DAYANANDA SAGAR COLLEGE OF ENGINEERING
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DEPARTMENT OF MATHEMATICS

Course Material

COURSE	OPTIMIZATION AND TRANSFORMS
COURSE CODE	21MAT31C
MODULE	2
MODULE NAME	NUMERICAL METHODS FOR ODE & PDE
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Objectives:

At the end of this Module, student will be able to analyze

- Numerical solution of Ordinary Differential Equations by Taylor's series method
- Numerical solution of Ordinary Differential Equations Runge-Kutta method of fourth order.
- Numerical solution of Partial Differential Equations: Finite difference approximations to derivatives.
- Numerical solution of one-dimensional heat equation by Schmidt method
- And by Crank-Nicholson Method,
- Numerical solution of one-dimensional wave equation.
- Application Problems

Numerical solution of Ordinary Differential Equations

Introduction:

We have studied various analytical methods for finding the solution of the equation of the form f(x, y(x), y^1(x))= 0, there exist large number of ODE's whose solution cannot be obtained by the known analytical methods. But Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems leads to differential equations. So, it is important for engineers and scientists to know how to set up differential equations and solve them. In such cases, we use numerical methods to get an approximate solution of a given differential equation under the prescribed conditions.

Consider the following differential equation $\frac{dy}{dx}$ = f(x, y) with initial condition x = x_0, y = y_0 is called **Initial value problem** and it can be solved numerically by these following methods

Single step Methods:
1) Taylor's series method
2) Runge-Kutta method of fourth order (RK4)

Taylor's series method:

Consider the one dimensional initial value problem

$$\frac{dy}{dx} = \mathbf{f(x, y)}, \quad \mathbf{y(x_0) = y_0}$$

where **f** is a function of two variables **x** and **y** and **(x_0 , y_0)** is a known point on the solution curve.

If the existence of all higher order partial derivatives is assumed for **y** at $\mathbf{x = x_0}$, then by Taylor series the value of **y** can be written as

$$\mathbf{y(x) = y(x_0) + h\, y'(x_0) + h^2/2\, y''(x_0) + h^3/3!\, y'''(x_0) + \; . \; . \; . \; . \; . \; .}$$

Where $h = (x-x_0)$, Since at $\mathbf{x_0}$, $\mathbf{y_0}$ is known, **y'** at $\mathbf{x_0}$ can be found by computing $\mathbf{f(x_0,y_0)}$.

Similarly higher derivatives of **y** at $\mathbf{x_0}$ also can be computed by making use of the relation $\frac{dy}{dx}$ **= f(x, y)**

Problems:-

1) Using Taylor series method, find y(0.1) for $\frac{dy}{dx} = x - y^2$, y(0) = 1 correct upto four decimal places.

Ans: Since at $x_0 = 0, y_0 = 1$;

Given $\frac{dy}{dx} = y' = f(x, y) = \mathbf{x - y^2}$

$y' = x - y^2$ $\qquad y_0' = x_0 - y_0^2 = 0 - 1^2 = -1$,

$y'' = 1 - 2yy'$ $\qquad y''_0 = 1 - 2y_0y_0' = 1 - 2(1)(-1) = 3$

$y''' = -2yy'' - 2y'^2$ $\qquad y'''_0 = -2y_0y''_0 - 2y_0'^2 = -2(1)(3) - 2(-1)^2 = -8$

$y^{iv} = -2yy''' - 6y'y''$

$y^{iv}_0 = -2y_0y'''_0 - 6y_0'y''_0 = -2(1)(-8) - 6(-1)(3) = 34$

Similarly $\quad y^{v} = -2yy^{iv} - 8y'y''' - 6y''^2$ $\quad y^{v}_0 = -186$

The forth order Taylor's formula is

$y(x) = y(x_0) + (x-x_0)\, y'(x_0,) + (x-x_0)^2\, y''(x_0)/2! + (x-x_0)^3\, y'''(x_0)/3! + (x-x_0)^4\, y^{iv}(x_0)/4! +$

$(x-x_0)^5\, y^{v}(x_0)/5! + . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$

$= 1 - x + 3\, x^2/2! - 8\, x^3/3! + 34\, x^4/4! - 186\, x^5/5!$ $\qquad$ (since $x_0 = 0$)

$= 1 - x + 3\, x^2/2 - 4\, x^3/3 + 17\, x^4/12 - 31\, x^5/20$

$y(0.1) = 1 - (0.1) + 3\,(0.1)^2/2 - 4\,(0.1)^3/3 + 17\,(0.1)^4/12 - 31\,(0.1)^5/20 = 0.9138$

2) Solve the initial value problem $y' = -2xy^2$, $y(0) = 1$ for y at x = 0.2 with step length 0.2 using Taylor series method of order four.

Ans:- Given $\frac{dy}{dx} = y' = f(x,y) = -2xy^2$,

$y' = f(x,y) = -2xy^2$ $\qquad y'_0 = -2(0)(1)^2 = 0.0$

$y'' = -2y^2 - 4xyy'$ $\qquad y''_0 = -2(1)^2 - 4(0)(1)(0) = -2$

$y''' = -8yy' - 4xy'^2 - 4xyy''$ $\qquad y'''_0 = -8(1)(0) - 4(0)(0)^2 - 4(0)(1)(-2) = 0.0$

$y^{iv} = -12y'^2 - 12yy'' - 12xy'y'' - 4xyy'''$

$y^{iv} = -12(0)^2 - 12(1)(-2) - 12(0)(0)(-2) - 4(0)(1)(0) = 24$

The 4^{th} order Taylor's formula is

$y(x_i+h) = y(x_0) + h\,y'(x_0) + h^2\,y''(x_0)/2! + h^3\,y'''(x_0)/3! + h^4\,y^{iv}(x_0)/4! + \ldots$

given at x=0, y=1 and with h = 0.2 we have

Where $h=x-x_0$

$y(0.2) = 1 + 0.2\,(0) + 0.2^2\,(-2)/2! + 0 + 0.2^4\,(24)/4! = 0.9615$

now at x = 0.2 we have y = 0.9615

Runge-Kutta method of fourth order:

Consider the initial value problem $\frac{dy}{dx}$ =y' = f(x,y) , $y(x_0)=y_0$

We need to find $y(x_0+h)$ where h is the step size

We have to compute k_1, k_2, k_3, k_4 by the following formulae

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)$

Where

$k_1 = h\, f(x_n, y_n)$

$k_2 = h\, f(x_n + h/2, y_n + k_1/2)$

$k_3 = h\, f(x_n + h/2, y_n + k_2/2)$

$k_4 = h\, f(x_n + h, y_n + k_3)$

Problems:-

1) Given the IVP $\frac{dy}{dx} = x+y^2$, y(0)=1 estimate the value of y(0.2) with h=0.1 by Runge-kutta 4th order method

