CHAPTER: 04
3-D TRANSFORMATION

We follow notation

$$P' = T \cdot P$$

where $P' = \begin{bmatrix} x_1' & x_2' & \cdots & x_n' \\ y_1' & y_2' & \cdots & y_n' \\ z_1' & z_2' & \cdots & z_n' \\ 1 & 1 & \cdots & 1 \end{bmatrix}$

$$P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and T is transformation matrix.

Same as 2D there are 5 transformations

1. Translation

$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Scaling

$$S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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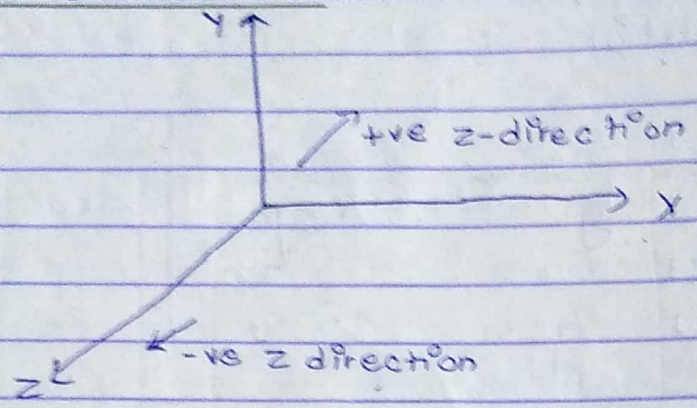
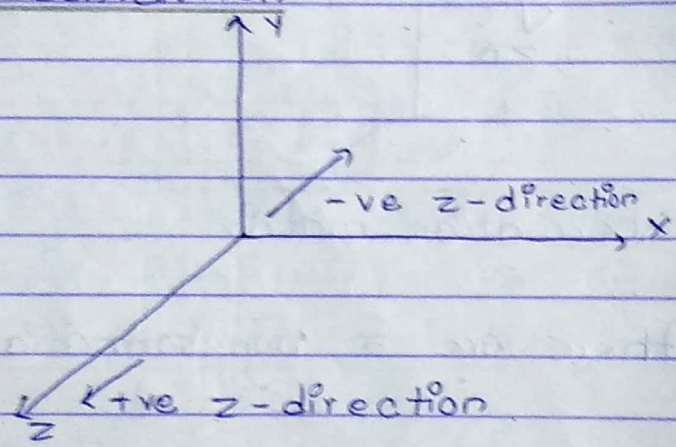
Note :- In 3D there are two conventions.

① LHS convention



② RHS convention



3. Shearing

Assume a = shear constant along x-axis
b = shear constant along y-axis
c = shear constant along z-axis

→ z-axis shear (across xy-plane)

$$x' = x + az$$
$$y' = y + bz$$
$$z' = z$$

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$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$P' = SHz \cdot P$

→ x-axis shear (across yz-plane)

$x' = x$

$y' = y + bx$

$z' = z + cx$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$P' = SHx \cdot P$

→ y-axis shear (across xz-plane)

$y' = y$

$x = x + ay$

$z = z + cy$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Reflection of points relative to xy plane

$P' = Rf_{xy\ plane}\ P$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$x' = x \;,\; y' = y \;,\; z' = -z$

Relative to xz plane

$P' = Rf_{xz\,plane} \cdot P$

$x' = x \;,\; y' = -y \;,\; z' = z$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Relative to yz plane

$P' = Rf_{yz\,plane} \cdot P$

$x' = -x \;,\; y' = y \;,\; z' = z$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Three-dimensional Composite transformation.

- successive translation is additive.
- successive rotation is additive.
- successive scaling is multiplicative.

