Nonlinear Dynamics: Two-Dimensional Systems

One-dimensional system: $\dot{x} = f(x)$

Two-dimensional system: $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$

A Two-Dimensional, Autonomous Linear System:

$$\boxed{\dot{x} = ax + by \quad \& \quad \dot{y} = cx + dy}$$

a, b, c, and d are fixed parameters.

Write $\boxed{x \longrightarrow x_1}$ and $\boxed{y \longrightarrow x_2}$.

$$\therefore \boxed{\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2 \end{aligned}}$$

Autonomous and Linearly Coupled.

Matrix Form: (Matrix Multiplication)

$$\boxed{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

$$\Rightarrow \boxed{\dot{\vec{x}} = \tilde{A}\,\vec{x}} \text{ where } \tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\vec{x} \longrightarrow$ Vector field.

Any linear combination of x_1 and x_2 is also a solution.

The Oscillator Equation

$$\boxed{m\ddot{x} = -kx} \Rightarrow \ddot{x} + (k/m)x = 0 .$$

Define $\boxed{\omega^2 = k/m}$ $\Rightarrow$ $\boxed{\ddot{x} + \omega^2 x = 0}$.

Write $\boxed{\dot{x} = v}$ and $\boxed{\dot{v} = -\omega^2 x}$.

Vector field on a Phase plane: $\boxed{(\dot{x}, \dot{v}) \equiv (v, -\omega^2 x)}$

Now $\dfrac{\dot{v}}{\dot{x}} = \dfrac{-\omega^2 x}{v}$ $\Rightarrow$ $\boxed{\dfrac{dv/dt}{dx/dt} = \dfrac{dv}{dx} = \dfrac{-\omega^2 x}{v}}$

$$\Rightarrow \int v\,dv + \int \omega^2 x\,dx = \text{constant} .$$

$$\Rightarrow \frac{v^2}{2} + \omega^2 \frac{x^2}{2} = \frac{c^2}{2}$$

$$\Rightarrow \boxed{\frac{v^2}{c^2} + \frac{x^2}{c^2/\omega^2} = 1}$$

Fixed point is at $(0,0)$.

Equation of an ellipse.

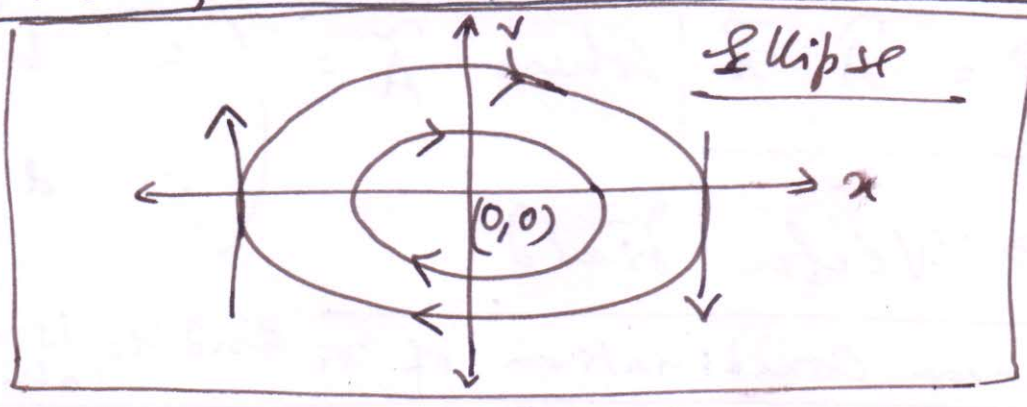
Now $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ $\Rightarrow$ when $v = 0$, $(\dot{x}, \dot{v}) = (0, -\omega^2 x)$.

$\therefore$ ~~when~~ $x > 0$, $\dot{v} < 0$, and ~~when~~ $x < 0$, $\dot{v} > 0$.

Similarly, when $x = 0$, $(\dot{x}, \dot{v}) = (v, 0)$.

$\therefore$ $v > 0$, $\dot{x} > 0$, and $v < 0$, $\dot{x} < 0$

Phase trajectories



Fixed point at the Centre, $(0,0)$

Example: An Uncoupled System

$\boxed{\dot{x} = ax}$, and $\boxed{\dot{y} = -y}$ (0,0) is the fixed point. $\downarrow$ $\dot{x} = 0$, $\dot{y} = 0$

The System is $\boxed{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$

Solve $\boxed{\dot{x} = ax} \Rightarrow \frac{dx}{dt} = ax \Rightarrow \boxed{x = x_0 e^{at}}$

and $\dot{y} = -y \Rightarrow \frac{dy}{dt} = -y \Rightarrow \boxed{y = y_0 e^{-t}}$

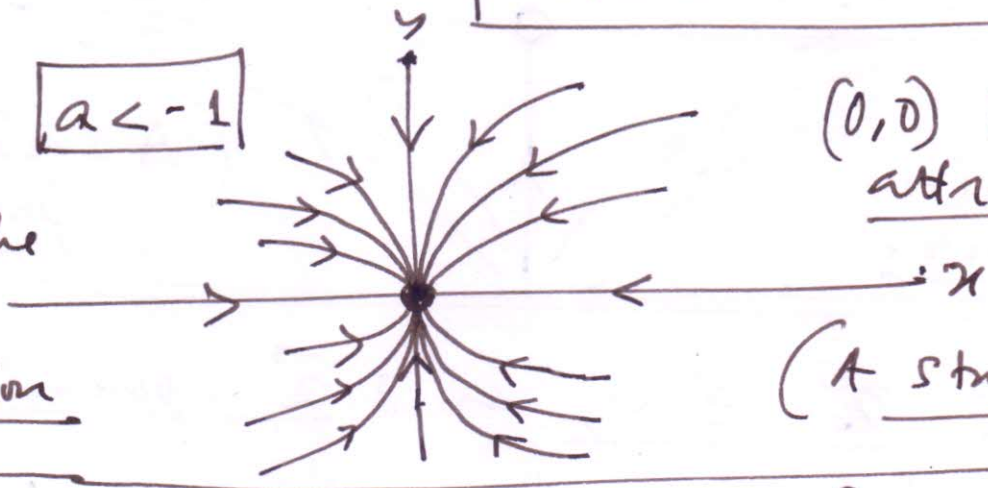
$$\therefore \boxed{\left(\frac{x}{x_0}\right)\left(\frac{y}{y_0}\right)^a = e^{at} e^{-at} = 1}$$

