Indiana University-Purdue University–Indianapolis
Qualifying Exam
Real Analysis (MATH 54400)

Solutions

These are **NOT the official solutions**. They may not all be perfectly written.
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January 2025

Throughout the exam, λ denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{L})$.

Problem 1. Show that the set of all real numbers that have decimal expansion with the digit 5 appearing infinitely often is a Borel set.

Proof. Because the collection of all Borel sets is a σ-algebra, it is closed under taking complements, we will show that the set of all real numbers that have decimal expansion with digit 5 appearing finitely many often is a Borel set.

Furthermore, since a σ-algebra is closed under taking countable unions,

and $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [n, n+1]$, and $[n, n+1] = n + [0, 1]$ for any $n \in \mathbb{Z}$, we will show that the set of all numbers in the interval $[0, 1]$ that have decimal expansion with digit 5 appearing finitely many often is a Borel set.

For $k \in \mathbb{N}_0$, define

$$B_k := \{x \in [0, 1] : \text{decimal expansion of } x \text{ has exactly } k \text{ number of 5's}\}.$$

Then,

$$B_1 = \frac{5 + B_0}{10} \cup \bigcup_{\substack{a_1=0\\ a_1\neq 5}}^{9} \frac{a_1 5 + B_0}{10^2} \cup \bigcup_{\substack{a_1,a_2=0\\ a_1,a_2\neq 5}}^{9} \frac{a_1 a_2 5 + B_0}{10^3} \cup \bigcup_{\substack{a_1,a_2,a_3=0\\ a_1,a_2,a_3\neq 5}}^{9} \frac{a_1 a_2 a_3 5 + B_0}{10^4} \cup \cdots$$

$$B_2 = \frac{5 + B_1}{10} \cup \bigcup_{\substack{a_1=0\\ a_1\neq 5}}^{9} \frac{a_1 5 + B_1}{10^2} \cup \bigcup_{\substack{a_1,a_2=0\\ a_1,a_2\neq 5}}^{9} \frac{a_1 a_2 5 + B_1}{10^3} \cup \bigcup_{\substack{a_1,a_2,a_3=0\\ a_1,a_2,a_3\neq 5}}^{9} \frac{a_1 a_2 a_3 5 + B_1}{10^4} \cup \cdots$$

$$B_3 = \frac{5 + B_2}{10} \cup \bigcup_{\substack{a_1=0\\ a_1\neq 5}}^{9} \frac{a_1 5 + B_2}{10^2} \cup \bigcup_{\substack{a_1,a_2=0\\ a_1,a_2\neq 5}}^{9} \frac{a_1 a_2 5 + B_2}{10^3} \cup \bigcup_{\substack{a_1,a_2,a_3=0\\ a_1,a_2,a_3\neq 5}}^{9} \frac{a_1 a_2 a_3 5 + B_2}{10^4} \cup \cdots$$

$$\vdots$$

Here by $a_1 a_2$, for example, we mean the two digits a_1 and a_2 of the number $a_1 a_2$; not the multiplication $a_1 \times a_2$.

Thus, it suffices to show that B_0 is Borel. Let

$$
\begin{aligned}
I_0 &= [0,1] \\
I_1 &= I_0 \backslash [0.5, 0.6) = \bigcup_{\substack{m=0\\ m\neq 5}}^{9} \frac{m+I_0}{10} \\
I_2 &= \bigcup_{\substack{m=0\\ m\neq 5}}^{9} \frac{m+I_1}{10} \\
&\cdot \\
&\cdot \\
&\cdot \\
I_{n+1} &= \bigcup_{\substack{m=0\\ m\neq 5}}^{9} \frac{m+I_n}{10} \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}
$$

Then, $B_0 = \bigcap_{n=0}^{\infty} I_n$. Thus, B_0 is Borel. This completes the proof. $\square$

Problem 2. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h \in \mathcal{L}^1(\mu)$. Prove for any $c > 0$ that

$$\mu\left(\{x \in X : |h(x)| \geq c\}\right) \leq \frac{1}{c}||h||_1.$$

Proof. Let $c > 0$. Then, $X = A \sqcup B$ where

$$A := \{x \in X : |h(x)| < c\} \quad \text{and} \quad B := \{x \in X : |h(x)| \geq c\}.$$

Thus,

$$||h||_1 = ||h_{|A}||_1 + ||h_{|B}||_1 \geq ||h_{|B}||_1 = \int_B |h| \; d\mu \geq \int_B c \; d\mu = c \cdot \mu(B).$$

Therefore, $\mu(B) \leq \frac{1}{c}||h||_1$. $\square$

Problem 3. Prove the existence of the limit

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{1+x^2} \cdot \frac{e^{nx}}{1+e^{nx}} \; d\lambda(x)$$

and compute it.

Proof. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) := \dfrac{1}{1+x^2} \cdot \dfrac{e^{nx}}{1+e^{nx}}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \dfrac{1}{1+x^2}$. Then, $f_n \nearrow f$. Furthermore, we have the improper integral $\displaystyle\int_{\mathbb{R}} \frac{1}{1+x^2}\, d\lambda(x) = \pi$. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{1+x^2} \cdot \frac{e^{nx}}{1+e^{nx}}\, d\lambda(x) = \pi.$$

□

Problem 4. A sequence of measurable functions $f_n : [0,1] \to [-\infty, \infty]$ is said to **converge in measure** to a measurable function $f : [0,1] \to [-\infty, \infty]$ if

$$\text{for any } \delta > 0, \ \lim_{n\to\infty} \lambda\{x \in [0,1] : |(f_n - f)(x)| > \delta\} = 0.$$

Suppose that for each $n \in \mathbb{Z}^+$ we have $f_n \in \mathcal{L}^1([0,1])$ and also that $f \in \mathcal{L}^1([0,1])$. If $f_n \to f$ in measure, does it imply that

$$\lim_{n\to\infty} \int_{[0,1]} f_n\, d\lambda = \int_{[0,1]} f\, d\lambda?$$

Either prove it or give an explicit counterexample.

Answer. Recall. "Pointwise Convergence" $\Rightarrow$"Convergence in Measure".

Let $f_n := n\chi_{[0,\frac{1}{n}]}$. Then, $f_n \to f \equiv 0$. Hence, $f_n \xrightarrow{\mu} f$. But

$$\int_{[0,1]} f_n\, d\mu = n \int_{[0,1]} \chi_{[0,\frac{1}{n}]}\, d\mu = n \cdot \frac{1}{n} = 1 \not\twoheadrightarrow 0 = \int_{[0,1]} f\, d\mu.$$

Note. Without using the "Recall" above, one can directly show $f_n \xrightarrow{\mu} f$ as follows. Here, as mentioned above, $f \equiv 0$.

Notice that $\{x \in [0,1] : |(f_n - f)(x)| > \delta\} = \{x \in [0,1] : n\chi_{[0,\frac{1}{n}]} > \delta\} \subseteq \left[0, \dfrac{1}{n}\right]$. The inclusion "$\subseteq$" becomes equal whenever $0 \le \delta < 1$. Thus,
$\mu\left(\{x \in [0,1] : |(f_n - f)(x)| > \delta\}\right) = \dfrac{1}{n} \to 0.$

Problem 5. Let $C([0,1])$ denote the vector space of all continuous functions $f : [0,1] \to \mathbb{R}$ with the operations of addition and scalar multiplication done pointwise. Prove that $C([0,1])$ with the norm defined by $\displaystyle\int_0^1 |f|$ is not a Banach space.

Proof. Define the sequence $(f_n)_{n=1}^{\infty}$ of functions by $f_n(x) := nx\chi_{[0,\frac{1}{n}]}(x) + \chi_{[\frac{1}{n},1]}(x)$. That is:

$$f_n(x) = \begin{cases} nx, & \text{if } 0 \le x \le \frac{1}{n}, \\ 1, & \text{if } \frac{1}{n} < x \le 1. \end{cases}$$

Also, define the function $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } 0 < x \le 1. \end{cases}$$

Then,

$$\int_0^1 f_n \, d\mu = \int_0^{\frac{1}{n}} nx \, d\mu + \int_{\frac{1}{n}}^1 1 \, d\mu = \frac{1}{2n} + 1 - \frac{1}{n} \to 1 = \int_0^1 f \, d\mu.$$

$\square$

But $f \notin C([0,1])$ even though $f_n \in C([0,1])$ for all $n \in \mathbb{N}$.

Problem 6. Let $\mathcal{L}^1(\mathbb{R})$ denote the Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $||f||_1 := \displaystyle\int |f| \, d\lambda < \infty$. Ignoring that there exist non-zero functions $f \in \mathcal{L}^1(\mathbb{R})$ with $||f||_1 = 0$, let us consider $\big(\mathcal{L}^1(\mathbb{R}), ||\cdot||_1\big)$ as a normed linear space.

(a). Show for Lebesgue almost every $x \in \mathbb{R}$ that

$$\int f(x-y)e^{-y^2} \, d\lambda(y)$$

is a finite number.

(b). Define a new function $Af : \mathbb{R} \to [-\infty, \infty]$ by $Af(x) = \displaystyle\int f(x-y)e^{-y^2} \, d\lambda(y)$. Prove that $Af \in \mathcal{L}^1(\mathbb{R})$.

(c). Show that A defines a bounded linear transformation $A : \mathcal{L}^1(\mathbb{R}) \to \mathcal{L}^1(\mathbb{R})$ and show that $||A|| \le \sqrt{\pi}$.

Proof. (a). By change of variables we get

$$\begin{aligned} \left| \int f(x-y)e^{-y^2} \, d\lambda(y) \right| &= \left| \int f(y)e^{-(x-y)^2} \, d\mu(y) \right| \\ &\le \int |f(y) \, d\mu(y)| \, , \;\because\; |e^{-(x-y)^2}| \le 1 \\ &< \infty. \end{aligned}$$

Thus, the given integral is finite for almost every $x \in \mathbb{R}$.
Note. The *almost every* part comes from the fact:
"$f \in \mathcal{L}^1(\mathbb{R}) \Rightarrow f(x) < \infty$ for almost every $x \in \mathbb{R}$".

(b).

$$\begin{aligned}
\int |Af| \ d\lambda(x) &= \int \left| \int f(x-y)e^{-y^2} \ d\lambda(y) \right| \ d\lambda(x) \\
&< \int \int \left| f(x-y)e^{-y^2} \right| \ d\lambda(y) \ d\lambda(x) \\
&= \int \int |f(x-y)| \cdot e^{-y^2} \ d\lambda(y) \ d\lambda(x) \\
&= \int \int |f(x-y)| \cdot e^{-y^2} \ d\lambda(x) \ d\lambda(y), \text{ by part } (a) \text{ and Tonelli's Theorem} \\
&= \int e^{-y^2} \left[\int |f(x-y)| \ d\lambda(x) \right] \ d\lambda(y) \\
&= \int e^{-y^2} ||f||_1 \ d\lambda(y) \\
&= ||f||_1 \int e^{-y^2} \ d\lambda(y) \\
&= ||f||_1 \sqrt{\pi}.
\end{aligned}$$

Note. To see why $\int e^{-y^2} \ d\lambda(y) = \sqrt{\pi}$.

Let $I := \int e^{-y^2} \ d\lambda(y)$. Then

$$\begin{aligned}
I^2 &= \left(\int e^{-y^2} \ d\lambda(y) \right) \times \left(\int e^{-x^2} \ d\lambda(x) \right) \\
&= \int \int e^{-(x^2+y^2)} \ d\lambda(y) \ d\lambda(x) \\
&= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \ dr \ d\theta, \text{ where } r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}. \\
&= \int_0^{2\pi} \left[\frac{e^{-r^2}}{2} \right]_{\infty}^{0} \ d\theta = \frac{1}{2} \int_0^{2\pi} \ d\theta \\
&= \pi.
\end{aligned}$$

Thus, $I = \sqrt{\pi}$.

(c). For $f, g \in \mathcal{L}^1(\mathbb{R})$ and $K \in \mathbb{R}$ we have

$$\begin{aligned}
A(f + Kg) &= \int [f(x-y) + Kg(x-y)] \, e^{-y^2} \ d\lambda(y) \\
&= \int f(x-y)e^{-y^2} \ d\lambda(y) + K \int g(x-y)e^{-y^2} \ d\lambda(y), \because \text{ integration is linear}
\end{aligned}$$

Thus, A is linear. From part (b) we have $||Af|| \leq \sqrt{\pi}||f||_1$ for any $f \in \mathcal{L}^1(\mathbb{R})$. Thus, $||A|| \leq \sqrt{\pi}$.

$\square$

January 2024

Below $m^*()$ and $m(\cdot)$ are the outer Lebesgue and Lebesgue measures on the real line.

Problem 1. Let $\{E_n\}_n$ be pairwise disjoint measurable sets, $E = \cup_n E_n$, and A be any set. Show that

$$m^*(E \cap A) = \sum_n m^*(E_n \cap A).$$

Proof. By the countable sub-additivity,

$$m^*(E \cap A) \leq \sum_n m^*(E_n \cap A).$$

Thus if we can show the opposite inequality, then we are done. Because E_1 is measurable, by the definition of measurability and the pairwise disjointness of $\{E_n\}_{n=1}$ we get

$$m^*(E \cap A) = m^*(E_1 \cap A) + m^*\Big(\bigsqcup_{n=2}(E_n \cap A)\Big).$$

This implies for any $N \in \mathbb{N}$,

$$m^*(E \cap A) = \sum_{n=1}^{N} m^*(E_n \cap A) + m^*\Big(\bigsqcup_{n=N+1}(E_n \cap A)\Big).$$

Therefore for any $N \in \mathbb{N}$,

$$m^*(E \cap A) \geq \sum_{n=1}^{N} m^*(E_n \cap A).$$

Then by taking the limit as $N \to \infty$ we get

$$m^*(E \cap A) \geq \sum_{n=1} m^*(E_n \cap A).$$

This completes the proof. □

Problem 2. Let f be a continuous function on $[0,1]$. Show that there exists a measurable subset $E \subset [0,1]$ such that $f(E)$ is not measurable if and only if there exists $A \subset [0,1]$ such that $m(A) = 0$ and $m^*(f(A)) > 0$.

