Deep Learning Using TensorFlow

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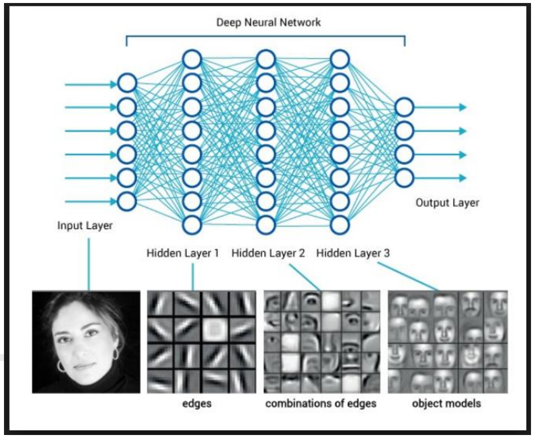
Section 2

Lesson 2.1: Gradient Descent Algorithm

Backpropagation Algorithm:



- Conceptually the backpropagation algorithm is very simple

- Algorithm
  - Assign random values to all the weights of the NN
  - Take the first observed data
    - Forward Propagation: Compute Output
    - Compute error = $(Computed\ Output\ - Observed\ Output)^2$
    - Backpropagation: adjust weights to reduce error
    - Repeat forward, backward propagation, till error is minimized
  - Repeat the previous step for the next sample till all samples are processed
  - The final weights of the NN will be used for prediction

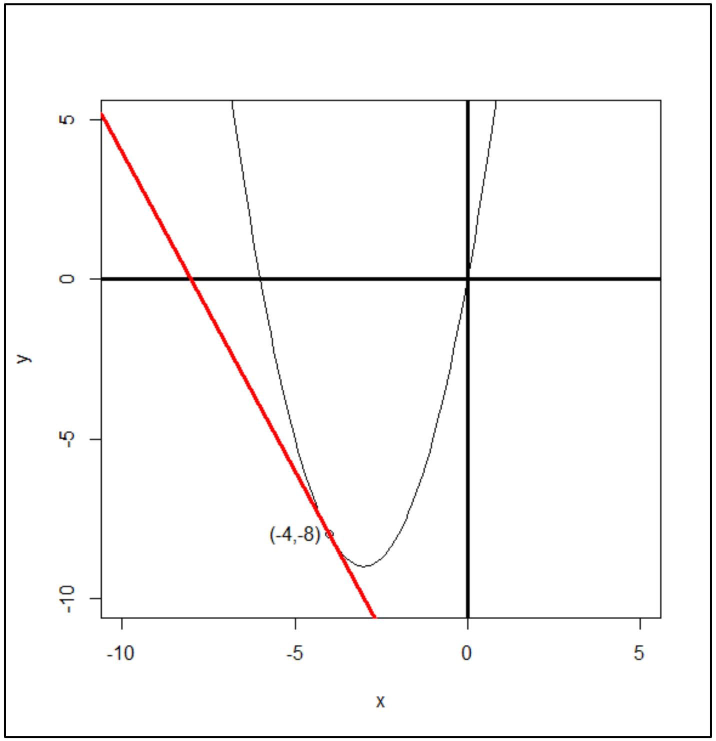
What is a Gradient?

Gradient Vector: 2 Variables

- Definition: 2 variables x, y
  - The gradient vector of a function y=f(x)
  - $\nabla y = \nabla f(x) = \frac{\partial f}{\partial x} = \frac{df}{dx}$
- Gradient property
  - Gradient vector gives the direction of fastest increase (or decrease) of function f(x)

Example: Gradient

- $y = x^2 + 6x$
  - $\frac{dy}{dx} = 2x + 6$
  - $x = -4$
  - $y = (-4)^2 + 6(-4) = 16 - 24 = -8$
  - $\frac{dy}{dx} = 2x + 6 = 2 * (-4) + 6 = -2$
  - Gradient at point(-4,-8) = -2



What is Gradient Descent Algorithm?

Who Invented Gradient Descent Algorithm?

- Gradient Descent algorithm was invented by **Cauchy** in 1847
- Méthode générale pour la résolution **des** systèmes d'équations simultanées. pp. 536–538

Augustin-Louis Cauchy

Cauchy around 1840. Lithography by Zéphirin Belliard after a painting by Jean Roller.

Born	21 August 1789 Paris, France
Died	23 May 1857 (aged 67) Sceaux, France
Nationality	French
Alma mater	École Nationale des Ponts et Chaussées
Known for	See list
Spouse(s)	Aloise de Bure
Children	Marie Françoise Alicia, Marie Mathilde

What is Gradient Descent Algorithm?

- Suppose a function $y = f(x)$ is given
  - Gradient Descent algorithm allows us to find the values of 'x' where the 'y' value becomes minimum or maximum values
- The Gradient Descent algorithm can be extended to any function with 2 or more variables $'z = f(x, y)'$

- Function y=f(x)
- Gradient Descent Algorithm: Minimum
  - Initialize the value of x
  - Learning rate = η
  - While NOT converged:
    - $x^{t+1} \leftarrow x^t - \eta \frac{\partial y}{\partial x} \|_{x^t}$

- Function z=f(x,y)
- Gradient Descent Algorithm: Minimum
  - Initialize the value of x and y
  - Learning rate = η
  - While NOT converged:
    - $x^{t+1} \leftarrow x^t - \eta \frac{\partial z}{\partial x} \|_{x^t, y^t}$
    - $y^{t+1} \leftarrow y^t - \eta \frac{\partial z}{\partial y} \|_{x^t, y^t}$

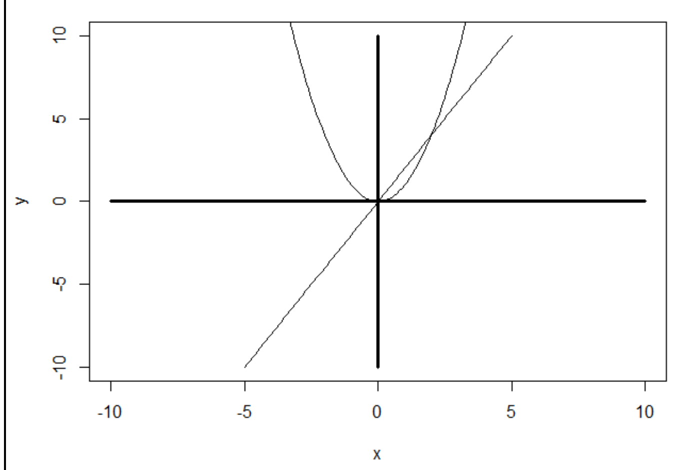
Finding Minimum of a Function Using Gradient Descent Algorithm

2 Variables (x, y) Function

Example 1
Find the value of ‘x’ where the value of ‘y’ is minimum

- $y = x^2$
- To find minimum point, equate the first derivative = 0
  - $\frac{dy}{dx} = 2x = 0$
  - $x = 0$
- To find the value of ‘x’
  - Where the value of ‘y’ is minimum
  - Correct answer: x = 0

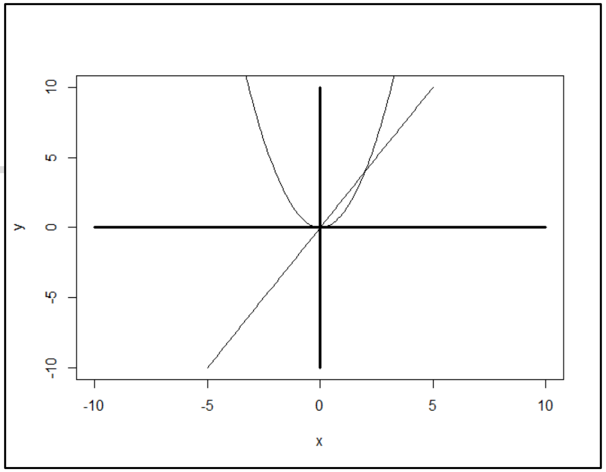
```
> x = seq(-5,5,0.1)
> y = x^2
> dy_dx = function (w1) { 2*w1 }
> plot(x,y,type='l',xlim=c(-10,10),ylim=c(-10,10))
> lines(x,dy_dx(x))
> lines(c(0,0),c(-10,10),lwd=3)
> lines(c(-10,10),c(0,0),lwd=3)
```

