Lecture - 18

BLUE :-

$$\hat{\theta} = \frac{s^T C^{-1} x}{s^T C^{-1} s}$$

$$\mathrm{var}(\hat{\theta}) = \frac{1}{s^T C^{-1} s}.$$

\# $x[n] = A + w[n]$

[$w[n]$: zero mean uncorrelated noise with variance σ_n^2]

$$\hat{A} = \frac{\sum_n \frac{x[n]}{\sigma^2[n]}}{\sum_n \frac{1}{\sigma^2[n]}} \qquad \mathrm{var}(\hat{A}) = \frac{1}{\sum_{n=0}^{N-1} \frac{1}{\sigma_n^2}}.$$

Vector Parameter :-

$$\hat{\theta} = \vec{A}\vec{x}$$

$$E[\hat{\theta}] = A\,E[x] = A\theta.$$

$$E[x] = H\theta$$

$$\left[H = \begin{bmatrix} s[0] \\ s[1] \\ \vdots \\ s[N-1] \end{bmatrix} \right]$$

$$AH = I.$$

$$\vec{C}_\theta = A^T \vec{C} A \quad \text{s.t.} \quad AH = I$$

[$\vec{C}$ = Covariance matrix of $\vec{x}$]

$$\hat{\theta} = Ax = (H^T C^{-1} H)^{-1} H^T C^{-1} x$$

$$C_\theta = (H^T C^{-1} H)^{-1}$$

$$\boxed{\begin{aligned} x &= H\theta + w \\ \hat{\theta} &= \end{aligned}}$$

Gauss-Markov Theorem:-

If the data are of general linear model form $\vec{x} = \vec{H}\vec{\theta} + \vec{w}$

$\vec{H}$: $N \times P$ observation matrix
$\vec{\theta}$: $P \times 1$ vector
$\vec{w}$: $N \times 1$ noise with zero mean and covariance $\vec{C}$

then the BLUE of $\hat{\theta}$ is $\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} x$

$$\mathrm{var}(\hat{\theta}_i) = \left[(H^T C^{-1} H)^{-1} \right]_{ii}$$

$$\mathrm{cov}(\hat{\theta}) = (H^T C^{-1} H)^{-1}$$

Maximum-Likelihood Estimation (MLE):-

$*$ $x[n] = A + w[n]$ $\quad [w[n] \sim \mathcal{N}(0, A)]$

(i) MVUE using CRLB:-

$$p(x; A) = \frac{1}{(2\pi A)^{N/2}} \exp\left[-\frac{1}{2A} \sum_{n=0}^{N-1} (x[n] - A)^2 \right]$$

$$\frac{\partial \ln p(x; A)}{\partial A} = -\frac{N}{2A} + \frac{1}{A} \sum_n (x[n] - A) + \frac{1}{2A^2} \sum_n (x[n] - A)^2$$

$\stackrel{?}{=} I(\theta)(g(x) - \theta) \stackrel{?}{=} I(A)(\hat{A} - A)$

$$\mathrm{var}(\hat{A}) \geq B \qquad [B = ?]$$

(ii) Using Sufficient Statistic

$$\frac{1}{A}\sum_n \left(x[n]-A\right)^2 = \frac{1}{A}\sum_n x^2[n] - 2N\bar{x} + NA$$

$$p(x;A) = \underbrace{\frac{1}{(2\pi A)^{N/2}} \exp\left[-\frac{1}{2}\left(\frac{1}{A}\sum_n x^2[n] + NA\right)\right]}_{g\left(\sum_n x^2[n],\, A\right)} \underbrace{\exp\left[N\bar{x}\right]}_{h(x)}$$

$$T(x) = \sum_n x^2[n]$$

Assuming $T(x)$ is complete

(i) $$E\left[g\left(T(x)\right)\right] = A$$

$$E\left[g\left(\sum_n x^2[n]\right)\right] = A$$

$$E\left[\sum_n x^2[n]\right] = N\, E\left[x^2[n]\right]$$
$$= N\left[\mathrm{var}\left(x[n]\right) + \left(E\left[x[n]\right]\right)^2\right]$$
$$= N\left[A + A^2\right] \neq A$$

(ii) $$E\left(\check{A} \mid T(x)\right)$$ [$\check{A}$: unbiased estimator]

If $\check{A} = x[0]$

$$E\left(x[0] \mid \sum_n x^2[n]\right)$$

For MLE:

If $N \to \infty$: $E(\hat{A}) \to A$: $\mathrm{Var}(\hat{A}) \to$ CRLB

$$\hat{\theta} = \arg\max_{\theta} p(x;\theta) \quad \text{for } x \text{ fixed.}$$

\# $$p(x;A) = \frac{1}{(2\pi A)^{N/2}} \exp\left[-\frac{1}{2A}\sum_n (x[n]-A)^2\right]$$

$$\frac{\partial \ln p(x;A)}{\partial A} = -\frac{N}{2A} + \frac{1}{A}\sum_n (x[n]-A) + \frac{1}{2A^2}\sum_n (x[n]-A)^2 \longrightarrow 0$$

$ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\hat{A}^2 + \hat{A} - \frac{1}{N}\sum_n x^2[n] = 0$$

$$\hat{A} = -\frac{1}{2} \pm \sqrt{\frac{1}{N}\sum_n x^2[n] + \frac{1}{4}}$$

$$\hat{A} = -\frac{1}{2} + \sqrt{\frac{1}{N}\sum_n x^2[n] + \frac{1}{4}} \quad \text{to make } A>0$$

* $$x[n] = A + w[n] \qquad [w[n] \sim \mathcal{N}(0,\sigma^2)]$$

$$p(x;A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} (x[n]-A)^2\right]$$

$$\frac{\partial \ln p(x;A)}{\partial A} = \frac{1}{\sigma^2}\sum_n (x[n]-A) \longrightarrow 0$$

$$\hat{A} = \frac{1}{N}\sum_n x[n]$$

* If efficient exists, the MLE will produce it.

Assymptotic Property of the MLE:

If the PDF $p(x;\theta)$ satisfies some regularity conditions, then the MLE of the unknown parameter θ is "assymptotically distributed according to"

$$\hat{\theta} \overset{a}{\sim} \mathcal{N}(\theta, I^{-1}(\theta))$$

$I(\theta)$: Fisher information evaluated at θ.

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = I(\theta)(g(x) - \theta)$$

$$\hat{\theta}_{MVUE} = g(x)$$

$$I(\theta)(g(x) - \theta) = 0$$

$$\hat{\theta}_{MLE} = g(x) = \hat{\theta}_{MVUE}$$