Rotation

- Rotation about z-axis (on xy-plane)

$$\left.\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}\right\} \text{------} \quad (1)$$

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$$P' = R_z(\theta) \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation about x-axis (yz-plane)

In eqn (i) replace x by y, y by z and z by x.

$$\left.\begin{aligned} y' &= y\cos\theta - z\sin\theta \\ z' &= y\sin\theta + z\cos\theta \\ x' &= x \end{aligned}\right\} \text{———— (ii)}$$

$$P' = R_x(\theta) \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation about y-axis (zx plane)

In eqn (ii), replace x by y, y by z & z by x

$$z' = z\cos\theta - x\sin\theta$$

$$x' = z\sin\theta + x\cos\theta$$

$$y' = y$$

$$P' = R_y(\theta) \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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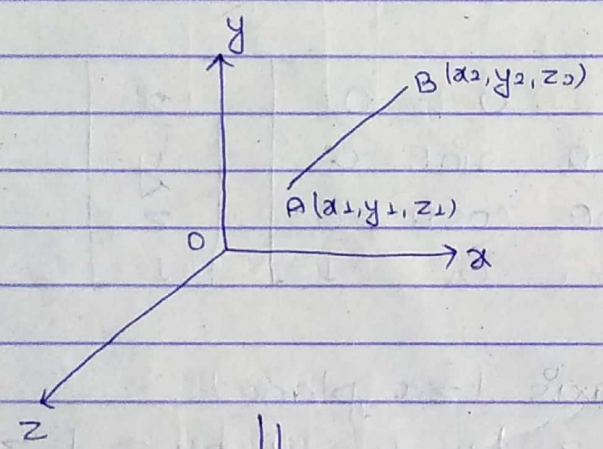
Rotation about an axis parallel to any one standard axis.

(i) Translate the axis to standard axis
(ii) Perform desired rotation.
(iii) Inverse translation.

$$T_\theta = T^{-1} \cdot R_\theta \cdot T$$

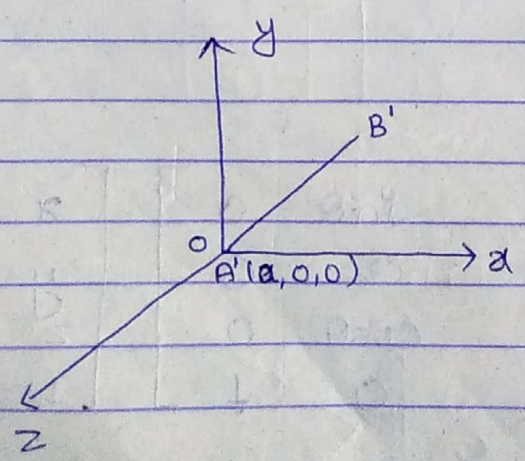
Rotation about an axis not parallel to any standard axis.

Step 1:



Translate point A to origin

i.e,
$$\begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



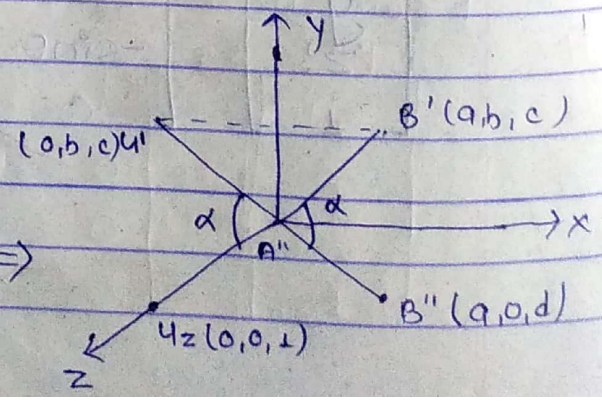
$A' = (0,0,0)$

$B' = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Step 2: Perform rotation about angle α to rotate $A'B'$ to $A''B''$

$u' = (0,b,c)$, $u_z = (0,0,1)$

$\Rightarrow$



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For step 2.1

$$u' \cdot u_z = \left(\frac{0}{\sqrt{b^2+c^2}}, \frac{b}{\sqrt{b^2+c^2}}, \frac{c}{\sqrt{b^2+c^2}} \right) \cdot (0,0,1)$$

$$= \frac{c}{\sqrt{b^2+c^2}}$$

$$|u'| \cdot |u_z| = \sqrt{\left(\frac{b}{\sqrt{b^2+c^2}}\right)^2 + \left(\frac{c}{\sqrt{b^2+c^2}}\right)^2} \times 1$$

$$= 1$$

$$\cos\alpha = \frac{u' \odot u_z}{|u_1||u_2|}$$

$$\boxed{\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}}$$ where, $d = \sqrt{b^2+c^2}$

$$\sin\alpha = \frac{u' \otimes u_z}{|u'||u_z|} \hat{n} \quad \text{——— (i)}$$

$$u' \otimes u_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b/d & c/d \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{b\hat{i}}{d}$$

