

Flows on the line - (first-order systems)

One-Dimensional System: $\rightarrow \boxed{\dot{x} = f(x)}$

$\boxed{\dot{x} = dx/dt} \Rightarrow \boxed{x \equiv x(t)}$ \downarrow Autonomous System

- i) $x(t)$ is a real valued function of t (time).
- ii) $f(x)$ is a smooth real-valued function of x .

Existence and Uniqueness Theorem:

for the initial-value problem, $\boxed{\dot{x} = f(x)}$,

$\boxed{x \equiv x(t)}$ and $\boxed{x(0) = x_0}$ (initial condition).

- i/. $f(x)$ and $\boxed{f'(x) = df/dx}$ are finite-valued and smooth.
- ii/. Continuous on an open interval R of the x -axis (in the one-dimensional system).
- iii/. x_0 is a point in R (on the x -axis).

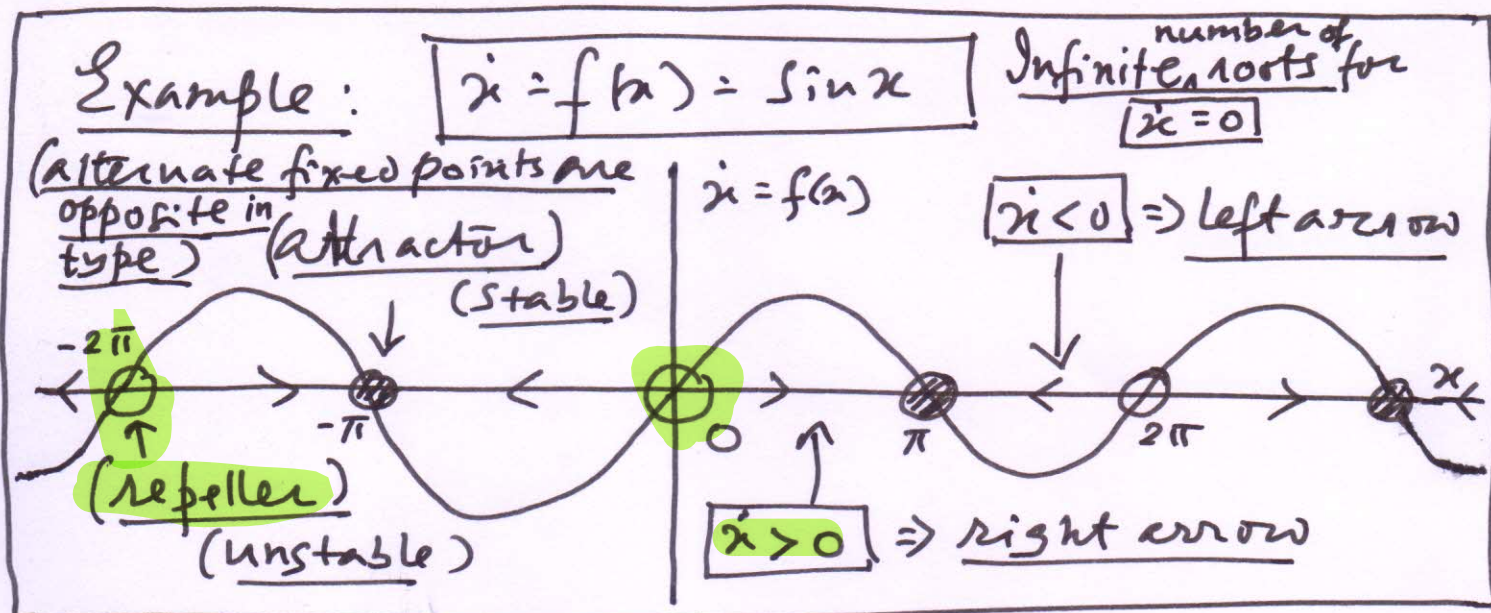
Fulfilling the foregoing conditions, the initial-value problem has a solution on some time interval $(-\tau, \tau)$ about $t=0$.

This solution exists and is unique.

Note: i/. effectively the solution is single-valued.
 2/. The solution may not exist forever.

$$\dot{x} = f(x)$$

Phase Portraits: Plotting \dot{x} versus x in,



For $x \equiv x(t)$, an autonomous first-order system is given by $\dot{x} = f(x)$. The fixed point (or equilibrium point) of such a system is obtained when $\dot{x} = f(x) = 0$.

Hence, $f(x_c) = 0$ gives the fixed point (equilibrium point) on the line $\dot{x} = 0$ at $x = x_c$ (in which x_c is the fixed point).

This is not a turning point for $x \equiv x(t)$.

$$f'(x) = \frac{d^2x}{dt^2} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d\dot{x}}{dt} = \dot{x} = \frac{df}{dx} \frac{dx}{dt} \quad (\text{applying chain rule})$$

