Assignment 3

Diya Goyal
Roll no. CS20BTECH11014

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https://github.com/diya−goyal−29/AI1103/blob/main/Assignment%203/Assignment%203.tex

1 CSIR-UGC-NET-Exam-JUNE-2015 Question 60 :

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables having an exponential distribution with mean $\frac{1}{\lambda}$.Let $S_n = X_1 + X_2 + \cdots + X_n$ and $N = inf\{n \geq 1 : S_n > 1\}$. Then $Var(N)$ is

1) 1
2) λ
3) λ^2
4) ∞

2 Solution :

$Var(N) = E[N^2] - E[N]^2$
Replacing n with n+1

Definition 2.1 (infimum)**.** N is the greatest lower bound of the subset S of T (where T is the set of all natural numbers and S is the set of all natural numbers such that $S_n \geq 1$).
($N = inf\{n \geq 1 : S_n > 1\}$)
The event $\{N \leq n\}$ implies $\{S_{n+1} > 1\}$ (or equivalently, $\{N > n\}$ implies $\{S_{n+1} \leq 1\}$).

Lemma 2.1. ***Tail Sum formula :*** *A random variable X which takes values only in N, then* $E[X] = \sum_{k=1}^{\infty} Pr(X)$

Proof.

$$E[X] = \sum_{k=1}^{\infty} kPr(X = k) \tag{2.0.1}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{k} Pr(X = k) \tag{2.0.2}$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} Pr(X = k) \tag{2.0.3}$$

$$= \sum_{n=1}^{\infty} Pr(X \geq n) \tag{2.0.4}$$

□

By the tail-sum formula, we can thus get the expectation of N as

$$E[N] = \sum_{n=0}^{\infty} Pr(N > n) = \sum_{n=0}^{\infty} Pr(S_n \leq 1) \tag{2.0.5}$$

S_n follows Erlang distribution.

$$\text{P.D.F of } S_n = f(x)_{S_n} = \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} \tag{2.0.6}$$

$$Pr(S_n \leq 1) = \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.7}$$

P.D.F of Erlang distribution :

Definition 2.2.

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \tag{2.0.8}$$

where,
k is the shape
λ is the rate
To prove the above equation we can make use of mathematical induction.

Proof.

$$S_1 = X_1 \tag{2.0.9}$$
$$f_{S_1}(x) = \lambda e^{[-\lambda x]} \tag{2.0.10}$$

Basic step proved.
Assuming S_k to be true we need to prove that S_{k+1} is true.

$$S_{k+1} = X_1 + X_2 + X_3 + \cdots + X_{k+1} \tag{2.0.11}$$
$$= S_k + X_{k+1} \tag{2.0.12}$$
$$f_{S_{k+1}}(x) = f_{S_k} + X_{k+1} \tag{2.0.13}$$
$$= \int_0^x \lambda e^{-\lambda(x-t)} \cdot \lambda^k e^{-\lambda t} \frac{t^{k-1}}{(k-1)!} \tag{2.0.14}$$
$$= \lambda^{k+1} e^{-\lambda x} \int_0^x \frac{t^{k-1}}{(k-1)!} \tag{2.0.15}$$
$$= \lambda^{k+1} e^{-\lambda x} \frac{x^k}{(k)!} \tag{2.0.16}$$

□

Substituting the value

$$E[N] = \sum_{n=0}^{\infty} \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.17}$$
$$= \int_0^1 \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.18}$$
$$e^{x\lambda} = \sum_{n=0}^{\infty} \frac{(x\lambda)^n}{(n)!} \tag{2.0.19}$$
$$E[N] = \lambda \int_0^1 e^{\lambda x} e^{-\lambda x} dx \tag{2.0.20}$$
$$= \lambda \int_0^1 dx \tag{2.0.21}$$
$$= \lambda \tag{2.0.22}$$

By tail-sum formula the expectation of N^2 is

$$E[N^2] = \sum_{n=0}^{\infty} Pr(N^2 \geq n) \tag{2.0.23}$$
$$= \sum_{n=0}^{\infty} [(n+1)^2 - n^2] Pr(N \geq n) \tag{2.0.24}$$
$$= \sum_{n=0}^{\infty} [2n+1] Pr(S_{n+1} \leq 1) \tag{2.0.25}$$
$$= \sum_{n=0}^{\infty} \int_0^1 [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.26}$$
$$= \int_0^1 \sum_{n=0}^{\infty} [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.27}$$
$$= \int_0^1 2\lambda^2 x e^{\lambda x} e^{-\lambda x} + \lambda e^{\lambda x} e^{-\lambda x} dx \tag{2.0.28}$$
$$= \lambda^2 + \lambda \tag{2.0.29}$$

$$Var(N) = E[N^2] - E[N]^2 \tag{2.0.30}$$
$$= \lambda^2 + \lambda - (\lambda)^2 \tag{2.0.31}$$
$$= \lambda \tag{2.0.32}$$

Thus Option 2 is correct.