From (i) $\sin\alpha = \frac{b\hat{i}}{d} / 1 \times \hat{i}$

$$\boxed{\sin\alpha = \frac{b}{d}}$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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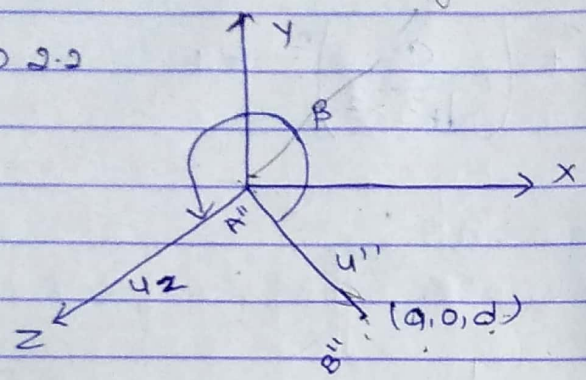
$$R^{-1}x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & b/d & 0 \\ 0 & -b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2.2



⇒ Rotate A''B'' about angle β in anticlockwise direction

$$u'' = \frac{a}{\sqrt{a^2+d^2}}\vec{i} + 0\vec{j} + \frac{d}{\sqrt{a^2+d^2}}\vec{k}$$

$$\sqrt{a^2+d^2} = \sqrt{a^2+b^2+c^2} = l$$

$$\cos\beta = \frac{u'' \odot u_z}{|u''||u_z|}$$

$$u'' \odot u_z = \left(\frac{a}{l}, 0, \frac{d}{l}\right) \cdot (0,0,1)$$

$$= \frac{d}{l}$$

$$|u''| \cdot |u_z| = \sqrt{\frac{a^2}{l^2} + \frac{d^2}{l^2}} \cdot 1$$

$$= \sqrt{\frac{a^2+d^2}{l^2}}$$

$$= \frac{\sqrt{a^2+d^2}}{l} = \frac{\cancel{l}}{\cancel{l}} = 1$$

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$$\boxed{\cos\beta = \frac{d}{l}}$$

$$\sin\beta = \frac{u'' \otimes uz}{|u''|\,|uz|} \cdot \hat{n} \quad \text{——— (ii)}$$

$$u'' \otimes uz = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a/l & 0 & d/l \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -\frac{a}{l}\vec{j}$$

From (ii) $\sin\beta = \dfrac{\left(-\dfrac{a\vec{j}}{l}\right)}{1} \cdot \vec{j}$

$$\boxed{\sin\beta = -\frac{a}{l}}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y^{-1}(\beta) = \begin{bmatrix} d/l & 0 & a/l & 0 \\ 0 & 1 & 0 & 0 \\ -a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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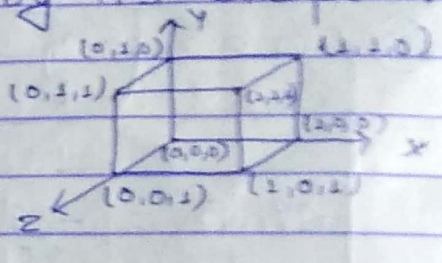
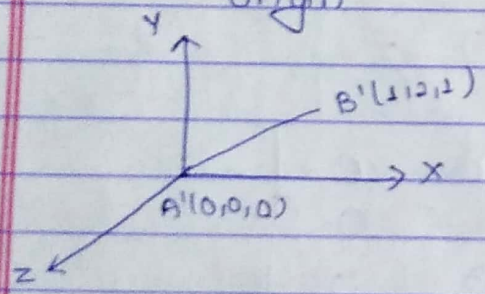
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta) = T^{-1} R_x^{-1}(\alpha) R_y^{-1}(\beta) R_z(\theta) R_y(\beta) R_x(\alpha) T$$