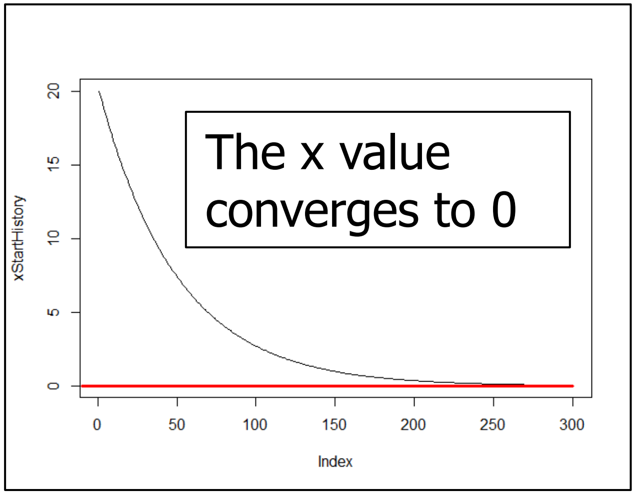


Example 1: R Code

- $y = x^2$
- To find minimum point, equate the first derivative = 0
  - $\frac{dy}{dx} = 2x = 0$
  - $x = 0$
- Gradient Descent Algorithm
  - Initialize the value of x
  - Learning rate = η
  - While NOT converged:
    - $x^{t+1} \leftarrow x^t - \eta \frac{\partial y}{\partial x} \|_{x^t}$

```
> dy_dx = function (w1) { 2*w1 }
> xStart = 20
> learningRate = 0.01
> maxLimit = 300
> xStartHistory = rep(0,maxLimit)
> for ( i in 1:maxLimit )
+ {
+   xStartHistory[i] = xStart
+   xStart = xStart - learningRate * dy_dx(xStart)
+ }
> plot(xStartHistory,type='l')
> lines(c(-10,10),c(0,0),lwd=3,col='red')
> lines(c(maxLimit,maxLimit),c(0,0),lwd=3,col='red')
> lines(c(0,maxLimit),c(0,0),lwd=3,col='red')
```





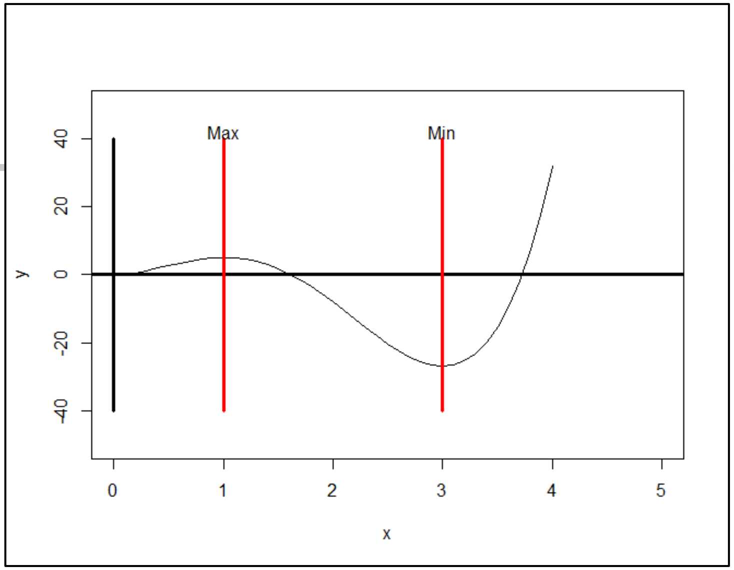
Gradient Descent Algorithm Parameters

- Parameters
  - Initial value of 'x'
  - Learning Rate
- If the choice of initial value of 'x' and learning rate is different
  - The Gradient Descent algorithm may not converge

Finding Minimum + Maximum of a Function Using Gradient Descent Algorithm

2 Variables (x,y) Function

Example 3
Find the value of 'x' where the value of 'y' is
* Minimum
* Maximum



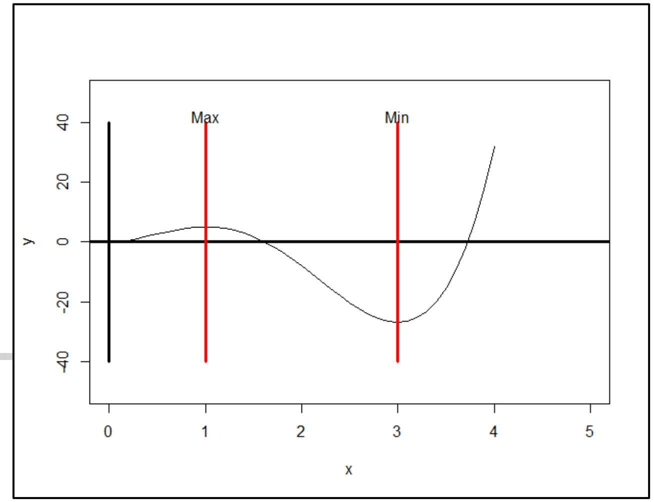
- $y = 3x^4 - 16x^3 + 18x^2$
- $\frac{dy}{dx} = 12x^3 - 48x^2 + 36x$
- To find the value of 'x'
  - Where the value of 'y' is local minimum
  - Correct answer: x = 3, y = -27

- To find the value of 'x'
  - Where the value of 'y' is local maximum
  - Correct answer: x = 1, y = 5

```
> x = seq(0,4,0.1)
> y = 3*x^4 - 16*x^3 + 18*x^2
> dy_dx = function (w1) { 12*w1^3 - 48*w1^2 + 36*w1 }
> plot(x,y,type='l',xlim=c(0,5),ylim=c(-50,50))
> #lines(x,dy_dx(x))
> lines(c(0,0),c(-40,40),lwd=3)
> lines(c(-10,10),c(0,0),lwd=3)
> lines(c(3,3),c(-40,40),lwd=3,col='red')
> lines(c(1,1),c(-40,40),lwd=3,col='red')
> text(1,42,"Max")
> text(3,42,"Min")
```

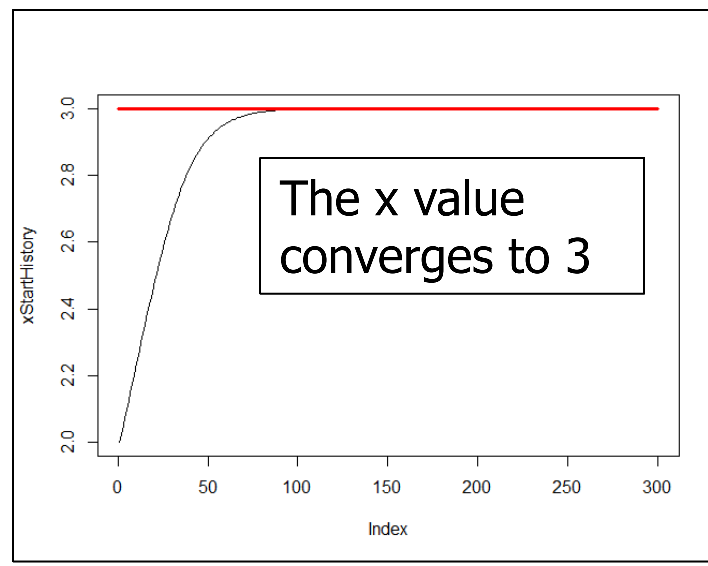
Example 3: Minimum

- $y = 3x^4 - 16x^3 + 18x^2$
- $\frac{dy}{dx} = 12x^3 - 48x^2 + 36x$
- Gradient Descent Algorithm: Minimum
  - Initialize the value of x
  - Learning rate = η
  - While NOT converged:
    - $x^{t+1} \leftarrow x^t - \eta \frac{\partial y}{\partial x} \|_{x^t}$



- To find the value of 'x'
  - Where the value of 'y' is local minimum
  - Correct answer: x = 3, y = -27