Define $\boxed{Y = y/y_0}$ and $\boxed{X = x/x_0}$ $\therefore$ $\boxed{Y = X^{-1/a}}$

Plotting 5 Cases: $\boxed{a < -1,\ a = -1,\ -1 < a < 0,\ a = 0 \text{ and } a > 0.}$

Case I: $\boxed{a < -1}$

y is the slower direction

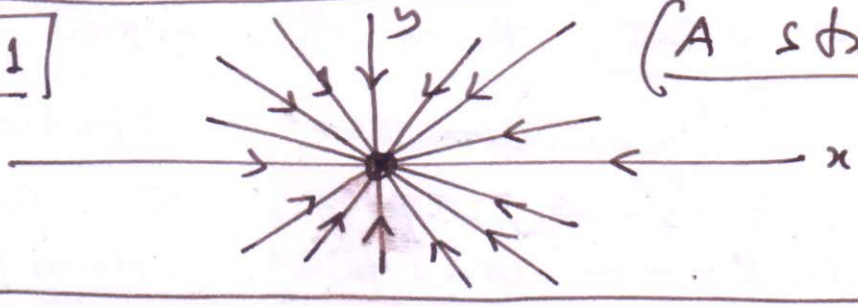


(0,0) is an attractor

(A stable Node)

Case II: $\boxed{a = -1}$

(0,0) is an attractor

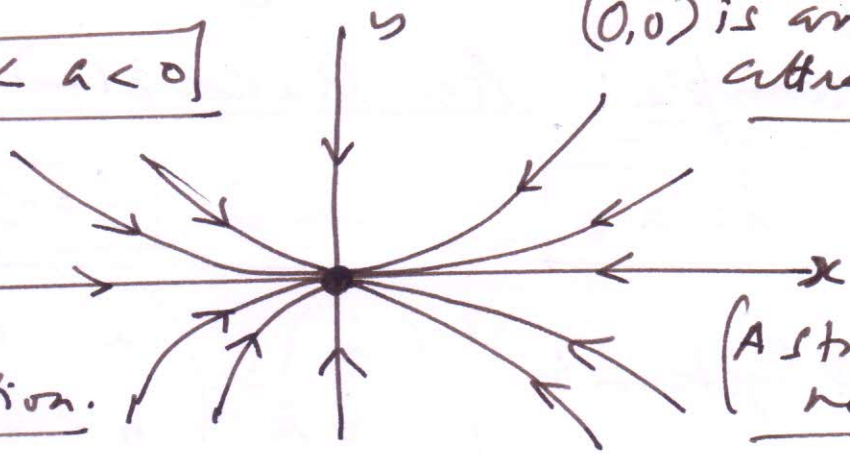


(A stn)

Case III : $\boxed{-1 < a < 0}$

(0,0) is an attractor

Approach to the fixed point is tangential to the slower direction.

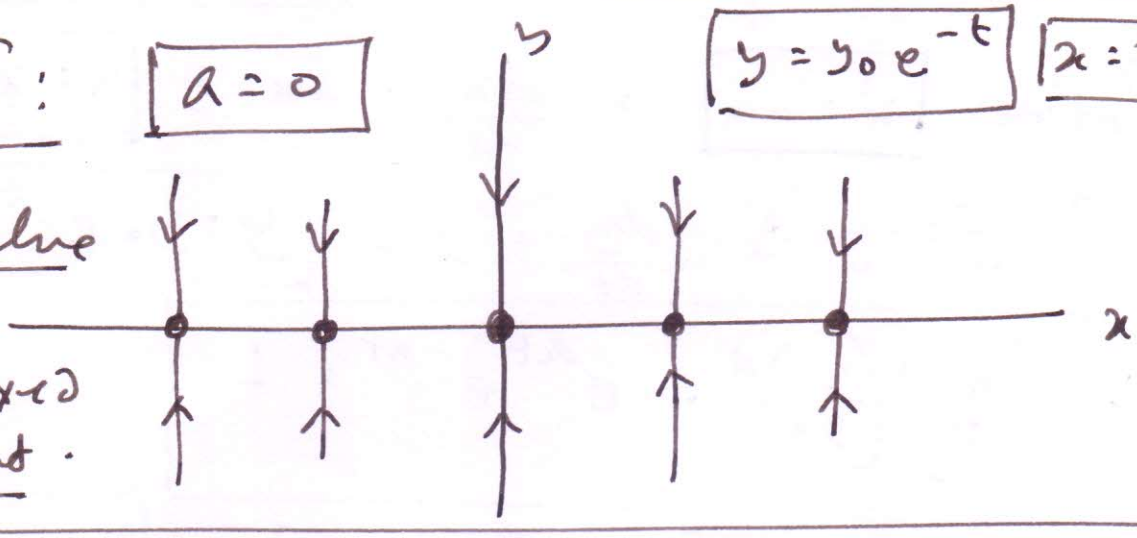


(A stable node)

Case IV : $\boxed{a = 0}$

$\boxed{y = y_0 e^{-t}}$ $\boxed{x = x_0}$

Any value of x is a fixed point.