Q. Find the new co-ordinates of a unit cube 90°-rotated about an axis defined by its endpoint A(2,1,0) and B(3,3,1)

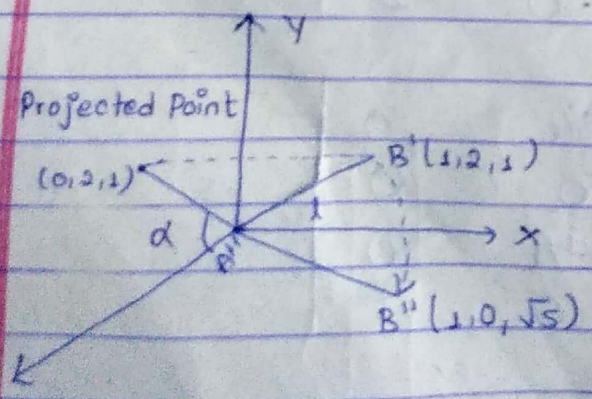


Step 1: Translate point A to origin



$$T = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: Rotate axis A'B' about the x-axis by and angle α, until it lies on the xz plane.



$$\sin\alpha = \frac{2}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} = \frac{b}{d}$$

$$\cos\alpha = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \frac{c}{d}$$

$$l = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

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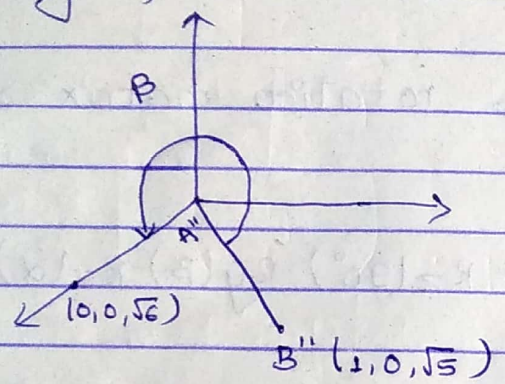
$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{5}/5 & -2\sqrt{5}/5 & 0 \\ 0 & 2\sqrt{5}/5 & \sqrt{5}/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Step 3</u>: Rotate axis $A'B''$ about the y-axis by and angle ϕ, until it coincides with z-axis



$$\sin\beta = -\frac{a}{l} = -\frac{1}{\sqrt{6}} = -\frac{\sqrt{6}}{6}$$

$$\cos\beta = \frac{d}{l} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

$$R_y(\beta) = \begin{bmatrix} \sqrt{30}/6 & 0 & -\sqrt{6}/6 & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{6}/6 & 0 & \sqrt{30}/6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^{-1}y(\beta) = \begin{bmatrix} \sqrt{30}/6 & 0 & \sqrt{6}/6 & 0 \\ 0 & 1 & 0 & 0 \\ -\sqrt{6}/6 & 0 & \sqrt{30}/6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R^{-1}x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{5}/5 & 2\sqrt{5}/5 & 0 \\ 0 & -2\sqrt{5}/5 & \sqrt{5}/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Step 4: Rotate the cube 90° about z-axis

$$R_z(90^\circ) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally the concatenated rotation matrix about arbitrary axis becomes.

$$R(\theta) = T^{-1} R_x^{-1}(\alpha)\, R_y^{-1}(\beta)\, R_z(90^\circ)\, R_y(\beta)\, R_x(\alpha)\, T$$