Ans:- $\frac{dy}{dx} = f(x,y)=x+y^2$ $x_0=0$,$y_0=1$ h=0.1

Step 1: n=0, $x_0=0$,$y_0=1$

$k_1 = h\, f(x_n, y_n)=0.1(x_0+y_0^2) = 0.1$

$k_2 = h\, f(x_n + h/2, y_n + k_1/2) = 0.1152$

$k_3 = h\, f(x_n + h/2, y_n + k_2/2) = 0.1168$

$k_4 = h\, f(x_n + h, y_n + k_3) = 0.1347$

$y_1 = y_0 + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)=1.1164$

step 2 n=1, $x_1 = 0.1$, $y_1 = 1.1164$

$k_1 = h\ f(x_n, y_n)=0.1(x_1+y_1^2) = 0.1346$

$k_2 = h\ f(x_n + h/2, y_n + k_1/2) = 0.1551$

$k_3 = h\ f(x_n + h/2, y_n + k_2/2) = 0.1575$

$k_4 = h\ f(x_n + h, y_n + k_3) = 0.1823$

$y_2 = y_1 + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)=1.2735$

$y(0.2)=1.2735$

2) Given the IVP $\frac{dy}{dx} = 1+y^2$, y(0)=0 estimate the value of y(0.6) with h=0.2 by Runge-kutta 4th order method

Ans:- $\frac{dy}{dx} = f(x,y)=1+y^2$ $x_0=0$,$y_0=0$, h=0.2

Step 1 n=0, $x_0=0$, $y_0=0$

$k_1 = h\ f(x_n, y_n)=0.2(1+y_0^2) = 0.2$

$k_2 = h\ f(x_n + h/2, y_n + k_1/2) = 0.202$

$k_3 = h\ f(x_n + h/2, y_n + k_2/2) = 0.2020$

$k_4 = h\ f(x_n + h, y_n + k_3) = 0.2081$

$y_1 = y_0 + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)=0.2027$

step 2 n=1, $x_1 = 0.2$, $y_1 = 0.2027$

$k_1 = h\ f(x_n, y_n)=0.2(1+y_1^2) = 0.2082$

$k_2 = h\, f(x_n + h/2, y_n + k_1/2) = 0.2188$

$k_3 = h\, f(x_n + h/2, y_n + k_2/2) = 0.2194$

$k_4 = h\, f(x_n + h, y_n + k_3) = 0.2356$

$y_2 = y_1 + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)=0.4227$

step 3 n=3, $x_1 = 0.4$, $y_1 = 0.4227$

$k_1 = h\, f(x_n, y_n)=0.2(1+y_1^2) = 0.2357$

$k_2 = h\, f(x_n + h/2, y_n + k_1/2) = 0.2584$

$k_3 = h\, f(x_n + h/2, y_n + k_2/2) = 0.2609$

$k_4 = h\, f(x_n + h, y_n + k_3) = 0.2934$

$y_2 = y_1 + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)=0.6841$

$y(0.6)=0.6841$

Numerical solution of Partial Differential Equations

Introduction:

Partial differential equations arise in the study of many branches of applied mathematics, e.g. in fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electromagnetic theory.

Only a few of these equations can be solved by analytical methods which are also complicated by requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. Of all the numerical methods available for the solution of partial differential equations, the method of finite differences is most commonly used.

In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equations are changes to a system of linear equations which are solved by iterative procedures. This process is slow but produces good results in many boundary value problems.

An added advantage of this method is that the computation can be carried by electronic computers. To accelerate the solution, some times the method of relaxation proves quite effective.

The general linear partial differential equation of the second order in two independent variables is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \qquad (1)$$

Such a partial differential equation is said to be

I. Elliptic if $B^2 - 4AC < 0$
II. Parabolic if $B^2 - 4AC = 0$
III. Hyperbolic if $B^2 - 4AC > 0$

Note: A Partial equation is classified according to the region in which it is desired to be solved. For instance, the partial differential equation $f_{xx} + f_{yy} = 0$ is elliptic if $y > 0$, parabolic if $y = 0$ and hyperbolic if $y < 0$.

For example: Classify the following equation

1. $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$
   Here, $B^2 - 4AC = 0$, Where $A = 1, B = 4, C = 4$. So the equation is parabolic.

2. $x^2\frac{\partial^2 u}{\partial x^2} + (1 - y^2)\frac{\partial^2 u}{\partial y^2} = 0$, $-\infty < x < \infty, -1 < y < 1$.

Since $A = x^2, B = 0, C = 1 - y^2, , B^2 - 4AC = 4x^2(y^2 - 1)$

For all x, x^2 is positive and for all y between -1 and 1, $y^2 < 1$. Therefore, $B^2 - 4AC < 0$. Hence the equation is elliptic.

3. $(1 + x^2)\frac{\partial^2 u}{\partial x^2} + (5 + 2x^2)\frac{\partial^2 u}{\partial x \partial t} + (4 + x^2)\frac{\partial^2 u}{\partial t^2} = 0$
   Since , $B^2 - 4AC = 9 > 0$, the equation is hyperbolic.

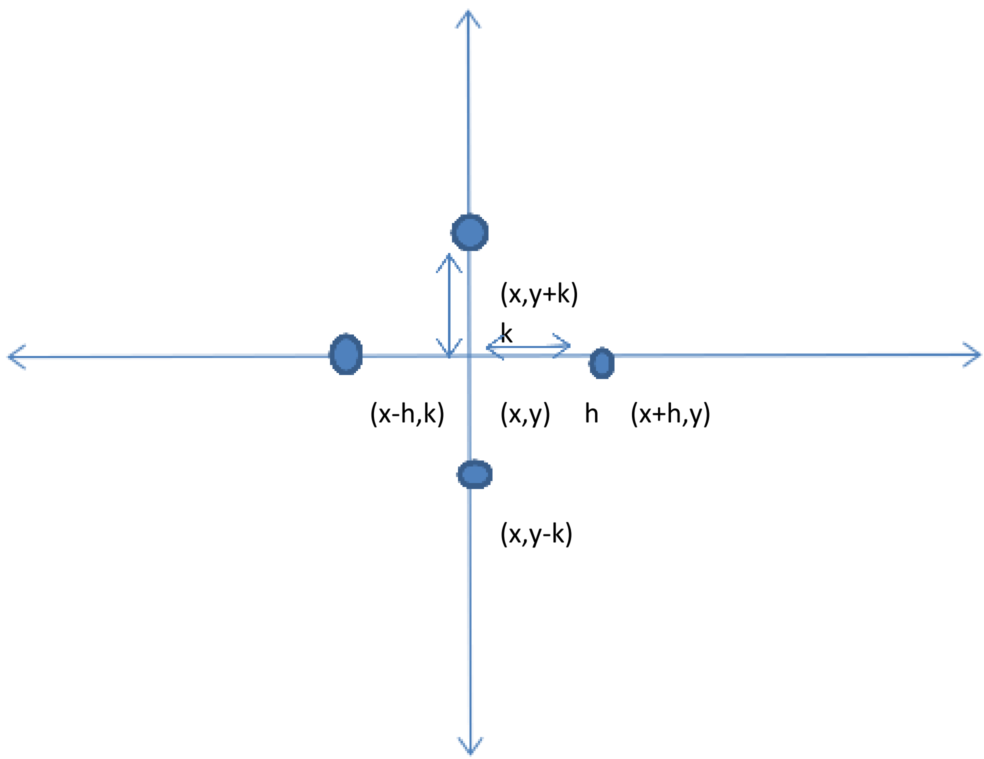
Finite difference approximation to partial derivative:

If you recall that, every derivation can be replaced by algebraic equations, and that algebraic equations are called finite difference approximation.

Every derivative, either it is y' or y'', we obtain the algebraic form for these derivative substituted back in the ordinary differential equation, then solve it is reduce to a system of linear equation.

Also, we can solve a PDE with boundary value problem using finite difference approximation. But PDE contains partial derivatives, so you would have more than one dependent variable.
So, what are these algebraic expressions of finite difference approximation that we have to define for partial derivative.
Let $u = u(x, y)$, possible partial derivatives up to the second order are,
First order: $u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}$
Second order: $u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, u_{yy} = \frac{\partial^2 u}{\partial y^2}$

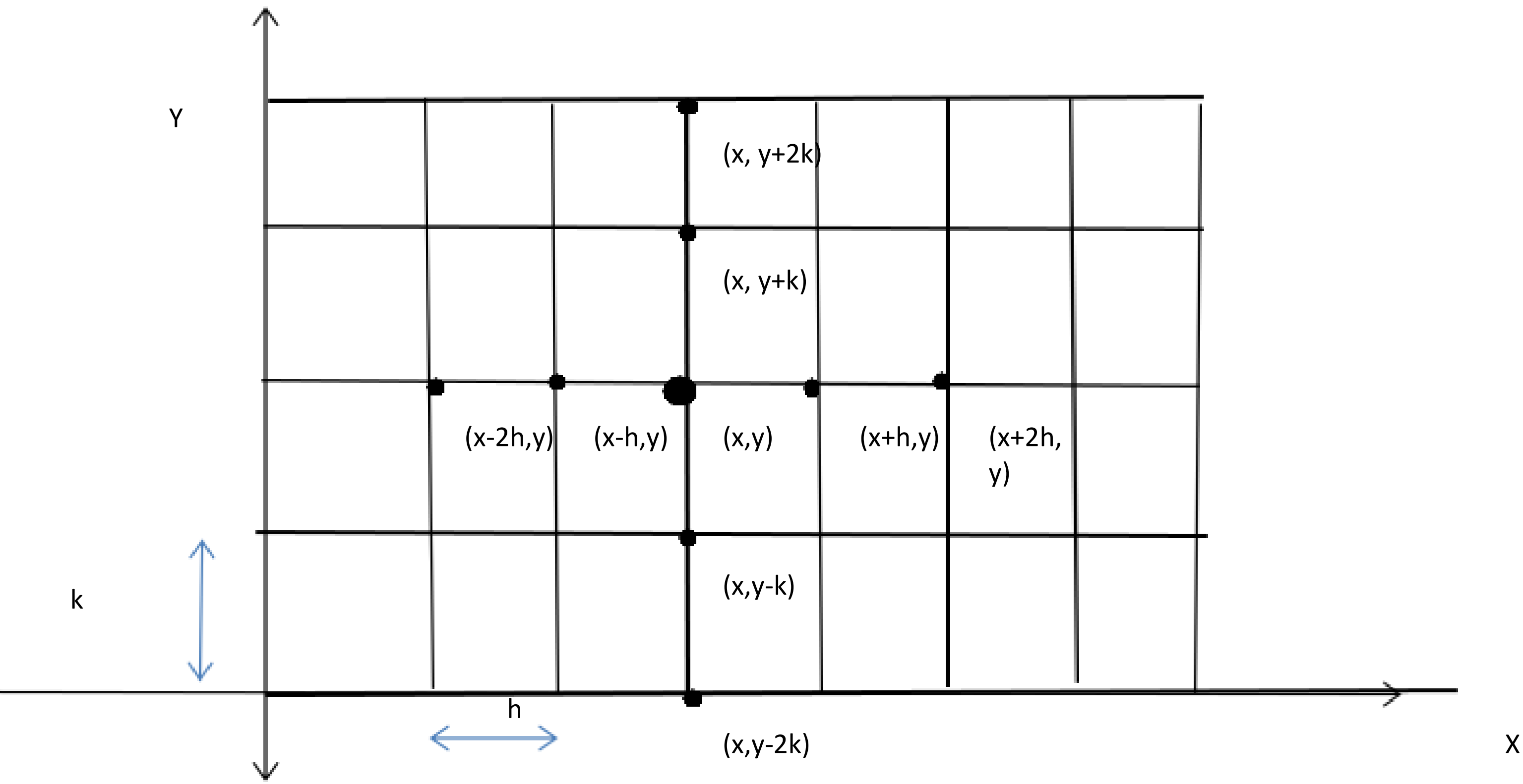
Let h and k be the step sizes in the x and y directions respectively, we will discuss about the co-ordinate.



If the increment in the x-direction is h, the next co-ordinate will be the $(x + h, y)$, and the y-co-ordinate is remain same. Similarly the increment in the y-direction is k, the next co-ordinate will be the $(x, y + k)$ and the x −co-ordinate is remain same.

Directional Representation of co-ordinate:

Here we will extend the co-ordinates to a wider rectangular region.



Co-ordinate in Finite difference form:-

We already know in ODE $u(x)$ is written as u_i. In two dimensional case, how to write the co-ordinate in finite difference form, Let $u = u(x, y)$.

To write in the finite difference form replace x by ih, or y by jk, then $u = u(x, y) = u(ih, jk) = u_{i,j}$.