Proof. ($\Rightarrow$). Suppose there is a measurable set $E \subset [0,1]$ such that $f(E)$ is not measurable. Then $E = G \cap A$ where G is a G_δ set and A is a measurable set such that A^c is a null set. Because the continuous image of a G_δ set is measurable, we must then have $f(A)$ is non-measurable. Hence $m^*(f(A)) > 0$ as every null set is measurable.

($\Leftarrow$) Suppose there exists $A \subset [0,1]$ such that $m(A) = 0$ and $m^*(f(A)) > 0$. Then by Vitali's theorem there is $F \subset f(A)$ such that F is non-measurable. Let $E = A \cap f^{-1}(F)$. Then E is measurable as $m(E) = 0$ and $f(E) = F$ is non-measurable. □

Problem 3. Let $\{f_n\}_n$ be a sequence of measurable functions on E, $m(E) < \infty$. Assume that for every $x \in E$ there exists a constant M_x such that $|f_n(x)| \leq M_x$ for all n. Show that for every $\epsilon > 0$ there exists a closed set $F_\epsilon \subseteq E$ and a constant M_ϵ such that $|f_n(x)| \leq M_\epsilon$, for all n and $x \in F_\epsilon$, and $m(E \backslash F_\epsilon) < \epsilon$.

Proof. Let $\epsilon > 0$ be given. Because $m(E) < \infty$, we can find $N_\epsilon \in \mathbb{N}$ such that $m(E \backslash [-N_\epsilon, N_\epsilon]) < \epsilon/2$. Let $F := E \cap [-N_\epsilon, N_\epsilon]$. Then by Lusin's theorem we can find a decreasing sequence $(F_n)_{n=1}$ of compact subsets of F such that for each $n \in \mathbb{N}$, f_n is continuous on F_n and $m(F \backslash F_n) \leq \epsilon \sum_{k=1}^{n} 1/2^{k+1}$. Let $F_\epsilon = \bigcap_{n=1} F_n$. Then $m(E \backslash F_\epsilon) \leq \epsilon/2 + \epsilon \sum_{k=1} 1/2^{k+1} = \epsilon/2 + \epsilon/2 = \epsilon$.

It remains to show that there is a uniform constant M_ϵ such that $|f_n(x)| \leq M_\epsilon$, for all n and $x \in F_\epsilon$. For a contradiction let's assume that it is not the case. Then the decreasing sequence $(A_j)_{j=1}$ of compact sets defined by

$$A_j := \overline{\{x \in F_\epsilon : |f_{n_j}(x)| \geq j \text{ for some } n_j \in \mathbb{N}\}}$$

has non-empty intersection by the Cantor's intersection theorem. Then for any $x_0 \in \bigcap_{j=1} A_j$, we have $M_{x_0} \geq |f_{n_j}(x_0)| \geq j$ for all $j \in \mathbb{N}$. This implies $M_{x_0} = \infty$. Contradiction. □

Problem 4. Let $\{f_n\}_n$ be sequence of functions on $[0, 1]$ whose total variations are uniformly bounded. If $f_n \to f$ pointwise on $[0, 1]$ as $n \to \infty$, show that f is a function of bounded variation. Will the conclusion remain true if we simply assume that each f_n is a function of bounded variation? (justify your answer)

Proof. Let $0 < \epsilon < 1$ be given. Let $\mathcal{P} := \{0 = x_0, x_1 \cdots x_{m-1} = 1\}$ be a partition of $[0, 1]$. Then for each $j \in \{0, \cdots m-1\}$, there is f_{n_j} such that $|f_{n_j}(x_j) - f(x_j)| < \frac{\epsilon}{2m}$. Let $M \geq \max\{n_j : j = 0, \cdots, m-1\}$. Then $|f_M(x_j) - f(x_j)| < \frac{\epsilon}{2m}$ for all $j = 0, \cdots, m-1$. Let $K > 0$ be a uniform bound such that $TV(f_n) < K$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
V(f, \mathcal{P}) &= \sum_{j=1}^{m-1} |f(x_j) - f(x_{j-1})| \\
&\leq \sum_{j=1}^{m-1} |f(x_j) - f_M(x_j)| + \sum_{j=1}^{m-1} |f_M(x_j) - f_M(x_{j-1})| + \sum_{j=1}^{m-1} |f_M(x_{j-1}) - f(x_{j-1})| \\
&\leq \epsilon/2 + K + \epsilon/2 = \epsilon + K < 1 + K.
\end{aligned}$$

Hence $f \in \mathcal{F}_{BV}([0, 1])$. □

For the second part: No. Let f be any (almost everywhere) continuous function on $[0, 1]$ which is not of bounded variation. By Weierstrass polynomial approximation theorem, there is a sequence $(P_n)_{n=1}$ of polynomial functions on $[0, 1]$ that converges uniformly to f. But each P_n is of bounded variation as any polynomial on a compact interval is Lipschitz.

Problem 5. Let $\{f_n\}_n$ be sequence of measurable functions on a measurable set E that converges almost everywhere on E to an integrable function f. Show that

$$\int_E |f_n - f| \to 0 \text{ if and only if } \int_E |f_n| \to \int_E |f|$$

(both limits taking place as $n \to \infty$).

Proof. ($\Rightarrow$). Because $\big||f_n| - |f|\big| \leq |f_n - f|$, we have

$$\int_E |f| - \int_E |f_n - f| \leq \int_E f_n \leq \int_E |f| + \int_E |f_n - f|.$$

Then by taking the limit we get

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

Note: By the second inequality and the hypothesis we have that for all but finitely many $n \in \mathbb{N}$, $f_n \in L^1([0,1])$.

($\Leftarrow$). Notice that $|f_n - f| \to 0$, and $\int_E |f_n - f| \leq \int_E |f_n| + \int_E |f| < \infty$ for all but finitely many $n \in \mathbb{N}$ ($\because$ $\int_E |f_n| \to \int_E |f|$). Then by the General Lebesgue Dominated Convergence Theorem we get the claim $\int_E |f_n - f| \to 0$. $\square$

Problem 6. Let $f \in L^p[0,1]$ for some $p > 1$. Show that $f \in L^1[0,1]$ and that for any $c \in (0, ||f||_1)$ it holds that

$$m(\{x \in [0,1] : |f(x)| > c\}) \geq \Big(\frac{||f||_1 - c}{||f||_p}\Big)^q,$$

where q is the conjugate exponent of p.

Proof. Let $E_1 := \{x \in [0,1] : |f(x)| > 1\}$. Then

$$||f||_1 \leq \int_{E_1} |f| + \int_{[0,1]\setminus E_1} |f| \leq ||f||_p + 1 < \infty.$$

Thus $f \in L^1[0,1]$.

If $||f||_p = 0$, then $f \overset{a.e}{=} 0$. Thus we have the claim vacuously as such a $0 < c < 0$ does not exist. So, suppose $||f||_p \neq 0$. Because $c \in (0, ||f||_1)$, $||f||_1 - c > 0$. Let $E_c := \{x \in [0,1] : |f(x)| > c\}$. Then

$$||f||_1 \leq c + \int_{[0,1]} |f| \cdot \chi_{E_c} \leq c + ||f||_p \cdot m(E_c)^{1/q}.$$

Hence

$$m(E_c) \geq \Big(\frac{||f||_1 - c}{||f||_p}\Big)^q.$$

$\square$

August 2023

In what follows, $m()$ is the Lebesgue measure on the real line and E is measurable.

Problem 1. Let E be a measurable set for which there exists $\delta \in (0,1)$ such that $m(E \cap I) > \delta m(I)$ for any interval $I \subset (-\infty, \infty)$. Show that $m((-\infty,\infty)\backslash E) = 0$.

Proof. Let $F \subset (-\infty,\infty)\backslash E$ be such that $m(F) < \infty$. Then for any $\epsilon > 0$, there is a countable collection $\{I_n\}_{n=1}$ of disjoint open intervals such that $\bigsqcup_{n=1} I_n \supset F$ and $\sum_{n=1} m(I_n) < m(F) + \delta\epsilon$. Hence

$$\delta \sum_{n=1} m(I_n) < m(E \cap \bigsqcup_{n=1} I_n) < m(\Big(\bigsqcup_{n=1} I_n\Big)\backslash F) = \sum_{n=1} m(I_n) - m(F) = \delta\epsilon.$$

Because $\epsilon > 0$ is arbitrary, this implies $\sum_{n=1} m(I_n) = 0$. Hence $m(F) = 0$ as $F \subset \bigsqcup_{n=1} I_n$. Because F is arbitrary, and any measurable set with infinite measure always have a measurable subset with finite measure, we have the claim. $\square$

Problem 2. A sequence of measurable functions f_n on E converges to a measurable function f *in measure* on E if $\lim_{n\to\infty} m\{x : |(f_n - f)(x)| > \delta\} = 0$ for any $\delta > 0$. Show that

(a) almost everywhere convergence implies convergence in measure if $m(E) < \infty$;

(b) almost everywhere convergence does not imply in general convergence in measure if $m(E) = \infty$;

(c) converge in measure does not imply in general almost everywhere convergence even if $m(E) < \infty$.

Proof. (a) Suppose $f_n \overset{a.e}{\to} f$ on E. Let $\epsilon > 0$. Then by Egoroff's theorem, there is closed $F \subset E$ such that $f_n \rightrightarrows f$ on F and $m(E\backslash F) < \epsilon$. Therefore for any $\delta > 0$, we have

$$\begin{aligned} m\{x \in E : |(f_n - f)(x)| > \delta\} &= m\{x \in F : |(f_n - f)(x)| > \delta\} + \\ &\quad m\{x \in E\backslash F : |(f_n - f)(x)| > \delta\} \\ &\le m\{x \in F : |(f_n - f)(x)| > \delta\} + \epsilon. \end{aligned}$$

Because $\epsilon > 0$ is arbitrary and $\lim_{n\to\infty} m\{x \in F : |(f_n - f)(x)| > \delta\} = 0$, we have the claim.

(b) Let $E = [0,\infty)$ and define the sequence of measurable functions $(f_n)_{n=1}$ on E by $f_n = \chi_{[n,\infty)}$. Thus for any $x \in E$, $f_n(x) \to 0$. But, for example,

$$\lim_{n\to\infty} m\{x : |(f_n - 0)(x)| > 1/2\} = \lim_{n\to\infty} m([n,\infty)) = \infty.$$

(c) Define a collection $\{E_{k,m}\}_{0\le k\le m-1,\ m=1}$ of intervals of $[0,1]$ by $E_{k,m} := [k/m, (k+1)/m]$. Define the sequence $(f_n)_{n=0}$ of measurable functions on $[0,1]$ by $f_n = \chi_{E_{k,m}}$ where

$n = m(m-1)/2 + k$, $n \in \mathbb{N}$, $0 \le k \le m-1$. In other words, $(f_n)_{n=1}$ are the indicator functions of the sequence of sets:

$$\Big([0,1], [0, \frac{1}{2}], [\frac{1}{2}, 1], [0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1], \cdots\Big).$$

Then for any $x \in [0,1]$, we can find sub sequences $(f_{n_i})_{i=1}$ and $(f_{n_j})_{j=1}$, say, such that $f_{n_i}(x) = 0$ and $f_{n_j}(x) = 1$ (simply by choosing k, m so that $x \notin E_{k,m}$ and $x \in E_{k,m}$ respectively). Thus $(f_n)_{n=1}$ does not converge point-wise anywhere in $[0,1]$. But for any $0 < \delta < 1$ we have

$$m\big(\{x \in [0,1] : f_n(x) > \delta\}\big) = m(E_{k,m}) = 1/m < 1/\sqrt{n},$$

and for any $\delta \ge 1$ we have $m\big(\{x \in [0,1] : f_n(x) > \delta\}\big) = 0$. Thus $(f_n)_{n=1}$ converges to $f \equiv 0$ in measure on $[0,1]$.

□

See the following *Problem* 3 for a slightly different example.

Problem 3. Show that if functions f_n are non-negative and measurable on E and $\int_E f_n \le 1/n^2$, then $f_n \to 0$ almost everywhere on E. Will the claim remain true if $1/n^2$ is replaced by $1/n$?

Proof. (See Problem 5 of August 2021 for a different proof)

Let $F := \{x \in E : f_n(x) \nrightarrow 0\}$. Then

$$F = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}.$$

With the Chebychev's inequality and the continuity of the Lebesgue measure we have

$$\begin{aligned}
m(F) &= m\big(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}\big) \\
&\le \sum_{k=1}^{\infty} \lim_{N\to\infty} m\big(\bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}\big) \\
&\le \sum_{k=1}^{\infty} \lim_{N\to\infty} \sum_{n=N}^{\infty} m\big(\{x \in E : f_n(x) > 1/k\}\big) \\
&\le \sum_{k=1}^{\infty} \lim_{N\to\infty} \sum_{n=N}^{\infty} k \int_E f_n \le \sum_{k=1}^{\infty} \lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{k}{n^2} = 0.
\end{aligned}$$

□

No. We can inductively construct a sequence $(a_n)_{n=0}$ of numbers in $[0,1]$ with following properties:

i. $a_0 = 0$ and for any $n \in \mathbb{N}$, $a_n \neq a_{n+1}$.

ii. For $n \in \mathbb{N}$, if $a_n + 1/n \leq 1$, then let $a_{n+1} := a_n + 1/n$.

iii. For $n \in \mathbb{N}$, if $a_n < 1$ and $a_n + 1/n > 1$, then let $a_{n+1} = 1$ and $a_{n+2} = 0$.