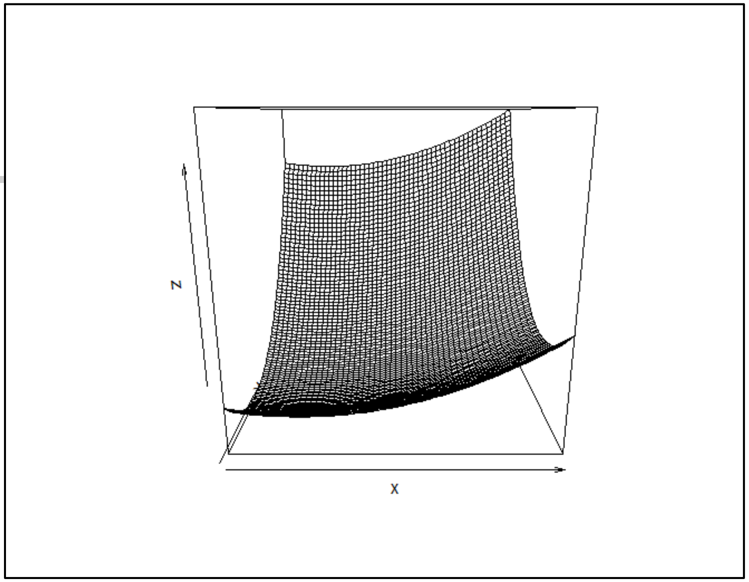
```
> dy_dx = function (w1) { 12*w1^3 - 48*w1^2 + 36*w1 }
> xStart = 2
> learningRate = 0.001
> maxLimit = 300
> xStartHistory = rep(0,maxLimit)
> for ( i in 1:maxLimit )
+ {
+   xStartHistory[i] = xStart
+   xStart = xStart - learningRate * dy_dx(xStart) #
For minimum 'subtract'
+   ## xStart = xStart + learningRate * dy_dx(xStart)
# maximum 'add'
+
+ }
> plot(xStartHistory,type='l')
> lines(c(0,maxLimit),c(3,3),lwd=3,col='red')
```



Finding Minimum of a Function Using Gradient Descent Algorithm

3 Variables (x,y,z) Function

Example 4
Find the value of ‘x’ where the value of ‘y’ is
* Minimum

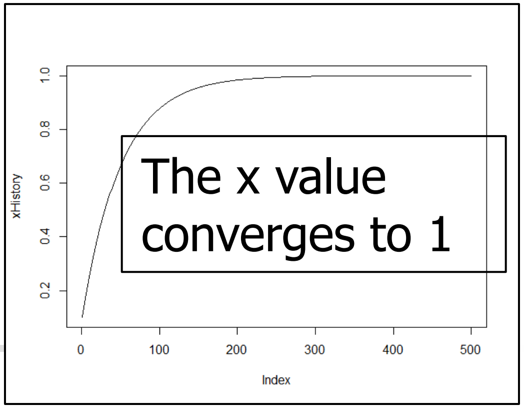


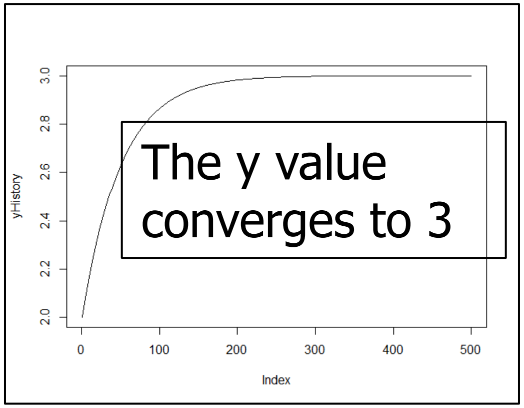
- $z = f(x,y) = x^2 + y^2 - 2x - 6y + 14$
- $\frac{\partial z}{\partial x} = 2x - 2$
- $\frac{\partial z}{\partial y} = 2y - 6$
- To find the value of ‘x’ and ‘y’
  - Where the value of ‘z’ is local minimum
  - Correct answer: x = 1, y = 3
  - $\frac{\partial z}{\partial x} = 2x - 2 = 0; x = 1$
  - $\frac{\partial z}{\partial y} = 2y - 6 = 0; y = 3$

```
> x = seq(0,5,0.1)
> y = seq(0,10,0.1)
> z = function (x,y) { x^2 + y^2 - 2*x - 6*y + 14 }
> dz_dx = function (x1,y1) { 2*x1 - 2 }
> dz_dy = function (x1,y1) { 2*y1 - 6 }
> z<-outer(x,y,z)
> persp(x, y, z)
> contour(z)
```

Example 4: Minimum

- $z = f(x,y) = x^2 + y^2 - 2x - 6y + 14$
- $\frac{\partial z}{\partial x} = 2x - 2; \quad \frac{\partial z}{\partial y} = 2y - 6$
- To find the value of 'x' and 'y'
  - Where the value of 'z' is local minimum
  - Correct answer: x = 1, y = 3
  - $\frac{\partial z}{\partial x} = 2x - 2 = 0; x = 1$
  - $\frac{\partial z}{\partial y} = 2y - 6 = 0; y = 3$






```
> dz_dx = function (x1,y1) { 2*x1 - 2 }
> dz_dy = function (x1,y1) { 2*y1 - 6 }
> xStart = 0.1;  yStart = 2
> learningRate = 0.01;  maxLimit = 500
> xHistory = yHistory = rep(0,maxLimit)
> for ( i in 1:maxLimit)
+ {
+   xHistory[i] = xStart
+   yHistory[i] = yStart
+   dW = dz_dx(xStart,yStart)
+   db = dz_dy(xStart,yStart)
+
+   xStart = xStart - learningRate * dW
+   yStart = yStart - learningRate * db
+
+ }
> plot(xHistory,type='l')
> plot(yHistory,type='l')
```

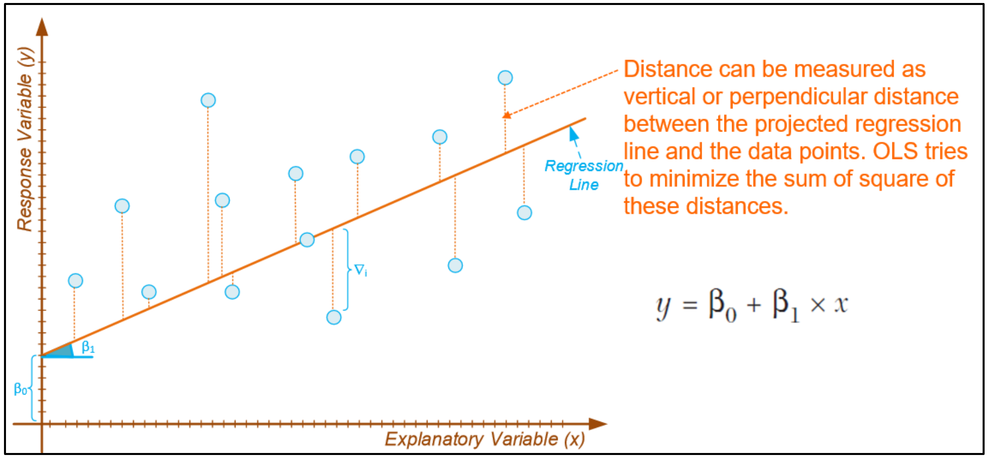
- Function z=f(x,y)
- Gradient Descent Algorithm: Minimum
  - Initialize the value of x and y
  - Learning rate = η
  - While NOT converged:
    - $x^{t+1} \leftarrow x^t - \eta \frac{\partial z}{\partial x} \|_{x^t, y^t}$
    - $y^{t+1} \leftarrow y^t - \eta \frac{\partial z}{\partial y} \|_{x^t, y^t}$

Solving Regression Problem Using Gradient Descent Algorithm – 2 Vraiables

- $y = mx + b$
- $y\ is\ the\ explanatory\ variable$
- $x\ is\ the\ pedictor\ variable$
- $m\ is\ the\ slope\ of\ the\ line$
- $b\ is\ the\ intercept$