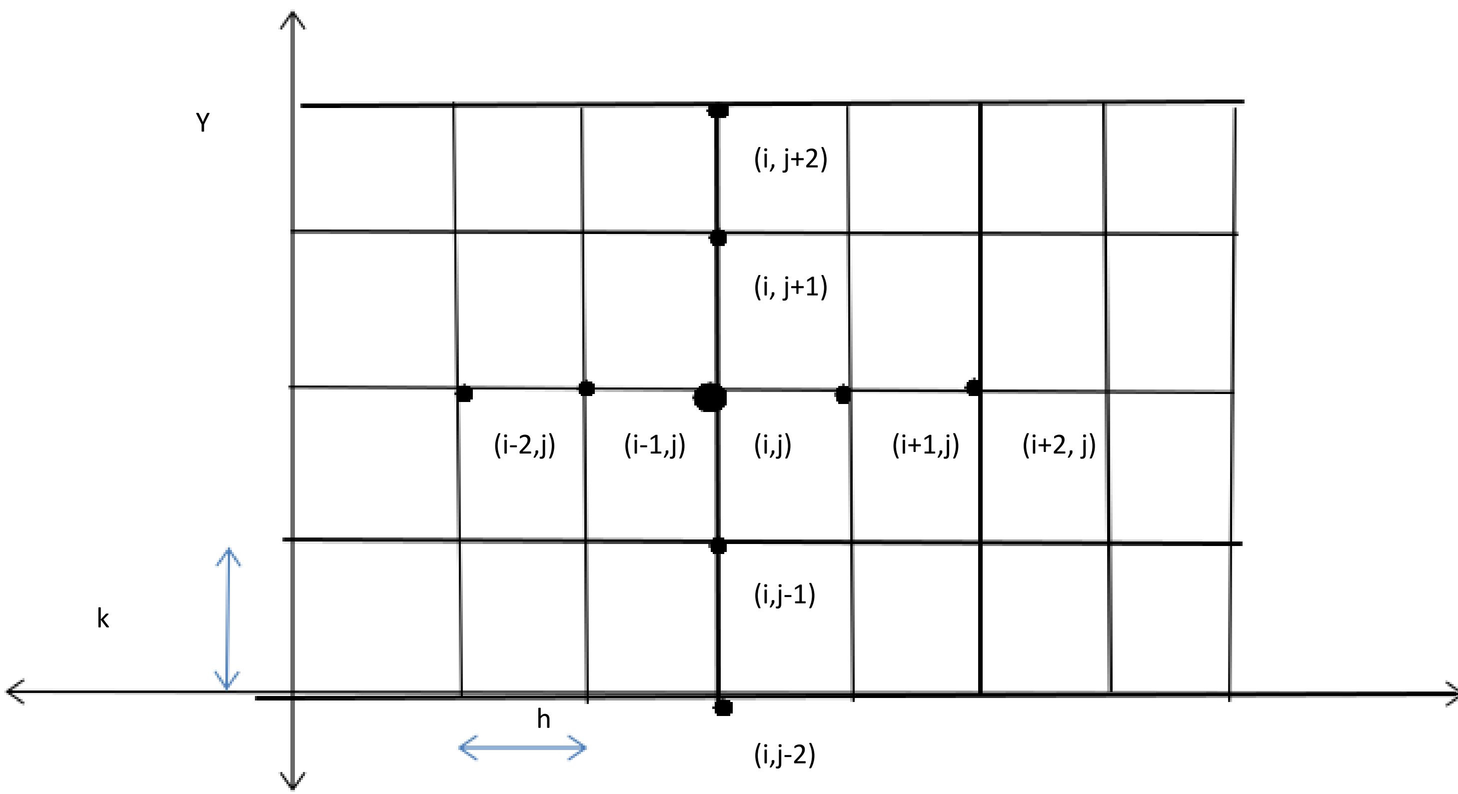
So, $u(x, y) = u_{i,j}$.

Similarly, $u(x+h,y) = u(ih+h,jk) = u\big((i+1)h,jk\big) = u_{i+1,j}$ which is the finite difference form of $u(x+h,y)$.

Hence, $u(x-h,y) = u(ih-h,jk) = u\big((i-1)h,jk\big) = u_{i-1,j}$

$$u(x-h,y-k) = u(ih-h,jk-k) = u\big((i-1)h,(j-1)k\big) = u_{i-1,j-1}$$

<u>Representation of co-ordinates in finite difference form</u>:-



It is easier to handle this type of co-ordinate system rather than having argument with us.

So, let us continue now, to understand how to rewrite the first and second order derivative in finite difference form, that we have exactly study in ODE.

Finite Difference Approximations of partial derivative:-

First we will revised, what we derived in the earlier, for the first order derivative

$$y'(x) = \begin{cases} \dfrac{y(x+h) - y(x)}{h} & (Forward\ approximation) \\ \dfrac{y(x) - y(x-h)}{h} & (backward\ approximation) \\ \dfrac{y(x+h) - y(x-h)}{2h} & (centra\ approximation) \end{cases}$$

Same expression here we will use to find the expression for partial derivative.

$$u_x = \frac{\partial u}{\partial x} = \begin{cases} \dfrac{u_{i+1,j} - u_{i,j}}{h} & (forward) \\ \dfrac{u_i - u_{i-1,j}}{h} & (backward) \\ \dfrac{u_{i+1,j} - u_{i-1,j}}{2h} & (central) \end{cases}$$

$$u_y = \frac{\partial u}{\partial y} = \begin{cases} \dfrac{u_{i,j+1} - u_{i,j}}{k} & (forward) \\ \dfrac{u_i - u_{i,j-1}}{k} & (backward) \\ \dfrac{u_{i,j+1} - u_{i,j-1}}{2k} & (central) \end{cases}$$

Let us now we derive the central difference approximation for the 2nd order derivative.

For ODE, $y''(x) = \frac{y(x+h)-2y(x)+y(x-h)}{h^2}$

Now, in partial derivative case

$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$ (central approximation)

$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}$ (central approximation)

<u>Solution of One dimensional heat equation</u>:-

The heat equation in one dimension is a typical parabolic partial differential equation and is a time variable problem.

If we consider a long thin insulated rod and equate the amount of heat absorbed to the difference between the amount of heat entering a small element and that leaving the element in time Δt, we obtain the partial differential equation

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

$C^2 = \frac{k}{s\rho}$, where k is coefficient of conductivity of the material,

ρ is its density. s is its specific heats.

