Bifurcations

[BIFURCATIONS IN NONLINEAR SYSTEMS]

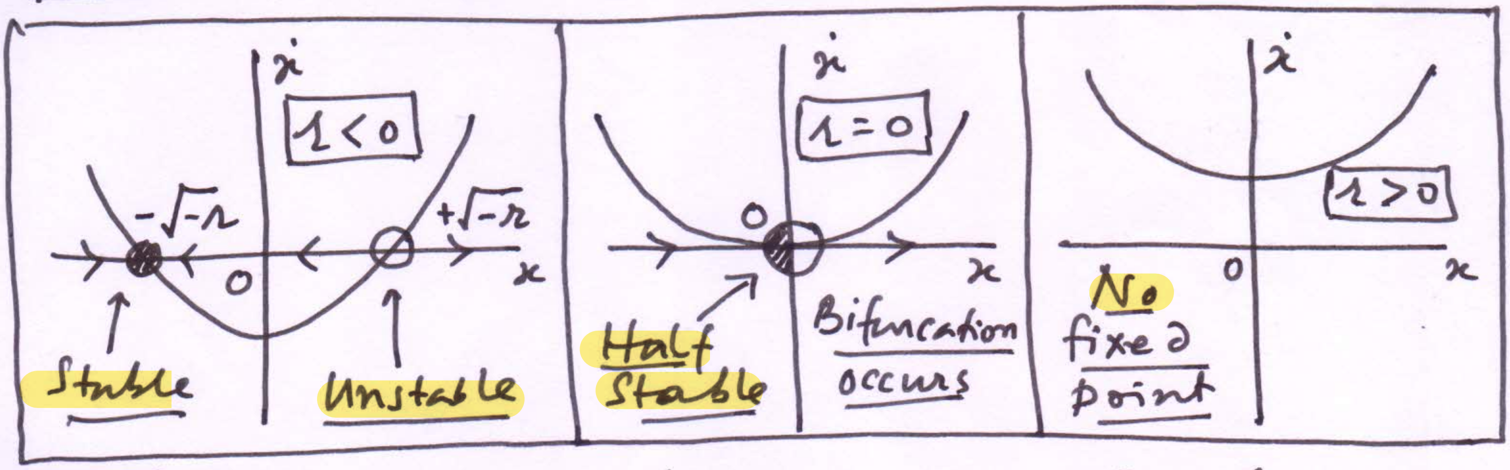
1/. Qualitative changes in the dynamics due to changes in parameters.

2/. Result of multiple fixed points (nonlinear).

Saddle-Node Bifurcation

$\boxed{\dot{x} = f(x, r) = r + x^2}$ $r \rightarrow$ Bifurcation parameter.

Alternative names: 1/. Fold Bifurcation, 2/. Turning-point bifurcation, 3/. Blue-Sky Bifurcation.



At $\boxed{r=0}$, bifurcation occurs. Two fixed points collide and annihilate each other.

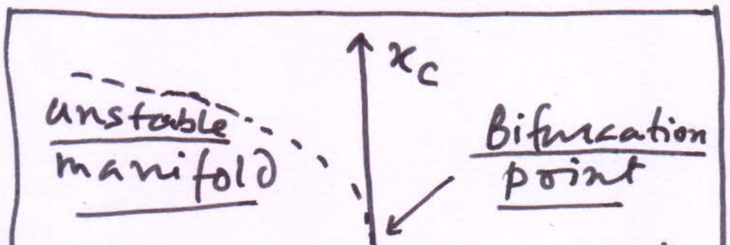
Bifurcation Diagram:

Solve $\boxed{\dot{x} = f(x_c, r) = 0}$

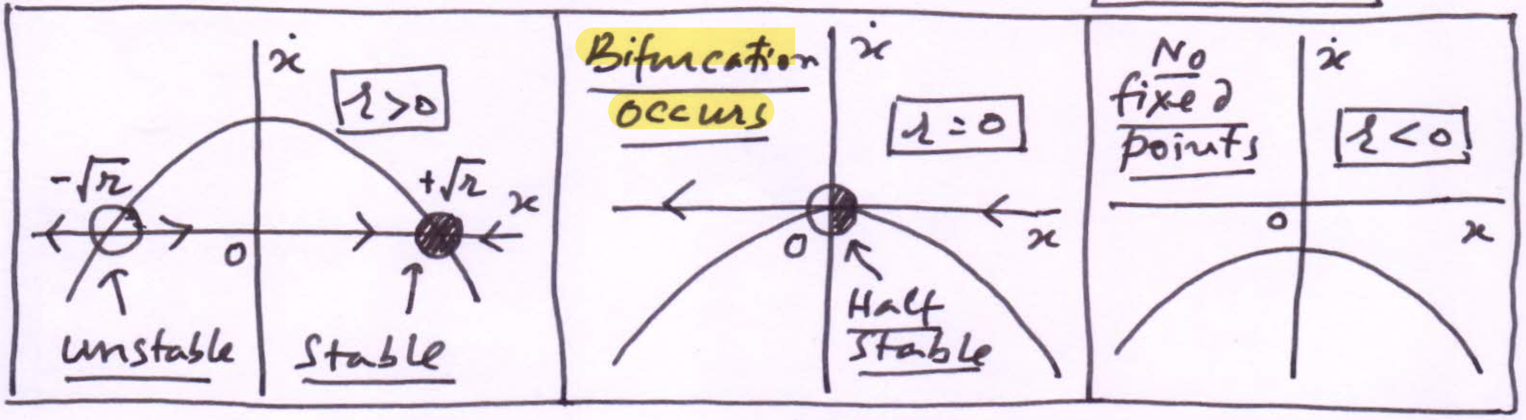
$\Rightarrow \boxed{x_c^2 + r = 0} \Rightarrow \boxed{x_c = \pm\sqrt{-r}}$

Real x_c only for $\boxed{r<0}$.

No fixed point for $\boxed{r>0}$.



Example: $\boxed{\dot{x} = f(x,r) = r - x^2}$ When $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = \pm\sqrt{r}}$

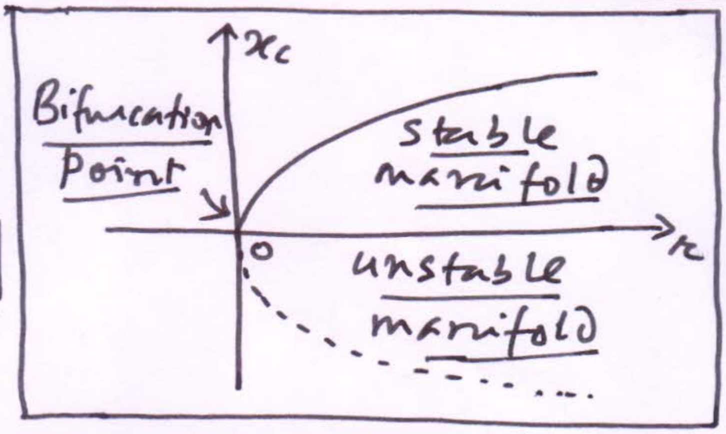


Bifurcation Diagram:

Solve $\boxed{\dot{x} = f(x_c, r) = 0}$

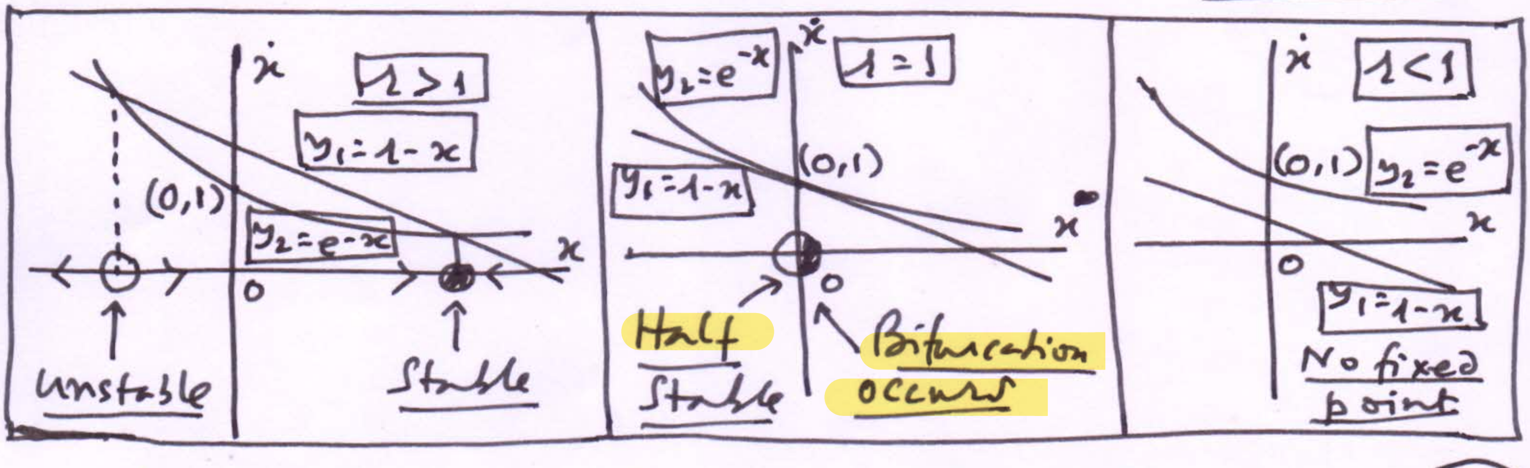
$\Rightarrow$ $\boxed{r - x_c^2 = 0}$ $\Rightarrow$ $\boxed{x_c = \pm\sqrt{r}}$

Real fixed points only for $\boxed{r>0}$.



Example: $\boxed{\dot{x} = r - x - e^{-x}}$ Define $\boxed{y_1 = r - x}$ $\boxed{y_2 = e^{-x}}$.

