

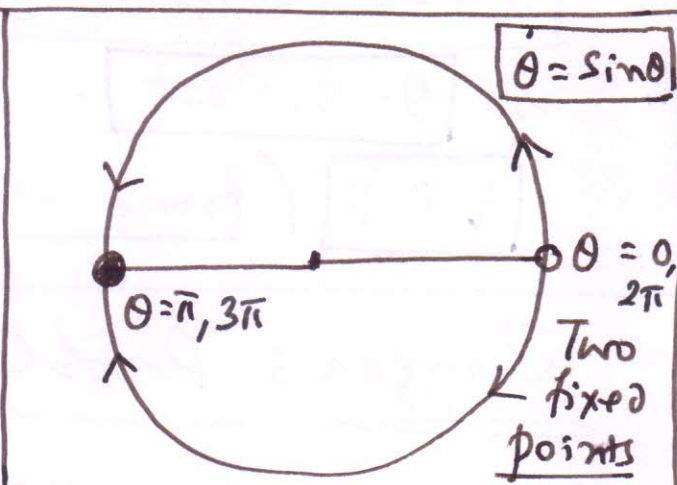
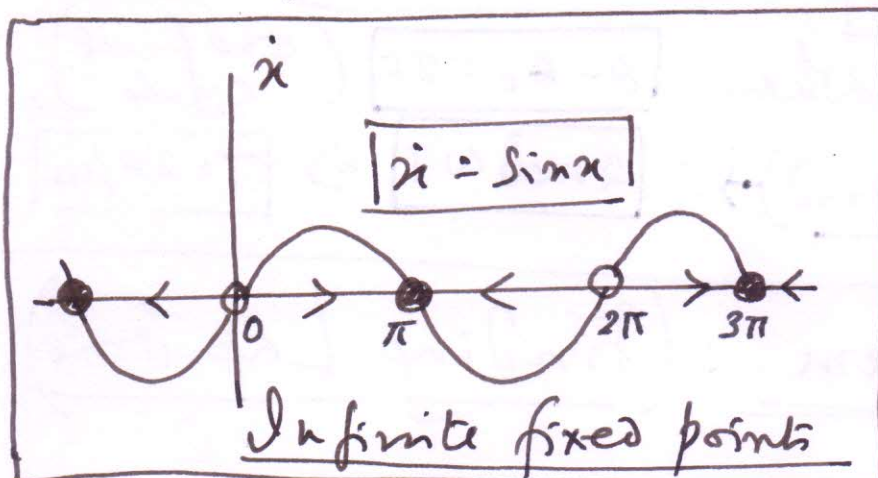
# Nonlinear Dynamics:

## Flows on the Circle

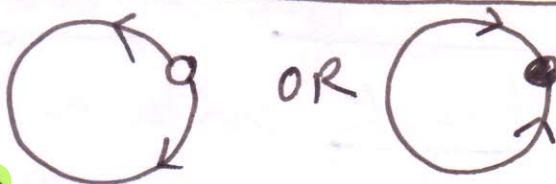
$\dot{x} = f(x)$  is a flow on the line.

$\dot{\theta} = f(\theta)$  is a flow on the circle.  
(also a one-dimensional system).

Example:  $\dot{x} = \sin x$  and  $\dot{\theta} = \sin \theta$



Impossible situations:



There cannot be an odd number of fixed points on the circle.

Exception: Odd number of half-stable points is allowed.



Nature of  $f(\theta)$ : 1/.  $f(\theta)$  is real 2/.  $f(\theta)$  is  $2\pi$  periodic.

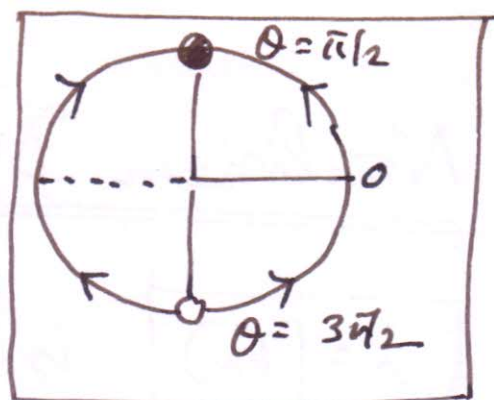
Example:  $\dot{\theta} = \theta$  is NOT a  $2\pi$  periodic function.

