Roll No. 20200 3025

Autumn Semester

Academic Year 2022-23

Introduction to Nonlinear Dynamics (SC401)

Second In-Semester Examination

Dhirubhai Ambani Institute of Information and Communication Technology

Time: 2 Hours

Total Marks: 35

All questions are compulsory. Answer all the sub-parts of a question together. Marks for each question are indicated next to it. All terms and symbols carry their standard textbook meaning.

1. For $\dot{x} = f(x)$, the following two functions are given: A. $f(x) = x^3 - x$ B. $f(x) = e^x + x - 1$
   For both functions, graphically indicate the possible fixed points and their stability. [3+2=5]

2. Given $\dot{x} = f(x) = x[1 - (x-2)^2]$.
   A. Identify all the fixed points. B. Classify the stability of the fixed points by linear stability analysis. C. Plot $\dot{x} = f(x)$, showing all the fixed points clearly. [1.5+2+1.5=5]

3. Consider the equation $\dot{x} = f(x) = a - bx$, in which $a, b > 0$.
   A. Plot $\dot{x}$ versus x and clearly show the fixed point behaviour. B. Integrate to obtain $x \equiv x(t)$ under the initial condition $x = 0$ at $t = 0$. Argue that when $t \ll b^{-1}$, x varies linearly with t. C. Plot $x(t)$ for all t with clear labels. [1.5+2+1.5=5]

4. For a parachutist in free fall through air, the downward motion is governed by the autonomous equation $\dot{v} = f(v)$, in which $f(v)$ is a nonlinear function of v.
   A. Using physical arguments, write the formula of $f(v)$. Plot $\dot{v}$ versus v, clearly indicating the terminal velocity. B. Rescale the equation $\dot{v} = f(v)$ to a parameter-free form and find its integral solution under the initial condition $v = 0$ at $t = 0$. C. Plot the integral solution, showing its early and late behaviour in time. [2+2+1=5]

5. Consider the logistic equation $\dot{x} = f(x) = ax - bx^2$ (with $a, b > 0$).
   A. Plot $\dot{x}$ versus x, clearly showing the stable and unstable fixed points. B. Rescale $\dot{x} = f(x)$ and integrate it to obtain $x \equiv x(t)$ under a proper initial condition. Plot $x(t)$ for all t with clear labels. C. Obtain the nonlinear time scale from the integral solution. [1+3+1=5]

6. Given a first-order autonomous equation $\dot{x} = f(x)$,
   A. Carry out a Taylor expansion about the fixed point of the equation. Retaining only the linear order in the Taylor expansion, derive the approximate formula of $x(t)$ about the fixed point. B. State the conditions for the stability and the instability of fixed points under a linear stability analysis. C. Argue that overshooting of a stable fixed point cannot occur. [3+1+1=5]

7. A. State the three laws of social dynamics. Discuss how the Malthusian law and the logistic equation follow from the laws of social dynamics. B. The radioactive decay of carbon-14 follows the mathematical principle $\dot{N} = -\lambda N$, with $\lambda > 0$. Using the concept of half life, mathematically discuss how the age of ancient civilizations is measured. [3+2=5]

Roll No. 20200 3025

Academic Year 2022-23

Autumn Semester

Introduction to Nonlinear Dynamics (SC401)
End-Semester Examination
Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar

Time: 3 Hours | Total Marks: 45

All questions are compulsory. Answer all the sub-parts of a question together. Marks for each question are indicated next to it. All terms and symbols carry their standard textbook meaning.

1. Classify the bifurcation in the following three cases by obtaining their normal forms. Find the value of the parameter r, when bifurcation occurs. *Hint:* Use the given series expansion formulae. [1.5+1.5+2=5]
   A. $\dot{x} = r - x + \ln(1+x)$ B. $\dot{x} = x + r(1 - e^{x})$ C. $\dot{x} = x - r\sin x$
   $\ln(1+x) = x - x^2/2 + x^3/3 - \cdots$, $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, $\sin x = x - x^3/3! + x^5/5! - \cdots$

2. (a) For the system $\dot{x} = 3x - 4y$ and $\dot{y} = x - y$, identify the fixed point. Classify its stability by finding the eigenvalues of the matrix.
   (b) For the system $\dot{x} = x - x^3 + y$ and $\dot{y} = -y$, identify the fixed points. Classify their stability by finding the eigenvalues of the Jacobean matrix. [2+3=5]

3. (a) With mathematical reasons argue whether or not $\ddot{x} + \dot{x}^4 + \exp(-x^2) = 0$ is a reversible system.
   (b) For the system $\ddot{x} = -4x + x^3$, find all the equilibrium points and classify their stability. Also find the potential function of the system. [2.5+2.5=5]

