Lecture - 14

$H = DB$

$y = DBx + w$

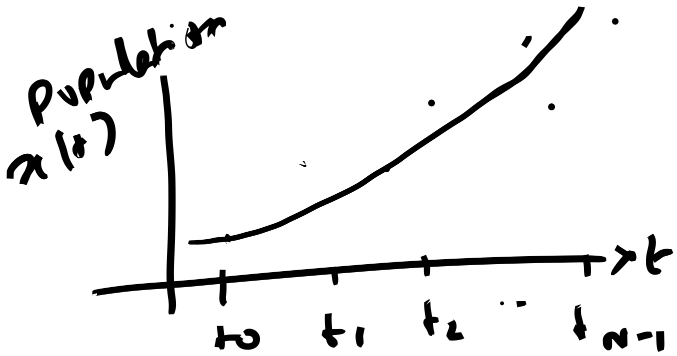
$$x = H\theta + w$$

$$\hat{\theta}_{MVUE} = (H^T H)^{-1} H^T x$$

$$I(\theta) = \frac{H^T H}{\sigma^2}$$

$$C_{\hat{\theta}} = \sigma^2 (H^T H)^{-1}$$

#1.



$$x(t_n) = \theta_1 + \theta_2 t_n + \theta_3 t_n^2 + w(t_n)$$

$n = 0, 1, \dots, N-1$

$$\vec{x} = \vec{H}\vec{\theta} + \vec{w}$$

$$\vec{x} = \begin{bmatrix} x(t_0) \\ x(t_1) \\ \vdots \\ x(t_{N-1}) \end{bmatrix} \qquad \vec{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$H = \begin{array}{c} n=0 \rightarrow \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ \vdots & \vdots & \\ 1 & t_{N-1} & t_{N-1}^2 \end{bmatrix}_{N \times 3} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)(N-1)} \end{bmatrix}$$

$$W = e^{-j\frac{2\pi}{N}}$$

$$\hat{\theta} = (H^T H)^{-1} H^T x$$

$$\hat{s}(t) = \sum_{i=1}^{3} \hat{\theta}_i t^{i-1}$$

$\downarrow$

(curve)

#2.

$$x[n] = \sum_{k=1}^{m} a_k \cos\left(\frac{2\pi k n}{N}\right) + \sum_{k=1}^{m} b_k \sin\left(\frac{2\pi k n}{N}\right) + w[n]$$

$$\theta = [a_1\ a_2 \cdots a_m, b_1, b_2 \cdots b_m]^T$$

$$\vec{H} = \begin{bmatrix} 1 & 1 & 1 \cdots & 1 & 0 & 0 & 0 \cdots & 0 \\ \cos\left(\frac{2\pi}{N}\right) & \cdots & \cos\left(\frac{2\pi m}{N}\right) & \sin\left(\frac{2\pi}{N}\right) & \cdots & \sin\left(\frac{2\pi m}{N}\right) \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \end{bmatrix}_{N \times 2m}$$

$$\vec{H} = [\vec{h}_1\ \vec{h}_2 \cdots \vec{h}_{2m}]$$

$$\vec{h}_i^T \vec{h}_j = 0 \quad \text{for } i \neq j$$

$$H^T H = \begin{bmatrix} h_1^T \\ \vdots \\ h_{2m}^T \end{bmatrix} [h_1\ h_2 \cdots h_{2m}]$$

$$= \begin{bmatrix} h_1^T h_1 & h_1^T h_2 & \cdots & h_1^T h_{2m} \\ h_2^T h_1 & h_2^T h_2 & \cdots & h_2^T h_{2m} \\ \vdots & & & \\ h_{2m}^T h_1 & h_{2m}^T h_2 & \cdots & h_{2m}^T h_{2m} \end{bmatrix}$$

0

0

$$\sum_{n=0}^{N-1} \cos\left(\frac{2\pi i n}{N}\right) \cos\left(\frac{2\pi j n}{N}\right) = \frac{N}{2}\delta_{ij}$$

$$\sum_{n=0}^{N-1} \sin\left(\frac{2\pi i n}{N}\right) \sin\left(\frac{2\pi j n}{N}\right) = \frac{N}{2}\delta_{ij}$$

$$\boxed{\begin{aligned} &\int_0^T \cos n\omega_0 t \, dt = 0 \\ &\int_0^T \sin n\omega_0 t \, dt = 0 \end{aligned}}$$

$$H^T H = \begin{bmatrix} \frac{N}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{N}{2} & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots & \frac{N}{2} \end{bmatrix} = \frac{N}{2} I$$

$$\hat{\theta} = (H^T H)^{-1} H^T x = \frac{2}{N} H^T x = \frac{2}{N} \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_{2m}^T \end{bmatrix} x$$

$$= \begin{bmatrix} \frac{2}{N} h_1^T x \\ \frac{2}{N} h_2^T x \\ \vdots \\ \frac{2}{N} h_{2m}^T x \end{bmatrix}$$

$$\hat{a}_k = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi k n}{N}\right)$$

$$\hat{b}_k = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \sin\left(\frac{2\pi k n}{N}\right)$$

$$E[\hat{a}_k] = a_k$$

$$E[\hat{b}_k] = b_k$$

$$C_{\hat{\theta}} = \sigma^2 (H^T H)^{-1} = \sigma^2 \left(\frac{N}{2} I\right)^{-1} = \frac{2\sigma^2}{N} I$$