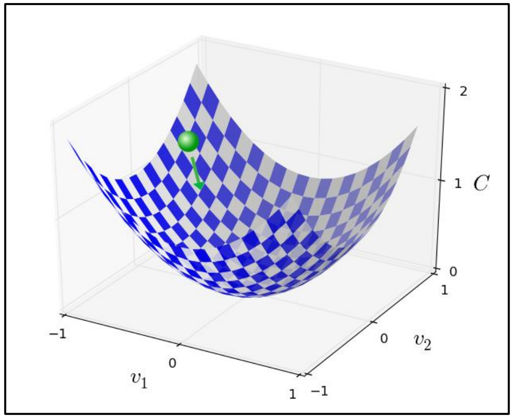
Computing the Regression Line Compute: Intercept and Slope

- Residual = Observed value – Computed Value
- Suppose regression equation is
  - $y = mx + b$
  - $y\ is\ the\ explanatory\ variable$
  - $x\ is\ the\ pedictor\ variable$
  - $m\ is\ the\ slope\ of\ the\ line$
  - $b\ is\ the\ intercept$
- $Residual = \ y_i - (mx_i + b)$
- $Residual^2 = \ (y_i - (mx_i + b))^2$
- $Residuals\ Sum\ of\ Squares = (RSS) = \sum_{i=1}^{N}(y_i - (mx_i + b))^2$



Residual Sum of Squares

- $Residuals\ Sum\ of\ Squares = (RSS) = \sum_{i=1}^{N}(y_i - (mx_i + b))^2$
- To find the minimum point of this function,
  - we will take the partial derivative of RSS with respect to 'm' and 'b' and set that to zero.
- The RSS is a convex function and it has a minimum point



Partial Derivatives of the RSS w.r.t. Intercept and Slope

- $Residuals\ Sum\ of\ Squares = (RSS) = \sum_{i=1}^{N}(y_i - (mx_i + b))^2$
- To find the minimum point of this function,
  - we will take the partial derivative of RSS with respect to 'm' and 'b' and set that to zero.

- $RSS = \sum_{i=1}^{N}(y_i - (mx_i + b))^2$
- $\frac{\partial RSS(m,b)}{\partial b} = \sum_{i=1}^{N}\frac{\partial}{\partial b}(y_i - (mx_i + b))^2$
- $\frac{\partial RSS(m,b)}{\partial b} = -2\sum_{i=1}^{N}(y_i - (mx_i + b))$

- $RSS = \sum_{i=1}^{N}(y_i - (mx_i + b))^2$
- $\frac{\partial RSS(m,b)}{\partial m} = \sum_{i=1}^{N}\frac{\partial}{\partial m}(y_i - (mx_i + b))^2$
- $\frac{\partial RSS(m,b)}{\partial m} = -2\sum_{i=1}^{N}(y_i - (mx_i + b))x_i$

$$\nabla RSS(b,m) = \begin{vmatrix} \frac{\partial RSS(m,b)}{\partial b} \\ \frac{\partial RSS(m,b)}{\partial m} \end{vmatrix} = \begin{vmatrix} -2\sum_{i=1}^{N}(y_i - (mx_i + b)) \\ -2\sum_{i=1}^{N}(y_i - (mx_i + b))x_i \end{vmatrix} = 0$$

Regression
Closed Form Solution

- To Compute ‘m’ and ‘b’
  - SET GRADIENT = 0
- $\nabla\, RSS(b,m) = \begin{vmatrix} \frac{\partial RSS(m,b)}{\partial b} \\ \frac{\partial RSS(m,b)}{\partial m} \end{vmatrix} = \begin{vmatrix} -2\sum_{i=1}^{N}(y_i - (mx_i + b)) \\ -2\sum_{i=1}^{N}(y_i - (mx_i + b))x_i \end{vmatrix} = 0$
- -----------------------------------
- Top term
- $b = \left(\frac{\sum y_i}{N} - m\frac{\sum x_i}{N}\right) = \mu_y - m\mu_x$
- --------------------------------------
- Bottom term
- $m = \frac{\sum y_i x_i - \frac{\sum y_i \sum x_i}{N}}{\sum x_i^2 - \frac{\sum x_i \sum x_i}{N}} = r\frac{\sigma_y}{\sigma_x} = Correlation\,\frac{Std\ Dev\ of\ y}{Std\ Dev\ of\ x}$
- --

Closed Form Solution

$$m = \frac{\sum y_i x_i - \frac{\sum y_i \sum x_i}{N}}{\sum x_i^2 - \frac{\sum x_i \sum x_i}{N}}$$

$$b = \left(\frac{\sum y_i}{N} - m\frac{\sum x_i}{N}\right)$$

	A	B	C	D	E	F	G
1							
2							
3		X	Y		X*Y		X^2
4		0	1		0		0
5		1	3		3		1
6		2	7		14		4
7		3	13		39		9
8		4	21		84		16
9							
10	SUM	10	45		140		30
11	AVERAGE	2	9		28		6
12	StdDev	1.58113883	8.124038405				
13	Correlation	0.97312368					
14							

Regression Equation
$y = 5x - 1$

C16 =E10-(B10*C10/5)

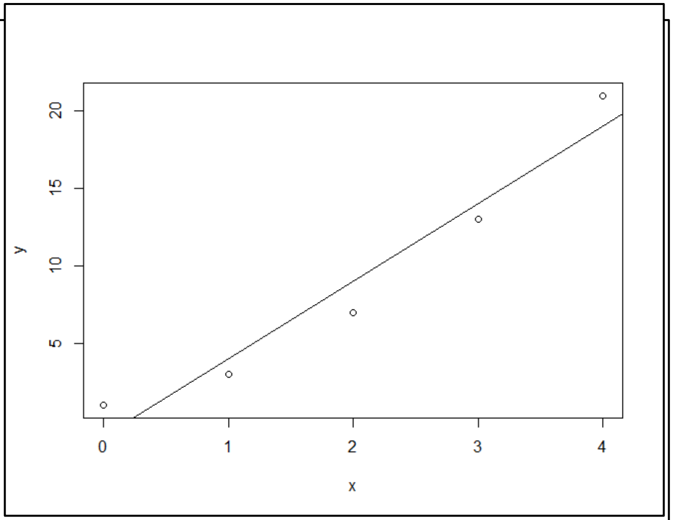
	A	B	C	D	E	F	G
14							
15	Closed Form	Slope : Using SUM					
16		Numerator	50		(Sum of X*Y) - (1/N)*((Sum of X) * (Sum of Y))		
17		Denominator	10		(Sum of X^2) - (1/N)*((Sum of X * Sum of X))		
18		Slope	5				
19							
20		Intercept	-1		(Mean of Y) - slope * (Mean of X)		
21							

Systems of Linear Equations Multi Variables

- Systems of Linear Equations (Variables = m, Observations = n)
  - $y_1 = (b + m_1 x_{11} + m_2 x_{12} + \cdots + m_m x_{1m}) + e_1$
  - $y_2 = (b + m_1 x_{21} + m_2 x_{22} + \cdots + m_m x_{2m}) + e_2$
  - …
  - $y_n = (b + m_1 x_{n1} + m_2 x_{n2} + \cdots + m_m x_{nm}) + e_n$
- $Y = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1m} \\ 1 & x_{12} & \ldots & x_{2m} \\ \ldots & \ldots & \cdots & \ldots \\ 1 & x_{1n} & \ldots & x_{nm} \end{bmatrix} \quad A = \begin{bmatrix} b \\ m_1 \\ \ldots \\ m_m \end{bmatrix} \quad E = \begin{bmatrix} e_1 \\ e_2 \\ \ldots \\ e_n \end{bmatrix}$
- $\boldsymbol{A = (X^T X)^{-1} X^T Y}$

Regression
Using R 'lm' command

```
> x = c(0,1,2,3,4)
> y = c(1,3,7,13,21)
> plot(x,y)
> model = lm(y~x)
> summary(model)

Call:
lm(formula = y ~ x)

Residuals:
 1  2  3  4  5
 2 -1 -2 -1  2

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -1.0000     1.6733  -0.598  0.59220
x             5.0000     0.6831   7.319  0.00527 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.16 on 3 degrees of freedom
Multiple R-squared:  0.947,   Adjusted R-squared:  0.9293
F-statistic: 53.57 on 1 and 3 DF,  p-value: 0.005268

> abline(model)
>
```