Analytical solutions of equation (2), obtained by the method of separation of variables are given by

$$u(x,t) = e^{-\rho^2 c^2 t}(C_1 \cos \rho x + C_2 \sin \rho x)$$

$$\} \qquad (3)$$

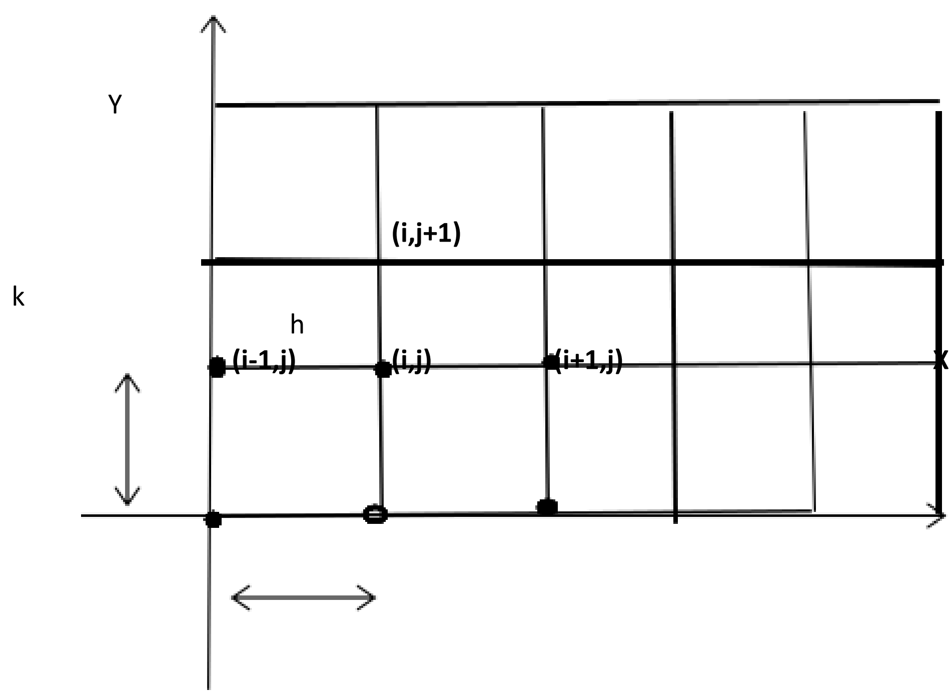
$$u(x,t) = e^{\rho^2 c^2 t}(C_3\, e^{\rho x} + C_4\, e^{-px}\,)$$

From equation (3), the appropriate form of solution should be chosen depending upon the boundary condition given. It is clear that to solve equation (2), we need one initial condition and two boundary conditions. Now we will discuss the finite difference approximations to this equations.

We can solve the 1-D heat conduction equation by two method

1) Explicit Method
2) Implicit Method

Consider a rectangular mesh in the $x - t$ plane with spacing h along x −direction and k along time t −direction.



Denoting a mesh point $(x, t) = (ih, jk)$ as simply (i, j), we have

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

$=> \frac{u_{i,j+1} - u_{i,j}}{k} = C^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$

$$=> u_{i,j+1} - u_{i,j} = \frac{kC^2}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

Take $\propto = \frac{kC^2}{h^2}$

$$=> u_{i,j+1} - u_{i,j} = [\propto u_{i+1,j} - 2 \propto u_{i,j} + \propto u_{i-1,j}]$$

$$=> u_{i,j+1} = [\propto u_{i+1,j} - 2 \propto u_{i,j} + \propto u_{i-1,j} + u_{i,j}]$$

$$=> u_{i,j+1} = \propto u_{i+1,j} + (1 - 2 \propto)u_{i,j} + \propto u_{i-1,j}$$

$$=> u_{i,j+1} = \propto u_{i-1,j} + (1 - 2 \propto)u_{i,j} + \propto u_{i+1,j} \qquad (4)$$

Where $\propto = \frac{kC^2}{h^2}$ is mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j + 1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels $j + 1$ and j and therefore, called a 2-level formula. This equation (4) is called the Schmidt explicit formula which is valid only for $0 < \propto \leq 1/2$. (which is called the stability condition for the explicit formula).

In particular, when $\propto = 1/2$, equation (4) reduces to

$$u_{i,j+1} = \frac{1}{2}u_{i-1,j} + \frac{1}{2}u_{i+1,j}$$

$=> u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$ (5)

Which shows that the value of u at x_i at time t_{j+1} is the mean of the u −values at x_{i-1} and x_{i+1} at time t_j. This relation is known as Bender-Schmidt recurrence relation, gives the values of u at the internal mesh points with the help of boundary condition.

Q.1 Solve the boundary value problem $\boldsymbol{u_t = u_{xx}}$ under the conditions $\boldsymbol{u(0,t) = u(1,t) = 0}$ and $\boldsymbol{u(x,0) = sin\pi x}$, $\boldsymbol{0 \leq x \leq 1}$ using Schmidt method. (Take h = 0.2 and $\propto = 1/2$).

Sol: $C^2 = 1, h = 0.2, \alpha = 1/2$

We know that $\propto = \frac{kC^2}{h^2} => k = 0.02$

Since $\alpha = 1/2$, we use Bender-Schmidt relation

$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$

We have $u(0,t) = u(1,t) = 0$ and $u(x,0) = sin\pi x, 0 \leq x \leq 1$

Since step size in x-direction is 0.2 so, the values of $x = 0, 0.2, 0.4, 0.6, 0.8, 1$.

$u(x,0) = sin\pi x$, here $t = 0$

$$u(0.2,0) = 0.5878, u(0.4,0) = 0.95111, u(0.6,0) = 0.95111$$

$u(0.8,0) = 0.5878$, $u(1,0) = \sin\pi = 0$

Other value can be calculated by using the formula of equation (5) as shown in the table below

$x \rightarrow$		0	0.2	0.4	0.6	0.8	1
$t \downarrow$	j \ i	0	1	2	3	4	5
0	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

Q2: Find the values of $\boldsymbol{u(x,t)}$ satisfying the parabolic equation $\boldsymbol{\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}}$ and the boundary conditions $\boldsymbol{u(0,t) = 0 = u(8,t)}$ and $\boldsymbol{u(x,0) = 4x - \frac{1}{2}x^2}$ at the points $\boldsymbol{x = i,\ \ i = 0,1,2,\dots,7}$ and $\boldsymbol{t = \frac{1}{8}j:\ \ j = 0,1,2,3,4,5.}$

Sol: Here $C^2 = 4, h = 1\ , k = \frac{1}{8}$. Then $\propto = \frac{kC^2}{h^2} = \frac{1}{2}$.

$\therefore$ we have Bender-Schmidt's recurrence relation

$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$ (1)

Now since, $u(0,t) = 0 = u(8,t)$

$\therefore u_{0,j} = 0, u_{8,j} = 0,$ When $x = 0$ and $8,$ for all values of $j,$ i.e. the entries in the first and last column are zero.

