



Indiana University-Purdue University–Indianapolis
Qualifying Exam
Real Analysis (MATH 54400)

Solutions

These are **NOT** the official solutions. They may not all be perfectly written.

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January 2025

Throughout the exam, λ denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{L})$.

Problem 1. Show that the set of all real numbers that have decimal expansion with the digit 5 appearing infinitely often is a Borel set.

Proof. Because the collection of all Borel sets is a σ -algebra, it is closed under taking complements, we will show that the set of all real numbers that have decimal expansion with digit 5 appearing finitely many often is a Borel set.

Furthermore, since a σ -algebra is closed under taking countable unions,

and $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1]$, and $[n, n+1] = n + [0, 1]$ for any $n \in \mathbb{Z}$, we will show that the

set of all numbers in the interval $[0, 1]$ that have decimal expansion with digit 5 appearing finitely many often is a Borel set.

For $k \in \mathbb{N}_0$, define

$$B_k := \{x \in [0, 1] : \text{decimal expansion of } x \text{ has exactly } k \text{ number of 5's}\}.$$

Then,

$$\begin{aligned} B_1 &= \frac{5 + B_0}{10} \cup \bigcup_{\substack{a_1=0 \\ a_1 \neq 5}}^9 \frac{a_1 5 + B_0}{10^2} \cup \bigcup_{\substack{a_1, a_2=0 \\ a_1, a_2 \neq 5}}^9 \frac{a_1 a_2 5 + B_0}{10^3} \cup \bigcup_{\substack{a_1, a_2, a_3=0 \\ a_1, a_2, a_3 \neq 5}}^9 \frac{a_1 a_2 a_3 5 + B_0}{10^4} \cup \dots \\ B_2 &= \frac{5 + B_1}{10} \cup \bigcup_{\substack{a_1=0 \\ a_1 \neq 5}}^9 \frac{a_1 5 + B_1}{10^2} \cup \bigcup_{\substack{a_1, a_2=0 \\ a_1, a_2 \neq 5}}^9 \frac{a_1 a_2 5 + B_1}{10^3} \cup \bigcup_{\substack{a_1, a_2, a_3=0 \\ a_1, a_2, a_3 \neq 5}}^9 \frac{a_1 a_2 a_3 5 + B_1}{10^4} \cup \dots \\ B_3 &= \frac{5 + B_2}{10} \cup \bigcup_{\substack{a_1=0 \\ a_1 \neq 5}}^9 \frac{a_1 5 + B_2}{10^2} \cup \bigcup_{\substack{a_1, a_2=0 \\ a_1, a_2 \neq 5}}^9 \frac{a_1 a_2 5 + B_2}{10^3} \cup \bigcup_{\substack{a_1, a_2, a_3=0 \\ a_1, a_2, a_3 \neq 5}}^9 \frac{a_1 a_2 a_3 5 + B_2}{10^4} \cup \dots \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Here by $a_1 a_2$, for example, we mean the two digits a_1 and a_2 of the number $a_1 a_2$; not the multiplication $a_1 \times a_2$.

Thus, it suffices to show that B_0 is Borel. Let

$$\begin{aligned}
I_0 &= [0, 1] \\
I_1 &= I_0 \setminus [0.5, 0.6) = \bigcup_{\substack{m=0 \\ m \neq 5}}^9 \frac{m + I_0}{10} \\
I_2 &= \bigcup_{\substack{m=0 \\ m \neq 5}}^9 \frac{m + I_1}{10} \\
&\cdot \\
&\cdot \\
&\cdot \\
I_{n+1} &= \bigcup_{\substack{m=0 \\ m \neq 5}}^9 \frac{m + I_n}{10} \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}$$

Then, $B_0 = \bigcap_{n=0}^{\infty} I_n$. Thus, B_0 is Borel. This completes the proof. \square

Problem 2. Suppose (X, \mathcal{S}, μ) is a measure space and $h \in \mathcal{L}^1(\mu)$. Prove for any $c > 0$ that

$$\mu(\{x \in X : |h(x)| \geq c\}) \leq \frac{1}{c} \|h\|_1.$$

Proof. Let $c > 0$. Then, $X = A \sqcup B$ where

$$A := \{x \in X : |h(x)| < c\} \quad \text{and} \quad B := \{x \in X : |h(x)| \geq c\}.$$

Thus,

$$\|h\|_1 = \|h|_A\|_1 + \|h|_B\|_1 \geq \|h|_B\|_1 = \int_B |h| \, d\mu \geq \int_B c \, d\mu = c \cdot \mu(B).$$

Therefore, $\mu(B) \leq \frac{1}{c} \|h\|_1$. \square

Problem 3. Prove the existence of the limit

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{1}{1+x^2} \cdot \frac{e^{nx}}{1+e^{nx}} \, d\lambda(x)$$

and compute it.

Proof. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ by $f_n(x) := \frac{1}{1+x^2} \cdot \frac{e^{nx}}{1+e^{nx}}$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) := \frac{1}{1+x^2}$. Then, $f_n \nearrow f$. Furthermore, we have the improper integral $\int_{\mathbb{R}} \frac{1}{1+x^2} d\lambda(x) = \pi$. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{1}{1+x^2} \cdot \frac{e^{nx}}{1+e^{nx}} d\lambda(x) = \pi.$$

□

Problem 4. A sequence of measurable functions $f_n : [0, 1] \rightarrow [-\infty, \infty]$ is said to **converge in measure** to a measurable function $f : [0, 1] \rightarrow [-\infty, \infty]$ if

$$\text{for any } \delta > 0, \lim_{n \rightarrow \infty} \lambda\{x \in [0, 1] : |(f_n - f)(x)| > \delta\} = 0.$$

Suppose that for each $n \in \mathbb{Z}^+$ we have $f_n \in \mathcal{L}^1([0, 1])$ and also that $f \in \mathcal{L}^1([0, 1])$. If $f_n \rightarrow f$ in measure, does it imply that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n d\lambda = \int_{[0,1]} f d\lambda?$$

Either prove it or give an explicit counterexample.

Answer. Let $f_n := n\chi_{[0, \frac{1}{n}]}$ and $f \equiv 0$. Notice that

$$\{x \in [0, 1] : |(f_n - f)(x)| > \delta\} = \left\{x \in [0, 1] : n\chi_{[0, \frac{1}{n}]} > \delta\right\} \subseteq \left[0, \frac{1}{n}\right] \text{ for large enough } n.$$

Thus,

$$\mu(\{x \in [0, 1] : |(f_n - f)(x)| > \delta\}) = \frac{1}{n} \rightarrow 0.$$

This implies, $f_n \xrightarrow{\mu} f \equiv 0$.

