Differential Equations

Contain Derivatives

Consider the equation of a straight line $\boxed{y = mx + c}$, with m and c being fixed parameters.

Taking the first derivative we get,

$$\boxed{\frac{dy}{dx} = m}$$

and the second derivative gives us $\boxed{\frac{d^2y}{dx^2} = 0}$.

i/. Successive derivatives reduce the number of fixed parameters. This implies greater generalisation and more universal relevance.

ii/. Derivatives capture changes, and are relevant for evolving systems.

These are the two advantages of working with differential equations.

Changes in an independent variable, t, ("time", but it can be anything else).

We use a differential equation to express changes of a variable, x, in time, t.

dependent variable, x $\longrightarrow$ Population, Capital, height, position, etc.

$\boxed{\dfrac{dx}{dt}}$ $\longrightarrow$ Rate at which x changes with t.

Since $\boxed{x \equiv x(t)}$, i.e. x depends on ONLY one variable, we get a full derivative (or ordinary derivative) in t. This requires an ordinary differential equation.

Orders of a differential equation:

A.) First-order: Highest derivative is $\boxed{\dfrac{dx}{dt}}$.

B.) Second-order: Highest derivative is $\boxed{\dfrac{d^2x}{dt^2}}$.

Examples:

A.) First-order ordinary differential equation

$$\boxed{\frac{dx}{dt} = x}$$ Eg. compound interest.

B.) Second-order ordinary differential equation.

$$\boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0}$$ Eg. damped oscillator.

Order of the differential equation ~~derivatives~~ = the number of initial (or boundary) conditions required in an integral solution.

If there are more than one independent variables, as in $\boxed{\psi(x,t)}$, then we have a partial differential equation, such as The Diffusion (or Heat) Equation:

$$\boxed{\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial x^2}}$$

which requires one initial condition (first order in t) and two boundary conditions (second order in space).

Order = Initial   Order = Boundary

The Wave Equation : $$\boxed{\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2}}$$

Requires two initial conditions and two boundary conditions, because it is second order in t and also second order in x.

Second-order Differential Equations

Consider Newton's Second Law.

$$\boxed{F = k\,m\,a} \Rightarrow \boxed{F = m\frac{d^2x}{dt^2}} \quad (k = 1)$$

Now we write $$\boxed{\frac{d^2x}{dt^2} = \frac{F(x,t)}{m}},$$

in which we substitute,

$$\boxed{\frac{dx}{dt} = v} \Rightarrow \boxed{x = x(t)}$$

and $$\boxed{\frac{dv}{dt} = \frac{F(x,t)}{m}} \Rightarrow \boxed{v = v(t)}$$

At ~~at~~ a given time, $t = t_0$, two initial conditions are required, $x(t_0)$ (an initial position) and $v(t_0)$ (an initial velocity). The former specifies the state and the latter the rate at which the state is changing (velocity).

Rate $\propto$ State : $\boxed{\dfrac{dx}{dt} \propto x}$

First - Order System

We consider a system $\boxed{\dfrac{dx}{dt} = \pm a x}$ in which $\boxed{a > 0}$. (geometric growth for $+a$)

$\boxed{+ \text{ sign} \Rightarrow \text{growth}}$ $\boxed{- \text{ sign} \Rightarrow \text{decay}}$

Rescaling : $\boxed{\dfrac{dx}{d(at)} = \pm x}$

~~Now~~ Now we rescale $\boxed{T = at}$, and get $\boxed{\dfrac{dx}{dT} = \pm x}$ $\boxed{\boxed{x=0} \text{ is a trivial solution.}}$

Separation of Variables : $\boxed{\displaystyle\int \frac{dx}{x} = \pm \int dT}$

$\Rightarrow \boxed{\ln x = \ln A \pm \ln e^{T}}$ $\boxed{A \to \text{integral constant}}$

$\Rightarrow \boxed{x = A e^{\pm T}} \Rightarrow \boxed{x = A e^{\pm at}}$

A ~~linear~~ First-Order Autonomous Ordinary Differential Equation : $\boxed{\dfrac{dx}{dt} = f(x)}$

$\boxed{\dfrac{dx}{dt} = f(x) = a \pm bx}$ $\boxed{a, b > 0}$

(An autonomous form)

$\boxed{\boxed{\dfrac{dx}{dt} = f(x,t)} \rightarrow \text{is in a NON-AUTONOMOUS form.}}$

Transformation of variables:

Write $\boxed{y = a \pm bx}$. $\Rightarrow$ $\boxed{\dfrac{dy}{dt} = \pm b \dfrac{dx}{dt}}$

But $\boxed{\dfrac{dx}{dt} = a \pm bx = y}$.

Hence, $\boxed{\dfrac{dy}{dt} = \pm by}$, which we

rescale to get $\boxed{\dfrac{dy}{d(bt)} = \pm y}$ $\boxed{T = bt}$.

and, therefore, $\boxed{\dfrac{dy}{dT} = \pm y}$. This

equation is in the rate $\propto$ state form.

Its solution is $\boxed{y = c\, e^{\pm T}}$, as before.

$\Rightarrow$ $\boxed{a \pm bx = c\, e^{\pm bt}}$ $c \to$ Integration Constant

$\Rightarrow$ $\mp bx = a - c e^{\pm bt}$

$\Rightarrow$ $\mp x = \dfrac{a}{b} - \dfrac{c}{b} e^{\pm bt}$

$\Rightarrow$ $\boxed{x = \mp \left(\dfrac{a}{b} - \dfrac{c}{b} e^{\pm bt} \right)}$.

The choice of the lower sign gives $\boxed{x = \dfrac{a}{b} - \dfrac{c}{b} e^{-bt}}$ from $\boxed{\dfrac{dx}{dt} = a - bx}$

Solving $\boxed{\dfrac{dx}{dt} = a - bx}$ where $\boxed{a, b > 0}$

Separation of variables: $\boxed{\dfrac{dx}{f(x)} = dt}$

$$\Rightarrow \boxed{\frac{dx}{a-bx} = dt} \Rightarrow \boxed{\int \frac{d(-bx)}{a-bx} = \int d(-bt)}$$

$$\Rightarrow \boxed{\ln(a-bx) = \ln c - bt = \ln c + \ln e^{-bt}}$$

$$\Rightarrow \boxed{a - bx = c\,e^{-bt}} \Rightarrow bx = a - c\,e^{-bt}$$

$$\Rightarrow \boxed{x = \frac{a}{b} - \frac{c}{b}\,e^{-bt}}$$

$c \rightarrow$ Integration Constant

Since we started with a first-order differential equation in t, we require ONE INITIAL condition, which is when $\boxed{t = 0, \quad x = 0}$. $\Rightarrow \boxed{0 = \dfrac{a}{b} - \dfrac{c}{b}e^{-b0}}$

$\Rightarrow \boxed{c = a}$, by which we get.

$$\boxed{x = \frac{a}{b}\left(1 - e^{-bt}\right)}$$

. We now define a scale for x as $\boxed{x_0 = a/b}$ and a scale for t as $\boxed{\tau = 1/b}$. Using these scales

We can write $\boxed{x = x_0 \left(1 - e^{-t/\tau}\right)}$

(Alternative approach)

<u>Rescaling</u> $\boxed{X = \frac{x}{x_0}}$ and $\boxed{T = \frac{t}{\tau}}$, we get $\boxed{X = 1 - e^{-T}}$. We can also perform a <u>rescaling</u> on $\boxed{\frac{dx}{dt} = a - bx}$ to obtain $\boxed{X = 1 - e^{-T}}$. This can be done as $\boxed{\frac{1}{b}\frac{dx}{dt} = \frac{a}{b} - x}$.

$\Rightarrow \boxed{\frac{dx}{d(bt)} = \frac{a}{b} - x}$. Since $\boxed{T = bt}$ and $\boxed{x_0 = a/b}$

we write $\boxed{\frac{dx}{dT} = x_0 - x}$. <u>($x_0$ and τ are NATURAL scales)</u>

$\Rightarrow \boxed{\frac{d(x/x_0)}{dT} = 1 - (x/x_0)}$. Since $\boxed{X = \frac{x}{x_0}}$

we finally get $\boxed{\frac{dX}{dT} = 1 - X}$, a <u>rescaled</u> <u>(parameter free)</u> <u>differential equation</u> whose <u>solution</u> is as before, $\boxed{X = 1 - e^{-T}}$. The <u>limiting cases</u> of this solution are when $\boxed{T = 0, \quad X = 0}$ and when <u>$T \to \infty$</u>, <u>$X \to 1$</u>, which is a <u>convergence</u> to a <u>finite value</u>.

Plotting $\boxed{X = 1 - e^{-T}}$ (Plotting by hand)

i.) We know that when $\boxed{T = 0}$, $\boxed{X = 0}$.

Now, when $\boxed{0 < T \ll 1}$, we expand

$$\boxed{e^{-T} = 1 - T + \frac{T^2}{2!} - \frac{T^3}{3!} + \cdots}$$ Infinite Series

Successive terms in this series diminish very rapidly since $T \ll 1$. Hence,

$$\boxed{e^{-T} \simeq 1 - T} \text{ when } T \ll 1$$ (Small time limit)

$$\therefore \boxed{X = 1 - e^{-T} \simeq \cancel{1} - (\cancel{1} - T)}$$

$$\Rightarrow \boxed{X \simeq T} \Rightarrow \boxed{\frac{x}{x_0} \simeq \frac{t}{\tau}}.$$

$$\therefore x \simeq x_0 \frac{t}{\tau} = \frac{a}{\cancel{b}} \cancel{b} t \simeq at$$

Hence, for (small time) $t \ll \tau$ (or $T \ll 1$), $\boxed{x \simeq at}$.

ii.) In the opposite limit when $T \to \infty$. (long time limit)

$$\boxed{X = 1 - e^{-\infty} = 1} \Rightarrow X \to 1, \text{ when } T \to \infty.$$

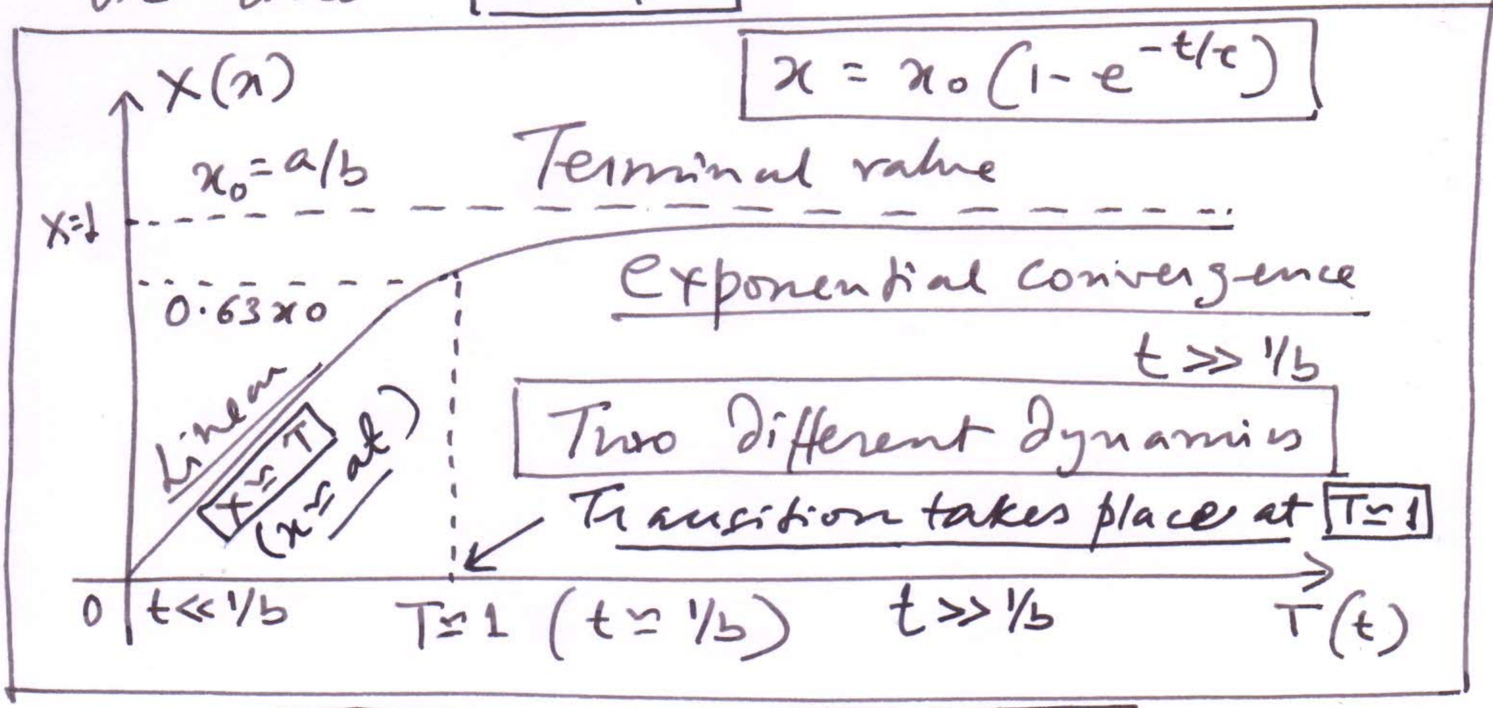
$\Rightarrow \boxed{x \to x_0 = a/b}$, when $\boxed{t \to \infty}$. So for long time scale, x converges towards a limiting value of a/b.

iii.) We can obtain the derivative of $\boxed{x = 1 - e^{-T}}$, as $\boxed{\frac{dx}{dT} = e^{-T}}$. If $\boxed{\frac{dx}{dT} = 0}$ $\Rightarrow$ $\boxed{T \to \infty}$.

The second derivative is $\boxed{\frac{d^2x}{dT^2} = -e^{-T}}$

when $\boxed{T \to \infty, \ \frac{d^2x}{dT^2} = 0}$. Hence this is not a turning point ↗ $\boxed{\text{At } x = 1}$.

iv.) The transition from the linear behaviour $\boxed{x \simeq T}$ to an exponential convergence of $\boxed{x = 1 - e^{-T}}$ takes place when $\boxed{T \simeq 1}$ or when $\boxed{t \simeq 1/b}$ (natural time scale).



When $\boxed{t = \tau}$, $\boxed{x = x_0(1 - e^{-1})}$ $\Rightarrow$ $\boxed{x \simeq 0.63x_0}$

There are two different dynamics on two different time scales. Eg. Growth of humans or the inflationary Universe.

Systems of the form: $\boxed{\dfrac{dx}{dt} = a + bx}$

$\boxed{a, b > 0}$

We know when $\boxed{\dfrac{dx}{dt} = a - bx}$ (with $a, b > 0$)

the solution is $\boxed{x = \frac{a}{b}\left(1 - e^{-bt}\right)}$. When

$\boxed{\dfrac{dx}{dt} = a + bx = a - (-b)x}$, we make the transformation $\boxed{b \to -b}$

Hence, $\boxed{x = \dfrac{a}{-b}\left(1 - e^{bt}\right)}$

$\Rightarrow \boxed{x = \frac{a}{b}\left(e^{bt} - 1\right)}$ is the solution of $\boxed{\dfrac{dx}{dt} = a + bx}$.

Writing $\boxed{x_0 = a/b}$ and $\boxed{\tau = 1/b}$, we get

$\boxed{x = x_0\left(e^{t/\tau} - 1\right)}$ OR $\boxed{X = e^{T} - 1}$ $\left(\begin{array}{c} X = x/x_0 \\ \text{and } T = t/\tau \end{array}\right)$

$\tau = 1/b$ is the natural time scale.

Limiting behaviour:

$\left(e^{t/\tau} = 1 + t/\tau + t^2/2!\tau^2 + \dots\right)$

i.) when $\boxed{t \ll \tau}$, $\boxed{e^{t/\tau} \simeq 1 + t/\tau}$ (linear order only)
(small time)

$\therefore x \simeq x_0\left(\cancel{1} + \frac{t}{\tau} - \cancel{1}\right) = x_0 \frac{t}{\tau} = at$.

$\Rightarrow \boxed{x = at}$ (early growth is linear).

ii.) when $t \gg \tau$, $e^{t/\tau} - 1 \simeq e^{t/\tau}$ (for long time).

$\therefore \boxed{x = x_0 e^{t/\tau}}$ (late growth is exponential)

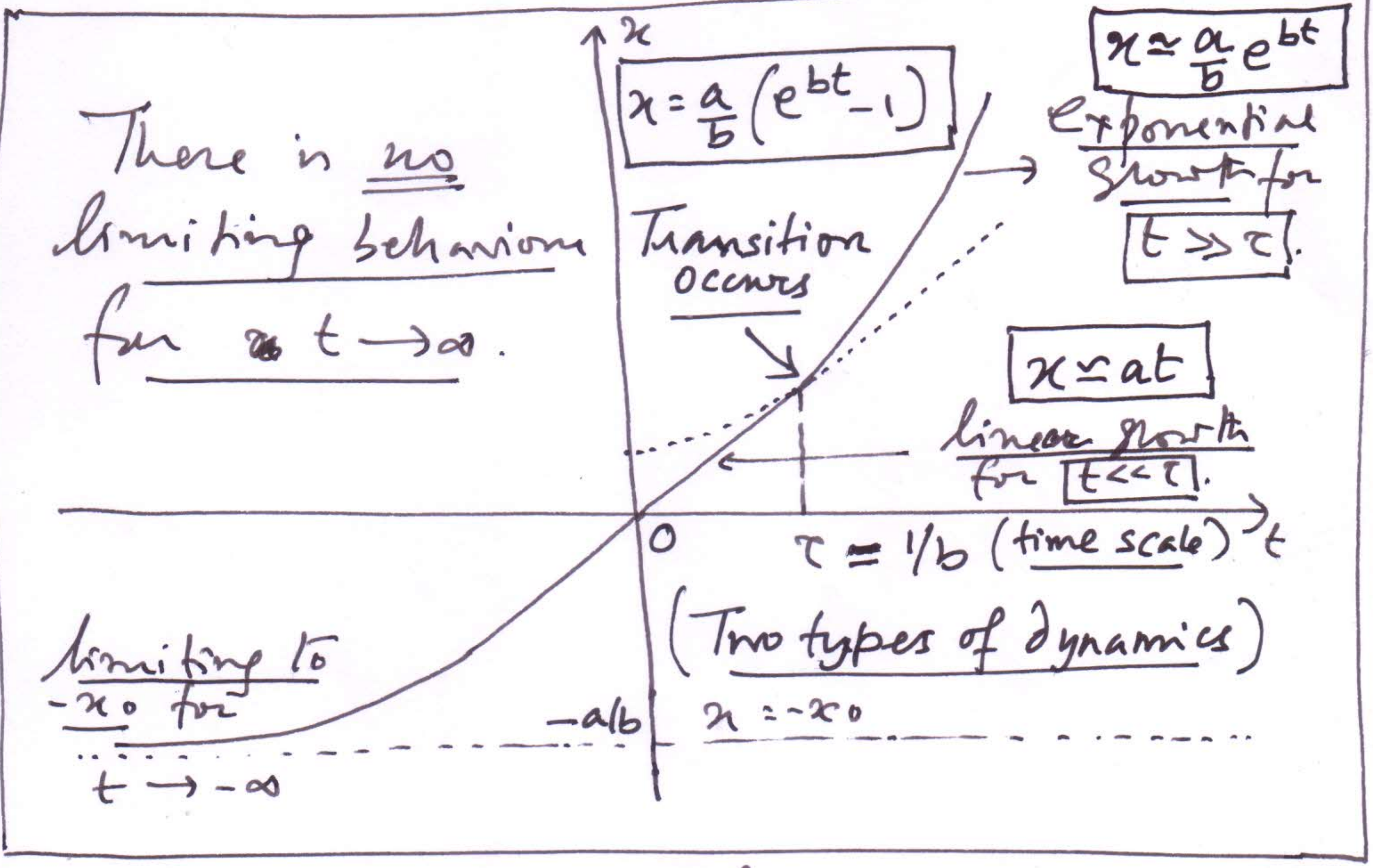
Consider a hypothetical case when $\boxed{t<0}$.

iii) For $t \to -\infty$, $\boxed{x \to -x_0}$ (limiting terminal value)

iv) For $|t| \ll \tau$, $\boxed{e^{t/\tau} \simeq 1 + t/\tau}$ (linear order)

$\Rightarrow \quad x \simeq x_0 t/\tau \quad \Rightarrow \quad \boxed{x \simeq at}$ (linear)

Plotting: $\boxed{x = x_0\left(e^{t/\tau} - 1\right)}$ $\boxed{x_0 = a/b}$ $\boxed{\tau = 1/b}$



There is an exchange of the functional behaviour from the first to the third quadrant as $\boxed{\frac{dx}{dt} = a - bx}$ goes to $\boxed{\frac{dx}{dt} = a + bx}$.

The Logistic Equation (Second Order of Nonlinearity)

Write an equation $\boxed{c\dfrac{dx}{dt} = Ax - Bx^2}$

$\Rightarrow$ $\boxed{\dfrac{dx}{dt} = ax - bx^2}$, $\boxed{a, b > 0}$ $\boxed{\begin{array}{c} a = A/c \\ b = B/c \end{array}}$

When $x \to 0$ $\boxed{\dfrac{dx}{dt} \simeq ax}$ (Rate $\propto$ State)

$\Rightarrow$ $\boxed{x \simeq x_0 e^{at}}$ $\Rightarrow$ Early growth is exponential

When x is large, $-bx^2$ inhibits and saturates growth (as in population growth)

Rescaling of variables : $\boxed{\dfrac{dx}{dt} = ax\left(1 - \dfrac{bx}{a}\right)}$

Define $\boxed{K = a/b}$ $\to$ (Carrying capacity)

$\Rightarrow$ $\boxed{\dfrac{dx}{dt} = ax\left(1 - \dfrac{x}{K}\right)}$

$\Rightarrow$ $\dfrac{d}{d(at)}\left(\dfrac{x}{K}\right) = \left(\dfrac{x}{K}\right)\left(1 - \dfrac{x}{K}\right)$

i) x is scaled by K.
ii) t is scaled by $1/a$.

Define $\boxed{X = x/K}$ and $\boxed{T = at = \dfrac{t}{1/a}}$

$\Rightarrow$ $\boxed{\dfrac{dX}{dT} = X(1-X)}$ The rescaled logistic equation

Integral Solution:
(Separation of variables)

$$\boxed{\int \frac{dx}{x(1-x)} = \int dT}$$

Now, by the method of partial fractions,

$$\boxed{\frac{1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x}} \Rightarrow \boxed{1 \equiv A(1-x) + Bx}$$

i.) When $x = 1$, $B = 1$, ii) When $x = 0$, $A = 1$

$$\Rightarrow \boxed{\int \frac{dx}{x(1-x)} = \int \frac{dx}{x} + \int \frac{dx}{1-x} = \int dT}$$

$$\Rightarrow \int \frac{dx}{x} - \int \frac{d(-x)}{1-x} = \int dT$$

C is an integration constant

$$\Rightarrow \boxed{\ln x - \ln(1-x) = \ln e^T + \ln C}$$

$$\Rightarrow \frac{x}{1-x} = Ce^T \Rightarrow x = Ce^T - xCe^T$$

$$\Rightarrow \boxed{x(1 + Ce^T) = Ce^T}$$

$$\Rightarrow \boxed{x = \frac{Ce^T}{1+Ce^T} = \frac{1}{1+C^{-1}e^{-T}}}$$

When $\boxed{T = 0 \ (\text{i.e. } t = 0)}$, $\boxed{x = x_0 \ (\text{or } x = x_0)}$.
(The initial value must NOT be zero)

$$\Rightarrow \boxed{x_0 = \frac{1}{1+C^{-1}}} \boxed{\Rightarrow 1 + C^{-1} = \frac{1}{x_0}}$$

$$\Rightarrow \quad \frac{1}{C} = \frac{1}{X_0} - 1 \quad \Rightarrow \quad \boxed{\frac{1}{C} = \frac{1 - X_0}{X_0}}$$

$$\Rightarrow \quad \boxed{C = \frac{X_0}{1 - X_0} = \frac{x_0/k}{1 - x_0/k} = \frac{x_0}{k - x_0}}.$$

Returning to variables x and t we get,

$$\boxed{X = \frac{x}{k} = \frac{1}{1 + c^{-1} e^{-T}} = \frac{1}{1 + c^{-1} e^{-at}}} \Rightarrow \boxed{x = \frac{k}{1 + c^{-1} e^{-at}}}$$

i.) When $t \rightarrow \infty$, $x \rightarrow k$ (The limiting carrying capacity). (OR $X \rightarrow 1$) (for ANY initial value)

Further, $$\boxed{x = \frac{k\, e^{at}}{\left(\frac{k - x_0}{x_0}\right) + e^{at}} = \frac{x_0 k\, e^{at}}{(k - x_0) + x_0 e^{at}}}$$

$$\Rightarrow \quad \boxed{x = \frac{x_0 k\, e^{at}}{k + x_0 (e^{at} - 1)} = \frac{x_0\, e^{at}}{1 + \frac{x_0}{k}(e^{at} - 1)}}.$$

ii.) When $\boxed{t \ll a^{-1}}$, $(t \rightarrow 0)$ in the early growth stage.

(linear order only)

$$e^{at} - 1 \simeq \cancel{1} + at \cancel{- 1} \simeq at \rightarrow 0.$$

Hence, e^{at} in the numerator determines the dynamics, compared to $e^{at} - 1$ in the denominator.

$\Rightarrow \boxed{x \simeq x_0 e^{at}}$ in the early growth, but this also appears as if due to $\boxed{k \rightarrow \infty}$, i.e. an infinite carrying capacity ($b = 0$).

Going back to $\boxed{\dfrac{dx}{dT} = x(1-x) = f(x)}$

we see that starting from $\boxed{x = x_0}$ and tending towards $\boxed{x \to 1}$ (the upper limit), $\boxed{\dfrac{dx}{dT} > 0}$, i.e. there is always growth.

Now $\boxed{\dfrac{d^2x}{dT^2} = \dfrac{df}{dT} = \dfrac{df}{dx} \cdot \dfrac{dx}{dT}}$ $\quad \dfrac{df}{dx} = 0 \Rightarrow \boxed{x = 1/2}$

$\boxed{f(x) = x(1-x) = x - x^2} \Rightarrow \boxed{\dfrac{df}{dx} = 1 - 2x}$.

i.) when $x < 1/2$, $\dfrac{df}{dx} > 0$,

ii.) when $x > 1/2$, $\dfrac{df}{dx} < 0$.

f(x) has a TURNING POINT at $x = 1/2$

Since $\boxed{\dfrac{dx}{dT} > 0}$ for any FINITE value of T,

we see that $\boxed{\text{for } x < 1/2, \ \dfrac{d^2x}{dT^2} > 0}$, i.e.

growth occurs at an increasing rate. On the other hand $\boxed{\text{for } x > 1/2, \ \dfrac{d^2x}{dT^2} < 0}$,

i.e. growth occurs at a decreasing rate. This means that before $x = 1/2$, the growth is exponential, and beyond $x = 1/2$, the growth starts slowing down towards the carrying capacity.

Hence, $\boxed{X = 1/2}$ is the point where the NONLINEAR effect starts to be functional.

The corresponding time scale T_{nl} (the nonlinear time scale) can be obtained by

$$\boxed{X = \frac{1}{2} = \frac{1}{1 + c^{-1} e^{-T_{nl}}}} \Rightarrow 2 = 1 + c^{-1} e^{-T_{nl}}$$

$$\Rightarrow c^{-1} e^{-T_{nl}} = 1 \Rightarrow \boxed{c\, e^{T_{nl}} = 1}.$$

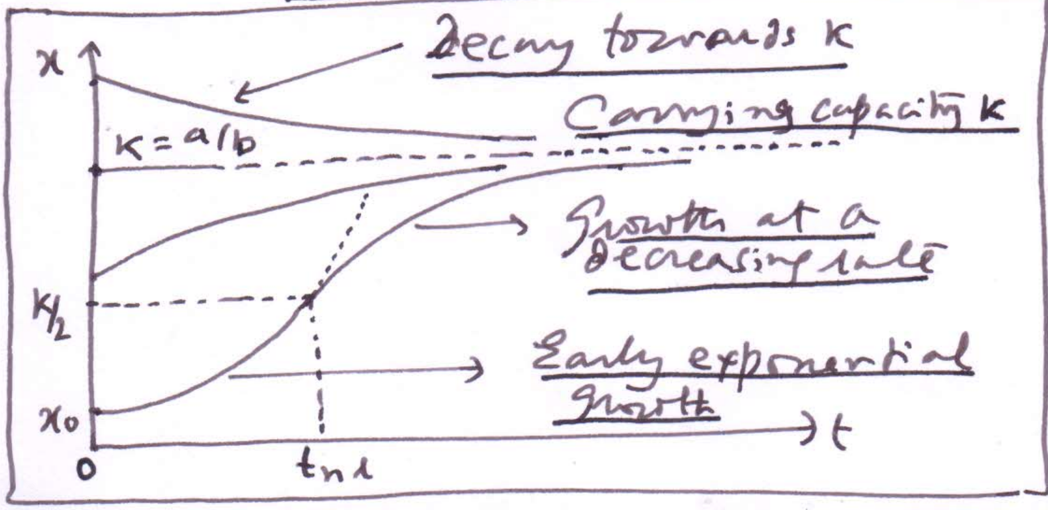
$$\Rightarrow \boxed{T_{nl} = \ln\left(\frac{1}{c}\right) = \ln\left(\frac{1 - X_0}{X_0}\right)}$$

Hence $\boxed{a\, t_{nl} = \ln\left(\frac{1 - x_0/K}{x_0/K}\right) = \ln\left(\frac{K - x_0}{x_0}\right)}$

$\Rightarrow \boxed{t_{nl} = \frac{1}{a} \ln\left(\frac{K}{x_0} - 1\right)}$ Realistically $t_{nl} > 0$.