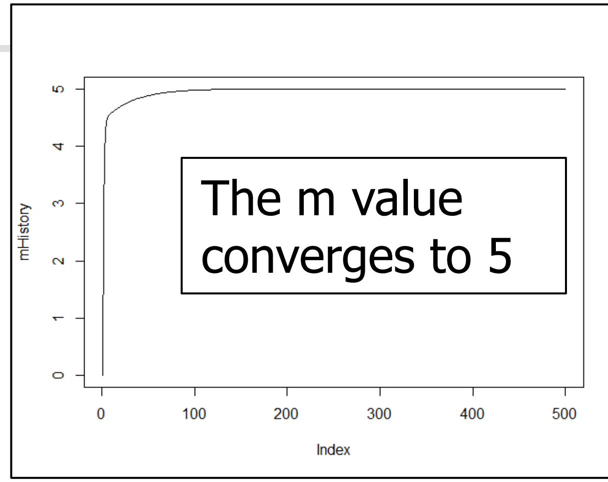


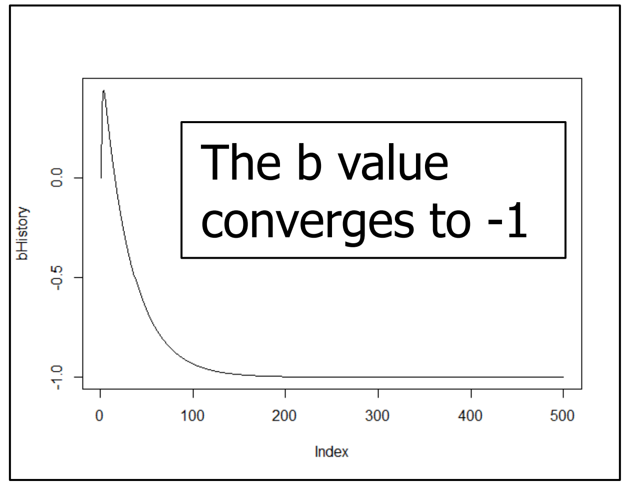
Regression Equation
$y = 5x - 1$

Regression
Gradient Descent Algorithm Approach

Regression Equation
$y = 5x - 1$

```
> dRSS_dm = function (m,b) {-2*sum((y-m*x-b)*x)   }
> dRSS_db = function (m,b) { -2*sum(y-m*x-b) }
> mStart = bStart = 0
> learningRate = 0.01;  maxLimit = 500
> mHistory = bHistory = rep(0,maxLimit)
> for ( i in 1:maxLimit )
+ {
+   mHistory[i] = mStart
+   bHistory[i] = bStart
+
+   dW = dRSS_dm(mStart,bStart)
+   db = dRSS_db(mStart,bStart)
+
+   mStart = mStart - learningRate * dW
+   bStart = bStart - learningRate * db
+ }
> plot(mHistory,type='l')
> plot(bHistory,type='l')
```





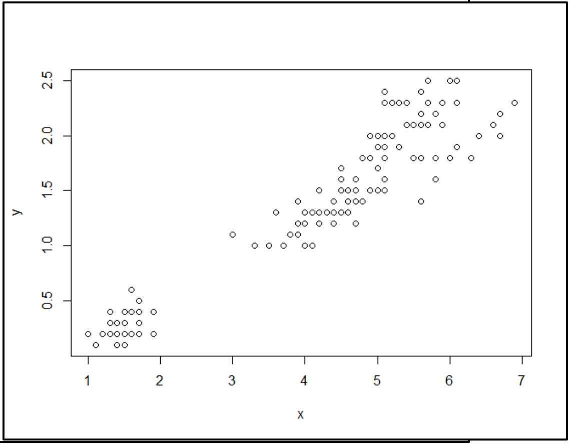
Solving Regression Problem Using Gradient Descent Algorithm – 2 Variables

Iris Dataset

- $y = mx + b$
  - $\boldsymbol{petal.width = m * petal.length + intercept}$
- $y\ is\ the\ explanatory\ variable$
- $x\ is\ the\ pedictor\ variable$
- $m\ is\ the\ slope\ of\ the\ line$
- $b\ is\ the\ intercept$

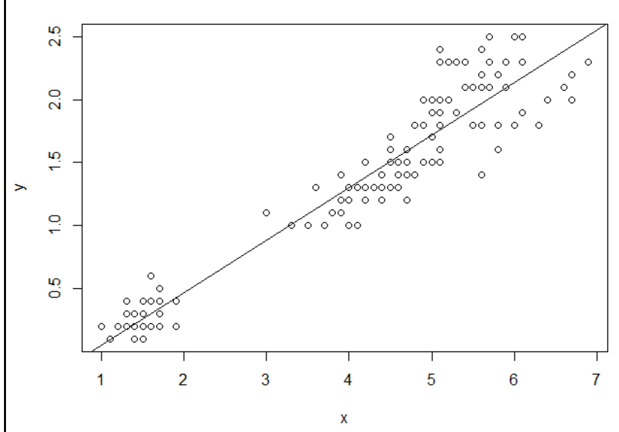
Read the Iris Dataset R Code

```
> data(iris)
> ########################
> dim(iris)
[1] 150   5
> summary(iris)
  Sepal.Length    Sepal.Width     Petal.Length    Petal.Width          Species  
 Min.   :4.300   Min.   :2.000   Min.   :1.000   Min.   :0.100   setosa    :50  
 1st Qu.:5.100   1st Qu.:2.800   1st Qu.:1.600   1st Qu.:0.300   versicolor:50  
 Median :5.800   Median :3.000   Median :4.350   Median :1.300   virginica :50  
 Mean   :5.843   Mean   :3.057   Mean   :3.758   Mean   :1.199                  
 3rd Qu.:6.400   3rd Qu.:3.300   3rd Qu.:5.100   3rd Qu.:1.800                  
 Max.   :7.900   Max.   :4.400   Max.   :6.900   Max.   :2.500                  
> head(iris)
  Sepal.Length Sepal.Width Petal.Length Petal.Width Species
1          5.1         3.5          1.4         0.2  setosa
2          4.9         3.0          1.4         0.2  setosa
3          4.7         3.2          1.3         0.2  setosa
4          4.6         3.1          1.5         0.2  setosa
5          5.0         3.6          1.4         0.2  setosa
6          5.4         3.9          1.7         0.4  setosa
> x = iris$Petal.Length
> y = iris$Petal.Width
> plot(x,y)
```



Regression: R
Using R ‘lm’ command

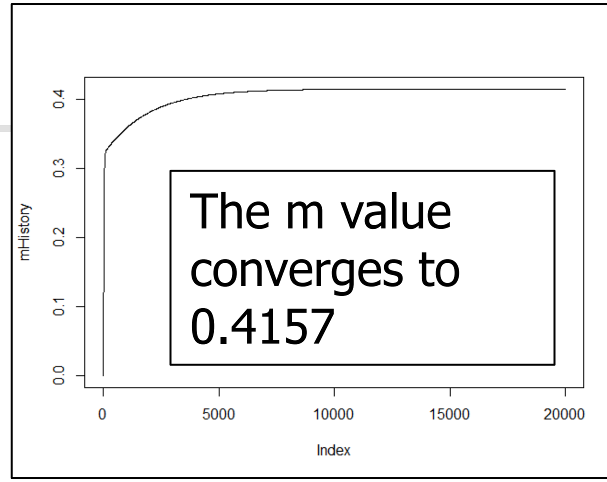
```
> model = lm(y~x)
> summary(model)

Call:
lm(formula = y ~ x)

Residuals:
     Min       1Q   Median       3Q      Max
-0.56515 -0.12358 -0.01898  0.13288  0.64272

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.363076   0.039762  -9.131  4.7e-16 ***
x            0.415755   0.009582  43.387  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.2065 on 148 degrees of freedom
Multiple R-squared:  0.9271,  Adjusted R-squared:  0.9266
F-statistic:  1882 on 1 and 148 DF,  p-value: < 2.2e-16

> abline(model)
```