$\boxed{\dot{x} = f(x) = y_1 - y_2}$. When $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{y_1 = y_2}$.



1/. Reduce r so that the fixed points approach each other.

2/. At $\boxed{r = 1}$ the fixed points coalesce (coincide) and the tangents of y_1 and y_2 are equal.

(P.T.O.)

Bifurcation condition: $\boxed{\dot{x} = f(x, r) = 0}$

$\Rightarrow \boxed{y_1 = y_2} \Rightarrow \boxed{r - x_c = e^{-x_c}}$. Further,

since the tangents are the same, $\boxed{\frac{dy_1}{dx} = \frac{dy_2}{dx}}$

$\Rightarrow \frac{d}{dx}(r - x) = \frac{d}{dx}(e^{-x}) \Rightarrow \boxed{\frac{d}{dx}(r - x - e^{-x}) = 0}$.

Since $\boxed{f(x, r) = r - x - e^{-x}} \Rightarrow \boxed{\frac{df}{dx} = 0} \Rightarrow \boxed{f'(x) = 0}$

Since the first derivative, $\boxed{f'(x) = 0}$ at bifurcation, $\Rightarrow$ the bifurcation point is neither stable, nor unstable.

$\therefore \boxed{f'(x) = -1 + e^{-x} = 0} \Rightarrow \boxed{x_c = 0} \Rightarrow \boxed{f(x_c) = 0}$

$\Rightarrow \boxed{r - x - e^{-x} = 0} \Rightarrow \boxed{r - 0 - e^{-0} = 0} \Rightarrow \boxed{r_b = 1}$

At the bifurcation point, $\boxed{r = r_b = 1}$.

Normal form of the Saddle-Node Bifurcation

$\boxed{\dot{x} = r \pm x^2} \longrightarrow$ The normal form.

Example: $\boxed{\dot{x} = r - x - e^{-x}}$.

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{r_b = 1}$

Expand about $\boxed{x_c = 0}$. $\boxed{e^{-x} = 1 - x + \frac{x^2}{2!} + \cdots}$

$\Rightarrow \dot{x} = (r - x) - \left(1 - x + \frac{x^2}{2!} + \cdots\right)$

$\Rightarrow \boxed{\dot{x} = r - \cancel{x} - 1 + \cancel{x} - \frac{x^2}{2!} + \cdots}$

Expand and go only up to the second order.

(P.T.O.)

$$\Rightarrow \boxed{\dot{x} = (r-1) - \frac{x^2}{2}}$$ Up to the second order. (Parabolic form).

For rescaling we write $\boxed{x = ku}$ [k is a ~~constant~~ parameter]

$$\Rightarrow \boxed{\dot{x} = k\dot{u} = r-1 - \frac{k^2u^2}{2}}$$ Choose $\boxed{k=2}$ to get

$$\Rightarrow \boxed{\dot{u} = \left(\frac{r-1}{k}\right) - \frac{ku^2}{2}} \Rightarrow \boxed{\dot{u} = \frac{r-1}{2} - u^2}$$

$$\Rightarrow \boxed{\dot{u} = R - u^2}$$ in which $\boxed{R = \frac{r-1}{2}}$ (Normal form)

When $\boxed{R=0} \Rightarrow r = 1$ or $\boxed{r_b = 1}$. At $\boxed{u=0}$ (also $x=0$) bifurcation occurs for $\boxed{r = r_b = 1}$.

Taylor Expansion (about the bifurcation point)

$\boxed{\dot{x} = f(x, r)}$. Bifurcation occurs when $\boxed{\dot{x}=0}$ at $\boxed{x = x_c}$ for $\boxed{r = r_b}$.

$\therefore$ $\boxed{f(x_c, r_b) = 0}$ and $\boxed{f'(x_c, r_b) = 0}$ are the conditions required for bifurcation.

Expand about x_c as $\boxed{x = x_c + \epsilon}$ and about r_b as $\boxed{r = r_b + \rho}$. Hence,

$$\boxed{\begin{aligned}\dot{x} &= f(x,r) = f(x_c, r_b) \\ &+ \left.\frac{\partial f}{\partial x}\right|_{x_c, r_b}\epsilon + \left.\frac{\partial f}{\partial r}\right|_{x_c, r_b}\rho + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c, r_b}\epsilon^2 \\ &+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial r\,\partial x}\right|_{x_c, r_b}\epsilon\rho + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x\,\partial r}\right|_{x_c, r_b}\rho\epsilon + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial r^2}\right|_{x_c, r_b}\rho^2 \\ &+ \dots\end{aligned}}$$

(P.T.O.)

Since the bifurcation conditions are $\boxed{f(x_c, \lambda_b) = 0}$ and $\boxed{\left.\frac{\partial f}{\partial x}\right|_{x, \lambda_b} = 0}$, we retain only the ρ and ϵ^2 terms. Since, $\boxed{\epsilon = x - x_c}$ and $\boxed{\rho = \lambda - \lambda_b}$, we have

$$\boxed{\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x_c, \lambda_b} (\lambda - \lambda_b) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c, \lambda_b} (x - x_c)^2}$$

Writing $\boxed{a = \left.\frac{\partial f}{\partial x}\right|_{x_c, \lambda_b}}$ and $\boxed{b = \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c, \lambda_b}}$

gives $\boxed{\dot{x} = a(\lambda - \lambda_b) + b(x - x_c)^2}$. $(a, b \neq 0)$.

Transform $\boxed{y = x - x_c}$ $\Rightarrow$ $\boxed{\dot{x} = \dot{y}}$.

$\Rightarrow$ $\boxed{\dot{y} = a(\lambda - \lambda_b) + b y^2}$ Now scale as $\boxed{y = ku}$.

$\Rightarrow$ $\dot{y} = k\dot{u} = a(\lambda - \lambda_b) + b k^2 u^2$

$\Rightarrow$ $\boxed{\dot{u} = \frac{a(\lambda - \lambda_b)}{k} + b k u^2}$

To derive the normal form.

Choose $\boxed{k = \frac{1}{|b|}}$, so that,

$\boxed{\dot{u} = a|b|(\lambda - \lambda_b) \pm u^2}$ Define $\boxed{R = a|b|(\lambda - \lambda_b)}$

$\Rightarrow$ $\boxed{\dot{u} = R \pm u^2}$ This is the normal form of the saddle-node bifurcation.

When $\boxed{R = 0}$, $\Rightarrow$ $\boxed{\mu = \lambda_b}$. Bifurcation occurs for ~~[illegible]~~ $\boxed{\dot{u} = 0}$ at $\boxed{u = 0}$ ..

Some Examples of Saddle-Node Bifurcation

Example 1: $\boxed{\dot{x} = f(x,r) = 1 + rx + x^2}$

$\Rightarrow \dot{x} = 1 + 2\frac{r}{2}x + x^2 + \frac{r^2}{4} - \frac{r^2}{4}$

$\Rightarrow \boxed{\dot{x} = \left(x + \frac{r}{2}\right)^2 + \left(1 - \frac{r^2}{4}\right)}$

Set $\boxed{y = x + \frac{r}{2}}$ $\Rightarrow$ $\boxed{\dot{x} = \dot{y}}$ define $\boxed{R = 1 - \frac{r^2}{4}}$

$\Rightarrow \boxed{\dot{y} = R + y^2}$. Bifurcation occurs when $\boxed{R = 0}$ $\Rightarrow 1 - \frac{r^2}{4} = 0 \Rightarrow \boxed{r_b = \pm 2}$

Example 2: $\boxed{\dot{x} = f(x,r) = r + x - \ln(1+x)}$

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{r_b = 0}$.

Now $\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots}$ (Expand about $x = 0$)

$\Rightarrow \boxed{\dot{x} = r + \cancel{x} - \left(\cancel{x} - \frac{x^2}{2} + \cdots\right)}$ Going up to the second order.

$\Rightarrow \boxed{\dot{x} = r + \frac{x^2}{2}}$ Scale as $\boxed{x = ku}$.

$\Rightarrow k\dot{u} = r + \frac{k^2u^2}{2} \Rightarrow \boxed{\dot{u} = \frac{r}{k} + \frac{k}{2}u^2}$

Choose $\boxed{k = 2}$ $\Rightarrow \boxed{\dot{u} = \frac{r}{2} + u^2}$, in the normal form $\boxed{\dot{u} = R + u^2}$. Bifurcation occurs when $\boxed{R = 0}$ or $\boxed{r = 0}$ at $\boxed{u = 0}$ or $\boxed{x = 0}$

Example 3: $\boxed{\dot{x} = f(x,r) = r + \frac{x}{2} - \frac{x}{1+x}}$

$\Rightarrow \boxed{\dot{x} = r + \frac{x}{2} - x(1+x)^{-1}}$

By binomial expansion $\boxed{(1+x)^n = 1 + nx + \cdots}$

$\Rightarrow \boxed{\dot{x} = r + \frac{x}{2} - x(1 - x + \cdots)}$

$\Rightarrow \dot{x} = r + \frac{x}{2} - x + x^2 = r - \frac{x}{2} + x^2$

$\Rightarrow \dot{x} = r - 2x\left(\frac{1}{4}\right) + x^2 + \frac{1}{16} - \frac{1}{16}$

$\Rightarrow \boxed{\dot{x} = \left(r - \frac{1}{16}\right) + \left(x - \frac{1}{4}\right)^2}$ Set $\boxed{y = x - \frac{1}{4}}$

$\Rightarrow \boxed{\dot{x} = \dot{y}}$

$\Rightarrow \boxed{\dot{y} = R + y^2}$ in which $\boxed{R = r - 1/16}$.

Bifurcation occurs for $\boxed{R = 0} \Rightarrow \boxed{r_b = 1/16}$.