So  $\theta = 0 \Rightarrow \dot{\theta} = 0$ , and  $\theta = 2\pi \Rightarrow \dot{\theta} \neq 0$  (not unique) at  $\theta = 0, 2\pi, \dots$



Example :  $\dot{\theta} = \cos \theta$

The fixed points are shifted by a phase of  $\pi/2$ .



## Uniform Oscillator

$\theta \rightarrow$  phase, angle or phase angle.

$$\dot{\theta} = \omega \text{ (constant)} \Rightarrow \theta = \omega t + \theta_0$$

$\Rightarrow \theta - \theta_0 = \omega t$ . When  $\theta - \theta_0 = 2\pi$  (one full cycle),

$$t = T \text{ (time period)} \Rightarrow 2\pi = \omega T \Rightarrow T = 2\pi/\omega$$

## Jogger's Problem : (Finding Lap Time)

1. Faster jogger Speedy runs on a circular track with a time period of  $T_1$ .

2. Slower jogger Pokey has a time period  $T_2$ .

$\therefore T_1 < T_2$ . Also  $\dot{\theta}_1 = \omega_1$  and  $\dot{\theta}_2 = \omega_2$   
(both have uniform jogging rate).

Phase Difference,  $\phi = \theta_1 - \theta_2$

$$\Rightarrow \dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2$$

$$\Rightarrow \left( \frac{d\phi}{dt} = \omega_1 - \omega_2 \right) \Rightarrow dt = \frac{d\phi}{\omega_1 - \omega_2}$$

(P.T.O.)

$$\Rightarrow \int_0^{T_{\text{lap}}} dt = \int_0^{2\pi} \frac{d\phi}{\omega_1 - \omega_2}$$

(continued)

Now  $\boxed{\omega_1 = 2\pi/T_1}$   
and  $\boxed{\omega_2 = 2\pi/T_2}$

$$\Rightarrow \boxed{T_{\text{lap}} = \frac{2\pi}{\omega_1 - \omega_2}}$$

One full lap implies  $2\pi$  phase ~~and~~ difference

$$\Rightarrow \boxed{T_{\text{lap}} = \frac{1}{T_1^{-1} - T_2^{-1}} = \left( \frac{1}{T_1} - \frac{1}{T_2} \right)^{-1}} \quad (\text{Beats})$$

## Non-Uniform Oscillator

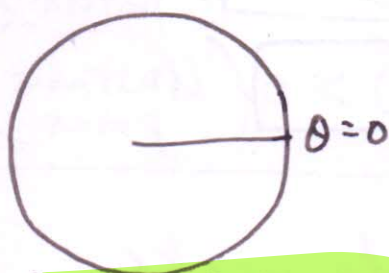
$$\dot{\theta} = \omega - a \sin \theta$$

with  $\underline{a > 0}, \underline{\omega > 0}$ .