When $\dot{x} = 0 \Rightarrow \dot{x} = \frac{df}{dx} \dot{x} = 0$. Hence

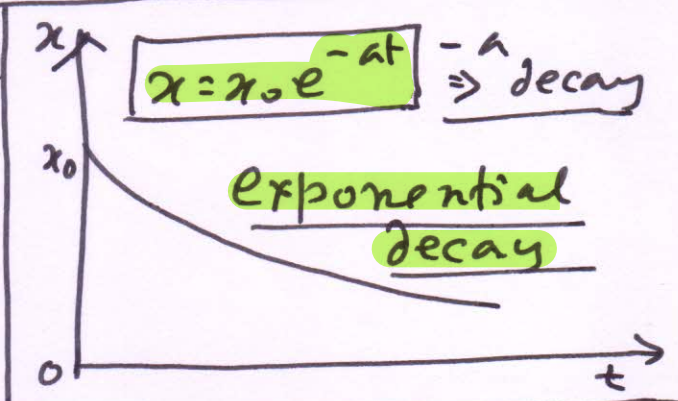
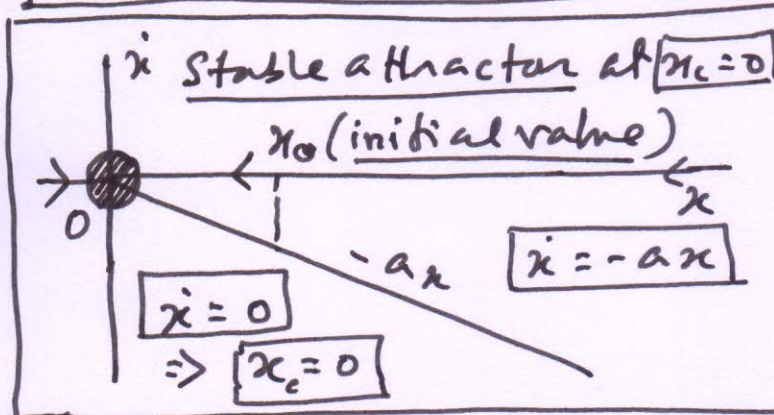
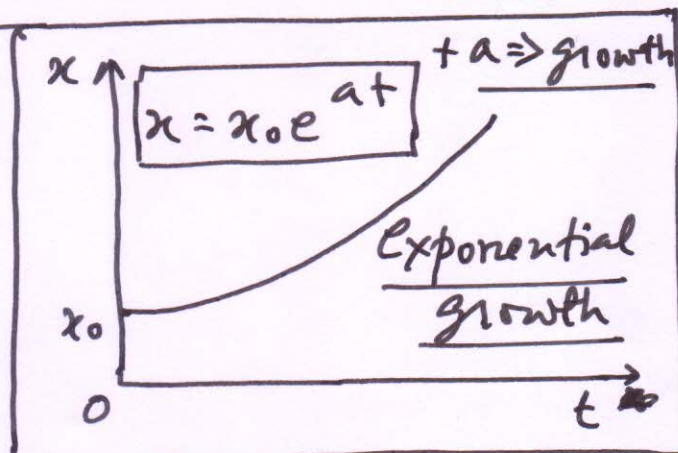
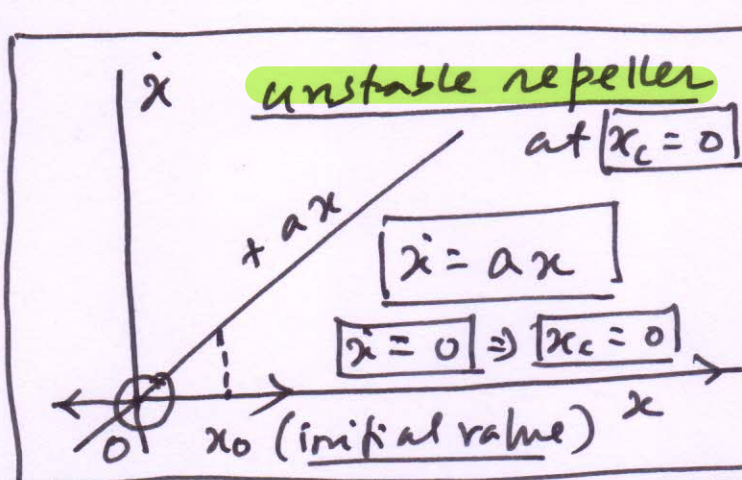
both $\dot{x} = 0$ and $\ddot{x} = 0$ at the same time.

(Turning points usually have non-zero second derivatives.)

Phase Portraits of Polynomial Forms

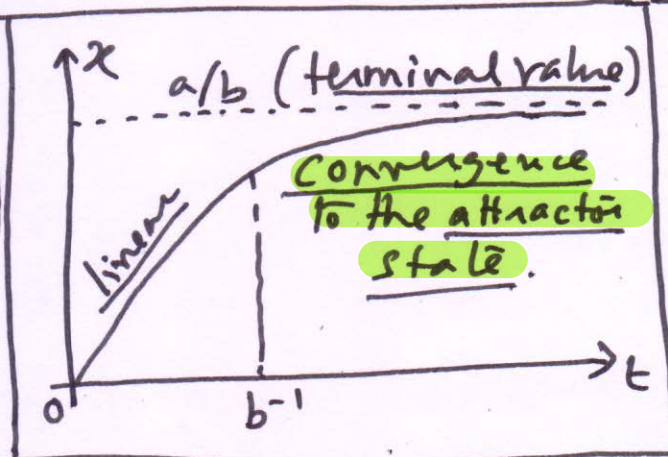
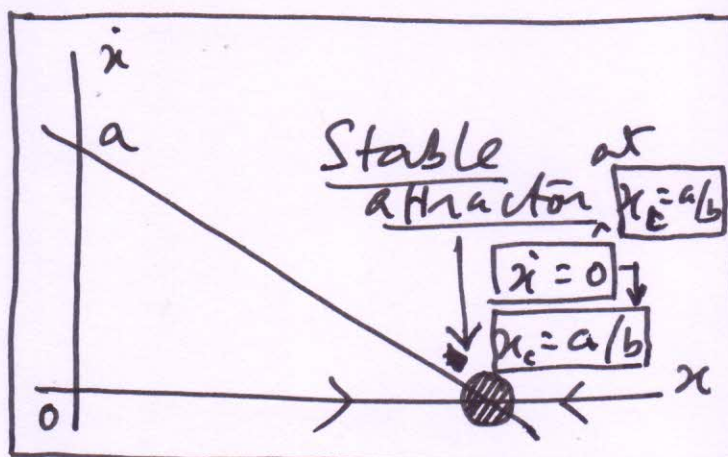
Example: Consider $\dot{x} = \frac{dx}{dt} = f(x) = \pm ax$ ($a > 0$).

Integral Solution: $x(t) = x_0 e^{\pm at}$ $\frac{x_0 \text{ is integration Constant.}}$



