

# Assignment 3

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Download latex-tikz codes from

<https://github.com/diya-goyal-29/AI1103/blob/main/Assignment%203/Assignment%203.tex>

*Proof.*

1 CSIR-UGC-NET-EXAM-JUNE-2015 QUESTION 60  
:

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables having an exponential distribution with mean  $\frac{1}{\lambda}$ . Let  $S_n = X_1 + X_2 + \dots + X_n$  and  $N = \inf\{n \geq 1 : S_n > 1\}$ . Then  $\text{Var}(N)$  is

- 1) 1
- 2)  $\lambda$
- 3)  $\lambda^2$
- 4)  $\infty$

2 SOLUTION :

$$\text{Var}(N) = E[N^2] - E[N]^2$$

Replacing  $n$  with  $n+1$

**Definition 2.1** (infimum).  $N$  is the greatest lower bound of the subset  $S$  of  $T$  (where  $T$  is the set of all natural numbers and  $S$  is the set of all natural numbers such that  $S_n \geq 1$ ).

$(N = \inf\{n \geq 1 : S_n > 1\})$

The event  $\{N \leq n\}$  implies  $\{S_{n+1} > 1\}$  (or equivalently,  $\{N > n\}$  implies  $\{S_{n+1} \leq 1\}$ ).

**Lemma 2.1. Tail Sum formula :** A random variable  $X$  which takes values only in  $N$ , then  $E[X] = \sum_{k=1}^{\infty} \Pr(X)$

$$E[X] = \sum_{k=1}^{\infty} k \Pr(X = k) \quad (2.0.1)$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^k \Pr(X = k) \quad (2.0.2)$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \Pr(X = k) \quad (2.0.3)$$

$$= \sum_{n=1}^{\infty} \Pr(X \geq n) \quad (2.0.4)$$

□

By the tail-sum formula, we can thus get the expectation of  $N$  as

$$E[N] = \sum_{n=0}^{\infty} \Pr(N > n) = \sum_{n=0}^{\infty} \Pr(S_n \leq 1) \quad (2.0.5)$$

$S_n$  follows Erlang distribution.

$$\text{P.D.F of } S_n = f(x)_{S_n} = \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} \quad (2.0.6)$$

$$\Pr(S_n \leq 1) = \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.7)$$

**P.D.F of Erlang distribution :**

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad (2.0.8)$$

where,

$k$  is the shape

$\lambda$  is the rate

**Proof :** We can use mathematical induction to prove this

$$S_1 = X_1 \quad (2.0.9)$$

$$f_{S_1}(x) = \lambda e^{-\lambda x} \quad (2.0.10)$$

Basic step proved.

Assuming  $S_k$  to be true we need to prove that  $S_{k+1}$  is true.

$$S_{k+1} = X_1 + X_2 + X_3 + \cdots + X_{k+1} \quad (2.0.11)$$

$$= S_k + X_{k+1} \quad (2.0.12)$$

$$f_{S_{k+1}}(x) = f_{S_k} + X_{k+1} \quad (2.0.13)$$

$$= \int_0^x \lambda e^{-\lambda(x-t)} \cdot \lambda^k e^{-\lambda t} \frac{t^{k-1}}{(k-1)!} \quad (2.0.14)$$

$$= \lambda^{k+1} e^{-\lambda x} \int_0^x \frac{t^{k-1}}{(k-1)!} \quad (2.0.15)$$

$$= \lambda^{k+1} e^{-\lambda x} \frac{x^k}{(k)!} \quad (2.0.16)$$

Substituting the value

$$E[N] = \sum_{n=0}^{\infty} \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.17)$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.18)$$

$$e^{x\lambda} = \sum_{n=0}^{\infty} \frac{(x\lambda)^n}{(n)!} \quad (2.0.19)$$

$$E[N] = \lambda \int_0^1 e^{\lambda x} e^{-\lambda x} dx \quad (2.0.20)$$

$$= \lambda \int_0^1 dx \quad (2.0.21)$$

$$= \lambda \quad (2.0.22)$$

By tail-sum formula the expectation of  $N^2$  is

$$E[N^2] = \sum_{n=0}^{\infty} Pr(N^2 \geq n) \quad (2.0.23)$$

$$= \sum_{n=0}^{\infty} [(n+1)^2 - n^2] Pr(N \geq n) \quad (2.0.24)$$

$$= \sum_{n=0}^{\infty} [2n+1] Pr(S_{n+1} \leq 1) \quad (2.0.25)$$

$$= \sum_{n=0}^{\infty} \int_0^1 [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.26)$$

$$= \int_0^1 \sum_{n=0}^{\infty} [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.27)$$

$$= \int_0^1 2\lambda^2 x e^{\lambda x} e^{-\lambda x} + \lambda e^{\lambda x} e^{-\lambda x} dx \quad (2.0.28)$$

$$= \lambda^2 + \lambda \quad (2.0.29)$$

$$Var(N) = E[N^2] - E[N]^2 \quad (2.0.30)$$

$$= \lambda^2 + \lambda - (\lambda)^2 \quad (2.0.31)$$

$$= \lambda \quad (2.0.32)$$

Thus Option 2 is correct.