Second Order Systems: Examples and Applications

Richardson's Theory of Conflict (Lewis Fry Richardson)

(The British Journal of Psychology: Generalised Foreign Politics, 1939)

$x(t) \rightarrow$ War potential or armaments of a nation.
$y(t) \rightarrow$ War potential of an enemy nation.

1. For x: $\frac{dx}{dt} = ky + g - \alpha x$ in which, $k, g, \alpha > 0$. k indicates the war-readiness of y, g indicates the grievance x feels towards y, and α is the cost of armaments incurred by x (which restrains growth of x).

2. For y: $\frac{dy}{dt} = lx + h - \beta y$ in which $l, h, \beta > 0$. l indicates the war readiness of x, h indicates the grievance y feels towards x, and β is the cost of armaments incurred by y.

$k, g, l, h \rightarrow$ hawk parameters, $\alpha, \beta \rightarrow$ dove parameters

k, l (Thucydides, Grey), g, h (Leo Amery)

Case I: Mutual Disarmament without Animosity and Grievance.

$\boxed{g = h = 0}$ $\Rightarrow$ $\boxed{\frac{dx}{dt} = -\alpha x + Ky}$ and

$\boxed{\frac{dy}{dt} = lx - \beta y}$ $\Rightarrow$ Equilibrium $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$

$\Rightarrow$ $\boxed{x_e = 0}$ and $\boxed{y_e = 0}$ are equilibrium solutions.

For a system $\boxed{\frac{dx}{dt} = Ax + By}$ and $\boxed{\frac{dy}{dt} = Cx + Dy}$,

we can get $\boxed{\frac{d^2x}{dt^2} - (A+D)\frac{dx}{dt} + (AD - BC)x = 0}$.

(The same applies for y). Here $\boxed{\tau = A + D}$ and $\boxed{\Delta = AD - BC}$.

Use a solution $\boxed{x = x_0 e^{\omega t}}$, to get,

$\boxed{\frac{dx}{dt} = \omega x}$ and $\boxed{\frac{d^2x}{dt^2} = \omega^2 x}$. From these

we can write $\boxed{\omega^2 - \tau\omega + \Delta = 0}$, which

implies $\boxed{\omega_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$ Now $A = -\alpha$, $B = K$, $C = l$, $D = -\beta$

Hence, $\boxed{\omega_{1,2} = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha\beta - Kl)}}{2}}$.

If $\boxed{\alpha\beta > Kl}$, then the discriminant of the quadratic above will be less than $(\alpha+\beta)$. Hence both roots of ω will be negative, i.e. $\boxed{x = 0}$ and $\boxed{y = 0}$ will be stable equilibrium solutions. This state represents mutual disarmament. Q.E.D.

Hence with $x=0$ and $y=0$ (mutual disarmament) and with both roots of $\omega_{1,2} < 0$, peace prevails for all time.

Example: Canada / U.S., Norway / Sweden.

Case II: Mutual Disarmament without Satisfaction of Grievance.

Initially $x = y = 0$ (mutual disarmament)

but $g, h \neq 0$ (Grievance continues)

Hence $\frac{dx}{dt} = g$ and $\frac{dy}{dt} = h$. Since both $g, h > 0$, x and y will grow in time.

Case III: Unilateral disarmament.

Initially $y = 0$ but $x \neq 0$ (Unilateral disarmament)

Hence, $\frac{dy}{dt} = lx + h$ Since, x, l and h are all positive.

$\frac{dy}{dt} > 0 \Rightarrow$ y will grow again.

Example: German rearmament before World War II

Can be reduced by reducing grievance and building confidence. Eg. Germany and Japan after World War II.

Case IV: Arms Race

Initially set $\alpha = \beta = 0$ $\Rightarrow$ No restraint on armament.

Also $g = h = 0$ $\Rightarrow$ No history of animosity.

$\Rightarrow$ $\frac{dx}{dt} = ky$ and $\frac{dy}{dt} = lx$. Equilibrium is obtained for $\frac{dx}{dt} = \frac{dy}{dt} = 0$ $\Rightarrow$ $x_e = y_e = 0$.