This can only happen if $\boxed{\frac{K}{x_0} - 1 > 1}$

$\Rightarrow \boxed{\frac{K}{x_0} > 2} \Rightarrow \boxed{x_0 < K/2}$ Needed for strong growth



i.) For $\boxed{\frac{K}{2} < x_0 < K}$ there will be ONLY growth at a decreasing rate.

ii) For $\boxed{x_0 > K}$, there will be ONLY DECAY

Higher Orders of Nonlinearity: Logistic-Type Equation

$$\boxed{\frac{dx}{dt} = ax - bx^{\alpha+1}} \quad \boxed{\alpha \geq 2}, \quad \boxed{\alpha \in \mathbb{Z}}$$

$$\Rightarrow \frac{dx}{dt} = ax\left(1 - \frac{bx^\alpha}{a}\right) = ax\left(1 - \frac{x^\alpha}{a/b}\right)$$

Now transform $\boxed{\xi = x^\alpha}$ $\Rightarrow$ $\boxed{d\xi = \frac{\alpha x^\alpha}{x}\, dx}$

$$\Rightarrow \boxed{\frac{d\xi}{dt} = \alpha \frac{\xi}{x} \frac{dx}{dt}} \Rightarrow \boxed{\frac{dx}{dt} = \frac{d\xi}{dt} \cdot \frac{x}{\alpha \xi}}$$

Hence $\frac{d\xi}{dt} \frac{\cancel{x}}{\alpha \xi} = a\cancel{x}\left(1 - \frac{\xi}{k}\right)$ $\boxed{k = a/b}$

$$\Rightarrow \boxed{\frac{d\xi}{dt} = a\alpha\,\xi\left(1 - \frac{\xi}{k}\right)}$$

Now rescale $\boxed{X = \xi/k}$

and $\boxed{T = a\alpha t}$,

$$\Rightarrow \boxed{\frac{d}{d(a\alpha t)}\left(\frac{\xi}{k}\right) = \frac{\xi}{k}\left(1 - \frac{\xi}{k}\right)}$$

$$\Rightarrow \boxed{\frac{dX}{dT} = X(1-X)}$$

is a familiar rescaled form. (logistic equation)

$$\Rightarrow \boxed{X = \frac{\xi}{k} = \frac{1}{1 + C^{-1}e^{-T}} = \frac{1}{1 + C^{-1}e^{-a\alpha t}}}$$

$$\boxed{C = \frac{X_0}{1 - X_0} = \frac{\xi_0/k}{1 - \xi_0/k} = \frac{\xi_0}{k - \xi_0} = \frac{x_0^\alpha}{k - x_0^\alpha}}$$

$$\Rightarrow \boxed{\xi = \frac{k e^{a\alpha t}}{C^{-1} + e^{a\alpha t}}} \quad \boxed{C^{-1} = \frac{k - x_0^\alpha}{x_0^\alpha} = \frac{k - \xi_0}{\xi_0}}$$

$$\Rightarrow \boxed{x^\alpha = \frac{k\, e^{a\alpha t}}{\left(\frac{k - x_0^\alpha}{x_0^\alpha}\right) + e^{a\alpha t}} = \frac{k\, x_0^\alpha e^{a\alpha t}}{(k - x_0^\alpha) + x_0^\alpha e^{a\alpha t}}}$$

$$\Rightarrow \boxed{x^\alpha = \frac{x_0^\alpha e^{a\alpha t}}{1 + \frac{x_0^\alpha}{k}\left(e^{a\alpha t} - 1\right)}}$$

$$\Rightarrow \boxed{x = \frac{x_0\, e^{at}}{\left[1 + \frac{x_0^\alpha}{k}\left(e^{a\alpha t} - 1\right)\right]^{1/\alpha}}}$$

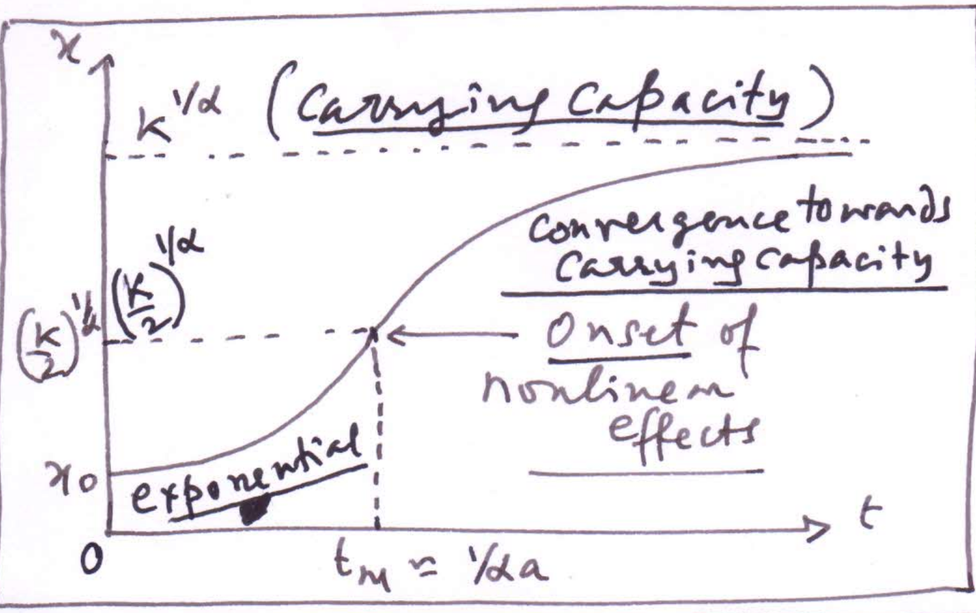
For $\boxed{t \to 0}$ Exponential Early Growth as $\boxed{x \simeq x_0 e^{at}}$

From $\boxed{x^\alpha = \frac{k}{1 + c^{-1} e^{-a\alpha t}}}$, we see that for $\boxed{t \to \infty}$, $\boxed{x \longrightarrow k^{1/\alpha}}$, i.e. the Carrying capacity has been reduced to $k^{1/\alpha}$.

Nonlinear Time Scale: $\boxed{T_{nl} = \ln\left(\frac{1}{c}\right)}$.

$$\Rightarrow \boxed{t_{nl} = \frac{1}{\alpha a} \ln\left(\frac{k - x_0^\alpha}{x_0^\alpha}\right) = \frac{1}{a\alpha} \ln\left(\frac{k}{x_0^\alpha} - 1\right)}$$

Realistically for $t_{nl} > 0$, $\boxed{\frac{k}{x_0^\alpha} - 1 > 1} \Rightarrow \boxed{x_0 < \left(\frac{k}{2}\right)^{1/\alpha}}$



For $\alpha \geq 2$, the Carrying Capacity is $k^{1/\alpha}$ in x. In ξ it is κ, and in X it is 1.

i/. The Carrying Capacity is reduced.

ii/. The nonlinear time is ~~also~~ also reduced.

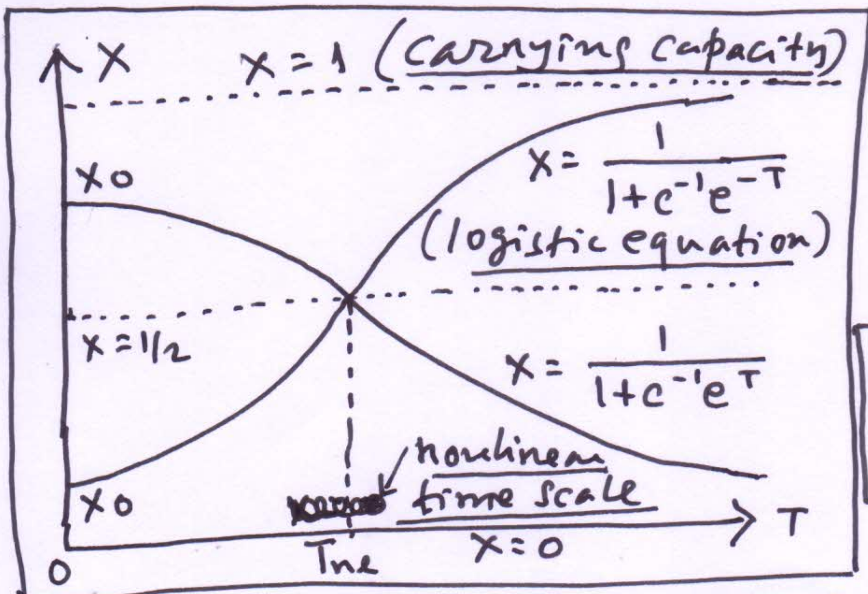
An Equation of the form $\boxed{\frac{dx}{dT} = -x(1-x)}$

$\Rightarrow \boxed{\frac{dx}{d(-T)} = x(1-x)}$ → This equation is the equivalent of $\boxed{T \longrightarrow -T}$ in the logistic equation, $\boxed{\frac{dx}{dT} = x(1-x)}$. Its solution is $\boxed{x = \frac{1}{1+c^{-1}e^{-T}}}$

∴ Transforming the rescaled time $\boxed{T \longrightarrow -T}$.

gives $\boxed{x = \frac{1}{1+c^{-1}e^{T}}}$ ⇒ when $\boxed{T \longrightarrow \infty}$, $\boxed{x \longrightarrow 0}$.

At $\boxed{T = 0}$, $\boxed{x = x_0}$ (the initial condition).



This solution can be compared with the Fermi-Dirac Distribution function.

$$\boxed{f(\epsilon) = \frac{1}{1+e^{(\epsilon-\epsilon_F)/k_BT}}}$$

$T \rightarrow$ temperature (constant) (not to be confused with rescaled time)

$\epsilon_F \rightarrow$ Parameter (constant) (Fermi Energy)

$$\therefore \boxed{f(\epsilon) = \frac{1}{1+e^{-\epsilon_F/k_BT}\, e^{\epsilon/k_BT}}}$$

Inverting the solution of the logistic equation gives the Fermi-Dirac type solution.

An Equation of the form $\boxed{\dfrac{dx}{dt} = a - bx^2}$

We write $\boxed{\dfrac{1}{a}\dfrac{dx}{dt} = 1 - \dfrac{x^2}{a/b}}$ and

define $\boxed{X = \dfrac{x}{\sqrt{a/b}}}$ $\Rightarrow$ $\dfrac{\sqrt{a/b}}{a}\dfrac{dX}{dt} = 1 - X^2$

Now also define $\boxed{T = \sqrt{ab}\, t}$, to get

$$\boxed{\frac{dX}{dT} = 1 - X^2} \Rightarrow \boxed{\int \frac{dX}{(1-X)(1+X)} = \int dT}$$

Using the method of partial fractions,

$$\boxed{\frac{1}{(1-X)(1+X)} = \frac{A}{1-X} + \frac{B}{1+X}} \Rightarrow 1 = A(1+X) + B(1-X)$$

i.) When $X = 1$. $\Rightarrow$ $1 = A.2$ $\Rightarrow$ $\boxed{A = 1/2}$.

ii.) When $X = -1$. $\Rightarrow$ $1 = B.2$ $\Rightarrow$ $\boxed{B = 1/2}$.

$$\Rightarrow \int \frac{dX}{(1-X)(1+X)} = \frac{1}{2}\int \frac{dX}{1-X} + \frac{1}{2}\int \frac{dX}{1+X} = \int dT$$

$$\Rightarrow \int \frac{dX}{1+X} - \int \frac{d(-X)}{1+(-X)} = 2\int dT$$

$$\Rightarrow \boxed{\ln(1+X) - \ln(1-X) = 2T + c}$$

The initial condition CAN be $x = 0$ at $t = 0$.

When $\boxed{t = 0,\ \text{i.e., } T = 0}$ and $\boxed{x = 0,\ \text{i.e. } X = 0}$,

$\boxed{c = 0}$ under this initial condition.

$$\Rightarrow \boxed{\ln\left(\frac{1+x}{1-x}\right) = 2T = \ln e^{2T}}.$$

$$\Rightarrow \boxed{\frac{1+x}{1-x} = e^{2T}} \Rightarrow 1+x = e^{2T} - x e^{2T}$$

$$\Rightarrow x\left(1+e^{2T}\right) = e^{2T}-1 \Rightarrow \boxed{x = \frac{e^{2T}-1}{e^{2T}+1}}$$

$$\Rightarrow \boxed{x = \frac{e^{2T}-1}{e^{2T}+1} = \frac{(e^{T}-e^{-T})/2}{(e^{T}+e^{-T})/2}}$$

Now $\boxed{\sinh(T) = \frac{e^{T}-e^{-T}}{2}}$, $\boxed{\cosh(T) = \frac{e^{T}+e^{-T}}{2}}$

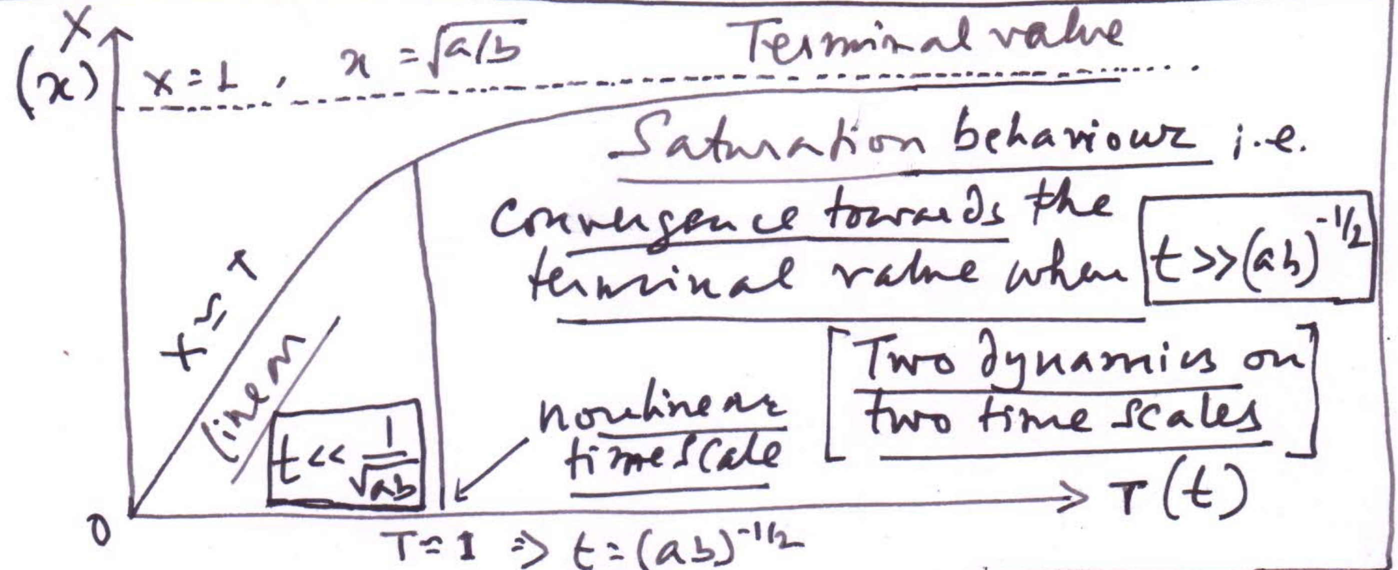
Hence $\boxed{x = \tanh(T)} \Rightarrow \boxed{x = \sqrt{\frac{a}{b}}\tanh\left(\sqrt{ab}\,t\right)}$.

i.) When $\underline{T \ll 1}$, $\boxed{e^{T} \simeq 1+T}$ and $\boxed{e^{-T} \simeq 1-T}$.

$$\therefore\ x \simeq \frac{(1+T)-(1-T)}{(1+T)+(1-T)} \simeq \frac{2T}{2} \simeq T \quad \text{(linear) (on small time)}$$

ii.) When $\underline{T \to \infty}$, $x = \frac{1-e^{-2T}}{1+e^{-2T}} \longrightarrow 1$ (on long time)

i.e $\boxed{x \longrightarrow \sqrt{a/b}}$ (approach towards this terminal value)



Modifications of the Logistic Equation

$$\boxed{\frac{dx}{dt} = ax - bx^2 + c} \text{ where } \boxed{a, b, c > 0}$$

(adding a constant to the right hand side)

$$\Rightarrow \frac{dx}{dt} = -(\sqrt{b}\,x)^2 + 2\sqrt{b}\,x\,\frac{a}{2\sqrt{b}} + c + \frac{a^2}{4b} - \frac{a^2}{4b}$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -\left[(\sqrt{b}\,x)^2 - 2(\sqrt{b}\,x)\left(\frac{a}{2\sqrt{b}}\right) + \frac{a^2}{4b}\right] + \left(\frac{a^2}{4b} + c\right)}$$

$$\Rightarrow \boxed{\frac{dx}{dt} = \left(\frac{a^2}{4b} + c\right) - \left(\sqrt{b}\,x - \frac{a}{2\sqrt{b}}\right)^2}$$

This term is a perfect square

$$\Rightarrow \boxed{\frac{dx}{dt} = \left(c + \frac{a^2}{4b}\right) - b\left(x - \frac{a}{2b}\right)^2}$$

define $\boxed{\alpha^2 = \frac{a^2}{4b} + c}$ and $\boxed{y = x - \frac{a}{2b}}$,

to get. $\boxed{\frac{dy}{dt} = \alpha^2 - by^2}$ $\boxed{\text{Since, } \frac{dx}{dt} = \frac{dy}{dt}}$

$\Rightarrow \boxed{\frac{1}{\alpha^2}\frac{dy}{dt} = 1 - \frac{y^2}{\alpha^2/b}}$, Now define $\boxed{X = \frac{y}{\alpha/\sqrt{b}}}$

$\Rightarrow \boxed{\frac{1}{\alpha^2}\cdot\frac{\alpha}{\sqrt{b}}\frac{dX}{dt} = 1 - X^2} \Rightarrow \boxed{\frac{dX}{dT} = 1 - X^2}$,

when $\boxed{T = \alpha\sqrt{b}\,t}$. The solution of this equation, as earlier

is $\boxed{\frac{1+X}{1-X} = Ae^{2T}} \Rightarrow \boxed{X = \frac{Ae^{2T} - 1}{Ae^{2T} + 1}}$ A is an integration Constant

Power Laws in Non-Autonomous Systems

Consider a non-autonomous equation $\boxed{\frac{dx}{dt} = \alpha \frac{x}{t}}$.

Integral Solution: $\boxed{\int \frac{dx}{x} = \alpha \int \frac{dt}{t}} \Rightarrow \boxed{\ln x = \alpha \ln t - \alpha \ln c}$

$\therefore \boxed{x = \left(\frac{t}{c}\right)^{\alpha}}$ When $\alpha < 0$, for $t \to \infty$, $x \to 0$ and for $t \to 0$, $x \to \infty$.

To prevent this divergence we translate $\boxed{t \to t + t_0}$.

Hence $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$. We write an equation as $\boxed{\frac{dx}{dt} = \alpha \frac{x}{t + t_0}}$, which we transform as $\boxed{\frac{dx}{dT} = \alpha \frac{x}{T}}$. The integral solution of this equation is $\boxed{x = \left(\frac{t + t_0}{c}\right)^{\alpha}}$, in which when $t \to 0$ (for $\alpha < 0$), the divergence on x is contained by $\boxed{x \to (t_0/c)^{\alpha}}$.

A Nonlinear Generalisation: Consider now $\boxed{(t + t_0)\frac{dx}{dt} = \alpha x - b x^{\mu + 1}}$, which is a nonlinear, non-autonomous equation.

Substitute $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$, and $\boxed{\xi = x^{\mu}}$.

$\therefore$ We get, $\boxed{T \frac{dx}{dT} = \alpha x \left(1 - \frac{x^{\mu}}{\alpha / b}\right)}$. $\boxed{k = \frac{\alpha}{b}}$

Now $\frac{d\xi}{dT} = \mu \frac{x^{\mu}}{x} \frac{dx}{dT} \Rightarrow \boxed{\frac{dx}{dT} = \frac{x}{\mu \xi} \frac{d\xi}{dT}}$

$$T\frac{dx}{dT} = \frac{T\cancel{x}}{\mu\,\mathcal{E}_s}\frac{d\mathcal{E}_s}{dT} = \alpha\cancel{x}\left(1-\frac{\mathcal{E}_s}{K}\right)$$

$$\Rightarrow \boxed{\frac{d\mathcal{E}_s}{dT} = \alpha\mu\,\frac{\mathcal{E}_s}{T}\left(1-\frac{\mathcal{E}_s}{K}\right)}$$

. Now rescale $\boxed{X = \mathcal{E}_s/K}$.

$$\Rightarrow \frac{d(\mathcal{E}_s/K)}{dT} = \alpha\mu\,\frac{(\mathcal{E}_s/K)}{T}\left(1-\frac{\mathcal{E}_s}{K}\right)$$

$$\Rightarrow \boxed{\frac{dX}{dT} = \alpha\mu\,\frac{X}{T}(1-X)}$$

. We integrate this equation by the method of separation of variables and partial fractions.

$$\Rightarrow \boxed{\int\frac{dX}{X(1-X)} = \alpha\mu\int\frac{dT}{T}}$$

. Now $\boxed{\frac{1}{X(1-X)} \equiv \frac{A}{X} + \frac{B}{1-X}}$

$$\Rightarrow \boxed{1 \equiv A(1-X) + BX}$$

Now when $X=0$, $A=1$. and when $X=1$, $B=1$.

$$\therefore \int\frac{dX}{X(1-X)} = \int\frac{dX}{X} - \int\frac{d(-X)}{1-X} = \alpha\mu\int\frac{dT}{T}$$

Integral constant $C>0$

$$\Rightarrow \boxed{\ln X - \ln(1-X) = \alpha\mu\ln T - \alpha\mu\ln C}$$

$$\Rightarrow \boxed{\ln\left(\frac{X}{1-X}\right) = \ln\left(\frac{T}{C}\right)^{\alpha\mu}} \Rightarrow \boxed{\frac{X}{1-X} = \left(\frac{T}{C}\right)^{\alpha\mu}}.$$

$$\Rightarrow X = \left(\frac{T}{C}\right)^{\alpha\mu} - X\left(\frac{T}{C}\right)^{\alpha\mu} \Rightarrow \boxed{X\left[1+\left(\frac{T}{C}\right)^{\alpha\mu}\right] = \left(\frac{T}{C}\right)^{\alpha\mu}}$$

$$\Rightarrow \boxed{X = \frac{(T/C)^{\alpha\mu}}{1+(T/C)^{\alpha\mu}}} \Rightarrow \boxed{X = \frac{1}{1+(T/C)^{-\alpha\mu}}}.$$

$$\boxed{X = \frac{x^{\mu}}{K}} \Rightarrow \boxed{x^{\mu} = \frac{K\,(T/C)^{\alpha\mu}}{1+(T/C)^{\alpha\mu}}}$$

in which $\boxed{T = t + t_0}$.

Case I: $\boxed{\mu = 1}$ and $\boxed{\alpha > 0}$ and $\boxed{t_0 = 0}$.

$\therefore \boxed{x = \dfrac{k(t/c)^{\alpha}}{1+(t/c)^{\alpha}}}$ i.) When $\boxed{t \to 0}$, $\boxed{1 + \left(\frac{t}{c}\right)^{\alpha} \simeq 1}$

$\Rightarrow \boxed{x \simeq k(t/c)^{\alpha}}$ for small values of t.

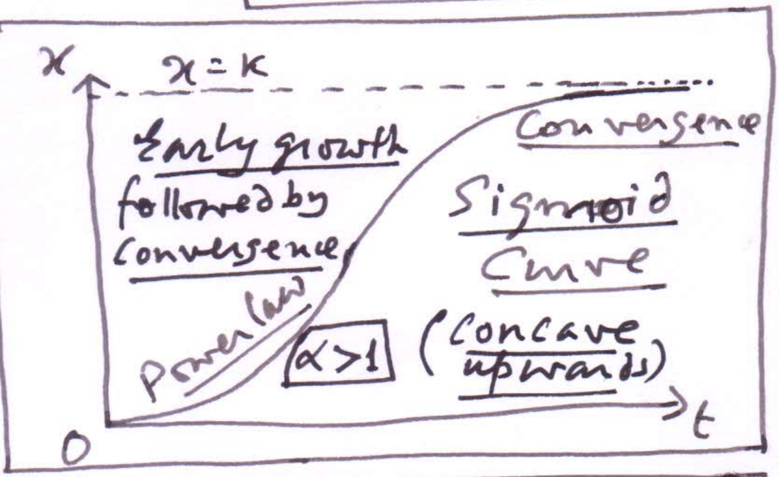
$\Rightarrow$ When $\boxed{t = 0, \ x = 0}$.

ii.) When $\boxed{t \to \infty}$,

$\boxed{x = \dfrac{k}{1+(t/c)^{-\alpha}}}$.

$\Rightarrow \boxed{x \longrightarrow k}$ (limiting value)

(x starts at $x = 0$)



Case II: $\boxed{\mu = -1}$ and $\boxed{\alpha < 0}$ and $\boxed{t_0 \neq 0}$.

We write $\boxed{k^{-1} = \eta}$ in $\boxed{x^{\mu} = \dfrac{1}{k^{-1} + k^{-1}(T/c)^{-\alpha\mu}}}$

and $\boxed{\dfrac{1}{k} \cdot \dfrac{1}{c^{-\alpha\mu}} = \dfrac{1}{c_1^{-\alpha\mu}}}$, to get,

$$\boxed{x = \left[\frac{1}{\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}}\right]^{1/\mu}} \Rightarrow \boxed{x = \left[\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}\right]^{-1/\mu}}$$

When $\boxed{\mu = -1}$, $\boxed{x = \eta + \left(\dfrac{t+t_0}{c_1}\right)^{\alpha}}$

We know $\boxed{\alpha < 0}$. For the special case of $\boxed{\alpha = -2}$ (Zipf's law), (GEORGE KINGSLEY ZIPF)

$\boxed{x = \eta + \left(\dfrac{c_1}{t+t_0}\right)^{2}}$ When $\boxed{t \to \infty}$ $\boxed{x \longrightarrow \eta}$.

Conservative Systems

From Newton's Second Law

$$\boxed{F = ma} \Rightarrow \boxed{m\frac{dv}{dt} = F \equiv F(x)} \text{ (say)}$$

We write F (force) as $\boxed{F = -\frac{d\psi}{dx}}$,
in which $\psi \equiv \psi(x)$ is a potential function.

Multiplying throughout by v we get,

$$\boxed{mv\frac{dv}{dt} = -\frac{d\psi}{dx}v} \quad \text{Now } \boxed{v = \frac{dx}{dt}} \text{ (velocity)}$$

$$\Rightarrow \quad mv\frac{dv}{dt} + \frac{\cancel{dx}}{dt}\frac{d\psi}{\cancel{dx}} = 0$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}\left(\frac{1}{2}mv^2 + \psi\right) = 0} \Rightarrow \boxed{\frac{1}{2}mv^2 + \psi = \mathcal{E}}.$$

in which $\mathcal{E}$ (total energy) is constant in time.

Reversible Systems

All conservative systems are reversible.

Write $\boxed{m\frac{dv}{dt} = m\frac{d^2x}{dt^2}}$ since $\boxed{v = \frac{dx}{dt}}$.

$$\Rightarrow \quad \boxed{m\frac{d^2x}{dt^2} = F(x)}$$

This equation is symmetric under $\boxed{t \longrightarrow -t}$.

The time reversal symmetry makes the system $\boxed{m\frac{d^2x}{dt^2} = F(x)}$ reversible in time.

Dissipation and ~~[illegible]~~ Irreversibility

Friction (or viscosity) is effective in opposing motion. This dissipates energy and the conservative condition is lost.

Further, reversibility is also lost.

Since friction (and dissipation) acts only when there is motion, we can write dissipation as a function of velocity, $\boxed{D \equiv D(v)}$. Hence, we get $\boxed{m\frac{d^2x}{dt^2} = F(x) - D(v)}$ or

$\boxed{m\frac{dv}{dt} = F(x) - D(v)}$. The simplest possible way to write this function is by a linear formula $\boxed{D(v) = kv}$, in which k is a proportional constant.

Now since $\boxed{v = dx/dt}$ we can write

$\boxed{m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} + F(x)}$, with the negative sign indicating an opposition to motion. Further the transformation of $\boxed{t \longrightarrow -t}$ is no longer symmetric. The system is IRREVERSIBLE.

The Problem of Atomic Waste Disposal

The <u>v-t equation</u> is $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$.

i.) When $t \ll t_0$ $\boxed{v \simeq v_T \frac{t}{t_0}}$ (linear <u>limit</u>).

ii.) When $t \to \infty$ $(t \gg t_0)$ $\boxed{v \simeq v_T}$ (<u>constant</u>).

Since $\boxed{v = dz/dt}$, when <u>$t \ll t_0$</u>, we get $\boxed{\frac{dz}{dt} \simeq \frac{v_T}{t_0}\, t}$ $\Rightarrow$ $\boxed{z \simeq \left(\frac{v_T}{t_0}\right)\frac{t^2}{2}}$ (<u>parabolic</u>).

And when <u>$t \to \infty$</u>. $\boxed{\frac{dz}{dt} \simeq v_T}$ $\Rightarrow$ $\boxed{z \simeq v_T t}$ (<u>linear</u>).