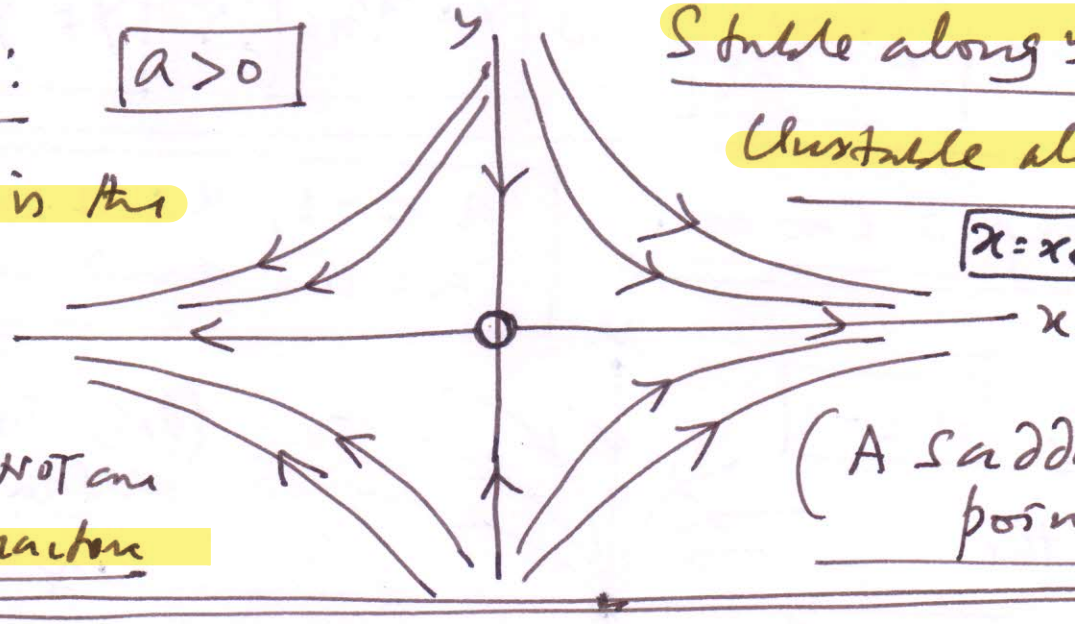
Case V : $\boxed{a > 0}$

Stable along y

Unstable along x

$\boxed{x = x_0 e^{at}}$

(0,0) is the fixed point but NOT an attractor



(A saddle point)

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1. When $t \to \infty$, and $\vec{x}$ approaches the fixed point, then the fixed point is attracting. This defines a stable manifold.
2. When $t \to -\infty$, approach to a fixed point defines an unstable manifold.

Classification of Linear Systems

Write a solution $\boxed{\vec{x}(t) = \vec{v}\, e^{\lambda t}}$.

$$\Rightarrow \boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v_1 & e^{\lambda t} \\ v_2 & e^{\lambda t} \end{pmatrix}}$$

$\vec{v}$ is a fixed vector, and $\vec{v} \neq \vec{0}$.

$$\therefore \quad \dot{\vec{x}} = \lambda \vec{v}\, e^{\lambda t} = \lambda \vec{x}.$$

Further $\boxed{\dot{\vec{x}} = \tilde{A}\, \vec{x}}$ The matrix relation

$$\Rightarrow \quad \lambda \vec{v}\, \cancel{e^{\lambda t}} = \tilde{A}\, \vec{v}\, \cancel{e^{\lambda t}} \Rightarrow \tilde{A}\vec{v} = \lambda \vec{v}$$

$$\Rightarrow \quad \tilde{A}\vec{v} - \lambda \vec{v} = \vec{0} \Rightarrow \boxed{(\tilde{A} - \lambda \tilde{I})\vec{v} = \vec{0}}$$

Since $\vec{v} \neq \vec{0} \Rightarrow \boxed{\tilde{A} - \lambda \tilde{I} = \tilde{0}}$ $\quad \boxed{\tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$\vec{v} \rightarrow$ Eigenvector

$\lambda \rightarrow$ Eigenvalue, $\quad \boxed{\vec{x}(t) = \vec{v}\, e^{\lambda t}} \rightarrow$ Eigensolution.

Characteristic Equation of a Linear Two-Dimensional System.

$$\boxed{\det(\tilde{A} - \lambda \tilde{I}) = 0}$$

(Fixed point is at $(0,0)$)

where $\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\lambda \tilde{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

A two-dimensional system is described by a 2×2 matrix.

$$\Rightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\Rightarrow ad - \lambda d - a\lambda + \lambda^2 - bc = 0.$$

$$\Rightarrow \boxed{\lambda^2 - (a+d)\lambda + (ad-bc) = 0}$$

This is like

$$\boxed{\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}}$$

Fixed point is due to

$$\boxed{\dot{x} = \dot{y} = 0}$$

$\Rightarrow$ $x = 0$ and $y = 0$

$\therefore$ $(0,0)$ is the fixed point.

$\boxed{\text{Trace, } \tau = a + d}$, $\boxed{\text{Determinant } \Delta = ad - bc}$.

Quadratic Characteristic Equation:

$$\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$$

$$\Rightarrow \boxed{\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

Two solution.

$$\boxed{\begin{aligned} \lambda_1 &= \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \\ \text{and} \quad \lambda_2 &= \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \end{aligned}}$$

Two eigenvalues

Linear Superposition of two solutions:

$$\boxed{\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}}$$

Different Cases of Δ and τ :

Case 1: $\boxed{\Delta < 0}$ $\Rightarrow (-\Delta) > 0$

$\therefore$ λ_1, λ_2 are real with opposite signs.

The fixed point is a saddle point.

Case II: $\boxed{\Delta > 0}$ There are 4 subcases.

Subcase i: $\boxed{\tau^2 - 4\Delta < 0}$

$\therefore$ $\lambda_1 \lambda_2$ are complex conjugates

If $\tau > 0$, $\Rightarrow$ unstable spiral
If $\tau < 0$, $\Rightarrow$ stable spiral
} Fixed points.

Subcase ii: $\boxed{\tau^2 - 4\Delta > 0}$

$\therefore$ λ_1, λ_2 are real with same signs.

because $\tau > \sqrt{\tau^2 - 4\Delta}$

Fixed points

If $\tau > 0$, $\lambda_1, \lambda_2 > 0$ $\Rightarrow$ unstable node
If $\tau < 0$, $\lambda_1, \lambda_2 < 0$ $\Rightarrow$ stable node