Example 4: $\boxed{\dot{x} = f(x,r) = r - \cosh x}$ Bifurcation at $\boxed{x_c = 0}$ and $\boxed{r_b = 1}$

Now $\boxed{\cosh x = \frac{e^x - e^{-x}}{2}}$. Series expansion

$\Rightarrow \dot{x} = r - \frac{1}{2}(e^x - e^{-x}) = r - \frac{1}{2}\left[\left(1 + \cancel{x} + \frac{x^2}{2!} + \cdots\right) + \left(1 - \cancel{x} + \frac{x^2}{2!} + \cdots\right)\right]$

$\Rightarrow \boxed{\dot{x} = r - \frac{1}{2}(2 + x^2) = (r - 1) - \frac{x^2}{2}}$

Scale $\boxed{x = ku}$

$\Rightarrow k\dot{u} = r - 1 - \frac{k^2u^2}{2} \Rightarrow \boxed{\dot{u} = \frac{r-1}{k} - \frac{ku^2}{2}}$

Choose $\boxed{k = 2} \Rightarrow \boxed{\dot{u} = \left(\frac{r-1}{2}\right) - u^2}$ define $\boxed{R = \frac{r-1}{2}}$

$\Rightarrow \boxed{\dot{u} = R - u^2}$. Bifurcation occurs for

$\boxed{R = 0}$, $\Rightarrow \boxed{\frac{r-1}{2} = 0} \Rightarrow \boxed{r_b = 1}$ at $\boxed{u = 0}$

$\Rightarrow \boxed{x = 0}$.

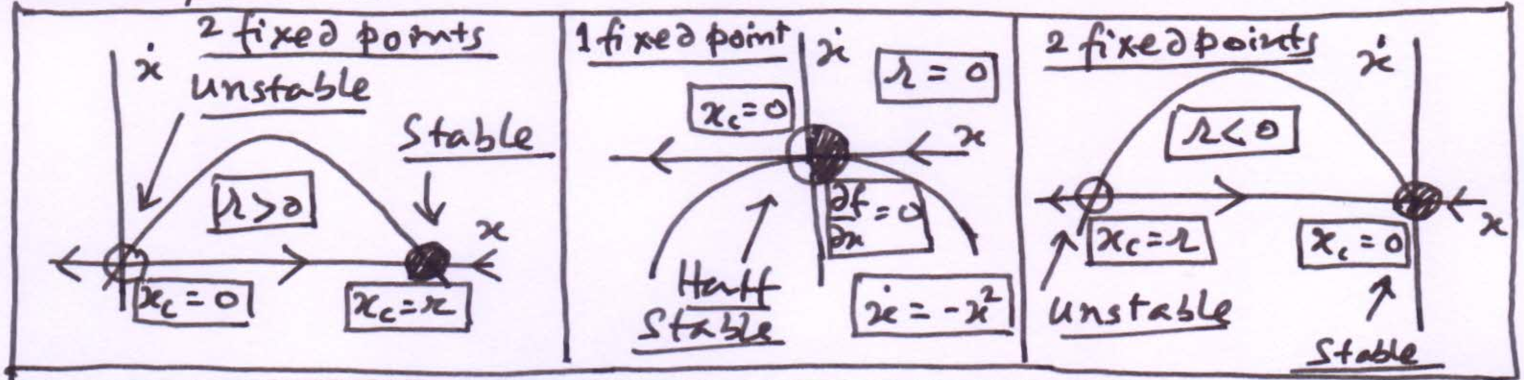
Transcritical Bifurcation

1/. Two fixed points ALWAYS exist.

2/. They exchange their stability properties.

$$\boxed{\dot{x} = f(x, r) = rx - x^2}$$ Normal form of transcritical bifurcation.

Fixedpoints: When $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = 0}$, $\boxed{x_c = r}$.



1/. Bifurcation occurs when $\boxed{r=0}$ at $\boxed{x_c=0}$.

2/. The two fixed points exchange their stabilities.

3/. At the bifurcation point, $\boxed{\frac{\partial f}{\partial x} = 0}$

4/. No fixed point is eliminated (unlike saddle-node)

5/. Resembles the logistic equation: $\boxed{\dot{x} = ax - bx^2}$

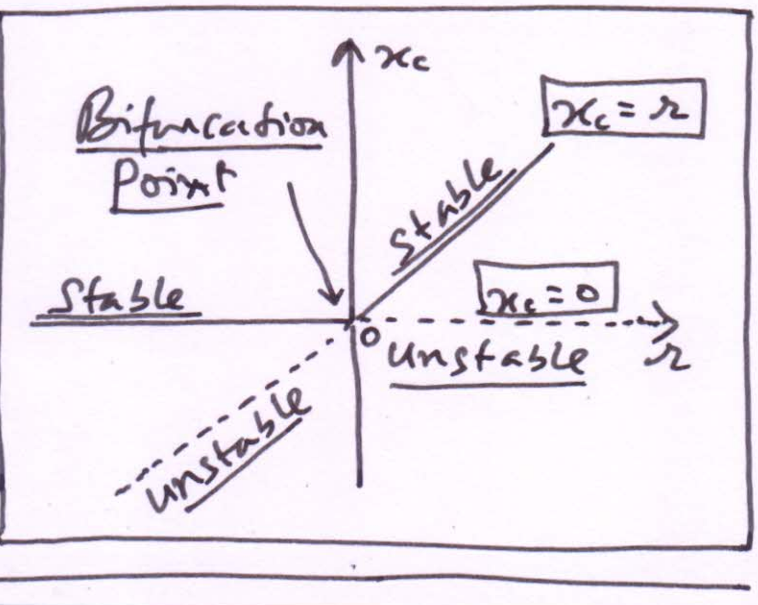
Bifurcation Diagram:

Solve $\boxed{\dot{x} = f(x_c, r) = 0}$

$\Rightarrow$ $\boxed{r x_c - x_c^2 = 0}$ $\Rightarrow$ $\boxed{x_c = 0}$

and $\boxed{x_c = r}$ (Straight lines).

Both fixed points occur for all real values of r



Some Examples of Transcritical Bifurcation

Example 1: $\boxed{\dot{x} = x(1-x^2) - a(1-e^{-bx})}$

$\boxed{\dot{x} = f(x,a,b)}$ For all a, b, $\boxed{x_c = 0}$ is a fixed point.

Expand $\boxed{e^{-bx} = 1 - bx + \frac{b^2x^2}{2!} + \cdots}$ about $x = 0$.

$\Rightarrow \boxed{\dot{x} = x - x^3 - a\left(bx - \frac{b^2x^2}{2}\right)}$ up to the second order.

$\Rightarrow \boxed{\dot{x} = x(1-ab) + \frac{ab^2x^2}{2} - x^3}$ The cubic order is to be neglected.

$\Rightarrow \boxed{\dot{x} = x(1-ab) + \frac{ab^2}{2}x^2}$ when $\boxed{\dot{x} = 0}$ $\boxed{x_c = 0}$ and $\boxed{x_c = \frac{2(ab-1)}{ab^2}}$.

We scale $\boxed{x = ku}$

$\Rightarrow \boxed{k\dot{u} = ku(1-ab) + \frac{ab^2k^2u^2}{2}}$ $\Rightarrow \boxed{\dot{u} = u(1-ab) + \frac{ab^2ku^2}{2}}$

Choose $\boxed{k = -2/ab^2}$ and define $\boxed{R = 1 - ab}$.

$\Rightarrow \boxed{\dot{u} = Ru - u^2}$ Transcritical bifurcation occurs when $\boxed{R = 0}$ $\Rightarrow \boxed{ab = 1}$ (at $\boxed{u = 0} \Rightarrow \boxed{x = 0}$).

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,a,b)}$

$\boxed{f(x,a,b) = x(1-x^2) - a(1-e^{-bx})}$

i.) $\boxed{\frac{\partial f}{\partial a} = -(1-e^{-bx})}$ $\therefore$ When $\boxed{x = 0,\ \frac{\partial f}{\partial a} = 0}$

ii.) $\boxed{\frac{\partial f}{\partial b} = ae^{-bx}\cdot(-x)}$ $\therefore$ When $\boxed{x = 0,\ \frac{\partial f}{\partial b} = 0}$

All the first derivatives vanish

iii.) $\boxed{\frac{\partial f}{\partial x} = f'(x) = 1 - 3x^2 + ae^{-bx}(-b)}$ $\therefore$ When $\boxed{x = 0}$, $\boxed{\frac{\partial f}{\partial x} = 1 - ab}$

Since bifurcation occurs when $\boxed{ab = 1}$ $\Rightarrow \boxed{\frac{\partial f}{\partial x} = 0}$.

Example 2: $\boxed{\dot{x} = f(x,r) = r\ln x + x - 1}$

When $\boxed{\dot{x} = 0}$, $\boxed{x_c = 1}$ → the fixed point.

Transform $\boxed{y = x - 1}$. $\Rightarrow$ $\boxed{x = 1 + y}$ & $\boxed{\dot{x} = \dot{y}}$.

$\therefore$ $\boxed{\dot{y} = r\ln(1+y) + y}$ Now $\boxed{\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + \cdots}$ (series expansion)

$\Rightarrow$ $\boxed{\dot{y} = r\left(y - \frac{y^2}{2}\right) + y}$ ← up to the second order

$\Rightarrow$ $\boxed{\dot{y} = (r+1)y - r\frac{y^2}{2}}$ Scaling $\boxed{y = ku}$,

$\Rightarrow$ $\boxed{k\dot{u} = ku(r+1) - r\frac{k^2u^2}{2}}$ $\Rightarrow$ $\boxed{\dot{u} = (r+1)u - r\frac{ku^2}{2}}$

Choose $\boxed{k = 2/r}$ and $\boxed{R = r+1}$

$\Rightarrow$ $\boxed{\dot{u} = Ru - u^2}$ (Normal form of the transcritical bifurcation.)

