Problem Set 1.1

18.01 OCW Discord Due : October 2^{nd}, 2021
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1. Continuity of Functions and Limits

1.(a) $\frac{x}{x^2+1}$

* Answer : The function is continuous everywhere for $x \in (-\infty, \infty)$, because the denominator $x^2+1 \neq 0$ for all x.

1.(b) $\frac{1}{\sqrt{x}}$

* Answer : The function is continuous for $x \in (0, \infty)$. $x=0$ doesn't work due to division by zero, and $x<0$ is undefined for $\sqrt{x}$.

2.(a) $\lim\limits_{x \to 0} \frac{\sin x}{3\sqrt{x}}$

Note: The limit doesn't exist from LHS because $\frac{1}{\sqrt{x}}$ is undefined. Below I will find limit for RHS, i.e. $\lim\limits_{x \to 0^+} \frac{\sin x}{3\sqrt{x}}$.

$$\lim_{x \to 0^+} \frac{\sin(x)}{3\sqrt{x}} = \lim_{x \to 0^+} 3x^{-1/2} \cdot \lim_{x \to 0^+} \sin(x) = 0 \cdot 0 = 0$$

2.(b) $\lim\limits_{x \to 0} \frac{3x^2 + \sin x}{x}$

$$= \lim_{x \to 0} \frac{3x^2}{x} + \lim_{x \to 0} \frac{\sin x}{x} = \underbrace{\lim_{x \to 0} 3x}_{\text{Easy limit}} + \underbrace{\lim_{x \to 0} \frac{\sin x}{x}}_{\text{Learned from class}} = 3 \cdot (0) + 1 = 1$$

2. Derivatives

3. Show $y = 6x^2 + 5x - 3$ has no tangent line w/ slope 4.

$$\frac{dy}{dx} = 12x + 5 = 4 \qquad x = -\frac{1}{12}$$

* Update : $y = 6x^3 + 5x - 3$

$$\frac{dy}{dx} = 18x^2 + 5 = 4 \qquad x^2 = \frac{-1}{18}$$

Since there is no x s.t. $x^2 = \frac{-1}{18}$, there is no tangent line w/ slope 4 for $y = 6x^3 + 5x - 3$

2. Derivatives

4. (a) $f(t) = \sqrt{t} - \frac{1}{\sqrt{t}}$

Method #1: $f(t) = t^{1/2} - t^{-1/2}$

$$\frac{df}{dt} = \frac{1}{2}t^{-1/2} + \frac{1}{2}t^{-3/2} = \frac{1}{2}\left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t^3}}\right)$$

Method #2: $f(t) = \sqrt{t}\left(1 - \frac{1}{t}\right)$

Product Rule: $u = t^{1/2}$ $\quad u' = \frac{1}{2}t^{-1/2}$ $\quad (uv)' = u'v + uv'$

$v = 1 - t^{-1}$ $\quad v' = t^{-2}$

$$\frac{df}{dt} = \left(\frac{1}{2}t^{-1/2}\right)(1 - t^{-1}) + t^{1/2}(t^{-2})$$

$$= \frac{1}{2}t^{-1/2} - \frac{1}{2}t^{-3/2} + t^{-3/2} = \frac{1}{2}t^{-1/2} + \frac{1}{2}t^{-3/2} = \frac{1}{2}\left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t^3}}\right)$$

Method #3: Using Chain Rule $\quad u = \sqrt{t}$

$f(u) = u - \frac{1}{u}$

$$\frac{df}{dt} = \frac{df}{du} \cdot \frac{du}{dt} = (1 + u^{-2}) \cdot \frac{1}{2}t^{-1/2}$$

$$= (1 + t^{-1})\left(\frac{1}{2}t^{-1/2}\right)$$

$$= \frac{1}{2}t^{-1/2} + \frac{1}{2}t^{-3/2} = \frac{1}{2}\left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{t^3}}\right)$$

4. (b) $y = \dfrac{x^2 - 2\sqrt{x}}{x}$

Method 1 : Quotient Rule

$u = x^2 - 2\sqrt{x}$ $\qquad du = 2x - x^{-1/2}$

$v = x$ $\qquad dv = 1$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{(2x - x^{-1/2})(x) - (x^2 - 2\sqrt{x})}{x^2}$$

$$= \frac{2x^2 - \sqrt{x} - x^2 + 2\sqrt{x}}{x^2} = \frac{x^2 + \sqrt{x}}{x^2} = 1 + \frac{\sqrt{x}}{x^2} = 1 + x^{-3/2}$$

Method 2 : Simplify first.

$$y = \frac{x^2 - 2\sqrt{x}}{x} = x - \frac{2}{\sqrt{x}} = x - 2x^{-1/2}$$

Method 2.1 : $\dfrac{dy}{dx} = 1 + x^{-3/2}$

Method 2.2 : Chain rule using $u = x^{-1/2}$

$y(u) = u^{-2} - 2u$ $\qquad \dfrac{dy}{du} = -2u^{-3} - 2$ $\qquad \dfrac{du}{dx} = \dfrac{-1}{2}x^{-3/2}$

$$\frac{dy}{dx} = \left(-2\underline{(x^{-1/2})^{-3}} - 2\right)\left(\underline{\frac{-1}{2}x^{-3/2}}\right)$$

$$= 1 + x^{-3/2}$$

4. (c) $y = \sqrt{x}(x-1)$

Method 1: Product Rule $\quad (uv)' = u'v + uv'$

$u = \sqrt{x} \qquad du = \frac{1}{2}x^{-1/2}$

$v = x-1 \qquad dv = 1$

$$\frac{dy}{dx} = \left(\frac{1}{2}x^{-1/2}\right)(x-1) + \sqrt{x} = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2} + x^{1/2} = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$$

Method 2: Simplify First

$$y = \sqrt{x}(x-1) = x^{3/2} - x^{1/2}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$$

Method 3: Chain Rule

$u = \sqrt{x} \qquad y(u) = u(u^2-1) = u^3 - u$

$\frac{dy}{du} = 3u^2 - 1 \qquad \frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$$\frac{dy}{dx} = (3x-1)\left(\frac{1}{2}x^{-1/2}\right) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$$