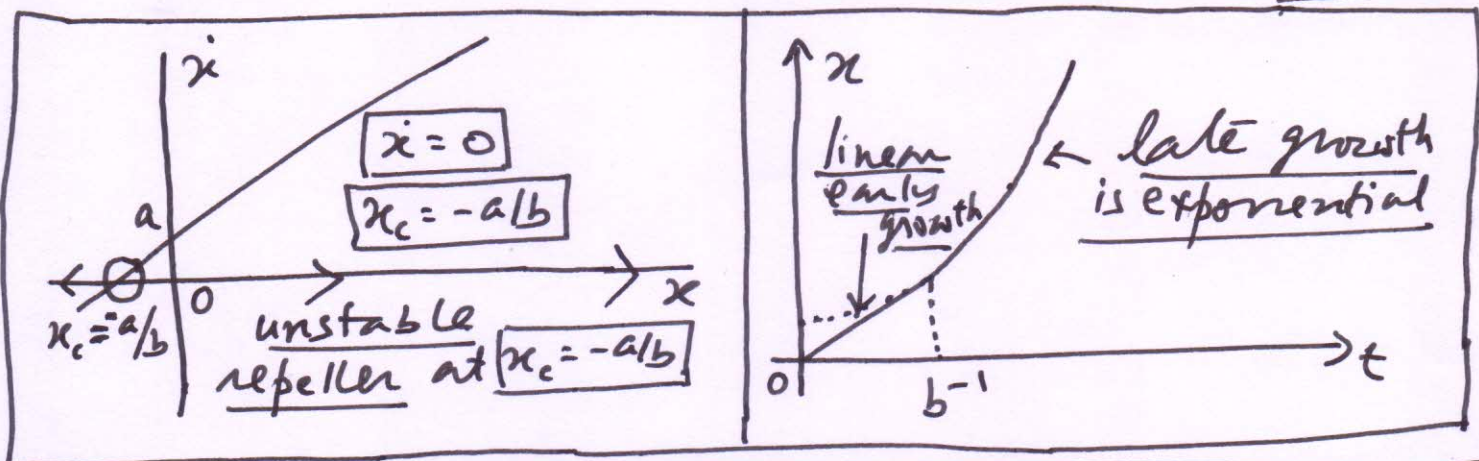
Example: $\dot{x} = f(x) = a - bx$ ($a, b > 0$).

Integral Solution: $x(t) = \frac{a}{b} (1 - e^{-bt})$ $\text{for } x=0 \text{ at } t=0.$

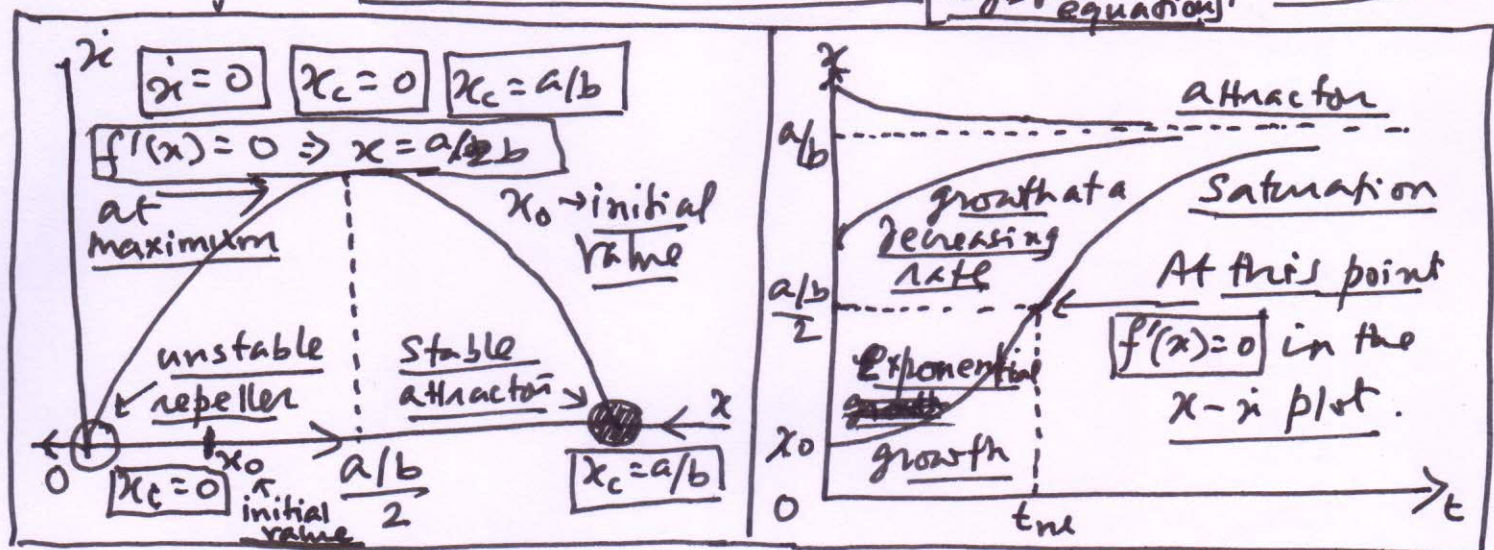


Example: $\dot{x} = f(x) = a + bx$ ($a, b > 0$)

Integral solution: $x(t) = \frac{a}{b} (e^{bt} - 1)$ for $x=0$ at $t=0$



Example: $\dot{x} = f(x) = ax - bx^2$ ($a, b > 0$) Nonlinear function
logistic equation



Rescaling $X = \frac{x}{a/b}$ and $T = at$, we get a parameter-free equation $\frac{dX}{dT} = X - X^2 \Rightarrow \frac{dX}{dT} = F(X)$

i) When $\frac{dX}{dT} = F(X) = 0$, $X = 0$ and $X = 1$ (fixed points)

ii) $\frac{dF}{dX} = 1 - 2X \Rightarrow \frac{dF}{dX} = 0$ is at $X = \frac{1}{2}$ (turning point for $F(X)$)

iii) $\frac{d^2 X}{dT^2} = \frac{dF}{dX} \frac{dX}{dT} \Rightarrow \frac{d^2 X}{dT^2} = (1 - 2X) \frac{dX}{dT}$ (P.T.O.)
chain rule

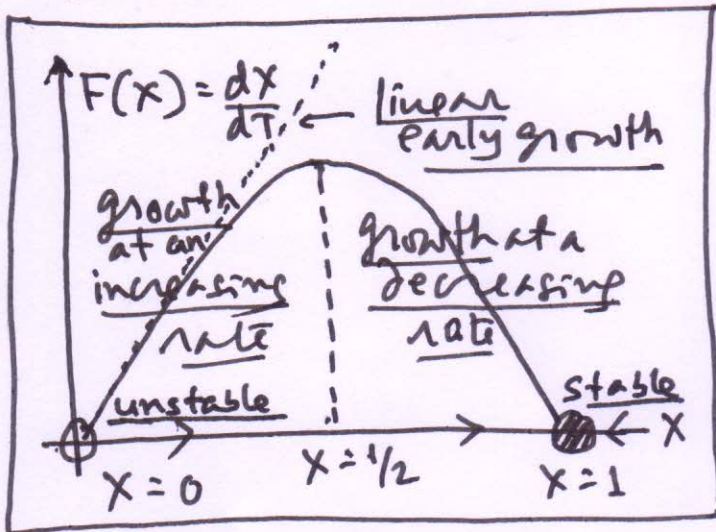
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Now for $0 < x \leq 1$, $\frac{dx}{dt} > 0 \Rightarrow$ x always grows with T.