For bifurcation, $\boxed{R = 0}$ $\Rightarrow$ $\boxed{r_b = -1}$ at $\boxed{u = y = 0}$ or $\boxed{x = 1}$.

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,r)}$

$\boxed{f(x,r) = r\ln x + x - 1}$ $\begin{bmatrix} x \to \text{variable} \\ r \to \text{parameter} \end{bmatrix}$.

i.) $\boxed{\frac{\partial f}{\partial x} = f'(x) = \frac{r}{x} + 1}$ $\therefore$ When $\boxed{\boxed{r = -1}, \boxed{x = 1} \quad \boxed{\frac{\partial f}{\partial x} = 0}}$

ii.) $\boxed{\frac{\partial f}{\partial r} = \ln x}$ $\therefore$ When $\boxed{x = 1, \; \frac{\partial f}{\partial r} = 0}$ First derivatives vanish.

iii.) $\boxed{\frac{\partial^2 f}{\partial x \partial r} = \frac{1}{x} = \frac{\partial^2 f}{\partial r \partial x}}$ At $\boxed{x = 1, \; \frac{\partial^2 f}{\partial r \partial x} = 1 = \frac{\partial^2 f}{\partial x \partial r}}$

The mixed second derivative is not zero.

Example 3: $\boxed{\dot{x} = f(x,\lambda) = \lambda x + x^2}$

Scale $\boxed{x = ku}$ $\Rightarrow$ $\boxed{k\dot{u} = \lambda k u + k^2 u^2}$

$\Rightarrow$ $\boxed{\dot{u} = \lambda u + k u^2}$ Choose $\boxed{k = -1}$ to get

$\boxed{\dot{u} = \lambda u - u^2}$. Bifurcation occurs for $\boxed{\lambda_b = 0}$ at $\boxed{u = x = 0}$.

- The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,\lambda)}$

$\boxed{f(x,\lambda) = \lambda x + x^2}$ $\Rightarrow$ $\boxed{\frac{\partial f}{\partial x} = \lambda + 2x}$, $\boxed{\frac{\partial f}{\partial \lambda} = x}$

when $\boxed{\lambda = 0}$ and $\boxed{x = 0}$ $\boxed{\frac{\partial f}{\partial x} = 0}$ $\boxed{\frac{\partial f}{\partial \lambda} = 0}$.

$\boxed{\frac{\partial^2 f}{\partial \lambda \partial x} = 1 = \frac{\partial^2 f}{\partial x \partial \lambda}}$ first derivatives vanish. The mixed second derivative survives.

Example 4: $\boxed{\dot{x} = f(x,\lambda) = \lambda x - \ln(1+x)}$

$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \dots}$ up to the second order.

$\Rightarrow$ $\boxed{\dot{x} = \lambda x - \left(x - \frac{x^2}{2}\right)}$ $\Rightarrow$ $\boxed{\dot{x} = (\lambda - 1)x + \frac{x^2}{2}}$

Scale $\boxed{x = ku}$, $\Rightarrow$ $\boxed{k\dot{u} = (\lambda - 1)ku + \frac{k^2 u^2}{2}}$.

$\Rightarrow$ $\boxed{\dot{u} = (\lambda - 1)u + \frac{k u^2}{2}}$ Choose $\boxed{k = -2}$ ~~to get~~ and define $\boxed{R = \lambda - 1}$.

$\Rightarrow$ $\boxed{\dot{u} = Ru - u^2}$ $\rightarrow$ The normal form of transcritical bifurcation

When $\boxed{R = 0}$, $\boxed{\lambda_b = 1}$ at $\boxed{u = x = 0}$

- The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,\lambda)}$

(i) $\boxed{f(x,\lambda) = \lambda x - \ln(1+x)}$ $\Rightarrow$ $\boxed{\frac{\partial f}{\partial x} = \lambda - \frac{1}{1+x}}$

and $\boxed{\frac{\partial f}{\partial \lambda} = x}$. $\boxed{\frac{\partial^2 f}{\partial x \partial \lambda} = 1 = \frac{\partial^2 f}{\partial \lambda \partial x}}$. When $\boxed{x = 0}$, $\boxed{\lambda_b = 1}$. $\boxed{\frac{\partial f}{\partial x} = 0}$, $\boxed{\frac{\partial f}{\partial \lambda} = 0}$

Both first derivatives vanish.

Example 5: $\boxed{\dot{x} = f(x,r) = x(r - e^{x})}$

$\boxed{e^{x} = 1 + x + \frac{x^2}{2!} + \cdots} \Rightarrow \boxed{\dot{x} = rx - x\left(1 + x + \frac{x^2}{2!} + \cdots\right)}$

$\Rightarrow \boxed{\dot{x} = (r-1)x - x^2}$ up to the second order.

Define $\boxed{R = r - 1} \Rightarrow \boxed{\dot{x} = Rx - x^2}$ (Normal form)

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{R = 0} \Rightarrow \boxed{r_b = 1}$

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,r)}$

$\boxed{f(x,r) = rx - xe^{x}} \Rightarrow \boxed{\frac{\partial f}{\partial r} = x}$ $\boxed{\frac{\partial f}{\partial x} = r - xe^{x} - e^{x}}$

When $\boxed{x = 0}$, $\boxed{r = 1}$ $\boxed{\frac{\partial f}{\partial r} = 0 = \frac{\partial f}{\partial x}}$ First derivatives vanish.

$\boxed{\frac{\partial^2 f}{\partial r \partial x} = 1 = \frac{\partial^2 f}{\partial x \partial r}}$. The mixed second derivative is non-zero.

Example 6: $\boxed{\dot{x} = f(x,r) = x - rx(1-x)}$

$\Rightarrow \boxed{\dot{x} = (1-r)x + rx^2}$ Scale $\boxed{x = ku}$.

$\Rightarrow k\dot{u} = (1-r)ku + rk^2u^2 \Rightarrow \boxed{\dot{u} = (1-r)u + rku^2}$

Choose $\boxed{k = -1/r}$, Define $\boxed{R = 1 - r} \Rightarrow \boxed{\dot{u} = Ru - u^2}$

(Normal form)

Bifurcation occurs at $\boxed{u = 0}$

$\Rightarrow \boxed{x = 0}$ for $\boxed{R = 0} \Rightarrow \boxed{r_b = 1}$.

The derivatives at the Bifurcation: $\boxed{\dot{x} = f(x,r)}$

$\boxed{\frac{\partial f}{\partial x} = 1 - r + 2rx}$, $\boxed{\frac{\partial f}{\partial r} = -x + x^2}$. When $\boxed{x = 0}$, $\boxed{r = 1}$ $\boxed{\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial r}}$

$\boxed{\frac{\partial^2 f}{\partial r \partial x} = -1 + 2x = \frac{\partial^2 f}{\partial x \partial r}}$ At the bifurcation the mixed second derivative is non-zero

Both the first derivatives vanish at bifurcation.

Taylor Expansion (about the bifurcation point)

$\boxed{\dot{x} = f(x,\lambda)}$ Bifurcation occurs when $\boxed{\dot{x}=0}$ at $\boxed{x=x_c}$ for $\boxed{\lambda=\lambda_b}$.

The conditions at the bifurcation are

$$\boxed{f(x_c,\lambda_b)=0}, \quad \boxed{\left.\frac{\partial f}{\partial x}\right|_{x_c,\lambda_b} = f'(x_c,\lambda_b) = 0}, \quad \boxed{\left.\frac{\partial f}{\partial \lambda}\right|_{x_c,\lambda_b} = 0}.$$

All first derivatives vanish for transcritical bifurcations.

Expand about x_c as $\boxed{x = x_c + \epsilon}$ and about λ_b as $\boxed{\lambda = \lambda_b + \rho}$ $\Rightarrow$ $\boxed{\begin{array}{l}\epsilon = x - x_c \\ \rho = \lambda - \lambda_b\end{array}}$

$$\boxed{\begin{aligned}\dot{x} = f(x,\lambda) &= f(x_c,\lambda_b) + \left.\frac{\partial f}{\partial x}\right|_{x_c,\lambda_b}\epsilon + \left.\frac{\partial f}{\partial \lambda}\right|_{x_c,\lambda_b}\rho \\ &+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,\lambda_b}\epsilon^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial \lambda\partial x}\right|_{x_c,\lambda_b}\epsilon\rho + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x\partial \lambda}\right|_{x_c,\lambda_b}\rho\epsilon + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial \lambda^2}\right|_{x_c,\lambda_b}\rho^2 + \dots\end{aligned}}$$

$\boxed{\text{Now } f(x_c,\lambda_b) = \left.\frac{\partial f}{\partial x}\right|_{x_c,\lambda_b} = \left.\frac{\partial f}{\partial \lambda}\right|_{x_c,\lambda_b} = 0}$ which are the conditions for the transcritical bifurcation.