<u>Fugitive Elasticity</u> (<u>Maxwell</u>)

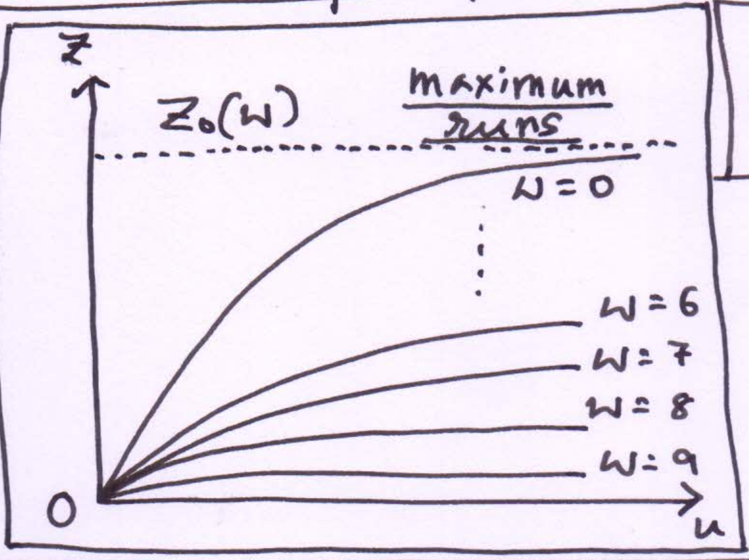
<u>Kelvin's Viscoelastic formula:</u> $\boxed{\frac{d\epsilon}{dt} = \frac{\sigma}{\eta} - \frac{Y}{\eta}\epsilon}$.

Or $\boxed{\sigma = Y\epsilon + \eta \frac{d\epsilon}{dt}}$. The <u>right hand side</u> must have <u>dimensional compatibility</u> in a system that shows <u>visco-elastic</u> <u>behaviour</u>.

Hence $\boxed{Y\epsilon \sim \eta \frac{d\epsilon}{dt}}$. We now write $\boxed{t = T t_0}$,

in which <u>T is dimensionless</u> and <u>t_0</u> is <u>a time scale</u>. Hence, $\boxed{Y\epsilon \sim \frac{\eta}{t_0}\frac{d\epsilon}{dT}}$, which gives $\left[\frac{\eta}{Y}\right]$ a <u>time dimension</u>. <u>Viscosity behaves like elasticity</u>. $\boxed{\eta \sim Y t_0}$.

The Graph of the Duckworth-Lewis Equation



$$\boxed{Z(u, w) = Z_0(w)\left[1 - e^{-b(w)u}\right]}$$

With larger values of w (wickets lost), values of b increase. Convergence is quicker.

Changes in Population (Discrete/Continuous)

Population changes in discrete step of unity (1). If a population size is x, and it changes (grows) by Δx, then the per capita growth is $\boxed{\dfrac{\Delta x}{x}}$ and the per capita growth rate is $\boxed{\dfrac{1}{x}\dfrac{\Delta x}{\Delta t}}$, in which Δt is the time taken for the growth.

If x is very large and $\boxed{\Delta x \ll x}$, then the discrete quantities can be replaced by continuously changing quantities.

$$\Rightarrow \boxed{\frac{1}{x}\frac{\Delta x}{\Delta t} \equiv \frac{1}{x}\frac{dx}{dt}}$$

Now x is continuously differentiable with respect to t.

Plotting of Equations like $\boxed{\dfrac{dx}{dT} = -x(1-x)}$

$$\boxed{\frac{d^2x}{dT^2} = \frac{df}{dx}\,\frac{dx}{dT}}$$ in which $\boxed{f(x) = -x + x^2}$

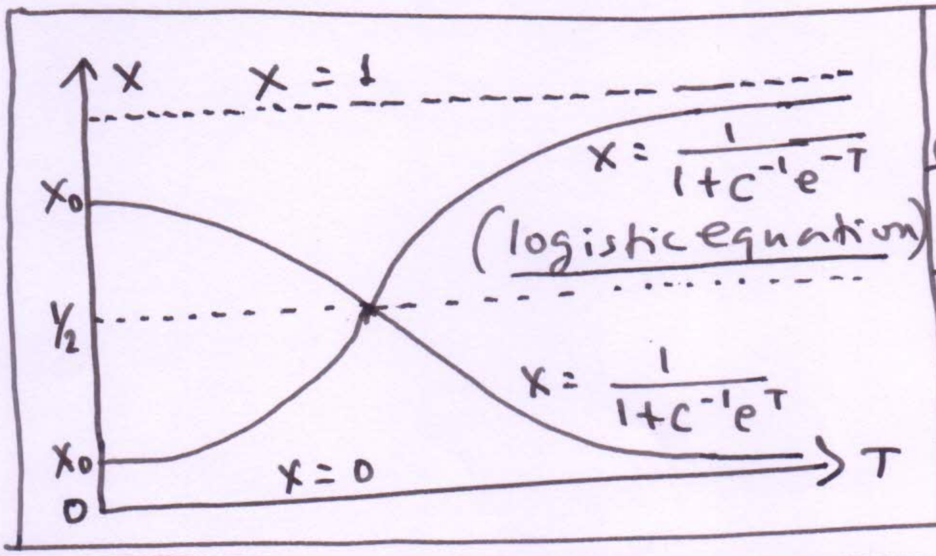
$\Rightarrow \boxed{\dfrac{df}{dx} = f'(x) = -1 + 2x}$

i.) If $\boxed{x < 1/2}$, $\boxed{\dfrac{d^2x}{dT^2} > 0}$ $\left(\because \dfrac{dx}{dT} < 0 \text{ and } \dfrac{df}{dx} < 0\right)$

This means $x(T)$ decreases at an increasing rate.

ii.) If $\boxed{x > 1/2}$, $\boxed{\dfrac{d^2x}{dT^2} < 0}$ $\left(\because \dfrac{dx}{dT} < 0 \text{ and } \dfrac{df}{dx} > 0\right)$,

i.e. $x(T)$ decreases at a decreasing rate.



For $\dfrac{dx}{dT} = x(1-x)$,

Solution: $\boxed{x = \dfrac{1}{1+c^{-1}e^{-T}}}$

For $\dfrac{dx}{dT} = -x(1-x)$

Solution $\boxed{x = \dfrac{1}{1+c^{-1}e^{T}}}$

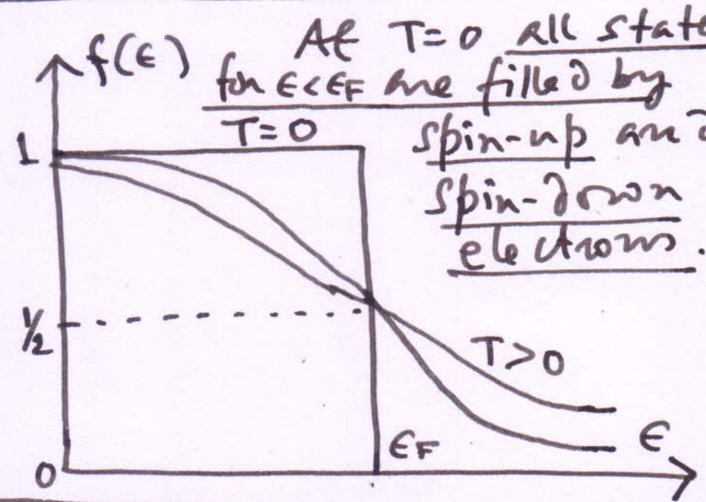
Fermi Function:

$$\boxed{f(\epsilon) = \frac{1}{1 + e^{(\epsilon - \epsilon_F)/k_B T}}}$$

Let $T = 0 \Rightarrow$ For $\epsilon < \epsilon_F$,

$f(\epsilon) = \dfrac{1}{1+e^{-\infty}} = 1$. And for

$\epsilon > \epsilon_F$ $f(\epsilon) = \dfrac{1}{1+e^{\infty}} = 0$



At $T=0$ all states for $\epsilon < \epsilon_F$ are filled by spin-up and spin-down electrons.

Power Laws and Their Properties

$\boxed{y = f(x) = Ax^{\gamma}}$ Scale $\boxed{x \longrightarrow \lambda x}$

$\therefore \boxed{f(x) \longrightarrow f(\lambda x) = A(\lambda x)^{\gamma} = A\lambda^{\gamma} x^{\gamma} = y\lambda^{\gamma}}$

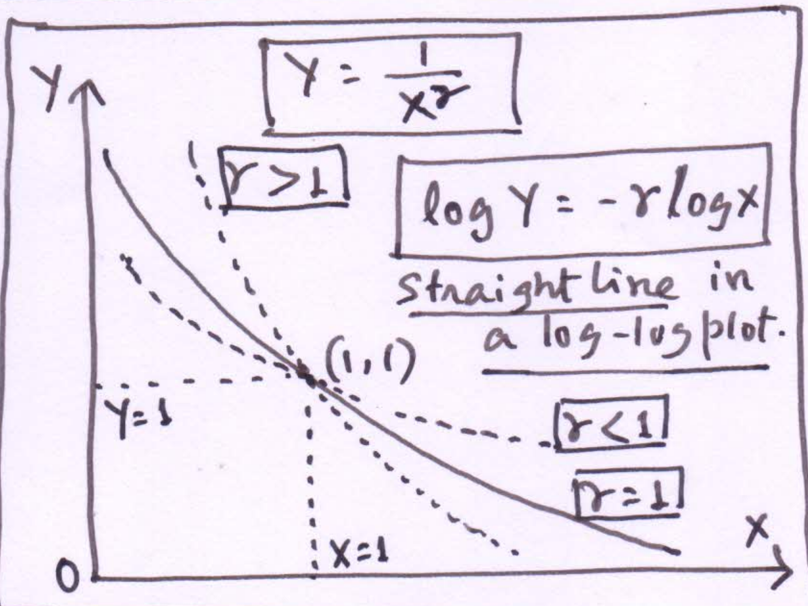
$\Rightarrow$ y is scaled as $\boxed{y\lambda^{\gamma}}$ (Scale invariance)

Inverse Power-Laws

$\boxed{y^{m} x^{n} = c}$ $\Rightarrow$ $\boxed{y x^{n/m} = c^{1/m} = a \text{ (say)}}$

$\Rightarrow$ $\boxed{\frac{y}{a} x^{n/m} = 1}$ Rescale $\boxed{Y = y/a}$, $\boxed{X = x}$ and $\boxed{\gamma = n/m}$, ($\gamma > 0$).

$\Rightarrow$ $\boxed{YX^{\gamma} = 1}$ (as in $\boxed{PV^{\gamma} = \text{constant}}$).



1/. All the curves pass through (1,1).

2/. As $x \to \infty$, the decay is faster for higher values of γ.

3/. For finite values of X and Y, no curve touches $\boxed{X=0}$ or $\boxed{Y=0}$.

4/. Any part of a curve is self-similar to any other part — scale-invariant.

Fall of a Parachutist | Free Fall

The equation $\boxed{m\frac{dv}{dt} = mg - Kv^2}$ is used to describe the free-fall of a parachutist from a height of about 30,000 ft to about 2,000 ft. After that the parachute is opened (no longer in free fall).

Bernoulli Equation

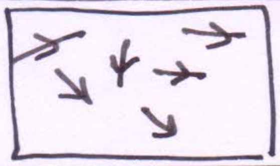
$$\boxed{\frac{V^2}{2} + \frac{P}{\rho} + gz = \text{Constant}}$$

$z \rightarrow$ height

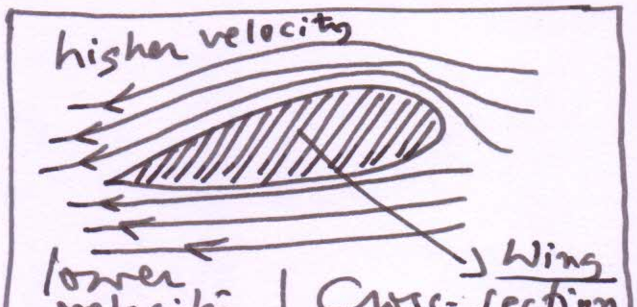
$V \rightarrow$ velocity

$P \rightarrow$ Pressure

$\rho \rightarrow$ Density, $g \rightarrow$ acceleration due to gravity.

i.) Streamline Motion:		Smooth and laminar
ii.) Turbulent Motion:		Random and chaotic

Lift of an Aircraft



i.) Above the wing closer stream-lines have higher velocity. Hence pressure is lower.

ii.) Below the wing the streamlines have lower velocity. Hence at nearly the same height the pressure is higher. This gives the lift.

Item Response Theory: Additional Points

i.) Item discrimination: $\boxed{P = c + \frac{1-c}{1+e^{-(\theta-b)\omega}}}$.

when $\boxed{\omega = \infty}$, for $\boxed{\theta > b, \quad P = c + 1 - c = 1}$, and for $\boxed{\theta < b}$, $\boxed{P = c}$ (probability that a candidate with low ability responds correctly).

$\Rightarrow$ P varies between c (non-zero lowerbound) and unity (completely perfect response).

ii.) Item difficulty: The parameter b sets a scale for ability (θ). Higher ability to respond to an item is $\boxed{\theta > b}$, and low ability is $\boxed{\theta < b}$.

Sigmoid Activation Function

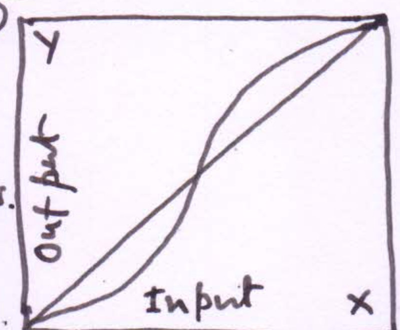
Biological neurons have a floor and ceiling of activity. This is expressed by the logistic function $\boxed{y_i = \frac{1}{1+e^{-x_j}} - a}$.

$x_j \rightarrow$ input to j-th unit. $y_i \rightarrow$ Activation response.

The Hill Function $\boxed{Y = \frac{1}{1 + (X/\theta)^{-N}}}$ Power law

$N \rightarrow$ Hill coefficient, $\theta \rightarrow$ Threshold (constant)

Used for Positive Cooperativity, in haemoglobin, which has four monomers. Binding of one monomer with oxygen, increases the affinity for binding in the other three. After that it saturates.



Taylor Expansion in Multiple Variables

I/. One Variable: $\boxed{f \equiv f(x)}$ expanded about $\boxed{x = x_c}$.

$$\Rightarrow \boxed{f = f(x_c) + \left.\frac{df}{dx}\right|_{x_c}(x - x_c) + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x_c}(x - x_c)^2 + \cdots}$$

II/. Two Variables: $\boxed{f \equiv f(x,y)}$ about $\boxed{(x_c, y_c)}$.

$$\Rightarrow \; f = f(x_c, y_c) \longrightarrow \underline{1 \text{ zero-order term}} \; (2^0)$$

$$+ \left.\frac{\partial f}{\partial x}\right|_{x_c,y_c}(x - x_c) + \left.\frac{\partial f}{\partial y}\right|_{x_c,y_c}(y - y_c) \longrightarrow \underline{2 \text{ first-order terms } (2^1)}$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,y_c}(x - x_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{x_c,y_c}(x - x_c)(y - y_c)$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_c,y_c}(y - y_c)(x - x_c) + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_c,y_c}(y - y_c)^2 + \cdots$$

$$\longrightarrow \underline{4 \text{ second-order terms } (2^2)}$$

III/. Three Variables: $\boxed{f \equiv f(x,y,z)}$ about $\boxed{(x_c, y_c, z_c)}$.

$$\Rightarrow f = f(x_c, y_c, z_c) \longrightarrow \underline{1 \text{ zero-order term} (3^0)}$$

$$+ \left.\frac{\partial f}{\partial x}\right|_{x_c,y_c,z_c}(x - x_c) + \left.\frac{\partial f}{\partial y}\right|_{x_c,y_c,z_c}(y - y_c) + \left.\frac{\partial f}{\partial z}\right|_{x_c,y_c,z_c}(z - z_c) \longrightarrow \underline{3 \text{ first-order terms } (3^1)}$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,y_c,z_c}(x - x_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_c,y_c,z_c}(y - y_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial z^2}\right|_{x_c,y_c,z_c}(z - z_c)^2$$

$$+ \frac{2}{2!}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_c,y_c,z_c}(x - x_c)(y - y_c) + \frac{2}{2!}\left.\frac{\partial^2 f}{\partial y \partial z}\right|_{x_c,y_c,z_c}(y - y_c)(z - z_c)$$

$$+ \frac{2}{2!}\left.\frac{\partial^2 f}{\partial z \partial x}\right|_{x_c,y_c,z_c}(z - z_c)(x - x_c) + \cdots \longrightarrow \underline{9 \text{ second-order terms } (3^2)},$$

with $\underline{6 \text{ mixed terms}}$.

Additional Discussions on the Spread of Industrial Innovations (E. Mansfield)

$\boxed{\lambda = f\left(p, s, \frac{x}{N}\right)}$. Following a Taylor expansion we are able to write $\boxed{\lambda = \left(a_4 + a_8 p + a_9 s\right)\frac{x}{N}}$.

In λ, we have p and s as variables.

Writing $\boxed{\lambda = k(x/N)}$, where $\boxed{k = a_4 + a_8 p + a_9 s}$.

we use it in $\boxed{\frac{dx}{dt} = k\frac{x}{N}(N-x)}$. In this equation, $\boxed{k \equiv k(p,s)}$ has p and s as free parameters, with their values fixed at the beginning.

Nonlinear Time Scale in Mansfield's Equation

Given $\boxed{x = \frac{N}{1+(N-1)e^{-k(t-t_0)}}}$, which is the solution of the logistic equation, we set $\boxed{x = N/2}$, the scale of nonlinearity in time, $\boxed{(t-t_0)|_{nl}}$.

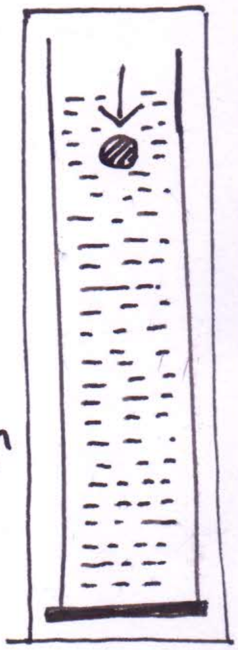
$\therefore \boxed{\frac{N}{2} = \frac{N}{1+(N-1)e^{-k(t-t_0)|_{nl}}}} \Rightarrow 2 = 1+(N-1)e^{-k(t-t_0)|_{nl}}$

$\Rightarrow \boxed{(N-1)\,e^{-k(t-t_0)|_{nl}} = 1} \Rightarrow \boxed{(N-1) = e^{k(t-t_0)|_{nl}}}$.

$\therefore k(t-t_0)|_{nl} = \ln(N-1) \Rightarrow \boxed{(t-t_0)|_{nl} = \frac{1}{k}\ln(N-1)}$.

The nonlinear time.

Examples and Applications



Stokes' Law of Terminal Velocity

For a heavy sphere (or any other shape) falling through a long column of viscous liquid, there are three forces acting on it, namely,

i) gravity, $\boxed{mg}$, ii) buoyancy, $\boxed{\rho_l V g}$, where ρ_l is the liquid density and V is the volume of the sphere, and iii) viscous drag, $\boxed{kv}$, where $\boxed{k = 6\pi\eta r}$, r being the radius of the sphere, η the viscosity and v the velocity.

Hence,

$$\boxed{m\frac{dv}{dt} = \overset{\downarrow}{mg} - \overset{\uparrow}{\rho_l V g} - \overset{\uparrow}{kv}}.$$

Writing $\boxed{m = \rho V}$, where ρ is the density of the sphere, and dividing throughout by m we get, $\boxed{\frac{dv}{dt} = \bar{g} - \frac{k}{m}v}$, in which $\boxed{\bar{g} = g\left(1 - \frac{\rho_l}{\rho}\right)}$. The above equation is in the form $\boxed{\frac{dx}{dt} = a - bx}$, with the equivalence $\boxed{a \longrightarrow \bar{g}}$ and $\boxed{b \longrightarrow k/m}$.

Atomic Waste Disposal

$z \to$ Depth of the sea

Following the principle of the problem of Stokes's law of terminal velocity, we write $\boxed{m\dfrac{d^2z}{dt^2} = F = \underset{\downarrow}{W} - \underset{\uparrow}{B} - \underset{\uparrow}{D}}$, where $\boxed{W = mg}$ is the weight, $\boxed{B = (\rho_w V g)}$ is the buoyancy and $\boxed{D = kv}$ is the drag. Here ρ_w is the density of water, and k is the drag coefficient. The drag is proportional to the velocity, $\boxed{D \propto v}$. Noting $\boxed{\dfrac{dz}{dt} = v}$,

we get $\boxed{\dfrac{dv}{dt} = g\left(1 - \dfrac{\rho_w V}{m}\right) - \dfrac{k}{m}v}$, in

which we further write, $\boxed{m = \bar{\rho} V}$, where $\bar{\rho}$ is the average density of the fuel drum and V is its volume. Hence, $\boxed{\bar{g} = g\left(1 - \dfrac{\rho_w}{\bar{\rho}}\right)}$

Using which we get $\boxed{\dfrac{dv}{dt} = \bar{g} - \dfrac{k}{m}v}$. The solution of this equation is $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$

where $\boxed{v_T = \dfrac{m\bar{g}}{k}}$ and $\boxed{t_0 = \dfrac{m}{k}}$ under the initial condition at $\boxed{t=0,\ v=0}$.

Clearly $\boxed{v_T = \bar{g}\,t_0}$ which is the terminal velocity obtained when $t \to \infty$, in $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$. Experimentally $\boxed{k = 0{\cdot}08 \text{ (in fps units)}}$, which gives the value of $\boxed{v_T \simeq 714 \text{ ft s}^{-1}}$. This is far greater than the tolerance velocity $\boxed{v_{tol} \simeq 40 \text{ ft s}^{-1}}$ at which the drums would break upon impact with the sea floor.

Since $\boxed{v_T > v_{tol}}$, the v-t equation does not guarantee that v_{tol} may not be overcome. Hence, we need to look at the v-z equation, which can be obtained from $\boxed{\dfrac{dv}{dt} = \dfrac{dv}{dz}\dfrac{dz}{dt} = v\dfrac{dv}{dz}}$

$$\therefore \boxed{v\frac{dv}{dz} = \bar{g} - \frac{v}{t_0}} \quad \text{Since } \boxed{t_0 = m/k},$$

$$\Rightarrow \boxed{t_0\, v\frac{dv}{dz} = \bar{g}\,t_0 - v = v_T - v}.$$

$$\Rightarrow \boxed{-\frac{v\,dv}{v_T - v} = -\frac{dz}{t_0}}$$ Separation of variables.

$$\Rightarrow \quad \frac{v_T - v - v_T}{v_T - v}\, dv = -\frac{dz}{t_0}$$

$$\Rightarrow \quad \boxed{\int dv + \int \frac{v_T\, d(-v)}{v_T - v} = -\int \frac{dz}{t_0}}$$

$$\Rightarrow \quad v + v_T \int \frac{d(-v/v_T)}{1 + (-v/v_T)} = -\frac{z}{t_0}$$

$$\Rightarrow \quad \boxed{v + v_T \ln\left(1 - \frac{v}{v_T}\right) = -\frac{z}{t_0} + c}\ .$$

When $\boxed{z = 0}$ (at the surface of the sea), $\boxed{v = 0}$. For this initial condition, $\boxed{c = 0}$.

$$\Rightarrow \quad \boxed{z = -t_0\left[v + v_T \ln\left(1 - \frac{v}{v_T}\right)\right]}\ .$$

This is a transcendental equation and a solution of $\boxed{v \equiv v(z)}$ cannot be found in closed form. Therefore, we invert the problem. First we write $\boxed{v = v_{tol} = 40\ \text{ft s}^{-1}}$. The depth at which this velocity is to be reached is z_{tol}. The weight of a drum, $\boxed{W = 527.4\ \text{lbs}}$. Hence, $\boxed{m = \frac{W}{g} = \frac{527.4}{32.2} = 16.38\ \text{slugs}}$.

$\Rightarrow \boxed{t_0 = \frac{m}{k} = \frac{16.38}{0.08}}$ in fps unit, $\boxed{v_T = 714\ \text{ft s}^{-1}}$.

Hence, $\boxed{z_{tol} = -\frac{16.38}{0.08}\left[40 + 714 \ln\left(1 - \frac{40}{714}\right)\right]}$ (ft)

$$\Rightarrow \boxed{Z_{tot} = \frac{-16.38}{0.08} \times -1.1644 = 238\,ft}$$

Since the actual sea depth is 300 ft, at the point of impact, $V > V_{tol}$ $\Rightarrow$ $\boxed{\text{Drums will break}}$

To check if the depth, z, is a monotonic function of t, we consider $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$, in which we write $\boxed{v = \frac{dz}{dt} = v_T\left(1 - e^{-t/t_0}\right)}$.

$$\Rightarrow Z = v_T t - \frac{v_T e^{-t/t_0}}{-1/t_0} + c = v_T t + v_T t_0 e^{-t/t_0} + c$$

when $\boxed{t = 0,\ z = 0}$ $\Rightarrow$ $\boxed{c = -v_T t_0}$.

$$\Rightarrow \boxed{Z = v_T t + v_T t_0\left(e^{-t/t_0} - 1\right)}$$

define $\boxed{\varsigma = z/v_T t_0}$ and $\boxed{\tau = t/t_0}$.

Hence, we get $\boxed{\varsigma = (\tau - 1) + e^{-\tau}}$ $\Rightarrow$ $\boxed{\frac{d\varsigma}{d\tau} = 1 - e^{-\tau}}$

$\boxed{\frac{d^2\varsigma}{d\tau^2} = e^{-\tau} = 1 > 0}$ (minimum)

$\boxed{\frac{d\varsigma}{d\tau} = 0}$ only when $\boxed{\tau = 0}$. Hence ς (or z) increases monotonically for τ (or t) > 0.

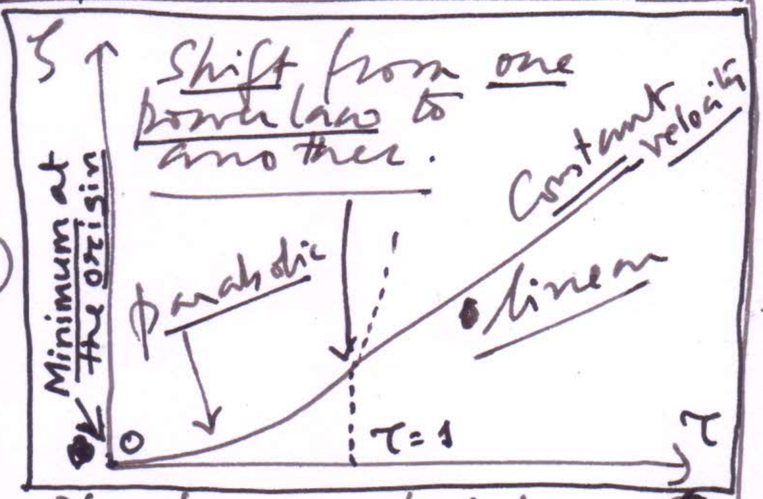
i.) When $\tau \to 0$,

$$\varsigma = \cancel{\tau} - \cancel{1} + \left(\cancel{1} - \cancel{\tau} + \frac{\tau^2}{2} + \cdots\right)$$

$$\Rightarrow \boxed{\varsigma \simeq \frac{\tau^2}{2}} \quad \text{(parabolic)}$$

ii.) When $\tau \to \infty$,

$$\boxed{\varsigma \simeq \tau} \quad \text{(linear)}$$



No turning points.

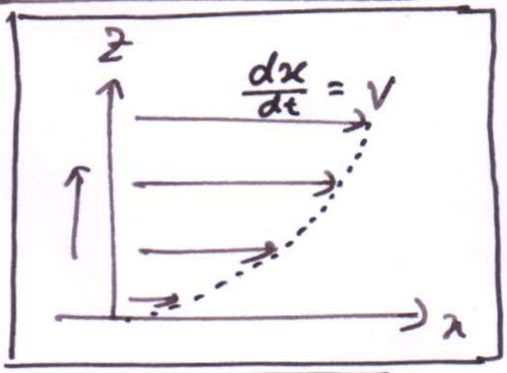
Monotonic → Also velocity increases with time.

Kelvin's Viscoelastic Deformation of Rocks

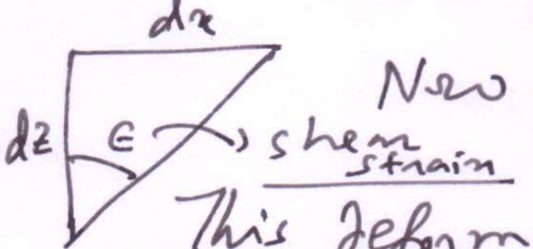
$\boxed{\sigma \rightarrow \text{Stress}}$. $\boxed{\epsilon \rightarrow \text{Strain}}$. For a Solid $\boxed{\sigma \propto \epsilon} \Rightarrow \boxed{\sigma = Y\epsilon}$ where Y is the Young's modulus (an elastic property)

For a liquid $\boxed{\sigma = \eta \dfrac{dv}{dz}}$

where $\eta \rightarrow$ Coefficient of viscosity.



Now $\boxed{\sigma = \eta \dfrac{d}{dz}\left(\dfrac{dx}{dt}\right) = \eta \dfrac{d}{dt}\left(\dfrac{dx}{dz}\right)}$



Now $\boxed{\dfrac{dx}{dz} = \tan\epsilon \approx \epsilon}$ for small deformation.