But

$$\int_{[0,1]} f_n d\mu = n \int_{[0,1]} \chi_{[0, \frac{1}{n}]} d\mu = n \cdot \frac{1}{n} = 1 \not\rightarrow 0 = \int_{[0,1]} f d\mu.$$

Problem 5. Let $C([0, 1])$ denote the vector space of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ with the operations of addition and scalar multiplication done pointwise. Prove that $C([0, 1])$ with the norm defined by $\int_0^1 |f|$ is not a Banach space.

Proof. Define the sequence $(f_n)_{n=1}^\infty$ of functions by $f_n(x) := nx\chi_{[0, \frac{1}{n}]}(x) + \chi_{[\frac{1}{n}, 1]}(x)$. That is:

$$f_n(x) = \begin{cases} nx, & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 1, & \text{if } \frac{1}{n} < x \leq 1. \end{cases}$$

Also, define the function $f : [0, 1] \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } 0 < x \leq 1. \end{cases}$$

Then,

$$\int_0^1 f_n d\mu = \int_0^{\frac{1}{n}} nx d\mu + \int_{\frac{1}{n}}^1 1 d\mu = \frac{1}{2n} + 1 - \frac{1}{n} \rightarrow 1 = \int_0^1 f d\mu.$$

□

But $f \notin C([0, 1])$ even though $f_n \in C([0, 1])$ for all $n \in \mathbb{N}$.

Problem 6. Let $\mathcal{L}^1(\mathbb{R})$ denote the Lebesgue measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\|f\|_1 := \int |f| d\lambda < \infty$. Ignoring that there exist non-zero functions $f \in \mathcal{L}^1(\mathbb{R})$ with $\|f\|_1 = 0$, let us consider $(\mathcal{L}^1(\mathbb{R}), \|\cdot\|_1)$ as a normed linear space.

(a). Show for Lebesgue almost every $x \in \mathbb{R}$ that

$$\int f(x-y)e^{-y^2} d\lambda(y)$$

is a finite number.

(b). Define a new function $Af : \mathbb{R} \rightarrow [-\infty, \infty]$ by $Af(x) = \int f(x-y)e^{-y^2} d\lambda(y)$. Prove that $Af \in \mathcal{L}^1(\mathbb{R})$.

(c). Show that A defines a bounded linear transformation $A : \mathcal{L}^1(\mathbb{R}) \rightarrow \mathcal{L}^1(\mathbb{R})$ and show that $\|A\| \leq \sqrt{\pi}$.

Proof. (a). By change of variables we get

$$\begin{aligned} \left| \int f(x-y)e^{-y^2} d\lambda(y) \right| &= \left| \int f(y)e^{-(x-y)^2} d\mu(y) \right| \\ &\leq \int |f(y)| d\mu(y), \because |e^{-(x-y)^2}| \leq 1 \\ &< \infty. \end{aligned}$$

Thus, the given integral is finite for almost every $x \in \mathbb{R}$.

Note. The *almost every* part comes from the fact:

“ $f \in \mathcal{L}^1(\mathbb{R}) \Rightarrow f(x) < \infty$ for almost every $x \in \mathbb{R}$ ”.

(b).

$$\begin{aligned}
\int |Af| \, d\lambda(x) &= \int \left| \int f(x-y)e^{-y^2} \, d\lambda(y) \right| \, d\lambda(x) \\
&< \int \int |f(x-y)e^{-y^2}| \, d\lambda(y) \, d\lambda(x) \\
&= \int \int |f(x-y)| \cdot e^{-y^2} \, d\lambda(y) \, d\lambda(x) \\
&= \int \int |f(x-y)| \cdot e^{-y^2} \, d\lambda(x) \, d\lambda(y), \text{ by part (a) and Tonelli's Theorem} \\
&= \int e^{-y^2} \left[\int |f(x-y)| \, d\lambda(x) \right] \, d\lambda(y) \\
&= \int e^{-y^2} \|f\|_1 \, d\lambda(y) \\
&= \|f\|_1 \int e^{-y^2} \, d\lambda(y) \\
&= \|f\|_1 \sqrt{\pi}.
\end{aligned}$$

Note. To see why $\int e^{-y^2} \, d\lambda(y) = \sqrt{\pi}$.

Let $I := \int e^{-y^2} \, d\lambda(y)$. Then

$$\begin{aligned}
I^2 &= \left(\int e^{-y^2} \, d\lambda(y) \right) \times \left(\int e^{-x^2} \, d\lambda(x) \right) \\
&= \int \int e^{-(x^2+y^2)} \, d\lambda(y) \, d\lambda(x) \\
&= \int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta, \text{ where } r^2 = x^2 + y^2, \tan \theta = \frac{y}{x}. \\
&= \int_0^{2\pi} \left[\frac{e^{-r^2}}{2} \right]_\infty^0 \, d\theta = \frac{1}{2} \int_0^{2\pi} d\theta \\
&= \pi.
\end{aligned}$$

Thus, $I = \sqrt{\pi}$.

(c). For $f, g \in \mathcal{L}^1(\mathbb{R})$ and $K \in \mathbb{R}$ we have

$$\begin{aligned}
A(f + Kg) &= \int [f(x-y) + Kg(x-y)] e^{-y^2} \, d\lambda(y) \\
&= \int f(x-y)e^{-y^2} \, d\lambda(y) + K \int g(x-y)e^{-y^2} \, d\lambda(y), \because \text{integration is linear}
\end{aligned}$$

Thus, A is linear. From part (b) we have $\|Af\| \leq \sqrt{\pi}\|f\|_1$ for any $f \in \mathcal{L}^1(\mathbb{R})$. Thus, $\|A\| \leq \sqrt{\pi}$.

□

January 2024

Below $m^*(\cdot)$ and $m(\cdot)$ are the outer Lebesgue and Lebesgue measures on the real line.

Problem 1. Let $\{E_n\}_n$ be pairwise disjoint measurable sets, $E = \cup_n E_n$, and A be any set. Show that

$$m^*(E \cap A) = \sum_n m^*(E_n \cap A).$$

Proof. By the countable sub-additivity,

$$m^*(E \cap A) \leq \sum_n m^*(E_n \cap A).$$

Thus if we can show the opposite inequality, then we are done. Because E_1 is measurable, by the definition of measurability and the pairwise disjointness of $\{E_n\}_{n=1}$ we get

$$m^*(E \cap A) = m^*(E_1 \cap A) + m^*\left(\bigsqcup_{n=2} (E_n \cap A)\right).$$

This implies for any $N \in \mathbb{N}$,

$$m^*(E \cap A) = \sum_{n=1}^N m^*(E_n \cap A) + m^*\left(\bigsqcup_{n=N+1} (E_n \cap A)\right).$$

Therefore for any $N \in \mathbb{N}$,

$$m^*(E \cap A) \geq \sum_{n=1}^N m^*(E_n \cap A).$$

Then by taking the limit as $N \rightarrow \infty$ we get

$$m^*(E \cap A) \geq \sum_{n=1} m^*(E_n \cap A).$$

This completes the proof. \square

Problem 2. Let f be a continuous function on $[0, 1]$. Show that there exists a measurable subset $E \subset [0, 1]$ such that $f(E)$ is not measurable if and only if there exists $A \subset [0, 1]$ such that $m(A) = 0$ and $m^*(f(A)) > 0$.