Retain only the ϵ^2 term or the $(x-x_c)^2$ term and the two $\rho\epsilon$ terms or the $(x-x_c)(\lambda-\lambda_b)$ terms.

$$\therefore \boxed{\dot{x} = \left[\left.\frac{\partial^2 f}{\partial \lambda\partial x}\right|_{x_c,\lambda_b}(\lambda-\lambda_b)\right](x-x_c) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,\lambda_b}(x-x_c)^2}$$

Writing $\boxed{a = \left.\frac{\partial^2 f}{\partial \lambda\partial x}\right|_{x_c,\lambda_b}}$ and $\boxed{b = \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,\lambda_b}}$

gives. $\boxed{\dot{x} = a(\lambda-\lambda_b)(x-x_c) + b(x-x_c)^2}$ $(a, b \neq 0)$

Transform $\boxed{y = x - x_c}$ $\Rightarrow$ $\boxed{\dot{x} = \dot{y}}$ (P.T.O.)

$$\boxed{\dot{y} = a(r - r_b)y + by^2}$$ Scaling as $\boxed{y = ku}$

$\Rightarrow k\dot{u} = a(r - r_b)ku + bk^2u^2$

$\Rightarrow \boxed{\dot{u} = a(r - r_b)u + bku^2}$ Choose $\boxed{k = -\frac{1}{b}}$ and define $\boxed{R = a(r - r_b)}$

$\therefore \boxed{\dot{u} = Ru - u^2}$ Normal form of the transcritical bifurcation.

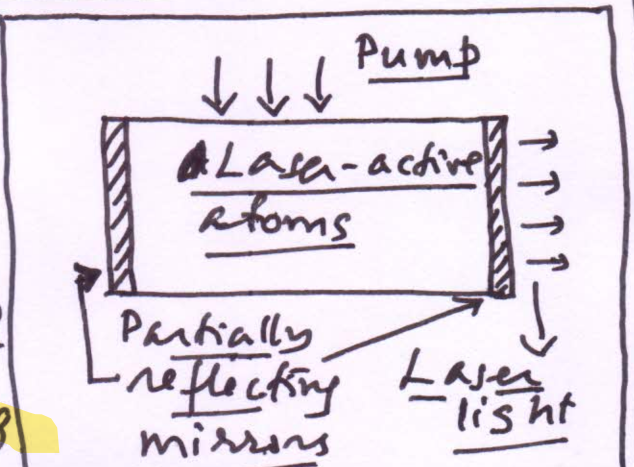
Bifurcation occurs when $\boxed{R = 0} \Rightarrow \boxed{r = r_b}$ at $\boxed{u = 0}$

Laser Threshold

1/. Laser action happens only above a threshold pumping.

2/. Below the threshold the laser is like an ordinary lamp

3/. The process is self-organising due to cooperative interaction (among the atoms).



i.) $n(t)$ is the photon number in the laser.

ii.) $N(t)$ is the number of stimulated atoms.

iii.) $\boxed{\frac{dn}{dt} = \text{gain} - \text{loss}}$. Gain is due to stimulated emission. Loss is due to leakage.

iv.) Gain happens after the random encounters between photons and stimulated atoms.

v.) $\boxed{\text{Gain} \propto nN}$, → (Jointly proportional).

vi.) $\boxed{\text{Gain} = GnN}$ $\boxed{G > 0}$ → Gain Coefficient.

(P.T.O.)

(continued)

vii.) $\boxed{\text{Loss} \propto n}$ $\Rightarrow$ $\boxed{\text{Loss} = kn}$ $\boxed{k > 0}$ $\rightarrow$ Loss coefficient.

viii.) $\boxed{\frac{dn}{dt} = \dot{n} = GnN - kn}$

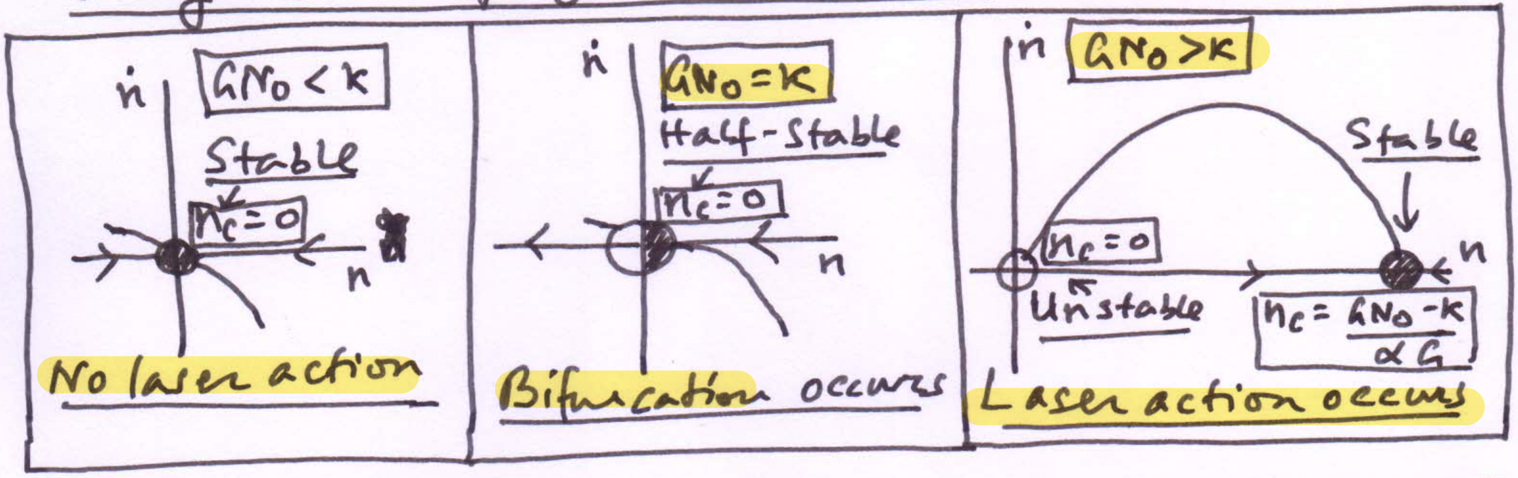
ix.) $\boxed{N(t) = N_0 - \alpha n}$ N_0 is the fixed number of atoms in the absence of lasing action. α is the rate at which atoms return to the ground state. $(\alpha > 0)$

$$\therefore \boxed{\dot{n} = Gn(N_0 - \alpha n) - kn = (GN_0 - k)n - \alpha G n^2}$$

(Like the logistic equation leading to transcritical bifurcation)

Fixed points: $\boxed{\dot{n} = 0}$ $\Rightarrow$ $\boxed{n_c = 0}$ and $\boxed{n_c = \frac{GN_0 - k}{\alpha G}}$

Only $n > 0$ is physically valid. (Two fixed points)

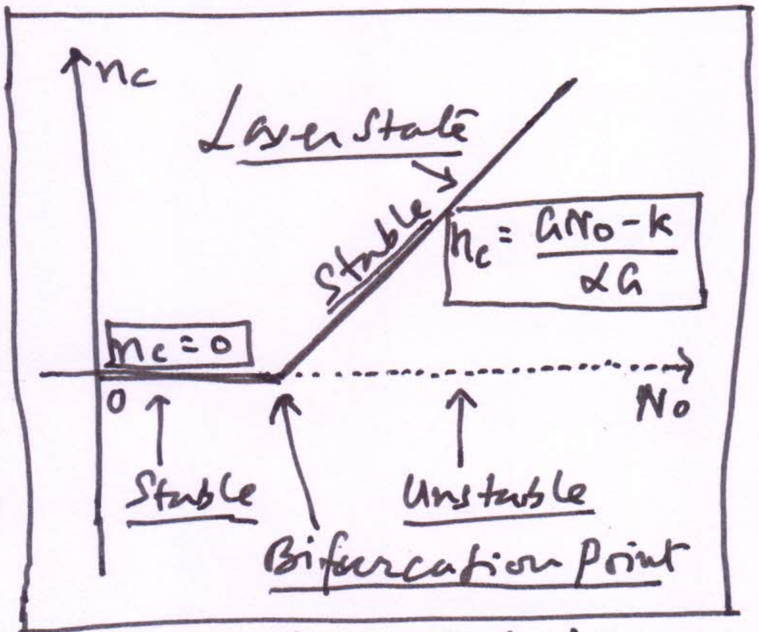


Bifurcation Diagram:

Solve $\boxed{\dot{n} = f(n_c, N_0) = 0}$

(N_0 is the parameter)

$\Rightarrow$ $\boxed{n_c = 0}$ and $\boxed{n_c = \frac{GN_0 - k}{\alpha G}}$



1/. Only $\boxed{n_c > 0}$ gives laser action

2/. Laser action begins at the transcritical bifurcation point. For $\boxed{GN_0 > k}$ the laser state is an attractor.

Pitchfork Bifurcation

1/. Either one fixed point or three fixed points.
2/. At the bifurcation, two fixed points annihilate each other (saddle-node bifurcation).
3/. Symmetric (invariant) for $\boxed{x \to -x}$.

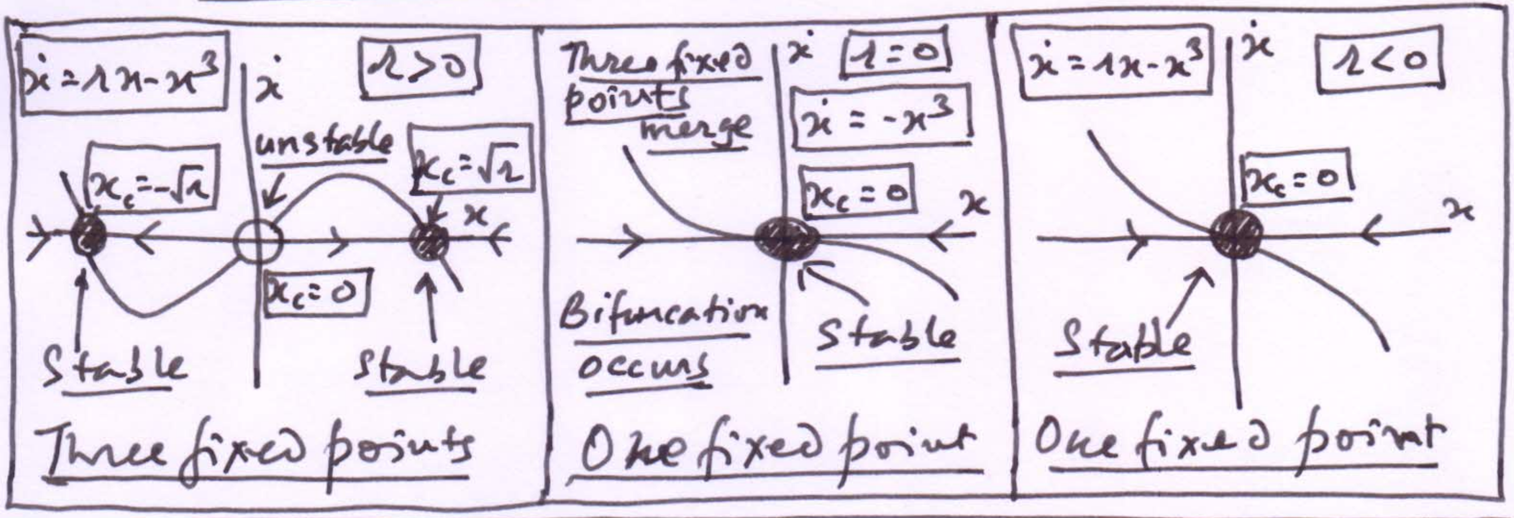
Supercritical Pitchfork Bifurcation

$\boxed{\dot{x} = f(x,r) = rx - x^3}$ Normal form of the supercritical pitchfork bifurcation.