This deformation of a highly viscous liquid is named as FUGITIVE ELASTICITY by Maxwell. $\Rightarrow \boxed{\sigma = \eta \dfrac{d\epsilon}{dt}}$. Hence for a constant stress, σ, we can write

$\boxed{\sigma = Y\epsilon + \eta \dfrac{d\epsilon}{dt}} \rightarrow$ Viscoelastic (Both viscosity and elasticity)

$\Rightarrow \boxed{\dfrac{d\epsilon}{dt} = \dfrac{\sigma}{\eta} - \dfrac{Y}{\eta}\epsilon}$ like $\boxed{\dfrac{dx}{dt} = a - bx}$

$\boxed{a \rightarrow \sigma/\eta,\ b \rightarrow Y/\eta}$.

Solid rocks FLOW OUT under the weight of the Earth matter above it.

Duckworth - Lewis Method (in cricket)

$$\boxed{Z(u, w) = Z_0(w)\left[1 - e^{-b(w)u}\right]}$$

$w \to$ No. of wickets lost. $u \to$ No. of overs left.
$Z(u,w) \longrightarrow$ No. of runs obtainable.
(Compare with $\boxed{x = x_0\left(1 - e^{-t/\tau}\right)}$).
w is to be treated as a parameter.

Reduce the Duckworth - Lewis equation to an autonomous system. We write

$$\boxed{\frac{dz}{du} = -z_0 e^{-bu} \times -b = \left(z_0 e^{-bu}\right) b}.$$

But $\boxed{z_0 e^{-bu} = z_0 - z}$. Therefore,

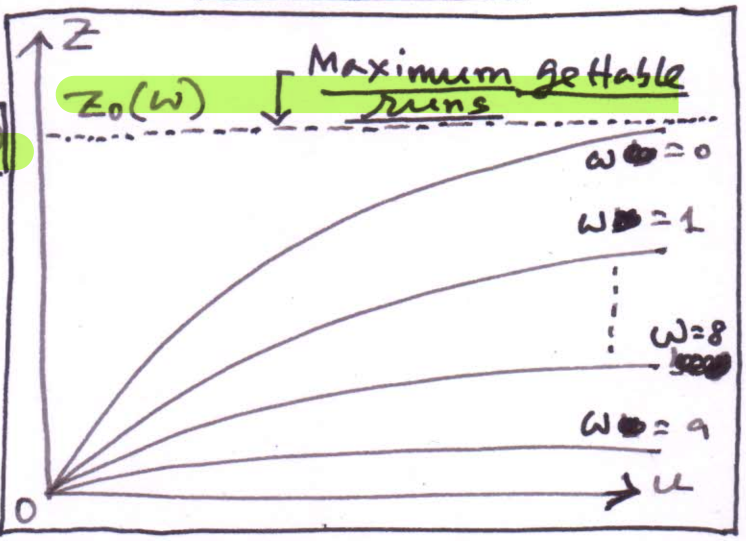
$$\boxed{\frac{dz}{du} = b(z_0 - z) = bz_0 - bz} \longrightarrow \boxed{\frac{dz}{du} = f(z) \quad \text{Autonomous}}$$

Now compare with $\boxed{\frac{dx}{dt} = a - bx}$.

We see $\boxed{a \to bz_0}$ and $\boxed{b \to b(w)}$.

The limiting value is $\boxed{a/b \to bz_0/b = z_0(w)}$

As b increases (and also w), more wickets are lost. Hence less will be the gettable runs.



Van Meegeren Art Forgery Case

Radioactivity: $\boxed{\text{Rate} \propto \text{State}}$

$$\boxed{\frac{dN}{dt} = -\lambda N}$$

$\boxed{\lambda > 0}$, $\lambda \rightarrow$ Decay constant, radioactive DECAY

Integrate: $\Rightarrow$ $\boxed{\ln N = -\lambda t + c}$.

Initial condition is when $\boxed{t = t_0,\ N = N_0}$.

$\therefore$ $\boxed{c = \ln N_0 + \lambda t_0}$ $\Rightarrow$ $\boxed{\ln N - \ln N_0 = -\lambda(t - t_0)}$

$\Rightarrow$ $\boxed{N = N_0\, e^{-\lambda(t - t_0)}}$ $\rightarrow$ Exponential decay.

Half-life: $\Rightarrow$ $\boxed{N = N_0/2}$

Time taken to decay to half the initial amount.

$\Rightarrow$ $\frac{N}{N_0} = 2^{-1} = e^{-\lambda(t - t_0)}$

Write $t - t_0 = T_{1/2}$

$\Rightarrow$ $-\lambda(t - t_0) = -\ln 2$

$$\Rightarrow \boxed{t - t_0 = T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}}$$

Eg.
$T_{1/2}\left(\begin{matrix}\text{Carbon}\\ C_{14}\end{matrix}\right) = 5568$ years, $T_{1/2}\left(\begin{matrix}\text{Uranium}\\ U_{238}\end{matrix}\right) = 4.5 \times 10^9$ years

Actual Age: $\boxed{t - t_0 = \frac{1}{\lambda} \ln(N_0/N)}$

OR $\boxed{t - t_0 = \frac{T_{1/2}}{\ln 2} \ln(N_0/N)}$

1/. N and λ can be measured.

2/. The difficulty is in knowing N_0 (the initial amount).

All paints contain white lead (lead oxide).
White lead contains radioactive Pb-210, with a half life of approximately $\boxed{22 \text{ years}}$, in which time it decays to Pb-206 (non-radioactive).
Let $\boxed{x_0 = x(t_0)}$ be the amount of Pb-210 ~~per~~ contained per gram of white lead, at the time of manufacture of the pigment.
The decay rate of Pb-210 is given by

$$\boxed{\frac{dx}{dt} = -\lambda x + r(t)}$$

, in which $r(t)$ is the rate at which Pb 210 is replenished due to the radioactive decay of Ra-226 per minute per gram of white lead. If R is the amount of radium (Ra-226) at time t, with a half life of $\boxed{T_{R1/2} = 1600 \text{ years}}$, we write the decay equation of Ra-226 as $\boxed{R = R_0 \, e^{-\lambda_R (t - t_0)}}$.
We expand this as $\boxed{R = R_0 \left[1 - \lambda_R (t - t_0) + \cdots\right]}$.
Now, $\boxed{t - t_0 = 300 \text{ years}}$ at most, which is the age of the original painting. Further $\boxed{\lambda_R = \frac{\ln 2}{T_{R1/2}}}$

Hence,

$$\boxed{\lambda_R (t - t_0) = \frac{\ln 2}{T_{R1/2}} (t - t_0) \simeq 0.13 \ll 1}.$$

Therefore, we neglect all the higher powers in the expansion and retain only,

$$\boxed{R \simeq R_0 \left[1 - \frac{\ln 2}{T_{R1/2}}(t - t_0)\right]}$$

. The decay rate of Ra-226 is

$$\boxed{\frac{dR}{dt} \simeq -\frac{R_0 \ln 2}{T_{R1/2}} = -r(t)}$$

, which is constant. Hence, the rate of replenishment of Pb 210, $r(t)$ is also constant.

$$\Rightarrow \boxed{r(t) = \frac{R_0 \ln 2}{T_{R1/2}}}$$

. The decay rate of Pb 210 is given now as

$$\boxed{\frac{dx}{dt} = r - \lambda x}$$

, which, with $\boxed{r, \lambda > 0}$, is now in the form $\boxed{\frac{dx}{dt} = a - bx}$.

Integration: $\boxed{\frac{dx}{r - \lambda x} = dt}$ Separation of variables.

$$\Rightarrow \boxed{\int \frac{d(-\lambda x)}{r - \lambda x} = -\lambda \int dt} \Rightarrow \boxed{\ln(r - \lambda x) = -\lambda t + C}$$

The initial condition is when $\boxed{t = t_0, \quad x = x_0}$.

$$\Rightarrow \boxed{C = \lambda t_0 + \ln(r - \lambda x_0)}$$

. Using this we get

$$\boxed{\ln\left(\frac{r - \lambda x}{r - \lambda x_0}\right) = -\lambda(t - t_0)}$$

$$\Rightarrow r - \lambda x = (r - \lambda x_0)\, e^{-\lambda(t - t_0)}$$

$$\Rightarrow r - \lambda x_0 = (r - \lambda x)\, e^{+\lambda(t - t_0)}$$

$$\Rightarrow \boxed{x_0 = \frac{r}{\lambda} - \left(\frac{r}{\lambda} - x\right) e^{\lambda(t - t_0)}}$$

.

Only x and t are variables.

$$\boxed{x_0 = \frac{1}{\lambda} + \left(x - \frac{1}{\lambda}\right) e^{\lambda(t-t_0)}}$$. In this equation, both λ and 1 are fixed known quantities. x can be measured. For a new painting x is large and $t-t_0$ is small, and for an old painting, x is small and $t-t_0$ is large. x_0 is ALWAYS fixed.

i/. When $t-t_0 = 300$ years, $\lambda(t-t_0) = 9.45$
ii/. When $t-t_0 = 20$ years, $\lambda(t-t_0) = 0.63$.

For measured values of x, using $\boxed{t-t_0 = 300 \text{ yrs}}$ makes the value of x_0 absurdly high. x_0 is acceptably small when $\boxed{t-t_0 = 20 \text{ years}}$.

Hence, the painting is a forgery.

Radio-Carbon Dating: Age of Ancient Cultures.

$$\boxed{{}^1n + {}^{14}N \longrightarrow {}^{14}C + {}^1p} \rightarrow \text{Willard Libby}$$

$$\boxed{N = N_0 e^{-\lambda(t-t_0)}} \Rightarrow \boxed{\frac{N_0}{N} = e^{\lambda(t-t_0)}} .$$

$$\frac{dN}{dt} = \dot{N} = N_0 e^{-\lambda(t-t_0)} \times -\lambda = -\lambda N . \left(\begin{matrix}\text{late} \\ \text{a state}\end{matrix}\right)$$

At $\boxed{t = t_0}$, $\boxed{\frac{dN}{dt} = \dot{N}(t_0) = -\lambda N_0}$, $(N_0 = N(t_0))$

$$\Rightarrow t - t_0 = \frac{1}{\lambda} \ln\left(\frac{N_0}{N}\right) = \frac{1}{\lambda} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]$$

$$\Rightarrow \boxed{t - t_0 = \frac{T_{1/2}}{\ln 2} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]} \quad \boxed{T_{1/2} = 5568 \text{ years}}$$

Exercise 1: For livingwood $\dot{N}(t_0) = 6.68$ unit

For a charcoal sample $\dot{N}(t) = 4.09$ unit

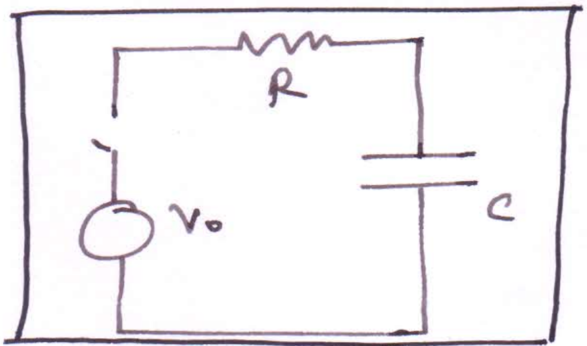
$$\Rightarrow \boxed{t - t_0 = \frac{5568}{\ln 2} \ln\left(\frac{6.68}{4.09}\right)} \qquad t = 1950 \text{ A.D.}$$

$$\Rightarrow t_0 = (1950) - 3940 \simeq 2000 \text{ B.C.}$$

Exercise 2: $\dot{N}(t_0) = 6.68$ unit, $\dot{N}(t) = 0.97$ unit, $t = 1950$ A.D.

$$\Rightarrow \boxed{t_0 = 1950 - \frac{5568}{\ln 2} \ln\left(\frac{6.68}{0.97}\right) \simeq 13{,}500 \text{ B.C.}}$$

Q-R-C Circuit



$\boxed{Q = VC}$ $\Rightarrow$ $\boxed{V = Q/C}$

and $\boxed{V = IR}$

For the full circuit $\boxed{V_0 = IR + Q/C}$.

Further $\boxed{I = \frac{dQ}{dt}}$ $\Rightarrow$ $\boxed{R\frac{dQ}{dt} = V_0 - \frac{Q}{C}}$

$$\Rightarrow \boxed{\frac{dQ}{dt} = \frac{V_0}{R} - \frac{Q}{RC}}$$

in the form $\boxed{\frac{dx}{dt} = a - bx}$

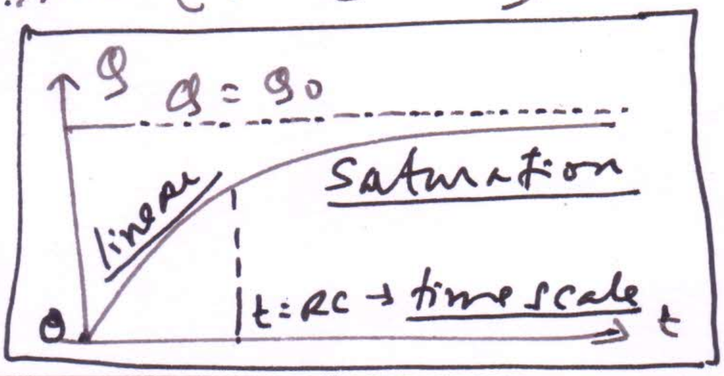
$\boxed{a \to V_0/R}$, $\boxed{b \Rightarrow \frac{1}{RC}}$

$\boxed{x = (a/b)\left[1 - e^{-bt}\right]}$

Solution is $Q = \frac{V_0}{\cancel{R}} \cdot \cancel{R}C\left(1 - e^{-t/RC}\right)$

$$\Rightarrow \boxed{Q = Q_0\left(1 - e^{-t/RC}\right)}$$

where $\boxed{Q_0 = CV_0}$ limiting value



Population Dynamics

Use a differential equation, i.e, by a continuum description (differentiable), $x(t)$.

Rate of per capita growth ~~rate~~ of the population is

$$\boxed{\frac{\Delta x}{x \Delta t} = r(x,t)}$$

$r \rightarrow$ Difference between growth rate and death rate.

By assuming a continuously differentiable function, $x(t)$.

$$\boxed{\frac{1}{x}\frac{dx}{dt} = r(x,t)}$$

Initially (for simplicity), assume that $\boxed{r = a \text{ (constant)}}$. Hence, $\boxed{\frac{dx}{dt} = ax}$ (autonomous)

$(a > 0) \Rightarrow$ growth.

$$\Rightarrow \boxed{\int \frac{dx}{x} = \int a\,dt} \Rightarrow \boxed{\ln x = at + \ln A}.$$

when $\boxed{t = t_0\,,\; x = x_0}$ $\Rightarrow \boxed{\ln A = \ln x_0 - at_0}$.

$$\Rightarrow \boxed{x = x_0\, e^{a(t-t_0)}}$$

Malthusian Law of Population Growth.

THOMAS ROBERT MALTHUS: An Essay on the Principle of Population.

This law shows an exponential growth.

Between 1700 A.D - 1961 A.D, world population Doubled every 35 years, approximately.

In 1961 A.D., $x_0 = 3.06\times10^9$ and $a = 2\% = 0.02$.

i) a was measured from $\boxed{\frac{\Delta x}{x}\cdot\frac{1}{\Delta t} = a}$ which is the percentage increase rate ($t \to$ in years)

ii) For a population size to double, $\boxed{x = 2x_0}$.

Hence, $\boxed{T = t - t_0 = \frac{1}{a}\ln\left(\frac{x}{x_0}\right) = \frac{\ln 2}{a}}$.

$\Rightarrow \boxed{T = \frac{1}{0.02}\ln 2 = 50\ln 2 \simeq 35 \text{ years}}$ Doubling time

Growth at this rate cannot be sustained in the long run. The Malthusian law fails obviously, when long term growth is considered.

The Logistic Model: (PIERRE FRANÇOIS VERHULST).

(introduce $-bx$ on the R.H.S.)

$$\boxed{\frac{\Delta x}{x\Delta t} = r(x) = a - bx}$$

i) $a, b > 0 \to$ vital coefficients

ii) $r(x)$ becomes small for large x.

$$\Rightarrow \boxed{\frac{dx}{dt} = x(a-bx) = ax\left(1 - \frac{x}{a/b}\right)}$$ The Logistic Equation

Define $\boxed{K = a/b} \to$ The carrying capacity and set $\boxed{x = \frac{K}{1+c^{-1}e^{-at}}}$. For $t\to\infty$, $x \to K$ (The upper limit).

Practical Examples of Population Dynamics

I) The World Population : $\boxed{\frac{1}{x}\frac{dx}{dt} = r = a - bx}$

Ⓐ Here $r = r(x) = 0.02$ per annum in 1961 A.D.
Ⓑ $a = 0.029$ (ecological estimates). Ⓒ $x = 3.06 \times 10^9$

Hence $\boxed{r = a - bx}$ $\Rightarrow$ $0.02 = 0.029 - b(3.06 \times 10^9)$

$\Rightarrow \boxed{b = \frac{0.009}{3.06 \times 10^9} \simeq 3 \times 10^{-12}}$. Numerically b is much smaller than a.

Carrying capacity of the world population, $(K = a/b)$, is $\boxed{K \simeq \frac{a}{b} = \frac{0.029}{3 \times 10^{-12}} \simeq 10^{10} \text{ (10 billion)}}$ Estimate of 1961 A.D.

II.) Population of the U.S.A. : $\boxed{x = \frac{K}{1 + c^{-1} e^{-at}}}$

Write $\boxed{c^{-1} = e^{at_0}}$ (constant) $\Rightarrow \boxed{x = \frac{K}{1 + e^{-a(t - t_0)}}}$ Three unknown parameters, a, K, t_0.

Therefore, census data were taken for 3 years, 1790 A.D., 1850 A.D. and 1910 A.D. by Pearl and Reed (1920 A.D.).

$\boxed{a \simeq 0.03,\ b \simeq 1.6 \times 10^{-10}}$ $\boxed{\text{Carrying capacity } K = a/b \simeq 200 \text{ million.}}$

But the present U.S. population is more than 300 million.

How? Pearl and Reed estimated in 1920. But after World War II, the vital coefficients changed; a increased and b decreased. (Belgium showed similar changes). France, however, gave a good match with predictions.

Policy Implications: $\boxed{\frac{1}{x}\frac{dx}{dt} = r(x) = a\left(1-\frac{x}{k}\right)}$

Percentage growth rate $\boxed{r = a\left(1-\frac{x}{k}\right) = a\left(\frac{k-x}{k}\right)}$

i.) When $x \ll k$, $r \simeq a$, ii.) when $x \to k$, $r \to 0$, i.e. $\boxed{\frac{k-x}{k}}$, the fractional space for growth, is reduced. Members within the population come in their way.

To maintain a high value of r, either (A) reduce x or (B) increase k (by reducing the value of b).

How? War instincts: Lebensraum, ethnic cleansing, external invasion, increasing national wealth by war and colonisation, preventing immigration.

India is a fertile land, and hence can sustain large populations (in the Ganga Valley)

Criticisms (and Scope for improvement):

i.) Technology, environment and sociological factors are changing rapidly, affecting a and b very rapidly as well. So they need re-calibration more frequently.

ii) Model by subdividing groups according to age and gender.

iii.) Large populations live in congested conditions and suffer outbreaks of epidemics. Population sizes can fluctuate, not according to the logistic law.

The Laws of Social Dynamics

Elliott W. Montroll

(Analogous to Newton's Laws of Mechanics)

1/. "First Law": In the absence of any social, economic or ~~ec~~ ecological force, $\boxed{\frac{1}{x}\frac{dx}{dt} = \text{constant}}$ $\boxed{x \equiv x(t) \text{ is the population size.}}$

2/. "Second Law": The constancy of $\boxed{\frac{1}{x}\frac{dx}{dt}}$ is violated when a force (social, economic or ecological) is applied. "Force" causes "replacements". Constancy of $\boxed{\frac{1}{x}\frac{dx}{dt}}$ is the Malthusian law. The simplest form of the replacing "force" is the linear function: $\boxed{a - bx}$.

$\Rightarrow \boxed{\frac{1}{x}\frac{dx}{dt} = a - bx}$ (No longer a constant).

$\Rightarrow \boxed{\frac{dx}{dt} = x(a - bx)} \rightarrow$ The Logistic Equation

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3/. "Third Law": Evolution is the natural response to a replacement. The "force" brings about change. (Eg. Genetic mutation brings about extinction and replacement of species).

 (Example)

Problem of Sharks and Salmon

$$\boxed{\frac{dx}{dt} = ax - bx^2 - c} \; , \quad \boxed{a, b, c > 0}$$

(c is an additive constant)

$\Rightarrow \boxed{\dfrac{dx}{dt} = ax - bx^2 + (-c)}$. Now we already know that a system like $\boxed{\dfrac{dx}{dt} = ax - bx^2 + c}$ can be transformed to a form $\boxed{\dfrac{dy}{dt} = \alpha^2 - by^2}$, in which $\boxed{y = x - \dfrac{a}{2b}}$ and $\boxed{\alpha^2 = \dfrac{a^2}{4b} + c}$. We, thus replace all $\boxed{\text{"c" with "}-c\text{"}}$, i.e. $\boxed{c \longrightarrow -c}$, and rescale further by $\boxed{X = \dfrac{y}{\alpha/\sqrt{b}}}$ and $\boxed{T = \alpha\sqrt{b}\, t}$ to get $\boxed{\dfrac{dX}{dT} = 1 - X^2}$, whose integral solution is

$$\boxed{X = \frac{A - e^{-2T}}{A + e^{-2T}}}$$

$\boxed{\text{$A$ is an integration constant}}$

$\boxed{c = 0 \Rightarrow \text{logistic equation}}$

This is then written as,

$$\boxed{x = \frac{a}{2b} + \frac{\alpha}{\sqrt{b}}\left[\frac{A - e^{-2\alpha\sqrt{b}t}}{A + e^{-2\alpha\sqrt{b}t}}\right]}$$

When $t \to \infty$,

$$\boxed{x \longrightarrow \frac{a}{2b} + \frac{1}{\sqrt{b}} \cdot \sqrt{\frac{a^2}{4b} - c}} \Rightarrow \boxed{x \longrightarrow \frac{a}{2b}\left(1 + \sqrt{1 - \frac{4bc}{a^2}}\right)}$$

This limiting value of the population does not depend on the value of A.

(Example) A Critical Case

Population of New York City:

$$\boxed{\frac{dx}{dt} = ax - bx^2 - c} \quad \boxed{a, b, c > 0}.$$

a → growth parameter, b, c → decline parameters

$$a = \frac{1}{25} = 4\times10^{-2}, \quad b = \frac{1}{25\times10^6} = 4\times10^{-8}, \quad c = 10^4$$

$$\therefore \boxed{\frac{4bc}{a^2} = \frac{4\times4\times10^{-8}\times10^4}{4\times4\times10^{-4}} = 1} \Rightarrow \boxed{a^2 = 4bc} \Rightarrow \boxed{\frac{a^2}{4b} - c = 0}$$

$$\Rightarrow \boxed{\alpha^2 = 0} \Rightarrow \boxed{\frac{dy}{dt} = -by^2} \Rightarrow \boxed{\int y^{-2}\,dy = -b\int dt}$$

$$\Rightarrow \boxed{-y^{-1} = -bt + \text{Constant}}. \Rightarrow \boxed{\frac{1}{y} = bt + A}$$ A is the integration constant.

$$\Rightarrow \boxed{y = \frac{1}{bt+A}} \Rightarrow \boxed{x = \frac{a}{2b} + \frac{1}{bt+A}}$$ Power-law convergence.

When $t \to \infty$, $y \to 0$, and $\boxed{x \to \frac{a}{2b}}$.

This is the limiting value of the population, $\boxed{x \to \frac{4\times10^{-2}}{2\times4\times10^{-8}}}$, $x \to 0.5$ million.

This convergence is slow as in a power-law. This happens in all critical phenomena, such as phase transitions. Power laws are also seen in gas laws, Zipf's law (GEORGE KINGSLEY ZIPF) and Pareto's law in income and wealth distributions (VILFREDO PARETO). They are SCALE-FREE.

Free (Parachute is not opened)

Fall of a Parachutist

Also $\boxed{Re = \frac{\rho l v}{\eta}}$

Reynold's Number : $\rightarrow \boxed{Re = \frac{l v}{\nu}}$

$l \rightarrow$ characteristic length, $v \rightarrow$ characteristic velocity.

$\boxed{\nu = \eta/\rho} \longrightarrow$ Kinematic viscosity, in which ρ is the density, η is the dynamic viscosity.

$\boxed{\nu_{water} \sim 10^{-2} \text{ S.I. units}}$. $\boxed{\nu_{air} \sim 10^{-4} \text{ S.I. units}}$.

$\boxed{\text{Drag force}, \quad D \propto v^{\gamma}}$, where v is the velocity.

i/. When $\boxed{R_e \sim 10}$ (low v, high ν), $\boxed{\gamma = 1}$.
ii/. When $\boxed{Re \sim 10^3}$ (high v, low ν), $\boxed{\gamma = 2}$.
(This is due to turbulence in air).
iii/. $10 < R < 10^3$, γ is uncertain. $\gamma = ?$

For a freely falling parachutist, $\boxed{\gamma = 2}$. $\boxed{D = kv^2}$

$\Rightarrow \boxed{m\frac{dv}{dt} = mg - kv^2}$, $\boxed{k > 0}$. We now get, $\boxed{\frac{dv}{dt} = g - \frac{k}{m}v^2}$.

$\Rightarrow \boxed{\frac{1}{g}\frac{dv}{dt} = 1 - \frac{v^2}{(\sqrt{mg/k})^2}}$ We rescale, $\boxed{X = \frac{v}{\sqrt{mg/k}}}$.

$\Rightarrow \frac{\sqrt{mg/k}}{g}\frac{dX}{dt} = 1 - X^2$. Now rescale $\boxed{T = \sqrt{kg/m}\, t}$.

$\Rightarrow \boxed{\frac{dX}{dT} = 1 - X^2}$ The initial condition is $\boxed{t = 0, \ v = 0}$.

$\Rightarrow \boxed{T = 0}$ and $\boxed{X = 0}$ are the rescaled initial conditions.

The integral solution is $\boxed{X = \tanh(T)}$.

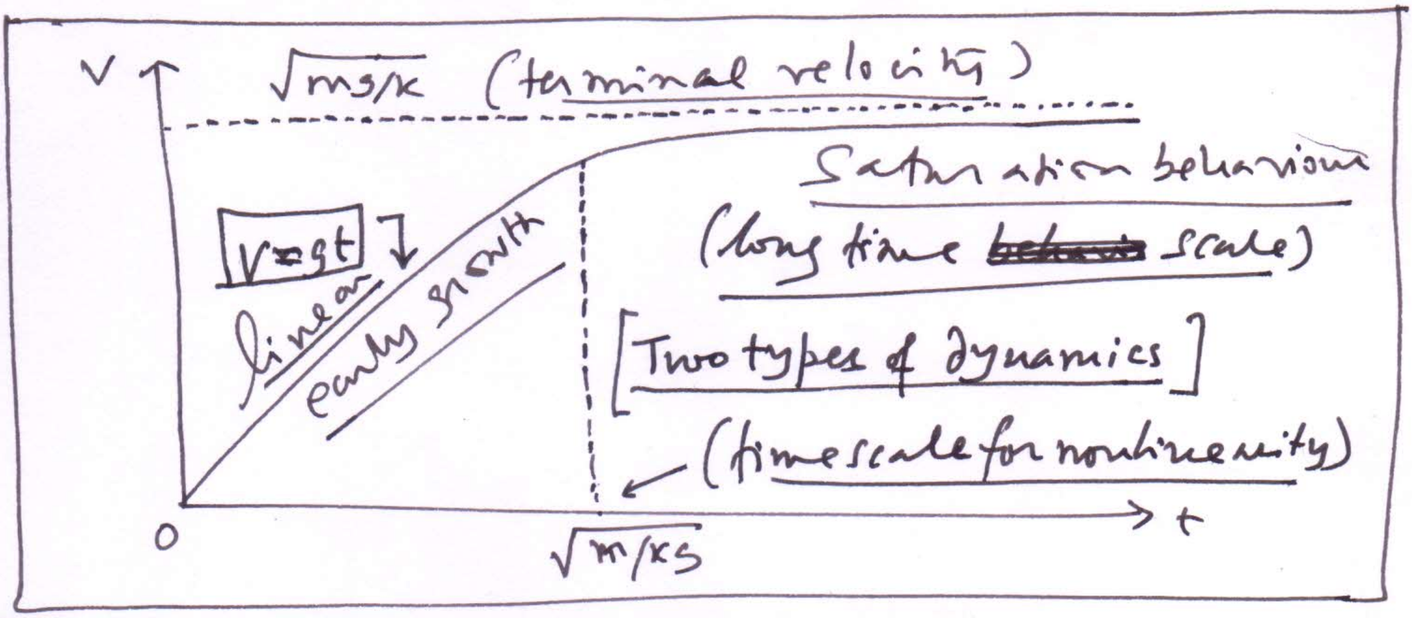
$$\Rightarrow \boxed{\frac{V}{\sqrt{mg/k}} = \tanh\left(\sqrt{kg/m}\,t\right)}$$

$$\Rightarrow \boxed{V = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right)}$$

When $t \to \infty$, $V \to \sqrt{\frac{mg}{k}}$.

