Lecture - 15

$x = H\theta + w$

$\hat{\theta} = (H^T H)^{-1} H^T x$; $I(\theta) = \frac{H^T H}{\sigma^2}$

* **General Linear Model :-**

(i) Noise is not white $\Rightarrow$ $w \sim \mathcal{N}(0, C)$

(ii) $x = H\theta + s + w$ s is signal vector (known)

(i) C is not a scaled identity matrix.

C is +ve definite matrix.

C^{-1} : +ve " "

$C^{-1} = D^T D$ [D = non singular matrix]

$$C = E[w w^T]$$

$$= E[(Dw)(Dw)^T] = E[D w w^T D^T]$$

$$= D\, E[w w^T] D^T$$

$$= D C D^T$$

$$= D (D^T D)^{-1} D^T$$

$$= D D^{-1} (D^T)^{-1} D^T$$

$$= I.$$

$$x = H\theta + w$$

$$Dx = DH\theta + Dw$$

$$x' = H'\theta + w' \qquad [w' \sim \mathcal{N}(0, I)]$$

$$\hat{\theta} = (H'^T H')^{-1} H'^T x'$$

$$= (H^T D^T D H)^{-1} H^T D^T D x$$

MVUE: $\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} x$

$$C_{\hat{\theta}} = (H'^T H')^{-1} = (H^T D^T D H)^{-1} = (H^T C^{-1} H)^{-1}$$

* 2nd Extension: $\quad x = H\theta + s + w.$

$$x - s = H\theta + w.$$

$$x' = H\theta + w.$$

$$\hat{\theta} = (H^T H)^{-1} H^T x' = (H^T H)^{-1} H^T (x - s)$$

$$C_{\hat{\theta}} = \sigma^2 (H^T H)^{-1}$$

#1. $\quad x[n] = A + r^n + w[n] \quad$ for $n = 0, 1, \cdots, N-1$

r is known, A needs to be estimated.

$w[n] \sim \mathcal{N}(0, \sigma^2)$

$$\vec{x} = A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \vec{s} + \vec{w}$$

$$\vec{s} = [1 \ r \ r^2 \dots r^{N-1}]^T$$

$$\hat{\theta} = (H^T H)^{-1} H^T x' \qquad [x' = x - s]$$

$$\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} (x[n] - r^n)$$

$$\text{var}(\hat{A}) = \frac{\sigma^2}{N}.$$

$(N > p)$

$$* \quad \vec{x} = \vec{H}\vec{\theta} + \vec{s} + \vec{w}$$

$$\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} (x - s)$$

$$C_{\hat{\theta}} = (H^T C^{-1} H)^{-1}$$

- $\vec{x}$: $N \times 1$: Observation vector
- $\vec{H}$: $N \times p$: Observation matrix
- $\vec{\theta}$: $p \times 1$: vector Param.
- $\vec{s}$: $N \times 1$: Known Signal
- $\vec{w}$: $N \times 1$ $\rightarrow$ GN $\sim \mathcal{N}(0, C)$

RBLS : (Rao-Blackwell-Lehmann-Scheffe)

Sufficient Statistic :

$$x[n] = A + w[n] \qquad [w[n] \sim \mathcal{N}(0, \sigma^2)]$$

$$\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \qquad \text{minimum variance } \frac{\sigma^2}{N}.$$

$S_1 = \{x[0], x[1], \dots, x[N-1]\}$

$S_2 = \{x[0]+x[1], x[2], \dots, x[N-1]\}$

$S_3 = \{\sum_{n=0}^{N-1} x[n]\}$ $\Rightarrow$ minimal sufficient statistic.

$$P(x;A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]$$

$$T(x) = \sum_{n=0}^{N-1} x[n] = T_0$$

$$P(x \mid T(x) = T_0 ; A) = \underbrace{P(x \mid T(x) = T_0)}_{\text{independent of parameter}}$$

<u>Neyman - Fisher Factorization:</u>

$$P(\vec{x};\theta) = g(T(\vec{x}), \theta)\, h(\vec{x})$$

g : fn depending on x only through $T(x)$

h : " " " x.

$T(x)$: sufficient statistic for θ.

#1.

$$p(x;A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right\}$$

$$\sum_{n=0}^{N-1}(x[n]-A)^2 = \sum_{n=0}^{N-1} x^2[n] - 2A\sum_{n=0} x[n] + NA^2$$

$$p(x;A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \underbrace{\exp\left[-\frac{1}{2\sigma^2}\left(NA^2 - 2A\sum_n x[n]\right)\right]}_{g(T(x),A)} \underbrace{\exp\left[-\frac{1}{2\sigma^2}\sum_n x^2[n]\right]}_{h(x)}$$

$$T(x) = \sum_{n=0}^{N-1} x[n]$$

$$g(T(x),\theta)\,h(x)$$

#2.

$$x[n] = w[n]$$

power $= \sigma^2$

$$p(x;\sigma^2) = \underbrace{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right]}_{g(T(x),\sigma^2)} \cdot \underbrace{1}_{h(x)}$$

$$T(x) = \sum_{n=0}^{N-1} x^2[n].$$

#3.

$$x[n] = \underbrace{A}_{\text{known}} \cos(2\pi \underbrace{f_0}_{\text{known}} n + \underbrace{\phi}_{\text{unknown phase}}) + \underbrace{w[n]}_{\text{known variance } \sigma^2}$$

$n = 0, 1, \ldots, N-1$