Proof. (\Rightarrow). Suppose there is a measurable set $E \subset [0, 1]$ such that $f(E)$ is not measurable. Then $E = G \cap A$ where G is a G_δ set and A is a measurable set such that A^c is a null set. Because the continuous image of a G_δ set is measurable, we must then have $f(A)$ is non-measurable. Hence $m^*(f(A)) > 0$ as every null set is measurable.

(\Leftarrow) Suppose there exists $A \subset [0, 1]$ such that $m(A) = 0$ and $m^*(f(A)) > 0$. Then by Vitali's theorem there is $F \subset f(A)$ such that F is non-measurable. Let $E = A \cap f^{-1}(F)$. Then E is measurable as $m(E) = 0$ and $f(E) = F$ is non-measurable. \square

Problem 3. Let $\{f_n\}_n$ be a sequence of measurable functions on E , $m(E) < \infty$. Assume that for every $x \in E$ there exists a constant M_x such that $|f_n(x)| \leq M_x$ for all n . Show that for every $\epsilon > 0$ there exists a closed set $F_\epsilon \subseteq E$ and a constant M_ϵ such that $|f_n(x)| \leq M_\epsilon$, for all n and $x \in F_\epsilon$, and $m(E \setminus F_\epsilon) < \epsilon$.

Proof. Let $\epsilon > 0$ be given. Because $m(E) < \infty$, we can find $N_\epsilon \in \mathbb{N}$ such that $m(E \setminus [-N_\epsilon, N_\epsilon]) < \epsilon/2$. Let $F := E \cap [-N_\epsilon, N_\epsilon]$. Then by Lusin's theorem we can find a decreasing sequence $(F_n)_{n=1}$ of compact subsets of F such that for each $n \in \mathbb{N}$, f_n is continuous on F_n and $m(F \setminus F_n) \leq \epsilon \sum_{k=1}^n 1/2^{k+1}$. Let $F_\epsilon = \bigcap_{n=1}^\infty F_n$. Then $m(E \setminus F_\epsilon) \leq \epsilon/2 + \epsilon \sum_{k=1}^\infty 1/2^{k+1} = \epsilon/2 + \epsilon/2 = \epsilon$.

It remains to show that there is a uniform constant M_ϵ such that $|f_n(x)| \leq M_\epsilon$, for all n and $x \in F_\epsilon$. For a contradiction let's assume that it is not the case. Then the decreasing sequence $(A_j)_{j=1}$ of compact sets defined by

$$A_j := \overline{\{x \in F_\epsilon : |f_{n_j}(x)| \geq j \text{ for some } n_j \in \mathbb{N}\}}$$

has non-empty intersection by the Cantor's intersection theorem. Then for any $x_0 \in \bigcap_{j=1}^\infty A_j$, we have $M_{x_0} \geq |f_{n_j}(x_0)| \geq j$ for all $j \in \mathbb{N}$. This implies $M_{x_0} = \infty$. Contradiction. \square

Problem 4. Let $\{f_n\}_n$ be sequence of functions on $[0, 1]$ whose total variations are uniformly bounded. If $f_n \rightarrow f$ pointwise on $[0, 1]$ as $n \rightarrow \infty$, show that f is a function of bounded variation. Will the conclusion remain true if we simply assume that each f_n is a function of bounded variation? (justify your answer)

Proof. Let $0 < \epsilon < 1$ be given. Let $\mathcal{P} := \{0 = x_0, x_1 \cdots x_{m-1} = 1\}$ be a partition of $[0, 1]$. Then for each $j \in \{0, \dots, m-1\}$, there is f_{n_j} such that $|f_{n_j}(x_j) - f(x_j)| < \frac{\epsilon}{2m}$. Let $M \geq \max\{n_j : j = 0, \dots, m-1\}$. Then $|f_M(x_j) - f(x_j)| < \frac{\epsilon}{2m}$ for all $j = 0, \dots, m-1$. Let $K > 0$ be a uniform bound such that $TV(f_n) < K$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} V(f, \mathcal{P}) &= \sum_{j=1}^{m-1} |f(x_j) - f(x_{j-1})| \\ &\leq \sum_{j=1}^{m-1} |f(x_j) - f_M(x_j)| + \sum_{j=1}^{m-1} |f_M(x_j) - f_M(x_{j-1})| + \sum_{j=1}^{m-1} |f_M(x_{j-1}) - f(x_{j-1})| \\ &\leq \epsilon/2 + K + \epsilon/2 = \epsilon + K < 1 + K. \end{aligned}$$

Hence $f \in \mathcal{F}_{BV}([0, 1])$. \square

For the second part: No. Let f be any (almost everywhere) continuous function on $[0, 1]$ which is not of bounded variation. By Weierstrass polynomial approximation theorem, there is a sequence $(P_n)_{n=1}$ of polynomial functions on $[0, 1]$ that converges uniformly to f . But each P_n is of bounded variation as any polynomial on a compact interval is Lipschitz.

Problem 5. Let $\{f_n\}_n$ be sequence of measurable functions on a measurable set E that converges almost everywhere on E to an integrable function f . Show that

$$\int_E |f_n - f| \rightarrow 0 \text{ if and only if } \int_E |f_n| \rightarrow \int_E |f|$$

(both limits taking place as $n \rightarrow \infty$).

Proof. (\Rightarrow). Because $||f_n| - |f|| \leq |f_n - f|$, we have

$$\int_E |f| - \int_E |f_n - f| \leq \int_E f_n \leq \int_E |f| + \int_E |f_n - f|.$$

Then by taking the limit we get

$$\lim_{n \rightarrow \infty} \int_E |f_n| = \int_E |f|.$$

Note: By the second inequality and the hypothesis we have that for all but finitely many $n \in \mathbb{N}$, $f_n \in L^1([0, 1])$.