So the sequence has infinite number of finite strings of consecutive terms starting from 0 and ending at 1, again and again. This is possible as $\sum_{n=1} 1/n = \infty$ implies for any $k \in \mathbb{N}$, there is $K \in \mathbb{N}$ such that $\sum_{n=k}^{K} 1/n \geq 1 = m([0,1])$.

Then we can define a sequence $I_n := [b_n, b_{n+1}]$ of closed sub-intervals of $[0,1]$ where $b_n = a_{n+K_n}$ for some $K_n \in \mathbb{N}$ such that $b_{n+1} - b_n \leq 1/n$, and for any $n \in \mathbb{N}$, there exist $M_n \leq n < N_n$ such that $[0,1] = \bigcup_{n=M_n}^{N_n} I_n$. Here are the first few intervals of such a sequence:

$$\begin{gathered}[0,1],\\ \left[0,\frac{1}{2}\right], \left[\frac{1}{2},\frac{5}{6}\right], \left[\frac{5}{6},1\right]\\ \left[0,\frac{1}{5}\right], \left[\frac{1}{5},\frac{11}{30}\right], \left[\frac{11}{30},\frac{107}{210}\right], \cdots\end{gathered}$$

Therefore for any $x \in [0,1]$, we can find two sub-sequences $(I_{n_i})_{i=1}$ and $(I_{n_j})_{j=1}$ such that $x \in I_{n_i}$ for all $i \in \mathbb{N}$ and $x \notin I_{n_j}$ for all $j \in \mathbb{N}$. Hence the sequence $(f_n)_{n=1}$ of functions on $[0,1]$ defined by $f_n := \chi_{I_n}$ does not converge pointwise anywhere in $[0,1]$. But

$$\int_{[0,1]} |f_n| = m(I_n) \leq \frac{1}{n}.$$

Problem 4. Let f be an integrable function on a bounded measurable set E. Find $\lim_{p\to 0^+} \int_E |f|^p$.

Answer. We prove

$$\lim_{p\to 0^+} \int_E |f|^p = m(E \backslash E^*).$$

Here $E^* := f^{-1}(\{0,\infty\})$.

Let $E_1 := \{x \in E : 0 < |f(x)| \leq 1\}$ and $E_2 := \{x \in E : \infty > |f| > 1\}$. Then for all $0 < p < 1$, $|f\chi_{E_1}|^p \leq 1$, and $\lim_{p\to 0^+} |f\chi_{E_2}(x)|^p = 1$. Because $f \in L^1(E)$, $m(f^{-1}(\{\infty\})) = 0$. Thus by the Lebesgue Dominated Convergence Theorem,

$$\lim_{p\to 0^+} \int_E |f|^p = \lim_{p\to 0^+} \int_{E_1} |f|^p + \lim_{p\to 0^+} \int_{E_2} |f|^p = m(E_1) + m(E_2) = m(E \backslash E^*).$$

Q: What if E is an unbounded measurable set with finite measure?

An idea: $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n := E \cap [n, n+1]$.

Problem 5. Let $E \subset [0,1]$ be a Borel set such that $0 < m(E \cap I) < m(I)$ for any subinterval of $[0,1]$. Show that $f(x) = m([0,x] \cap E) - m([0,x] \backslash E)$ is absolutely continuous on $[0,1]$ but it is not monotone on any subinterval of $[0,1]$.

Proof. Because $f(x) = \int_0^x (\chi_E - \chi_{E^c})$, an indefinite integral, $f \in AC[0,1]$ by the fundamental theorem of Lebesgue integral calculus.

<u>For the second part:</u>

• **Proof 1:** The hypothesis "$0 < m(E \cap I) < m(I)$ *for any subinterval of* $[0,1]$" says E is dense in $[0,1]$, and does not contain any intervals (hence totally disconnected). This, together with the fact "$f'(x) = \chi_E - \chi_{E^c}$ for almost every $x \in [0,1]$", imply that for any sub-interval $J \subset [0,1]$, there exist points $a, b \in J$ such that $f'(a) = 1 > 0$ and $f'(b) = -1 < 0$. Thus f can not be monotone on J.

• **Proof 2.** Notice that, because any Borel set is Lebesgue measurable,

$$f(x) = 2m([0,x] \cap E) - x.$$

<u>Case 1:</u> Suppose there is an interval J on which f is monotonically decreasing. Then for any $x_1, x_2 \in J$, $x_1 < x_2$ we have $f(x_1) - f(x_2) \geq 0$. Which implies $(x_2 - x_1)/m([x_1, x_2] \cap E) \geq 2$. For any given $\epsilon > 0$, choose $\{I_k\}_{k=1}$, $I_k \subset J$ so that $\bigsqcup_{k=1} I_k \supset E \cap J$ and $\sum_{k=1} m(I_k) < m(E \cap J) + \epsilon$. Then

$$m(E \cap J) + \epsilon > \sum_{k=1} m(I_k) = \sum_{k=1} \frac{m(I_k)}{m(I_k \cap E \cap J)} m(I_k \cap E \cap J) \geq 2 \sum_{k=1} m(I_k \cap E \cap J) = 2m(E \cap J).$$

Because $\epsilon > 0$ is arbitrary, this implies $m(E \cap J) = 0$. Contradiction.

<u>Case 2:</u> Suppose there is an interval J on which f is strictly increasing. Then for any $x_1, x_2 \in J$, $x_1 < x_2$ we have $f(x_2) - f(x_1) > 0$. Which implies $m([x_1, x_2] \cap E) > (x_2 - x_1)/2$. Then by *Problem* 1 (replacing $(-\infty, \infty)$ with J), $m(J \backslash E) = 0$. Which implies $m(J \cap E) = m(J)$. Contradiction.

Because J is an arbitrary interval, f is not monotone on any sub-interval of $[0,1]$.

• **Proof 3.** If you are allowed, use the Lebesgue Density Theorem. □

Q: Can E be a Lebesgue measurable but non-Borel set (with positive measure, of course)?

Q: Does there exist a Lebesgue measurable but non-Borel set $E \subset [0,1]$ such that for any interval $I \subset [0,1]$, $E \cap I$ is non-Borel?

An idea: Any Lebesgue measurable set is a Borel set minus a set of measure zero.

January 2023

Problem 1. Let A be a set such that $A \cap K \neq \emptyset$ for any compact subset K of real line of positive measure. Show that A has infinite outer Lebesgue measure.

Proof. If $m(A) < \infty$, then $m(A^c) > 0$. Hence by the inner regularity of the Lebesgue measure there is a compact set $K \subset A^c$, say, with $m(K) > 0$. This contradicts the hypothesis as $A \cap K = \emptyset$. $\square$

Problem 2. Show that if f is continuous almost everywhere on $[0, 1]$, then it is measurable.

Proof. Let $E \subset [0, 1]$ be the set of all points at where f is continuous. Then $m(E^c) = 0$, and therefore E and $f|_E$ ($\because$ the inverse image of any open set under $f|_E$ is the intersection between an open set and E) are measurable. Because $f^{-1}((a, \infty)) \overset{a.e}{=} f|_E^{-1}((a, \infty))$ for any $a \in \mathbb{R}$, f is measurable. $\square$

Problem 3. For a non-negative measurable function f on a measurable set E define

$$m_f(t) = m(\{f > t\}), t > 0,$$

where m is the Lebesgue measure. Show that $\int_E \varphi = \int_0^\infty m_\varphi$ for any non-negative simple function φ.

Proof. Let $\varphi = \sum_{j=1}^n c_k \chi_{E_j}$ with $c_1 < \cdots < c_n$. Then

$$m_\varphi(t) = \chi_{[0,c_1)}(t) \cdot \sum_{j=1}^n m(E \cap E_j) + \sum_{k=1}^{n-1} \Big[\chi_{[c_k,c_{k+1})}(t) \cdot \sum_{j=k+1}^n m(E \cap E_j)\Big].$$

Thus

$$\begin{aligned}\int_0^\infty m_\varphi &= c_1 \sum_{j=1}^n m(E \cap E_j) + \sum_{k=1}^{n-1} \Big[(c_{k+1} - c_k) \sum_{j=k+1}^n m(E \cap E_j)\Big] \\ &= \sum_{j=1}^n c_j \cdot m(E \cap E_j) = \int_E \varphi.\end{aligned}$$

$\square$

Problem 4. In the setting of the previous problem and assuming its statement is true, show that $\int_E f = \int_0^\infty m_f$ for any non-negative measurable function f.

Proof. By the simple approximation theorem, we can find an increasing sequence $(\varphi_n)_{n=1}$ of positive simple functions such that $\varphi_n \nearrow f$ on E. By the monotone convergence theorem $\int_E \varphi_n \nearrow \int_E f$, and by the previous problem $\int_E \varphi_n = \int_0^\infty m_{\varphi_n}$ for all $n \in \mathbb{N}$. Thus if we can show $\lim_{n\to\infty} \int_0^\infty m_{\varphi_n} = \int_0^\infty m_f$, then we are done.

Notice that for any $t > 0$, $x \in \{x \in \mathbb{R} : f(x) > t\}$ if and only if $x \in \{x \in \mathbb{R} : \varphi_n(x) > t\}$ for all but finitely many $n \in \mathbb{N}$. Thus

$$\{x \in \mathbb{R} : f(x) > t\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\}.$$

Then

$$\begin{aligned}
m_f(t) &:= m(\{x \in \mathbb{R} : f(x) > t\}) \\
&= m\Big(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\} \Big) \\
&= \lim_{k\to\infty} m\Big(\bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\} \Big), \because \text{continuity of Lebesgue measure} \\
&= \lim_{k\to\infty} m(\{x \in \mathbb{R} : \varphi_k(x) > t\}), \because \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\} = \{x \in \mathbb{R} : \varphi_k(x) > t\} \\
&= \lim_{n\to\infty} m(\{x \in \mathbb{R} : \varphi_n(x) > t\}) = \lim_{n\to\infty} m_{\varphi_n}(t).
\end{aligned}$$

Because $t > 0$ is arbitrary and $(m_{\varphi_n})_{n=1}$ is an increasing sequence of measurable functions, by the monotone convergence theorem we get

$$\lim_{n\to\infty} \int_0^{\infty} m_{\varphi_n} = \int_0^{\infty} m_f.$$

This completes the proof. □

Problem 5. Let f be integrable function of the real line. Show that $f = 0$ almost everywhere if $\int_I f = 0$ for any interval I of length 1.

Proof. For the sake of contradiction let's assume that there is $\epsilon > 0$ and a measurable set E_1^+ inside of an interval of length 1 such that $m(E_1^+) > 0$ and $f|_{|E_1^+} > \epsilon$ (choose E_1^+ to be maximal, up to measure zero set, within a unit interval). Then the hypothesis implies that there is another set E_1^-, within an interval of length 1 to E_1^+, such that $m(E_1^+) = m(E_1^-)$ and $f|_{E_1^-} < -\epsilon$. By continuing in this way we can find a sequence $(E_n^+)_{n=1}$ of disjoint measurable sets with same (positive) measure such that $f|_{|E_n^+} > \epsilon$ for all $n \in \mathbb{N}$. This violates the integrebility of f as

$$\int |f| > \int_{\bigsqcup_{n=1}^{\infty} E_n^+} f \geq \sum_{n=1}^{\infty} \epsilon \cdot m(E_n^+) = \epsilon \cdot \infty = \infty.$$

□

Problem 6. Let $f \in L^p[0, 1]$, $p \in (1, \infty)$, and F be an indefinite integral of f. Show that for each $x \in (0, 1)$

$$\lim_{h\to 0^+} \frac{F(x + h) - F(x)}{h^{1/q}} = 0,$$

where q is the conjugate exponent of p.

