Lecture 17

$$\hat{\theta} = E\left[\check{\theta} \mid T(x)\right] = g(T(x))$$

#1. $$x[n] = A + w[n]$$

$$T(x) = \sum_n x[n]$$

$$E\left[g(T(x))\right] = A$$

#2. $$x[0] = A + w[0]$$ $$\left[w[0] \sim \mathcal{U}\left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \right]$$

$$E\left[g[x[0]]\right] = A$$

$$h \stackrel{?}{=} g$$

$$E\left[h[x[0]]\right] = A$$

$$T(x) = x[0]$$

$$v(T) = g(T) - h(T)$$

$$E\left[v(T)\right] = 0$$

$$\int_{-\infty}^{\infty} v(T)\, p(x; A)\, dx = 0 \quad \forall A.$$

$$x = x[0] = T$$

$$\int_{-\infty}^{\infty} v(T)\, p(T; A)\, dT = 0 \quad \forall A.$$

$$\int_{A-\frac{1}{2}}^{A+\frac{1}{2}} v(T)\, dT = 0 \qquad \forall A$$

$$v(T) = \sin 2\pi T$$

$$g(T) - h(T) = \sin 2\pi T$$

$$g(T) \neq h(T)$$

* A sufficient statistic is complete if

$$\int_{-\infty}^{\infty} v(T)\, p(T;\theta)\, dT = 0 \qquad \text{for all } \theta.$$

is satisfied only by $v(T) = 0$ for all T.

* (i) Find sufficient statistic for θ by using NF factorization Theorem.

(ii) Check if $T(x)$ is complete. If yes then proceed. If not this approach can not be applied.

(iii) $$\hat{\theta} = E\left(\check{\theta} \mid T(x)\right)$$

or

$$\hat{\theta} = g\left(T(x)\right)$$

Best Linear Unbiased Estimator (BLUE)

$$\{x[0], x[1], \ldots, x[N-1]\} \quad \theta$$

Linear: $$\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n]$$

Unbiased: $$E[\hat{\theta}] = \theta$$

$$E\left[\sum_{n=0}^{N-1} a_n x[n]\right] = \theta \quad \text{— (1)}$$

Best: $$\mathrm{var}(\hat{\theta}) = E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right]$$

$$= E\left[\left(\sum_n a_n x[n] - E\left[\sum_n a_n x[n]\right]\right)^2\right]$$

$$\vec{a} = [a_0 \; a_1 \; \cdots \; a_{N-1}]^T$$

$$\mathrm{var}(\hat{\theta}) = E\left[\left(\vec{a}^T\vec{x} - a^T E[\vec{x}]\right)^2\right]$$

$$= E\left[\left(\vec{a}^T\left(\vec{x} - E[\vec{x}]\right)\right)^2\right]$$

$$= E\left[\vec{a}^T \underbrace{\left(\vec{x} - E[\vec{x}]\right)\left(\vec{x} - E[\vec{x}]\right)^T}\vec{a}\right]$$

$$\mathrm{var}(\hat{\theta}) = \vec{a}^T C \vec{a} \quad \text{— (2)}$$

$$E[x[n]] = S[n]\,\theta \qquad \text{— (3)} \qquad [s[n] \text{ known}]$$

minimize $\operatorname{var}(\hat{\theta}) = \vec{a}^T \vec{C} \vec{a}$ s.t. $E(\hat{\theta}) = \sum_n a_n E[x[n]] = \theta$.

The constraint: $\sum_{n=0}^{N-1} a_n E[x[n]] = \theta$.

$$\sum_{n=0}^{N-1} a_n s[n]\theta = \theta$$

$$\sum_{n=0}^{N-1} a_n s[n] = 1.$$

$$\vec{a}^T \vec{s} = 1 \qquad \left[\vec{s} = \begin{bmatrix} s[0] \\ s[1] \\ \vdots \\ s[N-1] \end{bmatrix} \right.$$

The minimization problem—

$$J = \vec{a}^T \vec{C} \vec{a} + \lambda(\vec{a}^T \vec{s} - 1) \qquad \boxed{J: \text{Lagrangian } f^n}$$

$$\frac{\partial J}{\partial a} = 2\vec{C}\vec{a} + \lambda \vec{s} \longrightarrow 0$$

$$\vec{a} = -\frac{\lambda}{2} \vec{C}^{-1} \vec{s} \qquad \text{— (4)}$$

$$\vec{a}^T \vec{s} = 1$$

$$\left(-\frac{\lambda}{2} C^{-1} s\right)^T \vec{s} = 1$$

$$-\frac{\lambda}{2} s^T C^{-1} s = 1$$

$$-\frac{\lambda}{2} = \frac{1}{\vec{s}^T C^{-1} s} \quad — \quad 5$$

$$a_{opt} = \frac{\vec{C}^{-1}\vec{s}}{\vec{s}^T \vec{C}^{-1}\vec{s}}$$

$$\boxed{\hat{\theta} = \vec{a}^T \vec{x} = \frac{\vec{s}^T \vec{C}^{-1} \vec{x}}{\vec{s}^T C^{-1} \vec{s}}}$$

$$\boxed{var(\hat{\theta}) = \vec{a}_{opt}^T \vec{C} \, \vec{a}_{opt} = \frac{1}{\vec{s}^T \vec{C}^{-1} \vec{s}}}$$

$$\boxed{E[\hat{\theta}] = \frac{s^T C^{-1} E[x]}{s^T C^{-1} s} = \frac{s^T C^{-1} \theta s}{s^T C^{-1} s} = \theta}$$

unbiased

1. s : scaled mean
2. C : covariance matrix.

entire PDF is not known

#1. $x[n] = A + w[n]$

$w[n] \to$ unspecified PDF σ^2.

$n = 0, 1, \ldots, N-1$

$$E[x[n]] = A$$

$$E[x[n]] = s[n]\,\theta \qquad \begin{bmatrix} \theta = A \\ s[n] = 1 \end{bmatrix}$$

$$\hat{A} = \frac{\vec{s}^T \vec{C}^{-1} \vec{x}}{\vec{s}^T \vec{C}^{-1} \vec{s}} = \frac{\vec{1} \frac{1}{\sigma^2} \vec{I} \vec{x}}{1^T \frac{1}{\sigma^2} \vec{I} \, 1^T}$$

$$= \frac{\vec{1}^T \vec{x}}{\vec{1}^T \vec{1}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$$\mathrm{Var}(\hat{A}) = \frac{1}{\vec{1}^T \frac{1}{\sigma^2} I \vec{1}} = \frac{\sigma^2}{\vec{1}^T \vec{1}} = \frac{\sigma^2}{N}.$$

#2. uncorrelated noise σ_n^2

$$\vec{s} = \vec{1}$$

$$\hat{A} = \frac{\vec{1}^T C^{-1} \vec{x}}{\vec{1}^T C^{-1} \vec{1}} \qquad \mathrm{Var}(\hat{A}) = \frac{1}{\vec{1}^T C^{-1} \vec{1}}.$$

$$C = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} \frac{1}{\sigma_0^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_1^2} & \cdots & & \\ 0 & \cdots & & & \frac{1}{\sigma_{N-1}^2} \end{bmatrix}$$

$\hat{A} =$ ________ $\qquad \mathrm{Var}(\hat{A}) =$ ________