Solution: Logic Review

CS 536: Science of Programming, Fall 2023

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Problems

1. $(p \rightarrow q)$ is meant by (a) p is sufficient for q , (b) p only if q , and (d) p is necessary for q .
   $(q \rightarrow p)$ is meant by (c) p if q .
2. For expressions e_1 and e_2,
   a. Yes: $e_1 \neq e_2$ does imply $e_1 \not\equiv e_2$. This is the contrapositive of ($e_1 \equiv e_2 \rightarrow e_1 = e_2$), which holds because $e_1 \equiv e_2$ only differ in redundant parentheses.
   b. No, $e_1 = e_2$ doesn't imply $e_1 \equiv e_2$, Simple example: $1+2 = 3$ but $1+2 \not\equiv 3$.
3. (State for expression)
   a. $\{v = 5,\ z = 6\}$ and $v + 0 * w$ *well-formed, improper (no binding for w)*
   b. $\{v = -4,\ w = 6\}$ and $sqrt(v) * sqrt(w)$ *well-formed, proper, sqrt(–4) gets r/t error*
   c. $\{y = 18,\ z = 2\}$ and $y * y / (z + 4)$ *well-formed, proper, evaluates successfully*

4. (Prove a tautology)
   (The parenthesized comments aren't necessary for correctness.)

	$p \wedge \neg(q \wedge r) \rightarrow q \wedge r \rightarrow \neg p$	
$\Leftrightarrow$	$p \wedge \neg(q \wedge r) \rightarrow (\neg(q \wedge r) \vee \neg p)$	Defn $\rightarrow$ (the second $\rightarrow$)
$\Leftrightarrow$	$\neg(p \wedge \neg(q \wedge r)) \ \vee$ $(\neg(q \wedge r) \vee \neg p)$	Defn $\rightarrow$ (the first $\rightarrow$)
$\Leftrightarrow$	$(\neg p \vee \neg\neg(q \wedge r)) \ \vee$ $(\neg(q \wedge r) \vee \neg p)$	DeMorgan's law ($\neg$ of $\wedge$)
$\Leftrightarrow$	$\neg p \vee T \vee \neg p$	Excluded middle (on $\neg(q \wedge r) \vee \neg\neg(q \wedge r)$)
$\Leftrightarrow$	T	Domination, twice

   Note as discussed in class, we omitted the associativity and commutativity step for changing $(\neg p \vee \neg\neg(q \wedge r)) \vee (\neg(q \wedge r) \vee \neg p)$ to $(\neg p \vee (\neg(q \wedge r) \vee \neg\neg(q \wedge r)) \vee \neg p)$ before applying excluded middle. If you want to use $(q \wedge r) \vee \neg(q \wedge r)$ for the excluded middle step, you first have to change $\neg\neg(q \wedge r)$ to $(q \wedge r)$ using the $\neg\neg$ rule.

5. (Simplify and remove all ¬)

	$\neg(\forall x.(\exists y.x \le y) \vee \forall z.x \ge z)$	
⇔	$\exists x.\neg((\exists y.x \le y) \vee \forall z.x \ge z)$	DeMorgan's law (¬ of ∀)
⇔	$\exists x.(\neg \exists y.x \le y) \wedge \neg \forall z.x \ge z)$	DeMorgan's law (¬ of ∨)
⇔	$\exists x.(\forall y.\neg(x \le y)) \wedge \neg \forall z.x \ge z)$	DeMorgan's law (¬ of ∃)
⇔	$\exists x.(\forall y.x > y) \wedge \neg \forall z.x \ge z)$	Negation of ≤
⇔	$\exists x.(\forall y.x > y) \wedge \exists z.\neg(x \ge z)$	DeMorgan's law (¬ of ∀)
⇔	$\exists x.(\forall y.x > y) \wedge \exists z.x < z$	Negation of ≥

6. (Full parenthesizations)
   a. $(_1(_2(_3(_4\ p \wedge (_5 \neg r)_5)_4 \vee s)_3 \to (_6(_7(_8 \neg q)_8 \wedge r)_7 \to (_9 \neg p)_9)_6)_2 \leftrightarrow (_{10}\ s \to t)_{10})_1$
   b. $(_1 \forall m.(_2 0 < m < n)_2 \wedge (_3 \exists j.(_4 0 \le j < m_4) \to (_5(_6 b[0])_6 \le (_7 b[j])_7 \le (_8 b[m])_8)_5)_3)_1$

   b. $(\exists j.(((0 \le j) \wedge (j < m)) \wedge (\forall k.(((m \le k) \wedge (k < n)) \to (b[j] < b[k])))))$. (This predicate asks "Is there a value in $b[0..m-1]$ < every value in $b[m..n-1]$?)
   c. $(\forall x.((\exists y.(p \wedge q)) \to (\forall z.(p \to (q \wedge r)))))$