When $t \to 0$, $\boxed{\tanh\left(\sqrt{\frac{kg}{m}}\,t\right) \simeq \sqrt{\frac{kg}{m}}\,t}$

$$\Rightarrow \boxed{V \simeq \sqrt{\frac{\cancel{m}}{\cancel{k}}}\sqrt{g}\cdot\sqrt{\frac{\cancel{k}}{\cancel{m}}}\sqrt{g}\,t \simeq gt} \quad (\text{linear})$$



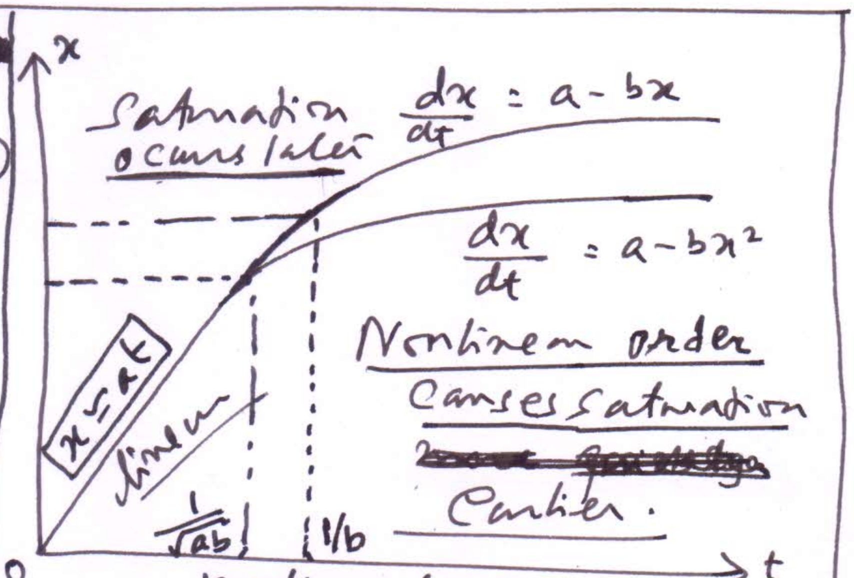
Comparison of $\boxed{\frac{dx}{dt} = a - bx}$ (linear)

and $\boxed{\frac{dx}{dt} = a - bx^2}$

(Nonlinear).

For $t \to 0$,

$\boxed{x \simeq at}$ for both (linear).



Item Response Theory

Singh, Pathak & Pandey

$$\boxed{P(\theta) = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$$ Item Response Function

$\theta \to$ Ability, $P(\theta) \to$ Performance Index.
$c \to$ Probability that a candidate with low ability will respond correctly to an item.
$\omega \to$ Item discrimination parameter.
$b \to$ Item difficulty parameter.

Define $\boxed{\phi = P(\theta) - c} \Rightarrow \boxed{\phi = (1-c)\left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-1}}$

$$\Rightarrow \frac{d\phi}{d\theta} = (1-c).\times -1\left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-2} \times e^{-\left(\frac{\theta-b}{\omega}\right)} \times -\frac{1}{\omega}$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{(1-c)}{\omega} \cdot \frac{\phi^2}{(1-c)^2} \cdot e^{-\left(\frac{\theta-b}{\omega}\right)}.$$

Now $\boxed{\left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-1} = \frac{\phi}{1-c}} \Rightarrow \boxed{e^{-\frac{\theta-b}{\omega}} = \frac{1-c}{\phi} - 1}$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{(1-c)}{\omega} \cdot \frac{\phi^2}{(1-c)^2} \cdot \left[-1 + \frac{1-c}{\phi}\right]$$ ← Autonomous form ↓

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{\cancel{(1-c)}}{\omega} \cdot \frac{\phi}{\cancel{1-c}}\left[1 - \frac{\phi}{1-c}\right] \qquad \boxed{\frac{d\phi}{d\theta} = f(\phi)}$$

$$\Rightarrow \boxed{\frac{d\phi}{d\theta} = \frac{\phi}{\omega}\left[1 - \frac{\phi}{1-c}\right]}$$ → The logistic equation.

Compare with $\boxed{\frac{dx}{dt} = ax\left(1 - \frac{x}{k}\right)}$. $\Rightarrow$ The limiting value of ϕ is $1-c$ (like carrying capacity).

By definition

$\Rightarrow \boxed{P(\theta) - c = \phi \rightarrow 1 - c} \rightarrow$ when $\theta \rightarrow \infty$.

$\therefore \boxed{P(\theta) - c = 1 - c} \Rightarrow \boxed{P(\theta) \rightarrow 1}$ when $\theta \rightarrow \infty$.

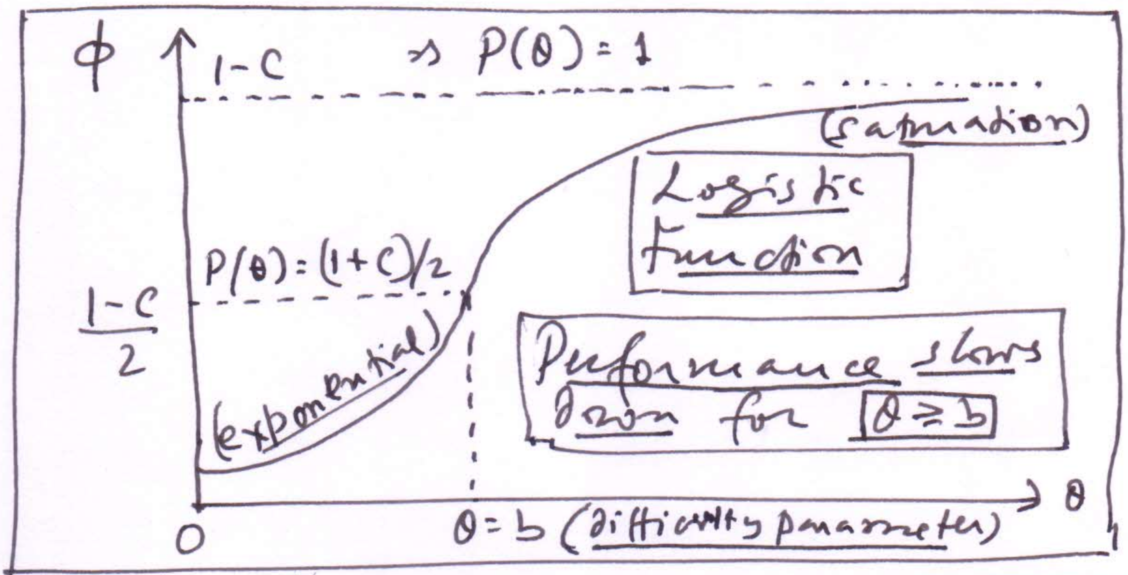
(Absolutely perfect performance when ability is infinite).

When $\boxed{\phi = \frac{1-c}{2}}$,

$\Rightarrow P - c = \frac{1-c}{2}$

$\Rightarrow \boxed{P = \frac{1+c}{2}}$

Performance starts to decelerate after this value.



Using the $P(\theta)$ function, we write $\boxed{\frac{1+c}{2} = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$

$\Rightarrow \boxed{\frac{\cancel{1-c}}{2} = \frac{\cancel{1-c}}{1+e^{-(\theta-b)/\omega}}} \Rightarrow \boxed{e^{-(\theta-b)/\omega} = 1}$

$\Rightarrow \boxed{\theta = b}$ when $\boxed{P = \frac{1+c}{2}}$.

$\Rightarrow$ Beyond $\boxed{\theta = b}$ (item difficulty parameter), performance increases at a decreasing rate (i.e. slows down)

Item discrimination parameter, ω: When ω is small the logistic function is step-like and steep.

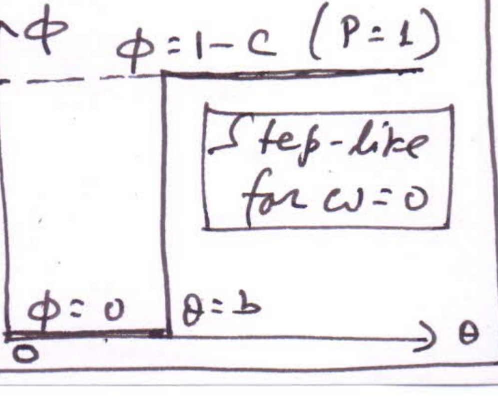
Let $\boxed{\omega = 0}$, in $\boxed{P = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$

i/. when $\theta > b$, $e^{-(\theta-b)/\omega} = e^{-\infty} = 0$

and $\boxed{P = 1, \phi = 1-c}$

ii/. when $\theta < b$,

$e^{-(\theta-b)/\omega} = e^{\infty} \rightarrow \infty \Rightarrow$ and $\boxed{P = c, \phi = 0}$



The Spread of Technological Innovations

Agricultural Innovation among Farmers

i.) Total number of farmers in a farming community is $\underline{\underline{N}}$ (fixed value).
ii.) $x(t) \rightarrow$ Number of farmers who have adopted an innovation.
iii.) $N - x(t) \rightarrow$ Number of farmers who have not adopted the innovation.

$\boxed{\Delta x \propto \Delta t}$, $\boxed{\Delta x \propto x}$ and $\boxed{\Delta x \propto (N-x)}$

$\therefore$ $\boxed{\Delta x \propto x(N-x)\Delta t}$ (jointly proportional) $\Rightarrow$ $\boxed{\dfrac{\Delta x}{\Delta t} \equiv \dfrac{dx}{dt} = c\, x(N-x)}$.

$(c > 0)$ (proportional constant)

The initial condition is $\boxed{x(0) = 1}$

Rescale: $\boxed{\dfrac{d(x/N)}{dt} = cN \dfrac{x}{N}\left(1 - \dfrac{x}{N}\right)}$ $\Rightarrow$ the logistic equation,

define $\boxed{X = x/N}$ and $\boxed{T = cNt}$ to get

$\boxed{\dfrac{dX}{dT} = X(1-X)}$ whose solution is $\boxed{X = \dfrac{1}{1 + A^{-1}e^{-T}}}$.

Hence $\boxed{x = \dfrac{N}{1 + A^{-1}e^{-cNt}}}$. when $\boxed{t = 0,\ x = 1}$.

$\Rightarrow 1 = \dfrac{N}{1 + A^{-1}} \Rightarrow \boxed{A^{-1} = N - 1}$.

$$\Rightarrow \boxed{x = \frac{N}{1 + (N-1)e^{-cNt}} = \frac{N e^{cNt}}{(N-1) + e^{cNt}}}.$$

i.) A discrepancy arises due to not accounting for information obtained through the mass media.
ii.) The ~~~~ slowing of the growth rate happens later than expected.

Modification: $\boxed{\Delta x = c'(N-x)\Delta t}$ $(c'>0)$ proportional constant

(Correction due to impersonal communication.)

The total effect is $\boxed{\Delta x = cx(N-x)\Delta t + c'(N-x)\Delta t}$.

$$\Rightarrow \boxed{\frac{\Delta x}{\Delta t} = (cx + c')(N-x)} \Rightarrow \boxed{\frac{dx}{dt} = N(cx+c')\left(1-\frac{x}{N}\right)}.$$

$$\Rightarrow \boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1 - \frac{x}{N}\right)}, \boxed{\frac{c'}{c} > 0}.$$

Early growth: When $\boxed{x \ll N}$, $\boxed{\frac{dx}{dt} \simeq Nc\left(x + \frac{c'}{c}\right)}$.

i.) Quicker than exponential, if $c'/c > 0$.
ii.) Slower than exponential, if $c'/c < 0$.

Since $c', c > 0$, the non-human intervention boosts early growth of the function, $x(t)$.

$$\Rightarrow \boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c} - \frac{x^2}{N} - \frac{c'}{c}\frac{x}{N}\right)}$$

$$\Rightarrow \frac{dx}{dt} = Ncx + Nc' - cx^2 - c'x$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -\left[cx^2 - (Nc - c')x - Nc'\right]}$$

Accounting for both human intervention and the mass media

$$\Rightarrow \frac{dx}{dt} = -\left[(\sqrt{c}\,x)^2 - 2\,\frac{1}{2}\cdot\frac{\sqrt{c}\,x}{\sqrt{c}}(Nc - c') + \frac{(Nc-c')^2}{4c} - \frac{(Nc-c')^2}{4c} - Nc'\right]$$

$$\Rightarrow \frac{dx}{dt} = -\left[\sqrt{c}\,x - \frac{(Nc-c')}{2\sqrt{c}}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -c\left[x - \frac{(Nc-c')}{2c}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]}$$

Define $\boxed{y = x - \frac{(Nc-c')}{2c}}$ and $\boxed{\alpha^2 = Nc' + \frac{(Nc-c')^2}{4c}}$

Hence $\boxed{\frac{dy}{dt} = \frac{dx}{dt}}$ $\Rightarrow$ $\boxed{\frac{dy}{dt} = \alpha^2 - cy^2}$

$\Rightarrow$ $\boxed{\frac{1}{\alpha^2}\frac{dy}{dt} = 1 - \frac{y^2}{\alpha^2/c}}$. Rescaling now $\boxed{X = \frac{y}{\alpha/\sqrt{c}}}$ and $\boxed{T = \alpha\sqrt{c}\,t}$,

we get $\boxed{\frac{dX}{dT} = 1 - X^2}$, whose solution is known to be $\boxed{\left(\frac{1+X}{1-X}\right) = Ae^{2T}}$. The initial condition is when $\boxed{t=0,\ x=0}$. $\Rightarrow$ $\boxed{y_0 = -\frac{Nc-c'}{2c} = \frac{1}{2}\left(\frac{c'}{c} - N\right)}$

Hence, $\boxed{X_0 = \frac{y_0}{\alpha/\sqrt{c}}}$. Making X the subject of T,

we get, $\boxed{X = \frac{Ae^{2T}-1}{Ae^{2T}+1}}$ $\Rightarrow$ $\boxed{y = \frac{\alpha}{\sqrt{c}}\left(\frac{Ae^{2\alpha\sqrt{c}t}-1}{Ae^{2\alpha\sqrt{c}t}+1}\right)}$.

Now $4c\alpha^2 = 4Ncc' + (Nc-c')^2$ (from the definition of α)

$\Rightarrow$ $4c\alpha^2 = 4Ncc' + N^2c^2 - 2Ncc' + c'^2 = (Nc+c')^2$

$\Rightarrow$ $\boxed{2\alpha\sqrt{c} = (Nc+c')}$ & $\frac{\alpha}{\sqrt{c}} = \frac{2\alpha\sqrt{c}}{2c} = \frac{Nc+c'}{2c}$.

$\Rightarrow$ $\boxed{\frac{\alpha}{\sqrt{c}} = \frac{1}{2}\left(N + \frac{c'}{c}\right)}$. Further, when

$t=0$ $(T=0)$ and $x=0$ $(X=X_0)$, $\boxed{A = \frac{1+X_0}{1-X_0}}$.

Therefore, $\boxed{A = \frac{1+y_0/\alpha\sqrt{c}}{1-y_0/\alpha\sqrt{c}} = \frac{\alpha/\sqrt{c}+y_0}{\alpha/\sqrt{c}-y_0}}$

Now, $\boxed{y_0 + \frac{\alpha}{\sqrt{c}} = \frac{1}{2}\frac{c'}{c} - \cancel{\frac{N}{2}} + \cancel{\frac{N}{2}} + \frac{c'}{2c} = \frac{c'}{c}}$.

Also,

$$\boxed{\frac{\alpha}{\sqrt{c}} - y_0 = \frac{N}{2} + \cancel{\frac{c'}{2c}} - \cancel{\frac{1}{2}\frac{c'}{c}} + \frac{N}{2} = N}$$

Hence, $\boxed{A = \frac{\alpha/\sqrt{c} + y_0}{\alpha/\sqrt{c} - y_0} = \frac{c'}{cN}}$. Hence we write the full integral solutions as $y = x - \frac{(Nc - c')}{2c} = \frac{\alpha}{\sqrt{c}}\left(\frac{Ae^{2\alpha\sqrt{c}t} - 1}{Ae^{2\alpha\sqrt{c}t} + 1}\right)$

Substituting for A, $2\alpha\sqrt{c}$, and $\alpha/\sqrt{c}$, we get,

$$\boxed{x = \frac{Nc - c'}{2c} + \frac{1}{2}\left(N + \frac{c'}{c}\right) \cdot \frac{(c'/Nc)e^{(Nc+c')t} - 1}{(c'/Nc)e^{(Nc+c')t} + 1}}.$$

$$\Rightarrow x = \frac{Nc - c'}{2c} + \frac{Nc + c'}{2c} \cdot \frac{c'e^{(Nc+c')t} - Nc}{c'e^{(Nc+c')t} + Nc}$$

$$\Rightarrow x = \frac{(Nc - c')\left[c'e^{(Nc+c')t} + Nc\right] + (Nc + c')\left[c'e^{(Nc+c')t} - Nc\right]}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{Ncc'e^{(Nc+c')t} - \cancel{c'^2 e^{(Nc+c')t}} + \cancel{(Nc)^2} - Ncc' + Ncc'e^{(Nc+c')t} + \cancel{c'^2 e^{(Nc+c')t}} - Ncc' - \cancel{(Nc)^2}}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{\cancel{2}N\cancel{c}c'e^{(Nc+c')t} - \cancel{2}N\cancel{c}c'}{\cancel{2c}\left[c'e^{(Nc+c')t} + Nc\right]} = \frac{Nc'e^{(Nc+c')t} - Nc'}{Nc + c'e^{(Nc+c')t}}$$

$$\Rightarrow \boxed{x = \frac{Nc'\left[e^{(Nc+c')t} - 1\right]}{cN + c'e^{(Nc+c')t}}} \rightarrow \text{The integral solution of } \boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1 - \frac{x}{N}\right)}.$$

The above solution is recast as

$$\boxed{x = \frac{Nc'\left[1 - e^{-(Nc+c')t}\right]}{c' + cNe^{-(Nc+c')t}}}$$

$\therefore$ When $t \to \infty$, $\boxed{x \to \frac{Nc'}{c'} = N}$ The maximum value.

Industrial Innovations: (Edwin Mansfield)

(Study on Coal, iron and steel, brewing and railroads).

i.) Total number of firms in an industry is N.
ii.) $x(t) \rightarrow$ Number of firms that have adopted a technological innovation.

$$\boxed{\Delta x \propto \Delta t} \text{ and } \boxed{\Delta x \propto (N-x)} \Rightarrow \boxed{\Delta x \propto (N-x)\Delta t}$$

Jointly, we write $\boxed{\Delta x = \lambda (N-x)\Delta t}$,

in which $\lambda \rightarrow$ proportional factor (not constant)

$\boxed{\lambda = \lambda\left(p, s, \frac{x}{N}\right)}$, in which (with N being fixed)

i.) $p \rightarrow$ profitability in investing in an innovation.
ii.) $s \rightarrow$ Investing ability to acquire innovation, as a percentage of the total assets.
iii.) $\frac{x}{N} \rightarrow$ Percentage of firms that have already adopted the innovation.

Edwin Mansfield's Study: (To determine λ).

i.) Carry out a Taylor expansion of λ about some equilibrium values of p, s and x/N, represented with a subscript c $\left(p_c, s_c, \frac{x}{N}\Big|_c\right)$.
ii.) Limit the Taylor expansion only up to the second order, (i.e. orders of p^2, s^2, $\left(\frac{x}{N}\right)^2$.).
iii.) Gather all the coefficients of zeroth, first and second orders.

Accordingly $\boxed{\lambda = f\left(p, s, \frac{x}{N}\right)}$ is Taylor expanded as,

$$\boxed{\begin{aligned}\lambda &= f\left(p_c, s_c, \tfrac{x}{N}\Big|_c\right) \\ &+ \frac{\partial f}{\partial p}\Big|_c (p-p_c) + \frac{\partial f}{\partial s}\Big|_c (s-s_c) + \frac{\partial f}{\partial (x/N)}\Big|_c \left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) \\ &+ \frac{1}{2!}\frac{\partial^2 f}{\partial p^2}\Big|_c (p-p_c)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial s^2}\Big|_c (s-s_c)^2 + \frac{\partial^2 f}{\partial^2 (x/N)}\Big|_c \left(\frac{x}{N} - \frac{x}{N}\Big|_c\right)^2 \\ &+ \frac{2}{2!}\frac{\partial^2 f}{\partial p \partial s}\Big|_c (p-p_c)(s-s_c) + \frac{2}{2!}\frac{\partial^2 f}{\partial p \partial (x/N)}\Big|_c (p-p_c)\left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) \\ &+ \frac{2}{2!}\frac{\partial^2 f}{\partial s \partial (x/N)}\Big|_c (s-s_c)\left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) + \cdots\end{aligned}}$$

In deriving the above expression we have used the mathematical principle $\boxed{\dfrac{\partial^2 \psi}{\partial x \partial y} = \dfrac{\partial^2 \psi}{\partial y \partial x}}$.

We get, on collecting ~~Collecting~~ all the terms of the same order,

$$\boxed{\begin{aligned}\lambda = a_1 + a_2 p + a_3 s + a_4\left(\frac{x}{N}\right) + a_5 p^2 + a_6 s^2 + a_7 ps \\ + a_8 p\left(\frac{x}{N}\right) + a_9 s\left(\frac{x}{N}\right) + a_{10}\left(\frac{x}{N}\right)^2\end{aligned}},$$

in which all a_i are constants, depending on the equilibrium values of p, s and $\frac{x}{N}$, and their derivatives.

Edwin Mansfield's study shows $\boxed{a_{10} = 0}$ and

$$\boxed{a_1 + a_2 p + a_3 s + a_5 p^2 + a_6 s^2 + a_7 ps = 0}.$$

The remaining terms $\boxed{\lambda = (a_4 + a_8 p + a_9 s)\frac{x}{N}}$

Define $\boxed{k = a_4 + a_8 p + a_9 s} \Rightarrow \boxed{\lambda = k\frac{x}{N}}$.

$\boxed{K \equiv k(p,s)}$, i.e, k depends on profitability and investing power. (k is not to be confused with the carrying capacity in the logistic equation)

$\therefore$ $\boxed{\Delta x = k \frac{x}{N}(N-x)\Delta t}$ $\Rightarrow$ $\boxed{\frac{\Delta x}{\Delta t} = k\frac{x}{N}(N-x)}$

$\Rightarrow$ $\boxed{\frac{dx}{dt} = k\frac{x}{N}(N-x)}$ $\Rightarrow$ $\boxed{\frac{d(x/N)}{dt} = \frac{kx}{N}\left(1-\frac{x}{N}\right)}$

Define $\boxed{X = \frac{x}{N}}$ and $\boxed{T = kt}$, to get

$\boxed{\frac{dX}{dT} = X(1-X)}$, which is the logistic equation.

The solution is $\boxed{X = \frac{1}{1+A^{-1}e^{-T}}}$ $\Rightarrow$ $\boxed{x = \frac{N}{1+A^{-1}e^{-kt}}}$.

Initial condition: when $\boxed{t = t_0,\ x = 1}$.

$\Rightarrow$ $1 = \frac{N}{1+A^{-1}e^{-kt_0}}$ $\Rightarrow$ $\boxed{A^{-1} = (N-1)e^{kt_0}}$

$\Rightarrow$ $\boxed{x = \frac{N}{1+(N-1)e^{-k(t-t_0)}}}$ The integral solution for spread of industrial innovations.

This solution was used to study:

i.) The spread of twelve innovations such as the shuttle car, trackless mobile loaders, mining machines, coke ovens, wide strip mills, etc.

ii.) Across ~~four~~ major industries like coal, iron and steel, brewing and railroads.

The Dynamics of Free-Living Dividing Cell Growth

$\boxed{x \equiv x(t)} \rightarrow$ Volume of dividing cells at time t.

The growth rate equation is $\boxed{\dfrac{dx}{dt} = \lambda x}$ $(\lambda > 0)$.

At $\boxed{t = t_0 \,,\ x = x_0} \longrightarrow$ Initial condition.

$\Rightarrow \boxed{x(t) = x_0 \exp\left[\lambda(t - t_0)\right]}$. Cell doubling happens when $\boxed{x = 2x_0} \Rightarrow$ Doubling time $\boxed{t - t_0 = \dfrac{\ln 2}{\lambda}}$

Gompertz Law of Tumour Growth

$$\boxed{\frac{dx}{dt} = f(x) = -ax\ln(bx)} \quad \boxed{a, b > 0}.$$

$x(t) \rightarrow$ Number of cells in a tumour.

Scale $\boxed{y = x/b^{-1}}$ and $\boxed{T = at}$.

Rescaling:

$$\Rightarrow \boxed{\frac{d\left(\frac{x}{b^{-1}}\right)}{d(at)} = -\left(\frac{x}{b^{-1}}\right)\ln\left(\frac{x}{b^{-1}}\right)} \Rightarrow \boxed{\frac{dy}{dT} = -y\ln y}$$

Integral Solution: Substitute $\boxed{y = e^X} \Rightarrow \boxed{X = \ln y}$

$$\therefore \boxed{\frac{dy}{dT} = e^X \frac{dX}{dT} = y\frac{dX}{dT} = -y\ln y = -yX}$$

$$\Rightarrow \cancel{y}\frac{dX}{dT} = -\cancel{y}X \quad \Rightarrow \boxed{\frac{dX}{dT} = -X}.$$

$$\Rightarrow \boxed{\int\frac{dX}{X} = -\int dT} \quad \Rightarrow \boxed{X = X_0 e^{-T}}$$

X_0 is the integration constant

$\Rightarrow$ $\boxed{\ln y = X_0 e^{-T}}$ $\Rightarrow$ $\boxed{\ln(bx) = X_0 e^{-at}}$

$\Rightarrow$ $\boxed{x = \frac{1}{b} \exp\left(X_0 e^{-at}\right)}$. Exponential of an exponential.

i.) When $\boxed{t \to \infty,\ x \to b^{-1}}$ (limiting ~~value~~ value)

ii.) When $\boxed{t = 0,\ x = x_0}$ (initial value).

$\therefore$ $\boxed{x_0 = \frac{1}{b} \exp(X_0)}$ $\Rightarrow$ $\boxed{e^{X_0} = x_0 b}$

Fixing the unknown X_0 $\Rightarrow$ $\boxed{X_0 = \ln(x_0 b)}$

Since $\boxed{x_0 < b^{-1}}$ (the tumour GROWS from x_0 to b^{-1}).

$\Rightarrow$ $\boxed{\frac{x_0}{b^{-1}} < 1}$ $\Rightarrow$ $\boxed{X_0 = \ln\left(\frac{x_0}{b^{-1}}\right) < 0}$

Hence, $\boxed{x = \frac{1}{b} \exp\left[\ln\left(\frac{x_0}{b^{-1}}\right) e^{-at}\right]}$,

the Gompertz formula for tumour growth, which satisfies over 1000-fold growth.

We differentiate the $x \equiv x(t)$ equation to get,

$$\boxed{\frac{dx}{dt} = \frac{1}{b} \exp\left[\ln\left(\frac{x_0}{b^{-1}}\right) e^{-at}\right] \cdot \ln\left(\frac{x_0}{b^{-1}}\right) e^{-at} \cdot (-a)}$$

Now $\boxed{\ln\left(\frac{x_0}{b^{-1}}\right) < 0}$ $\therefore$ $\boxed{-a \ln\left(\frac{x_0}{b^{-1}}\right) = -aX_0 > 0}$

We write $\boxed{\lambda = -aX_0 > 0}$ to get $\boxed{\frac{dx}{dt} = x \lambda e^{-at}}$.

This equation is in the form $\boxed{\frac{dx}{dt} = f(x,t)}$

The non-autonomous form $\boxed{\dfrac{dx}{dt} = f(x,t)}$, can be ~~[illegible]~~ cast in two ways. They are:

i.) $\boxed{\dfrac{dx}{dt} = \left(\lambda e^{-at}\right) x = \bar{\lambda}(t)\, x} \rightarrow \bar{\lambda}$ depends on t.

ii.) OR $\boxed{\dfrac{dx}{dt} = \lambda \left(x e^{-at}\right)} \rightarrow \lambda$ is a constant

First form: $\boxed{\dfrac{dx}{dt} = \bar{\lambda}(t)\, x}$ (Rate $\propto$ State)

The time scale for tumour generation is $\boxed{\bar{t} \sim \dfrac{1}{\lambda}}$ (On comparing with $\boxed{t - t_0 = \dfrac{\ln 2}{\lambda}}$ in free-living and dividing cells)

$\Rightarrow \boxed{\bar{t} \sim \lambda^{-1} e^{at}}$ $\Rightarrow$ As t increases, longer time is taken for the same amount of growth. The cells mature and divide more slowly.

Second form: $\boxed{\dfrac{dx}{dt} = \lambda \left(x e^{-at}\right)}$. λ is constant and now rate is proportional to a state, ~~[illegible]~~ $\boxed{x e^{-at}}$.

This effective state, contributing to the growth of the tumour, decreases due to necrosis at the core of the tumour, with lower number of living cells.

First form: Growth process slows down. [SUMMARY]

Second form: Number of cells in the growth is lower.

Bacteria versus Toxin: (A non-autonomous system)

$x(t) \rightarrow$	Number of bacteria at time, t.
$T(t) \rightarrow$	Amount of toxin at time, t.

i.) In the ~~presence~~ absence of toxins bacteria grow, $\boxed{\frac{dx}{dt} = bx}$. $(b > 0)$.

ii.) In the presence of toxins, bacteria die out. $\boxed{\frac{dx}{dt} = -axT}$ $(a > 0)$.

iii.) Growth rate of toxins. $\boxed{\frac{dT}{dt} = c}$. $(c > 0)$.

$\Rightarrow \boxed{T = ct + k}$ (linear function) $\leftarrow$ Initial condition: When $t = 0$, $T = 0$ $\Rightarrow \boxed{k = 0}$ $\Rightarrow \boxed{T = ct}$.