But $F(x)$ has a turning point at $x = 1/2$.

$$\frac{dF}{dx} = 1 - 2x \Rightarrow \frac{d^2F}{dx^2} = -2 < 0 \therefore \text{The turning point at } x = 1/2 \text{ is a maximum.}$$

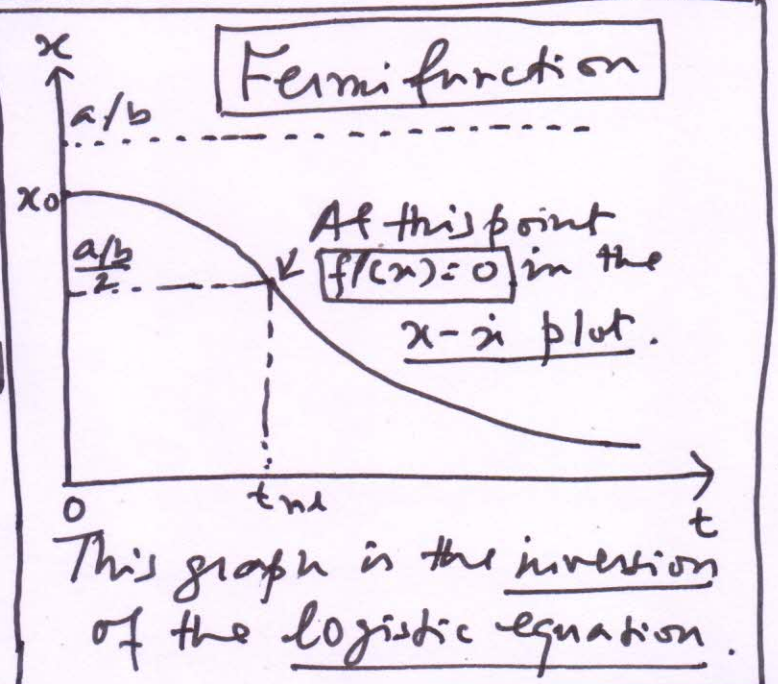
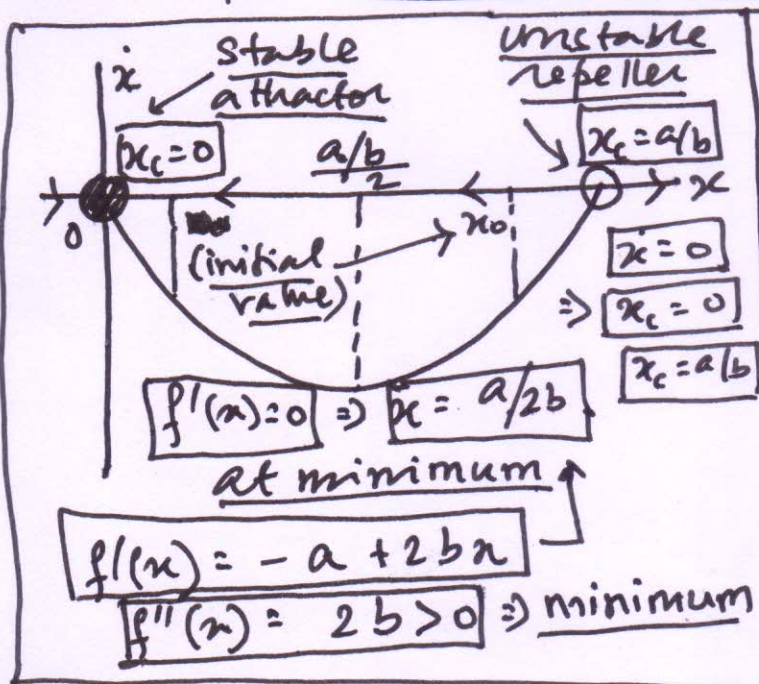


iv.) When $x < 1/2$, $\frac{dF}{dx} > 0 \Rightarrow \frac{d^2x}{dt^2} > 0$. Hence, for $x < 1/2$, growth occurs at an increasing rate.

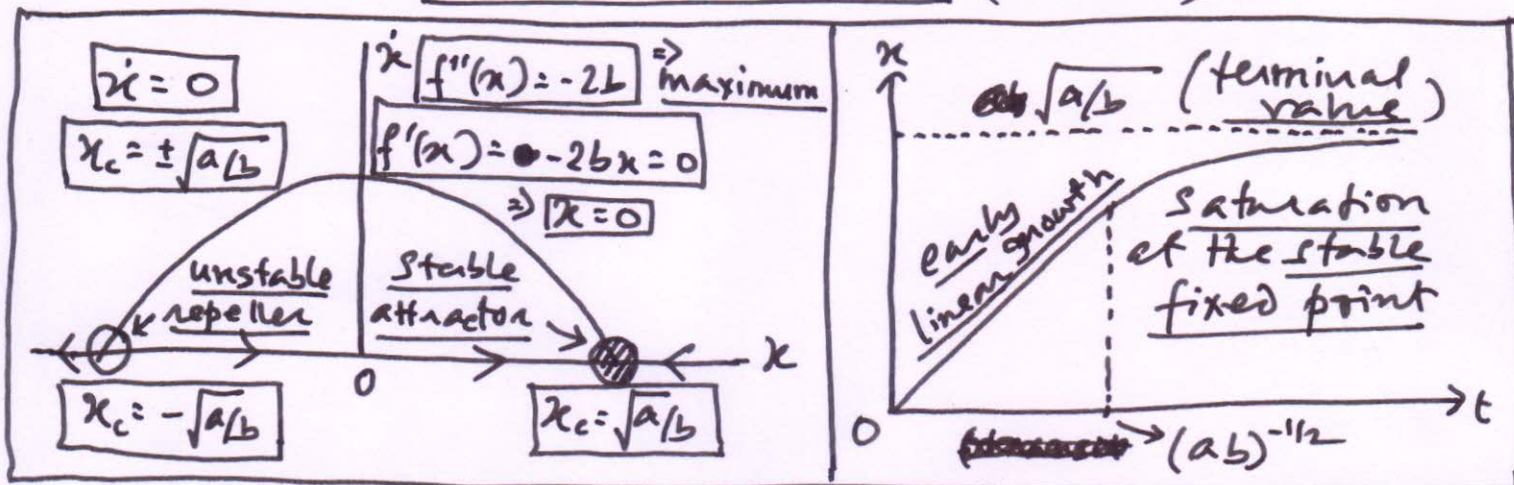
v.) When $x > 1/2$, $\frac{dF}{dx} < 0$

$\Rightarrow \frac{d^2x}{dt^2} < 0$. Hence, for $x > 1/2$, growth occurs at a decreasing rate. The slowing down of growth due to the nonlinear (x^2) term starts at $x = 1/2$.