Since $(x,0) = 4x - \frac{1}{2}x^2$, where $t = 0.$

$\therefore$ $u_{i,0} = 4i - \frac{1}{2}i^2,$ $i = 0,1,2,3,4,5,6,7$

$= 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$ at $t = 0.$

These are the entries of the first row. Putting $j = 0$ in the equation (1), we have $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

Taking $i = 1,2,3,4,5,6,7$ successively, we get

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(6 + 8) = 7$$

Similarly, $u_{4,1} = 7.5,$ $u_{5,1} = 7,$ $u_{6,1} = 5.5,$ $u_{7,1} = 3.$ These are the entries in the second row.

Putting $j = 1$ in the equation (1), the entries of the third row are given by $u_{i,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1})$

Similarly putting $j = 2,3,4$ successively in equation (1), the entries of the fourth, fifth, and sixth rows are obtained. Hence the values of $u_{i,j}$ are as given in the following table.

x →		0	1	2	3	4	5	6	7	8
t ↓	i \ j	0	1	2	3	4	5	6	7	8
0	0	0	3.5	6	7.5	8	7.5	6	3.5	0
1/8	1	0	3	5.5	7	7.5	7	5.5	3	0
2/8	2	0	2.75	5	6.5	7	6.5	5	2.75	0
3/8	3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4/8	4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5/8	5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

Q3: Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, subject to the conditions $(x, 0) = sin\pi x$, $0 \le x \le 1$; $u(0, t) = u(1, t) = 0$. Carry out computations for two levels taking $= \frac{1}{3}$, $k = \frac{1}{36}$.

Sol: Here $C^2 = 1, h = \frac{1}{3}, k = \frac{1}{36}$ so that $\propto = \frac{kC^2}{h^2} = 1/4$.

Also $u_{1,0} = \sin \pi/3 = \frac{\sqrt{3}}{2}, u_{2,0} = \sin 2\pi/3 = \frac{\sqrt{3}}{2}$ and all the boundary values are zero. Schmidt's formula,

$$u_{i,j+1} = \propto u_{i-1,j} + (1 - 2 \propto) u_{i,j} + \propto u_{i+1,j}$$

Becomes $u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right]$

For $i = 1,2; j = 0$:

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right] = \frac{1}{4}\left(0 + 2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = 0.65$$

$$u_{2,1} = \frac{1}{4}\left[u_{1,0} + 2u_{2,0} + u_{3,0}\right] = \frac{1}{4}\left(\frac{\sqrt{3}}{2} + 2 \times \frac{\sqrt{3}}{2} + 0\right) = 0.65$$

For $= 1,2;\ j = 1$:

$$u_{1,2} = \frac{1}{4}\left[u_{0,1} + 2u_{1,1} + u_{2,1}\right] = 0.49$$

$$u_{2,2} = \frac{1}{4}\left[u_{1,1} + 2u_{2,1} + u_{3,1}\right] = 0.49.$$

CRANK-NICOLSON METHOD:-

We have seen that the Schmidt scheme is computationally simple and for convergent results $\alpha \leq 1/2$.

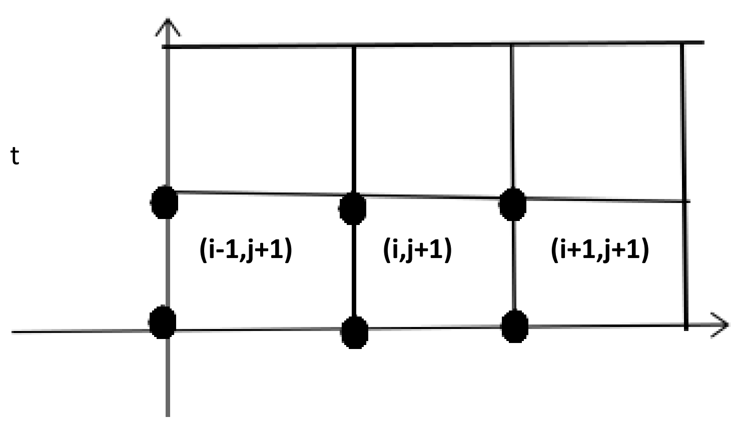
$$i.e \ \propto = \frac{kC^2}{h^2} \ \ => k = \frac{\propto h^2}{C^2} = \frac{h^2}{2C^2} \ (\propto \leq \frac{1}{2})$$

$$=> k \leq \frac{h^2}{2C^2}$$

To obtain more accurate results, h should be small $i.e\ k$ is necessary very small. This makes the computations exceptionally lengthy as more time levels would be required to cover the region.

So, we introduce another method that does not restrict $\propto$ and also reduces the volume of calculation was proposed by Crank and Nicolson in 1947.

According to this method, $\frac{\partial^2 u}{\partial x^2}$ is replaced by the average of its central difference approximation on the jth and $(j+1)$th time level.



(i-1,j) **(i,j)** **(i+1,j)** X

We obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right]$$

Now, the one dimensional heat equation,

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

$$\Rightarrow \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\mathrm{C}^2\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right] \qquad (2)$$

$$\Rightarrow u_{i,j+1} - u_{i,j} = \frac{1}{2\mathrm{h}^2}\mathrm{kC}^2\left[\{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\} + \{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\}\right]$$

Take $\propto = \frac{kC^2}{h^2}$

$$\Rightarrow 2u_{i,j+1} - 2u_{i,j} = \left[\{\propto u_{i-1,j} - 2\propto u_{i,j} + \propto u_{i+1,j}\} + \{\propto u_{i-1,j+1} - 2\propto u_{i,j+1} + \propto u_{i+1,j+1}\}\right]$$

Write the $(j+1)$th level term in left side,

$=> -\propto u_{i-1,j+1} + (2+2\propto)u_{i,j+1} - \propto u_{i+1,j+1} = \propto u_{i-1,j} + (2-2\propto)u_{i,j} + \propto u_{i+1,j}$ (3)

Clearly the left side of equation (3) contains three unknown values of u at the $(j+1)$th level while all the three values on the right are known values at the j^{th} level.

Thus equation (3) is a two level implicit relation and is known as Crank-Nicolson formula for the one dimensional heat equation and it is an implicit formula. It is convergent for all finite values of $\propto$.

If there are n −internal mesh points on each row, then the equation (3) gives n simultaneous equations for the n unknown values in terms of the known boundary values. These equations can be solved to obtain the values at these mesh points on all rows can be found. A method such as this in which the calculation of an unknown mesh value necessitates the solution of a set of simultaneous equations, is known as an implicit scheme.

Q4: Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $in\ 0 < x < 5,\ t \geq 0$ given that $u(x,0) = 20, u(0,t) = 0, u(5,t) = 100$. Compute u for the time-step with $h = 1$ by the Crank-Nicolson method.

Sol:

Here $C^2 = 1$ and $h = 1$.

