

Assignment 3

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Download latex-tikz codes from

<https://github.com/diya-goyal-29/AI1103/blob/main/Assignment%203/Assignment%203.tex>

CSIR-UGC-NET-Exam-JUNE-2015

Question 60 :

Let X_1, X_2, \dots, X_n be independent and identically distributed random variables having an exponential distribution with mean $\frac{1}{\lambda}$. Let $S_n = X_1 + X_2 + \dots + X_n$ and $N = \inf\{n \geq 1 : S_n > 1\}$. Then $\text{Var}(N)$ is

- 1) 1
- 2) λ
- 3) λ^2
- 4) ∞

Solution :

$$\text{Var}(N) = E[N^2] - E[N]^2$$

Replacing n with $n+1$

The event $\{N \leq n\}$ is the same as $\{S_{n+1} > 1\}$ (or equivalently, $\{N > n\}$ is the same as $\{S_{n+1} \leq 1\}$).

By the tail-sum formula, we can thus get the expectation of N as

$$E[N] = \sum_{n=0}^{\infty} P(N > n) = \sum_{n=0}^{\infty} P(S_n \leq 1) \quad (0.0.1)$$

S_n follows Erlang distribution.

$$\text{P.D.F of } S_n = \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} \quad (0.0.2)$$

$$P(S_n \leq 1) = \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (0.0.3)$$

Substituting the value in equation 0.0.1

$$E[N] = \sum_{n=0}^{\infty} \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (0.0.4)$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (0.0.5)$$

$$e^{x\lambda} = \sum_{n=0}^{\infty} \frac{(x\lambda)^n}{(n)!} \quad (0.0.6)$$

$$E[N] = \lambda \int_0^1 e^{\lambda x} e^{-\lambda x} dx \quad (0.0.7)$$

$$= \lambda \int_0^1 dx \quad (0.0.8)$$

$$= \lambda \quad (0.0.9)$$

$$E[N^2] = \sum_{n=0}^{\infty} P(N^2 \geq n) \quad (0.0.10)$$

$$= \sum_{n=0}^{\infty} [(n+1)^2 - n^2] P(N \geq n) \quad (0.0.11)$$

$$= \sum_{n=0}^{\infty} [2n+1] P(S_{n+1} \leq 1) \quad (0.0.12)$$

$$= \sum_{n=0}^{\infty} \int_0^1 [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (0.0.13)$$

$$= \int_0^1 \sum_{n=0}^{\infty} [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (0.0.14)$$

$$= \int_0^1 2\lambda^2 x e^{\lambda x} e^{-\lambda x} + \lambda e^{\lambda x} e^{-\lambda x} dx \quad (0.0.15)$$

$$= \lambda^2 + \lambda \quad (0.0.16)$$

$$Var(N) = E[N^2] - E[N]^2 \quad (0.0.17)$$

$$= \lambda^2 + \lambda - (\lambda)^2 \quad (0.0.18)$$

$$= \lambda \quad (0.0.19)$$

Thus Option 2 is correct.