Example: $\dot{x} = f(x) = -ax + bx^2$ ($a, b > 0$) (Inverse of the logistic equation)

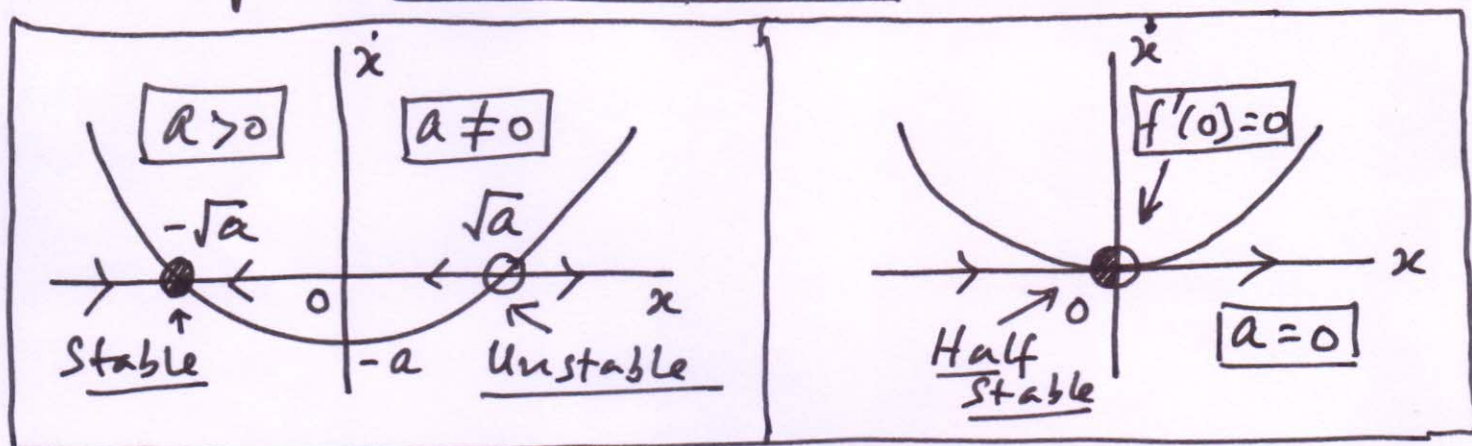


Example: $\dot{x} = f(x) = a - bx^2$ ($a, b > 0$)

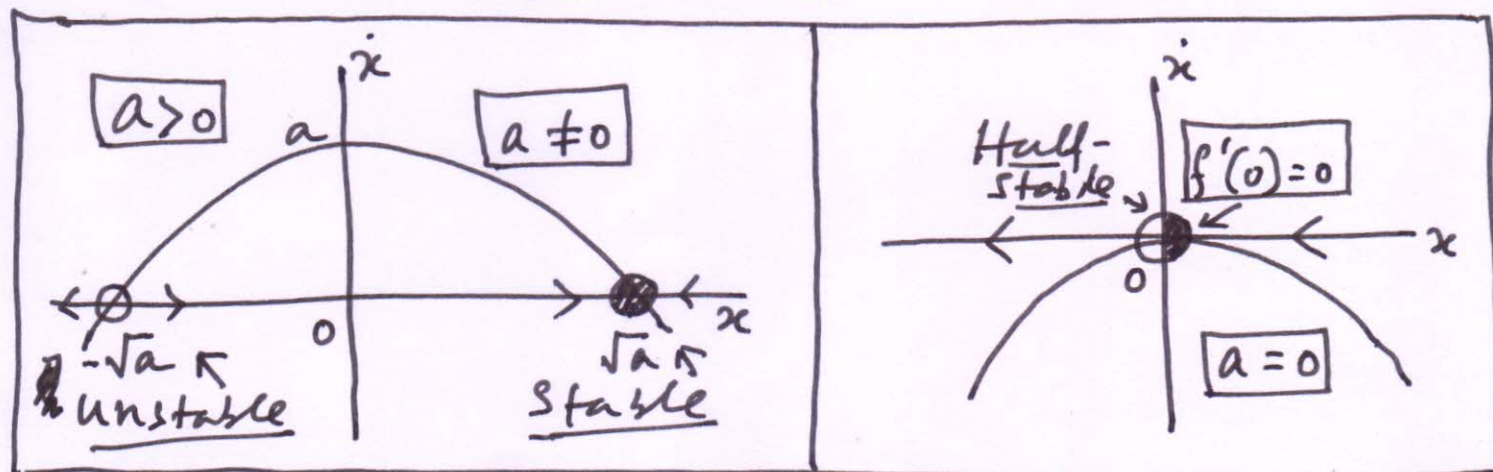


Critical Cases (Zero First Derivative)

Example: $\dot{x} = f(x) = x^2 - a$ $\Rightarrow f'(x) = 2x$ ($a > 0$)