(\Leftarrow). Notice that $|f_n - f| \rightarrow 0$, and $\int_E |f_n - f| \leq \int_E |f_n| + \int_E |f| < \infty$ for all but finitely many $n \in \mathbb{N}$ ($\because \int_E |f_n| \rightarrow \int_E |f|$). Then by the General Lebesgue Dominated Convergence Theorem we get the claim $\int_E |f_n - f| \rightarrow 0$. \square

Problem 6. Let $f \in L^p[0, 1]$ for some $p > 1$. Show that $f \in L^1[0, 1]$ and that for any $c \in (0, ||f||_1)$ it holds that

$$m(\{x \in [0, 1] : |f(x)| > c\}) \geq \left(\frac{||f||_1 - c}{||f||_p} \right)^q,$$

where q is the conjugate exponent of p .

Proof. Let $E_1 := \{x \in [0, 1] : |f(x)| > 1\}$. Then

$$||f||_1 \leq \int_{E_1} |f| + \int_{[0,1] \setminus E_1} |f| \leq ||f||_p + 1 < \infty.$$

Thus $f \in L^1[0, 1]$.

If $||f||_p = 0$, then $f \stackrel{a.e}{=} 0$. Thus we have the claim vacuously as such a $0 < c < 0$ does not exist. So, suppose $||f||_p \neq 0$. Because $c \in (0, ||f||_1)$, $||f||_1 - c > 0$. Let $E_c := \{x \in [0, 1] : |f(x)| > c\}$. Then

$$||f||_1 \leq c + \int_{[0,1]} |f| \cdot \chi_{E_c} \leq c + ||f||_p \cdot m(E_c)^{1/q}.$$

Hence

$$m(E_c) \geq \left(\frac{||f||_1 - c}{||f||_p} \right)^q.$$

\square

August 2023

In what follows, $m()$ is the Lebesgue measure on the real line and E is measurable.

Problem 1. Let E be a measurable set for which there exists $\delta \in (0, 1)$ such that $m(E \cap I) > \delta m(I)$ for any interval $I \subset (-\infty, \infty)$. Show that $m((-\infty, \infty) \setminus E) = 0$.

Proof. Let $F \subset (-\infty, \infty) \setminus E$ be such that $m(F) < \infty$. Then for any $\epsilon > 0$, there is a countable collection $\{I_n\}_{n=1}^\infty$ of disjoint open intervals such that $\bigcup_{n=1}^\infty I_n \supset F$ and $\sum_{n=1}^\infty m(I_n) < m(F) + \delta\epsilon$. Hence

$$\delta \sum_{n=1}^\infty m(I_n) < m(E \cap \bigcup_{n=1}^\infty I_n) < m((\bigcup_{n=1}^\infty I_n) \setminus F) = \sum_{n=1}^\infty m(I_n) - m(F) = \delta\epsilon.$$

Because $\epsilon > 0$ is arbitrary, this implies $\sum_{n=1}^\infty m(I_n) = 0$. Hence $m(F) = 0$ as $F \subset \bigcup_{n=1}^\infty I_n$. Because F is arbitrary, and any measurable set with infinite measure always have a measurable subset with finite measure, we have the claim. \square

Problem 2. A sequence of measurable functions f_n on E converges to a measurable function f in measure on E if $\lim_{n \rightarrow \infty} m\{x : |(f_n - f)(x)| > \delta\} = 0$ for any $\delta > 0$. Show that

- (a) almost everywhere convergence implies convergence in measure if $m(E) < \infty$;
- (b) almost everywhere convergence does not imply in general convergence in measure if $m(E) = \infty$;
- (c) converge in measure does not imply in general almost everywhere convergence even if $m(E) < \infty$.

Proof. (a) Suppose $f_n \xrightarrow{a.e.} f$ on E . Let $\epsilon > 0$. Then by Egoroff's theorem, there is closed $F \subset E$ such that $f_n \rightrightarrows f$ on F and $m(E \setminus F) < \epsilon$. Therefore for any $\delta > 0$, we have

$$\begin{aligned} m\{x \in E : |(f_n - f)(x)| > \delta\} &= m\{x \in F : |(f_n - f)(x)| > \delta\} + \\ &\quad m\{x \in E \setminus F : |(f_n - f)(x)| > \delta\} \\ &\leq m\{x \in F : |(f_n - f)(x)| > \delta\} + \epsilon. \end{aligned}$$

Because $\epsilon > 0$ is arbitrary and $\lim_{n \rightarrow \infty} m\{x \in F : |(f_n - f)(x)| > \delta\} = 0$, we have the claim.

- (b) Let $E = [0, \infty)$ and define the sequence of measurable functions $(f_n)_{n=1}^\infty$ on E by $f_n = \chi_{[n, \infty)}$. Thus for any $x \in E$, $f_n(x) \rightarrow 0$. But, for example,

$$\lim_{n \rightarrow \infty} m\{x : |(f_n - 0)(x)| > 1/2\} = \lim_{n \rightarrow \infty} m([n, \infty)) = \infty.$$

- (c) Define a collection $\{E_{k,m}\}_{0 \leq k \leq m-1, m=1}^\infty$ of intervals of $[0, 1]$ by $E_{k,m} := [k/m, (k+1)/m]$. Define the sequence $(f_n)_{n=0}^\infty$ of measurable functions on $[0, 1]$ by $f_n = \chi_{E_{k,m}}$ where

$n = m(m-1)/2 + k$, $n \in \mathbb{N}$, $0 \leq k \leq m-1$. In other words, $(f_n)_{n=1}$ are the indicator functions of the sequence of sets:

$$\left([0, 1], [0, \frac{1}{2}], [\frac{1}{2}, 1], [0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1], \dots\right).$$

Then for any $x \in [0, 1]$, we can find sub sequences $(f_{n_i})_{i=1}$ and $(f_{n_j})_{j=1}$, say, such that $f_{n_i}(x) = 0$ and $f_{n_j}(x) = 1$ (simply by choosing k, m so that $x \notin E_{k,m}$ and $x \in E_{k,m}$ respectively). Thus $(f_n)_{n=1}$ does not converge point-wise anywhere in $[0, 1]$. But for any $0 < \delta < 1$ we have

$$m(\{x \in [0, 1] : f_n(x) > \delta\}) = m(E_{k,m}) = 1/m < 1/\sqrt{n},$$

and for any $\delta \geq 1$ we have $m(\{x \in [0, 1] : f_n(x) > \delta\}) = 0$. Thus $(f_n)_{n=1}$ converges to $f \equiv 0$ in measure on $[0, 1]$. □

See the following *Problem 3* for a slightly different example.

Problem 3. Show that if functions f_n are non-negative and measurable on E and $\int_E f_n \leq 1/n^2$, then $f_n \rightarrow 0$ almost everywhere on E . Will the claim remain true if $1/n^2$ is replaced by $1/n$?

