Assignment 3

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Download latex-tikz codes from

https://github.com/diya−goyal−29/AI1103/blob/main/Assignment%203/Assignment%203.tex

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Question 60 :

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having an exponential distribution with mean $\frac{1}{\lambda}$.Let $S_n = X_1 + X_2 + \cdots + X_n$ and $N = inf\{n \geq 1 : S_n > 1\}$. Then $Var(N)$ is

1) 1
2) λ
3) λ^2
4) ∞

Solution :

$Var(N) = E[N^2] - E[N]^2$

Replacing n with n+1

The event $\{N \leq n\}$ is the same as $\{S_{n+1} > 1\}$ (or equivalently, $\{N > n\}$ is the same as $\{S_{n+1} \geq 1\}$).

By the tail-sum formula, we can thus get the expectation of N as

$$E[N] = \sum_{n=0}^{\infty} P(N > n) = \sum_{n=0}^{\infty} P(S_n \leq 1) \tag{0.0.1}$$

S_n follows Erlang distribution.

$$\text{P.D.F of } S_n = \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} \tag{0.0.2}$$

$$P(S_n \leq 1) = \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{0.0.3}$$

Substituting the value in equation 0.0.1

$$E[N] = \sum_{n=0}^{\infty} \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{0.0.4}$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{0.0.5}$$

$$e^{x\lambda} = \sum_{n=0}^{\infty} \frac{(x\lambda)^n}{(n)!} \tag{0.0.6}$$

$$E[N] = \lambda \int_0^1 e^{\lambda x} e^{-\lambda x} dx \tag{0.0.7}$$

$$= \lambda \int_0^1 dx \tag{0.0.8}$$

$$= \lambda \tag{0.0.9}$$

$$\tag{0.0.10}$$

$$
\begin{aligned}
E[N^2] &= \sum_{n=0}^{\infty} P(N^2 \geq n) && (0.0.11) \\
&= \sum_{n=0}^{\infty} [(n+1)^2 - n^2] P(N \geq n) && (0.0.12) \\
&= \sum_{n=0}^{\infty} [2n+1] P(S_{n+1} \leq 1) && (0.0.13) \\
&= \sum_{n=0}^{\infty} \int_0^1 [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx && (0.0.14) \\
&= \int_0^1 \sum_{n=0}^{\infty} [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx && (0.0.15) \\
&= \int_0^1 2\lambda^2 x e^{\lambda x} e^{-\lambda x} + \lambda e^{\lambda x} e^{-\lambda x} dx && (0.0.16) \\
&= \lambda^2 + \lambda && (0.0.17)
\end{aligned}
$$

$$
\begin{aligned}
Var(N) &= E[N^2] - E[N]^2 && (0.0.18) \\
&= \lambda^2 + \lambda - (\lambda)^2 && (0.0.19) \\
&= \lambda && (0.0.20)
\end{aligned}
$$

Thus Option 2 is correct.