Proof. We have $F(x) = F(0) + \int_0^x f$ for all $x \in [0,1]$. Therefore for any $x \in [0,1]$,

$$\begin{aligned}\lim_{h\to 0^+}\left|\frac{F(x+h)-F(x)}{h^{1/q}}\right| &\leq \lim_{h\to 0^+}\frac{\int_x^{x+h}|f|}{h^{1/q}}\\ &= \lim_{h\to 0^+}\frac{\int_{[0,1]}|f|\cdot\chi^2_{[x,x+h]}}{h^{1/q}}\\ &\overset{Holder}{\leq} \lim_{h\to 0^+}||f\chi_{[x,x+h]}||_p.\ (*)\end{aligned}$$

Let $h = 1/n$, $n \in \mathbb{N}$. For fixed $x \in [0,1]$, define the sequence $(f_{x,n})_{n=1}$ of $L^1[0,1]$ functions by

$$f_{x,n}(t) := |f(t)|^p \cdot \chi_{[x,x+1/n]}(t).$$

Thus $f_{x,n} \leq |f|^p$ for all $n \in \mathbb{N}$ and $f_{x,n}(t) \to 0$ for all $t \in [0,1]\backslash\{x\}$ as $n \to \infty$. Since the singletons have zero measure and $f^p \in L^1[0,1]$, the Lebesgue Dominated Convergence Theorem tells $\int_0^1 f_{x,n} \to 0$ for all $x \in [0,1]$. This together with the squeeze theorem give us

$$\lim_{h\to 0^+}||f\chi_{[x,x+h]}||_p = \lim_{n\to\infty}||f\chi_{[x,x+\frac{1}{n}]}||_p = \lim_{n\to\infty}\left(\int_0^1 f_{x,n}\right)^{1/p} = 0. \qquad (**)$$

By $(*)$ and $(**)$ we have the claim. $\square$

January 2022

Problem 1. Let $f \to [0, \infty)$ be a measurable function and define its distribution function $d_f : [0.\infty) \to [0, \infty]$ defined by $d_f(t) = |\{x \in \mathbb{R} : f(x) > t\}|$.

i. Prove or disprove the statement that the function d_f is left continuous, that is to say

$$\lim_{t \nearrow t_0} d_f(t) = d_f(t_0), \text{ for each } t_0 > 0.$$

ii. Prove or disprove the statement that the function d_f is right continuous, that is to say

$$\lim_{t \searrow t_0} d_f(t) = d_f(t_0), \text{ for each } t_0 \leq 0.$$

Proof. i. Let $t_0 = 1$ and $f(x) = \chi_{[0,1)}(x)$. Then $d_f(t_0) = 0 \neq 1 = \lim_{t \nearrow t_0} d_f(t)$. So, $d_{\chi_{[0,1)}}$ is not left continuous.

ii. Let $t_0 \geq 0$. We prove $\lim_{t \searrow t_0} d_f(t) = d_f(t_0)$. For $t > t_0$, notice that

$$m(f^{-1}(t_0, \infty)) = m(f^{-1}(t_0, t]) + m(f^{-1}(t, \infty)) \quad (*)$$

Because $f^{-1}(t_0, t_1] \subset f^{-1}(t_0, t_2]$ for any $t_1 < t_2$, $\lim_{t \searrow t_0} m(f^{-1}(t_0, t])$ exists (continuity of Lebesgue measure). If it is 0, then we are done. If not, then there is $M > 0$ and $N > t_0$ such that $\lim_{t \searrow t_0} m(f^{-1}(t_0, t]) \geq M$ for all $t \in (t_0, N)$. Let $t \in (t_0, N)$. Because $t_0 \notin (t_0, t]$, there is a countable union of disjoint (can be de-generated) intervals $(I_k)_{k=1}^{\infty}$ in $(t_0, t]$ such that $m(f^{-1}(I_k)) \geq M$ for each k (let the enumeration begins from right). This says $m(f^{-1}(t_0, t]) = \infty$. Because t is arbitrary, $\lim_{t \searrow t_0} m(f^{-1}(t, \infty)) \geq \lim_{n \to \infty} \sum_{k=1}^{n} m(I_k) \geq \lim_{n \to \infty} nM = \infty$. Here $\bigsqcup_{k=1}^{n} I_k \subset (t, N)$. So, $\lim_{t \searrow t_0} d_f(t) = \infty$. Then from $(*)$, $d_f(t_0) = \lim_{t \searrow t_0} d_f(t) = \infty$.

□

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Show the following.

i. if $F \subset \mathbb{R}$ is an F_σ set then $f(F) \subset \mathbb{R}$ is an F_σ set.

i. if $F \subset \mathbb{R}$ is a measurable set then $f(F) \subset \mathbb{R}$ is a measurable set.

Proof. i . Because $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1]$, $F \subset \mathbb{R}$ is an F_σ set if and only if it is a countable union of compact sets. Since the continuous image of a compact set is compact and $f(A \cup B) = f(A) \cup f(B)$ for any sets (let's say measurable) $A, B \subset \mathbb{R}$, we have that $f(F)$ is an F_σ set if F is an F_σ set.

ii . Suppose $F \subset \mathbb{R}$ be a measurable set. Then $F = E \sqcup (F \backslash E)$ where E is an F_σ set and $m(F \backslash E) = 0$. Hence $f(F) = f(E) \cup f(F \backslash E)$. So, if we can show $m(f(F \backslash E)) = 0$, then by part (i) we have that F is measurable. In-particular, if we can shoe $m(f(A)) \leq m(A)$ for any measurable set A, then we are done. Because the semi-ring of bounded intervals generates the collection of all Lebesgue measurable sets (and with the fact that singletons have measure zero), it is enough to show $m(f([a,b])) \leq M(b-a)$ for any closed interval $[a,b]$. Here $M>0$ is the Lipschitz constant of f. In-fact

$$m(f([a,b])) \leq \max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x) = f(\alpha) - f(\beta) \leq M(\alpha-\beta) \leq M(b-a).$$

Here $\alpha, \beta \in [a,b]$ are such that $f(\alpha) = \max_{x\in[a,b]} f(x)$ and $f(\beta) = \min_{x\in[a,b]} f(x)$.

□

Problem 3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function. Show that the following are equivalent:

(*) for each measurable set $E \subset \mathbb{R}$, the set $\phi^{-1}(E)$ is measurable and $|\phi^{-1}(E)| = |E|$;

(**) for each measurable function $f : \mathbb{R} \to \mathbb{R}$, the function $f \circ \phi : \mathbb{R} \to \mathbb{R}$ is measurable and

$$\int_{\mathbb{R}} f \circ \phi = \int_{\mathbb{R}} f.$$

Proof. $(* \Rightarrow **)$ Since ϕ is measurable by the hypothesis, $f \circ \phi$ is measurable. Because $f : \mathbb{R} \to \mathbb{R}$ be measurable, there is a sequence $(\varphi_n)_{n=1}^{\infty}$ of simple functions which converges pointwise to f on $\mathbb{R}$ with $|\varphi_n| \leq |f|$ for all $n \in \mathbb{N}$ (if $f > 0$, choose $(\varphi_n)_{n=1}^{\infty}$ to be increasing). Therefore if we can show the claim for any simple function, then we are done, as: If $f \in L^1(\mathbb{R})$, then use LDCT. If $f \notin L^1(\mathbb{R})$, then at-least one of $\int_{\mathbb{R}} f_-$ or $\int_{\mathbb{R}} f_+$ equals to ∞. Then use the monotone convergence theorem to this f_- or f_+.

Let $f(x) = \sum_{k=1}^{\infty} a_k \chi_{E_k}(x)$ be a simple function. Then

$$\int_{\mathbb{R}} f \circ \phi = \sum_{k=1}^{n} a_k m(\phi^{-1}(E_k)) = \sum_{k=1}^{n} a_k m(E_k) = \int_{\mathbb{R}} f.$$

$(* \Leftarrow **)$ Letting f be the identity we get $\phi : \mathbb{R} \to \mathbb{R}$ is measurable. And for any measurable set E, letting $f = \chi_E$, we get

$$m(\phi^{-1}(E)) = \int_{\mathbb{R}} f \circ \phi \overset{\text{Hypothesis}}{=} \int_{\mathbb{R}} f = m(E).$$

□

Problem 4. Let $E \subset \mathbb{R}$ be a measurable set with $0 < |E| < \infty$. Let $f \in Ł^{\infty}(E)$.

i. Show that $f \in L^p(E)$ for each $p \in [1, \infty)$

ii. Show that $||f||_p \to ||f||_\infty$ as $p \to \infty$.

Proof. i. Let $p \in [1,\infty)$. Then $||f||_p \leq ||f||_\infty \cdot m(E)^{1/p} < \infty$. Thus $f \in L^p(E)$.

ii. Because $||f||_p \leq ||f||_q$ for any $p \leq q$, the limit $L := \lim_{p \longrightarrow \infty} ||f||_p$ exists in $[0, \infty]$. But since $f \in L^\infty(E)$, $L \in [0, \infty)$. Notice that $||f||_p = \left(\int_E |f|^p \right)^{1/p} \leq m(E)^{\frac{1}{p}} ||f||_\infty$. Then by taking the limit as $p \to \infty$ we get $L \leq ||f||_\infty$. To show the reverse inequality: for $n \in \mathbb{N}$, define

$$E_n := \{x \in E : ||f||_\infty \Big(1 - \frac{1}{n}\Big) \leq |f(x)| \leq ||f||_\infty\}.$$

Then $m(E_n) > 0$. (This is true as $m(E) > 0$ and $|| \cdot ||_\infty$ is the *essential sup norm*. If it is just the *sup norm*, then the claim is false. For a counter example consider $f : [0, 1] \to [0, 1]$ defined by $f(x) = 1$ if $x = 1/2$, and $f(x) = 0$ else). Then

$$||f||_p^p = \int_E |f|^p = \int_{E_n} |f|^p + \int_{E \setminus E_n} |f|^p \geq ||f||_\infty^p \cdot \Big(1 - \frac{1}{n}\Big)^p \cdot m(E_n)$$

Hence $L = \lim_{p \longrightarrow \infty} ||f||_p \geq \lim_{p \longrightarrow \infty} ||f||_\infty \cdot \Big(1 - \frac{1}{n}\Big) \cdot m(E)^{\frac{1}{p}} = ||f||_\infty \cdot \Big(1 - \frac{1}{n}\Big)$. Since this is true for any $n \in \mathbb{N}$, we get $L \geq ||f||_\infty$, and therefore the claim.

□

Problem 5. Let $f \in L^1(\mathbb{R})$. For each $\lambda > 0$, put $f_\lambda(x) = \lambda f(\lambda x)$. Show that $f_\lambda \in L^1(\mathbb{R})$, and that $f_\lambda \to f$ in $L^1(\mathbb{R})$ as $\lambda \to 1$.

Proof. For $n \in \mathbb{N}$, we have $\int_{[-n,n]} |f_\lambda| = \int_{[-\lambda n, \lambda n]} |f| \leq ||f||_1 < \infty$. This is true for any $n \in \mathbb{N}$, $f_\lambda \in L^1(\mathbb{R})$. Define $g_\lambda := \frac{|f_\lambda| + |f| - |f_\lambda - f|}{2}$. Then $g_\lambda \geq 0$. By Lusin's theorem, for any $\epsilon > 0$, we can find a measurable set E_ϵ such that

(i). $||f \cdot \chi_{\mathbb{R} \setminus E_\epsilon}||_1 < \epsilon/2$,

(ii). $||f_\lambda \cdot \chi_{\mathbb{R} \setminus E_\epsilon}||_1 < \epsilon$ for all close enough λ to 1, and

(iii). $\lim_{\lambda \to 1} g_\lambda = |f|$ pointwise on E_ϵ.

Thus by Fatou's lemma,

$$||f_{E_\epsilon}||_1 \leq ||f_{E_\epsilon}||_1 - \frac{1}{2} \limsup_{\lambda \to 1} ||(f_\lambda - f)_{E_\epsilon}||_1.$$

Hence $\lim_{\lambda \to 1} ||(f_\lambda - f)_{E_\epsilon}||_1 = 0$. This together with (ii) gives us

$$\lim_{\lambda \to 1} ||f_\lambda - f)||_1 < \epsilon.$$

Because this is true for any $\epsilon > 0$, we have the claim. □

Problem 6. Assume $f_n \to f$ and $g_n \to g$ in $L^2(\mathbb{R})$. Show that $f_n g_n \to fg$ in $L^1(\mathbb{R})$.

Proof. We have that

$$\begin{aligned} ||f_n g_n - fg||_1 = ||(f_n - f)g_n + (g_n - g)f||_1 &\overset{\text{Minkowski}}{\leq} ||(f_n - f)g_n||_1 + ||(g_n - g)f||_1 \\ &\overset{\text{Holder}}{\leq} ||f_n - f||_2 \cdot ||g_n||_2 + ||g_n - g||_2 \cdot ||f||_2. \end{aligned}$$

Because $g_n \to g$ in $L^2(\mathbb{R})$, $g_n^2 \to g^2$ in $L^1(\mathbb{R})$. Thus by letting $n \to \infty$ in the above inequality we get the claim as $f, g \in L^2(\mathbb{R})$. □

August 2021

Problem 1. Let $n \in \mathbb{N}$ and $p_1, p_2, \cdots, p_n \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$. If $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \cdots, f_n \in L^{p_n}(\mathbb{R})$, then show that $f_1 f_2 \cdots f_n \in L^1(\mathbb{R})$.

Proof. We prove by induction on n. The base case (when $n = 2$) follows from the Holder's inequality. Let $k \in \mathbb{N}_{\geq 2}$ and suppose that the claim is true for all $n = 2, \cdots, k$. That is; if $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = 1$ and $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \cdots, f_k \in L^{p_k}(\mathbb{R})$, then $f_1 f_2 \cdots f_k \in L^1(\mathbb{R})$.

Need to show if $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} + \frac{1}{p_{k+1}} = 1$ and $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \cdots, f_k \in L^{p_k}(\mathbb{R}), f_{k+1} \in L^{p_{k+1}}(\mathbb{R})$, then $f_1 f_2 \cdots f_k f_{k+1} \in L^1(\mathbb{R})$. Let $q = \frac{p_k p_{k+1}}{p_k + p_{k+1}}$. Then $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{q} = 1$. This implies $q > 1$. Thus if we can show $f_k f_{k+1} \in L^q(\mathbb{R})$, then we are done by the induction hypothesis as it implies $f_1 f_2 \cdots (f_k f_{k+1}) \in L^1(\mathbb{R})$. Notice that

$$f_k \in L^{p_k}(\mathbb{R}) \Rightarrow f_k^{p_k} \in L^1(\mathbb{R}) \Rightarrow f_k^q \in L^{\frac{p_k}{q}}(\mathbb{R}).$$

Similarly

$$f_{k+1} \in L^{p_{k+1}}(\mathbb{R}) \Rightarrow f_k^{p_{k+1}} \in L^1(\mathbb{R}) \Rightarrow f_{k+1}^q \in L^{\frac{p_{k+1}}{q}}(\mathbb{R}).$$

Since $\frac{q}{p_k} + \frac{q}{p_{k+1}} = 1$, by the Holder's inequality, $||f_k^q f_{k+1}^q||_1 \leq ||f_k^q||_{\frac{p_k}{q}} + ||f_{k+1}^q||_{\frac{p_{k+1}}{q}} < \infty$. Hence $f_k f_{k+1} \in L^q(\mathbb{R})$. □

Problem 2. Let A and B be Lebesgue measurable subsets of $\mathbb{R}$, with $A \subset B$. Show that $m(A) = m(B)$ if and only if every subset C satisfying $A \subset C \subset B$ is Lebesgue measurable.