4. Show that $F = ma$ gives a conservative system when $F \equiv F(x)$. Establish the conditions for the fixed points, and prove that they can only be saddle or centre-type points. [5]

5. If $x(t)$ is the war potential of a nation and $y(t)$ is the war potential of its enemy nation, then Richardson's theory of conflict gives $\dot{x} = ky + g - \alpha x$ and $\dot{y} = lx + h - \beta y$. Discuss the mathematical conditions and outcomes for: A. Mutual disarmament without grievance. B. Mutual disarmament with grievance. C. Unilateral disarmament. D. Arms race (with a phase plot of y versus x). [1+1+1+2=5]

6. An x-force and a y-force are engaged in isolated combat. Use Lanchester's combat models.
   A. Write the equations for a modern conventional-conventional combat. Solve them to get Lanchester's square law, and discuss the implications of this law. Plot y versus x with labels. B. Write the equations for a conventional-guerilla combat. Solve them and plot y versus x with labels. [3.5+1.5=5]

7. Use proper mathematical and practical arguments to establish the following:
   (a) A coupled system for the competition between two similar species with populations x and y, according to the principle of competitive exclusion. State the optimal equilibrium solutions for both species.
   (b) A coupled system for the co-existence of prey fish and predator fish with populations x and y, respectively, according to Volterra's predator-prey model. State the optimal equilibrium solutions for both. Also show how moderate fishing can increase the equilibrium population of the prey fish. [2+3=5]

8. A few infected persons introduce an infectious disease in a large population. The disease has a short incubation period, and recovered individuals gain permanent immunity. The population is divided into three classes — the infected class x, the susceptible class y, and the recovered class z. A. Applying practical rules, write the time-rate equations of x, y and z. B. Obtain $x \equiv x(y)$ for $t = 0, x = x_0, y = y_0$ and $z = 0$. C. Plot x versus y. Discuss the condition for an epidemic to break out. [1.5+1.5+2=5]

9. If $x(t)$ is the number of cells in a tumour, then the Gompertz law of tumour growth is given in an autonomous form by $\dot{x} = f(x) = -ax\ln(bx)$ (with $a, b > 0$). A. Reasoning clearly, draw the phase plot of the equation. B. With proper scaling and variable substitution, obtain the integral solution of $x \equiv x(t)$, under the initial condition $t = 0, x = x_0$. Also indicate the limit of x when $t \longrightarrow \infty$. [2+3=5]

$y = bx$ $\quad dy = b\,dy$ $\quad \therefore b\frac{dy}{dt} = -a\frac{y}{b}\ln(y)$

Roll No. 202001405

Academic Year 2022-23

Winter Semester

Modelling and Simulation (CS302)

First In-Semester Examination

Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar

Time: 2 Hours

Total Marks: **15**

Note: All questions are compulsory. Provide answers with all relevant steps shown clearly. Answer all the sub-parts of a question together. Marks for each question are indicated next to it. All terms and symbols carry their standard textbook meaning. Use of scientific calculators is allowed.

1. A stream of clean water flows into a polluted lake at a constant flow rate F. The polluted water of the lake also drains out at the same rate, thus making the lake volume V a constant. If $F = 5 \times 10^8\,\mathrm{m}^3/\mathrm{day}$ and $V = 5 \times 10^{11}\,\mathrm{m}^3$, estimate the time it will take for the pollution level in the lake to be at 20% of its initial level. [2]

2. A population of a species grows according to the logistic model with a carrying capacity of 5×10^8 members. For low population the doubling time is 40 minutes. If the initial population is 10^8, estimate the population after 2 hours. [2]

3. Write the complete differential equation to model the spread of an agricultural innovation among farmers through both personal and impersonal communications. Mathematically discuss how the growth rate will be quicker than exponential in the early stages of the innovation spread. [2]

4. Briefly discuss the following with relevant mathematical arguments:

   A. Scale invariance of power laws. B. Pareto's law in income distribution. [1+1=2]

5. If $x(t)$ is the number of cells in a tumour, then the Gompertz law of tumour growth is modelled by an autonomous equation as $\dot{x} = f(x) = -ax\ln(bx)$ (with $a, b > 0$). **Note:** $\dot{x} \equiv \mathrm{d}x/\mathrm{d}t$.

   A. Obtain the integral solution $x \equiv x(t)$, under the initial condition $t = 0$, $x = x_0$. B. From the integral solution derive a non-autonomous equation $\dot{x} = f(x, t)$, and use it to give two alternative explanations for the slowing of tumour growth in the late stages. [1.5+1.5=3]

6. A single dose of a drug (medicine) is administered to a patient. Using compartment modelling

   A. Write the two rate equations of drug diffusion in the gastrointestinal (GI) tract and the blood stream along with their proper initial conditions. B. Find the integral solutions of the two equations under the condition that the rate constants are the same in the GI tract and the blood stream. C. Sketch the two integral solutions using proper mathematical arguments. [1+1.5+1.5=4]

Winter Semester

Roll No.____________

Academic Year 2017-18

Analysis of Multidisciplinary Problems (SC465)

End-Semester Examination

Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar

Time: 3 Hours

Total Marks: 40

All questions are compulsory. Answer all the sub-parts of a question together. Marks for each question are indicated next to it. All terms and symbols carry their standard textbook meaning. Overdots on variables imply a time derivative.

