sample z-Test for Proportions

Using a Two-Sample z-Test for the Difference Between Proportions

|	
In Words	*In Symbols*
6. Find the standardized test statistic.	$z = \dfrac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$
7. Make a decision to reject or fail to reject the null hypothesis.	If z is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
8. Interpret the decision in the context of the original claim.	

Two Sample z-Test for Proportions

Example:

A recent survey stated that male college students smoke less than female college students. In a survey of 1245 male students, 361 said they smoke at least one pack of cigarettes a day. In a survey of 1065 female students, 341 said they smoke at least one pack a day. At $\alpha = 0.01$, can you support the claim that the proportion of male college students who smoke at least one pack of cigarettes a day is lower then the proportion of female college students who smoke at least one pack a day?

$H_0: p_1 \geq p_2$

$H_a: p_1 < p_2$ (Claim)
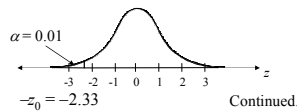
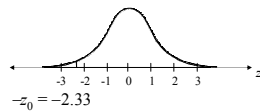$\alpha = 0.01$

$-z_0 = -2.33$

Continued.

Two Sample z-Test for Proportions

Example continued:

$H_0: p_1 \geq p_2$

$H_a: p_1 < p_2$ (Claim)

$-z_0 = -2.33$

$\bar{p} = \dfrac{x_1 + x_2}{n_1 + n_2} = \dfrac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \dfrac{361 + 341}{1245 + 1065} = \dfrac{702}{2310} \approx 0.304$

$\bar{q} = 1 - \bar{p} = 1 - 0.304 = 0.696$

Because 1245(0.304), 1245(0.696), 1065(0.304), and 1065(0.696) are all at least 5, we can use a two-sample z-test.

Continued.

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Two Sample z-Test for Proportions

Example continued:

H_0: $p_1 \geq p_2$

H_a: $p_1 < p_2$ (Claim)

$-z_0 = -2.33$
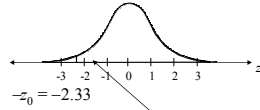
$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(0.29 - 0.32) - 0}{\sqrt{(0.304)(0.696)\left(\frac{1}{1245} + \frac{1}{1065}\right)}} \approx -1.56$$

Fail to reject H_0.

There is not enough evidence at the 1% level to support the claim that the proportion of male college students who smoke is lower then the proportion of female college students who smoke.