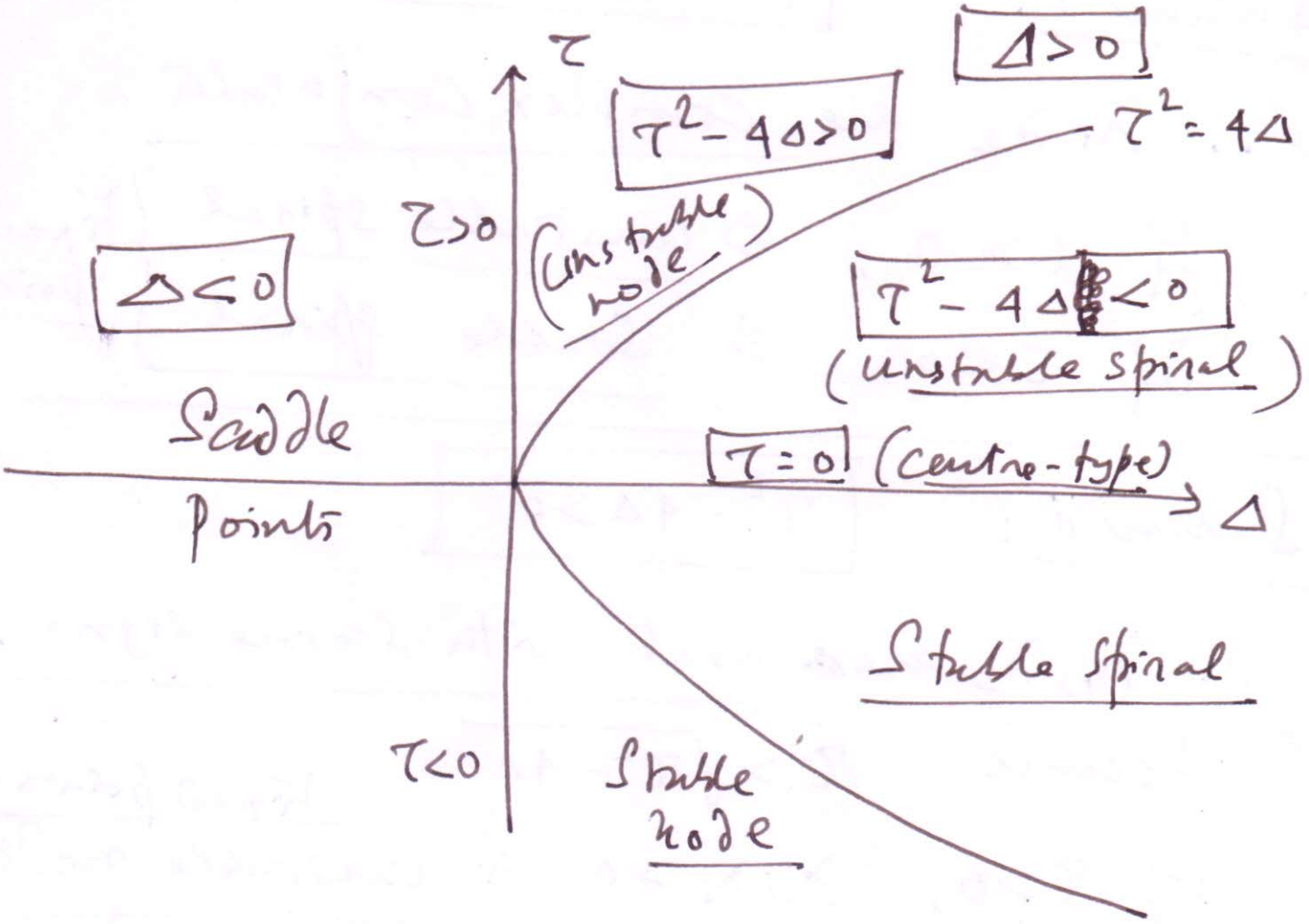
Subcase iii : $\boxed{\tau = 0}$

λ_1, λ_2 are purely imaginary.

$\Rightarrow$ The fixed point is centre-type.

Subcase iv : $\boxed{\tau^2 = 4\Delta}$ $\rightarrow$ Borderline case.

Parameter Space Plot



Borderline has stars and degenerate nodes.

Example: $\boxed{\dot{x} = x + y}$ and $\boxed{\dot{y} = 4x - 2y}$

The fixed point is: $\boxed{\dot{x} = \dot{y} = 0}$ $\boxed{\Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array}}$

$\boxed{x + y = 0}$ and $\boxed{4x - 2y = 0}$ $\Rightarrow$ $(x_c, y_c) = (0,0)$.

The matrix is $\tilde{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

Trace Determinant

$\Rightarrow$ $\boxed{\tau = -1}$ and $\boxed{\Delta = -2 - 4 = -6}$

The eigenvalues are $\boxed{\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm 5}{2} = 2, -3 .$$

Hence real eigenvalues with opposite signs $\Rightarrow$ The fixed point is a saddle point.

Finding Eigenvectors: $\boxed{(\tilde{A} - \lambda \tilde{I})\vec{v} = \vec{0}}$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Big| \quad \lambda = 2, -3$$

We get $\boxed{\begin{array}{l} (1-\lambda)v_1 + v_2 = 0 \\ 4v_1 - (2+\lambda)v_2 = 0 \end{array}}$ Two linear equations

and

i/. When $\lambda = 2$, we have $-v_1 + v_2 = 0$
and $4v_1 - 4v_2 = 0$.

$\Rightarrow v_2 = v_1 \quad \therefore$ $\boxed{\text{The eigenvectors are } (1,1)}$

ii/. When $\lambda = -3$, we have $4v_1 + v_2 = 0$
and $4v_1 + v_2 = 0$.