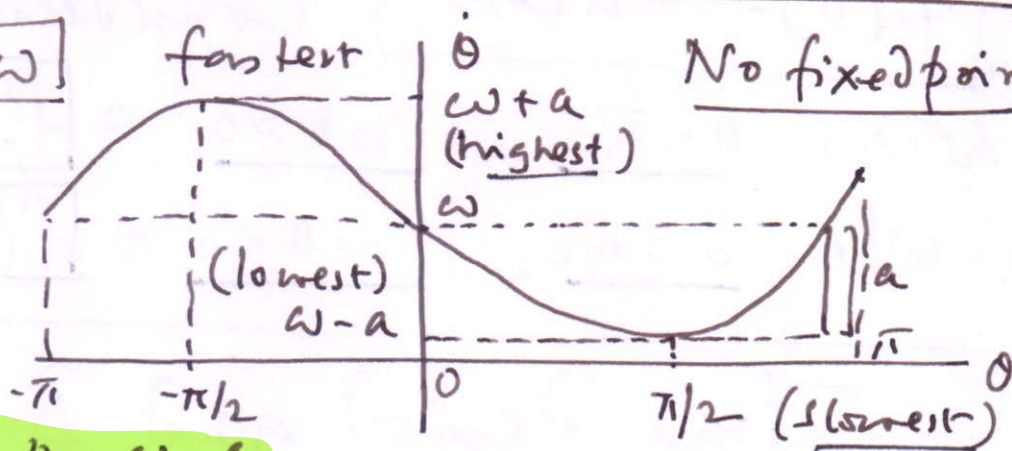
Case I:  $\boxed{a < \omega}$

fastest

No fixed point

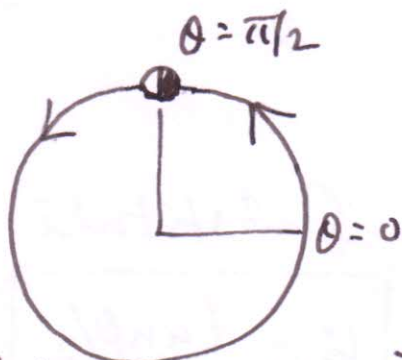


No fixed point on the circle.