$\therefore \boxed{\frac{dx}{dt} = -axct}$ in the presence of toxins.

Combined Equation: $\boxed{\frac{dx}{dt} = bx - axct}$

$\Rightarrow \boxed{\frac{dx}{dt} = f(x,t) = x(b - act)}$ Non-autonomous equation

Integral Solution: $\boxed{\int \frac{dx}{x} = \int (b - act)\,dt}$

$\Rightarrow \boxed{\ln x = \ln x_0 + bt - \frac{act^2}{2}}$ x_0 is an integration constant.

$\Rightarrow \boxed{x = x_0 \exp\left[bt - \frac{act^2}{2}\right]}$. From this solution we see that when $\boxed{t = 0,\ x = x_0}$ (initial condition). Further when $t \rightarrow \infty$, the square power dominates and $x \rightarrow 0$ (the limiting condition).

Now we write $\boxed{bt - \frac{act^2}{2} = \frac{2bt - act^2}{2}}$.

This ~~[illegible]~~ is $-\frac{1}{2}\left[(\sqrt{ac}\,t)^2 - 2\frac{b}{\sqrt{ac}}\sqrt{ac}\,t + \frac{b^2}{ac} - \frac{b^2}{ac}\right]$

which can be written as a square,

$$\boxed{-\frac{1}{2}\left[\left(\sqrt{ac}\,t - \frac{b}{\sqrt{ac}}\right)^2 - \frac{b^2}{ac}\right] = \frac{b^2}{2ac} - \frac{ac}{2}\left(t - \frac{b}{ac}\right)^2}$$

Hence $$\boxed{x = x_0\, e^{b^2/2ac} \times \exp\left[-\frac{ac}{2}\left(t - \frac{b}{ac}\right)^2\right]}$$

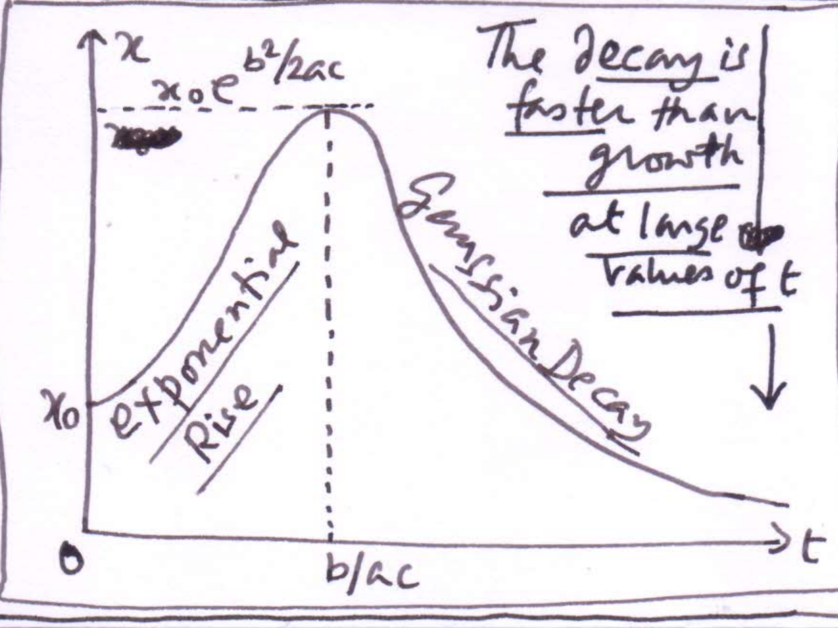
Clearly, i.) When $\boxed{t = 0,\ x = x_0}$, ii.) When $\boxed{t \to \infty,\ x \to 0}$, and iii.) When $\boxed{t = \frac{b}{ac}}$, $\boxed{x = x_0 e^{b^2/2ac} > x_0}$

Looking at $\boxed{\frac{dx}{dt} = x(b - act)}$, we see that when $\boxed{t = b/ac}$, $\boxed{\frac{dx}{dt} = 0}$.

Hence, $\boxed{t = b/ac}$ gives a turning point for $x(t)$.

The Second Derivative: $\boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt}(b - act) + x(-ac)}$

When $\boxed{t = b/ac}$, $\boxed{\frac{d^2x}{dt^2} = -acx < 0}$. This is the condition for a maximum value of $x(t)$.



Rescale: $\boxed{X = x/x_0}$ and $\boxed{T = t/(b/ac)}$. This gives

$$\boxed{X = e^{b^2/2ac} \times \exp\left[-\frac{b^2}{2ac} \times (T-1)^2\right]}.$$

i.) For $\boxed{T < 1}$, early growth is exponential.

ii.) For $\boxed{T > 1}$, the decay is Gaussian.

Flows on the Line - (First-Order Systems)

One-Dimensional System: $\rightarrow$ $\boxed{\dot{x} = f(x)}$ $\downarrow$ Autonomous System

$\boxed{\dot{x} = dx/dt}$ $\Rightarrow$ $\boxed{x \equiv x(t)}$

i.) $x(t)$ is a real valued function of t (time).
ii.) $f(x)$ is a smooth real-valued function of x.

Existence and Uniqueness Theorem:

For the initial-value problem, $\boxed{\dot{x} = f(x)}$,
$\boxed{x \equiv x(t)}$ and $\boxed{x(0) = x_0}$ (initial condition).

i/. $f(x)$ and $\boxed{f'(x) = df/dx}$ are finite-valued and smooth.
ii/. Continuous on an open interval R of the x-axis (in the one-dimensional system).
iii/. x_0 is a point in R (on the x-axis).

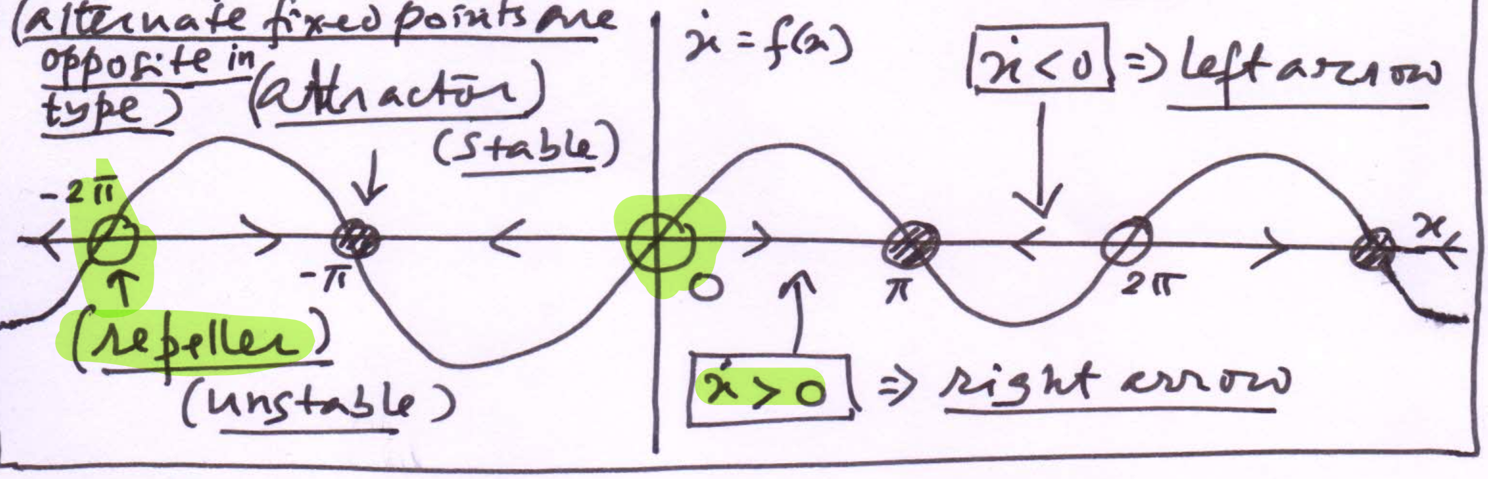
Fulfilling the foregoing conditions, the initial-value problem has a solution on some time interval $(-\tau, \tau)$ about $t = 0$.
This solution exists and is unique.

Note: 1/. Effectively the solution is single-valued.
2/. The solution may not exist forever.

$\boxed{\dot{x} = f(x)}$

Phase Portraits: Plotting $\dot{x}$ versus x in

Example: $\boxed{\dot{x} = f(x) = \sin x}$ Infinite number of roots for $\boxed{\dot{x} = 0}$



~~When~~ For $\boxed{x \equiv x(t)}$, an autonomous first-order system is given by $\boxed{\dot{x} = f(x)}$. The fixed point (or equilibrium point) of such a system is obtained when $\boxed{\dot{x} = f(x) = 0}$.

Hence, $\boxed{f(x_c) = 0}$ gives the fixed point (equilibrium point) on the line $\boxed{\dot{x} = 0}$ at $\boxed{x = x_c}$ (in which x_c is the fixed point).

This is not a turning point for $\boxed{x \equiv x(t)}$.

$\boxed{f'(x) = df/dx}$

$$\boxed{\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d\dot{x}}{dt} = \ddot{x} = \frac{df}{dx}\frac{dx}{dt}}$$ (applying chain rule).

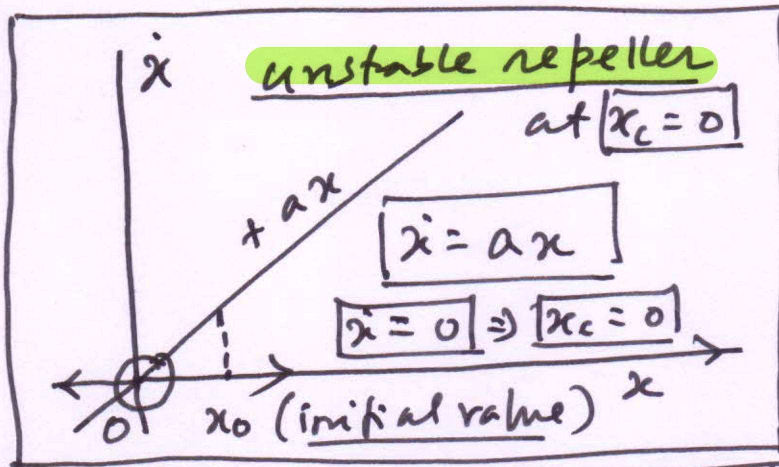
When $\boxed{\dot{x} = 0} \Rightarrow \boxed{\ddot{x} = \frac{df}{dx}\dot{x} = 0}$. Hence both $\boxed{\dot{x} = 0}$ and $\boxed{\ddot{x} = 0}$ at the same time.

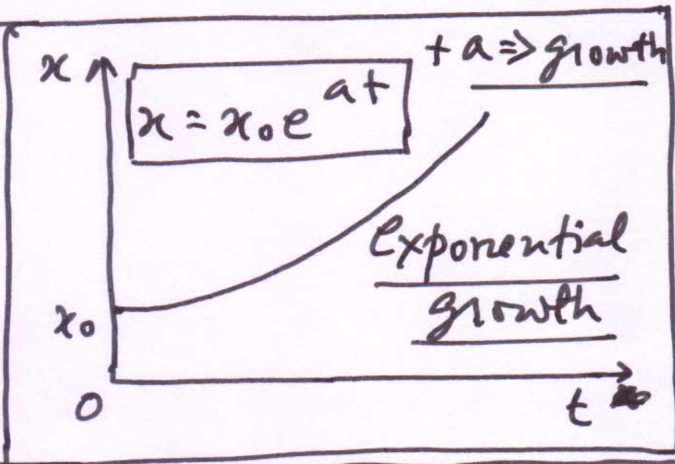
(Turning points usually have non-zero second derivatives.)

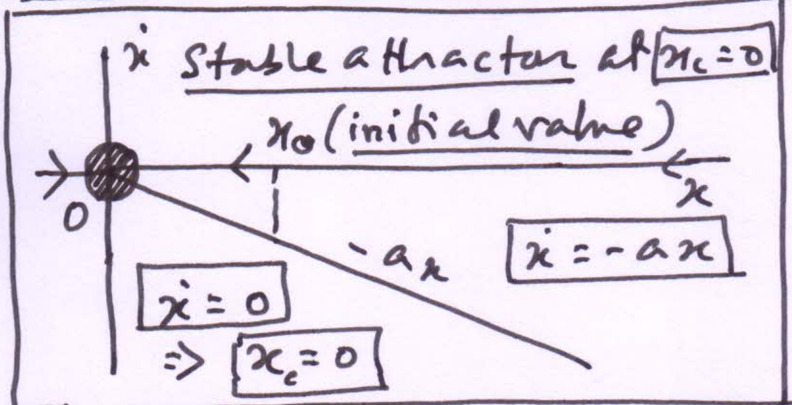
Phase Portraits of Polynomial Forms

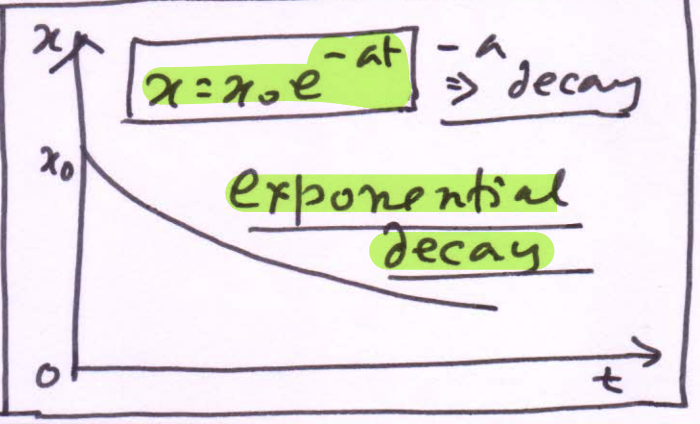
Example: Consider $\boxed{\dot{x} = \frac{dx}{dt} = f(x) = \pm a x}$ $(a > 0)$.

Integral Solution: $\boxed{x(t) = x_0 e^{\pm a t}}$ x_0 is integration constant.



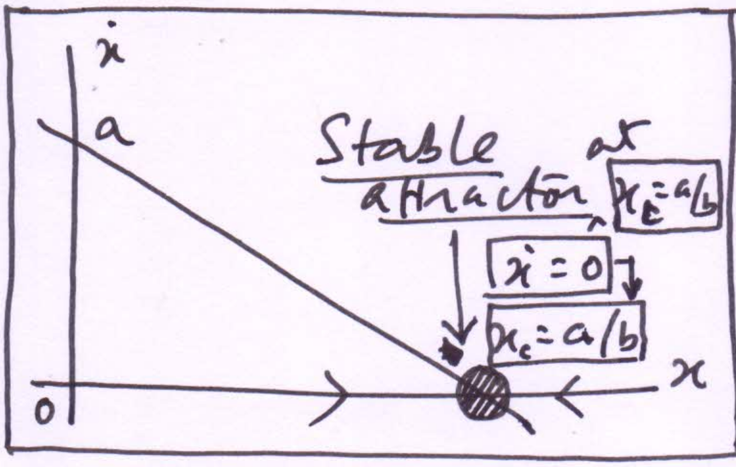


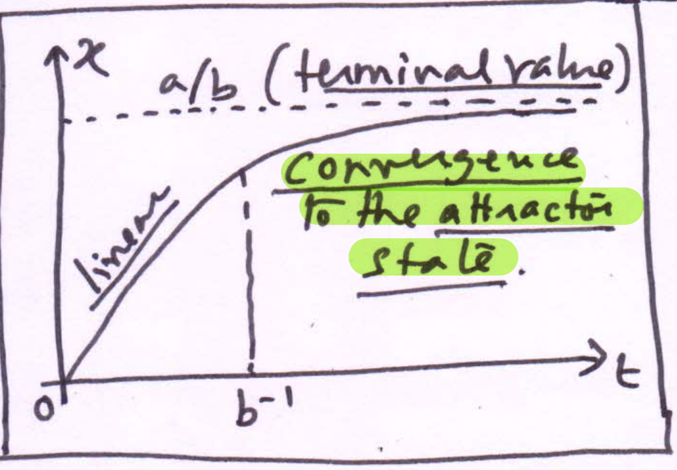




Example: $\boxed{\dot{x} = f(x) = a - b x}$ $(a, b > 0)$.

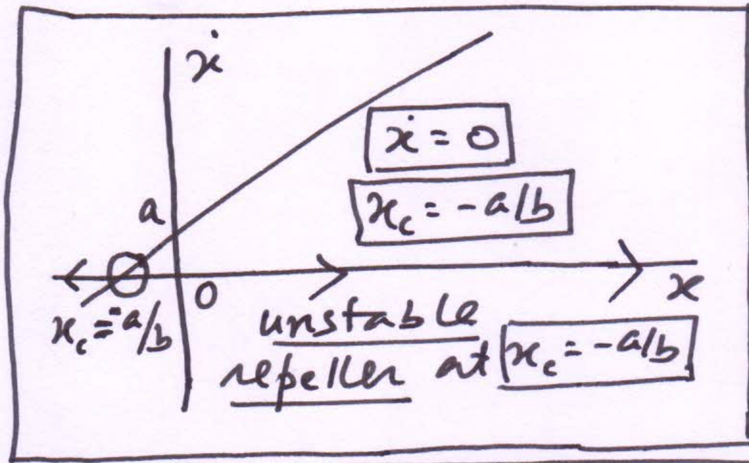
Integral Solution: $\boxed{x(t) = \frac{a}{b}\left(1 - e^{-bt}\right)}$ for $x = 0$ at $t = 0$.

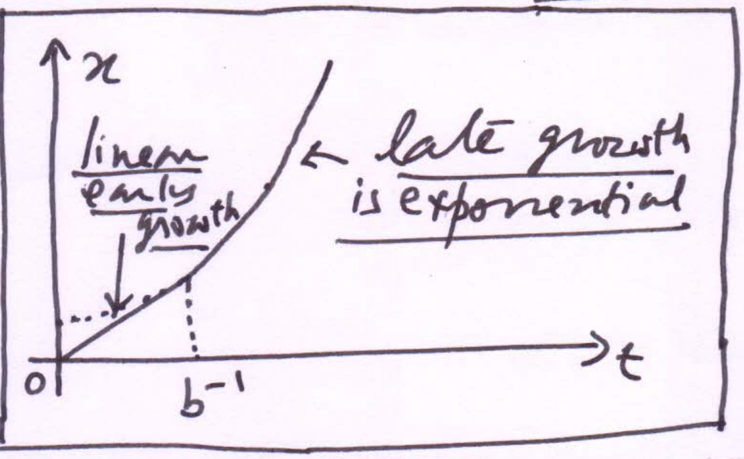




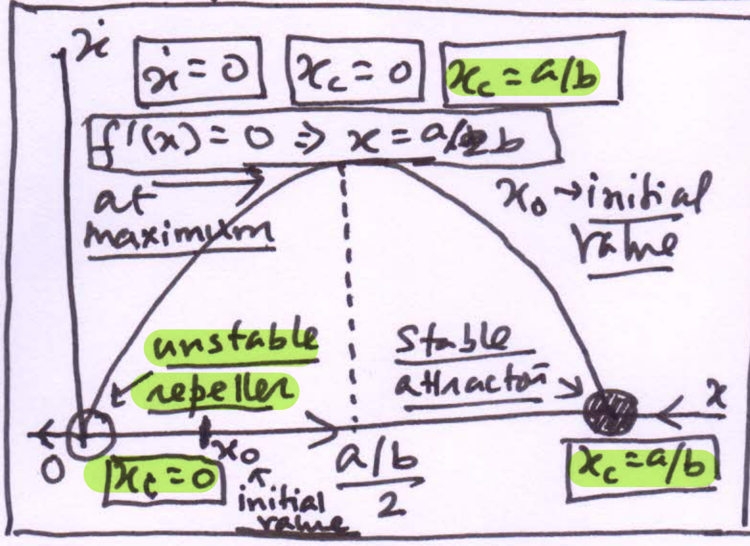
Example: $\boxed{\dot{x} = f(x) = a + bx}$ $(a, b > 0)$

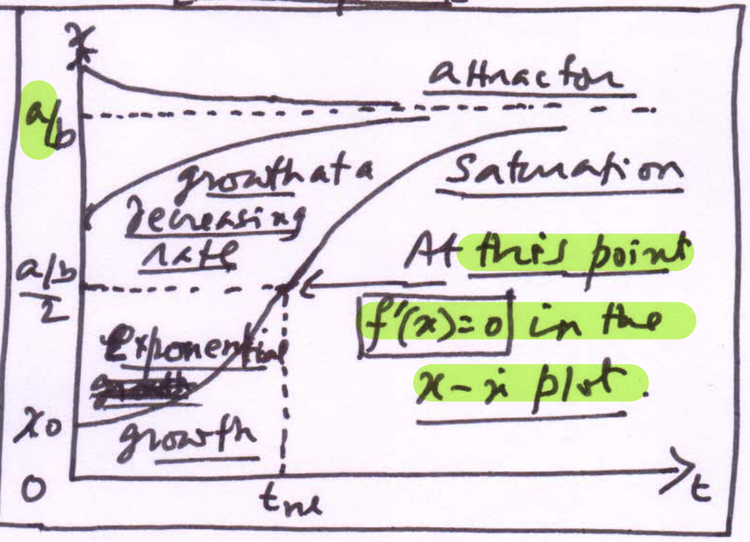
Integral solution: $\boxed{x(t) = \frac{a}{b}\left(e^{bt} - 1\right)}$ for $x = 0$ at $t = 0$





Example: $\boxed{\dot{x} = f(x) = ax - bx^2}$ $(a, b > 0)$ Nonlinear function [logistic equation]





Rescaling $\boxed{X = \frac{x}{a/b}}$ and $\boxed{T = at}$, we get a parameter-free equation $\boxed{\frac{dX}{dT} = X - X^2} \Rightarrow \boxed{\frac{dX}{dT} = F(X)}$

i.) When $\boxed{\frac{dX}{dT} = F(X) = 0}$, $\boxed{X = 0}$ and $\boxed{X = 1}$ (fixed points)

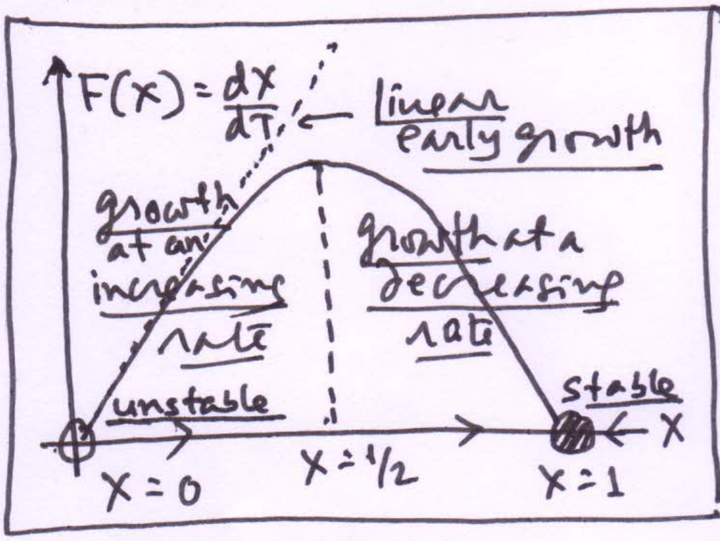
ii.) $\boxed{\frac{dF}{dX} = 1 - 2X} \Rightarrow \boxed{\frac{dF}{dX} = 0}$ is at $\boxed{X = \frac{1}{2}}$ (turning point for $F(X)$)

iii.) $\boxed{\frac{d^2X}{dT^2} = \frac{dF}{dX}\,\frac{dX}{dT}}$ $\Rightarrow$ chain rule $\boxed{\frac{d^2X}{dT^2} = (1 - 2X)\frac{dX}{dT}}$ (P.T.O.)

Now for $\boxed{0 < X \leq 1}$, $\boxed{\frac{dX}{dT} > 0}$ ⇒ X always grows with T.

But F(x) has a turning point at $\boxed{X = 1/2}$.

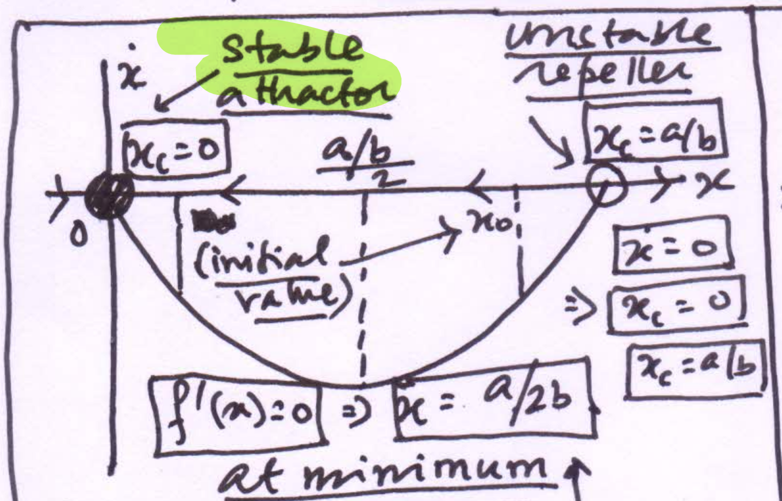
$\boxed{\frac{dF}{dx} = 1 - 2x}$ ⇒ $\boxed{\frac{d^2F}{dx^2} = -2 < 0}$ ∴ The turning point at $x = 1/2$ is a maximum.



iv.) When $\boxed{x < 1/2}$, $\boxed{\frac{dF}{dx} > 0}$ ⇒ $\boxed{\frac{d^2x}{dT^2} > 0}$. Hence, for $\boxed{x < \frac{1}{2}}$, growth occurs at an increasing rate.

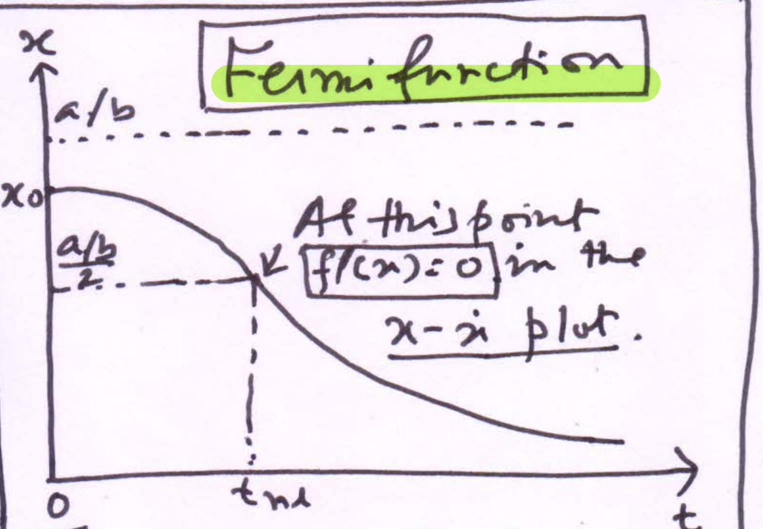
v.) when $\boxed{x > 1/2}$, $\boxed{\frac{dF}{dx} < 0}$ ⇒ $\boxed{\frac{d^2x}{dT^2} < 0}$. Hence, for $\boxed{x > 1/2}$, growth occurs at a decreasing rate. The slowing down of growth due to the nonlinear (x^2) term starts at $\boxed{x = 1/2}$.