Proof. ($\Rightarrow$) Suppose $m(A) = m(B)$. Then for any $C \subset \mathbb{R}$ with $A \subset C \subset B$, there is a set $E \subset \mathbb{R}$, say, of Lebesgue measure zero such that $C = A \sqcup E$. Hence C is measurable.
($\Leftarrow$) For contra-positive, assume $m(A) \lneq m(B)$. Then by Vitali's theorem there is a non-measurable set $F \subset B \backslash A$. Then $C := A \sqcup F$ is non-measurable with $A \subset C \subset B$. □

Problem 3.

(a). Prove that for every $\epsilon > 0$ there is an open subset of $[0, 1]$, whose closure is $[0, 1]$, and whose Lebesgue measure is less than ϵ.

(b). Prove that for every $a \in (0, 1]$ there is an open subset of $[0, 1]$, whose closure is $[0, 1]$, and whose Lebesgue measure is equal to a.

Proof. (a). If $\epsilon > 1$, then take $(0, 1)$ to be the open set. If $\epsilon \in (0, 1]$, then take the set below in part (b) when, for example, $a = \frac{\epsilon}{2}$.

(b). Let $a \in (0, 1]$. Consider the standard middle third Cantor set (let's denote it by $C_{\frac{1}{3}}$). Instead of removing middle thirds (the ratio of *"length of a removed interval in stage $n+1$"* to *"the length of the interval where it belonged as the middle third interval in stage n"* is $1 : 3$), let's remove the middle $(\frac{1+2a}{a})^{th}$. That is; the ratio of *"length of a removed interval in stage $n+1$"* to *"the length of the interval where it belonged as the middle $(\frac{1+2a}{a})^{th}$ interval in stage n"* is $1 : \frac{1+2a}{a}$. Let U_a be the disjoint union of those removed open intervals in this process. Hence U_a is open and clearly $\overline{U_a} = [0,1]$ as $[0,1]\backslash U_a$ does not contain any open interval. And

$$m(U_a) = \sum_{n=0}^{\infty} 2^n \left(\frac{a}{1+2a}\right)^{n+1} = a.$$

$\square$

Problem 4. Let $f, g : [0,1] \longrightarrow \mathbb{R}$. Prove that

$$TV(fg) \leq TV(f) \sup_{x\in[0,1]} |g(x)| + TV(g) \sup_{x\in[0,1]} |f(x)|,$$

where TV is the total variation. Give examples where the inequality is strict, and where the equality holds although none of the functions f, g is constant.

Proof. Let $\mathcal{P} = \{0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1\}$ be a partition of $[0,1]$. Then

$$\begin{aligned}
V(fg, \mathcal{P}) &= \sum_{k=1}^{n} |f(a_k)g(a_k) - f(a_{k-1})g(a_{k-1})| \\
&\leq \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})||g(a_k)| + |g(a_k) - g(a_{k-1})||f(a_{k-1})| \\
&\leq \sup_{x\in[0,1]} |g(x)| \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| + \sup_{x\in[0,1]} |f(x)| \sum_{k=1}^{n} |g(a_k) - g(a_{k-1})| \\
&= V(f, \mathcal{P}) \sup_{x\in[0,1]} |g(x)| + V(g, \mathcal{P}) \sup_{x\in[0,1]} |f(x)|.
\end{aligned}$$

Then by taking the suprimum over all the partitions of $[0,1]$, we get the claim. $\square$

For the strict inequality: Let $f(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$ and $g(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$. Then $fg = 0$.

Thus $TV(fg) = 0$ and $TV(f) \sup_{x\in[0,1]} |g(x)| + TV(g) \sup_{x\in[0,1]} |f(x)| = 1 \cdot 1 + 1 \cdot 1 = 2$.

For the equality: let $f(x)$ be as above and and let g be any function with $TV(g|_{[\frac{1}{2},1]}) = \infty$. Then the both sides of the inequality becomes ∞.

If need a finite equality: Let $f(x)$ be as above and $g(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$. Then $fg(x) := \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{else} \end{cases}$. Thus $TV(fg) = 2$ (for the partition, for example, $\{0 < \frac{1}{2} < 1\}$) and $TV(f) \sup_{x\in[0,1]} |g(x)| + TV(g) \sup_{x\in[0,1]} |f(x)| = 1 \cdot 1 + 1 \cdot 1 = 2$.

Problem 5. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,1])$, satisfying

$$\int_{[0,1]} |f_n| \leq \frac{1}{n^2}$$

for every $n \in \mathbb{N}$. Show that f_n converges to 0 pointwise a.e.

Proof. Let $\epsilon > 0$ be given. Then by the Chebychev's inequality,

$$m(\{x \in [0,1] : |f_n(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_{[0,1]} |f_n| \leq \frac{\epsilon}{n^2}.$$

Thus $\sum_{n=1}^{\infty} m(\{x \in [0,1] : |f_n(x)| \geq \epsilon\}) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{n^2} < \infty$. Therefore by the Borel-Cantelli lemma, for almost all $x \in [0,1]$ there is $N \in \mathbb{N}$ such that for each $n > N$, $x \notin \{x \in [0,1] : |f_n(x)| \geq \epsilon\}$. In other words: for almost all $x \in [0,1]$ there is $N \in \mathbb{N}$ such that for each $n > N$, $|f_n(x)| < \epsilon$. Hence the claim. $\square$

Problem 6. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function. Evaluate

$$\lim_{n \longrightarrow \infty} \int_0^1 nx^{n-1} f(x)dx.$$

Answer. For a fixed $a \in \mathbb{R}$ and $k \in \mathbb{N}$, we get $\lim_{n \longrightarrow \infty} \int_0^1 nx^{n-1} ax^k dx = a$. Thus by the linearity of the Lebesgue integral, $\lim_{n \longrightarrow \infty} \int_0^1 nx^{n-1} P(x)dx = P(1)$ for any polynomial $P(x)$ on $[0,1]$. Let f be a continuous function on $[0,1]$. By Weierstrass approximation theorem, there is a sequence $(P_m(x))_{m=1}^{\infty}$ of polynomial on $[0,1]$ such that $P_m \rightrightarrows f$. Because one of the limits is uniform, we can interchange the two limits. Thus

$$\begin{aligned}
f(1) = \lim_{m \longrightarrow \infty} P_m(1) &= \lim_{m \longrightarrow \infty} \Big(\lim_{n \longrightarrow \infty} \int_0^1 nx^{n-1} P_m(x)dx \Big) \\
&= \lim_{n \longrightarrow \infty} \Big(\lim_{m \longrightarrow \infty} \int_0^1 nx^{n-1} P_m(x)dx \Big) \\
&= \lim_{n \longrightarrow \infty} \int_0^1 \lim_{m \longrightarrow \infty} nx^{n-1} P_m(x)dx \\
&= \lim_{n \longrightarrow \infty} \int_0^1 nx^{n-1} f(x)dx.
\end{aligned}$$

Note: We can not start with $\lim_{n \longrightarrow \infty} \int_0^1 nx^{n-1} f(x)dx$ as we still don't know if the limit exists.

Winter 2021

1. Give an example of a measurable set $E \subset \mathbb{R}$ of finite measure such that $m(E \cap (E+r)) > 0$ for every $r \in \mathbb{R}$ (where $E + r = \{x + r : x \in E\}$).

Example 1.

$$\bigsqcup_{n=1}^{\infty}[-n - \frac{1}{2^n}, -n] \sqcup [-1, 1] \sqcup \bigsqcup_{n=1}^{\infty}[n, n + \frac{1}{2^n}].$$

Example 2. Let $(q_n)_{n\in\mathbb{N}}$ an enumeration of $\mathbb{Q}$. Then

$$\bigcup_{n=1}^{\infty}[q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}].$$

2. Let $f_n : [0, 1] \longrightarrow [0, \infty)$ $(n = 1, 2, 3, \cdots)$ be measurable functions. Prove that there exist numbers $a_n > 0$ $(n = 1, 2, 3, \cdots)$ such that the series $\sum_{n=1}^{\infty} a_n f_n(x)$ converges for almost every $x \in [0, 1]$.

Proof. Because f_n is finite, $\lim_{k\longrightarrow\infty} f_n^{-1}((k, \infty)) = 0$ for each $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$, there is $\delta(n) > 1$, say, such that $m(E_n) \leq 1/n^2$. Here $E_n := f_n^{-1}((\delta(n), \infty))$. Let $a_n = \frac{1}{\delta(n)\cdot 2^n}$. Then the series $\sum_{n=1}^{\infty} a_n f_n(x)$ diverges if and only if x belongs to infinitely many $E_n's$. That is $x \in \bigcap_{k=1}^{\infty} E_{n_k}$. Borel-Cantelli lemma says the set consists with such points has measure zero as $\sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} 1/n^2 < \infty$. $\square$

3. Let $f \in L^1(\mathbb{R})$. Prove that for every measurable set $E \subset \mathbb{R}$ of finite measure we have

$$\lim_{t\longrightarrow\infty} \int_E f(x + t)\, dx = 0.$$

Proof. Let $\epsilon > 0$ be given. Because $f \in L^1(\mathbb{R})$ there is $\delta > 0$ such that for any $F \subset \mathbb{R}$ with $m(F) < \delta$ we get $\int_F |f| < \frac{\epsilon}{2}$. Because $m(E) < \infty$, we can find an interval $[-a, a]$, say, such that $\int_{\mathbb{R}\setminus[-a,a]} |f| < \frac{\epsilon}{2}$ and $m(E\setminus[-a, a]) < \min\{\delta, \frac{\epsilon}{2}\}$. Because Lebesgue measure is invariant under translation, for all $t > 2a$

$$|\int_E f(x+t)| \leq \int_E |f(x+t)| \leq \int_{E\cap[-a,a]} |f(x+t)| + \frac{\epsilon}{2} = \int_{(E\cap[-a,a])+t} |f(x)| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

4. Assume that for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $C \in \mathbb{R}$, the total variation of f on every compact interval is smaller than or equal to C. Prove that

$$\int_{\mathbb{R}} |f(x + h) - f(x)|\, dx \leq C|h|.$$

Proof. Let $h \in \mathbb{R}$ be given. Define $g : \mathbb{R} \longrightarrow [0, \infty)$ by $g(x) := |f(x+h) - f(x)|$. For $i \in \mathbb{N}$, let $a_i := \sup\{i|h|, (i+1)|h|\}$. Then $\int_{i|h|}^{(i+1)|h|} g(x)\ dx \leq |h| \cdot g(a_i)$. Thus for any $n \in \mathbb{N}$,

$$\int_{[-n|h|, n|h|]} g(x)\ dx = \sum_{i=-n}^{n-1} \int_{[i|h|,(i+1)|h|]} g(x)\ dx \leq |h| \sum_{i=-n}^{n-1} g(a_i) = |h| \sum_{i=-n}^{n-1} |f(a_i + h) - f(a_i)|$$
$$\leq |h| V(f, \mathcal{P}) \leq |h| C.$$

Here $\mathcal{P}$ is a partition of $[-n|h|, n|h|]$ which includes the points $\{a_i, a_i + h\}_{i=-n}^{n-1}$. Because this is true for any $n \in \mathbb{N}$, we get the claim by the continuity of the Lebesgue integral. □

5. Let $f \in L^\infty([0, 1])$. Prove that

$$\lim_{p \longrightarrow \infty} ||f||_p = ||f||_\infty.$$

Proof. Because $||f||_p \leq ||f||_q$ for any $p \leq q$, the limit $L := \lim_{p \longrightarrow \infty} ||f||_p$ exists in $[0, \infty]$. But since $f \in L^\infty([0, 1])$, $L \in [0, \infty)$. Notice that $||f||_p = \Big(\int_{[0,1]} |f|^p \Big)^{1/p} \leq \Big(||f||_\infty^p \Big)^{1/p} = ||f||_\infty$. So, $L \leq ||f||_\infty$. To show the reverse inequality: for $n \in \mathbb{N}$, define

$$E_n := \{x \in [0, 1] : ||f||_\infty (1 - \frac{1}{n}) \leq |f(x)| \leq ||f||_\infty\}.$$

Then $m(E_n) > 0$. Thus

$$||f||_p^p = \int_{[0,1]} |f|^p = \int_{E_n} |f|^p + \int_{[0,1] \setminus E_n} |f|^p \geq ||f||_\infty^p \cdot \Big(1 - \frac{1}{n}\Big)^p \cdot m(E_n)$$

Hence $L = \lim_{p \longrightarrow \infty} ||f||_p \geq ||f||_\infty \Big(1 - \frac{1}{n}\Big)$. Since this is true for any $n \in \mathbb{N}$, we get $L \geq ||f||_\infty$ and therefore the claim. □

6. Give an example of a function $f : [0, 1] \longrightarrow \mathbb{R}$ such that $f \in L^p([0, 1])$ for every $p \in [1, \infty)$, but $f \notin L^\infty([0, 1])$.

Consider the function $f : [0, 1] \longrightarrow \mathbb{R}$ defined by

$$f(x) := \begin{cases} \sum_{n=1}^\infty n \chi_{\left(2^{\frac{1}{n+1}}, 2^{\frac{1}{n}}\right]}(x) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly it is unbounded and for any $p \geq 1$,

$$||f||_p^p = \sum_{n=1}^\infty \frac{n^p}{2^{n+1}} < \infty$$

by, for example, root test. Hence $f \in L^p([0, 1])$.