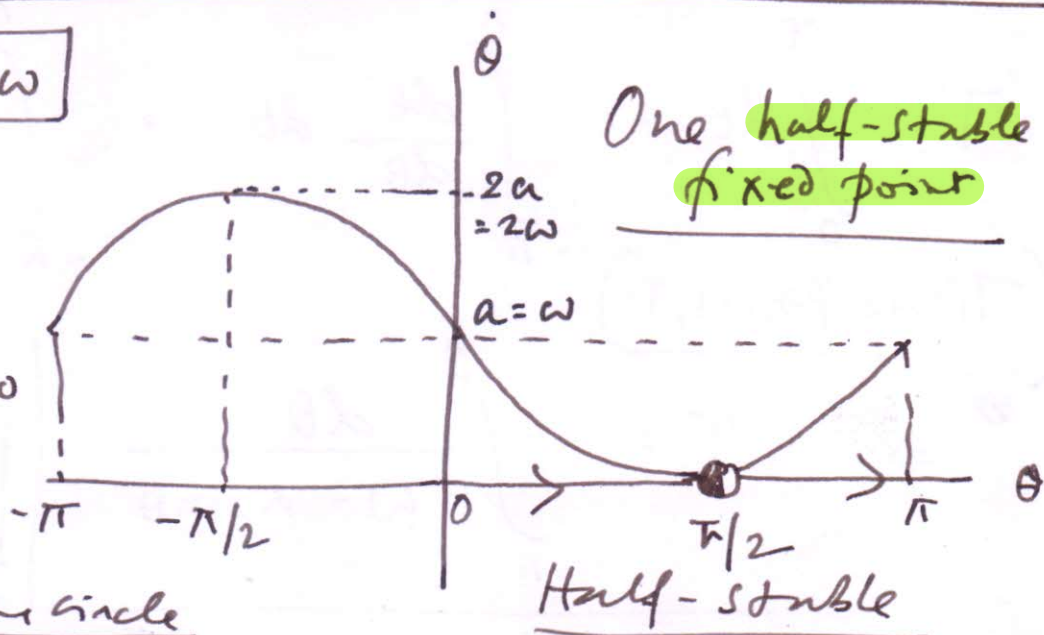


Case II:  $\boxed{a = \omega}$

One half-stable fixed point



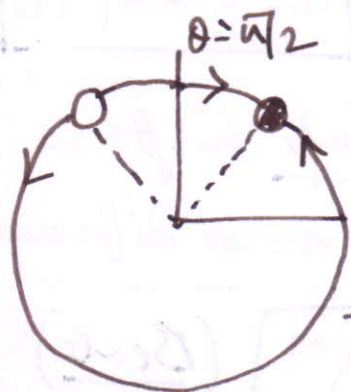
A half-stable fixed point on the circle



Half-stable

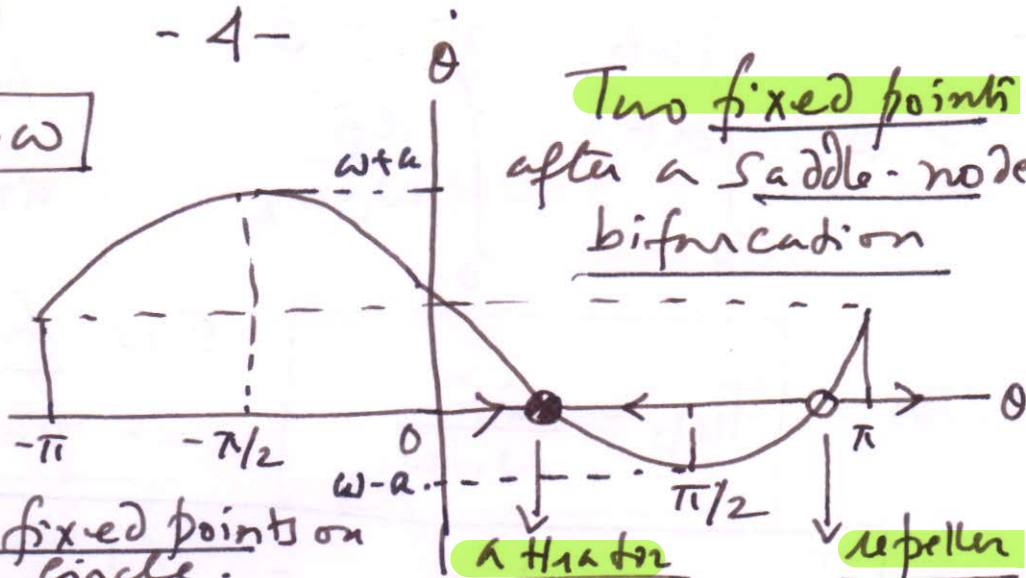


Case III:  $a > \omega$



$\theta = 0$

Two fixed points on the circle.



Two fixed points after a saddle-node bifurcation

## Linear Stability Analysis

$$\dot{\theta} = \omega - a \sin \theta$$

is the form

$$\dot{\theta} = f(\theta)$$

$(a > 0)$ .

$$f'(\theta) = -a \cos \theta$$

Considering Case III ( $a > \omega$ ),

1. When  $\theta < \pi/2$ ,  $\cos \theta > 0 \Rightarrow f'(\theta) < 0$  (Stable point).
2. When  $\theta > \pi/2$ ,  $\cos \theta < 0 \Rightarrow f'(\theta) > 0$  (Unstable point).

## Time Period and Bottle necks

$$T = \int_0^T dt = \int_{-\pi}^{\pi} \frac{dt}{d\theta} d\theta = \int_{-\pi}^{\pi} \frac{d\theta}{\dot{\theta}}$$

(Time Period, T)

$$\Rightarrow T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a \sin \theta}$$

Substitute

$$u = \tan \theta/2$$

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$$\Rightarrow \boxed{u = \frac{2 \sin \theta/2}{2 \cos \theta/2} \cdot \frac{\cos \theta/2}{\cos \theta/2}} \Rightarrow \boxed{\sin \theta = 2u \cos^2 \theta/2}$$

Further  $\boxed{du = \frac{1}{2} \sec^2 \theta/2 d\theta} \Rightarrow \boxed{d\theta = \frac{2 du}{\sec^2 \theta/2}}$

Substitute for  $\sin \theta$  and  $d\theta$  in the integral,

$$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a \sin \theta} = \int_{-\pi}^{\pi} \frac{2 du}{\sec^2 \theta/2 (\omega - 2au \cos^2 \theta/2)}$$

$$\Rightarrow \boxed{T = \int_{-\pi}^{\pi} \frac{2 du}{\omega \sec^2 \theta/2 - 2au}} \quad \left| \begin{array}{l} \text{When } \theta = \pm \pi, \\ u = \tan \theta/2 \text{ gives} \\ u \rightarrow \pm \infty. \\ (\text{limits of the integral}) \end{array} \right.$$