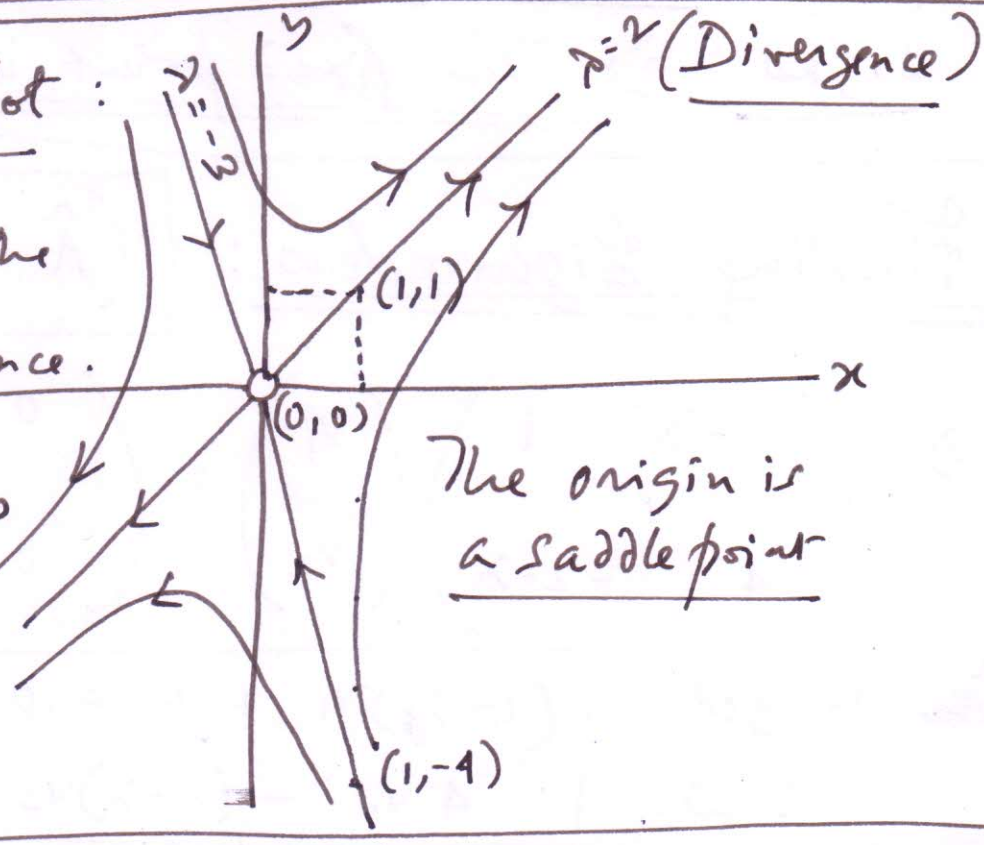
$\Rightarrow v_2 = -4v_1 \quad \therefore$ $\boxed{\text{The eigenvectors are } (1,-4)}$

In general for a coupled set of linear equations,
$\boxed{\dot{x} = ax + by}$ and $\boxed{\dot{y} = cx + dy}$, the
eigenvectors are given by,
$\boxed{(a-\lambda)v_1 + bv_2 = 0}$ and $\boxed{cv_1 + (d-\lambda)v_2 = 0}$.

The phase plot:

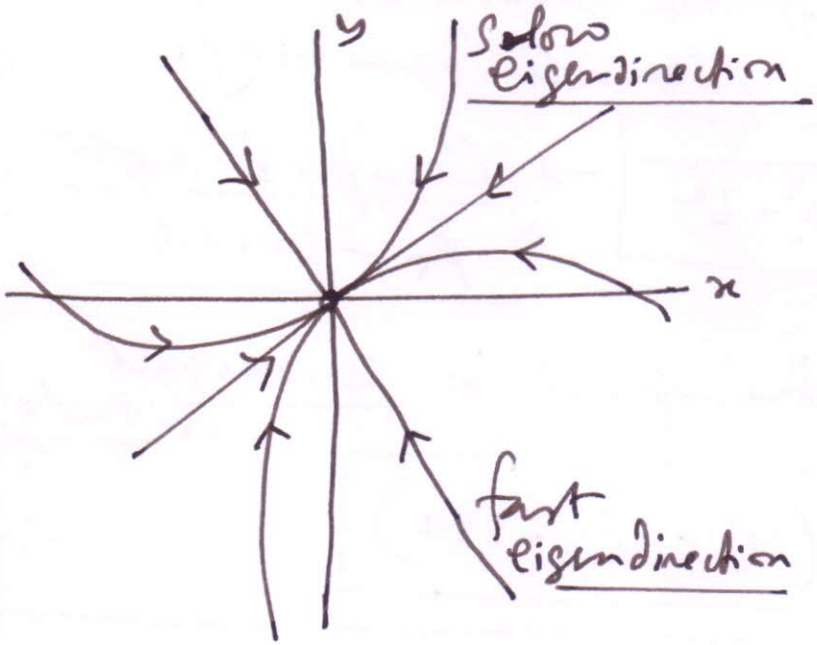
$\lambda = -3$ gives the direction of convergence.

The eigenvectors are $(1,1)$ and $(1,-4)$.



The origin is a saddle point

Example: If $\boxed{\lambda_1, \lambda_2 < 0}$ and $\boxed{\lambda_2 < \lambda_1 < 0}$, i.e, $\boxed{|\lambda_1| < |\lambda_2|}$ then we have a stable node, in which λ_1 gives the slow eigendirection.



Solutions approach tangentially along the slow eigendirection.

If $\lambda_1, \lambda_2 > 0$, the arrows reverse direction $\Rightarrow$ unstable

Example: A Damped Oscillator

$$\boxed{\ddot{x} + 2b\dot{x} + \omega^2 x = 0}$$

When $b = 0$, we get an undamped simple harmonic oscillator.

The fixed point conditions are $\boxed{\dot{x} = \dot{v} = 0}$, in the coupled system, $\boxed{\dot{x} = v}$ and $\boxed{\dot{v} = -\omega^2 x - 2bv}$.

The fixed points are $\boxed{v = 0 \text{ and } x = 0}$.

The matrix is $A \equiv \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2b \end{pmatrix}$ | Trace $\tau = -2b$ and $\Delta = \omega^2$.

The <u>eigenvalues</u> are $\boxed{\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$.

$$\Rightarrow \lambda = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2} = -b \pm \sqrt{b^2 - \omega^2}$$

Suppose $|b| < |\omega| \Rightarrow$ An oscillator with a small damping.

$$\Rightarrow \boxed{\lambda = -b \pm i\sqrt{\omega^2 - b^2}} \rightarrow \text{Complex roots}$$

$\Rightarrow$ A spiral

If $b > 0$, then the spiral is <u>convergent</u> (stable)

Now $\boxed{(\dot{x}, \dot{v}) \equiv (v, -\omega^2 x - 2bv)}$

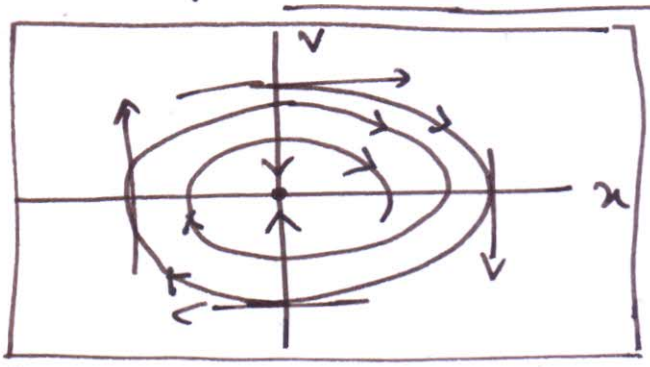
i/. When $v = 0$, $(\dot{x}, \dot{v}) \equiv (0, -\omega^2 x)$

$\Rightarrow$ If $x > 0$, $\dot{v} < 0$ and if $x < 0$, $\dot{v} > 0$.

ii/. When $x = 0$, $(\dot{x}, \dot{v}) \equiv (v, -2bv)$

$\Rightarrow$ If $v > 0$, $\dot{x} > 0$ and if $v < 0$, $\dot{x} < 0$.