Taking $\propto = 1.\ i.e. \frac{kC^2}{h^2} = 1 => k = 1$

Also we have

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Then Crank-Nicholson formula becomes

$$=> 4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$$

$\therefore\ 4u_1 = 0 + 20 + 0 + u_2 => 4u_1 - u_2 = 20$ (1)

$4u_2 = 20 + 20 + u_1 + u_3 => u_1 - 4u_2 + u_3 = -40$ (2)

$4u_3 = 20 + 20 + u_2 + u_4 => u_2 - 4u_3 + u_4 = -40$ (3)

$4u_4 = 20 + 100 + u_3 + 100 => u_3 - 4u_4 = -220$ (4)

Now (1)-4(2) gives $15u_2 - 4u_3 = 180$ (5)

4(3)+(4) gives $4u_2 - 15u_3 = -380$ (6)

Then 15(5)-4(6) gives $209\, u_2 = 4220 => u_2 = 20.2$

From equation (5), we get $u_3 = 30.75$

From (1), $u_1 = 10.05$, From (4), $u_4 = 62.69$

Thus the required values are 10.05, 20.2, 30.75 and 62.69.

Benefit of Crank-Nicolson's Method:-

Basic difference between the Explicit & Implicit method is in explicit method, previous time value are given where we find the next time value. In case of implicit method, value at $(j+1)$th level can found using the previous and next value of $(j+1)$th level and jth level, at a time.

Numerical solution of one-dimensional wave equation:

The wave equation $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. It solution $u(x,t)$ is the displacement function $u(x,t)$ defined for values of x from 0 to l and for t from 0 to ∞, satisfying the initial and boundary conditions.

The solution as for parabolic equations, advances in an open ended region. In the case of hyperbolic equations however, we have two initial conditions and two boundary conditions.

Such equations arises from convective type of problems in vibrations, wave mechanics and gas dynamics.

Solution of wave equation:-

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Subject to the initial conditions:

$$u = f(x), \frac{\partial u}{\partial t} = g(x), 0 \leq x \leq 1 \quad at\ t = 0 \qquad (2)$$

And the boundary conditions:

$$u(0,t) = \emptyset(t), \quad u(1,t) = \varphi(t) \tag{3}$$

Consider a rectangular mesh in the $x - t$ plane spacing h along $x -$ direction and k along time direction. Denoting a mesh point $(x,t) = (ih, jk)$ as simply (i,j) , we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Now replacing the derivatives in equation (1) by their above approximations, we obtain

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = C^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$=> u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{k^2 C^2}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

Take $\propto = \frac{k}{h}$

$$=> u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \propto^2 C^2[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

$$=> u_{i,j-1} - 2u_{i,j} + u_{i,j+1}$$
$$= \propto^2 C^2(u_{i-1,j}) - 2 \propto^2 C^2 u_{i,j} + \propto^2 C^2 u_{i+1,j}$$

$$=> u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \propto^2 C^2(u_{i-1,j}) - 2 \propto^2 C^2 u_{i,j} + \propto^2 C^2 u_{i+1,j}$$

$$=> u_{i,j+1} = 2(1 - \propto^2 C^2)u_{i,j} + \propto^2 C^2(u_{i-1,j} + u_{1+1,j}) - u_{i,j-1} \quad (4)$$

Now replacing the derivative in equation (2), by its central difference approximation, we get

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{\partial u}{\partial t} = g(x)$$

$$=> u_{i,j+1} - u_{i,j-1} = 2k\ g(x)$$

$$=> u_{i,j+1} = u_{i,j-1} + 2k\ g(x) \quad at\ t = 0 \quad (5)$$

$$=> u_{i,\ 1} = u_{i,\ -1} + 2k\ g(x) \quad for\ j = 0 \quad (6)$$

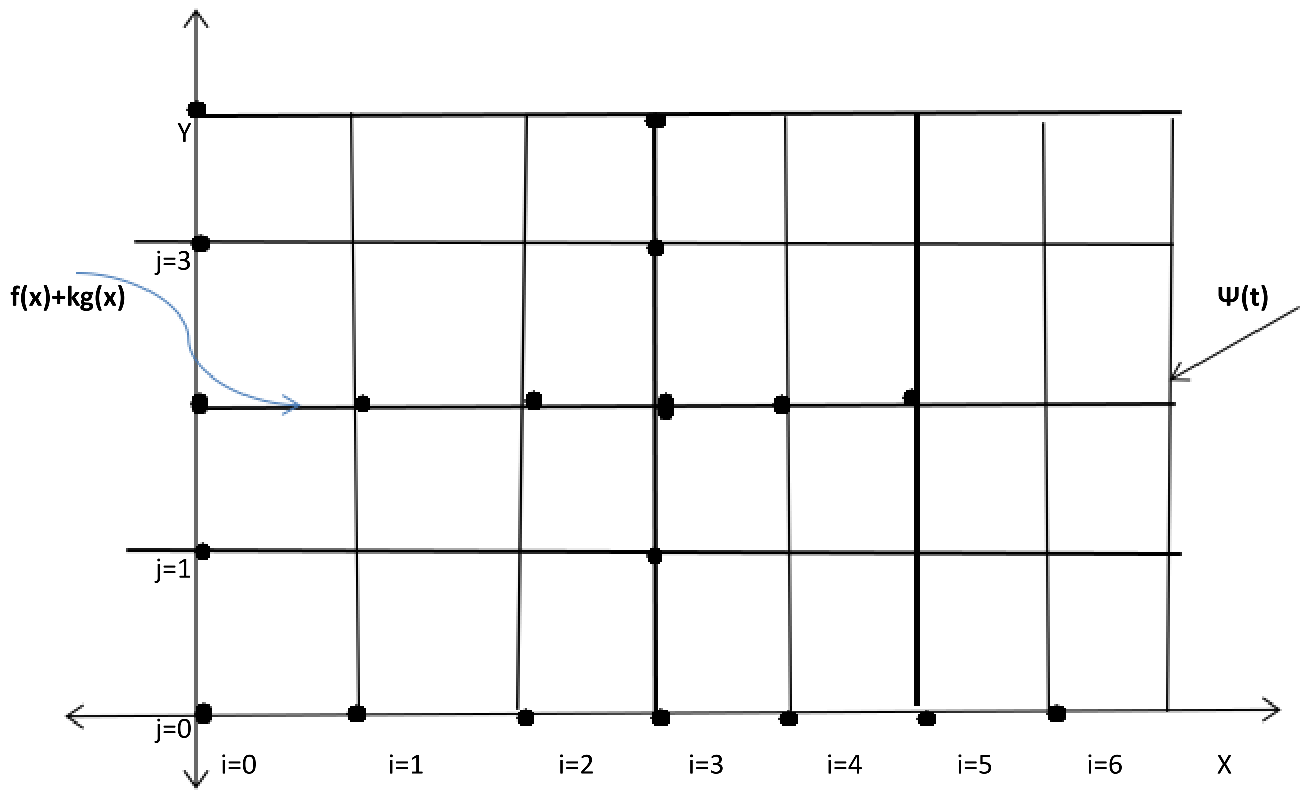
Also initial condition $u = f(x)$ at $t = 0$ becomes $u_{i,\ 0} = f(x)$ (7)

Combining (6) and (7) , we have

$$u_{i,\ 1} = f(x) + 2k\ g(x) \quad (8)$$

Also, equation (3) gives the values of $u_{i,j+1}$ at the $(j+1)$ th level when the nodal values at $(j-1)th$ and jth levels are known from equation (7) and (8), as shown in the figure.