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3D View Pipelining

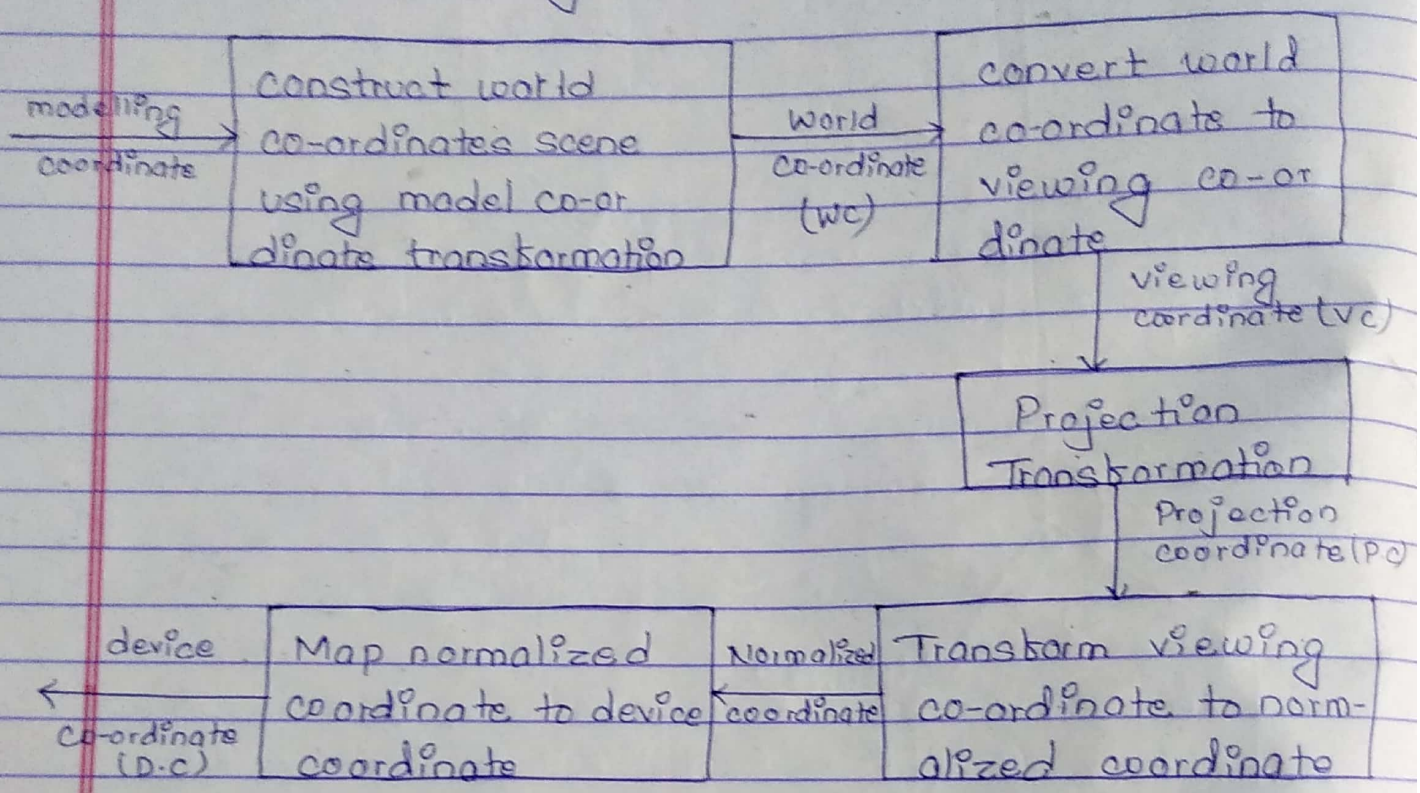

Fig. 3D View Pipelining

Once the scene has been modelled, WC are converted to VC which is used in graphics packages as a refrence for specifying observer viewing position and position of projection plane.

Projection operations are performed to convert VC to coordinate positions on projection plane which are normalized and then mapped to output devices.

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Normalized coordinates define which part will be visible and where it will appear on the display surface.

Transformation from world to viewing coordinate

1. Translate viewing coordinate to world co-ordinate origin.
2. Apply rotation to align x_v, y_v & z_v axes with world x_w, y_w & z_w axes.

The rotation can be either by rotating viewing coordinate axes along x_w, y_w, z_w axes or the composite rotation matrix can be obtained by setting up u, v, N vectors and forming a complete rotation matrix.

Given reference points P_0 and look at point P_{ref} and view up vector $V = (V_x, V_y, V_z)$

$$N = P_0 - P_{ref} = (N_x, N_y, N_z)$$

$$V = (V_x, V_y, V_z)$$

$$n = \frac{N}{|N|} = (n_x, n_y, n_z)$$

Again, unit vector along x_v axis

$$u \; \vartheta = \frac{V \times N}{|V \times N|} = \cancel{(\vartheta_x, \vartheta_y, \vartheta_z)} \; (u_x \; u_y \; u_z)$$

where, $V \times N = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V_x & V_y & V_z \\ N_x & N_y & N_z \end{vmatrix}$

and $\vartheta = n \times u = (\vartheta_x \; \vartheta_y \; \vartheta_z)$

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where, $n \times u = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ n_x & n_y & n_z \\ u_x & u_y & u_z \end{vmatrix}$