Example: $\boxed{\dot{x} = f(x) = -ax + bx^2}$ $(a, b > 0)$ (Inverse of the logistic equation)



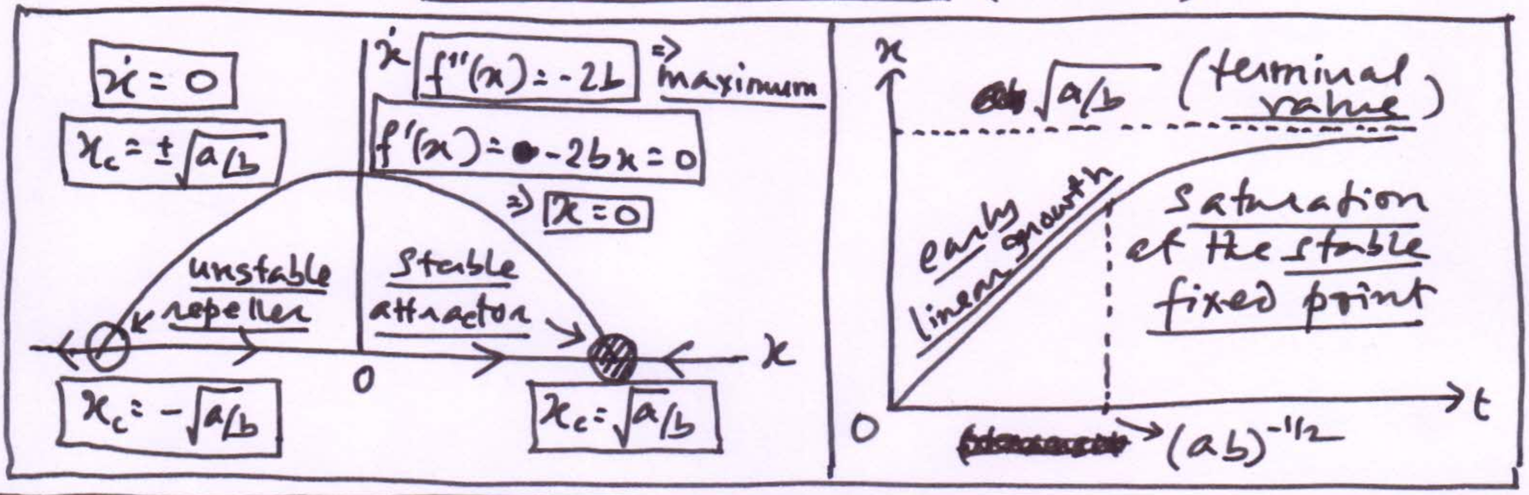
$\boxed{f'(x) = -a + 2bx}$

$\boxed{f''(x) = 2b > 0}$ ⇒ minimum



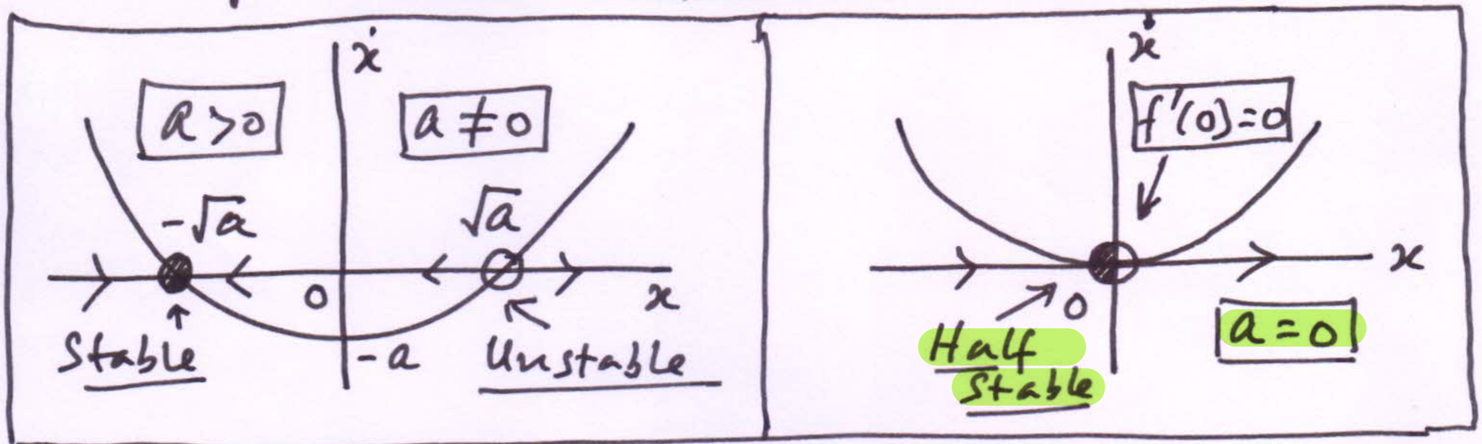
This graph is the inversion of the logistic equation.

Example: $\boxed{\dot{x} = f(x) = a - bx^2}$ ($a, b > 0$)

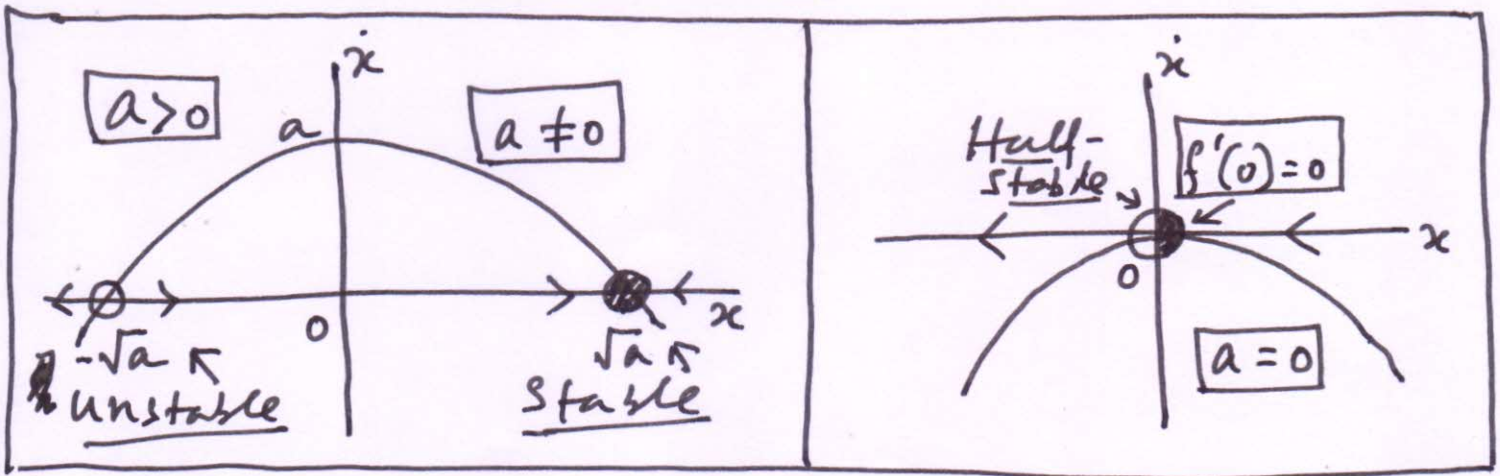


Critical Cases (Zero First Derivative)

Example: $\boxed{\dot{x} = f(x) = x^2 - a}$ $\Rightarrow$ $\boxed{f'(x) = 2x}$ ($a > 0$)



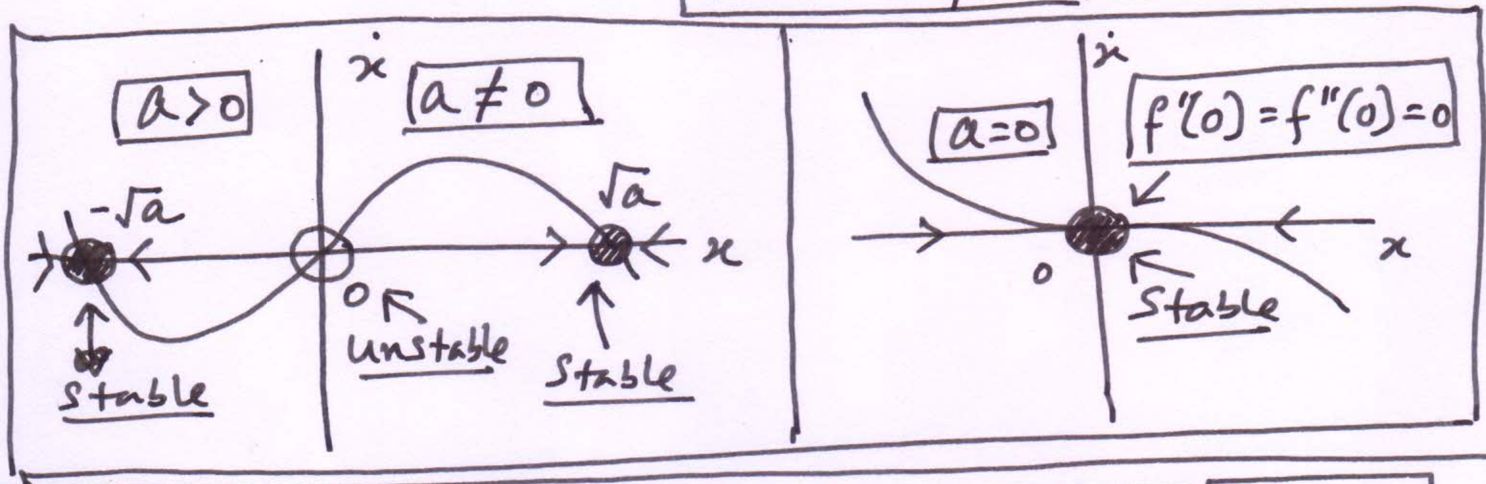
Example: $\boxed{\dot{x} = f(x) = a - x^2}$ $\Rightarrow$ $\boxed{f'(x) = -2x}$ ($a > 0$)



In the two cases above, the stable attractor and the unstable repeller exchange positions.

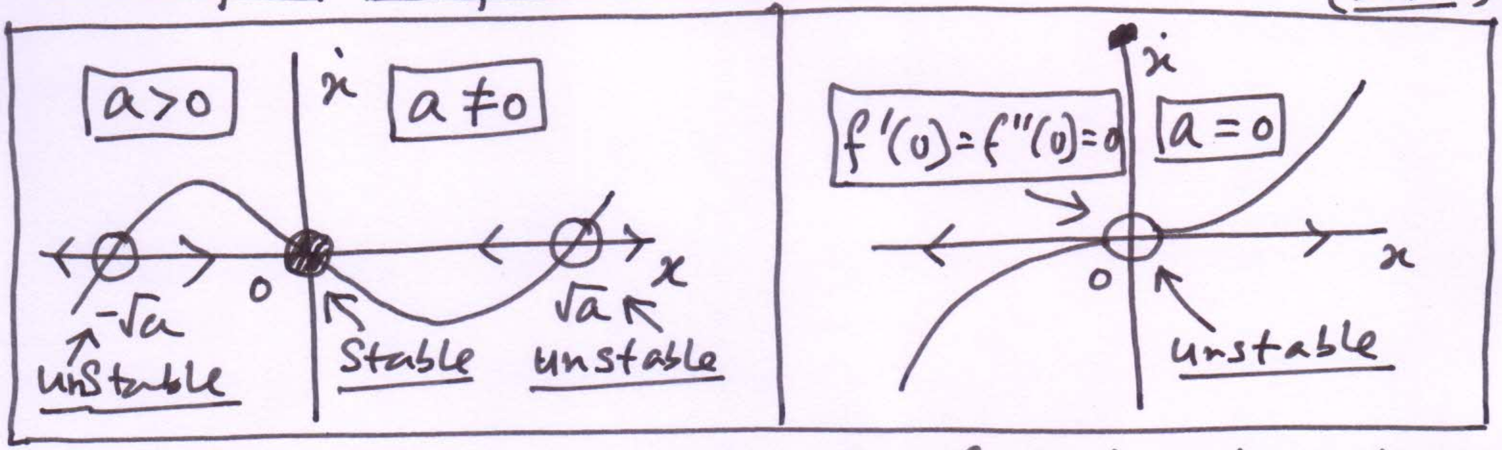
Example: $\boxed{\dot{x} = f(x) = ax - x^3}$ $\boxed{f'(x) = -3x^2}$ $(a>0)$

$\boxed{f''(x) = d^2f/dx^2}$ $\boxed{f''(x) = -6x}$



Example: $\boxed{\dot{x} = f(x) = -ax + x^3}$ $\boxed{f'(x) = 3x^2}$ $\boxed{f''(x) = 6x}$

$(a>0)$



In the two cases above, the fixed points exchange their stability properties.

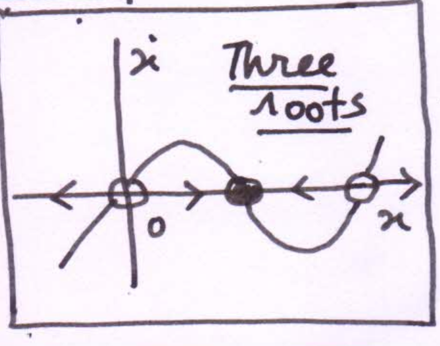
Plotting of a Polynomial Series

$$\boxed{\dot{x} = f(x) = a_0 x^0 \pm a_1 x \pm a_2 x^2 \pm a_3 x^3 \pm \cdots} \rightarrow \text{Polynomial Power Series.}$$

i.) Small powers dominate for small x.
ii.) Large powers dominate for large x.

Example: $\boxed{\dot{x} = f(x) = a_1 x - a_2 x^2 + a_3 x^3}$

i.) Negative sign $\Rightarrow$ Downward direction
ii.) Positive sign $\Rightarrow$ Upward direction



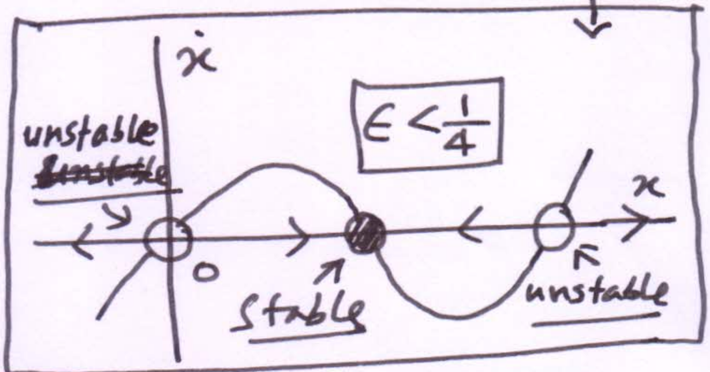
Plotting Cubic Polynomials

$\boxed{\dot{x} = f(x) = x - x^2 + \epsilon x^3}$ $(\epsilon > 0)$ Second-order inhibits, third-order enhances.

When $\boxed{\dot{x} = 0}$, $\boxed{x_c = 0}$ & $\boxed{1 - x_c + \epsilon x_c^2 = 0}$.

$\Rightarrow$ $\boxed{x_c = \dfrac{1 \pm \sqrt{1 - 4\epsilon}}{2\epsilon}}$. If $\boxed{\epsilon < \frac{1}{4}}$, there are three real fixed points.

$\boxed{x_c = \dfrac{1}{2\epsilon} \pm \dfrac{1}{2\epsilon}(1 - 4\epsilon)^{1/2}}$



When $\epsilon \to 0$ $\boxed{(1 - 4\epsilon)^{1/2} \simeq 1 - 2\epsilon}$

[Binomial theorem] $\rightarrow$ (approximation). $\rightarrow$ $\boxed{(1 + z^n) \simeq 1 + nz}$

$\Rightarrow$ $\boxed{x_c \simeq \dfrac{1}{2\epsilon} \pm \dfrac{1}{2\epsilon}(1 - 2\epsilon)}$ $\Rightarrow$ $\boxed{x_c \simeq \dfrac{1}{2\epsilon} \pm \dfrac{1}{2\epsilon} \mp 1}$

When $\boxed{\epsilon \to 0}$, choose lower sign $\Rightarrow$ $\boxed{x_c \to 1}$.

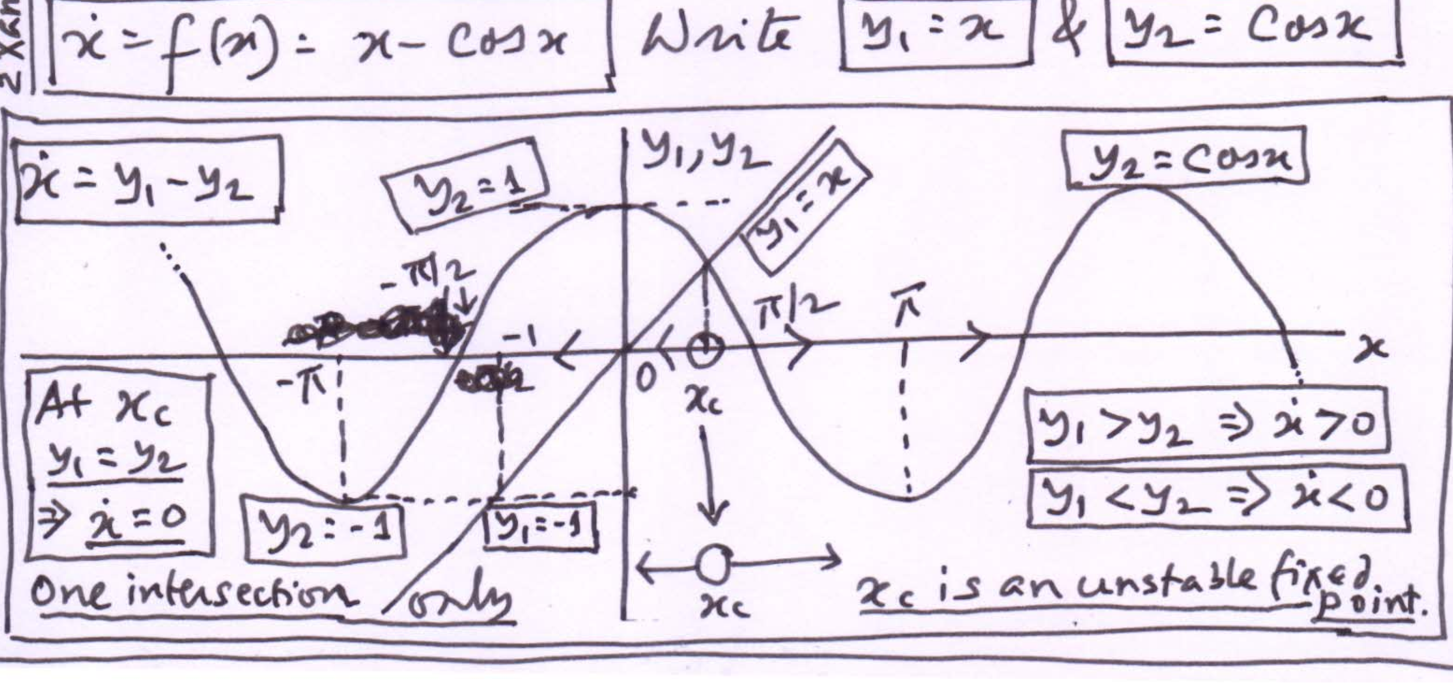
If the upper sign is chosen, $\boxed{x_c \simeq \dfrac{1}{\epsilon} - 1}$ $\Rightarrow$ $\boxed{x \to \infty}$.

The outer unstable fixed point is pushed to infinity.

Example

Fixed Points in Transcendental Functions

$\boxed{\dot{x} = f(x) = x - \cos x}$ Write $\boxed{y_1 = x}$ & $\boxed{y_2 = \cos x}$



Stability Analysis of Fixed Points

For a first-order autonomous dynamical system, $\boxed{\dot{x} = f(x)}$, the fixed point condition is $\boxed{\dot{x} = 0}$ at $\boxed{x = x_c}$ $\Rightarrow$ $\boxed{f(x_c) = 0}$ at $x = x_c$.

The fixed point coordinate x_c is perturbed by a small amount ϵ, i.e. $\epsilon \ll x_c$. Hence, we write $\boxed{x = x_c + \epsilon}$. Using this in $\boxed{\dot{x} = f(x)}$.

$$\boxed{\dot{x} = \dot{x}_c + \dot{\epsilon} = f(x) = f(x_c + \epsilon)}$$ Now $\boxed{\dot{x}_c = 0}$.

$$\therefore \boxed{\dot{\epsilon} = f(x_c + \epsilon) = f(x_c) + f'(x_c)\epsilon + \frac{f''(x_c)}{2!}\epsilon^2 + \cdots}$$

by a Taylor expansion. Now $\boxed{f(x_c) = 0}$, and truncating the Taylor expansion at the linear order (i.e. ignoring ϵ^2 and all higher order terms, due to the smallness of ϵ), we get $\boxed{\dot{\epsilon} \simeq f'(x_c)\epsilon}$, a linear differential equation in ϵ.

$$\Rightarrow \boxed{\frac{d\epsilon}{dt} = f'(x_c)\,\epsilon} \;\Rightarrow\; \boxed{\int \frac{d\epsilon}{\epsilon} = f'(x_c)\int dt} \quad \boxed{f'(x_c) \text{ is constant}}$$

$$\Rightarrow \boxed{\ln \epsilon = \ln A + f'(x_c)\,t} \Rightarrow \boxed{\epsilon = A\,e^{f'(x_c)t}}$$

Hence $\boxed{x \simeq x_c + A\,e^{f'(x_c)t}}$ (A is an integration constant).

The foregoing result is due to a LINEAR STABILITY ANALYSIS when $\boxed{f(x_c) = 0}$, $\boxed{f'(x_c) \neq 0}$

(P.T.O.)

(continued)

1/. For a stable fixed point, as $\boxed{t \to \infty}$, $\boxed{\epsilon \to 0}$.

$\therefore$ $\boxed{x \longrightarrow x_c}$, i.e. there is a convergence towards x_c.

This happens only when $\boxed{f'(x_c) < 0}$.

2/. For an unstable fixed point, as $\boxed{t \to \infty}$.

$\boxed{\epsilon \longrightarrow \infty}$, i.e. a divergence away from x_c occurs.

This happens only when $\boxed{f'(x) > 0}$.

$\Rightarrow$ i.) If $\boxed{f'(x_c) < 0}$, the fixed point is stable.

ii.) If $\boxed{f'(x_c) > 0}$, the fixed point is unstable.

3/. Now $\boxed{x = x_c + A\, e^{f'(x_c)t}} \Rightarrow \boxed{\epsilon = A\, e^{f'(x_c)t}}$

$$\Rightarrow \boxed{t = \frac{1}{f'(x_c)} \ln\left(\frac{\epsilon}{A}\right) = \frac{1}{f'(x_c)} \ln\left(\frac{x - x_c}{A}\right)}$$

For a stable fixed point, $\boxed{f'(x_c) < 0}$ and

$\boxed{x \longrightarrow x_c}$ or $\boxed{\epsilon \longrightarrow 0}$. Hence $\boxed{t \to \infty}$, for $\boxed{x \to x_c}$

The convergence to x_c takes infinitely long.

Hence, for a first-order system, there is

No overshoot of the stable fixed point, and no oscillation about the fixed point is possible.

Oscillations are only possible when $\boxed{f'(x_c)}$ is imaginary, but since $\boxed{\dot{x} = f(x)}$ is real, this is not allowed in a first-order system.

Critical Condition in the Stability Analysis

Given $\boxed{\dot{x} = f(x)}$, the fixed point is at $\boxed{\dot{x} = 0}$, i.e. $\boxed{f(x_c) = 0}$. Perturbing $\boxed{x = x_c + \epsilon}$, we get

$$\boxed{\dot{x} = \dot{\epsilon} = f(x_c) + f'(x_c)\,\epsilon + \frac{1}{2!} f''(x_c)\,\epsilon^2 + \cdots} \quad (\dot{x}_c = 0).$$

At the fixed point $\boxed{f(x_c) = 0}$. In addition if $\boxed{f'(x_c) = 0}$, then we have a critical condition (no longer linear).

$$\Rightarrow \boxed{\dot{\epsilon} \simeq \frac{f''(x_c)}{2!}\,\epsilon^2}$$ (ϵ^2 can no longer be neglected, but higher powers of ϵ^2 are neglected).

$$\Rightarrow \boxed{\frac{d\epsilon}{dt} \simeq \frac{f''(x_c)}{2}\,\epsilon^2} \Rightarrow \boxed{\int \frac{d\epsilon}{\epsilon^2} = \frac{f''(x_c)}{2} \int dt}.$$

$$\Rightarrow \boxed{\frac{\epsilon^{-1}}{-1} \simeq \frac{f''(x_c)}{2}(t - A)}$$ → A is an integration constant [$f''(x_c)$ is constant]

$$\Rightarrow \boxed{\epsilon \simeq \frac{-2}{f''(x_c)}\,\frac{1}{t - A}} \Rightarrow \boxed{x \simeq x_c - \frac{2}{f''(x_c)}\,\frac{1}{t - A}}$$

When $\boxed{t \to \infty}$, $\boxed{x \to x_c}$ (slow power-law convergence).

1/. When $\boxed{f'(x_c) < 0}$, $\boxed{x \simeq x_c + e^{f'(x_c)t}}$. As $t \to \infty$, the convergence towards x_c is a rapid exponential convergence.

2/. When $\boxed{f'(x_c) = 0}$ (critical condition),

$\boxed{x \simeq x_c + \frac{B}{t - A}}$ $\left(B = \frac{-2}{f''(x_c)}\right)$, the convergence ~~towards~~ towards x_c as $t \to \infty$, is a slow power-law convergence.

Testing Stability of Fixed Points $\boxed{\dot{x} = f(x)}$

1/. $\boxed{\dot{x} = f(x) = a - x^2}$ $(\underline{a > 0})$ $\boxed{f'(x) = -2x}$ $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = \pm\sqrt{a}}$

$\therefore$ $\boxed{f'(\sqrt{a}) = -2\sqrt{a} < 0 \ (\underline{\text{Stable}})}$, $\boxed{f'(-\sqrt{a}) = 2\sqrt{a} > 0 \ (\underline{\text{unstable}})}$

2/. $\boxed{\dot{x} = f(x) = x^2 - a}$ $(\underline{a > 0})$ $\boxed{f'(x) = 2x}$ $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = \pm\sqrt{a}}$

$\therefore$ $\boxed{f'(\sqrt{a}) = 2\sqrt{a} > 0 \ (\underline{\text{unstable}})}$, $\boxed{f'(-\sqrt{a}) = -2\sqrt{a} < 0 \ (\underline{\text{Stable}})}$

3/. $\boxed{\dot{x} = f(x) = a - bx}$ $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = a/b}$ $\boxed{f'(x) = -b}$

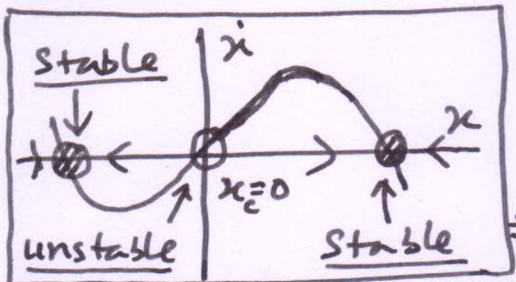
<u>If $b > 0$, x_c is stable</u>. <u>If $b < 0$, x_c is unstable</u>.

4/. $\boxed{\dot{x} = f(x) = ax - bx^2}$, $(a, b > 0)$. $\boxed{f'(x) = a - 2bx}$

$\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = 0,\ a/b}$. $\boxed{f'(0) = a > 0 \ (\underline{\text{unstable}})}$ $\boxed{f'(a/b) = -a < 0}$ $(\underline{\text{Stable}})$

5/. $\boxed{\dot{x} = f(x) = x + x^2 - x^3}$ (<u>Second order enhances, third order inhibits</u>).

$\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = 0}$ and $\boxed{x_c^2 - x_c - 1 = 0}$ $\Rightarrow$ $\boxed{x_c = \frac{1 \pm \sqrt{5}}{2}}$ (<u>3 roots</u>)



$\boxed{f'(x) = 1 + 2x - 3x^2}$ $\boxed{f'(0) = 1 > 0 \ (\underline{\text{unstable}})}$

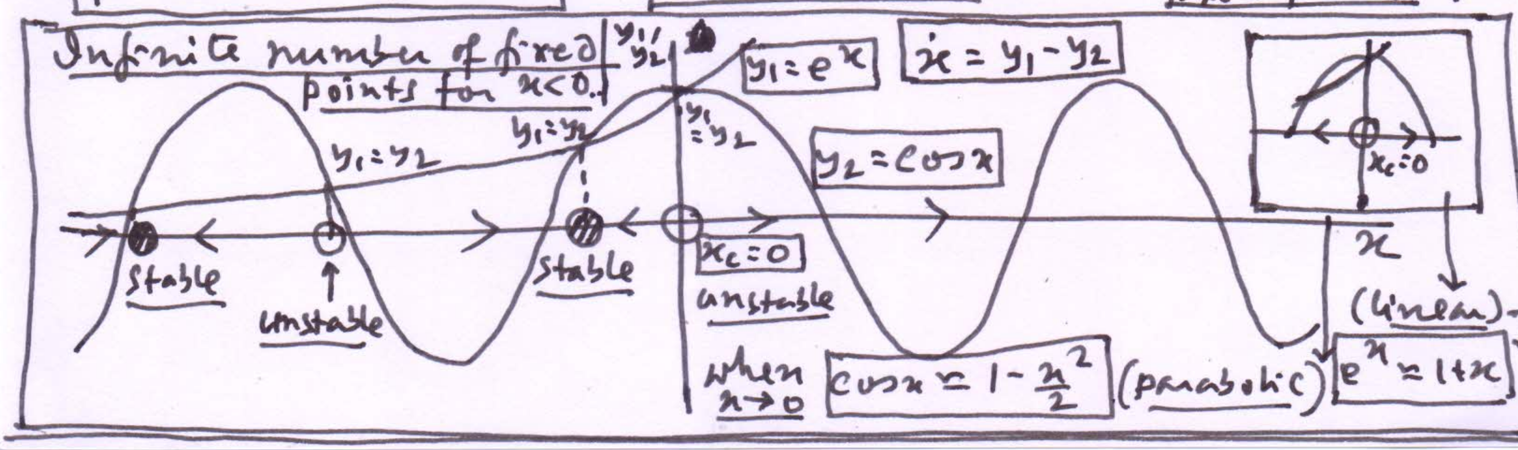
$$f'\left(\frac{1 \pm \sqrt{5}}{2}\right) = 1 + (1 \pm \sqrt{5}) - \frac{3}{4}(1 \pm \sqrt{5})^2$$

$\Rightarrow$ $\boxed{f'\left(\frac{1 \pm \sqrt{5}}{2}\right) = -\frac{\sqrt{5}}{2}(\sqrt{5} \pm 1) < 0}$ (Both outer roots are <u>Stable</u>).

6/. $\boxed{\dot{x} = f(x) = e^x - \cos x}$ Write $\boxed{y_1 = e^x}$ and $\boxed{y_2 = \cos x}$

At $\boxed{x = 0}$, $\boxed{y_1 = 1 = y_2}$ $\Rightarrow$ $\boxed{\dot{x} = 0}$ Hence $\boxed{x_c = 0}$ is a <u>fixed point</u>.

$\boxed{f'(x) = e^x + \sin x}$ $\Rightarrow$ $\boxed{f'(0) = 1 > 0}$ $\boxed{x_c = 0}$ is an <u>unstable fixed point</u>.