Summer 2020

Pavel Bleher

Problem 1. Let E be the set of all real numbers x on the interval $[0,1]$ such that in the decimal form of $x = 0.i_1 i_2 i_3 \cdots$ the number 5 appears infinitely many times. Prove that $mE = 1$.

Proof. Let $F := \{x \in [0,1] :$ the decimal expansion of x has no $5\}$. Then

$$F^c := [0,1]\backslash F = \frac{1}{10}(5,6) \sqcup \frac{1}{10^2}\bigsqcup_{\substack{i=0\\ i\neq 5}}^{9}(i5,i6) \sqcup \frac{1}{10^3}\bigsqcup_{\substack{i,j=0\\ i,j\neq 5}}^{9}(ij5,ij6) \sqcup \cdots$$

(Here we write 5 as $4.\bar{9}$ to make F compact)

Thus $m(F^c) = \frac{1}{9}\sum_{i=1}^{\infty}(\frac{9}{10})^i = 1$. This says $m(F) = 0$. For $n \in \mathbb{N} \cup \{0\}$, define

$$F_n := \{x \in [0,1] : \text{the decimal expansion of } x \text{ has no 5 after the } n^{th} \text{ place}\}.$$

Then $F_n \subset \frac{1}{10^n}\bigcup_{i=0}^{10^n-1}(i+F)$. Because the Lebesgue measure is invariant under translation, we then get $m(F_n) = 0$ and therefore $m(\bigcup_{n=1}^{\infty} F_n) = 0$. Since $E = [0,1]\backslash(\bigcup_{n=1}^{\infty} F_n)$, we have the claim. □

Problem 2. Prove that if functions f_1, f_2 are absolutely continuous on $[0,1]$, then the function $f(x) = \max\{e^{f_1(x)}, |f_2(x)|\}$ is absolutely continuous on $[0,1]$ as well.

Proof. Notice that e^x is Lipschitz on $[0,1]$ (with a Lipschitz constant, for an example, e. To see: the line $y = x+e$ lies above the curve $y = e^x$ over $[0,1]$. Or, the line $y = ex$ lies above the curve $y = e^x - 1$ over $[0,1]$.). Hence $e^{f_1(x)} \in AC[0,1]$. By the reverse triangle inequality, $|f_2| \in AC[0,1]$. This together with the fact $|f(a) - f(b)| \leq |e^{f_1(b)} - e^{f_1(a)}| + ||f_2(b)| - |f_2(a)||$ for any $a, b \in [0,1]$, we have $f \in AC[0,1]$. □

Problem 3. Prove that for any integrable function f on the interval $[a,b]$,

$$\lim_{n\longrightarrow\infty}\int_a^b f(x)\sin(\sin(nx))\ dx = 0.$$

Proof. Because the collection of all step functions on $[a,b]$ with rational values is dense (under $||\cdot||_1$ norm) in $L^1([a,b])$, and the linearity of the Lebesgue integral, it is enough to show

$$\lim_{n\longrightarrow\infty}\int_p^q \sin(\sin(nx))\ dx = 0 \text{ for any interval } [p,q] \subseteq [a,b].$$

By changing the variable, we get

$$\lim_{n\longrightarrow\infty}|\int_p^q \sin(\sin(nx))\ dx| = \lim_{n\longrightarrow\infty}\frac{1}{n}|\int_{np}^{nq}\sin(\sin(x))\ dx| \leq \lim_{n\longrightarrow\infty}\frac{2}{n} = 0.$$

Note: Over any interval, the area under the curve $y = \sin x$ is in between -2 and 2. □

Problem 4. Prove that for any $1 \leq p < \infty$ there is a constant $C_p > 0$ such that for any integrable function f on $[0,1]$,

$$|| \ln(1+|f|)||_p \leq C_p(1+||f||_1).$$

Proof. For a fixed $p \geq 1$, defined the function $\varphi_p : [1,\infty) \longrightarrow \infty$ by $\varphi_p(x) := (p+\ln x)^p$. Since $\varphi_p''(x) = -\frac{p}{x^2}(p+\ln x)^{p-2}(1+\ln x) < 0$ on $[1,\infty)$, φ_p is concave. Let $f \in L^1([0,1])$. Since $1+|f| \geq 1$, by the Jensen's inequality for concave functions we get

$$\int_{[0,1]} \varphi_p \circ (1+|f|) \leq \varphi_p\Big(\int_{[0,1]} 1+|f|\Big).$$

Because

$$|| \ln(1+|f|)||_p^p \leq \int_{[0,1]} (p+\ln(1+|f|))^p = \int_{[0,1]} \varphi_p \circ (1+|f|)$$

and

$$\begin{aligned} \varphi_p\Big(\int_{[0,1]} 1+|f|\Big) = \Big(p+\ln\big(\int_{[0,1]} 1+|f|\big)\Big)^p &= \Big(p+\ln(1+||f||_1)\Big)^p \\ &= \Big(\ln e^p(1+||f||_1)\Big)^p \\ &\leq [e^p(1+||f||_1)]^p, \end{aligned}$$

we get the claim by letting $C_p = e^p$. ☐

Problem 5. Let $f \in L^2[0,1]$ and $m = \int_{[0,1]} f$. Prove that

$$m^2 + \Big(\int_0^1 |f(x)-m|dx\Big)^2 \leq \int_0^1 f^2(x)dx.$$

Proof. Because $f \in L^2[0,1]$, $f \in L^1[0,1]$. Hence $|m| < \infty$ and

$$m^2 + \Big(\int_0^1 |f(x)-m|dx\Big)^2 \leq m^2 + \int_0^1 (f-m)^2 = m^2 + \int_0^1 f^2 - 2m^2 + m^2 = \int_0^1 f^2.$$

The inequality is due to Jensen. ☐

Problem 6. Let $f \in L^1(\mathbb{R})$ be such that

$$\int_a^{\frac{a+b}{2}} f = \int_{\frac{a+b}{2}}^b f, \forall \ -\infty < a < b < \infty.$$

Prove that then $f(x) = 0$ almost everywhere on $(-\infty,\infty)$.

Proof. We can assume $f \geq 0$. Suppose the set $E := f^{-1}((0,\infty))$ has a positive measure. Then there is $a > 0$ such that $M := \int_{-a}^a f > 0$. Then by the hypothesis $\int_{-a}^{3a} f = 2M$ and therefore by induction, for any $n \in \mathbb{N}$, $\int_{-a}^{(2n-1)a} f = nM$. Since $M > 0$, $\int_{\mathbb{R}} f = \infty$. This contradicts the fact f is integrable. Hence $m(E) = 0$. The general case follows by considering f_+ and f_- separately and $|f| = f_+ + f_-$. ☐

Winter 2020

Pavel Bleher

Problem 1. Let E be the set of real numbers x on the interval $[0,1]$ such that in the decimal expansion form of $x = 0.i_1 i_2 i_3 \cdots$, there is no 5, so that $i_k \neq 5$ for all k. Prove that

1. the set E is uncountable;
2. $m(E) = 0$, where $m(E)$ is the Lebesgue measure of E.

Proof. 1. The set $\{x \in [0,1] :$ the decimal expansion of x consists with only 0 and 1$\}$ is uncountable (by Cantor's diagonal argument) and is a subset of E.

2. Notice that

$$[0,1]\backslash E = \frac{1}{10}[5,6) \sqcup \frac{1}{10^2}\bigsqcup_{\substack{i=0\\ i\neq 5}}^{9}[i5, i6) \sqcup \frac{1}{10^3}\bigsqcup_{\substack{i,j=0\\ i,j\neq 5}}^{9}[ij5, ij6) \sqcup \frac{1}{10^4}\bigsqcup_{\substack{i,j,k=0\\ i,j,k\neq 5}}^{9}[ijk5, ijk6) \sqcup \cdots$$

Thus $m([0,1]\backslash E) = \frac{1}{9}\sum_{i=1}^{\infty}(\frac{9}{10})^i = 1$ and therefore $m(E) = 0$.

□

Problem 2. Let E be the same set as in Problem 1. Calculate the fractal dimension of E, defined as

$$d := \lim_{k\to\infty} \frac{\ln N(k)}{\ln k},$$

where $N(k)$ is the smallest number of intervals of length $\frac{1}{k}$ covering E. Prove rigorously the existence of the limit.

Answer. Since

$$[0,1]\backslash E = \frac{1}{10}[5,6) \sqcup \frac{1}{10^2}\bigsqcup_{\substack{i=0\\ i\neq 5}}^{9}[i5, i6) \sqcup \frac{1}{10^3}\bigsqcup_{\substack{i,j=0\\ i,j\neq 5}}^{9}[ij5, ij6) \sqcup \frac{1}{10^4}\bigsqcup_{\substack{i,j,k=0\\ i,j,k\neq 5}}^{9}[ijk5, ijk6) \sqcup \cdots,$$

for each $n \in \mathbb{N}$, E can be covered by 9^n number of intervals of length $1/10^n$. For $k \in \mathbb{N}\cup\{0\}$, let $n_k \in \mathbb{N}$ be such that $10^{n_k} \leq k < 10^{n_k+1}$. Then $9^{n_k} = N(10^{n_k}) \leq N(k) \leq N(10^{n_{k+1}}) = 9^{n_k+1}$. Hence

$$\frac{n_k \ln 9}{(n_k+1)\ln 10} \leq \frac{\ln N(k)}{\ln k} \leq \frac{(n_k+1)\ln 9}{n_k \ln 10}$$

and therefore by the squeeze theorem, $d = \frac{\ln 9}{\ln 10}$.

Problem 3. Let us enumerate all rational numbers on $[0,1]$, so that $\mathbb{Q}\cap[0,1] = \{r_1, r_2, r_3, \cdots\}$. Prove that for any $\alpha > -1$, the series

$$f(x) = \sum_{k=1}^{\infty} \frac{|x - r_k|^{\alpha}}{k^2}$$

is convergent for almost all x on $[0,1]$.

Proof. For $n \in \mathbb{N}$, define $f_n : [0,1] \longrightarrow [0,\infty)$ by $f_n(x) := \sum_{k=1}^{n} \frac{|x-r_k|^\alpha}{k^2}$. Then

$$\begin{aligned}
\int_{[0,1]} f_n(x)dx = \int_{[0,1]} \Big(\sum_{k=1}^{n} \frac{|x-r_k|^\alpha}{k^2}\Big)dx &= \sum_{k=1}^{n} \int_{[0,1]} \frac{|x-r_k|^\alpha}{k^2}dx \\
&= \sum_{k=1}^{n} \int_{[0,r_k]} \frac{(r_k-x)^\alpha}{k^2}dx + \int_{[r_k,1]} \frac{(x-r_k)^\alpha}{k^2}dx \\
&= \sum_{k=1}^{n} \frac{1}{k^2}\Big(\frac{r_k^{\alpha+1}}{\alpha+1} + \frac{(1+r_k)^{\alpha+1}}{\alpha+1}\Big) \\
&\leq \frac{1+2^{\alpha+1}}{\alpha+1} \sum_{k=1}^{\infty} \frac{1}{k^2} =: M < \infty. \qquad (0.1)
\end{aligned}$$

Since M does not depend on n, it is a uniform bound of the sequence $(\int_{[0,1]} f_n)_{n=1}^\infty$. Because $(f_n)_{n=1}^\infty$ is an increasing sequence, the pointwise limit $f(x) = \lim_{n\longrightarrow\infty} f_n(x)$ exists in $\overline{\mathbb{R}}$. Then by the Monotone Convergence Theorem and (0.1), it exists almost everywhere in $\mathbb{R}$. In other words, the given series converges for almost every $x \in [0,1]$. $\square$

Problem 4. Let $f(x)$ be the function defined in Problem 3. Prove that for any $\alpha > 0$, $f(x)$ is absolutely continuous on $[0,1]$.

Proof. Let f_n be as in Problem 3. Notice that each f_n is differentiable almost everywhere on $[0,1]$ (in fact, it is differentiable at all but finitely many points: $r_1, \cdots, r_n$) and $f_n'(x) = \sum_{k=1}^{n} \delta(r_k)\frac{\alpha(x-r_k)^{\alpha-1}}{k^2}$. Here $\delta(r_k) = 1$ if $x > r_k$ and $\delta(r_k) = -1$ if $x < r_k$. Notice that $|f_n'(x)| \leq \alpha \sum_{k=1}^{n} \frac{|x-r_k|^{\alpha-1}}{k^2}$. Because $\alpha > 0$, $\alpha - 1 > -1$. So, by replacing α with $\alpha - 1$ in Problem 3 we see that the series $g(x) := \sum_{k=1}^{\infty} \delta(r_k)\frac{\alpha(x-r_k)^{\alpha-1}}{k^2} (= \lim_{n\longrightarrow\infty} f_n'(x))$ converges absolutely for almost every $x \in [0,1]$. Hence

$$\begin{aligned}
f(x) = \lim_{n\longrightarrow\infty} f_n(x) &= \lim_{n\longrightarrow\infty} \Big(f_n(0) + \int_0^x f_n'(t)\ dt\Big) \\
&= \lim_{n\longrightarrow\infty} f_n(0) + \lim_{n\longrightarrow\infty} \int_0^x f_n'(t)\ dt \\
&= f(0) + \int_0^x \lim_{n\longrightarrow\infty} f_n'(t)\ dt, \text{ by general LDCT} \\
&= f(0) + \int_0^x g(t)\ dt.
\end{aligned}$$

By comparison test, $\lim_{n\longrightarrow 0} f_n(0) = f(0)$.