Further  $\boxed{\sec^2 \theta/2 = 1 + \tan^2 \theta/2 = 1 + u^2}$

$$\Rightarrow \boxed{T = \int_{-\infty}^{\infty} \frac{2 du}{\omega(1+u^2) - 2au} = \frac{2}{\omega} \int_{-\infty}^{\infty} \frac{du}{(1+u^2) - 2au/\omega}}$$

Now considering  $\boxed{1+u^2 - 2au/\omega}$ , we write it as,

$$1 + u^2 - 2u\left(\frac{a}{\omega}\right) + \frac{a^2}{\omega^2} - \frac{a^2}{\omega^2} = \left(u - \frac{a}{\omega}\right)^2 + \left[1 - \left(\frac{a}{\omega}\right)^2\right]$$

Next we substitute  $\boxed{z = u - \frac{a}{\omega}}$  and  $\boxed{\alpha^2 = 1 - \left(\frac{a}{\omega}\right)^2}$

$$\Rightarrow \boxed{dz = du}$$



$$T = \frac{2}{\omega} \int_{-\infty}^{\infty} \frac{dz}{z^2 + \alpha^2} = \frac{2}{\omega \alpha} \int_{-\infty}^{\infty} \frac{d(z/\alpha)}{(z/\alpha)^2 + 1}$$

$$\Rightarrow T = \frac{2}{\omega \alpha} \arctan\left(\frac{z}{\alpha}\right) \Big|_{-\infty}^{\infty} = \frac{2}{\omega \alpha} \left[ \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right]$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega \sqrt{1 - a^2/\omega^2}} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}} \rightarrow \text{Time Period to go from } -\pi \text{ to } \pi$$

Limiting Cases: 1/ When  $a = 0$ ,  $\boxed{T = 2\pi/\omega}$ .

2/ When  $a \rightarrow \omega$ ,  $T = \frac{2\pi}{\sqrt{(\omega+a)(\omega-a)}} \approx \frac{\sqrt{2}\pi}{\sqrt{\omega(\omega-a)}}$   
 (approximate  $\omega+a \approx 2\omega$ )  $\Rightarrow T \rightarrow \infty$  (becomes very large).

This is the BOTTLENECK TIME SCALE.

Saddle-Node Bifurcation near the Bottleneck

Start with  $\boxed{\dot{\theta} = \omega - a \sin \theta}$ .

The bifurcation occurs at  $\pi/2$  for  $\boxed{a = \omega}$ .

We write  $\boxed{\theta = \pi/2 + \phi}$  where  $\phi$  is a small angle ( $\phi \ll 1$ ).

$$\Rightarrow \dot{\theta} = \dot{\phi} = \omega - a \sin(\pi/2 + \phi)$$

$$\Rightarrow \dot{\phi} = \omega - a \sin[\pi/2 - (-\phi)] = \omega - a \cos(-\phi)$$

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$$\Rightarrow \dot{\phi} = \omega - a \cos \phi \quad (\because \cos \phi = \cos(-\phi))$$

$$\text{Now } \dot{\phi} \approx \omega - a \left(1 - \frac{\phi^2}{2} + \dots\right) \quad \left| \begin{array}{l} \text{Expansion} \\ \text{for small } \phi. \end{array} \right.$$

$$\Rightarrow \boxed{\dot{\phi} = (\omega - a) + \frac{a}{2} \phi^2}$$

Rescale  $\boxed{\phi = kx}$

$$\Rightarrow k \dot{x} = (\omega - a) + \frac{a}{2} k^2 x^2$$

$$\Rightarrow \boxed{\dot{x} = \frac{\omega - a}{k} + \frac{a}{2} k x^2}$$

Set  $k = \frac{2}{\lambda}$ .

$$\therefore \boxed{\dot{x} = \frac{a}{2} (\omega - a) + x^2}$$

Writing

$$\boxed{1 = \frac{a}{2} (\omega - a)}$$

We get,

$$\boxed{\dot{x} = 1 + x^2}$$

→ The normal form of a saddle-node bifurcation.