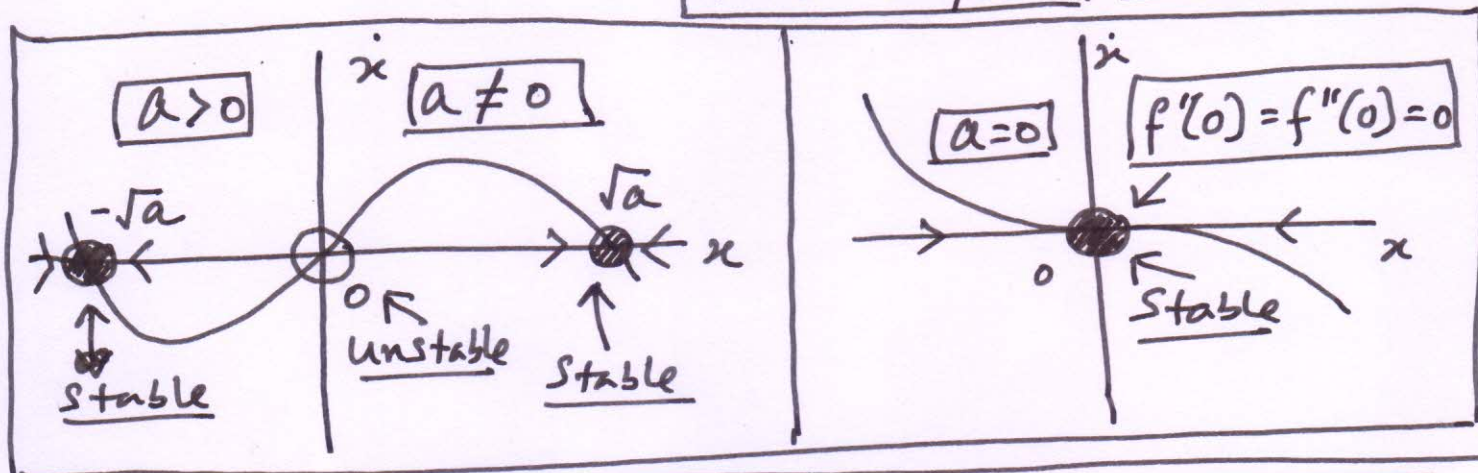


Example: $\dot{x} = f(x) = a - x^2$ $\Rightarrow f'(x) = -2x$ ($a > 0$)

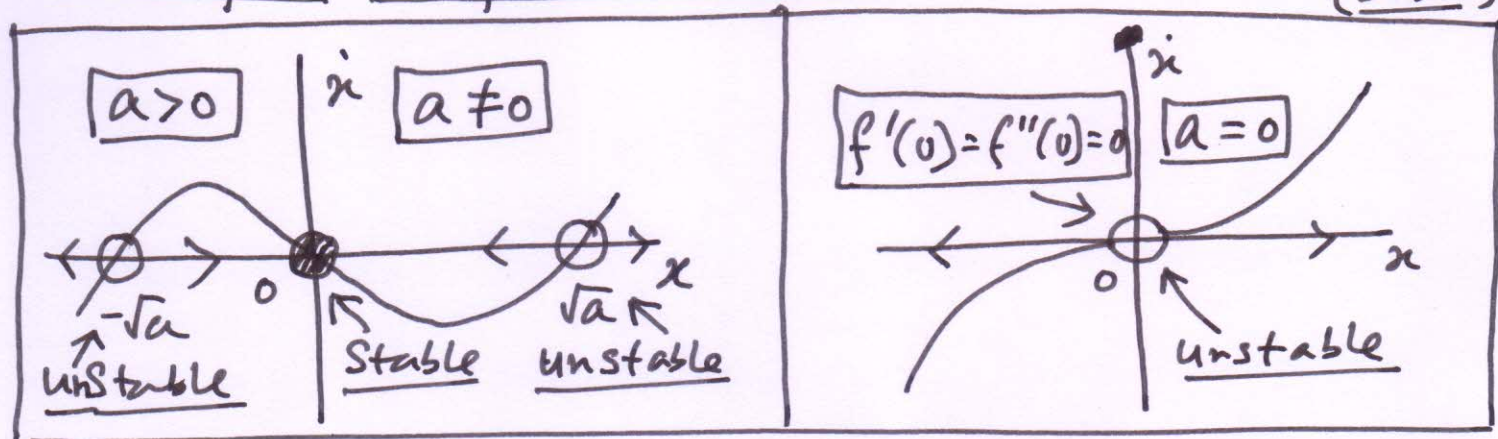


In the two cases above, the stable attractor and the unstable repeller exchange positions.

Example: $\dot{x} = f(x) = ax - x^3$ $f'(x) = -3x^2$ ($a > 0$)
 $f''(x) = d^2f/dx^2$ $f''(x) = -6x$



Example: $\dot{x} = f(x) = -ax + x^3$ $f'(x) = 3x^2$ $f''(x) = 6x$ ($a > 0$)



In the two cases above, the fixed points exchange their stability properties.

Plotting of a Polynomial Series

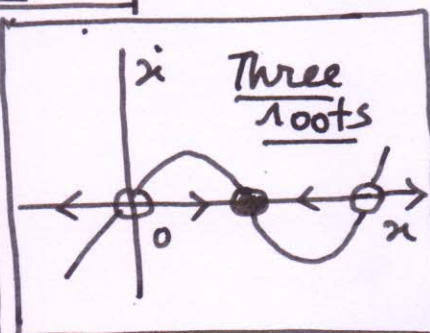
$\dot{x} = f(x) = a_0 x^0 \pm a_1 x \pm a_2 x^2 \pm a_3 x^3 \pm \dots$

Polynomial
→ Power
Series.

- i.) Small powers dominate for small x .
- ii.) Large powers dominate for large x .

Example: $\dot{x} = f(x) = a_1 x - a_2 x^2 + a_3 x^3$

- i.) Negative sign \Rightarrow Downward direction
- ii.) Positive sign \Rightarrow Upward direction



Plotting Cubic Polynomials

$$\dot{x} = f(x) = x - x^2 + \epsilon x^3 \quad (\epsilon > 0)$$

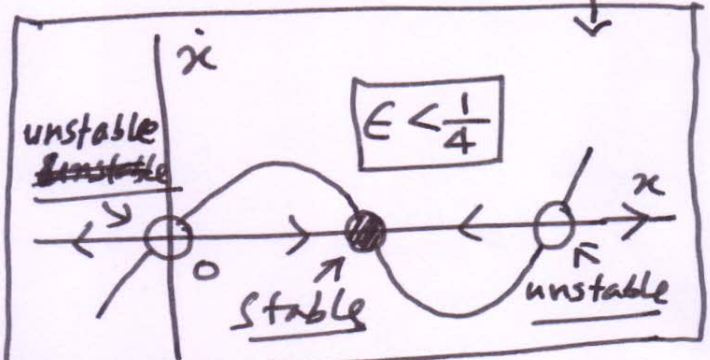
Second-order
inhibits,
third-order
enhances.

When $\dot{x} = 0$, $x_c = 0$ & $1 - x_c + \epsilon x_c^2 = 0$.