Fixed points: $\boxed{\dot{x} = 0} \Rightarrow \boxed{x_c = 0}$ and $\boxed{x_c = \pm\sqrt{r}}$

i.) Three fixed points when $\boxed{r > 0}$.
ii.) One fixed point when $\boxed{r < 0}$.

Bifurcation condition is $\boxed{r = 0}$



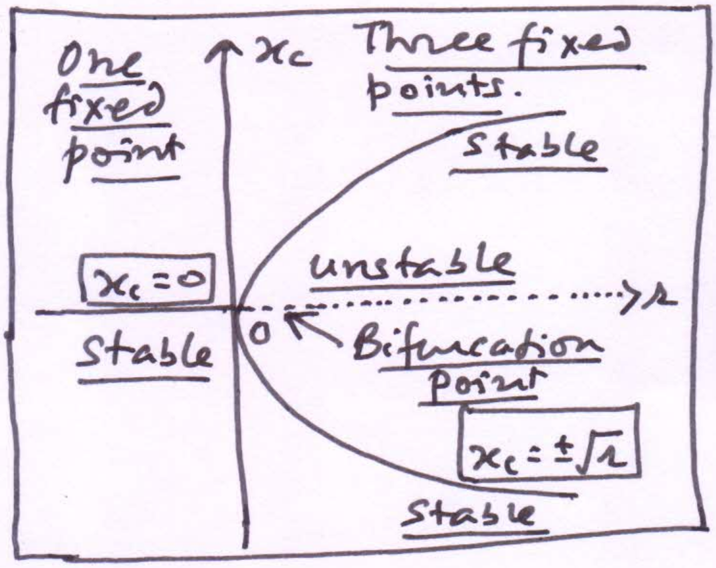
1/. Bifurcation occurs when $\boxed{r = 0}$ at $\boxed{x_c = 0}$.
2/. At the bifurcation, two fixed points disappear (as in a saddle-node bifurcation, they merge).
3/. After the bifurcation only one fixed point remains.
4/. At the bifurcation $\boxed{\frac{\partial f}{\partial x} = 0}$ and $\boxed{\frac{\partial^2 f}{\partial x^2} = 0}$.

Bifurcation Diagram:

Solve $\boxed{\dot{x} = f(x_c, \lambda) = 0}$

$\Rightarrow \boxed{\lambda x_c - x_c^3 = 0} \Rightarrow \boxed{x_c = 0}$

and $\boxed{x_c = \pm\sqrt{\lambda}}$ Two real roots for $\boxed{\lambda > 0}$. The bifurcation diagram looks like a pitchfork.



The bifurcation is called "supercritical" because the non-zero fixed points are above ("super") $(\lambda > 0)$ the bifurcation point at $\boxed{x_c = 0}$ for $\boxed{\lambda = \lambda_b = 0}$.

Example: $\boxed{\dot{x} = -x + \beta \tanh x}$ (β is a parameter).

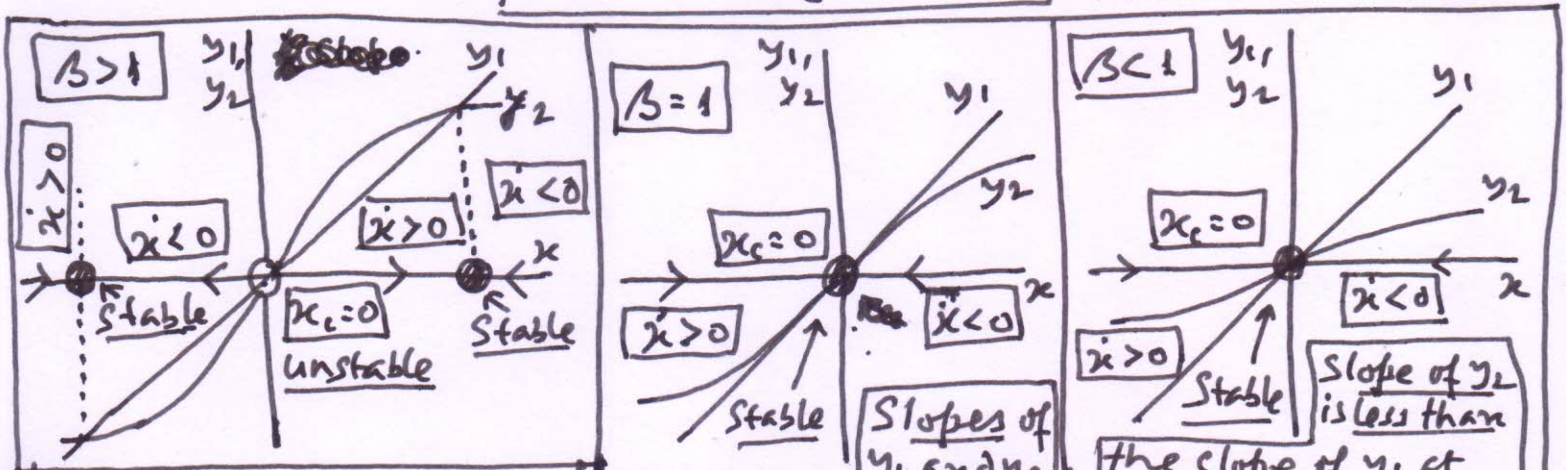
Define $\boxed{y_1 = x}$ and $\boxed{y_2 = \beta \tanh x} \Rightarrow \boxed{\dot{x} = y_2 - y_1}$

Now $\boxed{\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}$

i.) when $x \to 0$ $\boxed{e^x \simeq 1 + x}$ and $\boxed{e^{-x} \simeq 1 - x}$.

$\Rightarrow \boxed{\tanh x = \dfrac{1 + x - (1 - x)}{1 + x + 1 - x}} \Rightarrow \boxed{\tanh = \dfrac{2x}{2} \simeq x} \Rightarrow \boxed{y_2 \simeq \beta x}$

ii.) when $x \to \infty$, $\boxed{\tanh x = \dfrac{1 + e^{-2x}}{1 + e^{-2x}} \to 1} \Rightarrow \boxed{y_2 \to \beta}$ $\boxed{\text{When } x = 0,\ y_2 = 0}$



Slope of y_2 is greater than the slope of y_1 at $\boxed{x_c = 0}$.

Slopes of y_1 and y_2 are equal at $\boxed{x_c = 0}$

Slope of y_2 is less than the slope of y_1 at $\boxed{x_c = 0}$

(continued)

The normal form of $\boxed{\dot{x} = -x + \beta \tanh x}$

$$\boxed{\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}} \qquad \boxed{\begin{aligned} e^{x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \end{aligned}}$$

$$\Rightarrow \tanh x = \frac{(\cancel{1} + x + \cancel{x^2/2!} + x^3/3! + \cdots) - (\cancel{1} - x + \cancel{x^2/2!} - x^3/3! + \cdots)}{(1 + \cancel{x} + x^2/2! + \cancel{x^3/3!} + \cdots) + (1 - \cancel{x} + x^2/2! - \cancel{x^3/3!} + \cdots)}$$

$$\Rightarrow \boxed{\tanh x \simeq \frac{2x + 2x^3/6}{2 + x^2} = \frac{x + x^3/6}{1 + x^2/2}}$$ (up to the third order).

$$\Rightarrow \tanh x \simeq \left(x + \frac{x^3}{6}\right)\left(1 + \frac{x^2}{2}\right)^{-1} \simeq \left(x + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right)$$

$$\Rightarrow \boxed{\tanh x \simeq x + \frac{x^3}{6} - \frac{x^3}{2} = x + \frac{x^3}{6} - \frac{3x^3}{6} \simeq x - \frac{x^3}{3}}$$

$$\Rightarrow \boxed{\dot{x} = -x + \beta\left(x - \frac{x^3}{3}\right) = (\beta - 1)x - \frac{\beta}{3}x^3}$$

Rescale $\boxed{x = Ku}$ $\Rightarrow$ $\boxed{K\dot{u} = (\beta - 1)Ku - \frac{\beta}{3}K^3x^3}$

$\Rightarrow \boxed{\dot{u} = (\beta - 1)u - \frac{\beta}{3}K^2x^3}$ Choose $\boxed{K^2 = \frac{3}{\beta}}$ Define $\boxed{R = \beta - 1}$.

$\Rightarrow \boxed{\dot{u} = Ru - u^3}$ Normal form of the supercritical pitchfork bifurcation for $\boxed{\beta = 1}$.