Proof. (See Problem 5 of August 2021 for a different proof)

Let $F := \{x \in E : f_n(x) \not\rightarrow 0\}$. Then

$$F = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}.$$

With the Chebychev's inequality and the continuity of the Lebesgue measure we have

$$\begin{aligned} m(F) &= m\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}\right) \\ &\leq \sum_{k=1}^{\infty} \lim_{N \rightarrow \infty} m\left(\bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}\right) \\ &\leq \sum_{k=1}^{\infty} \lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} m(\{x \in E : f_n(x) > 1/k\}) \\ &\leq \sum_{k=1}^{\infty} \lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} k \int_E f_n \leq \sum_{k=1}^{\infty} \lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} \frac{k}{n^2} = 0. \end{aligned}$$

□

No. We can inductively construct a sequence $(a_n)_{n=0}$ of numbers in $[0, 1]$ with following properties:

- i. $a_0 = 0$ and for any $n \in \mathbb{N}$, $a_n \neq a_{n+1}$.
- ii. For $n \in \mathbb{N}$, if $a_n + 1/n \leq 1$, then let $a_{n+1} := a_n + 1/n$.
- iii. For $n \in \mathbb{N}$, if $a_n < 1$ and $a_n + 1/n > 1$, then let $a_{n+1} = 1$ and $a_{n+2} = 0$.

So the sequence has infinite number of finite strings of consecutive terms starting from 0 and ending at 1, again and again. This is possible as $\sum_{n=1}^{\infty} 1/n = \infty$ implies for any $k \in \mathbb{N}$, there is $K \in \mathbb{N}$ such that $\sum_{n=k}^K 1/n \geq 1 = m([0, 1])$.

Then we can define a sequence $I_n := [b_n, b_{n+1}]$ of closed sub-intervals of $[0, 1]$ where $b_n = a_{n+K_n}$ for some $K_n \in \mathbb{N}$ such that $b_{n+1} - b_n \leq 1/n$, and for any $n \in \mathbb{N}$, there exist $M_n \leq n < N_n$ such that $[0, 1] = \bigcup_{n=M_n}^{N_n} I_n$. Here are the first few intervals of such a sequence:

$$\begin{aligned} & [0, 1], \\ & \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{5}{6}\right], \left[\frac{5}{6}, 1\right] \\ & \left[0, \frac{1}{5}\right], \left[\frac{1}{5}, \frac{11}{30}\right], \left[\frac{11}{30}, \frac{107}{210}\right], \dots \end{aligned}$$

Therefore for any $x \in [0, 1]$, we can find two sub-sequences $(I_{n_i})_{i=1}$ and $(I_{n_j})_{j=1}$ such that $x \in I_{n_i}$ for all $i \in \mathbb{N}$ and $x \notin I_{n_j}$ for all $j \in \mathbb{N}$. Hence the sequence $(f_n)_{n=1}$ of functions on $[0, 1]$ defined by $f_n := \chi_{I_n}$ does not converge pointwise anywhere in $[0, 1]$. But

$$\int_{[0,1]} |f_n| = m(I_n) \leq \frac{1}{n}.$$

Problem 4. Let f be an integrable function on a bounded measurable set E . Find $\lim_{p \rightarrow 0^+} \int_E |f|^p$.

Answer. We prove

$$\lim_{p \rightarrow 0^+} \int_E |f|^p = m(E \setminus E^*).$$

Here $E^* := f^{-1}(\{0, \infty\})$.

Let $E_1 := \{x \in E : 0 < |f(x)| \leq 1\}$ and $E_2 := \{x \in E : \infty > |f| > 1\}$. Then for all $0 < p < 1$, $|f \chi_{E_1}|^p \leq 1$, and $\lim_{p \rightarrow 0^+} |f \chi_{E_2}(x)|^p = 1$. Because $f \in L^1(E)$, $m(f^{-1}(\{\infty\})) = 0$. Thus by the Lebesgue Dominated Convergence Theorem,

$$\lim_{p \rightarrow 0^+} \int_E |f|^p = \lim_{p \rightarrow 0^+} \int_{E_1} |f|^p + \lim_{p \rightarrow 0^+} \int_{E_2} |f|^p = m(E_1) + m(E_2) = m(E \setminus E^*).$$

Q: What if E is an unbounded measurable set with finite measure?

An idea: $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n := E \cap [n, n+1]$.

Problem 5. Let $E \subset [0, 1]$ be a Borel set such that $0 < m(E \cap I) < m(I)$ for any subinterval of $[0, 1]$. Show that $f(x) = m([0, x] \cap E) - m([0, x] \setminus E)$ is absolutely continuous on $[0, 1]$ but it is not monotone on any subinterval of $[0, 1]$.

Proof. Because $f(x) = \int_0^x (\chi_E - \chi_{E^c})$, an indefinite integral, $f \in AC[0, 1]$ by the fundamental theorem of Lebesgue integral calculus.

For the second part:

• **Proof 1:** The hypothesis “ $0 < m(E \cap I) < m(I)$ for any subinterval of $[0, 1]$ ” says E is dense in $[0, 1]$, and does not contain any intervals (hence totally disconnected). This, together with the fact “ $f'(x) = \chi_E - \chi_{E^c}$ for almost every $x \in [0, 1]$ ”, imply that for any subinterval $J \subset [0, 1]$, there exist points $a, b \in J$ such that $f'(a) = 1 > 0$ and $f'(b) = -1 < 0$. Thus f can not be monotone on J .

• **Proof 2.** Notice that, because any Borel set is Lebesgue measurable,

$$f(x) = 2m([0, x] \cap E) - x.$$

Case 1: Suppose there is an interval J on which f is monotonically decreasing. Then for any $x_1, x_2 \in J$, $x_1 < x_2$ we have $f(x_1) - f(x_2) \geq 0$. Which implies $(x_2 - x_1)/m([x_1, x_2] \cap E) \geq 2$. For any given $\epsilon > 0$, choose $\{I_k\}_{k=1}$, $I_k \subset J$ so that $\bigcup_{k=1} I_k \supset E \cap J$ and $\sum_{k=1} m(I_k) < m(E \cap J) + \epsilon$. Then

$$m(E \cap J) + \epsilon > \sum_{k=1} m(I_k) = \sum_{k=1} \frac{m(I_k)}{m(I_k \cap E \cap J)} m(I_k \cap E \cap J) \geq 2 \sum_{k=1} m(I_k \cap E \cap J) = 2m(E \cap J).$$

Because $\epsilon > 0$ is arbitrary, this implies $m(E \cap J) = 0$. Contradiction.