1. (a) A closed circuit has a d.c. power source of voltage V_0, a resistor R and a capacitor C. Obtain an equation for the charging of the capacitor, and plot it with clear labels.

   (b) The Duckworth–Lewis equation, used to reset targets in interrupted cricket matches, is given as $Z(u,w) = Z_0(w)\left[1 - e^{-b(w)u}\right]$, in which w is the number of wickets lost, u is the number of overs left and Z is the number of runs obtainable. Plot this equation for different values of w. Recast this equation as a first-order autonomous differential equation. [3+2=5]

2. If $x(t)$ is the war potential of a nation, and $y(t)$ is the war potential of its enemy nation, then Richardson's theory of conflict gives $\dot{x} = ky + g - \alpha x$ and $\dot{y} = lx + h - \beta y$. Giving a proper meaning of all the parameters used here, discuss the mathematical conditions and outcomes for:

   A. Mutual disarmament without grievance. B. Mutual disarmament with grievance. C. Unilateral disarmament. D. Arms race (with a plot of x versus y). [1+1+1+2=5]

3. If $x(t)$ is the number of cells in a tumour, then the Gompertz law of tumour growth in an autonomous form is $\dot{x} = -ax\ln(bx)$, in which $a, b > 0$.

   A. Reasoning clearly, draw the phase plot of the equation. B. With scaling and substitution, obtain the integral solution of $x \equiv x(t)$, under the initial condition $t = 0$, $x = x_0$. Also indicate the limit of x when $t \longrightarrow \infty$. [2+3=5]

4. Use proper mathematical and practical arguments to establish the following:

   (a) The general coupled equations for the co-existence of a prey species and a predator species with populations x and y, respectively, as per Volterra's predator-prey model. State the optimal equilibrium solutions for both.

   (b) The coupled equations representing the struggle for existence between two similar species with populations x and y, according to the principle of competitive exclusion. Express the mathematical conditions for intense competition and no competition. State the optimal equilibrium solutions for both. [2+3=5]

5. (a) Discuss: A. Clustering coefficient in small-world networks. B. Power-law degree distributions.

   (b) The radioactive decay of carbon-14 follows the equation, $\dot{N} = -\lambda N$, with $\lambda > 0$. Using the concept of half life, mathematically argue how this equation is used to measure the age of ancient civilisations. [2(1+1)+3=5]

6. On a field of battle, an x-force and a y-force are engaged in isolated combat. Use Lanchester's combat models.

   A. Obtain the equation for a conventional-conventional combat. Solve this equation to get Lanchester's square law, and state the implication of this law. Provide a plot of x versus y with clear labels. B. Obtain the equation for a conventional-guerilla combat. Solve this equation and provide a plot of x versus y. [3.5+1.5=5]

7. (a) Starting with the differential equation of a damped oscillator, obtain the eigenvalues for the overdamped, underdamped and critically damped conditions.

   (b) Juliet and Romeo are equally cautious about their feelings for each other. Write a set of coupled equation for their feelings, using a "cautiousness" parameter and a "responsiveness" parameter. Show mathematically how their love may either flourish or die out. [2+3=5]

8. A few infected persons introduce an infectious disease in a large population. The disease has a short incubation period, and recovered individuals gain permanent immunity. There are three classes of population — the infected class x, the susceptible class y, and the recovered class z.

   A. Write the time-rate equations of x, y and z following clear practical rules. B. Solve for $x \equiv x(y)$ under the initial condition $t = 0$, $x = x_0$, $y = y_0$ and $z = 0$. Identify the threshold of y for an epidemic to break out. C. Obtain an approximate formula for the number of susceptibles who get infected, if the initial number of susceptibles is slightly higher than the threshold. [1.5+1.5+2=5]

5

$x = b$

4

$-lxy$

2

3

Winter Semester

Roll No. 201501157.

Academic Year 2017-18

Analysis of Multidisciplinary Problems (SC465)

First In-Semester Examination

Dhirubhai Ambani Institute of Information and Communication Technology

Time: 1 Hour 30 Minutes

Total Marks: 20

All questions are compulsory. Answer all the sub-parts of a question together. Marks for each question are indicated next to it. All terms and symbols carry their standard textbook meaning.

Note: $\dot{x} \equiv \mathrm{d}x/\mathrm{d}t$.