The composite rotation matrix is then,

$$R = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, final transformation matrix is

$$V_c = R \cdot T$$

Q. Find viewing transformation matrix for which
Pref (look at position) = (0, 0, 0)
P_0 (eye position) = (1, 1, 1)
& view up vector, v = (0, 1, 0)

⇒ Soln:

We have,

$$N = P_0 - P_{ref} = (1, 1, 1)$$

$$n = \frac{N}{|N|} = \frac{(1, 1, 1)}{\sqrt{1^2+1^2+1^2}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

n_x, n_y, n_z

Given,

$$v = (0, 1, 0)$$

$$u = \frac{v \times N}{|v \times N|}$$

$$v \times N = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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$$V \times N = \vec{i}(1-0) - \vec{j}(0-0) + \vec{k}(-1)$$
$$= \vec{i} - \vec{k}$$

$$|V \times N| = \sqrt{2}$$

$$u = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

Now,

$$v = n \times u = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{vmatrix}$$

$$= -\frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$M_{wc,vc} = R \cdot T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Projection: Process of converting n dimension object into n-1 or less dimensions.

Taxonomy of Projection

- Projection
  - Parallel
    - Orthographic
      - Top (y) (xz plane)
      - Side (z) (xy plane)
      - Front (z) (yz plane)
      - Axinometric (multiple faces, doesnot lie in any plane)
    - Oblique
      - Cavilear
      - Cabinet
  - Perespective
    - viewplane 1pt
    - 2pt
    - 3pt

Projection line

Projection line

Parallel Projection: Projection lines are parallel to eachother.

- Relative size of object is maintained.
- Unrealistic

Orthographic: Projection lines are perpendicular to the projection plane view plane

Oblique: Projection lines are not perpendicular to view plane.

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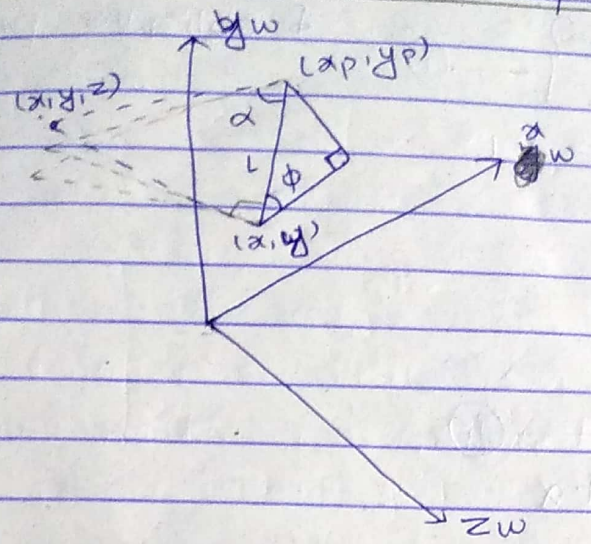
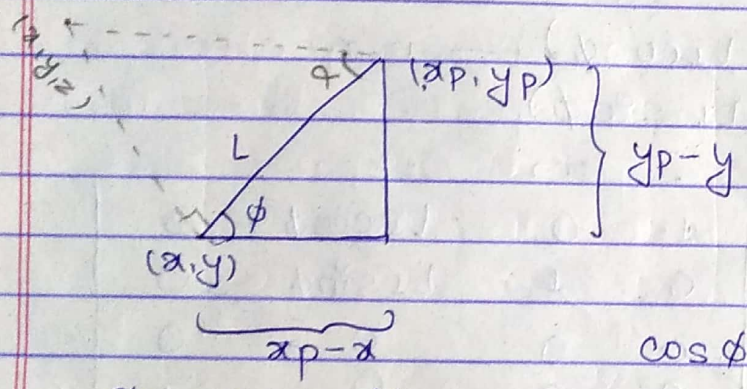
Composite matrix for oblique projection