And if $v > 0$, $\dot{v} < 0$ and if $v < 0$, $\dot{v} > 0$



With $b \neq 0$, the dissipative effect causes closing in of the trajectories, along the v axis.

Nonlinear Systems

A linear system:
$$\boxed{\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}}$$

Anything else is nonlinear.

Example:
$$\boxed{\begin{aligned}\dot{x} &= e^{-y} + xy + y^3 \\ \dot{y} &= \sin x + x^2/y + \ln y\end{aligned}}$$

Write a nonlinear system generally as

$$\boxed{\dot{x} = f(x,y)} \quad \text{and} \quad \boxed{\dot{y} = g(x,y)}$$

The fixed point condition is $\dot{x} = \dot{y} = 0$ at (x_c, y_c). $\Rightarrow$ $\boxed{f(x_c, y_c) = g(x_c, y_c) = 0}$

Perturb about the fixed point values as

$$\boxed{x = x_c + x'} \Rightarrow \boxed{\dot{x} = \dot{x}'} \quad (\because \dot{x}_c = 0)$$

and $$\boxed{y = y_c + y'} \Rightarrow \boxed{\dot{y} = \dot{y}'} \quad (\because \dot{y}_c = 0)$$

We perform a Taylor expansion as,

$$\boxed{\begin{aligned}\dot{x} = \dot{x}' &= f(x_c + x', y_c + y') \\ \dot{y} = \dot{y}' &= g(x_c + x', y_c + y')\end{aligned}}.$$

$$\therefore \quad \boxed{\begin{aligned} \dot{x}' &= f(x_c, y_c) + \left.\frac{\partial f}{\partial x}\right|_{x_c, y_c} x' + \left.\frac{\partial f}{\partial y}\right|_{x_c, y_c} y' + \cdots \\ \text{and} \quad \dot{y}' &= g(x_c, y_c) + \left.\frac{\partial g}{\partial x}\right|_{x_c, y_c} x' + \left.\frac{\partial g}{\partial y}\right|_{x_c, y_c} y' + \cdots \end{aligned}}$$

The above two equations only retain the <u>linear orders</u> of <u>x' and y'</u>.

$\because$ $\boxed{f(x_c, y_c) = g(x_c, y_c) = 0}$, we get.

$$\boxed{\begin{aligned} \dot{x}' &= \left.\frac{\partial f}{\partial x}\right|_{x_c, y_c} x' + \left.\frac{\partial f}{\partial y}\right|_{x_c, y_c} y' \\ \dot{y}' &= \left.\frac{\partial g}{\partial x}\right|_{x_c, y_c} x' + \left.\frac{\partial g}{\partial y}\right|_{x_c, y_c} y' \end{aligned}}$$

(<u>linear</u> and <u>coupled</u> set of equations)

which is in the form $\boxed{\begin{aligned} \dot{x}' &= a x' + b y' \\ \dot{y}' &= c x' + d y' \end{aligned}}$ and

We write the linear coupled set in matrix form as

$$\boxed{\begin{pmatrix} \dot{x}' \\ \dot{y}' \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial f}{\partial x}\right|_{x_c, y_c} & \left.\frac{\partial f}{\partial y}\right|_{x_c, y_c} \\ \left.\frac{\partial g}{\partial x}\right|_{x_c, y_c} & \left.\frac{\partial g}{\partial y}\right|_{x_c, y_c} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}}.$$

<u>In vector form</u> $\boxed{\dot{\vec{x}}' = \tilde{A}\, \vec{x}'}$, in which $\tilde{A}$ is known as the <u>JACOBEAN matrix</u>.

The Jacobian matrix is written as,

$$\boxed{\tilde{A} \equiv \begin{pmatrix} \left.\frac{\partial f}{\partial x}\right|_{x_c, y_c} & \left.\frac{\partial f}{\partial y}\right|_{x_c, y_c} \\ \left.\frac{\partial g}{\partial x}\right|_{x_c, y_c} & \left.\frac{\partial g}{\partial y}\right|_{x_c, y_c} \end{pmatrix}}$$

This is the equivalent of $f'(x_c)$ in first-order systems.

NOTE: Linearised systems correctly indicate saddle points, nodes and spirals. This may NOT be true for centre-type points.

Hence, saddle points, nodes and spirals are known as ROBUST cases.

Attractors: $\lambda_1, \lambda_2 > 0$, with positive real parts.

Repellers: $\lambda_1, \lambda_2 < 0$, with negative real parts.

Saddles: One λ is positive, the other is negative.

Stability/Instability $\Rightarrow$ Convergence/Divergence towards/away from fixed points.

Closed Orbits $\Rightarrow$ Periodic solutions without either convergence or divergence.

Existence and Uniqueness

1. f and g are continuous and are continuously differentiable for $x \equiv x(t)$ and $y \equiv y(t)$ to exist.
2. Trajectories never intersect on the phase plane. Otherwise the solutions passing through the intersection point will not be unique.
3. A trajectory inside a closed orbit is trapped forever.

Poincaré-Bendixson Theorem: A bounded trajectory inside a closed ~~fixed point~~ orbit, without any fixed points, will ultimately approach a closed orbit $\longrightarrow$ A limit cycle.

Nullclines: Either $\boxed{\dot{x} = 0}$ OR $\boxed{\dot{y} = 0}$, i.e. the flow is either purely horizontal or purely vertical.

Example:

$$\boxed{\begin{aligned} \dot{x} &= -x + x^3 = f(x,y) \\ \dot{y} &= -2y = g(x,y) \end{aligned}}$$

An uncoupled system.