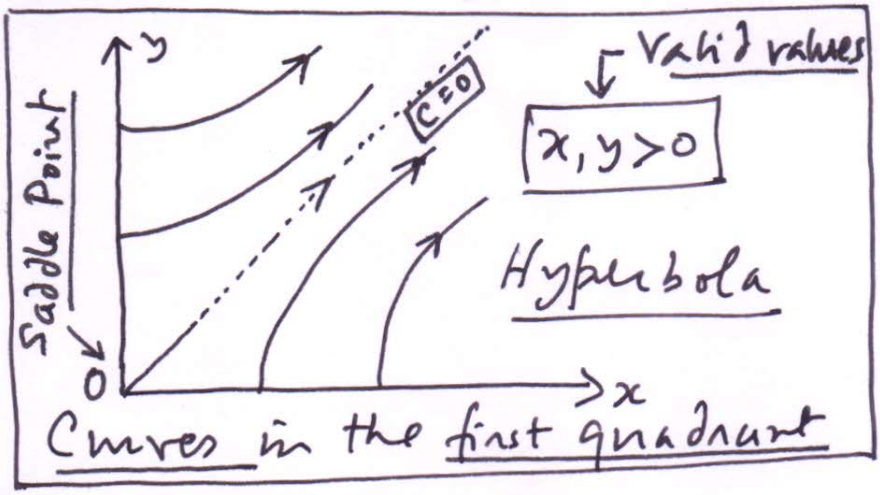
$\Rightarrow$ fixed point is a saddle point

$\Delta = -kl < 0$

Now $\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{lx}{ky}$ $\Rightarrow$ $\int ky\,dy = \int lx\,dx$

$\Rightarrow$ $\frac{lx^2}{2} - \frac{ky^2}{2} = \text{Constant } (c)$ $\Rightarrow$ $\frac{x^2}{2kc} - \frac{y^2}{2lc} = 1$

A hyperbola



Curves in the first quadrant

Whenever x grows, y will also grow, and vice-versa

$\Rightarrow$ an arms race

Example: USA/Soviet Union

$k, l > 0$

Now $\frac{d^2x}{dt^2} = k\frac{dy}{dt} = kl x = \omega^2 x$ where $\omega = \sqrt{kl}$

$\Rightarrow$ $x = Ae^{\sqrt{kl}\,t} + Be^{-\sqrt{kl}\,t}$. Since $y = \frac{1}{k}\frac{dx}{dt}$,

we get $y = A\sqrt{\frac{l}{k}}\,e^{\sqrt{kl}\,t} + B\sqrt{\frac{l}{k}}\,e^{-\sqrt{kl}\,t}$ after differentiating.

As $t \to \infty$, $x \to \infty$ and $y \to \infty$ (Uncontrolled growth)

The General Condition ($\alpha, \beta, k, l, h, g$ are all non-zero)

$$\frac{dx}{dt} = -\alpha x + ky + g \quad \text{and} \quad \frac{dy}{dt} = lx - \beta y + h$$

Equilibrium is obtained for $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

$$\Rightarrow \quad -\alpha x_c + k y_c + g = 0$$
$$\text{and} \quad l x_c - \beta y_c + h = 0$$

$$\Rightarrow \quad -l\alpha x_c + kl y_c + lg = 0$$
$$\text{and} \quad l\alpha x_c - \alpha\beta y_c + \alpha h = 0$$

$$\Rightarrow \quad y_c = \frac{\alpha h + lg}{\alpha\beta - lk}$$

Similarly we also get

$$-\alpha\beta x_c + \beta k y_c + \beta g = 0$$
$$\text{and} \quad kl x_c - k\beta y_c + kh = 0$$

$$\Rightarrow \quad x_c = \frac{kh + \beta g}{\alpha\beta - lk}$$

If $\alpha\beta > lk$, then $x_c, y_c > 0$. This is a permanent, fixed state of war preparedness.

Example: India / Pakistan, North Korea / South Korea

Estimation of the Parameters:

1/. α, β, k, l all have the dimension of inverse time.

2/. α^{-1} and β^{-1} → Life time of policy implementation.
(Example is life time of the parliament ≃ 5 years).
$\Rightarrow \alpha^{-1} \simeq 5$ yrs $\therefore \alpha = 0.2$ unit.

3/. k and l → Depends on the industrial capacity.

4/. g and h → Historical grievances are not constant in time but can change suddenly.