## Time Scales near the Saddle-Node Bifurcation

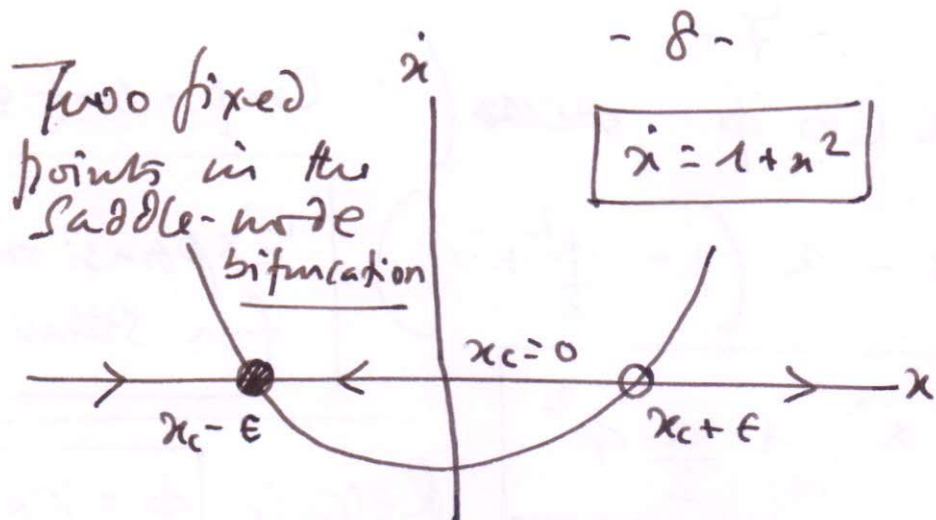
$$\boxed{\dot{x} = f(x, 1) = 1 + x^2}$$

The bifurcation occurs at  $\boxed{x_c = 0}$  for  $\boxed{1 = 0}$ .

$$\Rightarrow \boxed{\frac{dx}{dt} = 1 + x^2} \Rightarrow dt = \frac{dx}{1 + x^2}$$

$$\Rightarrow \tau = \int dt = \int_{x_c - \epsilon}^{x_c + \epsilon} \frac{dx}{1 + x^2} \quad \left| \begin{array}{l} \text{Time scale} \\ \text{to travel} \\ \text{through } x_c \end{array} \right.$$





The bifurcation takes place at  $x_c = 0$  for  $1 = 0$   $\pm \epsilon$  is a slight displacement from  $x_c$ .

$$\therefore T = \frac{1}{\sqrt{1}} \arctan\left(\frac{x}{\sqrt{1}}\right) \Big|_{x_c - \epsilon}^{x_c + \epsilon} \quad \text{Note } x_c = 0.$$

$$\Rightarrow T = \frac{1}{\sqrt{1}} \left[ \arctan\left(\frac{\epsilon}{\sqrt{1}}\right) - \arctan\left(\frac{-\epsilon}{\sqrt{1}}\right) \right]$$

Now  $1 = \frac{a}{2}(\omega - a) \Rightarrow$  when  $a \rightarrow \omega$ ,  $1 \rightarrow 0$ , and  $\pm \epsilon/\sqrt{1} \rightarrow \pm \infty$

$$\therefore T \approx \frac{1}{\sqrt{1}} \left[ \pi/2 - (-\pi/2) \right] \approx \frac{\pi}{\sqrt{1}} \rightarrow \infty$$

The time scale becomes very large.