7. (Minimal parenthesizations)
   a. $((\neg(p \vee q) \wedge r) \to (((\neg q) \vee r) \to ((p \vee (\neg r)) \wedge (q \wedge s))))$ minimizes to
   $\neg(p \vee q) \wedge r \to \neg q \vee r \to (p \vee \neg r) \wedge q \wedge s$
   b. $(\exists j.(((0 \le j) \wedge (j < m)) \wedge (\forall k.(((m \le k) \wedge (k < n)) \to (b[j] < b[k])))))$ minimizes to
   $\exists j.0 \le j \wedge j < m \wedge \forall k.m \le k \wedge k < n \to b[j] < b[k]$.
   c. $(\forall x.((\exists y.(p \wedge q)) \to (\forall z.(p \to (q \wedge r)))))$ minimizes to
   $\forall x.(\exists y.p \wedge q) \to \forall z.p \to q \wedge r$
8. (Syntactically equal?)
   a. The two are not ≡. They're both minimally parenthesized, so we can compare them as is, or we can compare the full parenthesizations:

      $\forall x.p \to \exists y.q \to r \ \equiv\ (\forall x.(p \to (\exists y.(q \to r))))$ but

      $((\forall x.p) \to (\exists y.q)) \to r \ \equiv\ (((\forall x.p) \to (\exists y.q)) \to r)$
   b. The two are not ≡. Comparing full parenthesization, we have

      $\exists x.p \wedge \exists y.(q \to r) \vee \exists z.r \to s \ \equiv\ (\exists x.(p \wedge (\exists y.((q \to r) \vee (\exists z.(r \to s))))))$

      $\exists x.p \wedge (\exists y.q \to r) \vee (\exists z.r \to s) \ \equiv\ (\exists x.(p \wedge ((\exists y.(q \to r)) \vee (\exists z.(r \to s)))))$
   c. The two are not ≡. They're both minimally parenthesized so we can compare them as is:

      $(\forall x.p \vee \forall y.q) \vee (\forall z.r) \to s$ vs $\forall x.p \vee (\forall y.q) \vee \forall z.r \to s$
   d. The two are not ≡. We can compare minimal parenthesizations.

      $p \wedge q \vee \neg r \to \neg p \to q$ is already minimally parenthesized

      but $((p \wedge q) \vee ((\neg r \to ((\neg p) \to q))))\ \equiv\ p \wedge q \vee (\neg r \to \neg p \to q)$.

9. (Tautology, contradiction, or contingency?)

   a. $((p \to q) \to r) \to (p \to (q \to r))$ is a tautology. We didn't ask for an argument, but here's one for reference's sake: For the proposition to $\Leftrightarrow F$, we need the antecedent $((p \to q) \to r) \Leftrightarrow T$ and the consequent $(p \to (q \to r)) \Leftrightarrow F$. For $(p \to (q \to r)) \Leftrightarrow F$, we need $(p \to (q \to r)) \Leftrightarrow (T \to F) \Leftrightarrow (T \to (T \to F))$, so we need $p = T,\ q = T, r = F$. In that state, $((p \to q) \to r) \Leftrightarrow ((T \to T) \to F) \Leftrightarrow (T \to F) \Leftrightarrow F$, but we needed it to $\Leftrightarrow T$. So the original implication is never $\Leftrightarrow (T \to F) \Leftrightarrow F$.

   b. $(p \to (q \to r)) \to ((p \to q) \to r)$ is a contingency. For an instance of satisfaction, if $p = T,\ q = T,\ r = T$, we get

$$\begin{aligned} &(p \to (q \to r)) \to ((p \to q) \to r) \\ &\quad \Leftrightarrow (T \to (T \to T)) \to ((T \to T) \to T) \\ &\quad \Leftrightarrow (T \to T) \to (T \to T) \\ &\quad \Leftrightarrow T \to T \\ &\quad \Leftrightarrow T. \end{aligned}$$

   For nonsatisfaction, if $p = F$, $q = F$, $r = F$, then

$$\begin{aligned} &(p \to (q \to r)) \to ((p \to q) \to r) \\ &\quad \Leftrightarrow (F \to (F \to F)) \to ((F \to F) \to F) \\ &\quad \Leftrightarrow (F \to T) \to (T \to F) \\ &\quad \Leftrightarrow T \to F \\ &\quad \Leftrightarrow F \end{aligned}$$

   ($p = F$, $q = T$, $r = F$ also works.*)*

   c. $(\forall x \in \mathbb{Z}.\ \forall y \in \mathbb{Z}.\ f(x, y) > 0) \to (\exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}.\ f(x, y) > 0)$.

   This is a tautology: The antecedent says that $f(x, y) > 0$ for all integers x and y, so it's certainly true for, say $x = y = 0$, so there exist x and y such that $f(x, y) > 0)$.

10. We want $GT(b, x, m, k)$ to be true when $k \geq m$ and $x > b[m],\ b[m+1],\ \ldots,\ b[k]$

    $GT(b, x, m, k) \ \equiv\ k \geq m \to \forall j.\ m \leq j \leq k \to x > b[j]$