When $\boxed{\dot{x} = \dot{y} = 0}$, we get ~~the~~ three fixed points at $\boxed{y = 0}$ and $\boxed{x(x^2-1) = 0}$ $\Rightarrow$ $\boxed{x = 0, \pm 1}$.

The Jacobian matrix $\tilde{A} \equiv \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$

1/. When $\boxed{x = 0}$, $\tilde{A} \equiv \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ $\Rightarrow$ $\boxed{\begin{aligned} \tau &= -3 \\ \Delta &= 2 \end{aligned}}$

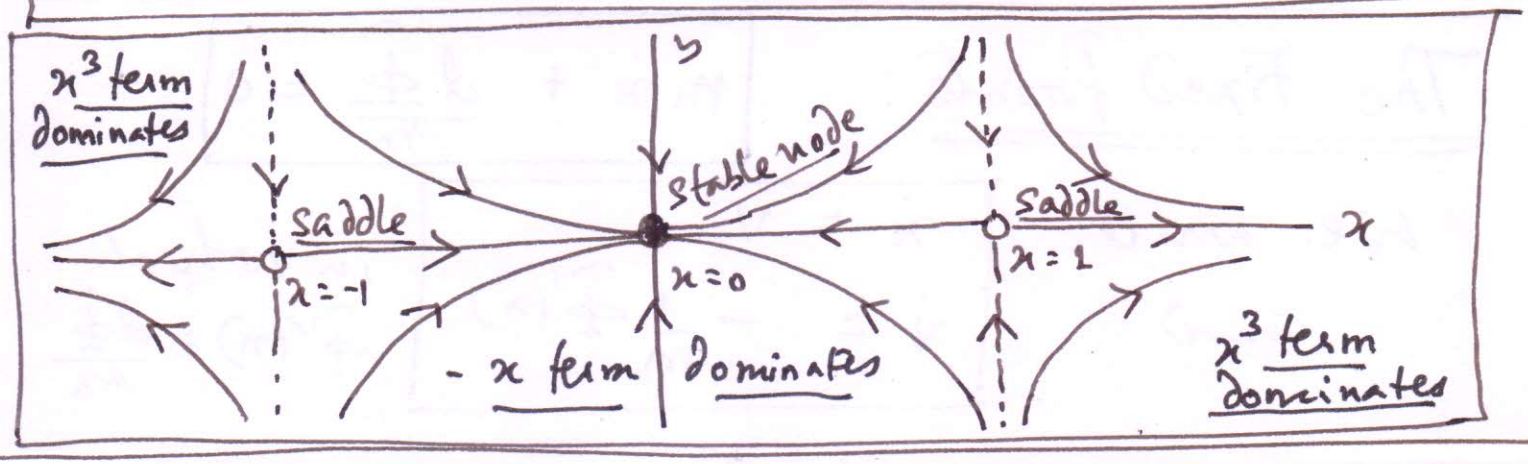
$$\Rightarrow \lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{-3 \pm 1}{2} \Rightarrow \lambda = -2, -1.$$

$\therefore$ $\boxed{x = 0}$ gives a stable node.

2/. When $\boxed{x = \pm 1}$ $\tilde{A} \equiv \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\Rightarrow$ $\boxed{\begin{aligned} \tau &= 0 \\ \Delta &= -4 \end{aligned}}$

$$\Rightarrow \lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \pm 4$$

$\therefore$ $\boxed{x = \pm 1}$ give two saddle points



Conservative Systems

Start with $\boxed{F = ma}$, where $\boxed{a = \frac{dv}{dt}}$

$$\Rightarrow \boxed{a = \frac{d^2x}{dt^2}} \Rightarrow \boxed{F = m\ddot{x}}$$

$$\therefore \boxed{m\dot{x}\ddot{x} - F\dot{x} = 0}$$ in which $\boxed{\dot{x} = v}$.

$$\Rightarrow m v \frac{dv}{dt} - F\frac{dx}{dt} = 0$$

Now we require $\boxed{F \equiv F(x) = -\frac{d\Phi}{dx}}$

$\Phi \rightarrow$ Potential function

$$\Rightarrow m v \frac{dv}{dt} + \frac{d\Phi}{\cancel{dx}}\frac{\cancel{dx}}{dt} = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left[\frac{1}{2}mv^2 + \Phi\right] = 0}$$

$$\Rightarrow \boxed{\frac{1}{2}mv^2 + \Phi(x) = \mathcal{E}}$$

$\mathcal{E} \rightarrow$ Constant in time

This is a <u>Conservative system</u> for ANY potential, $\Phi(x)$, starting from $\boxed{m\ddot{x} = F(x)}$.

The Fixed Points:

$$\boxed{m\ddot{x} + \frac{d\Phi}{dx} = 0}$$

We write

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{1}{m}\Phi'(x) \end{aligned}}$$

and

in which $\Phi'(x) = \frac{d\Phi}{dx}$

The <u>fixed points</u> are given by $\boxed{\dot{x} = \dot{v} = 0}$.

$\Rightarrow$ $\boxed{v = 0}$ and $\boxed{\Phi'(x_c) = 0}$.

The <u>fixed points</u> are given by the <u>turning points</u> of $\Phi(x)$ $[\because \Phi'(x) = 0]$.

<u>The Nature of the Fixed Points</u>: $\boxed{\begin{aligned} x &= x_c + x' \\ v &= 0 + v' \end{aligned}}$.

$\therefore \dot{x}' = v'$ and $\dot{v}' = -\frac{1}{m}\left[\Phi'(x_c) + \Phi''(x_c)x'\right]$

Since $\Phi'(x_c) = 0$, we get,

$$\boxed{\begin{aligned} \dot{x}' &= 0x' + v' \\ \dot{v} &= -\frac{1}{m}\Phi''(x_c)x' + 0v' \end{aligned}}$$

<u>Linear</u>, <u>coupled</u> equations in <u>x' and v'</u>.

$$\therefore \tilde{A} \equiv \begin{pmatrix} 0 & 1 \\ -\frac{\Phi''(x_c)}{m} & 0 \end{pmatrix} \Rightarrow \boxed{\begin{aligned} &\tau = 0 \text{ and} \\ &\Delta = \frac{\Phi''(x_c)}{m} \end{aligned}}$$

$$\boxed{\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \pm\sqrt{-\frac{\Phi''(x_c)}{m}}}$$

i/. If $\Phi''(x_c) > 0$, the fixed point is <u>centre-type</u>.

ii/. If $\Phi''(x_c) < 0$, the fixed point is a <u>saddle point</u>.

iii/. Centre-type points are <u>ROBUST</u> in a <u>conservative system</u>, given as $\boxed{\ddot{x} = f(x)}$.