Bifurcation Diagram:

Solve $\boxed{\dot{x} = f(x_c, \beta) = 0}$

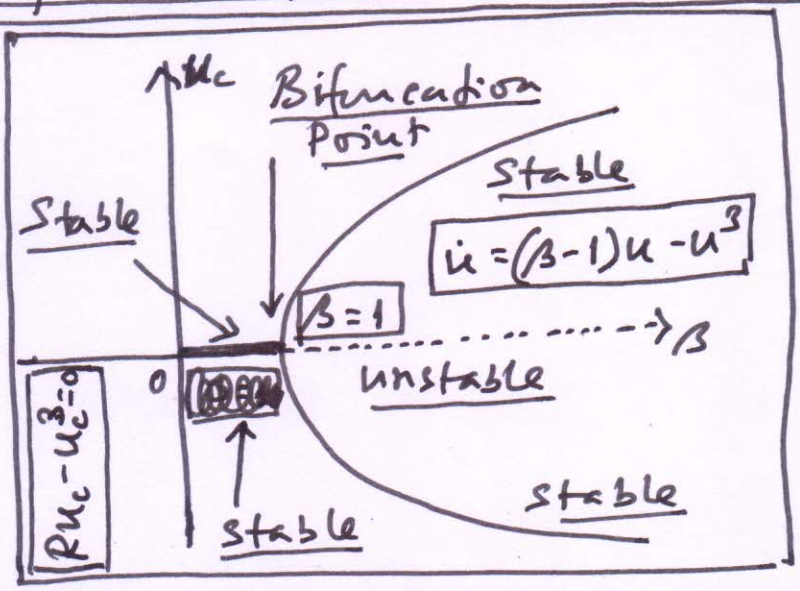
or $\boxed{\dot{u} = f(u_c, R) = 0}$ (in this problem)

$\Rightarrow \boxed{u_c = 0}$, $\boxed{u_c = \pm\sqrt{R}}$

$\Rightarrow \boxed{u_c = \pm\sqrt{\beta - 1}}$

i.) One fixed point for $\boxed{\beta < 1}$.

ii.) Three fixed points for $\boxed{\beta > 1}$.



Subcritical Pitchfork Bifurcation

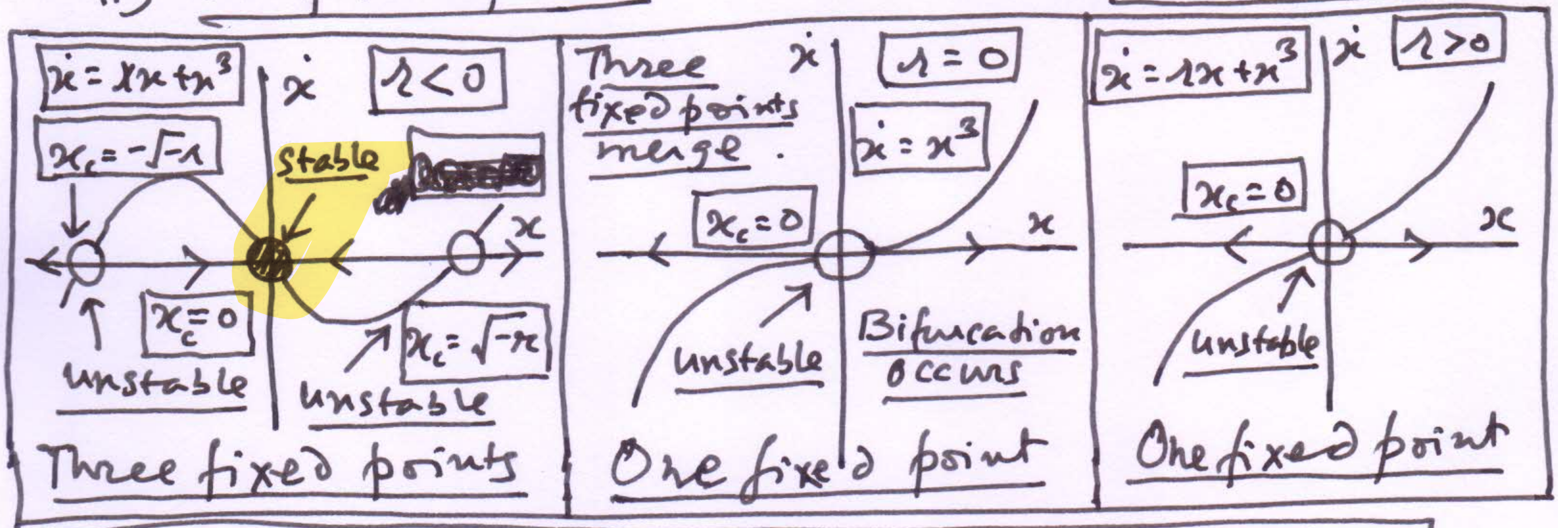
$\boxed{\dot{x} = f(x, r) = rx + x^3}$ Normal form of the subcritical pitchfork bifurcation

$\boxed{\text{The cubic term destabilizes for } x > 0.}$

Fixed points: $\boxed{\dot{x} = 0} \Rightarrow \boxed{x_c = 0}$ and $\boxed{x_c = \pm\sqrt{-r}}$.

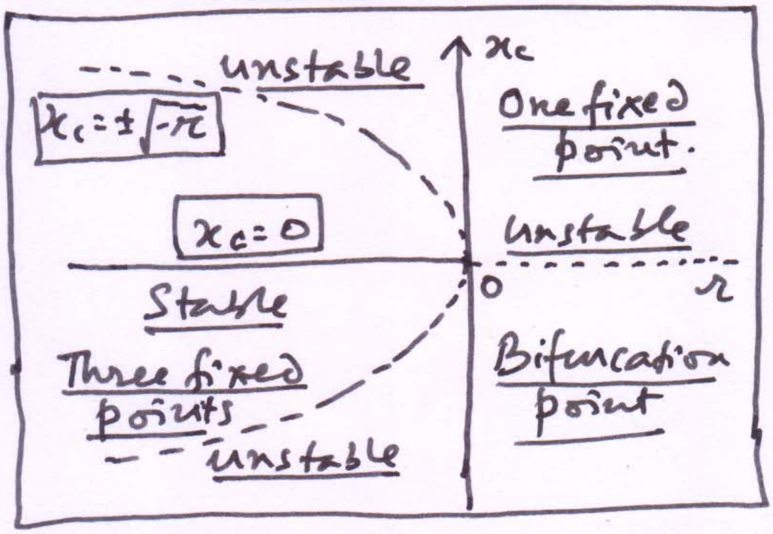
i.) Three fixed points when $\boxed{r < 0}$.
ii.) One fixed point when $\boxed{r > 0}$

Bifurcation condition is $\boxed{r = 0}$.



$\boxed{\text{All conditions for the supercritical pitchfork bifurcation apply in the subcritical case.}}$

Bifurcation Diagram:



Solve $\boxed{\dot{x} = f(x_c, r) = 0}$

$\Rightarrow \boxed{r x_c + x_c^3 = 0} \Rightarrow \boxed{x_c = 0}$

and $\boxed{x_c = \pm\sqrt{-r}}$. Two real roots for $\boxed{r < 0}$. The bifurcation is called "subcritical" because the non-zero fixed points are below ("sub") ($r < 0$) the bifurcation point at $\boxed{x_c = 0}$ for $\boxed{r = r_b = 0}$.

Some Examples of Pitchfork Bifurcation

Example 1: $\boxed{\dot{x} = f(x,r) = rx + 4x^3}$ Fixed point at $\boxed{x_c = 0}$

Scale $\boxed{x = ku}$ $\Rightarrow$ $\boxed{k\dot{u} = rku + 4k^3u^3}$.

$\Rightarrow$ $\boxed{\dot{u} = ru + 4k^2u^3}$. Choose $\boxed{k^2 = 1/4}$

$\Rightarrow$ $\boxed{\dot{u} = ru + u^3}$ Normal form of subcritical pitchfork bifurcation

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{r_b = 0}$.

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,r)}$

$\boxed{f(x,r) = rx + 4x^3}$ $\Rightarrow$ $\boxed{\frac{\partial f}{\partial r} = x}$, $\boxed{\frac{\partial^2 f}{\partial r^2} = 0}$

$\boxed{\frac{\partial f}{\partial x} = r + 12x^2}$, $\boxed{\frac{\partial^2 f}{\partial x^2} = 24x}$, $\boxed{\frac{\partial^2 f}{\partial r \partial x} = \frac{\partial^2 f}{\partial x \partial r} = 1}$

When $\boxed{x = 0,\ r = 0}$. $\boxed{\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial x^2} = 0}$

$\boxed{\frac{\partial^3 f}{\partial x^3} = 24}$. Only the mixed and the third derivatives are non-zero.

Example 2: $\boxed{\dot{x} = f(x,r) = rx - 4x^3}$ Fixed point at $\boxed{x_c = 0}$.

Scale $\boxed{x = ku}$ $\Rightarrow$ $\boxed{k\dot{u} = rku - 4k^3u^3}$.

$\Rightarrow$ $\boxed{\dot{u} = ru - 4k^2u^3}$. Choose $\boxed{k^2 = 1/4}$ $\Rightarrow$ $\boxed{\dot{u} = ru - u^3}$

This is the normal form of supercritical pitchfork bifurcation.

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{r_b = 0}$.

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,r)}$

$\boxed{f(x,r) = rx - 4x^3}$ $\Rightarrow$ $\boxed{\frac{\partial f}{\partial r} = x}$, $\boxed{\frac{\partial^2 f}{\partial r^2} = 0}$.