Thus equation (4) gives an implicit scheme for the solution of the wave equation.



A special case: The coefficient of $u_{i,j}$ in (4) will vanish if $\propto = \frac{1}{c}$ or $k = h/c$. Then equation (4) reduces to the simple form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \tag{8}$$

This result provides an explicit scheme for the solution of the wave equation.

<u>Observations</u>:

✓ For $\propto = \frac{1}{c}$, the solution of equation (4) is sable and coincides with the solution of equation (1).

- ✓ For $\propto < \frac{1}{c}$, the solution is stable but inaccurate.
- ✓ For $\propto > \frac{1}{c}$, the solution is unstable.
- ✓ The formula (4) converges for $\propto \leq 1$ $i.e.$ for $k \leq h$.

Q. Evaluate the pivotal values of the equation $\boldsymbol{u_{tt} = 16u_{xx}}$, taking $\boldsymbol{h \equiv 1}$ upto $\boldsymbol{t = 1.25}$. The boundary conditions are $\boldsymbol{u(0,t) = u(5,t) = 0, u_i(x,0) = 0}$ and $\boldsymbol{u(x,0) = x^2(5-x)}$.

Sol. Here $C^2 = 16$.

The difference equation for the given equation is

$u_{i,j+1} = 2(1 - 16 \propto^2)u_{i,j} + 16 \propto^2 (u_{i-1,j} + u_{1+1,j}) - u_{i,j-1}$ where $\propto = k/h$. (i)

Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have $16 \propto^2 = 1$ $i.e. \propto^2 = \frac{1}{16}, \frac{k^2}{h^2} = \frac{1}{16} => \frac{k}{h} = \frac{1}{4} i.e\ k = \frac{1}{4}$.

$\therefore$ (i) reduces to $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ (ii)

Which gives a convergent solution (since $\frac{k}{4} < 1$). Its solution coincides with the solution of the given differential equation.

Now since $u(0,t) = u(5,t) = 0$

$\therefore u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j, i.e. the entries in the first and last columns are zero.

Since $u_{(x,0)} = x^2(5-x)$.

$\therefore\ u_{i,0} = i^2(5-i) = 4, 12, 18, 16$ for $i = 1,2,3,4$ at $t = 0$.

These are the entries of the first row.

Finally, the initial condition $u_t(x,0) = 0$, becomes

$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$, when $j = 0$, giving $u_{i,1} = u_{i,-1}$ (iii)

Putting $j = 0$ in equation (ii), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

$$= u_{i-1,0} + u_{i+1,0} - u_{i,1} using\ (iii)$$

Or $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$ (iv)

Taking $i = 1,2,3,4$ successively, we obtain

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0+12) = 6$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(4+18) = 11$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(12+16) = 14$$

$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(18+0) = 9$$

These are the entries of the second row.

Putting $j = 1$ in equation (ii), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1,2,3,4$ successively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the third row.

Similarly putting $j = 2,3,4$ successively in (ii), the entries of the fourth, fifth, and sixth rows are obtained. Hence the values of $u_{i,j}$ are shown in the table below:

i \ j	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	-2	0
3	0	2	-2	-8	-7	0
4	0	-9	-14	-11	-6	0
5	0	-16	-18	-12	-4	0

Application Problem:

Q. The transverse displacement *u* of a point at a distance *x* from one end and at any time *t* of a vibrating string satisfies the equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ with boundary conditions *u* = 0 at *x* = 0, *t* > 0 and *u* = 0 at *x* = 4, *t* > 0 and initial conditions *u* = *x*(4 – *x*) and $\frac{\partial u}{\partial t} = 0$ at *t* = 0, $0 \le x \le 4$. Solve this equation numerically for one-half period of vibration, taking *h* = 1 and *k* = 1/2.

Solution: Here, $h/k = 2 = c$.

The difference equation for the given equation is
$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ (i)

which gives a convergent solution (since $k < h$).

Now since $u(0, t) = u(4, t) = 0$,

$\therefore u_{0,j} = 0 \;\; and \;\; u_{4,j} = 0$ for all values of *j*.
i.e., the entries in the first and last columns are zero.

Since $u_{(x,0)} = x(4 - x)$

$\therefore u_{i,0} = i(4 - i) = 3,4,3$ for $i = 1,2,3$ at $t = 0$.

These are the entries of the first row.

Also, $u_t(x, 0) = 0$ becomes

$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$, when $j = 0$, giving $u_{i,1} = u_{i,-1}$ (ii)

Putting $j = 0$ in equation (i), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

$$= u_{i-1,0} + u_{i+1,0} - u_{i,1} using\ (ii)$$

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \qquad \text{(iii)}$$

Taking $i = 1,2,3$, successively, we obtain

$$u_{1,1} = \frac{1}{2}\left(u_{0,0} + u_{1,0}\right) = 2$$

$$u_{2,1} = \frac{1}{2}\left(u_{1,0} + u_{3,0}\right) = 3$$

$$u_{3,1} = \frac{1}{2}\left(u_{2,0} + u_{4,0}\right) = 2$$

These are the entries of the 2nd row.

Putting $j = 1$ in equation (i), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1,2,3$, successively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0$$

These are the entries of the 3rd row and so on.

Now the equation of the vibrating string of length *l* is $u_{tt} = C^2 u_{xx}$.

∴ Its period of vibration $= \frac{2l}{c} = \frac{2\times 4}{2} = 4\ sec.$ $[l = 4\ and\ c = 2\ ]$.
This shows that we have to compute $u_{(x,t)}$ upto $t = 2$.
i.e. Similarly we obtain the values of $u_{i,2}$ (fourth row) and $u_{i,3}$ (fifth row).
Hence the values of $u_{i,j}$ are as shown in the table below :

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	-3	-2	0
4	0	-3	-4	-3	0

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