Example : $\boxed{\Phi(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4}$

$\Rightarrow \boxed{\Phi'(x) = -x + x^3}$ The fixed points are given by $\Phi(x_i) = 0$.

$\Rightarrow \Phi'(x) = 0 = -x + x^3 = 0$.

$\Rightarrow \boxed{x = 0}$ and $\boxed{x = \pm 1}$.

Now $\ddot{x} = f(x) = -\Phi'(x)$ (assuming $\boxed{m = 1}$).

$\Rightarrow \boxed{\dot{x} = v}$ and $\boxed{\dot{v} = -\Phi'(x)}$.

$$\boxed{\tilde{A} \equiv \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}} \text{ from } \tilde{A} \equiv \begin{pmatrix} 0 & 1 \\ -\frac{\Phi''(x_i)}{m} & 0 \end{pmatrix}$$

When $\boxed{x = 0}$

$$\tilde{A} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow \tau = 0, \quad \Delta = -1$

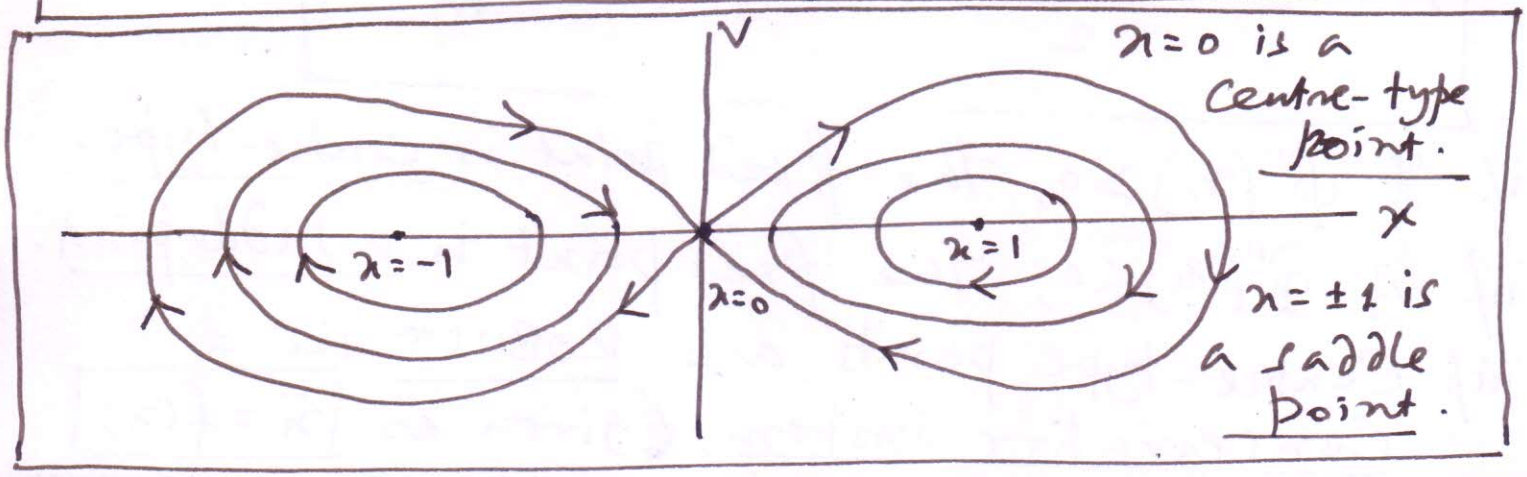
$\Rightarrow x = 0$ is a <u>saddle point</u>

When $\boxed{x = \pm 1}$

$$\tilde{A} \equiv \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

$\tau = 0, \quad \Delta = 2$

$\Rightarrow x = \pm 1$ is a <u>centre-type point</u>.



Reversible Systems

Example: $\boxed{m\ddot{x} = m\frac{d^2x}{dt^2} = F(x)}$ is an equation that remains invariant under the transformation $\boxed{t \to -t}$, i.e. there is time-reversal symmetry. $\boxed{\ddot{x} = f(x)}$

Further $\boxed{\dot{x} = y}$ and $\boxed{\dot{y} = \frac{F(x)}{m}}$ remain invariant, when $\boxed{t \to -t}$ and $\boxed{y \to -y}$.

This is a reversible system.

Generally, for $\boxed{\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}}$, there is time reversal symmetry $(t \to -t)$ when

$f(x,y)$ is an odd function of y,

i.e., $\boxed{f(x,-y) = -f(x,y)}$

and $g(x,y)$ is an even function of y,

i.e., $\boxed{g(x,-y) = g(x,y)}$.

This is the mathematical requirement of reversibility.

1/. All <u>Conservative systems</u> are <u>reversible systems</u>. $\boxed{\ddot{x} = f(x)}$

$\boxed{\dot{x} = y} \rightarrow$ <u>odd when $y \rightarrow -y$</u>

$\boxed{\dot{y} = f(x)} \rightarrow$ <u>Does not depend on y</u>.

$\therefore$ $\boxed{\ddot{x} = f(x) \text{ is reversible under } t \rightarrow -t}$.

2/. <u>Centre-type points</u> are <u>ROBUST</u> in a <u>reversible system</u>.

<u>Example</u>: <u>Damped system</u> (<u>Irreversible system</u>).

$\boxed{\ddot{x} + 2b\dot{x} + \omega^2 x = 0}$ (A <u>damped oscillator</u>)

Coupled system $\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2bv - \omega^2 x \end{aligned}}$ $\rightarrow$ The second equation is not <u>an even function in v</u>.

<u>Example</u>: $\boxed{\ddot{x} + 2b\dot{x}^2 + \omega^2 x = 0}$

Write a coupled system as,

$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2bv^2 - \omega^2 x \end{aligned}}$ $\rightarrow$ The second equation is now an <u>even function in v</u>.

This is a <u>reversible system</u>, but NOT a <u>Conservative system</u>.