Case 2: Suppose there is an interval J on which f is strictly increasing. Then for any $x_1, x_2 \in J$, $x_1 < x_2$ we have $f(x_2) - f(x_1) > 0$. Which implies $m([x_1, x_2] \cap E) > (x_2 - x_1)/2$. Then by *Problem 1* (replacing $(-\infty, \infty)$ with J), $m(J \setminus E) = 0$. Which implies $m(J \cap E) = m(J)$. Contradiction.

Because J is an arbitrary interval, f is not monotone on any sub-interval of $[0, 1]$.

• **Proof 3.** If you are allowed, use the Lebesgue Density Theorem. □

Q: Can E be a Lebesgue measurable but non-Borel set (with positive measure, of course)?

Q: Does there exist a Lebesgue measurable but non-Borel set $E \subset [0, 1]$ such that for any interval $I \subset [0, 1]$, $E \cap I$ is non-Borel?

An idea: Any Lebesgue measurable set is a Borel set minus a set of measure zero.

January 2023

Problem 1. Let A be a set such that $A \cap K \neq \emptyset$ for any compact subset K of real line of positive measure. Show that A has infinite outer Lebesgue measure.

Proof. If $m(A) < \infty$, then $m(A^c) > 0$. Hence by the inner regularity of the Lebesgue measure there is a compact set $K \subset A^c$, say, with $m(K) > 0$. This contradicts the hypothesis as $A \cap K = \emptyset$. \square

Problem 2. Show that if f is continuous almost everywhere on $[0, 1]$, then it is measurable.

Proof. Let $E \subset [0, 1]$ be the set of all points at where f is continuous. Then $m(E^c) = 0$, and therefore E and $f|_E$ (\cdot the inverse image of any open set under $f|_E$ is the intersection between an open set and E) are measurable. Because $f^{-1}((a, \infty)) \stackrel{a.e.}{=} f|_E^{-1}((a, \infty))$ for any $a \in \mathbb{R}$, f is measurable. \square

Problem 3. For a non-negative measurable function f on a measurable set E define

$$m_f(t) = m(\{f > t\}), t > 0,$$

where m is the Lebesgue measure. Show that $\int_E \varphi = \int_0^\infty m_\varphi$ for any non-negative simple function φ .

Proof. Let $\varphi = \sum_{j=1}^n c_j \chi_{E_j}$ with $c_1 < \dots < c_n$. Then

$$m_\varphi(t) = \chi_{[0, c_1)}(t) \cdot \sum_{j=1}^n m(E \cap E_j) + \sum_{k=1}^{n-1} \left[\chi_{[c_k, c_{k+1})}(t) \cdot \sum_{j=k+1}^n m(E \cap E_j) \right].$$

Thus

$$\begin{aligned} \int_0^\infty m_\varphi &= c_1 \sum_{j=1}^n m(E \cap E_j) + \sum_{k=1}^{n-1} \left[(c_{k+1} - c_k) \sum_{j=k+1}^n m(E \cap E_j) \right] \\ &= \sum_{j=1}^n c_j \cdot m(E \cap E_j) = \int_E \varphi. \end{aligned}$$

\square

Problem 4. In the setting of the previous problem and assuming its statement is true, show that $\int_E f = \int_0^\infty m_f$ for any non-negative measurable function f .

Proof. By the simple approximation theorem, we can find an increasing sequence $(\varphi_n)_{n=1}$ of positive simple functions such that $\varphi_n \nearrow f$ on E . By the monotone convergence theorem $\int_E \varphi_n \nearrow \int_E f$, and by the previous problem $\int_E \varphi_n = \int_0^\infty m_{\varphi_n}$ for all $n \in \mathbb{N}$. Thus if we can show $\lim_{n \rightarrow \infty} \int_0^\infty m_{\varphi_n} = \int_0^\infty m_f$, then we are done.

Notice that for any $t > 0$, $x \in \{x \in \mathbb{R} : f(x) > t\}$ if and only if $x \in \{x \in \mathbb{R} : \varphi_n(x) > t\}$ for all but finitely many $n \in \mathbb{N}$. Thus

$$\{x \in \mathbb{R} : f(x) > t\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\}.$$

Then

$$\begin{aligned} m_f(t) &:= m(\{x \in \mathbb{R} : f(x) > t\}) \\ &= m\left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\}\right) \\ &= \lim_{k \rightarrow \infty} m\left(\bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\}\right), \quad \because \text{continuity of Lebesgue measure} \\ &= \lim_{k \rightarrow \infty} m(\{x \in \mathbb{R} : \varphi_k(x) > t\}), \quad \because \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\} = \{x \in \mathbb{R} : \varphi_k(x) > t\} \\ &= \lim_{n \rightarrow \infty} m(\{x \in \mathbb{R} : \varphi_n(x) > t\}) = \lim_{n \rightarrow \infty} m_{\varphi_n}(t). \end{aligned}$$

Because $t > 0$ is arbitrary and $(m_{\varphi_n})_{n=1}$ is an increasing sequence of measurable functions, by the monotone convergence theorem we get

$$\lim_{n \rightarrow \infty} \int_0^{\infty} m_{\varphi_n} = \int_0^{\infty} m_f.$$

This completes the proof. \square

Problem 5. Let f be integrable function of the real line. Show that $f = 0$ almost everywhere if $\int_I f = 0$ for any interval I of length 1.

Proof. For the sake of contradiction let's assume that there is $\epsilon > 0$ and a measurable set E_1^+ inside of an interval of length 1 such that $m(E_1^+) > 0$ and $f|_{E_1^+} > \epsilon$ (choose E_1^+ to be maximal, up to measure zero set, within a unit interval). Then the hypothesis implies that there is another set E_1^- , within an interval of length 1 to E_1^+ , such that $m(E_1^+) = m(E_1^-)$ and $f|_{E_1^-} < -\epsilon$. By continuing in this way we can find a sequence $(E_n^+)_{n=1}$ of disjoint measurable sets with same (positive) measure such that $f|_{E_n^+} > \epsilon$ for all $n \in \mathbb{N}$. This violates the integrability of f as

$$\int |f| > \int_{\bigsqcup_{n=1}^{\infty} E_n^+} f \geq \sum_{n=1}^{\infty} \epsilon \cdot m(E_n^+) = \epsilon \cdot \infty = \infty.$$

\square

Problem 6. Let $f \in L^p[0, 1]$, $p \in (1, \infty)$, and F be an indefinite integral of f . Show that for each $x \in (0, 1)$

$$\lim_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h^{1/q}} = 0,$$

where q is the conjugate exponent of p .

