Lecture - 13

Vector parameter CRLB for Transformations:-

$$\vec{\alpha} = \vec{g}(\vec{\theta}) \qquad \vec{g} : r\text{-dimensional } f^n.$$

$$C_{\hat{\alpha}} - \frac{\partial \vec{g}(\vec{\theta})}{\partial \theta} \vec{I}^{-1}(\vec{\theta}) \frac{\partial \vec{g}(\vec{\theta})}{\partial \theta}^T \geq 0$$

$$\frac{\partial g(\vec{\theta})}{\partial \vec{\theta}} = \begin{bmatrix} \frac{\partial g_1(\vec{\theta})}{\partial \theta_1} & \frac{\partial g_1(\vec{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_1(\vec{\theta})}{\partial \theta_p} \\ \frac{\partial g_2(\vec{\theta})}{\partial \theta_1} & \frac{\partial g_2(\vec{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_2(\vec{\theta})}{\partial \theta_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_r(\vec{\theta})}{\partial \theta_1} & \frac{\partial g_r(\vec{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_r(\vec{\theta})}{\partial \theta_p} \end{bmatrix}$$

* CRLB for the General Gaussian Case:-

$$\vec{x} \sim \mathcal{N}\left(\vec{\mu}(\vec{\theta}), \vec{C}(\vec{\theta})\right)$$

Fisher information matrix

$$[I(\theta)]_{ij} = \left[\frac{\partial \vec{\mu}(\vec{\theta})}{\partial \theta_i}\right] \vec{C}^{-1}(\vec{\theta}) \left[\frac{\partial \vec{\mu}(\vec{\theta})}{\partial \theta_j}\right] + \frac{1}{2} \operatorname{tr}\left[C^{-1}(\theta) \frac{\partial C(\theta)}{\partial \theta_i} C^{-1}(\theta) \frac{\partial C(\theta)}{\partial \theta_j}\right]$$

$$\frac{\partial \vec{\mu}(\vec{\theta})}{\partial \theta_i} = \begin{bmatrix} \frac{\partial [\mu(\theta)]_1}{\partial \theta_i} \\ \frac{\partial [\mu(\theta)]_2}{\partial \theta_i} \\ \vdots \\ \frac{\partial [\mu(\theta)]_N}{\partial \theta_i} \end{bmatrix}$$

$$\frac{\partial C(\theta)}{\partial \theta_i} = \begin{bmatrix} \frac{\partial [C(\theta)]_{11}}{\partial \theta_i} & \frac{\partial [C(\theta)]_{12}}{\partial \theta_i} & \cdots & \frac{\partial [C(\theta)]_{1N}}{\partial \theta_i} \\ \vdots & \vdots & & \vdots \\ \frac{\partial [C(\theta)]_{N1}}{\partial \theta_i} & \cdots & \cdots & \cdots \end{bmatrix}$$

For the scalar parameter:-

$$x \sim \mathcal{N}\left(\vec{\mu}(\theta), \vec{C}(\theta)\right)$$

$$\vec{I}(\theta) = \left[\frac{\partial \vec{\mu}(\theta)}{\partial \theta}\right]^T \vec{C}^{-1}(\theta) \left[\frac{\partial \vec{\mu}(\theta)}{\partial \theta}\right] + \frac{1}{2} \operatorname{tr}\left[\left(\vec{C}^{-1}(\theta) \frac{\partial \vec{C}(\theta)}{\partial \theta}\right)^2\right]$$

$x[n] = s[n;\theta] + w[n]$ $\quad n = 0, 1, \ldots, N-1$

$w[N] \sim \mathcal{N}(0, \sigma^2)$

$$\vec{C} = \sigma^2 \vec{I}$$

$$I(\theta) = \frac{1}{\sigma^2}\left[\frac{\partial \mu(\theta)}{\partial \theta}\right]^T \left[\frac{\partial \mu(\theta)}{\partial \theta}\right]$$

$$= \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(\frac{\partial [\mu(\theta)]_n}{\partial \theta}\right)^2$$

$$I(\theta) = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2$$

$$\mathrm{var}(\hat{\theta}) \geq \frac{\sigma^2}{I(\theta)}$$

θ to be vector:-

$$[\vec{I}(\vec{\theta})]_{ij} = \left[\frac{\partial \vec{h}(\vec{\theta})}{\partial \theta_i}\right]^T \frac{\vec{I}}{\sigma^2}\left(\frac{\partial \vec{h}(\vec{\theta})}{\partial \theta_j}\right)$$

$$= \frac{1}{\sigma^2}\left[\frac{\partial \vec{h}(\vec{\theta})}{\partial \theta_i}\right]^T \left[\frac{\partial \vec{h}(\vec{\theta})}{\partial \theta_j}\right]$$

$$= \frac{1}{\sigma^2}\sum_{n=0}^{N-1} \frac{\partial s[n;\vec{\theta}]}{\partial \theta_i} \frac{\partial s[n;\vec{\theta}]}{\partial \theta_j}$$