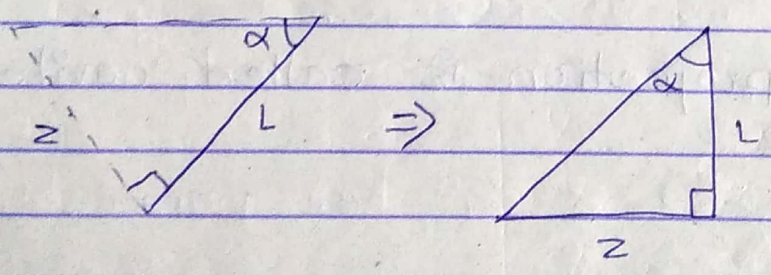


$$\cos\phi = \frac{x_p - x}{L}$$

$$x_p = x + \cos\phi L$$

$$y_p = y + \sin\phi L$$

$$\sin\phi = \frac{y_p - y}{L}$$



$$\tan\alpha = \frac{z}{L}$$

The figure shown is oblique projected projection of a point (x, y, z) to position (xp, yp) on viewplane.

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From fig.

$$\left.\begin{aligned} x_p &= x + L\cos\phi \\ y_p &= y + L\sin\phi \end{aligned}\right\} \text{—— (i)}$$

L = distance betn orthographic & oblique projected point

α depends on z & L

$$\tan\alpha = \frac{z}{L}$$

$$\text{or, } L = \frac{z}{\tan\alpha}$$

$$\text{or, } L = L_1 z \text{ —— (ii)}$$

where, $L_1 = \cot\alpha$

From (i) & (ii),

$$x_p = x + z(L_1\cos\phi)$$
$$y_p = y + z(L_1\sin\phi)$$

In matrix,

$$M_{parallel,\,oblique} = \begin{bmatrix} 1 & 0 & L_1\cos\phi & 0 \\ 0 & 1 & L_1\sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Common value for $\phi = 63.3^\circ$ or 45°

$\tan\alpha = 1$ or 2

If $\tan\alpha = 1$, projection is called cavilear projection

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Perspective Projection

For a perspective position, object positions are transformed to view plane along lines that converges to a point called projection refrence point (PRP) or center of projection (COP). The projected view of an object is determined by calculating the intersection of projection lines with a view plane.

A perspective projection produces realistic views but doesnot preserve relative proportions. Projections of distant objects are smaller than the projections of objects of same size that are closer to projection plane.

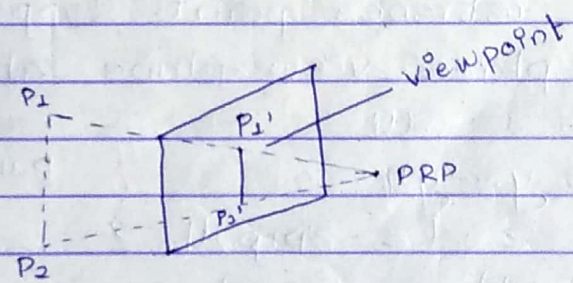
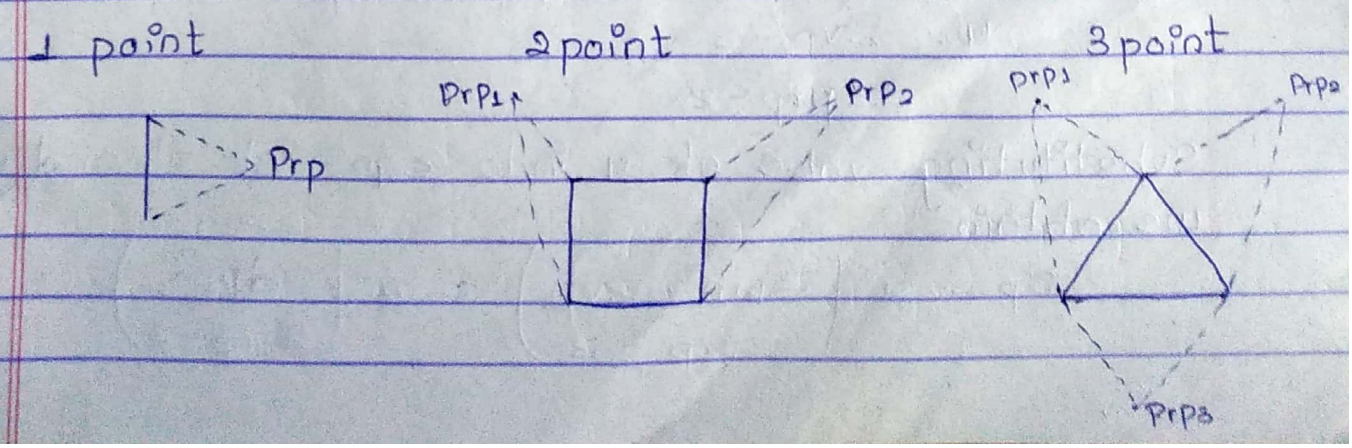