Proof. We have $F(x) = F(0) + \int_0^x f$ for all $x \in [0, 1]$. Therefore for any $x \in [0, 1]$,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \left| \frac{F(x+h) - F(x)}{h^{1/q}} \right| &\leq \lim_{h \rightarrow 0^+} \frac{\int_x^{x+h} |f|}{h^{1/q}} \\ &= \lim_{h \rightarrow 0^+} \frac{\int_{[0,1]} |f| \cdot \chi_{[x, x+h]}^2}{h^{1/q}} \\ &\stackrel{\text{Holder}}{\leq} \lim_{h \rightarrow 0^+} \|f \chi_{[x, x+h]}\|_p. \quad (*) \end{aligned}$$

Let $h = 1/n$, $n \in \mathbb{N}$. For fixed $x \in [0, 1]$, define the sequence $(f_{x,n})_{n=1}$ of $L^1[0, 1]$ functions by

$$f_{x,n}(t) := |f(t)|^p \cdot \chi_{[x, x+1/n]}(t).$$

Thus $f_{x,n} \leq |f|^p$ for all $n \in \mathbb{N}$ and $f_{x,n}(t) \rightarrow 0$ for all $t \in [0, 1] \setminus \{x\}$ as $n \rightarrow \infty$. Since the singletons have zero measure and $f^p \in L^1[0, 1]$, the Lebesgue Dominated Convergence Theorem tells $\int_0^1 f_{x,n} \rightarrow 0$ for all $x \in [0, 1]$. This together with the squeeze theorem give us

$$\lim_{h \rightarrow 0^+} \|f \chi_{[x, x+h]}\|_p = \lim_{n \rightarrow \infty} \|f \chi_{[x, x+\frac{1}{n}]\|_p = \lim_{n \rightarrow \infty} \left(\int_0^1 f_{x,n} \right)^{1/p} = 0. \quad (**)$$

By (*) and (**) we have the claim. □

January 2022

Problem 1. Let $f \rightarrow [0, \infty)$ be a measurable function and define its distribution function $d_f : [0, \infty) \rightarrow [0, \infty]$ defined by $d_f(t) = |\{x \in \mathbb{R} : f(x) > t\}|$.

- i. Prove or disprove the statement that the function d_f is left continuous, that is to say

$$\lim_{t \nearrow t_0} d_f(t) = d_f(t_0), \text{ for each } t_0 > 0.$$

- ii. Prove or disprove the statement that the function d_f is right continuous, that is to say

$$\lim_{t \searrow t_0} d_f(t) = d_f(t_0), \text{ for each } t_0 \leq 0.$$

Proof. i. Let $t_0 = 1$ and $f(x) = \chi_{[0,1)}(x)$. Then $d_f(t_0) = 0 \neq 1 = \lim_{t \nearrow t_0} d_f(t)$. So, $d_{\chi_{[0,1)}}$ is not left continuous.

- ii. Let $t_0 \geq 0$. We prove $\lim_{t \searrow t_0} d_f(t) = d_f(t_0)$. For $t > t_0$, notice that

$$m(f^{-1}(t_0, \infty)) = m(f^{-1}(t_0, t]) + m(f^{-1}(t, \infty)) \quad (*)$$

Because $f^{-1}(t_0, t_1] \subset f^{-1}(t_0, t_2]$ for any $t_1 < t_2$, $\lim_{t \searrow t_0} m(f^{-1}(t_0, t])$ exists (continuity of Lebesgue measure). If it is 0, then we are done. If not, then there is $M > 0$ and $N > t_0$ such that $\lim_{t \searrow t_0} m(f^{-1}(t_0, t]) \geq M$ for all $t \in (t_0, N)$. Let $t \in (t_0, N)$. Because $t_0 \notin (t_0, t]$, there is a countable union of disjoint (can be de-generated) intervals $(I_k)_{k=1}^\infty$ in $(t_0, t]$ such that $m(f^{-1}(I_k)) \geq M$ for each k (let the enumeration begins from right). This says $m(f^{-1}(t_0, t]) = \infty$. Because t is arbitrary, $\lim_{t \searrow t_0} m(f^{-1}(t, \infty)) \geq \lim_{n \rightarrow \infty} \sum_{k=1}^n m(I_k) \geq \lim_{n \rightarrow \infty} nM = \infty$. Here $\bigsqcup_{k=1}^n I_k \subset (t, N)$. So, $\lim_{t \searrow t_0} d_f(t) = \infty$. Then from $(*)$, $d_f(t_0) = \lim_{t \searrow t_0} d_f(t) = \infty$. □

Problem 2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function. Show the following.

- i. if $F \subset \mathbb{R}$ is an F_σ set then $f(F) \subset \mathbb{R}$ is an F_σ set.
ii. if $F \subset \mathbb{R}$ is a measurable set then $f(F) \subset \mathbb{R}$ is a measurable set.

Proof. i. Because $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1]$, $F \subset \mathbb{R}$ is an F_σ set if and only if it is a countable union of compact sets. Since the continuous image of a compact set is compact and $f(A \cup B) = f(A) \cup f(B)$ for any sets (let's say measurable) $A, B \subset \mathbb{R}$, we have that $f(F)$ is an F_σ set if F is an F_σ set.

- ii. Suppose $F \subset \mathbb{R}$ be a measurable set. Then $F = E \sqcup (F \setminus E)$ where E is an F_σ set and $m(F \setminus E) = 0$. Hence $f(F) = f(E) \cup f(F \setminus E)$. So, if we can show $m(f(F \setminus E)) = 0$, then by part (i) we have that F is measurable. In-particular, if we can show $m(f(A)) \leq m(A)$ for any measurable set A , then we are done. Because the semi-ring of bounded intervals generates the collection of all Lebesgue measurable sets (and with the fact

that singletons have measure zero), it is enough to show $m(f([a, b])) \leq M(b - a)$ for any closed interval $[a, b]$. Here $M > 0$ is the Lipschitz constant of f . In-fact

$$m(f([a, b])) \leq \max_{x \in [a, b]} f(x) - \min_{x \in [a, b]} f(x) = f(\alpha) - f(\beta) \leq M(\alpha - \beta) \leq M(b - a).$$

Here $\alpha, \beta \in [a, b]$ are such that $f(\alpha) = \max_{x \in [a, b]} f(x)$ and $f(\beta) = \min_{x \in [a, b]} f(x)$. □

Problem 3. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Show that the following are equivalent:

(*) for each measurable set $E \subset \mathbb{R}$, the set $\phi^{-1}(E)$ is measurable and $|\phi^{-1}(E)| = |E|$;

(**) for each measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$, the function $f \circ \phi : \mathbb{R} \rightarrow \mathbb{R}$ is measurable and

$$\int_{\mathbb{R}} f \circ \phi = \int_{\mathbb{R}} f.$$

Proof. ($* \Rightarrow **$) Since ϕ is measurable by the hypothesis, $f \circ \phi$ is measurable. Because $f : \mathbb{R} \rightarrow \mathbb{R}$ be measurable, there is a sequence $(\varphi_n)_{n=1}^{\infty}$ of simple functions which converges pointwise to f on \mathbb{R} with $|\varphi_n| \leq |f|$ for all $n \in \mathbb{N}$ (if $f > 0$, choose $(\varphi_n)_{n=1}^{\infty}$ to be increasing). Therefore if we can show the claim for any simple function, then we are done, as: If $f \in L^1(\mathbb{R})$, then use LDCT. If $f \notin L^1(\mathbb{R})$, then at-least one of $\int_{\mathbb{R}} f_-$ or $\int_{\mathbb{R}} f_+$ equals to ∞ . Then use the monotone convergence theorem to this f_- or f_+ .