$x[n] = w[n]$ $\quad \left[w[n] : WGN \sim (0, \sigma^2) \right)$

$$\theta : \sigma^2$$

$$C(\sigma^2) = \sigma^2 I \leftarrow \text{Identity matrix}$$

$$\vec{I}(\sigma^2) = \frac{1}{2}\operatorname{tr}\left[\left(C^{-1}(\theta) \frac{\partial C(\theta)}{\partial \theta}\right)^2\right]$$

$$= \frac{1}{2}\operatorname{tr}\left[\left(C^{-1}(\sigma^2) \frac{\partial C(\sigma^2)}{\partial \sigma^2}\right)^2\right]$$

$$= \frac{1}{2}\operatorname{tr}\left[\left(\frac{1}{\sigma^2} \frac{\partial \sigma^2 I}{\partial \sigma^2}\right)^2\right]$$

$$= \frac{1}{2} \operatorname{tr}\left[\frac{1}{\sigma^4} I\right]$$

$$= \frac{N}{2\sigma^4}.$$

$*$ $$x[n] = \underbrace{\underline{A} \cos\left(2\pi \underline{f_0} n + \underline{\phi}\right)} + w[n]$$

$n = 0, 1, \cdots, N-1$

$A > 0$

$0 < f_0 < \frac{1}{2}.$ $\quad \vec{\theta} = \left[A \;\; f_0 \;\; \phi\right]^T$

$$\left[I(\theta)\right]_{ij} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \frac{\partial s[n;\theta]}{\partial \theta_i} \frac{\partial s[n;\theta]}{\partial \theta_j}$$

$$\alpha = 2\pi f_0 n + \phi.$$

$$\left[I(\theta)\right]_{11} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \cos^2\alpha = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(\frac{1}{2} + \frac{1}{2}\cos 2\alpha\right)$$

$$\approx \frac{N}{2\sigma^2}.$$

$$\left[I(\theta)\right]_{12} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} 2\pi n \cos\alpha \left(-A \sin\alpha\right)$$

$$= -\frac{\pi A}{\sigma^2} \sum_{n=0}^{N-1} 2n \cos\alpha \sin\alpha$$

$$= -\frac{\pi A}{\sigma^2} \sum_{n=0}^{N-1} n \sin 2\alpha \approx 0$$

$\left[I(\theta)\right]_{13}$

$= ?$ ♡

$[I(\theta)]_{21} = ?$ 0 $\quad [I(\theta)]_{31} = ?$ 0

$[I(\theta)]_{22} = ? \quad [I(\theta)]_{32} = ?$

$[I(\theta)]_{23} = ? \quad [I(\theta)]_{33} = ?$

$I(\theta)$

$I^{-1}(\theta) = \begin{bmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{bmatrix}$

$var(\hat{\theta}) \geq$

$var(\hat{f_0}) \geq$

$var(\hat{\phi}) \geq$

Linear model:-

$$\vec{x} = \vec{H}\vec{\theta} + \vec{w}.$$

$\vec{x}$ = $N \times 1$ observation vector.

$\vec{H}$ = $N \times p$ " matrix

$\vec{\theta}$ = $p \times 1$ vector of parameters.

w : $N \times 1$ $\quad \mathcal{N}(0, \sigma^2 I)$

$$\hat{\theta}_{MVUE} = g(x)$$

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = I(\theta)\left(g(x) - \theta\right)$$

$$C_{\hat{\theta}} = I^{-1}(\theta)$$

$$p(x;\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}(x - H\theta)^T(x - H\theta)\right]$$

$$\ln p(x;\theta) = \left[-\ln(2\pi\sigma^2)^{N/2} - \frac{1}{2\sigma^2}(x - H\theta)^T(x - H\theta)\right]$$

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = -\frac{1}{2\sigma^2} \cdot \frac{\partial}{\partial \theta}\left[(x - H\theta)^T(x - H\theta)\right]$$

$$= -\frac{1}{2\sigma^2}\frac{\partial}{\partial \theta}\left[x^T x - 2x^T H\theta + \theta^T H^T H\theta\right]$$

$$\frac{\partial b^T\theta}{\partial \theta} = b \qquad = -\frac{1}{2\sigma^2}\left[-H^T x + H^T H\theta\right]$$

$$\frac{\partial \theta^T A\theta}{\partial \theta} = 2A\theta.$$

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = \frac{H^T H}{\sigma^2}\left[(H^T H)^{-1} H^T x - \theta\right]$$

$$I(\theta) = \frac{H^T H}{\sigma^2}$$

$$\hat{\theta}_{MVU} = g(x) = (H^T H)^{-1} H^T x.$$

$$C_{\hat{\theta}} = \sigma^2 (H^T H)^{-1}$$