Assignment 3

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Download latex-tikz codes from

https://github.com/diya−goyal−29/AI1103/blob/main/Assignment%203/Assignment%203.tex

1 CSIR-UGC-NET-Exam-JUNE-2015 Question 60 :

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having an exponential distribution with mean $\frac{1}{\lambda}$.Let $S_n = X_1 + X_2 + \cdots + X_n$ and $N = inf\{n \geq 1 : S_n > 1\}$. Then $Var(N)$ is

1) 1
2) λ
3) λ^2
4) ∞

2 Solution :

$Var(N) = E[N^2] - E[N]^2$
Replacing n with n+1
The event $\{N \leq n\}$ is the same as $\{S_{n+1} > 1\}$ (or equivalently, $\{N > n\}$ is the same as $\{S_{n+1} \geq 1\}$).
Tail Sum formula : A random variable X which takes values only in N, then $E[X] = \sum_{k=1}^{\infty} Pr(X)$
Proof :

$$E[X] = \sum_{x=1}^{\infty} xPr(X = x) \tag{2.0.1}$$
$$= \sum_{x=1}^{\infty} \sum_{k=1}^{x} Pr(X = x) \tag{2.0.2}$$
$$= \sum_{k=1}^{\infty} \sum_{x=k}^{\infty} Pr(X = x) \tag{2.0.3}$$
$$= \sum_{k=1}^{\infty} Pr(X \geq k) \tag{2.0.4}$$

By the tail-sum formula, we can thus get the expectation of N as

$$E[N] = \sum_{n=0}^{\infty} Pr(N > n) = \sum_{n=0}^{\infty} Pr(S_n \leq 1) \tag{2.0.5}$$

S_n follows Erlang distribution.

$$\text{P.D.F of } S_n = \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} \tag{2.0.6}$$
$$Pr(S_n \leq 1) = \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.7}$$

Substituting the value

$$E[N] = \sum_{n=0}^{\infty} \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.8}$$
$$= \int_0^1 \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.9}$$
$$e^{x\lambda} = \sum_{n=0}^{\infty} \frac{(x\lambda)^n}{(n)!} \tag{2.0.10}$$
$$E[N] = \lambda \int_0^1 e^{\lambda x} e^{-\lambda x} dx \tag{2.0.11}$$
$$= \lambda \int_0^1 dx \tag{2.0.12}$$
$$= \lambda \tag{2.0.13}$$

By tail-sum formula the expectation of N^2 is

$$E[N^2] = \sum_{n=0}^{\infty} Pr(N^2 \geq n) \tag{2.0.14}$$

$$= \sum_{n=0}^{\infty} [(n+1)^2 - n^2] Pr(N \geq n) \tag{2.0.15}$$

$$= \sum_{n=0}^{\infty} [2n+1] Pr(S_{n+1} \leq 1) \tag{2.0.16}$$

$$= \sum_{n=0}^{\infty} \int_0^1 [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.17}$$

$$= \int_0^1 \sum_{n=0}^{\infty} [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.18}$$

$$= \int_0^1 2\lambda^2 x e^{\lambda x} e^{-\lambda x} + \lambda e^{\lambda x} e^{-\lambda x} dx \tag{2.0.19}$$

$$= \lambda^2 + \lambda \tag{2.0.20}$$

$$Var(N) = E[N^2] - E[N]^2 \tag{2.0.21}$$

$$= \lambda^2 + \lambda - (\lambda)^2 \tag{2.0.22}$$

$$= \lambda \tag{2.0.23}$$

Thus Option 2 is correct.