This says f is an indefinite integral over $[0,1]$. Hence $f \in AC([0,1])$. $\square$

Problem 5. Prove that for any integrable function f on $(-\infty,\infty)$,

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} \cos(nx^2)\ f(x)\ dx = 0.$$

Proof. By the definition of Lebesgue integral of a non-negative function (we can assume $f \geq 0$. The general case follows by considering the positive and negative parts of f separately), if we can show the claim for any non-negative bounded function of finite support, then we are done. But any such a function can be approximated by (in this case by $||\cdot||_1$) step functions. Therefore by the linearity of the Lebesgue integral, if we can show the claim for the case $f(x) = 1$ then we are done. In fact

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\Big|\int_{\mathbb{R}} \cos(nx^2)\ dx\Big| &= \lim_{n\longrightarrow\infty}\Big|\int_{\mathbb{R}} \frac{\cos(x^2)}{\sqrt{n}}\ dx\Big|, \text{by letting } x = \frac{x}{\sqrt{n}}. \\
&= \lim_{n\longrightarrow\infty} \frac{1}{\sqrt{n}}\Big|\int_{\mathbb{R}} \cos(x^2)\ dx\Big| \\
&\leq \lim_{n\longrightarrow\infty} \frac{2}{\sqrt{n}} = 0.
\end{aligned}$$

This implies $\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} \cos(nx^2)\ dx = 0$, and therefore the claim. $\square$

Problem 6. Let $f(x)$ be an integrable function on $(-\infty, \infty)$ and

$$f_n(x) = n\int_x^{x+\frac{1}{n}} f(t)\ dt,\ n = 1, 2, 3, \cdots$$

Prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}} |f_n(x) - f(x)|dx = 0.$$

Proof. • **Proof 1** Let $n \in \mathbf{N}$. Notice that

$$\begin{aligned}
\int_{\mathbb{R}} |f_n(x)|\ dx = \sum_{k=-\infty}^{\infty}\int_0^{\frac{1}{n}} |f_n(x+\frac{k}{n})|\ dx &= \int_0^{\frac{1}{n}}\Big(\sum_{k=-\infty}^{\infty} |f_n(x+\frac{k}{n})|\Big)\ dx \\
&\leq \int_0^{\frac{1}{n}}\Big(\sum_{k=-\infty}^{\infty} n\int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} |f(t))|\ dt\Big)\ dx \\
&= \int_0^{\frac{1}{n}} n\int_{\mathbb{R}} |f|\ dx = \int_{\mathbb{R}} |f|\ dx.
\end{aligned}$$

Thus

$$||f_n||_1 \leq ||f||_1,\ \text{for all } n \in \mathbb{N}. \tag{0.2}$$

Let $s(x) := \sum_{i=1}^k a_i\chi_{I_1}(x)$ be a step function. Then

$$s_n(x) = n\int_x^{x+\frac{1}{n}}\sum_{i=1}^k a_i\chi_{I_i}(t)\ dt = n\sum_{i=1}^k a_i m(I_i\cap[x, x+\frac{1}{n}]).$$

Let x be a point in the interior of $\cup_{i=1}^k I_i$. WLOG assume $x \in int(I_j) = (p_j, q_j)$ for some $j \in \{1, \cdots, k\}$ and $p_j, q_j \in \mathbb{R}$. Then there is $N \in \mathbb{N}$ (for example take N such that $\frac{1}{N} \leq \min\{x - p_j, q_j - x\}$) such that for each $n > N$, $[x, x+\frac{1}{n}] \in I_j$. Hence for each $n > N$,

$s_n(x) = a_j = s(x)$. If $x \notin \bigcup_{i=1}^k I_j$, then clearly $s(x) = s_n(x) = 0$ for any $n \in \mathbb{N}$. And the set of end-points of the intervals I_i's are finite. This says $s_n \xrightarrow{a.e} s$. That is $(s_n - s) \xrightarrow{a.e} 0$. Then by the Lebesgue Dominated Convergence Theorem we have the claim for any step function:

$$\lim_{n\to\infty} ||s_n - s||_1 = 0. \tag{0.3}$$

Let $f \in L^1(\mathbb{R})$. Let $\epsilon > 0$ be given. Then we can find $s \in step(\mathbb{R})$ such that $||f-s||_1 < \frac{\epsilon}{2}$. Then by (0.2) we have $||f_n - s_n||_1 = ||(f_n - s_n)||_1 \leq ||f - s||_1 < \frac{\epsilon}{2}$. Thus

$$||f_n - f||_1 \leq ||f_n - s_n||_1 + ||s_n - s||_1 + ||s - f||_1 < ||s_n - s|| + \epsilon.$$

Then by taking the limit as $n \to 0$, we get $\lim_{n\to\infty} ||f_n - f||_1 = 0$ as $\epsilon > 0$ is arbitrary.

- **Proof 2** Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) := \int_0^x f(t)dt$. Because $f \in L^1(-\infty, \infty)$, this is well-defined and absolutely continuous. Thus $F'(x) = f(x)$ for almost every $x \in (-\infty, \infty)$. Notice that

$$\frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}} = f_n(x), \text{ for } n \in \mathbb{N}.$$

Thus $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \mathbb{R}$. By Minkowski's inequality and (0.2) we have $||f_n - f||_1 \leq ||f_n||_1 + ||f||_1 \leq 2||f||_1 < \infty$. Therefore by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)|dx = 0.$$

□

Summer 2019

Pavel Bleher

Problem 1. Let $E \subset \mathbb{R}$, $mE = 0$, and $P(x)$ a real polynomial of degree $d \geq 1$. Let

$$S = P^{-1}(E) = \{x \in \mathbb{R} : P(x) \in E\}.$$

Prove that $mS = 0$.

Proof. Let $I \subset \mathbb{R}$ be a bounded interval. Then $P^{-1}(I)$ consists with at-most d number of disjoint bounded intervals. Notice that if $l(I)$ goes to 0 (that is when I contracts to a point), so does $m(P^{-1}(I))$. Otherwise there will be a point a such that $P^{-1}(\{a\})$ has an interval. But that contradicts $d \geq 1$. Thus for a given $\epsilon > 0$, there is $d(\epsilon) > 0$ such that if $l(I) < d(\epsilon)$, then $m(P^{-1}(I)) < \epsilon$ (basically because P is continuous and P' is bounded on compact sets). Because $m(E) = 0$, we can find a countable collection $\{I_{i,\epsilon}\}_{i\in\mathbb{N}}$ of disjoint open intervals which forms an open cover of E such that $l(I_{i,\epsilon}) \leq d(\frac{\epsilon}{2^i})$. Then $S = P^{-1}(E) \subset P^{-1}(\bigsqcup_{i=1}^{\infty} I_{i,\epsilon}) = \bigsqcup_{i=1}^{\infty} P^{-1}(I_{i,\epsilon})$ and therefore $m(S) \leq \sum_{i=1}^{\infty} m(P^{-1}(I_{i,\epsilon})) \leq \epsilon$ (S is measurable as E and f are measurable). Because this is true for any $\epsilon > 0$, $m(S) = 0$. $\square$

Problem 2. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|\tan(nx)|^{1/2}}{n^2}$$

is finite almost everywhere on the interval $[0, \pi]$.

Proof. If we can prove $f \in L^1([0, \pi])$, then we are done. For $k \in \mathbb{N}$, define

$$f_k(x) := \sum_{n=1}^{k} \frac{|\tan(nx)|^{1/2}}{n^2}.$$

Then

$$\begin{aligned}
\int_0^{\pi} f_k(x)\, dx &= \sum_{n=1}^{k} \int_0^{\pi} \frac{|\tan(nx)|^{1/2}}{n^2}\, dx = \sum_{n=1}^{k} \int_0^{n\pi} \frac{|\tan(x)|^{1/2}}{n^3}\, dx \\
&\leq \sum_{n=1}^{k} \frac{1}{n^3} \sum_{i=0}^{n-1} \int_{i\pi}^{(i+1)\pi} \left|(i+\frac{1}{2})\pi - x\right|^{-\frac{1}{2}}\, dx \\
&\left(\because\ |\tan(nx)|^{1/2} \leq \left|\frac{\pi}{2} - x\right|^{-\frac{1}{2}} \text{ on } [0,\pi]\right) \\
&= \sum_{n=1}^{k} \frac{1}{n^3} \sum_{i=0}^{n-1} 2\sqrt{2\pi} = 2\sqrt{2\pi} \sum_{n=1}^{k} \frac{1}{n^2} \\
&\leq 2\sqrt{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} =: M < \infty.
\end{aligned}$$

Since this is true for any $k \in \mathbb{N}$, by the Monotone Convergence Theorem, $\int_0^{\pi} f(x) \leq M < \infty$. $\square$

Problem 3. Prove that for any integrable function f on the line,

$$\lim_{n\to\infty}\int_{\infty}^{\infty} f(x)\ [\sin(nx)]^n\ dx = 0.$$

Proof. • **Proof 1.** By the density of *step*$(\mathbb{R})$ in $L^1(\mathbb{R})$ under $||\cdot||_1$ and the linearity of the Lebesgue integral, it is enough to show the claim for the case when $f \in$ *step*$(\mathbb{R})$. So let $f(x) \overset{\text{a.e}}{=} \chi_{[a,b]}(x)$ for any interval $[a, b]$. By change of variables we get

$$\int_{-\infty}^{\infty} \chi_{[a,b]}(x)\cdot[\sin(nx)]^n\ dx = 1/n\int_{na}^{nb}(\sin x)^n\ dx.$$

Let $\alpha_n = 1-1/\ln(n+1)$. Then $\alpha_n \nearrow 1$ and $\lim_{n\to\infty}\alpha_n^n = 0$. For $n \in \mathbb{N}$ choose $E_n \subset [na, nb]$ so that $\alpha_n \geq \sup\{|\sin x| : x \in [na, nb]\backslash E_n\}$ and $\int_{E_n}|\sin x|^n\ dx \leq a_n \leq 1$.

(Idea: Depending on $n \in \mathbb{N}$, choose a finite collection of disjoint intervals around the points $[na, nb] \cap \sin^{-1}(\{\pm1\})$ with above given properties).

Then

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\int_{na}^{nb}|\sin x|^n\ dx &= \lim_{n\to\infty}\Big(\frac{1}{n}\int_{(na,nb)\backslash E_n}|\sin x|^n\ dx + \frac{1}{n}\int_{E_n}|\sin x|^n\ dx\Big)\\ &\leq \lim_{n\to\infty}\Big(\frac{1}{n}\cdot\alpha_n^n\cdot(nb-na)+\frac{1}{n}\cdot 1\Big)\\ &= \lim_{n\to\infty}\alpha_n^n(b-a)\\ &= 0.\end{aligned}$$

• **Proof 2.** Instead of creating the sequence $(E_n)_{n=1}$ of sets as above, use the fact $m\Big(\bigcup_{n=1}^{\infty}\sin^{-1}(\{\pm1\})/n\Big) = 0$ and the Lebesgue Dominated Convergence Theorem on $f_n(x) := \chi_{[a,b]}(x)\cdot\sin^n(nx)$. □

Problem 4. Let $f \geq 0$ be a measurable function on $E \subset \mathbb{R}$ and $mE < \infty$. Let

$$E_n = \{x \in E : f(x) \geq n\}.$$

Prove that f is integrable if and only if

$$\sum_{n=1}^{\infty} mE_n < \infty.$$

Proof. Notice that $\sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} n\cdot m(E_n\backslash E_{n+1})$ and

$$\sum_{n=1}^{\infty} n\cdot m(E_n\backslash E_{n+1}) \leq \sum_{n=1}^{\infty}\int_{E_n\backslash E_{n+1}} f \leq \sum_{n=1}^{\infty}(n+1)\cdot m(E_n\backslash E_{n+1}). \tag{0.4}$$

Because $m(E) < \infty$,

$$f \text{ is integrable iff } \int_{E_1} f < \infty \text{ iff } \sum_{n=1}^{\infty} \int_{E_n \setminus E_{n+1}} f < \infty.$$

Then by the comparison tests (standard and limit comparison tests) and (0.4), we get the claim. □

Problem 5. Let f be a measurable function on $[0, 1]$ and

$$A = \{x \in [0, 1] : f(x) \in \mathbb{Z}\}.$$

Prove that the set A is measurable and

$$\lim_{n\to\infty} \int_0^1 |\cos(\pi f(x))|^n \; dx = mA.$$

Proof. Because $A = \bigsqcup_{n\in\mathbb{Z}} f^{-1}(\{n\})$, it is measurable as countable union of measurable sets is measurable.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 |\cos(\pi f(x))|^n \; dx &= \lim_{n\to\infty} \int_{[0,1]\setminus A} |\cos(\pi f(x))|^n \; dx + \lim_{n\to\infty} \int_A |\cos(\pi f(x))|^n \; dx \\
&= 0 + \lim_{n\to\infty} \int_A 1 \; dx, \; \because |\cos(\pi f(x)\chi_{[0,1]\setminus A}(x))|^n \to 0 \text{ and by LDCT.} \\
&= m(A).
\end{aligned}$$

□

Problem 6. Let $f_n(x) = \cos(nx)$ on $[0, 2\pi]$. Prove that there is no sub-sequence f_{n_k} converging almost everywhere in $[0, 2\pi]$.