Hence, when  $a \approx \omega$ , we can write,

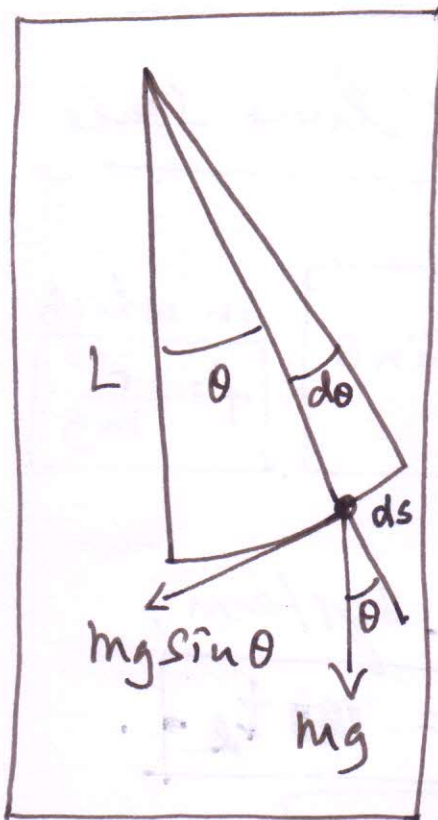
$T \approx \frac{\sqrt{2}\pi}{\sqrt{\omega}\sqrt{\omega-a}}$

This is the same as the bottle neck time.

$\therefore$  The bottle neck time is a consequence of the saddle-node bifurcation when  $a \rightarrow \omega$ .



# Overdamped Pendulum



$$\boxed{ds = L d\theta} \Rightarrow \frac{ds}{dt} = L \frac{d\theta}{dt} = L \dot{\theta}$$

But  $v = \frac{ds}{dt} = L \dot{\theta}$

$$\boxed{\ddot{\theta} = \frac{d^2\theta}{dt^2}}$$

Now  $\boxed{\text{Force} = m \frac{dv}{dt} = mL \ddot{\theta}}$

Hence  $\boxed{mL \ddot{\theta} = -mg \sin \theta}$

for an undamped pendulum.

For a pendulum that is damped,

a ~~periodic~~ regular force is necessary to maintain the oscillations.

$$\therefore \boxed{mL \ddot{\theta} = -mg \sin \theta + (F - b\dot{\theta})}$$

$F \rightarrow$  Force,  $b\dot{\theta} \rightarrow$  Damping force.

$$\therefore [b] = \frac{MLT^{-2}}{T^{-1}} = MLT^{-1} \rightarrow \text{Dimension of } b.$$

We define  $\boxed{t = \tau T}$  where  $T$  is ~~the~~ a constant relevant time scale.

Now dividing throughout by  $mg$  we get,

$$\boxed{\frac{L}{gT^2} \frac{d^2\theta}{d\tau^2} + \frac{b}{mgT} \frac{d\theta}{d\tau} = \frac{F}{mg} - \sin \theta}$$

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We define  $t_{os}^2 = \frac{L}{g}$  (Oscillatory time scale)

and  $t_d = \frac{b}{mg}$  (Damping time scale)

$$\therefore \left[ \left( \frac{t_{os}^2}{T^2} \right) \frac{d^2\theta}{d\tau^2} + \left( \frac{t_d}{T} \right) \frac{d\theta}{d\tau} = f - \sin\theta \right] \quad \text{in which} \quad \left[ f = \frac{F}{mg} \right]$$

For a Damping Dominated system, the natural time scale  $T = t_d$ .

$$\therefore \left[ \left( \frac{t_{os}}{t_d} \right)^2 \frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau} = f - \sin\theta \right]$$

$\therefore t_{os} \ll t_d \Rightarrow \left( \frac{t_{os}}{t_d} \right)^2 \ll 1$ . We approximate,

$$\left[ \frac{d\theta}{d\tau} \approx f - \sin\theta \right] \quad \text{by neglecting the second-order derivative.}$$

The above equation has the same form as  $\dot{\theta} = \omega - a \sin\theta \Rightarrow$  Bottleneck time scales are inherent to overdamped systems.

Large Damping: Due to 1. large b (damping coefficient) 2. small m (small inertial effects) in the formula  $t_d = b/mg$ .