$\boxed{\frac{\partial f}{\partial x} = r - 12x^2}$, $\boxed{\frac{\partial^2 f}{\partial x^2} = -24x}$, $\boxed{\frac{\partial^2 f}{\partial x \partial r} = \frac{\partial^2 f}{\partial r \partial x} = 1}$

(P.T.O.)

Example 2 (continued): When $\boxed{x=0}$, $\boxed{r=0}$,

$$\boxed{\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial x^2} = 0}\ ,\ \boxed{\frac{\partial^3 f}{\partial x^3} = -24}.$$

Only the mixed and the third derivatives are non-zero.

Example 3: $\boxed{\dot{x} = f(x,r) = x + \frac{rx}{1+x^2}}$ Fixed point at $\boxed{x_c = 0}$.

$\Rightarrow \boxed{\dot{x} = x + rx(1+x^2)^{-1} \simeq x + rx(1-x^2)}$ (Binomial expansion) $\boxed{(1+x)^n \simeq 1+nx}$

$\Rightarrow \boxed{\dot{x} = x + rx - rx^3 = (r+1)x - rx^3}$

Scale $\boxed{x = ku}$ $\Rightarrow \boxed{k\dot{u} = (r+1)ku - rk^3u^3}$

$\Rightarrow \boxed{\dot{u} = (r+1)u - rk^2u^3}$. Choose $\boxed{k^2 = 1/r}$.

Define $\boxed{R = r+1}$.

$\Rightarrow \boxed{\dot{u} = Ru - u^3}$ Normal form of the supercritical pitchfork bifurcation.

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{r_b = -1}$, $(R = 0)$.

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,r)}$.

$\boxed{f(x,r) = x + rx(1+x^2)^{-1}} \Rightarrow \boxed{\frac{\partial f}{\partial r} = \frac{x}{1+x^2}}\ ,\ \boxed{\frac{\partial^2 f}{\partial r^2} = 0}$.

$$\boxed{\frac{\partial f}{\partial x} = 1 + \frac{r}{1+x^2} + \frac{rx \times -1 \times 2x}{(1+x^2)^2} = 1 + \frac{r}{1+x^2} - \frac{2rx^2}{(1+x^2)^2}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-r.2x}{(1+x^2)^2} - 2r\left[\frac{2x}{(1+x^2)^2} + \frac{x^2 \times -2 \times 2x}{(1+x^2)^3}\right]$$

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = -\frac{2xr}{(1+x^2)^2} - 2r\left[\frac{2x}{(1+x^2)^2} - \frac{4x^3}{(1+x^2)^3}\right]}$$

When $\boxed{x = 0}$, $\boxed{r = -1}$ $\hookrightarrow (\boxed{x=0}, \boxed{r=-1})$ $\boxed{\frac{\partial f}{\partial r} = \frac{\partial^2 f}{\partial r^2} = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = 0}$ (P.T.O.)

$$\boxed{\frac{\partial^3 f}{\partial x^3} = \frac{-2r}{(1+x^2)^2} - \frac{2x2 \times -2 \times 2x}{(1+x^2)^3} - 2r\left[\frac{2}{(1+x^2)^2} + \frac{2x \times -2 \times 2x}{(1+x^2)^3} - \frac{4 \times 3x^2}{(1+x^2)^3} - \frac{4x^3 \times -3 \times 2x}{(1+x^2)^4}\right]}$$

When $\boxed{r=-1}$, $\boxed{x=0}$ $\boxed{\frac{\partial^3 f}{\partial x^3} = 2+4 = 6}$

$$\boxed{\frac{\partial^2 f}{\partial x \partial r} = \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} = \frac{\partial^2 f}{\partial r \partial x}}$$ When $\boxed{x=0}$ $\Rightarrow \boxed{\frac{\partial^2 f}{\partial r \partial x} = 1}$

Only the mixed and the third derivatives are non-zero.

Example 4: $\boxed{\dot{x} = f(x,r) = rx - \sinh x}$ fixed point at $\boxed{x_c = 0}$.

$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$ $\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$

$\boxed{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}$ Retaining only up to the cubic order.

$$\Rightarrow \dot{x} = rx - \frac{1}{2}\left[\left(\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cdots\right) - \left(\cancel{1} - x + \cancel{\frac{x^2}{2!}} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$\Rightarrow \boxed{\dot{x} = rx - \frac{1}{2}\left(2x + \frac{x^3}{3}\right) = (r-1)x - \frac{x^3}{6}}$$ Scale $\boxed{x = ku}$

$$\Rightarrow \boxed{k\dot{u} = (r-1)ku - \frac{k^3u^3}{6}}$$ Choose $\boxed{k^2 = 6}$ and define $\boxed{R = r-1}$.

$$\Rightarrow \boxed{\dot{u} = Ru - u^3}$$ Normal form of the supercritical pitchfork bifurcation.

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{R=0} \Rightarrow \boxed{r_b = 1}$.

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x,r)}$.

$\boxed{f(x,r) = rx - \left(\frac{e^x - e^{-x}}{2}\right)}$. $\boxed{\frac{\partial f}{\partial r} = x}$, $\boxed{\frac{\partial^2 f}{\partial r^2} = 0}$

(P.T.O.)

$$\boxed{\frac{\partial f}{\partial x} = r - \left(\frac{e^{x}+e^{-x}}{2}\right)}, \quad \boxed{\frac{\partial^2 f}{\partial x^2} = -\frac{1}{2}\left(e^{x}-e^{-x}\right)}$$

When $\boxed{x=0}$, $\boxed{r=1}$ $\boxed{\frac{\partial f}{\partial r} = \frac{\partial^2 f}{\partial r^2} = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = 0}$

$$\boxed{\frac{\partial^3 f}{\partial x^3} = -\frac{1}{2}\left(e^{x}+e^{-x}\right) = -1} \left(\begin{matrix} x=0 \\ r=1 \end{matrix}\right) \boxed{\frac{\partial^2 f}{\partial r \partial x} = 1 = \frac{\partial^2 f}{\partial x \partial r}}.$$

Only the mixed and the third derivatives are non-zero.

Taylor Expansion (about the bifurcation point)

$\boxed{\dot{x} = f(x,r)}$ Bifurcation occurs when $\boxed{\dot{x}=0}$ at $\boxed{x = x_c}$ for $\boxed{r = r_b}$.

The conditions at the bifurcation are

$$\boxed{f(x_c, r_b) = 0}, \quad \boxed{\left.\frac{\partial f}{\partial x}\right|_{x_c, r_b} = f'(x_c, r_b) = 0}, \quad \boxed{\left.\frac{\partial f}{\partial r}\right|_{x_c, r_b} = 0},$$

$$\boxed{\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c, r_b} = 0}, \quad \boxed{\left.\frac{\partial^2 f}{\partial r^2}\right|_{x_c, r_b} = 0}$$

All first and second derivatives vanish in a pitchfork bifurcation.

Expand about x_c as $\boxed{x = x_c + \epsilon}$ and about r_b as $\boxed{r = r_b + \rho}$ $\Rightarrow$ $\boxed{\begin{matrix}\epsilon = x - x_c \\ \rho = r - r_b\end{matrix}}$.

$$\boxed{\begin{aligned} \dot{x} = f(x,r) &= f(x_c, r_b) + \left.\frac{\partial f}{\partial x}\right|_{x_c, r_b}\epsilon + \left.\frac{\partial f}{\partial r}\right|_{x_c, r_b}\rho + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c, r_b}\epsilon^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x \partial r}\right|_{x_c, r_b}\rho\epsilon \\ &+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial r \partial x}\right|_{x_c, r_b}\epsilon\rho + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial r^2}\right|_{x_c, r_b}\rho^2 + \frac{1}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x_c, r_b}\epsilon^3 + \dots \end{aligned}}$$

Retain only the $\epsilon\rho$ or the $(x - x_c)(r - r_b)$ and the ϵ^3 or $(x - x_c)^3$ terms in the Taylor expansion.

(P.T.O.)

$$\therefore \boxed{\dot{x} = \left[\frac{\partial^2 f}{\partial x \partial r}\Big|_{x_c, r_b} (r - r_b)\right](x - x_c) + \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}\Big|_{x_c, r_b}(x - x_c)^3}$$

Writing $\boxed{a = \frac{\partial^2 f}{\partial x \partial r}\Big|_{x_c, r_b}}$ and $\boxed{b = \frac{1}{6}\frac{\partial^3 f}{\partial x^3}\Big|_{x_c, r_b}}$,

gives $\boxed{\dot{x} = a(r - r_b)(x - x_c) + b(x - x_c)^3}$ $(a, b \neq 0)$

Transform $\boxed{y = x - x_c}$ $\Rightarrow$ $\boxed{\dot{x} = \dot{y}}$

$\Rightarrow \boxed{\dot{y} = a(r - r_b)y + by^3}$ Scale $\boxed{y = ku}$

$\Rightarrow \boxed{k\dot{u} = a(r - r_b)ku + bk^3u^3}$ $\boxed{R = a(r - r_b)}$

$\Rightarrow \boxed{\dot{u} = a(r - r_b)u + bk^2u^3}$ Define ↑

Choose $k^2 = \frac{1}{|b|}$ (only the absolute value of b).

$\Rightarrow \boxed{\dot{u} = Ru \pm u^3}$ Normal form of the pitchfork bifurcation.

i.) If the sign of b is positive $(+)$, the bifurcation is subcritical.

ii.) If the sign of b is negative $(-)$, the bifurcation is supercritical.

Bifurcation occurs for $\boxed{\dot{u} = 0}$ at $\boxed{u = 0}$ (or $x = 0$), when $\boxed{R = 0}$ i.e. $\boxed{r = r_b}$.

NOTE: In all the cases of bifurcation, retain only the specifically relevant terms in the Taylor expansion about the bifurcation point. This will give the appropriate normal form.