Assignment 3

Diya Goyal
Roll no. CS20BTECH11014

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https://github.com/diya−goyal−29/AI1103/blob/main/Assignment%203/Assignment%203.tex

1 CSIR-UGC-NET-Exam-JUNE-2015 Question 60 :

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having an exponential distribution with mean $\frac{1}{\lambda}$.Let $S_n = X_1 + X_2 + \cdots + X_n$ and $N = inf\{n \geq 1 : S_n > 1\}$. Then $Var(N)$ is

1) 1
2) λ
3) λ^2
4) ∞

2 Solution :

$$Var(N) = E[N^2] - E[N]^2 \tag{2.0.1}$$

Replacing n with n+1

Definition 2.1 (infimum)**.** *N* is the greatest lower bound of the subset S of T (where T is the set of all natural numbers and S is the set of all natural numbers such that $S_n \geq 1$).
($N = inf\{n \geq 1 : S_n > 1\}$)
The event $\{N \leq n\}$ implies $\{S_{n+1} > 1\}$ (or equivalently, $\{N > n\}$ implies $\{S_{n+1} \leq 1\}$).

Lemma 2.1. *[Tail Sum formula] : A random variable X which takes values only in N, then* $E[X] = \sum_{k=1}^{\infty} Pr(X)$

Proof.

$$E[X] = \sum_{k=1}^{\infty} kPr(X = k) \tag{2.0.2}$$
$$= \sum_{k=1}^{\infty} \sum_{n=1}^{k} Pr(X = k) \tag{2.0.3}$$
$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} Pr(X = k) \tag{2.0.4}$$
$$= \sum_{n=1}^{\infty} Pr(X \geq n) \tag{2.0.5}$$

□

Corollary 2.1.1. *Summation of independent and identical exponential random variables follows Erlang Distribution.*

P.D.F [Erlang distribution] :

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \tag{2.0.6}$$

where,
k is the shape
λ is the rate

S_n = Erlang(n, λ)

$$\text{P.D.F of } S_n = f(x)_{S_n} = \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} \tag{2.0.7}$$
$$Pr(S_n \leq 1) = \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx \tag{2.0.8}$$

By the tail-sum formula(lemma 2.1), we can thus get the expectation of N as

$$E[N] = \sum_{n=0}^{\infty} Pr(N > n) = \sum_{n=0}^{\infty} Pr(S_n \leq 1) \quad (2.0.9)$$

Substituting the value

$$\begin{aligned} E[N] &= \sum_{n=0}^{\infty} \int_0^1 \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx & (2.0.10) \\ &= \int_0^1 \sum_{n=0}^{\infty} \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx & (2.0.11) \\ e^{x\lambda} &= \sum_{n=0}^{\infty} \frac{(x\lambda)^n}{(n)!} & (2.0.12) \\ E[N] &= \lambda \int_0^1 e^{\lambda x} e^{-\lambda x} dx & (2.0.13) \\ &= \lambda \int_0^1 dx & (2.0.14) \\ &= \lambda & (2.0.15) \end{aligned}$$

By tail-sum formula (lemma 2.1) the expectation of N^2 is

$$\begin{aligned} E[N^2] &= \sum_{n=0}^{\infty} Pr(N^2 \geq n) & (2.0.16) \\ &= \sum_{n=0}^{\infty} [(n+1)^2 - n^2] Pr(N \geq n) & (2.0.17) \\ &= \sum_{n=0}^{\infty} [2n+1] Pr(S_{n+1} \leq 1) & (2.0.18) \\ &= \sum_{n=0}^{\infty} \int_0^1 [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx & (2.0.19) \\ &= \int_0^1 \sum_{n=0}^{\infty} [2n+1] \frac{\lambda^{n+1} x^n e^{-\lambda x}}{(n)!} dx & (2.0.20) \\ &= \int_0^1 2\lambda^2 x e^{\lambda x} e^{-\lambda x} + \lambda e^{\lambda x} e^{-\lambda x} dx & (2.0.21) \\ &= \lambda^2 + \lambda & (2.0.22) \end{aligned}$$

Substituting the values of $E[N]$ and $E[N^2]$ in eq.(2.0.1)

$$\begin{aligned} Var(N) &= E[N^2] - E[N]^2 & (2.0.23) \\ &= \lambda^2 + \lambda - (\lambda)^2 & (2.0.24) \\ &= \lambda & (2.0.25) \end{aligned}$$

Thus Option 2 is correct.