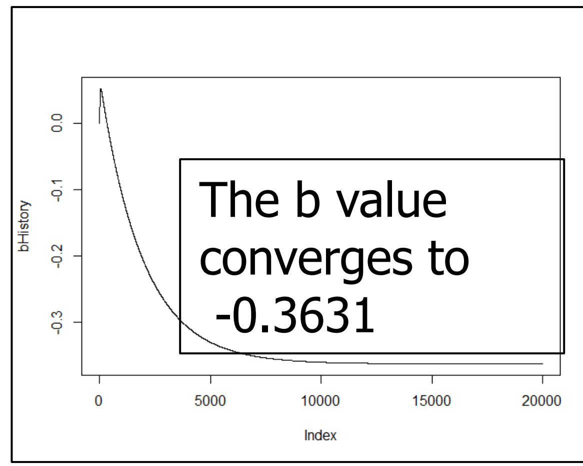


$$Regression\ Equation: R:\ Petal.Width = 0.4157 * Petal.Length - 0.3631$$

Regression: R
Gradient Descent Algorithm Approach

```
> dRSS_dm = function (m,b) {-2*sum((y-m*x-b)*x)   }
> dRSS_db = function (m,b) { -2*sum(y-m*x-b) }
> mStart = bStart = 0
> learningRate = 0.00001;   maxLimit = 20000
> mHistory = bHistory = rep(0,maxLimit)
> for ( i in 1:maxLimit )
+ {
+   mHistory[i] = mStart
+   bHistory[i] = bStart
+
+   dW = dRSS_dm(mStart,bStart)
+   db = dRSS_db(mStart,bStart)
+
+   mStart = mStart - learningRate * dW
+   bStart = bStart - learningRate * db
+
+ }
> plot(mHistory,type='l')
> plot(bHistory,type='l')
> mHistory[maxLimit]
[1] 0.4157522
> bHistory[maxLimit]
[1] -0.3630608
```

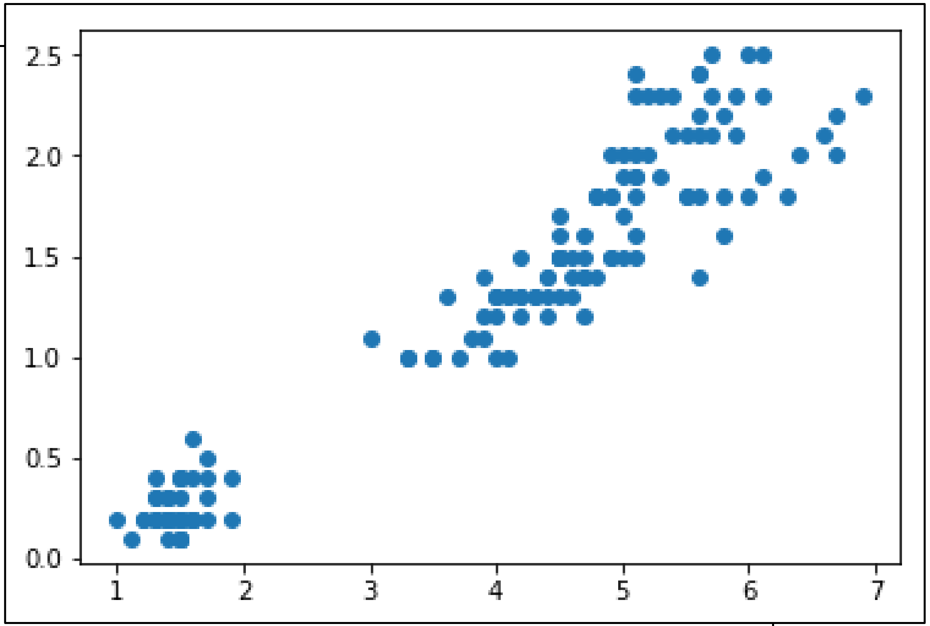




$$Regression\ Equation: R:\ Petal.Width = 0.4157 * Petal.Length - 0.3631$$
$$Regression\ Eq: R\ Grad\ Desc:\ Petal.Width = 0.4157 * Petal.Length - 0.3631$$

Read the Iris Dataset Python Code

```
####################################
# Load Libraries
#
from sklearn import linear_model
from sklearn import datasets
import matplotlib.pyplot as plt
#####################################
# 2. Read the Dataset
#
iris   = datasets.load_iris()

features = iris["data"]

petalLength = features[:,2]
petalLength[0:5]
Out[13]: array([ 1.4,  1.4,  1.3,  1.5,  1.4])

petalWidth = features[:,3]
petalWidth[0:5]
Out[15]: array([ 0.2,  0.2,  0.2,  0.2,  0.2])

plt.plot(petalLength,petalWidth,'o')
Out[16]: [<matplotlib.lines.Line2D at 0x16b604b01d0>]
```



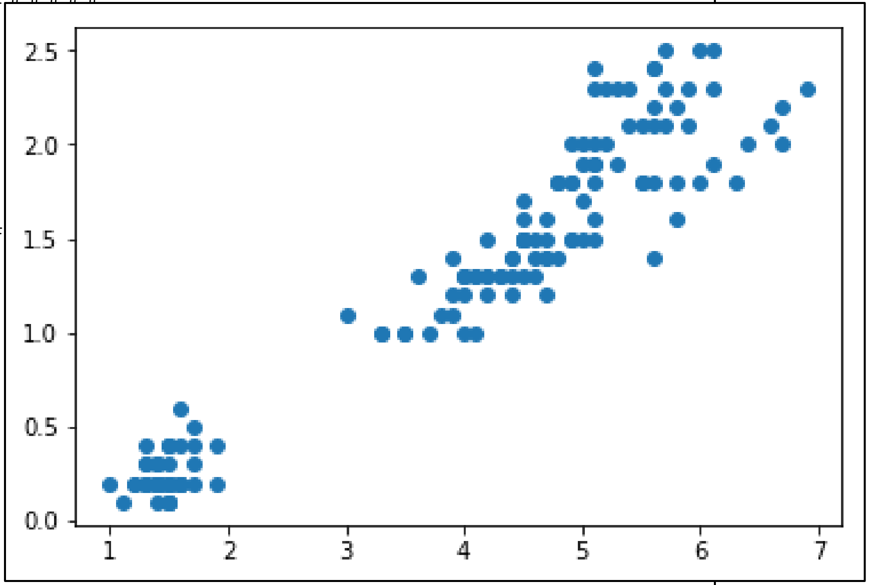
Regression
Using Python 'Scikit-Learn' Library

```
###########################################
# 3. Compute the regression equation using Scikit-Learn
#
petalLength = petalLength.reshape(-1,1)

petalWidth = petalWidth.reshape(-1,1)

linreg = linear_model.LinearRegression()

linreg.fit(petalLength, petalWidth)
Out[23]: LinearRegression(copy_X=True, fit_intercept=True, n_jobs=1,
normalize=False)

print (linreg.intercept_)
[-0.36651405]

print (linreg.coef_)
[[ 0.41641913]]
```

$$Regression\ Equation: R:\ Petal.Width = 0.4157 * Petal.Length - 0.3631$$
$$Regression\ Eq: R\ Grad\ Desc:\ Petal.Width = 0.4157 * Petal.Length - 0.3631$$
$$Regression\ Equation: Scikit:\ Petal.Width = 0.4164 * Petal.Length - 0.3665$$

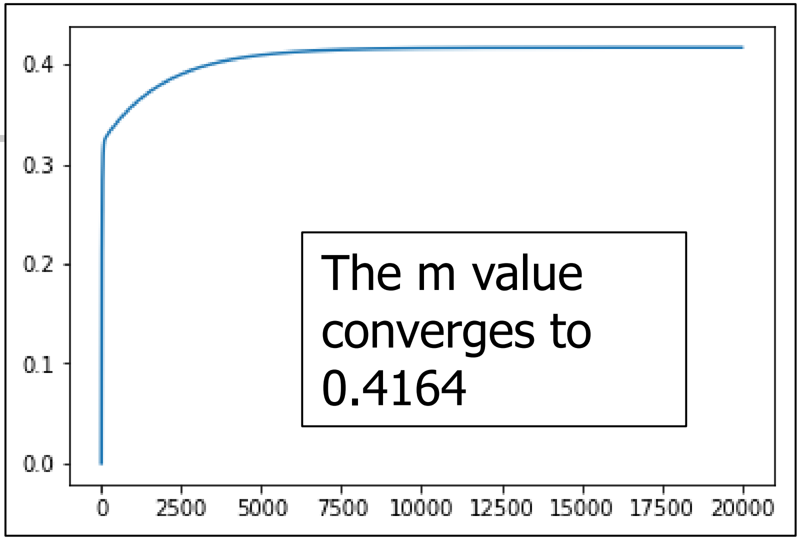
Regression: Python
Gradient Descent Algorithm Approach

