Lecture - 11 / 12

CRLB :-

Regularity Condition :- (i) $E\left[\frac{\partial \ln p(x;\theta)}{\partial \theta}\right] = 0 \quad \forall \theta.$

Bound (ii) $\text{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]}$

(iii) $\frac{\partial \ln p(x;\theta)}{\partial \theta} = I(\theta)\left(g(x) - \theta\right)$

MVUE $\hat{\theta} = g(x)$

$\text{var}(\hat{\theta}_{MVUE}) = \frac{1}{-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]} = \frac{1}{I(\theta)}$

$I(\theta)$ = Fisher information.

Proof :- (i) $E\left[\frac{\partial \ln p(x;\theta)}{\partial \theta}\right]$

$= \int \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta)\, dx.$

$$= \int \frac{1}{p(x;\theta)} \frac{\partial p(x;\theta)}{\partial \theta} p(x;\theta)\, dx$$

$$= \frac{\partial}{\partial \theta} \int p(x;\theta)\, dx$$

$$= 0$$

(ii) $\alpha = g(\theta)$

$$E[\hat{\alpha}] = \alpha = g(\theta)$$

$$\int \hat{\alpha}\, p(x;\theta)\, dx = g(\theta)$$

$$\int \hat{\alpha} \frac{\partial p(x;\theta)}{\partial \theta} dx = \frac{\partial g(\theta)}{\partial \theta}$$

$$* \quad \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta) = \frac{\partial p(x;\theta)}{\partial \theta}$$

$$\int \hat{\alpha} \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta) = \frac{\partial g(\theta)}{\partial \theta} \quad -(1)$$

$$E\left[\frac{\partial \ln p(x;\theta)}{\partial \theta}\right] = 0$$

$$\int \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta)\, dx = 0$$

$$\Rightarrow \int \alpha \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta)\, dx = 0 \quad — (2)$$

Eq. (2) − (1) :−

$$\int (\hat{\alpha} - \alpha) \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta)\, dx = \frac{\partial g(\theta)}{\partial \theta} \quad — (3)$$

* <u>Cauchy-Schwarz inequality condition:-</u>

$$\left[\int w(x)\, g(x)\, h(x)\, dx\right]^2 \leq \int w(x)\, g^2(x)\, dx \int w(x)\, h^2(x)\, dx$$

Bounding condition. $g(x) = c\,h(x)$ $\quad$ [c = constant] $\quad w(x) \geq 0 \;\; \forall x$

scalar fn

$$w(x) = p(x;\theta)$$

$$g(x) = \hat{\alpha} - \alpha$$

$$h(x) = \frac{\partial \ln p(x;\theta)}{\partial \theta}.$$

$$\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2 \leq \int (\hat{\alpha}-\alpha)^2 p(x;\theta)\, dx \int \left(\frac{\partial \ln p(x;\theta)}{\partial \theta}\right)^2 p(x;\theta)\, dx$$

$$\operatorname{var}(\hat{\alpha}) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2}{E\left[\left(\frac{\partial \ln p(x;\theta)}{\partial \theta}\right)^2\right]} \quad - (4)$$

we know

$$E\left[\frac{\partial \ln p(x;\theta)}{\partial \theta}\right] = 0$$

$$\int \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta)\, dx = 0$$

$$\frac{\partial}{\partial \theta} \int \frac{\partial \ln p(x;\theta)}{\partial \theta} p(x;\theta)\, dx = 0$$

$$\int \left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2} p(x;\theta) + \frac{\partial \ln p(x;\theta)}{\partial \theta} \cdot \frac{\partial p(x;\theta)}{\partial \theta}\right] dx = 0$$

$$-\int \frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2} p(x;\theta)\, dx = \int \frac{\partial \ln p(x;\theta)}{\partial \theta} \cdot \frac{\partial p(x;\theta)}{\partial \theta}\, dx$$

$$-E\left[-\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right] = \int \frac{\partial \ln p(x;\theta)}{\partial\theta} \cdot \frac{\partial \ln p(x;\theta)}{\partial\theta} p(x;\theta)\, dx$$

$$= E\left[\left(\frac{\partial \ln p(x;\theta)}{\partial\theta}\right)^2\right]$$

Thus $\operatorname{var}(\hat{\alpha}) \geq \dfrac{\left(\frac{\partial g(\theta)}{\partial\theta}\right)^2}{-E\left[\left(\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right)\right]}$.

