Lecture - 16

NF Factorization :-

$$p(x;\theta) = g\big(T(x), \theta\big)\, h(x)$$

$T(x)$ = sufficient statistic.

\# $$x[n] = \underbrace{A \cos(2\pi f_0}_{\text{known}} n + \underbrace{\phi}_{\text{unknown}}) + \underbrace{w[n]}_{\text{known } \sigma^2}$$

$$p(x;\phi) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left[x[n] - A\cos(2\pi f_0 n + \phi)\right]^2\right\}$$

The exponent :-

$$\sum_{n=0}^{N-1} x^2[n] - 2A\sum_{n=0}^{N-1}\cos(2\pi f_0 n + \phi)\, x[n] + \sum_{n=0}^{N-1} A^2\cos^2(2\pi f_0 n + \phi)$$

$$= \sum_{n=0}^{N-1} x^2[n] - 2A\underbrace{\left(\sum_{n=0}^{N-1} x[n]\cos 2\pi f_0 n\right)}_{T_1(x)}\cos\phi + 2A\underbrace{\left(\sum_{n=0}^{N-1} x[n]\sin 2\pi f_0 n\right)}_{T_2(x)}\sin\phi$$

$$+ \sum_{n=0}^{N-1} A^2\cos^2(2\pi f_0 n + \phi)$$

$$p(x;\phi) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{n=0}^{N-1} A^2\cos^2(2\pi f_0 n + \phi) - 2A\,T_1(x)\cos\phi + 2A\,T_2(x)\sin\phi\right]\right\} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right]$$

$T_1(x)$ & $T_2(x)$ are jointly sufficient statistic for ϕ.

* $$x[n] = A + w[n]$$

$$T(x) = \sum_{n=0}^{N-1} x[n]$$

(i) Find any unbiased estimator for A

$$\check{A} = x[0]$$

$$\hat{A} = E[\check{A} \mid T]$$

(ii) Find some fn g s.t. $\hat{A} = g(T)$ is an unbiased estimator of A.

(i) $$\check{A} = x[0]$$

$$\hat{A} = E\left[x[0] \,\middle|\, \sum_{n=0}^{N-1} x[n]\right]$$

* $\begin{bmatrix} x \\ y \end{bmatrix}$: Gaussian Random Vector

$$\vec{\mu} = \begin{bmatrix} E[x] \\ E[y] \end{bmatrix}$$

$$\vec{C} = \begin{bmatrix} \mathrm{Var}(x) & \mathrm{Cov}(x,y) \\ \mathrm{Cov}(y,x) & \mathrm{Var}(y) \end{bmatrix}$$

$$p(x|y) \sim \mathcal{N}(A, B)$$

$$A: E[x|y] = \int_{-\infty}^{\infty} x\, p(x|y)\, dx = \int_{-\infty}^{\infty} x \frac{p(x,y)}{p(y)}\, dx$$

$$= \int x \frac{p(x,y)}{\int p(x,y)\,dx}\, dx$$

$$A: E[x|y] = E[x] + \frac{\mathrm{cov}(x,y)}{\mathrm{var}(y)}\left(y - E[y]\right)$$

$$B: \mathrm{var}(x|y) = \mathrm{var}(x) - \frac{\mathrm{cov}^2(x,y)}{\mathrm{var}(y)}$$

$x : x[0]$

$y : \sum_n x[n]$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x[0] \\ \sum_n x[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\vec{x}}$$

The PDF of $[x\ y]^T$ is $\mathcal{N}(\vec{\mu}, \vec{C})$

$$\vec{\mu} : E[Lx] = \vec{L}\, E[x] = \vec{L} A \vec{1} = \begin{bmatrix} A \\ NA \end{bmatrix}$$

$$\vec{C} : \vec{L}\vec{L}^T \sigma^2 = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & N \end{bmatrix}$$

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$$\hat{A} = E[x|y] = E[x] + \frac{\operatorname{cov}(x,y)}{\operatorname{var}(y)}\left(y - E[y]\right)$$

$$= A + \frac{\sigma^2}{N\sigma^2}\left(\sum_{n=0}^{N-1} x[n] - NA\right)$$

$$= A + \frac{1}{N}\left(\sum_n x[n]\right) - A$$

$$\hat{A} = \frac{1}{N}\sum_n x[n] \quad \text{MVUE}$$

(ii) $\hat{A} = g\left(\sum_{n=0}^{N-1} x[n]\right)$

$$g(x) = \frac{x}{N}.$$

$$\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

* RBLS Theorem:-

If $\check{\theta}$ is an unbiased estimator of θ and $T(x)$ is a sufficient statistic for θ, then $\hat{\theta} = E[\check{\theta} | T(x)]$ is

(i) a valid estimator.

(ii) unbiased

(iii) of lesser or equal variance than that of $\check{\theta}$ for all θ.

Additionally if the sufficient statistic is complete then $\hat{\theta}$ is MVUE.

* A statistic is complete if there is only one fn of the statistic that is unbiased..

$$\hat{\theta} = E\left[\check{\theta} \mid T(x)\right]$$

$E\left[\check{\theta} \mid T(x)\right]$ is a solely fn of $T(x)$.

$$\hat{\theta} = E\left[\check{\theta} \mid T(x)\right] = \int \check{\theta}\, p\left(\check{\theta} \mid T(x)\right) d\check{\theta}$$

$$= g(T(x))$$

* $x[n] = A + w[n]$

$$T = \sum_{n=0}^{N-1} x[n]$$

$$E\left[g(T(x))\right] = A$$

$$E\left[h(T(x))\right] = A$$

[Suppose there exists another fn h for which $E[h(T(x))] = A$]

$$E\left[g(T) - h(T)\right] = 0 \quad \forall A$$

$$T \sim \mathcal{N}(NA, N\sigma^2)$$

$$\int_{-\infty}^{\infty} \left(g(T) - h(T)\right) \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left[-\frac{1}{2N\sigma^2}\left(T - N\theta\right)^2\right] dT = 0 \quad \forall \theta.$$

$$v(T) = g(T) - h(T)$$

$$\tau = \frac{T}{N} \quad \text{and} \quad v'(\tau) = v(N\tau) = v(T)$$

$$\int_{-\infty}^{\infty} v'(\tau) \frac{N}{\sqrt{2\pi N\sigma^2}} \exp\left[-\frac{N}{2\sigma^2}\left(\theta - \tau\right)^2\right] d\tau = 0 \quad \forall \theta$$

$$v'(\tau) * w(\tau) = 0 \quad \left[w(\tau) = \text{Gaussian pulse}\right]$$

$$v'(\tau) = 0 \quad \forall \tau$$

$$g(T) = h(T)$$