Proof. Let $(f_{n_k})_{k\in\mathbb{N}}$ be a sub-sequence of $(f_n)_{n\in\mathbb{N}}$. For the sake of contradiction let's assume that it converges almost everywhere on $[0, 2\pi]$. Then the sequence $\big((f_{n_k} - f_{n_{k+1}})^2\big)_{k=0}^{\infty}$ converges to 0 almost everywhere on $[0, 2\pi]$. Therefore by LDCT,

$$0 = \lim_{k\to\infty} \int_0^{2\pi} (f_{n_k} - f_{n_{k+1}})^2 = \lim_{k\to\infty} \int_0^{2\pi} \Big(\cos(n_k x) - \cos(n_{k+1} x)\Big)^2 \; dx = 2\pi.$$

Hence the contradiction. □

Winter 2019

Pavel Bleher

Problem 1. Let F be a bounded, closed set on the line, and

$$F_\epsilon = \bigcup_{x \in F} [x-\epsilon, x+\epsilon], \ \epsilon > 0.$$

Prove that $\lim_{\epsilon \to 0} mF_\epsilon = mF$, where m is the Lebesgue measure.

Proof. Because $F = \bigcap_{k=1} F_{1/k}$, we have the claim by the continuity of Lebesgue measure (and, if you want, the squeeze theorem). $\square$

Problem 2. Let $E_1 \subset E_2 \subset \cdots$ be an increasing sequence of Lebesgue measurable sets on the line, such that the set $E = \bigcup_{n=1}^\infty E_n$ has a finite Lebesgue measure, $mE < \infty$. Prove that for any set $A \subset \mathbb{R}$ (not necessarily measurable),

$$\lim_{n\to\infty} m^*(A \cap E_n) = m^*(A \cap E).$$

Proof. Because $E_n \cap A \subset E \cap A$ for all $n \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} m^*(A \cap E_n) \le m^*(A \cap E).$$

On the other hand, because $m^*(A \cap E) < m(E \backslash E_n) + m^*(A \cap E_n)$, we have

$$m^*(A \cap E) \le \liminf_{n\to\infty} m^*(A \cap E_n).$$

Thus

$$\lim_{n\to\infty} m^*(A \cap E_n) = m^*(A \cap E).$$

$\square$

Problem 3. Let $f \in L^1(-\infty, \infty)$. Find the limit,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\Big(\cos x + \frac{\sin^2 x}{2}\Big)^n \, dx,$$

and justify your answer.

Answer. Notice that $0 \le (1-\cos x)^2$ implies $\cos x + \sin^2 x/2 \le 1$. Hence the maximum value the function $g(x) := |\cos x + \sin^2 x/2|$ can obtain is 1, and $g^{-1}(\{1\})$ is countable. Because $f \in L^1(-\infty, \infty)$, $f^{-1}(\{\infty\})$ has measure zero. Therefore the set $E := g^{-1}(\{1\}) \cup f^{-1}(\{\infty\})$ also has measure zero. Define the sequence $(f_n)_{n=1}$ of measurable functions on $\mathbb{R}$ by $f_n(x) := f(x)\Big(\cos x + \frac{\sin^2 x}{2}\Big)^n$. Then it is integrable as $|f_n| \le |f|$, and $f_n \to 0$ for all $x \in \mathbb{R}\backslash E$. Thus by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\Big(\cos x + \frac{\sin^2 x}{2}\Big)^n \, dx = 0.$$

Problem 4. Prove that for any integrable function f on the interval $[a, b]$,

$$\lim_{n\to\infty} \int_a^b \frac{\cos(nx)}{1+\sin^2(nx)} f(x) \, dx = 0.$$

Proof. Due to the separability of $L^1[a,b]$ by $\mathcal{S}'[a,b]$ (the collection of all step functions on $[a,b]$ with rational values) under $L^1[a,b]$ norm, and the linearity of the Lebesgue integral, if we can show the claim when $f(x)=1$ on any sub-interval $[p,q]\subseteq[a,b]$, then we are done. In fact

$$\lim_{n\to\infty}\int_p^q \frac{\cos(nx)}{1+\sin^2(nx)}\,dx = \lim_{n\to\infty}\Big[\frac{\tan^{-1}(\sin(nx))}{n}\Big]_p^q \le \lim_{n\to\infty}\frac{\pi}{n}=0.$$

□

Problem 5. Let $E\subset[0,1]$ be a measurable set such that there exists $\epsilon>0$ such that

$$m(E\cap[a,b])\ge \epsilon|b-a|$$

for all $[a,b]\subset[0,1]$. Prove that $mE=1$.

Proof. We show $m(E^c)=m([0,1]\backslash E)=0$. For $k>1$, choose collection $\{I_n\}_{n=1}$ of open disjoint intervals such that $E^c\subset\bigsqcup_{n=1}I_n$ and $\sum_{n=1}m(I_n)<m(E^c)+\epsilon/k$. Then

$$\epsilon\sum_{n=1}m(I_n)\le\sum_{n=1}m(E\cap I_n)<m\big(\big(\bigsqcup_{n=1}I_n\big)\backslash E^c\big)=\sum_{n=1}m(I_n)-m(E^c)<\frac{\epsilon}{k}.$$

This implies $\sum_{n=1}m(I_n)\le 1/k$ for any $k>1$. Thus $\sum_{n=1}m(I_n)=0$. Because $E^c\subset\bigsqcup_{n=1}I_n$, $m(E^c)=0$. □

Problem 6. Prove that for any function $f\in L^2[0,1]$,

$$||\ln(1+|f|)||_{L^1[0,1]}\le||f||_{L^2[0,1]}.$$

Proof. By Jensen's inequality (on $\varphi(x)=x^2$),

$$||\ln(1+|f|)||^2_{L^1[0,1]}\le\int\big(\ln(1+|f|)\big)^2. \tag{0.5}$$

A fact:

$$\text{For any } x\ge0,\ \ln(1+x)\le x. \tag{0.6}$$

□

By combining (0.5) and (0.6), we get the claim.

Summer 2018

Pavel Bleher

Problem 1. Let E be the set of real numbers x on the interval $[0,1]$ such that in the decimal form of $x = 0.i_1 i_2 i_3 \cdots$ there is no string of four consecutive digits 2018. Prove that

1. the set E is uncountable
2. $mE = 0$, where mE is the Lebesgue measure of E.

Proof. 1. The set $\{x \in [0,1] :$ the decimal expansion of x consists with only 3 and 4$\}$ is uncountable (by Cantor's diagonal argument), and is a subset of E.

2. Notice that

$$[0,1]\backslash E = \frac{1}{10^4}[2018, 2019) \sqcup \frac{1}{10^8} \bigsqcup_{\substack{i=0 \\ i\neq 2018}}^{10^4-1} [i2018, i2019) \sqcup \frac{1}{10^{12}} \bigsqcup_{\substack{i,j=0 \\ i,j\neq 2018}}^{10^4-1} [ij2018, ij2019) \sqcup \cdots$$

Thus $m([0,1]\backslash E) = \frac{1}{10^4-1}\sum_{i=1}^{\infty}(\frac{10^4-1}{10^4})^i = 1$, and therefore $m(E) = 0$.

$\square$

Problem 2. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - n^{-1}|^{1/2}}{n^2}$$

is absolutely continuous on $[0,1]$.

Proof. Because $|x - 1/n| \leq 1$ for any $x \in [0,1]$, the series $\sum_{n=1}^{\infty} \frac{|x-n^{-1}|^{1/2}}{n^2}$ converges absolutely for all $x \in [0,1]$. Thus for any $a, b \in [0,1]$, $a < b$ we have

$$|f(b) - f(a)| = \Big|\sum_{n=1}^{\infty} \frac{|b - n^{-1}|^{1/2}}{n^2} - \sum_{n=1}^{\infty} \frac{|a - n^{-1}|^{1/2}}{n^2}\Big| \leq \sum_{n=1}^{\infty} \frac{b-a}{2n^{3/2}} \leq M(b-a).$$

Here $M = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. Thus f is Lipschitz and therefore absolutely continuous. $\square$

Problem 3. Prove that for any integrable function f on the interval $[a,b]$,

$$\lim_{n\to\infty} \int_a^b f(x)\sin^4(nx)dx = \frac{3}{8}\int_a^b f(x)\ dx.$$

Proof. We have that

$$\sin^4(nx) = \frac{3 - 4\cos(2nx) + \cos(4nx)}{8}.$$

Thus if we can show that

$$\lim_{n\to\infty} \int_a^b f(x)\cos(knx)dx = 0,$$

for any $f \in L^1[a,b]$ and any constant $k \in \mathbb{R}$, then we are done. Because $\mathcal{S}[a,b]$ - the collection of all step functions on $[a,b]$, is dense in $L^1[a,b]$ under $L^1[a,b]$ norm, and the Lebesgue integral is linear, if we can show that

$$\lim_{n\to\infty}\int_p^q \cos(nx)dx = 0 \text{ for any } p,q \in \mathbb{R}, p < q,$$

then we are done. In fact

$$\lim_{n\to\infty}\Big|\int_p^q \cos(nx)dx\Big| = \lim_{n\to\infty}\frac{1}{n}\Big|\int_{np}^{nq}\cos x dx\Big| \leq \lim_{n\to\infty}\frac{2}{n} = 0.$$

□

Problem 4. Let $f \in L^2(-\infty,\infty)$. Prove that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)f(x+n)dx = 0.$$

Proof. Because $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x+n)dx$ for any $n \in \mathbb{N}$, from Holder's inequality we have

$$\int_{\mathbb{R}} \big|f(x)f(x+n)\big|dx \leq ||f(x)||_2 \cdot ||f(x+n)||_2 = ||f(x)||_2^2 < \infty.$$

Thus $f(x)f(x+n) \in L^1(-\infty,\infty)$ for all $n \in \mathbb{N}$. Therefore if we can show

$$\lim_{n\to\infty}\int_{-N}^{N} f(x)f(x+n)dx = 0 \text{ for any } N \in \mathbb{N},$$

then we are done.

And if we can show this claim for any simple function on $[-N,N]$, then we are done by the simple approximation theorem. Let $\varphi(x) = \sum_{i=1}^k a_i\chi_{E_i}(x)$ where $E_i \subset [-N,N]$ for all $i = 1,\cdots k$. Then $\varphi(x+n) = \sum_{i=1}^k a_i\chi_{E_i - n}(x)$. Hence

$$\int_{[-N,N]} \varphi(x)\varphi(x+n)dx = \sum_{i=1,j=i}^{k} a_ia_jm(E_i \cap (E_j - n)).$$

Because each E_j is bounded, for large enough n (say $n > 2N$) we have that $E_i \cap (E_j - n) = \emptyset$ for all $i,j \in \{1,\cdots k\}$. Thus

$$\lim_{n\to\infty}\int_{[-N,N]} \varphi(x)\varphi(x+n)dx = 0.$$

□

Problem 5. Let $f \in L^3[0,\pi]$ and

$$g(x) = \frac{f(x)}{|\sin x|^{0.1}}.$$

Prove that $||g||_2 \leq 2||f||_3$.

Proof. Let $\theta \in [0, \pi/2]$ be such that $\theta^3 = \sin^2\theta$. Then simple calculations show that $\pi/4 < \theta < \pi/3$. So, $x^3 < \sin^2 x$ on $[0, \theta)$ and $\sin^2 x < x^3$ on $(\theta, \pi/2]$. Hence

$$\begin{aligned}
\int_0^\pi \frac{dx}{\sin^{0.6} x} &= 2\int_0^{\pi/2} \frac{dx}{\sin^{0.6} x} \quad \because\ y = 1/\sin^{0.6} x \text{ is symmetric around } x = \pi/2 \\
&\le 2\Big(\int_0^\theta \frac{dx}{x^{0.9}} + \int_\theta^{\pi/2} \frac{dx}{\sin^{0.6} x}\Big) \\
&\le 2\Big(\big[x^{0.1}\big]_0^\theta + \frac{\pi/2 - \theta}{\sin^{0.6}\theta}\Big) \le 2\Big(\theta^{0.1} + \frac{\pi/4}{\theta^{0.9}}\Big) \le 2\big((\pi/3)^{0.1} + (\pi/4)^{0.1}\big) < 2(2+2) \\
&< 8. \qquad (0.7)
\end{aligned}$$

This says that $\sin^{0.2} x \in L^3[0, \pi]$. Then by (generalized) Holder's inequality we have

$$||g||_2 = \Big(\int_{[0,\pi]} \frac{f(x)^2}{\sin^{0.2}(x)} dx\Big)^{1/2} \le ||f||_3 \Big(\int_{[0,\pi]} \frac{dx}{\sin^{0.6} x}\Big)^{1/6} \le ||f||_3 8^{1/6} < 2||f||_3.$$

□

Problem 6. Let $f(x)$ be an integrable function on $(-\infty, \infty)$ and $n \ge 1$ an integer. Define

$$f_n(x) = n\int_x^{x+\frac{1}{n}} f(t)dt.$$

Prove that $||f_n||_1 \le ||f||_1$.

Proof. • **Proof 1**. By Tonelli,

$$||f||_n \le \int_{\mathbb{R}} n \int_{\mathbb{R}} |f(t)| \cdot \chi_{[x, x+\frac{1}{n}]} dt dx = \int_{\mathbb{R}} n \int_{\mathbb{R}} |f(t)| \cdot \chi_{[x, x+\frac{1}{n}]} dx dt = \int_{\mathbb{R}} n|f(t)|\frac{1}{n} = ||f||_1.$$

• **Proof 2**.

$$\begin{aligned}
||f||_n = \int_{\mathbb{R}} |f_n(x)|\ dx &= \sum_{k=-\infty}^{\infty} \int_0^{\frac{1}{n}} |f_n(x + \frac{k}{n})|\ dx = \int_0^{\frac{1}{n}} \Big(\sum_{k=-\infty}^{\infty} \big|f_n(x + \frac{k}{n})\big|\Big)\ dx \\
&\le \int_0^{\frac{1}{n}} \Big(\sum_{k=-\infty}^{\infty} n \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} |f(t)|\ dt\Big)\ dx \\
&= \int_0^{\frac{1}{n}} n||f||_1\ dx = ||f||_1.
\end{aligned}$$

□