1. Consider the equation $\dot{x} = a - bx$, in which $a, b > 0$.

   A. Rescale and integrate it with the initial condition $x = 0$ at $t = 0$. B. Show when $t \ll b^{-1}$, x varies linearly with t. C. Plot the integral solution for all t with clear labels. [2+1+1=4]

2. Consider an object falling through a very long liquid column, with a velocity v at a depth z.

   A. Mentioning all the forces acting on the object, set down the v–t differential equation. Obtain the terminal velocity from this equation and a natural scale of t. B. Write the integral solution of the v–t equation. For both the small and large limits of t, indicate the approximate dependence of v on t. Integrate to get the approximate solutions of $z(t)$ in both limits. [2.5+2.5=5]

3. Rocks exhibit both elastic and viscous properties under the weight of earth matter.

   A. Write a relation between elastic stress and strain. B. Obtain a similar relation for the viscous effect. C. Taking both effects together, express the differential equation of the viscoelastic deformation of rocks. From the equation get the limiting value of the strain. [0.5+1.5+1=3]

4. Start with $F = ma$.

   A. Derive the formula for the conservation of total energy. B. Argue that conservation also allows time reversal. [2+1=3]

5. Lead ore contains radioactive lead-210 (Pb-210) which, with a half-life of 22 years, decays to non-radioactive lead-206 (Pb-206). On the other hand, Pb-210 is replenished by the radioactive decay of radium-226 (Ra-226), which has a half-life of 1600 years. For time scales that are much less than 1600 years, show how the decay rate of Pb-210 is approximated by the linear differential equation $\dot{x} = r - \lambda x$, in which r and λ are constants. [2.5]

6. Consider the equation $\dot{x} = ax - bx^{\alpha+1}$, in which $a, b > 0$ and $\alpha \geq 2$. Use a transformation to derive the standard logistic equation. From it obtain the carrying capacity of x. [2.5]

Roll No 201401185

Winter Semester

Academic Year 2016-17

Analysis of Multidisciplinary Problems (SC465)

First In-Semester Examination

Dhirubhai Ambani Institute of Information and Communication Technology

Time: 1 Hour 30 Minutes

Total Marks: 30

All questions are compulsory. Answer all the sub-parts of a question together. Marks for each question are indicated next to it. All terms and symbols carry their standard textbook meaning.

1. Consider the following equation, in which $a, b > 0$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = a - bx.$$

A. Give two examples where this equation is relevant. B. Rescale it and obtain its solution under the initial condition, $x = 0$ at $t = 0$. C. Show that when $t \ll b^{-1}$, x varies linearly with t. D. Plot the integral solution for all t with clear labels. [1+2+1+1=5]

2. Refer to Question 1, now with $f(x) = a + bx$.

A. Show how the rescaled integral solution will change. B. Discuss the behaviour of x when $t \gg b^{-1}$, and then plot x for all t with clear labels. [1+1.5=2.5]

3. A closed circuit has a d.c. power source of voltage V_0, a resistor R and a capacitor C. Obtain an equation for the charging of the capacitor, and plot it with clear labels. [2.5]

4. Consider an object falling through a very long liquid column, with a velocity v at a depth z.

A. Mentioning all the forces acting on the object, write the v–t integral solution. Clearly express the terminal velocity. B. Integrate the v–t equation to derive the z–t equation for the initial condition, $z = 0$ at $t = 0$. Rescale it and show that there is no turning point for $t > 0$. C. Sketch the rescaled z–t solution with clear labels. [2.5+1.5+1=5]

5. The Duckworth–Lewis equation, used to reset targets in interrupted cricket matches, is given as $Z(u, w) = Z_0(w)\left[1 - e^{b(w)u}\right]$, in which w is the number of wickets lost, u is the number of overs left and Z is the number of runs obtainable.

A. With clear labels, plot this equation for different values of b. B. Also recast this equation as a first-order autonomous differential equation. [1+1=2]

6. The radioactive decay of carbon-14 follows the mathematical principle,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N,$$

with $\lambda > 0$. Making use of the concept of half life, mathematically discuss how this equation may be used to measure the age of ancient civilisations. [3]

7. Consider the logistic equation (with $a, b > 0$),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = ax - bx^2.$$

(a) A. Rescale this equation and obtain its integral solution, with a proper initial condition. B. Identify the carrying capacity. Also for $t \longrightarrow 0$, show that the solution is approximately exponential. C. Obtain the nonlinear time scale and use it to find an upper limit to the initial condition for early exponential growth. D. Plot the full integral solution with clear labels. [2.5+2+1.5+1=7]

(b) Now with $f(x) = ax - bx^{\alpha+1}$, in which $\alpha \geq 2$, use a transformation to derive the standard logistic equation. Discuss how both the carrying capacity and the nonlinear time scale are reduced. [3]