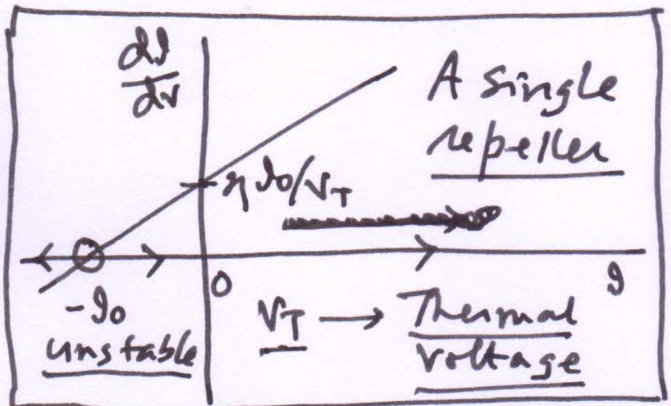


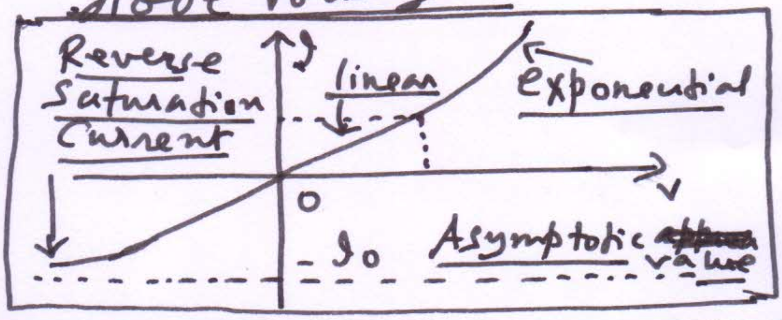
Practical Applications of Phase Portraits

In a p-n junction diode $\boxed{I = I_0\left(e^{\eta V/V_T} - 1\right)}$

$$\Rightarrow \boxed{\frac{dI}{dV} = I_0\, e^{\eta V/V_T} \cdot \frac{\eta}{V_T} = \frac{\eta}{V_T}\left(I + I_0\right)}, \quad \boxed{I \equiv I(V)}$$

in the form of $\boxed{\dot{x} = a + bx}$. I_0, η and V_T are fixed parameters of the system. I is the diode current, V is the diode voltage.





A few Physical Cases of $\boxed{\dot{x} = a - bx}$ $\boxed{\dot{x} = f(x)}$
(linear functions)

1/. Stokes' Law of Terminal Velocity : $\boxed{\dot{v} = f(v)}$

$$\boxed{\dot{v} = \bar{g} - \frac{k}{m}v}\,, \quad \boxed{\bar{g} = g\left(1 - \frac{\rho_f}{\rho}\right)} \quad \boxed{k, m > 0} \quad \boxed{g, \rho_f, \rho > 0}$$

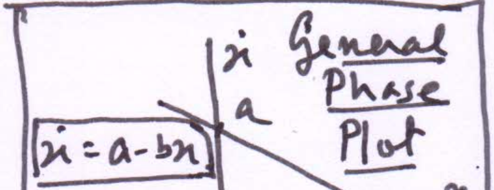
2/. Kelvin's Viscoelastic Formula: $\boxed{\dot{\epsilon} = f(\epsilon)}$

$$\boxed{\dot{\epsilon} = \frac{\sigma}{\eta} - \frac{Y}{\eta}\epsilon} \cdot \boxed{\sigma, \eta, Y > 0}$$ (for highly viscous materials)

3/. Q-R-C Circuit : $\boxed{\dot{Q} = f(Q)}$

$$\boxed{\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}} \quad \boxed{V_0, R, C > 0}$$

The foregoing physical systems have this common linear plot. →

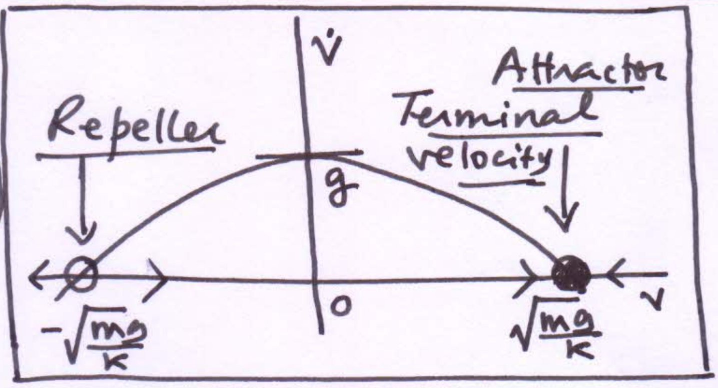


Free Fall of a Parachutist (Terminal Velocity)

$$\boxed{\dot{v} = g - \frac{k}{m} v^2}$$

$\boxed{g, k, m > 0}$ $\boxed{\dot{v} = f(v)}$

$\boxed{\dot{v} = 0} \Rightarrow \boxed{v_c = \pm\sqrt{\frac{mg}{k}}}$



Item Response Theory (Teaching-Learning Process)

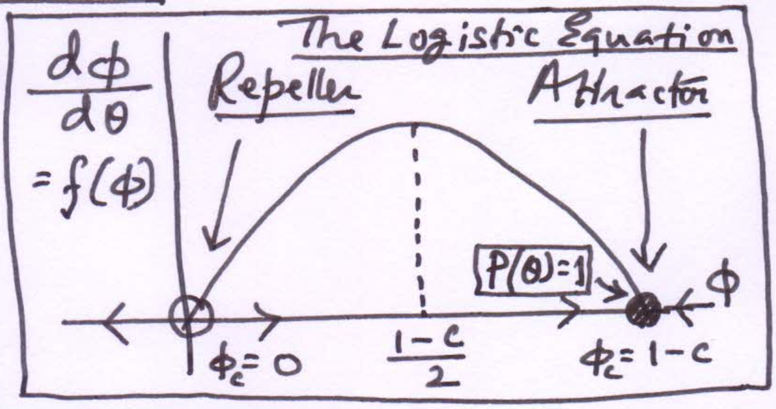
$$\boxed{P(\theta) = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$$

$\boxed{c, b, \omega > 0}$ (fixed parameters)

Define $\boxed{\phi = P(\theta) - c}$

$$\boxed{\frac{d\phi}{d\theta} = \frac{\phi}{\omega}\left[1 - \frac{\phi}{1-c}\right]}$$

$\boxed{\frac{d\phi}{d\theta} = 0} \Rightarrow \boxed{\phi_c = 0}$ $\boxed{\phi_c = 1-c}$



Spread of Agricultural Innovations

$$\boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1 - \frac{x}{N}\right)}$$

$\boxed{N, c, c' > 0}$ (Fixed parameters)

Rescale $\boxed{X = x/N}$, $\boxed{T = cNt}$ and $\boxed{a = c'/cN}$. $(a > 0)$

$\Rightarrow \boxed{\frac{dX}{dT} = (X+a)(1-X)}$ $\boxed{\frac{dX}{dT} = 0} \Rightarrow \boxed{X_c = -a}$ and $\boxed{X_c = 1}$

$\boxed{\frac{dX}{dT} = F(X)} \Rightarrow \boxed{\frac{d^2X}{dT^2} = \frac{dF}{dX}\frac{dX}{dT}} \rightarrow$ Chain Rule

$$\boxed{\frac{d(x/N)}{d(Nct)} = \left(\frac{x}{N} + \frac{c'}{cN}\right)\left(1 - \frac{x}{N}\right)} \rightarrow \text{Rescaling}$$

(P.T.O.)

Now, $\boxed{\frac{dX}{dT} = F(X) = (X+a)(1-X) = X + a - aX - X^2}$

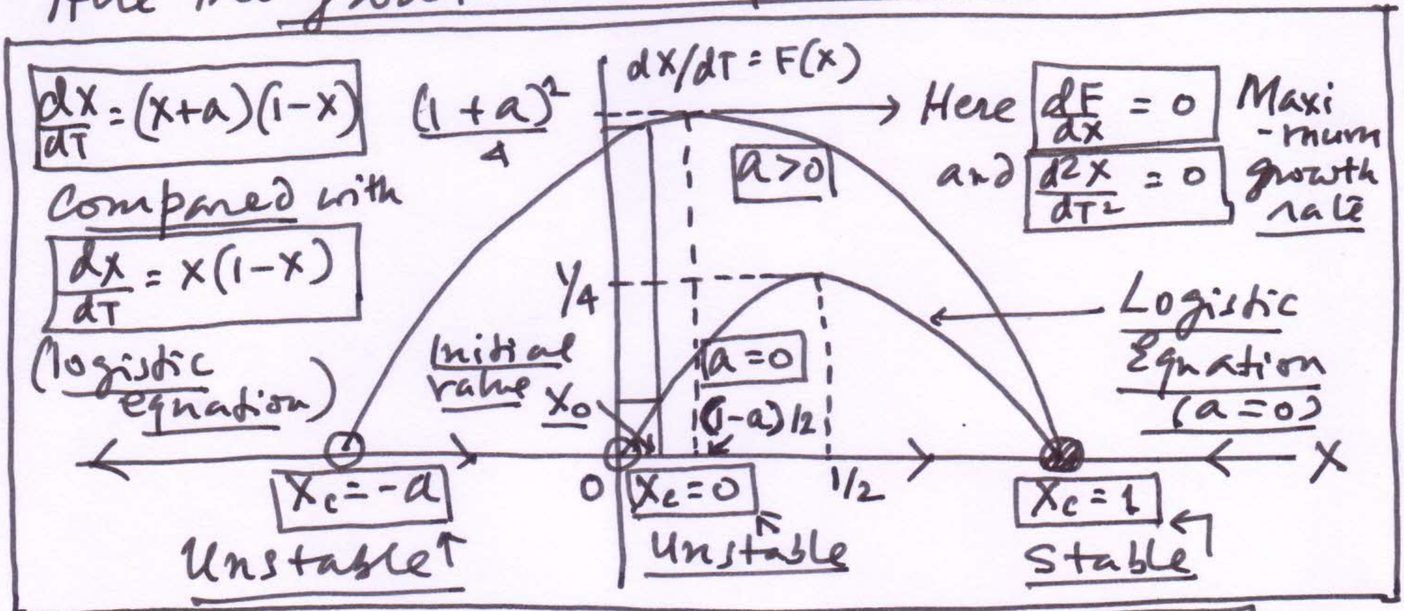
$\Rightarrow \boxed{F(X) = a + (1-a)X - X^2}$. We expect $\boxed{a<1}$, because $\boxed{a = \frac{c'/c}{N}}$ and since N, the number of farmers, has a large value, $\boxed{a<1}$. $\Rightarrow \boxed{(1-a)>0}$.

$\boxed{\frac{dF}{dX} = 1 - a - 2X}$ When $\boxed{\frac{dF}{dX} = 0} \Rightarrow \boxed{X = \frac{1-a}{2}}$

$$\boxed{F\left(\frac{1-a}{2}\right) = \left(\frac{1-a}{2} + a\right)\left(1 - \frac{1-a}{2}\right) = \frac{(1+a)^2}{4}}$$

At $\boxed{X = \frac{1-a}{2}}$, $\boxed{\frac{d^2X}{dT^2} = \frac{dF}{dX}\frac{dX}{dT} = 0}$ $\left(\because \frac{dF}{dX} = 0\right)$

Here the growth rate of X is highest.



i) For an initial value $\boxed{0 < X_0 < \frac{1-a}{2}}$, a much higher growth rate occurs than it would be for the logistic equation (a=0).

ii) For both cases, $\boxed{a>0}$ and $\boxed{a=0}$, the final attractor state is $\boxed{X_c=1} \Rightarrow \boxed{x = N}$. But for $\boxed{a>0}$, the early growth is much higher than for $\boxed{a=0}$.

Auto catalysis (In Chemical Reactions)

$$\boxed{A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X}$$

A is a catalyst, that aids chemical X to stimulate its own production — autocatalysis.

k_1 and k_{-1} are rate constants for the forward and backward reactions, respectively.

$\boxed{x = [X] \rightarrow \text{Concentration of } X, \quad a = [A] \rightarrow \text{Concentration of } A}$

For an unlimited amount of A, $\boxed{a = \text{constant}}$.

Law of mass action of chemical kinetics:
Rate of a chemical reaction is proportional to the product of the concentration of the reactants.

$A + X \xrightarrow{k_1} X + X$ (Depletion of X)
(Creation of X) $X + X \xrightarrow{k_{-1}} A + X$
(Rate balance)

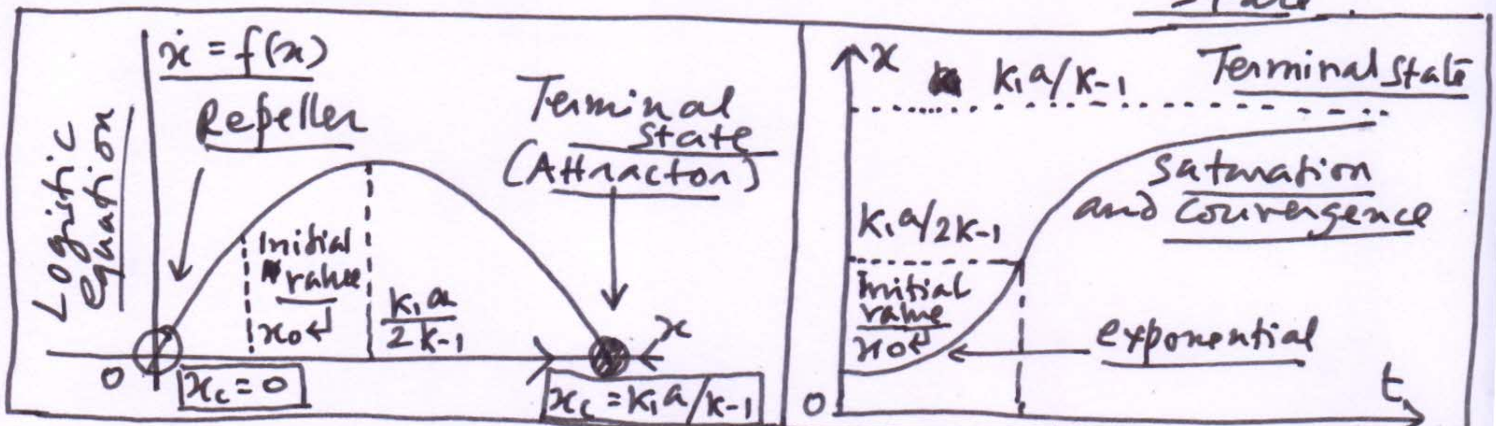
i) Forward reaction rate: $\boxed{\dot{x} \propto a x}$

ii) Backward reaction rate: $\boxed{\dot{x} \propto -x^2}$

[Negative sign ⇒ depletion]

$\dot{x} = f(x)$ ⇒ $\boxed{\dot{x} = k_1 a x - k_{-1} x^2}$ (in balance) → The Logistic Equation.

$\boxed{\dot{x} = 0} \Rightarrow \boxed{x_c = 0}$ and $\boxed{x_c = k_1 a / k_{-1}}$ → The stable terminal state.



Gompertz Law of Tumour Growth

$$\boxed{\dot{x} = \frac{dx}{dt} = f(x) = -ax\ln(bx)} \quad (\underline{a, b > 0})$$

$\boxed{\dot{x} = 0} \Rightarrow$ i.) $\ln(bx) = 0 \Rightarrow \boxed{bx_c = 1} \Rightarrow \boxed{x_c = b^{-1}}$ ← (fixed point)

ii.) $\boxed{\dot{x} = -\frac{a\ln(bx)}{x^{-1}}}$, when $\underline{x \to 0}$, $\boxed{\ln(bx) \to -\infty}$ and $\boxed{x^{-1} \to \infty}$.

Applying $\underline{\text{L'Hospital Rule}}$, when $\boxed{x \to 0}$,

$$\Rightarrow \boxed{\dot{x} = \frac{-ab(1/bx)}{-x^{-2}} = -\frac{ax^{-1}|_{x=0}}{-x^{-2}|_{x=0}} \to \frac{a\infty}{\infty^2} = 0}$$

$\Rightarrow$ when $\boxed{x \to 0}$, $\boxed{\dot{x} \to 0} \Rightarrow \boxed{x_c = 0}$ is a $\underline{\text{fixed point}}$.

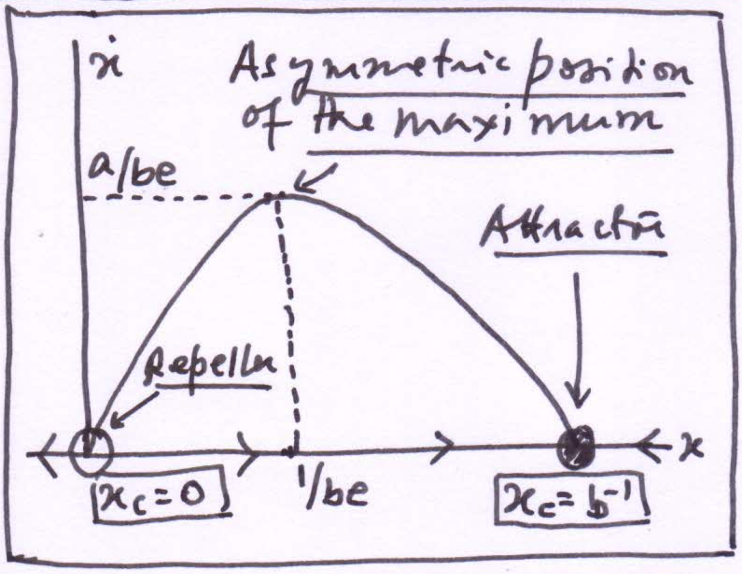
$\underline{\text{Turning point of } f(x)}$: $\boxed{f(x) = -ax\ln(bx)}$.

$$\Rightarrow f'(x) = -a\left[\ln(bx) + x \cdot \frac{1}{\cancel{bx}} \cdot \cancel{b}\right] = -a\left[1 + \ln(bx)\right]$$

$$\Rightarrow f'(x) = 0 \Rightarrow \boxed{\ln(bx) = -1} \Rightarrow bx = \frac{1}{e} \Rightarrow \boxed{x = \frac{1}{be}}$$

$$f''(x) = -a \cdot \frac{1}{bx} b = -\frac{a}{x} \Rightarrow \boxed{f''\left(\frac{1}{be}\right) = -abe} \quad (a, b, e > 0)$$

Since $\boxed{f''\left(\frac{1}{be}\right) < 0}$, the $\underline{\text{turning point}}$ is a $\underline{\text{maximum}}$.



i.) At the $\underline{\text{maximum}}$ $\boxed{x = \frac{1}{be}}$.

$$\Rightarrow \boxed{f\left(\frac{1}{be}\right) = -\frac{a}{be}\ln\left(\frac{1}{e}\right) = \frac{a}{be}}$$

ii.) Since $\boxed{e \simeq 2.72}$, the $\underline{\text{position of the maximum}}$ is $\underline{\text{asymmetric}}$ (not halfway between $\boxed{x_c = 0}$ and $\boxed{x_c = b^{-1}}$)

iii.) $\underline{\text{Tumour size is scaled by } b^{-1}!}$

The Allee Effect (Warder Clyde Allee)

[Effective growth rate of a species is highest for intermediate values of population size, x.]

$\boxed{\frac{\dot{x}}{x} = r - a(x-b)^2} \Rightarrow \boxed{\dot{x} = f(x) = x\left[r - a(x-b)^2\right]}$

Fixed points: $\boxed{\dot{x} = f(x_c) = 0} \Rightarrow \boxed{x_c = 0}$ and $(x_c - b)^2 = r/a \Rightarrow \boxed{x_c = b \pm \sqrt{r/a}}$. Three fixed points (r and a must have same signs for all the fixed points to be real).

Linear Stability Analysis: $\boxed{\dot{x} = f(x) = \left[r - a(x-b)^2\right]x}$

$\Rightarrow f(x) = x\left[r - a(x^2 - 2bx + b^2)\right] = x\left[r - ax^2 + 2abx - ab^2\right]$

$\Rightarrow \boxed{\dot{x} = f(x) = (r - ab^2)x + 2abx^2 - ax^3} \rightarrow$ cubic polynomial

$\Rightarrow \boxed{f'(x) = (r - ab^2) + 4abx - 3ax^2} \rightarrow$ quadratic

i) When $\boxed{x_c = 0}$: $\Rightarrow \boxed{f'(0) = r - ab^2}$. For $\boxed{x_c = 0}$ to be a stable fixed point, $\boxed{r - ab^2 < 0} \Rightarrow \boxed{\sqrt{r/a} < b}$.

ii) When $\boxed{x_c = b - \sqrt{\frac{r}{a}}}$: $\Rightarrow f'\left(b - \sqrt{\frac{r}{a}}\right) = (r - ab^2) + 4ab\left(b - \sqrt{\frac{r}{a}}\right) - 3a\left(b - \sqrt{\frac{r}{a}}\right)^2$

$\Rightarrow f'\left(b - \sqrt{\frac{r}{a}}\right) = r - ab^2 + 4ab^2 - 4b\sqrt{ar} - 3a\left(b^2 - 2b\sqrt{\frac{r}{a}} + \frac{r}{a}\right)$

$\Rightarrow f'\left(b - \sqrt{\frac{r}{a}}\right) = r - \cancel{ab^2} + \cancel{4ab^2} - 4b\sqrt{ar} - \cancel{3ab^2} + 6ab\sqrt{\frac{r}{a}} - 3r$

$\Rightarrow \boxed{f'\left(b - \sqrt{\frac{r}{a}}\right) = -2r + 2b\sqrt{ar} = 2\sqrt{ar}\left[b - \sqrt{\frac{r}{a}}\right]}$

If $\boxed{\sqrt{r/a} < b}$, then $\boxed{f'\left(b - \sqrt{r/a}\right) > 0} \Rightarrow$ unstable fixed point

(P.T.O.)

iii) When $\boxed{x_c = b+\sqrt{\frac{r}{a}}}$: $\Rightarrow f'\left(b+\sqrt{\frac{r}{a}}\right) = (r-ab^2) + 4ab\left(b+\sqrt{\frac{r}{a}}\right) - 3a\left(b+\sqrt{\frac{r}{a}}\right)^2$

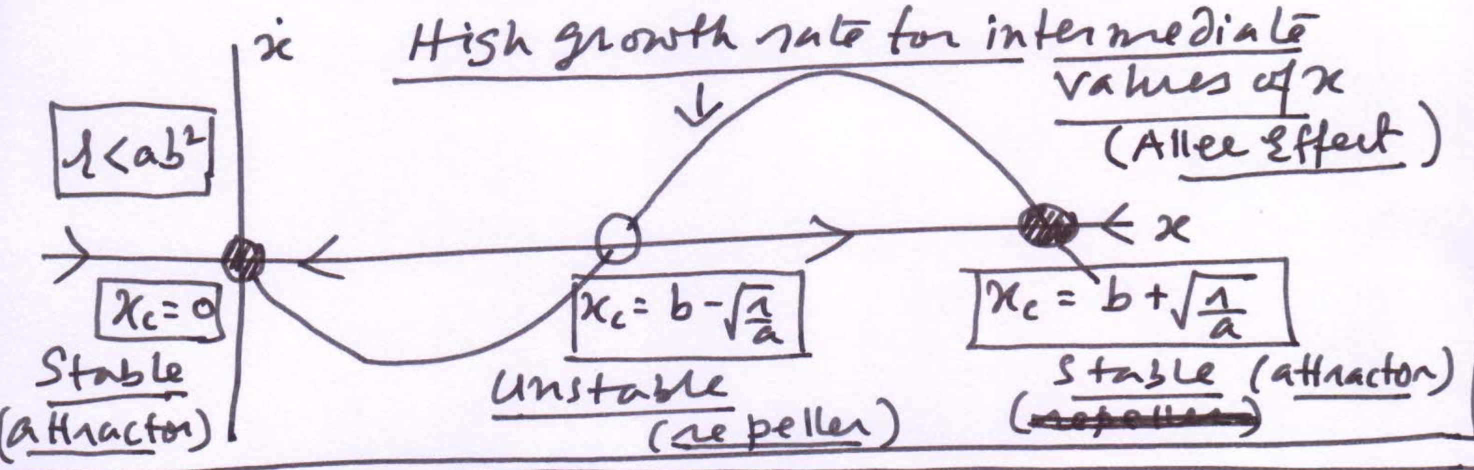
$$\Rightarrow f'\left(b+\sqrt{\tfrac{r}{a}}\right) = r - ab^2 + 4ab^2 + 4b\sqrt{ar} - 3a\left(b^2 + 2b\sqrt{\tfrac{r}{a}} + r/a\right)$$

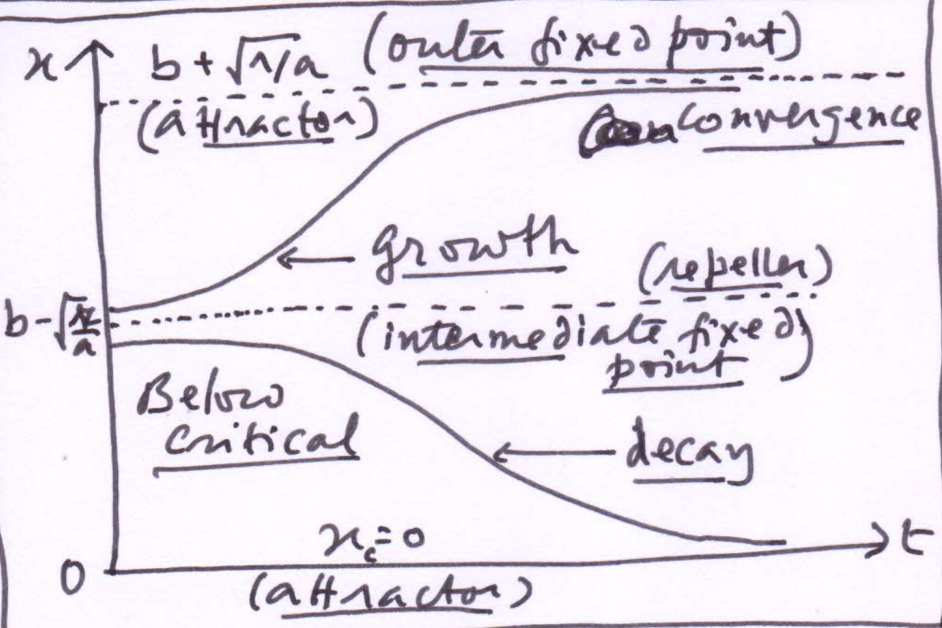
$$\Rightarrow f'\left(b+\sqrt{\tfrac{r}{a}}\right) = r + \cancel{3ab^2} + 4b\sqrt{ar} - \cancel{3ab^2} - 6b\sqrt{ar} - 3r$$

$$\Rightarrow \boxed{f'\left(b+\sqrt{\tfrac{r}{a}}\right) = -2r - 2b\sqrt{ar} = -2\sqrt{ar}\left[b+\sqrt{\tfrac{r}{a}}\right]}$$

If $\boxed{\left(b+\sqrt{\frac{r}{a}}\right) > 0}$, then $\boxed{f'\left(b+\sqrt{\frac{r}{a}}\right) < 0}$ $\Rightarrow$ Stable fixed point

Consider $\boxed{r - ab^2 < 0}$ $\boxed{\dot{x} = f(x) = (r-ab^2)x + 2abx^2 - ax^3}$





The difference between the logistic function and the Allee function is the existence of an intermediate state in the latter. Below this state there is only decay to $\boxed{x_c = 0}$.

1/. The Allee Effect is satisfied when $\boxed{r < ab^2}$.

2/. When x is large, $\boxed{\dot{x} \simeq -ax^3}$. (P.T.O.)

3/. Hence, for saturation at large values of x, $\boxed{a>0}$ (converge to the outer fixed point).

4/. For high growth rates at intermediate values of x, $\boxed{ab>0}$ $\because$ $\boxed{a>0}$, $\Rightarrow$ $\boxed{b>0}$.

5/. Since, $\boxed{b \pm \sqrt{r/a}}$ are real fixed points, r and a have same signs. $\because$ $\boxed{a>0}$, $\boxed{r>0}$

6/. Hence, all $\boxed{a, b, r > 0}$ (the fixed parameters).

Turning points of $f(x)$: $\Rightarrow$ $\boxed{f'(x) = 0}$

$$\Rightarrow \boxed{f'(x) = (r - ab^2) + 4abx - 3ax^2 = 0}$$ A quadratic equation

$$\Rightarrow \boxed{3ax^2 - 4abx - (r - ab^2) = 0}$$ Hence, two turning points exist.

$$x = \frac{4ab \pm \sqrt{16a^2b^2 + 4.3a(r - ab^2)}}{6a}$$

$$\Rightarrow x = \frac{2}{3}b \pm \frac{1}{6a}\sqrt{4a^2b^2 + 12ar\left(\frac{ab^2}{ab^2}\right)}$$

The discriminant is always positive.

$$\Rightarrow \boxed{x = \frac{2}{3}b \pm \frac{b}{3}\sqrt{1 + \frac{3r}{ab^2}} = \frac{b}{3}\left[2 \pm \left(1 + \frac{3r}{ab^2}\right)^{1/2}\right]}$$

If $\boxed{r \ll ab^2}$ we can expand the discriminant binomially, $\therefore$ $\boxed{\left(1 + \frac{3r}{ab^2}\right)^{1/2} \simeq 1 + \frac{3r}{2ab^2}}$

$$\Rightarrow \boxed{x \simeq \frac{b}{3}\left[2 \pm 1 \pm \frac{3r}{2ab^2}\right]} \Rightarrow \boxed{x \simeq b + b\left(\frac{r}{2ab^2}\right)}$$ (for upper sign)

and $\boxed{x \simeq \frac{b}{3} - b\left(\frac{r}{2ab^2}\right)}$ (for lower sign). Hence, one turning point is at $\boxed{x>b}$ and the other is at $\boxed{0<x<b}$.