

Assignment 3

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Download latex-tikz codes from

<https://github.com/diya-goyal-29/AI1103/blob/main/Assignment%203/Assignment%203.tex>

1 CSIR-UGC-NET-EXAM-JUNE-2015 QUESTION 60
:

Let X_1, X_2, \dots, X_n be independent and identically distributed random variables having an exponential distribution with mean $\frac{1}{\lambda}$. Let $S_n = X_1 + X_2 + \dots + X_n$ and $N = \inf\{n \geq 1 : S_n > 1\}$. Then $\text{Var}(N)$ is

- 1) 1
- 2) λ
- 3) λ^2
- 4) ∞

2 SOLUTION :

$$\text{Var}(N) = E[N^2] - E[N]^2$$

Replacing n with $n+1$

N is the greatest lower bound of the subset S of T (where T is the set of all natural numbers and S is the set of all natural numbers such that $S_n \geq 1$).

The event $\{N \leq n\}$ implies $\{S_{n+1} > 1\}$ (or equivalently, $\{N > n\}$ implies $\{S_{n+1} \leq 1\}$).

Tail Sum formula : A random variable X which takes values only in \mathbb{N} , then $E[X] = \sum_{k=1}^{\infty} \text{Pr}(X)$

Proof :

$$E[X] = \sum_{k=1}^{\infty} k \text{Pr}(X = k) \quad (2.0.1)$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^k \text{Pr}(X = k) \quad (2.0.2)$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \text{Pr}(X = k) \quad (2.0.3)$$

$$= \sum_{n=1}^{\infty} \text{Pr}(X \geq n) \quad (2.0.4)$$

By the tail-sum formula, we can thus get the expectation of N as

$$E[N] = \sum_{n=0}^{\infty} \text{Pr}(N > n) = \sum_{n=0}^{\infty} \text{Pr}(S_n \leq 1) \quad (2.0.5)$$

S_n follows Erlang distribution.

$$\text{P.D.F of } S_n = f(x)_{S_n} = \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} \quad (2.0.6)$$

$$\text{Pr}(S_n \leq 1) = \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.7)$$

P.D.F of Erlang distribution :

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad (2.0.8)$$

where,

k is the shape

λ is the rate

Proof : We can use mathematical induction to prove this

$$S_1 = X_1 \quad (2.0.9)$$

$$f_{S_1}(x) = \lambda e^{-\lambda x} \quad (2.0.10)$$

Basic step proved.

Assuming S_k to be true we need to prove that S_{k+1} is true.

$$Var(N) = E[N^2] - E[N]^2 \quad (2.0.30)$$

$$= \lambda^2 + \lambda - (\lambda)^2 \quad (2.0.31)$$

$$= \lambda \quad (2.0.32)$$

$$S_{k+1} = X_1 + X_2 + X_3 + \cdots + X_{k+1} \quad (2.0.11)$$

$$= S_k + X_{k+1} \quad (2.0.12)$$

$$f_{S_{k+1}}(x) = f_{S_k} + X_{k+1} \quad (2.0.13)$$

$$= \int_0^x \lambda e^{-\lambda(x-t)} \cdot \lambda^k e^{-\lambda t} \frac{t^{k-1}}{(k-1)!} \quad (2.0.14)$$

$$= \lambda^{k+1} e^{-\lambda x} \int_0^x \frac{t^{k-1}}{(k-1)!} \quad (2.0.15)$$

$$= \lambda^{k+1} e^{-\lambda x} \frac{x^k}{(k)!} \quad (2.0.16)$$

Thus Option 2 is correct.

Substituting the value

$$E[N] = \sum_{n=0}^{\infty} \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.17)$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.18)$$

$$e^{x\lambda} = \sum_{n=0}^{\infty} \frac{(x\lambda)^n}{(n)!} \quad (2.0.19)$$

$$E[N] = \lambda \int_0^1 e^{\lambda x} e^{-\lambda x} dx \quad (2.0.20)$$

$$= \lambda \int_0^1 dx \quad (2.0.21)$$

$$= \lambda \quad (2.0.22)$$

By tail-sum formula the expectation of N^2 is

$$E[N^2] = \sum_{n=0}^{\infty} Pr(N^2 \geq n) \quad (2.0.23)$$

$$= \sum_{n=0}^{\infty} [(n+1)^2 - n^2] Pr(N \geq n) \quad (2.0.24)$$

$$= \sum_{n=0}^{\infty} [2n+1] Pr(S_{n+1} \leq 1) \quad (2.0.25)$$

$$= \sum_{n=0}^{\infty} \int_0^1 [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.26)$$

$$= \int_0^1 \sum_{n=0}^{\infty} [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \quad (2.0.27)$$

$$= \int_0^1 2\lambda^2 x e^{\lambda x} e^{-\lambda x} + \lambda e^{\lambda x} e^{-\lambda x} dx \quad (2.0.28)$$

$$= \lambda^2 + \lambda \quad (2.0.29)$$