Finite Element Method (3D)

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1 Introduction

The force in every node of an element in the Finite Element formulation can be calculated using Equation 1.

$$F_i^\alpha = -\frac{1}{2}c_o A \left[\frac{\partial I_1}{\partial x_i^\alpha} - \frac{2}{J}\left(1 - \frac{K}{c_o}\log J\right)\frac{\partial J}{\partial x_i^\alpha}\right] \tag{1}$$

2 Shape Function Coefficients

The Shape function is given by Equation 2.

$$L_\alpha(\vec{X}) = \frac{V^\alpha(\vec{X})}{V(\vec{X})} \tag{2}$$

Where V is the volume of the tetrahedral. It can further simplified into Equation 3.

$$L_\alpha(\vec{X}) = a^\alpha + b_i^\alpha X_i \tag{3}$$

The coefficients a^α and b_i^α can be calculated by calculating the co-factors of the Matrix 4.

$$\begin{bmatrix} 1 & x^1 & y^1 & z^1 \\ 1 & x^2 & y^2 & z^2 \\ 1 & x^3 & y^3 & z^3 \\ 1 & x^4 & y^4 & z^4 \end{bmatrix} \tag{4}$$

where the co-factors divided by the (6 times) volume of each element correspond to the coefficients (Equation 3) as given in the Matrix 5.

$$\begin{bmatrix} a^1 & b_1^1 & b_2^1 & b_3^1 \\ a^2 & b_1^2 & b_2^2 & b_3^2 \\ a^3 & b_1^3 & b_2^3 & b_3^3 \\ a^4 & b_1^4 & b_2^4 & b_3^4 \end{bmatrix} \tag{5}$$

3 Deformation Gradient Tensor

The deformation gradient tensor is calculated using Equation 6.

$$F_{ij}(t) = \frac{\partial x_i}{\partial X_j} = \sum_{\alpha=1}^{4} b_j^\alpha x_i^\alpha(t) \tag{6}$$

This equation is implemented in MATLAB in form of matrix multiplication as shown in Equation 7. This operation is carried out for each finite element.

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} x^1 & x^2 & x^3 & x^4 \\ y^1 & y^2 & y^3 & y^4 \\ z^1 & z^2 & z^3 & z^4 \end{bmatrix} \begin{bmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \\ b_1^4 & b_2^4 & b_3^4 \end{bmatrix} \tag{7}$$

4 Derivative of the Jacobian $\partial J/\partial x_i^\alpha$

The derivative of the Jacobian is calculated as follows.

$$
\begin{aligned}
\frac{\partial J}{\partial x_m^\beta} &= \epsilon_{ijk} \frac{\partial}{\partial x_m^\beta} \left(F_{1i} F_{2j} F_{3k} \right) \\
&= \epsilon_{ijk} \left[F_{2j} F_{3k} \frac{\partial F_{1i}}{\partial x_m^\beta} + F_{1i} F_{2j} \frac{\partial F_{3k}}{\partial x_m^\beta} + F_{1i} F_{3k} \frac{\partial F_{2j}}{\partial x_m^\beta} \right]
\end{aligned}
$$

We have the following relation

$$
\frac{\partial F_{ij}}{\partial x_m^\beta} = \sum_{\alpha=1}^{3} \delta_{\alpha\beta} \delta_{im} b_j^\alpha
$$

Using the above relation we can simplify further.

$$
\begin{aligned}
\frac{\partial J}{\partial x_m^\beta} &= \epsilon_{ijk} \left[F_{2j} F_{3k} \sum_{\alpha=1}^{3} \delta_{\alpha\beta} \delta_{1m} b_i^\alpha + F_{1i} F_{2j} \sum_{\alpha=1}^{3} \delta_{\alpha\beta} \delta_{3m} b_k^\alpha + F_{1i} F_{3k} \sum_{\alpha=1}^{3} \delta_{\alpha\beta} \delta_{2m} b_j^\alpha \right] \\
&= \epsilon_{ijk} \left[F_{2j} F_{3k} \delta_{1m} b_i^\beta + F_{1i} F_{2j} \delta_{3m} b_k^\beta + F_{1i} F_{3k} \delta_{2m} b_j^\beta \right]
\end{aligned}
$$

For three dimensions we can write,

$$
\frac{\partial J}{\partial x_1^\beta} = \epsilon_{ijk} F_{2j} F_{3k} b_i^\beta \tag{8}
$$

$$
\frac{\partial J}{\partial x_2^\beta} = \epsilon_{ijk} F_{1i} F_{3k} b_j^\beta \tag{9}
$$

$$
\frac{\partial J}{\partial x_3^\beta} = \epsilon_{ijk} F_{1i} F_{2k} b_k^\beta \tag{10}
$$

This calculation can be performed in MATLAB using matrix multiplication as shown below. First we find the cofactor matrix, C of the deformation gradient tensor, F.

$$
\begin{bmatrix}
\partial J/\partial x_1^1 & \partial J/\partial x_1^2 & \partial J/\partial x_1^3 & \partial J/\partial x_1^4 \\
\partial J/\partial x_2^1 & \partial J/\partial x_2^2 & \partial J/\partial x_2^3 & \partial J/\partial x_2^4 \\
\partial J/\partial x_3^1 & \partial J/\partial x_3^2 & \partial J/\partial x_3^3 & \partial J/\partial x_3^4
\end{bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\begin{bmatrix}
b_1^1 & b_1^2 & b_1^3 & b_1^4 \\
b_2^1 & b_2^2 & b_2^3 & b_2^4 \\
b_3^1 & b_3^2 & b_3^3 & b_3^4
\end{bmatrix}
\tag{11}
$$