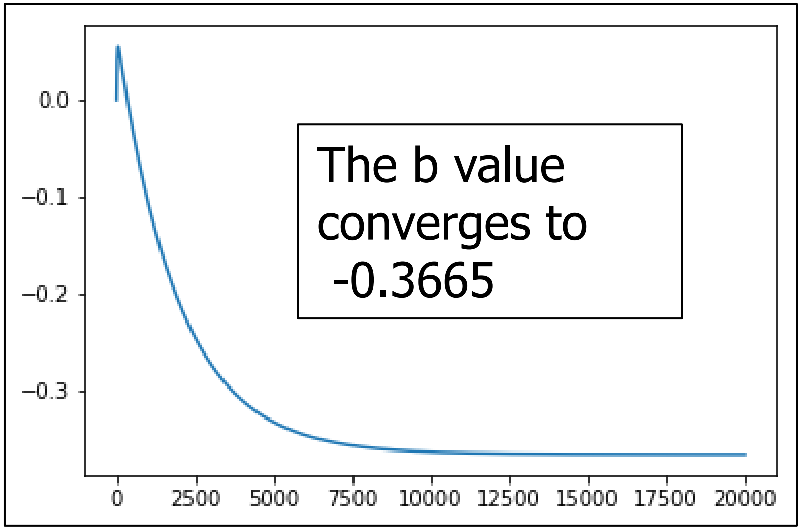
```
#####################################################
# 1. Load the libraries
#
import numpy as np
import matplotlib.pyplot as plt
from sklearn import datasets
############################################
# 2. Read the Dataset
#
iris    = datasets.load_iris()
features = iris["data"]

x = petalLength = features[:,2]
x[0:5]
Out[13]: array([ 1.4,  1.4,  1.3,  1.5,  1.4])

y = petalWidth = features[:,3]
y[0:5]
Out[15]: array([ 0.2,  0.2,  0.2,  0.2,  0.2])

plt.plot(x,y,'o')
```



Regression: Python
Gradient Descent Algorithm Approach

```
def dRSS_dm(m,b):
    return(-2*sum((y-m*x-b)*x))

def dRSS_db(m,b):
    return(-2*sum((y-m*x-b)))

mStart = 0
bStart = 0
learning_rate = 0.00001
maxLimit = 20000
mHistory = np.zeros(maxLimit)
bHistory = np.zeros(maxLimit)

for i in range(maxLimit):
    mHistory[i] = mStart
    bHistory[i] = bStart
    #print(mHistory[i], bHistory[i])

    dW = dRSS_dm(mStart,bStart)
    db = dRSS_db(mStart,bStart)

    mStart = mStart - learning_rate * dW
    bStart = bStart - learning_rate * db

print("mHistory=",mHistory[maxLimit-1])
mHistory= 0.416415833891

print("bHistory=",bHistory[maxLimit-1])
bHistory= -0.366499084269
```





Final Result

$$Regression\ Eq: R:\ Petal.Width = 0.4157 * Petal.Length - 0.3631$$
$$Regression\ Eq: R\ Grad\ Desc:\ Petal.Width = 0.4157 * Petal.Length - 0.3631$$

$$Regression\ Eq: Scikit:\ Petal.Width = 0.4164 * Petal.Length - 0.3665$$
$$Regression\ Eq: Python\ Grad\ Desc:\ Petal.Width = 0.4164 * Petal.Length - 0.3665$$

Solving Regression Problem Using Gradient Descent Algorithm

Multiple Predictor Variables

- $y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n + b$
- $y\ is\ the\ explanatory\ variable$
- $x_1, x_2, \ldots x_n\ are\ the\ pedictor\ variables$
- $m_1, m_2, \ldots\ m_n\ are\ \ the\ coefficients\ of\ the\ predictor\ variables$
- $b\ is\ the\ intercept$

Compute
Residual Sum of Squares

- Residual = Observed value – Computed Value
- Suppose regression equation is
  - $y = m_1x_1 + m_2x_2 + \cdots + m_nx_n + b$
  - $y\ is\ the\ explanatory\ variable$
  - $x_1, x_2, \ldots x_n\ are\ the\ pedictor\ variables$
  - $m_1, m_2, \ldots\ m_n\ are\ \ the\ coefficients\ of\ the\ predictor\ variables$
  - $b\ is\ the\ intercept$
- $Residual = \ y_i - (m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b)$
- $Residual^2 = (y_i - (m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b))^2$
- $Residuals\ Sum\ of\ Squares = (RSS) = \ \sum_{i=1}^{N}(y_i - (m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b))^2$

Partial Derivatives of the RSS w.r.t. Intercept and Slope

- $RSS = \sum_{i=1}^{N}(y_i - (m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b))^2$

- To find the minimum point of this function,
  - we will take the partial derivative of RSS with respect to
    - $m_1, m_2, \ldots\ m_n$ and
    - 'b' and set that to zero.

- $RSS = \sum_{i=1}^{N}(y_i - (m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b))^2$
- $\frac{\partial RSS}{\partial b} = \sum_{i=1}^{N}\frac{\partial}{\partial b}(y_i - (m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b))^2$
- $\frac{\partial RSS(m,b)}{\partial b} = -2\sum_{i=1}^{N}(y_i - m_1x_{1i} - m_2x_{2i} + \cdots - m_nx_{ni} - b)$

Partial Derivatives of the RSS w.r.t. Intercept and Slope

- $RSS = \sum_{i=1}^{N}(y_i-(m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b))^2$
- $\frac{\partial RSS}{\partial m_1} = \sum_{i=1}^{N}\frac{\partial}{\partial m_1}(\,y_i - (m_1x_{1i} + m_2x_{2i} + \cdots + m_nx_{ni} + b))^2$
- --
- $\frac{\partial RSS}{\partial m_1} = -2\sum_{i=1}^{N}(y_i - m_1x_{1i} - m_2x_{2i} - \cdots + m_nx_{ni} - b)x_{1i}$
- $\frac{\partial RSS}{\partial m_2} = -2\sum_{i=1}^{N}(y_i - m_1x_{1i} - m_2x_{2i} - \cdots + m_nx_{ni} - b)x_{2i}$
- …
- …
- $\frac{\partial RSS}{\partial m_n} = -2\sum_{i=1}^{N}(y_i - m_1x_{1i} - m_2x_{2i} - \cdots + m_nx_{ni} - b)x_{ni}$

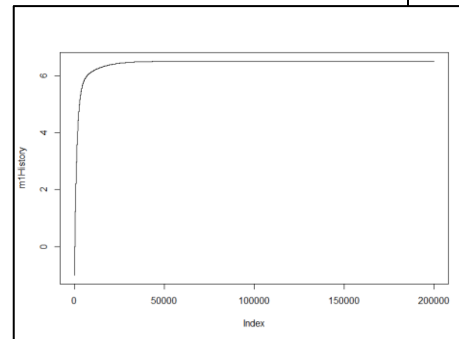
Data Set
Regression Using 'lm' function

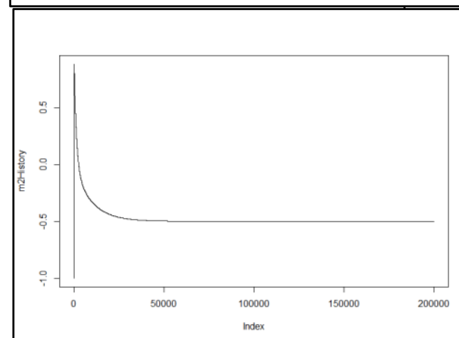
```
> ############################################
> x1 = c(0,1,2,3,4)
> x2 = c(5,10,15,20,15)
> y = c(1,3,7,13,21)
> model = lm(y~x1+x2)
> summary(model)

Call:
lm(formula = y ~ x1 + x2)

Residuals:
        1         2         3         4         5 
 1.00e+00 -1.00e+00 -1.00e+00  1.00e+00 -2.22e-16 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept)   2.5000     1.9105   1.309    0.321  
x1            6.5000     0.8062   8.062    0.015 *
x2           -0.5000     0.2236  -2.236    0.155  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.414 on 2 degrees of freedom
Multiple R-squared:  0.9848,  Adjusted R-squared:  0.9697 
F-statistic:    65 on 2 and 2 DF,  p-value: 0.01515
```