If $\alpha = g(\theta) = \theta$

$$\operatorname{var}(\hat{\theta}) \geq \frac{1}{I(\theta)}.$$

* condition for equality is

$$g(x) = c\, h(x)$$

$$h(x) = \frac{1}{c}\, g(x)$$

$$\frac{\partial \ln p(x;\theta)}{\partial\theta} = \frac{1}{c(\theta)}(\hat{\theta} - \theta)$$

c can depend on θ but not on x

$$\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2} = -\frac{1}{c(\theta)} + \frac{\partial\left(\frac{1}{c(\theta)}\right)}{\partial\theta}(\hat{\theta}-\theta)$$

$$E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right] = -E\left[\frac{1}{c(\theta)}\right] + E\left[\frac{\partial\left(\frac{1}{c(\theta)}\right)}{\partial\theta}(\hat{\theta}-\theta)\right]$$

$$-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right] = +\frac{1}{c(\theta)} \quad \left[\because E[\hat{\theta}] = \theta\right]$$

$$c(\theta) = \frac{1}{-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right]} = \frac{1}{I(\theta)}$$

* When CRLB is attained

$$\mathrm{var}(\hat{\theta}) = \frac{1}{I(\theta)} \quad \text{where } I(\theta) = -E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right]$$

CRLB for signals in WGN:-

Deterministic signal : $s[n;\theta]$

$$x[n] = s[n;\theta] + w[n] \quad \left[\text{signal is dependent on } \theta\right]$$

$$p(x;\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} \left(x[n] - s[n;\theta] \right)^2 \right\}$$

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(x[n] - s[n;\theta] \right) \frac{\partial s[n;\theta]}{\partial \theta}$$

$$\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left\{ \left(x[n] - s[n;\theta]\right) \frac{\partial^2 s[n;\theta]}{\partial \theta^2} - \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2 \right\}$$

$$-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right] = +\frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2$$

$$\operatorname{var}(\hat{\theta}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2}.$$

* <u>Transformation of parameter:</u> -

$$x[n] = A + w[n]$$

Power: A^2

$$\alpha = g(\theta)$$

$$\operatorname{var}(\hat{\alpha}) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2}{-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]}.$$

$$\alpha = A^2 = g(A)$$

$$\operatorname{var}(\hat{A^2}) \geq \frac{(2A)^2}{N/\sigma^2}$$

$$= \frac{4A^2\sigma^2}{N}.$$

$$\operatorname{var}(\hat{\theta}) \geq \frac{\sigma^2}{N}.$$

$$I(\hat{A}) = \frac{N}{\sigma^2}$$

$$\hat{A} = \bar{x}$$

$$\bar{x}^2$$

$E[\hat{x}^2] \stackrel{?}{=} \bar{x}^2 \leftarrow$ unbiased?

$E[\hat{x}^2] = var(\hat{x}) + E^2[\hat{x}]$

$var(x) = E[x^2] - E^2[x]$

$E[x^2] = var(x) + E^2[x]$

$= \frac{\sigma^2}{N} + A^2 \neq A^2$ Biased

* $g(\theta) = a\theta + b$

$\widehat{g(\theta)} = g(\hat{\theta}) = a\hat{\theta} + b$

$E[a\hat{\theta} + b] = a\theta + b = g(\theta)$

$\widehat{g(\theta)}$ is unbiased.

CRLB:-

$$var(\widehat{g(\theta)}) \geq \frac{\left(\frac{\partial g}{\partial \theta}\right)^2}{I(\theta)} = a^2 var(\hat{\theta})$$

* $A^2 \quad \bar{x}^2$

$\hat{x}^2$ is asymptotically unbiased.
for $N \to \infty$

$f(n) = n^2 + 2n$

$f(n) \sim n^2$

1 $\bar{x} \sim \mathcal{N}\left(A, \frac{\sigma^2}{N}\right)$

$$\mathrm{var}(\bar{x}^2) = E[\bar{x}^4] - E^2[\bar{x}^2]$$

$$\mathrm{var}(\bar{x}^2) = \frac{4A^2\sigma^2}{N} + \frac{2\sigma^4}{N^2}$$

$N \rightarrow \infty$

$$\mathrm{var}(\bar{x}^2) \sim \frac{4A^2\sigma^2}{N}$$

$\xi \sim \mathcal{N}(\mu, \sigma^2)$

$E(\xi^2) = \mu^2 + \sigma^2$

$E(\xi^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$

$$\boxed{\begin{aligned}\mathrm{var}(\xi^2) &= E[\xi^4] - E^2[\xi^2] \\ &= 4\mu^2\sigma^2 + 2\sigma^4\end{aligned}}$$

$$g(\bar{x}) \approx g(A) + \frac{dg(A)}{dA}(\bar{x} - A)$$

Taylor Series,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$E[g(\bar{x})] = g(A) = A^2$$

vector $\rightarrow$ lower case letter + bold fa..

* <u>vector parameter:-</u>

$$\vec{\theta} = [\theta_1 \ \theta_2 \ \cdots \ \theta_p]^T$$

<u>CRLB:-</u> $p(\vec{x}; \vec{\theta})$ satisfies regularity condition

$$E\left[\frac{\partial \ln p(\vec{x}; \vec{\theta})}{\partial \vec{\theta}}\right] = 0 \qquad \forall \vec{\theta}$$

The covariance matrix of any unbiased estimator $\hat{\theta}$ satisfies

$$\vec{C}_{\hat{\theta}} - I^{-1}(\vec{\theta}) \geq \vec{0}$$

$\vec{I}(\vec{\theta})$ = Fisher information matrix.

$$\vec{C}_{\hat{\theta}} = \vec{I}^{-1}(\vec{\theta})$$

$$\frac{\partial \ln p(\vec{x};\vec{\theta})}{\partial \vec{\theta}} = \vec{I}(\vec{\theta})\left(\hat{\vec{\theta}}(\vec{x}) - \vec{\theta}\right)$$