$\Rightarrow x_c = \frac{1 \pm \sqrt{1 - 4\epsilon}}{2\epsilon}$. If $\epsilon < \frac{1}{4}$, there are three real fixed points.

$$x_c = \frac{1}{2\epsilon} \pm \frac{1}{2\epsilon} (1 - 4\epsilon)^{1/2}$$

When $\epsilon \rightarrow 0$ $(1 - 4\epsilon)^{1/2} \approx 1 - 2\epsilon$
[Binomial theorem] \rightarrow (approximation). $\uparrow (1 + z^n) \approx 1 + nz$



$\Rightarrow x_c \approx \frac{1}{2\epsilon} \pm \frac{1}{2\epsilon} (1 - 2\epsilon) \Rightarrow x_c \approx \frac{1}{2\epsilon} \pm \frac{1}{2\epsilon} \mp 1$

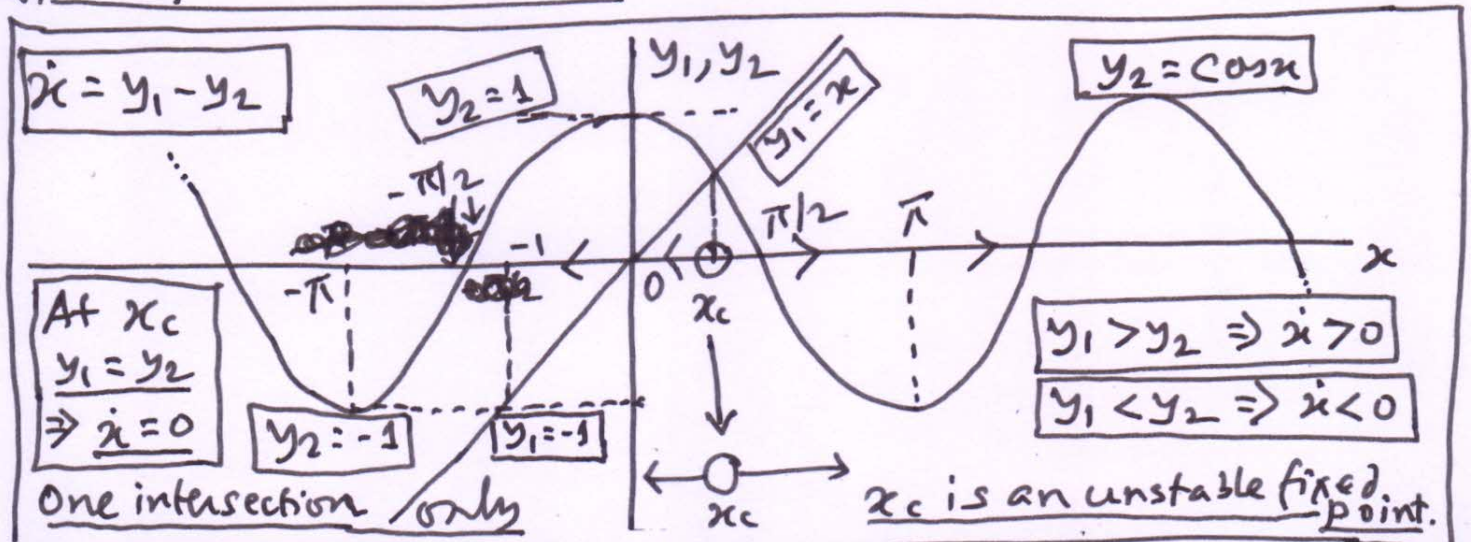
When $\epsilon \rightarrow 0$, choose lower sign $\Rightarrow x_c \rightarrow 1$.

If the upper sign is chosen, $x_c \approx \frac{1}{\epsilon} - 1 \Rightarrow x \rightarrow \infty$.

The only unstable fixed point is pushed to infinity.

Example Fixed Points in Transcendental Functions

$\dot{x} = f(x) = x - \cos x$ Write $y_1 = x$ & $y_2 = \cos x$



Stability Analysis of fixed Points

For a first-order autonomous dynamical system, $\dot{x} = f(x)$, the fixed point condition is $\dot{x} = 0$ at $x = x_c \Rightarrow f(x_c) = 0$ at $x = x_c$.

The fixed point coordinate x_c is perturbed by a small amount ϵ , i.e. $\epsilon \ll x_c$. Hence, we write $x = x_c + \epsilon$. Using this in $\dot{x} = f(x)$, ~~for~~ $\dot{x} = \dot{\epsilon} = f(x) = f(x_c + \epsilon)$ Now $\dot{x}_c = 0$.

$$\therefore \dot{\epsilon} = f(x_c + \epsilon) = f(x_c) + f'(x_c)\epsilon + \frac{f''(x_c)}{2!}\epsilon^2 + \dots$$

by a Taylor expansion. Now $f(x_c) = 0$, and truncating the Taylor expansion at the linear order (i.e. ignoring ϵ^2 and all higher order terms, due to the smallness of ϵ), we get $\dot{\epsilon} \approx f'(x_c)\epsilon$, a linear differential equation in ϵ .

$$\Rightarrow \frac{d\epsilon}{dt} = f'(x_c)\epsilon \Rightarrow \int \frac{d\epsilon}{\epsilon} = f'(x_c) \int dt \quad \left[\begin{array}{l} f'(x_c) \text{ is} \\ \text{constant} \end{array} \right]$$

$$\Rightarrow \ln \epsilon = \ln A + f'(x_c)t \Rightarrow \epsilon = A e^{f'(x_c)t}$$

Hence $x \approx x_c + A e^{f'(x_c)t}$ (A is an integration constant).

The foregoing result is due to a LINEAR STABILITY ANALYSIS when $f(x_c) = 0$, $f'(x_c) \neq 0$.
(P.T.O.)

(continued)

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1/ For a stable fixed point, as $t \rightarrow \infty$, $\epsilon \rightarrow 0$.

$\therefore x \rightarrow x_c$, i.e. there is a convergence towards x_c .

~~exp~~ This happens only when $f'(x_c) < 0$.

2/ For an unstable fixed point, as $t \rightarrow \infty$,

$\epsilon \rightarrow \infty$, i.e. a divergence away from x_c occurs.

This happens only when $f'(x) > 0$.

\Rightarrow i.) If $f'(x_c) < 0$, the fixed point is stable.
ii.) If $f'(x_c) > 0$, the fixed point is unstable.