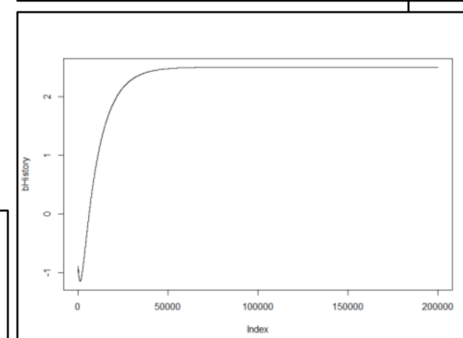
$$\textit{Regression Using 'lm' function:}$$
$$y = 6.5x_1 - 0.50x_2 + 2.5$$

Regression Using Gradient Descent

```
> dRSS_dm1 = function (m1,m2,b) {-2*sum((y - m1*x1 - m2*x2 - b)*x1)   }
> dRSS_dm2 = function (m1,m2,b) {-2*sum((y - m1*x1 - m2*x2 - b)*x2)   }
> dRSS_db  = function (m1,m2,b) {-2*sum (y - m1*x1 - m2*x2 - b)  }
> m1Start = m2Start = bStart = -1
> learningRate = 0.0001;    maxLimit = 200000
> m1History = m2History = bHistory = rep(0,maxLimit)
> for ( i in 1:maxLimit )
+ {
+   m1History[i] = m1Start
+   m2History[i] = m2Start
+   bHistory[i] = bStart
+
+   dW1 = dRSS_dm1(m1Start,m2Start,bStart)
+   dW2 = dRSS_dm2(m1Start,m2Start,bStart)
+   db = dRSS_db(m1Start,m2Start,bStart)
+
+   m1Start = m1Start - learningRate * dW1
+   m2Start = m2Start - learningRate * dW2
+   bStart = bStart - learningRate * db
+
+ }
> plot(m1History,type='l')
> plot(m2History,type='l')
> plot(bHistory,type='l')
> m1History[maxLimit]
[1] 6.5
> m2History[maxLimit]
[1] -0.5
> bHistory[maxLimit]
[1] 2.5
```







Regression Using Gradient Descent

$$y = 6.5x_1 - 0.50x_2 + 2.5$$

Result

$$Regression\ Using\ \text{'}lm\text{'}\ function: \qquad y = 6.5x_1 - 0.50x_2 + 2.5$$

$$Regression\ Using\ Gradient\ Descent: \qquad y = 6.5x_1 - 0.50x_2 + 2.5$$

Solving Regression Problem Using Gradient Descent Algorithm

Multiple Predictor Variables

Iris Dataset

- $y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n + b$
- $y \ is \ the \ explanatory \ variable$
- $x_1, x_2, \ldots x_n \ are \ the \ pedictor \ variables$
- $m_1, m_2, \ldots \ m_n \ are \ \ the \ coefficients \ of \ the \ predictor \ variables$
- $b \ is \ the \ intercept$

Iris Data Set

```
> ##########################################################
> # Gradient Decent:
> # 2 predictor variables + 1 response variable
> #
> rm(list=ls(all=TRUE))
> ####################################
> # Check out the iris dataset
> #
> data(iris)
> #########################
> dim(iris)
[1] 150   5
> summary(iris)
  Sepal.Length    Sepal.Width     Petal.Length    Petal.Width          Species  
 Min.   :4.300   Min.   :2.000   Min.   :1.000   Min.   :0.100   setosa    :50  
 1st Qu.:5.100   1st Qu.:2.800   1st Qu.:1.600   1st Qu.:0.300   versicolor:50  
 Median :5.800   Median :3.000   Median :4.350   Median :1.300   virginica :50  
 Mean   :5.843   Mean   :3.057   Mean   :3.758   Mean   :1.199                  
 3rd Qu.:6.400   3rd Qu.:3.300   3rd Qu.:5.100   3rd Qu.:1.800                  
 Max.   :7.900   Max.   :4.400   Max.   :6.900   Max.   :2.500                  
> head(iris)
  Sepal.Length Sepal.Width Petal.Length Petal.Width Species
1          5.1         3.5          1.4         0.2  setosa
2          4.9         3.0          1.4         0.2  setosa
3          4.7         3.2          1.3         0.2  setosa
4          4.6         3.1          1.5         0.2  setosa
5          5.0         3.6          1.4         0.2  setosa
6          5.4         3.9          1.7         0.4  setosa
```

$$\textit{Regression Using 'lm' function:}$$
$$Petal.Width = 0.449\ Petal.Length - 0.082\ Sepal.Length - 0.008$$

Regression Using ‘lm’ function

```
> x1 = iris$Petal.Length
> x2 = iris$Sepal.Length
> y = iris$Petal.Width
> model = lm(y~x1+x2)
> summary(model)

Call:
lm(formula = y ~ x1 + x2)

Residuals:
     Min       1Q   Median       3Q      Max
-0.60598 -0.12560 -0.02049  0.11616  0.59404

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.008996   0.182097  -0.049   0.9607
x1           0.449376   0.019365  23.205   <2e-16 ***
x2          -0.082218   0.041283  -1.992   0.0483 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.2044 on 147 degrees of freedom
Multiple R-squared:  0.929,   Adjusted R-squared:  0.9281
F-statistic: 962.1 on 2 and 147 DF,  p-value: < 2.2e-16
```

Regression Using Gradient Descent

```
> dRSS_dm1 = function (m1,m2,b) {-2*sum((y - m1*x1 - m2*x2 - b)*x1)  }
> dRSS_dm2 = function (m1,m2,b) {-2*sum((y - m1*x1 - m2*x2 - b)*x2)  }
> dRSS_db  = function (m1,m2,b) {-2*sum (y - m1*x1 - m2*x2 - b) }
> m1Start = m2Start = bStart = -1
> learningRate = 0.0001;   maxLimit = 200000
> m1History = m2History = bHistory = rep(0,maxLimit)
> for ( i in 1:maxLimit )
+ {
+   m1History[i] = m1Start
+   m2History[i] = m2Start
+   bHistory[i] = bStart
+
+   dW1 = dRSS_dm1(m1Start,m2Start,bStart)
+   dW2 = dRSS_dm2(m1Start,m2Start,bStart)
+   db = dRSS_db(m1Start,m2Start,bStart)
+
+   m1Start = m1Start - learningRate * dW1
+   m2Start = m2Start - learningRate * dW2
+   bStart = bStart - learningRate * db
+
+ }
> plot(m1History,type='l')
> plot(m2History,type='l')
> plot(bHistory,type='l')
> m1History[maxLimit]
[1] 0.4493761
> m2History[maxLimit]
[1] -0.08221782
> bHistory[maxLimit]
[1] -0.008995973
```

$$Regression\ Using\ 'lm'\ function:$$
$$Petal.Width = 0.449\ Petal.Length - 0.082\ Sepal.Length - 0.008$$

$$Regression\ Using\ Gradient\ Descent:$$
$$Petal.Width = 0.449\ Petal.Length - 0.082\ Sepal.Length - 0.008$$

Other Optimization Algorithms

- Stochastic Gradient Descent
- Momentum
- Nesterov Momentum
- AdaGrad
- RMS Prop
- Adam: Adaptive Momentum Estimation

Summary

- What is a Gradient?
- What is Gradient Descent Algorithm?
- Gradient Descent Algorithm
  - Minimum of a 2-Variable Function
  - Minimum & Maximum of a 2-Variable Function
  - Minimum of 3 Variable Function
  - Solving Regression problem – 2 Variables
  - Solving Regression problem – Iris Dataset (2 Variables)
  - Solving Regression problem – Multiple Variables
  - Solving Regression problem – Iris (Multiple Variables)