Fig. Perspective projection of a line of a viewplane

Types



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Composite transformation

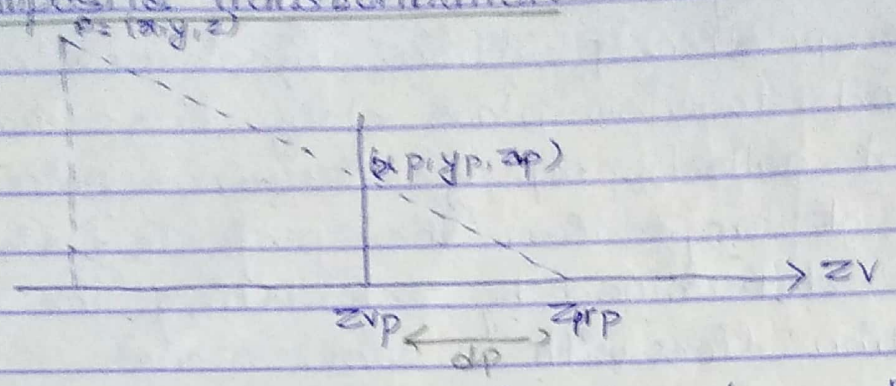

Fig. Perspective projection of a point p with co-ordinates (x, y, z) to position (xp, yp, zvp) on view plane

To obtain perspective ~~position~~ projection we transform points along projection lines that meet at projection refrence point. Suppose zprp is PRP and we place view plane at zvp.

$$\left.\begin{aligned} x' &= x - xu \\ y' &= y - yu \\ z' &= z - (z - z_{prp})u \end{aligned}\right\} \text{parametric form}$$

where (x', y', z') represents any point on projection line and u takes value from 0 to 1.

When $u=0$, we are at position $P(x, y, z)$ when $u=1$ we are at zprp $(0, 0, z_{prp})$ on view plane.

$$z' = z_{vp}$$

And we can solve z' eqn for parameter is,

$$u = \frac{z_{vp} - z}{z_{prp} - z}$$

Substituting value of u into eqns of x' and y', we obtain

$$x_p = x\left(\frac{z_{prp} - z_{vp}}{z_{prp} - z}\right) = x\left(\frac{d_p}{z_{prp} - z}\right) \rightarrow x \cdot \frac{1}{h} = x_p$$

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$$x_p = x - x\left(\frac{z_{vp} - z}{z_{prp} - z}\right)$$

$$\text{or, } x_p = \frac{x(z_{prp} - z) - x(z_{vp} - z)}{z_{prp} - z}$$

$$\text{or, } x_p = \frac{x(z_{prp} - z_{vp})}{z_{prp} - z}$$

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$$\text{and } y_p = y\left(\frac{z_{prp} - z_{vp}}{z_{prp} - z}\right) = y\left(\frac{d_p}{z_{prp} - z}\right)$$

where,

$d_p = z_{prp} - z_{vp}$ is distance of view plane from projection refrence point.

Using three dimentional homogeneous coordinate representation, we can write perspective projection transformation in matrix form as:

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -z_{vp}/d_p & z_{vp}(z_{prp}/d_p) \\ 0 & 0 & -1/d_p & z_{prp}/d_p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In this representation, homogeneous factor is,

$$h = \frac{z_{prp} - z}{d_p} = \frac{z_{prp}}{d_p} - \frac{z}{d_p} = -\left(\frac{1}{d_p}\right)z + \left(\frac{z_{prp}}{d_p}\right)$$

and projection coordinates on view plane are calculated from homogeneous coordinate as:

$$x_p = \frac{x_h}{h}, \quad y_p = \frac{y_h}{h}$$

where,

Original z-coordinate value would be obtained in projection co-ordinates for visible surface and other depth processing.

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