3/ Now $x = x_c + A e^{f'(x_c)t} \Rightarrow \epsilon = A e^{f'(x_c)t}$

$$\Rightarrow t = \frac{1}{f'(x_c)} \ln\left(\frac{\epsilon}{A}\right) = \frac{1}{f'(x_c)} \ln\left(\frac{x - x_c}{A}\right)$$

For a stable fixed point, $f'(x_c) < 0$ and

$x \rightarrow x_c$ or $\epsilon \rightarrow 0$. Hence $t \rightarrow \infty$, for $x \rightarrow x_c$

The convergence to x_c takes infinitely long.

Hence, for a first-order system, there is

No overshoot of the ^{stable} fixed point, and no oscillation about the fixed point is possible.

Oscillations are only possible when $f'(x_c)$ is imaginary, but since $\dot{x} = f(x)$ is real, this is not allowed in a first-order system.

Critical Condition in the Stability Analysis

Given $\dot{x} = f(x)$, the fixed point is at $\boxed{x=0}$, i.e. $\boxed{f(x_c)=0}$. Perturbing $\boxed{x = x_c + \epsilon}$, we get

$$\boxed{\dot{x} = \dot{\epsilon} = f(x_c) + f'(x_c)\epsilon + \frac{1}{2!}f''(x_c)\epsilon^2 + \dots} \quad (\dot{x}_c=0).$$

At the fixed point $\boxed{f(x_c)=0}$. In addition if $\boxed{f'(x_c)=0}$, then we have a critical condition.
(no longer linear).

$$\Rightarrow \boxed{\dot{\epsilon} \approx \frac{f''(x_c)}{2!} \epsilon^2} \quad \left(\epsilon^2 \text{ can no longer be neglected, but higher powers of } \epsilon^2 \text{ are neglected} \right).$$

$$\Rightarrow \boxed{\frac{d\epsilon}{dt} \approx \frac{f''(x_c)}{2} \epsilon^2} \Rightarrow \boxed{\int \frac{d\epsilon}{\epsilon^2} = \frac{f''(x_c)}{2} \int dt}$$

$$\Rightarrow \boxed{\frac{\epsilon^{-1}}{-1} \approx \frac{f''(x_c)}{2} (t-A)} \rightarrow \text{A is an integration constant} \quad \left[\frac{f''(x_c)}{2} \text{ is constant} \right]$$

$$\Rightarrow \boxed{\epsilon = \frac{-2}{f''(x_c)} \frac{1}{t-A}} \Rightarrow \boxed{x \approx x_c - \frac{2}{f''(x_c)} \frac{1}{t-A}} \quad \leftarrow$$

When $\boxed{t \rightarrow \infty}$, $\boxed{x \rightarrow x_c}$ (slow power-law convergence).

1/. Where $\boxed{f'(x_c) < 0}$, $\boxed{x \approx x_c + e^{f'(x_c)t}}$. As $t \rightarrow \infty$, the convergence towards x_c is a rapid exponential convergence.

2/. When $\boxed{f'(x_c)=0}$ (critical condition),

$$\boxed{x \approx x_c + \frac{B}{t-A}} \quad \left(B = \frac{-2}{f''(x_c)} \right), \text{ the convergence}$$

~~towards~~ x_c as $t \rightarrow \infty$, is a slow power-law convergence.

Testing Stability of Fixed Points $\dot{x} = f(x)$

1/ $\dot{x} = f(x) = a - x^2$ ($a > 0$) $f'(x) = -2x$ $\dot{x} = 0 \Rightarrow x_c = \pm\sqrt{a}$

$\therefore f'(\sqrt{a}) = -2\sqrt{a} < 0$ (stable), $f'(-\sqrt{a}) = 2\sqrt{a} > 0$ (unstable)

2/ $\dot{x} = f(x) = x^2 - a$ ($a > 0$) $f'(x) = 2x$ $\dot{x} = 0 \Rightarrow x_c = \pm\sqrt{a}$

$\therefore f'(\sqrt{a}) = 2\sqrt{a} > 0$ (unstable), $f'(-\sqrt{a}) = -2\sqrt{a} < 0$ (stable)

3/ $\dot{x} = f(x) = a - bx$ $\dot{x} = 0 \Rightarrow x_c = a/b$ $f'(x) = -b$

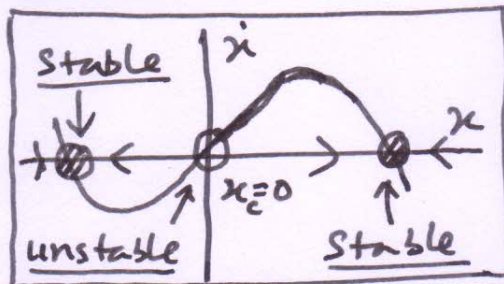
If $b > 0$, x_c is stable. If $b < 0$, x_c is unstable.

4/ $\dot{x} = f(x) = ax - bx^2$ ($a, b > 0$) $f'(x) = a - 2bx$ (stable)

$\dot{x} = 0 \Rightarrow x_c = 0, a/b$ $f'(0) = a > 0$ (unstable), $f'(a/b) = -a < 0$

5/ $\dot{x} = f(x) = x + x^2 - x^3$ (second order enhances, third order inhibits)

$\dot{x} = 0 \Rightarrow x_c = 0$ and $x_c^2 - x_c - 1 = 0 \Rightarrow x_c = \frac{1 \pm \sqrt{5}}{2}$ (3 roots)



$f'(x) = 1 + 2x - 3x^2$ $f'(0) = 1 > 0$ (unstable)

$f'(\frac{1 \pm \sqrt{5}}{2}) = 1 + (1 \pm \sqrt{5}) - \frac{3}{4}(1 \pm \sqrt{5})^2$
 $\Rightarrow f'(\frac{1 \pm \sqrt{5}}{2}) = -\frac{\sqrt{5}}{2}(\sqrt{5} \pm 1) < 0$ (Both are stable)

6/ $\dot{x} = f(x) = e^x - \cos x$ Write $y_1 = e^x$ and $y_2 = \cos x$

At $x=0$, $y_1 = 1 = y_2 \Rightarrow \dot{x} = 0$ Hence $x_c = 0$ is a fixed point.

$f'(x) = e^x + \sin x \Rightarrow f'(0) = 1 > 0$ $x_c = 0$ is an unstable fixed point.