Let $f(x) = \sum_{k=1}^{\infty} a_k \chi_{E_k}(x)$ be a simple function. Then

$$\int_{\mathbb{R}} f \circ \phi = \sum_{k=1}^n a_k m(\phi^{-1}(E_k)) = \sum_{k=1}^n a_k m(E_k) = \int_{\mathbb{R}} f.$$

($* \Leftarrow **$) Letting f be the identity we get $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is measurable. And for any measurable set E , letting $f = \chi_E$, we get

$$m(\phi^{-1}(E)) = \int_{\mathbb{R}} f \circ \phi \stackrel{\text{Hypothesis}}{=} \int_{\mathbb{R}} f = m(E).$$

□

Problem 4. Let $E \subset \mathbb{R}$ be a measurable set with $0 < |E| < \infty$. Let $f \in L^{\infty}(E)$.

i. Show that $f \in L^p(E)$ for each $p \in [1, \infty)$

ii. Show that $\|f\|_p \rightarrow \|f\|_{\infty}$ as $p \rightarrow \infty$.

Proof. i. Let $p \in [1, \infty)$. Then $\|f\|_p \leq \|f\|_{\infty} \cdot m(E)^{1/p} < \infty$. Thus $f \in L^p(E)$.

- ii. Because $\|f\|_p \leq \|f\|_q$ for any $p \leq q$, the limit $L := \lim_{p \rightarrow \infty} \|f\|_p$ exists in $[0, \infty]$. But since $f \in L^\infty(E)$, $L \in [0, \infty)$. Notice that $\|f\|_p = \left(\int_E |f|^p \right)^{1/p} \leq m(E)^{\frac{1}{p}} \|f\|_\infty$. Then by taking the limit as $p \rightarrow \infty$ we get $L \leq \|f\|_\infty$. To show the reverse inequality: for $n \in \mathbb{N}$, define

$$E_n := \{x \in E : \|f\|_\infty \left(1 - \frac{1}{n}\right) \leq |f(x)| \leq \|f\|_\infty\}.$$

Then $m(E_n) > 0$. (This is true as $m(E) > 0$ and $\|\cdot\|_\infty$ is the *essential sup norm*. If it is just the *sup norm*, then the claim is false. For a counter example consider $f : [0, 1] \rightarrow [0, 1]$ defined by $f(x) = 1$ if $x = 1/2$, and $f(x) = 0$ else). Then

$$\|f\|_p^p = \int_E |f|^p = \int_{E_n} |f|^p + \int_{E \setminus E_n} |f|^p \geq \|f\|_\infty^p \cdot \left(1 - \frac{1}{n}\right)^p \cdot m(E_n)$$

Hence $L = \lim_{p \rightarrow \infty} \|f\|_p \geq \lim_{p \rightarrow \infty} \|f\|_\infty \cdot \left(1 - \frac{1}{n}\right)^{\frac{1}{p}} \cdot m(E)^{\frac{1}{p}} = \|f\|_\infty \cdot \left(1 - \frac{1}{n}\right)$. Since this is true for any $n \in \mathbb{N}$, we get $L \geq \|f\|_\infty$, and therefore the claim. \square

Problem 5. Let $f \in L^1(\mathbb{R})$. For each $\lambda > 0$, put $f_\lambda(x) = \lambda f(\lambda x)$. Show that $f_\lambda \in L^1(\mathbb{R})$, and that $f_\lambda \rightarrow f$ in $L^1(\mathbb{R})$ as $\lambda \rightarrow 1$.

Proof. For $n \in \mathbb{N}$, we have $\int_{[-n, n]} |f_\lambda| = \int_{[-\lambda n, \lambda n]} |f| \leq \|f\|_1 < \infty$. This is true for any $n \in \mathbb{N}$, $f_\lambda \in L^1(\mathbb{R})$. Define $g_\lambda := \frac{|f_\lambda| + |f| - |f_\lambda - f|}{2}$. Then $g_\lambda \geq 0$. By Lusin's theorem, for any $\epsilon > 0$, we can find a measurable set E_ϵ such that

- (i). $\|f \cdot \chi_{\mathbb{R} \setminus E_\epsilon}\|_1 < \epsilon/2$,
- (ii). $\|f_\lambda \cdot \chi_{\mathbb{R} \setminus E_\epsilon}\|_1 < \epsilon$ for all close enough λ to 1, and
- (iii). $\lim_{\lambda \rightarrow 1} g_\lambda = |f|$ pointwise on E_ϵ .

Thus by Fatou's lemma,

$$\|f_{E_\epsilon}\|_1 \leq \|f_{E_\epsilon}\|_1 - \frac{1}{2} \limsup_{\lambda \rightarrow 1} \|(f_\lambda - f)_{E_\epsilon}\|_1.$$

Hence $\lim_{\lambda \rightarrow 1} \|(f_\lambda - f)_{E_\epsilon}\|_1 = 0$. This together with (ii) gives us

$$\lim_{\lambda \rightarrow 1} \|f_\lambda - f\|_1 < \epsilon.$$

Because this is true for any $\epsilon > 0$, we have the claim. \square

Problem 6. Assume $f_n \rightarrow f$ and $g_n \rightarrow g$ in $L^2(\mathbb{R})$. Show that $f_n g_n \rightarrow f g$ in $L^1(\mathbb{R})$.

Proof. We have that

$$\begin{aligned} \|f_n g_n - f g\|_1 &= \|(f_n - f)g_n + (g_n - g)f\|_1 \stackrel{\text{Minkowski}}{\leq} \|(f_n - f)g_n\|_1 + \|(g_n - g)f\|_1 \\ &\stackrel{\text{Holder}}{\leq} \|f_n - f\|_2 \cdot \|g_n\|_2 + \|g_n - g\|_2 \cdot \|f\|_2. \end{aligned}$$

Because $g_n \rightarrow g$ in $L^2(\mathbb{R})$, $g_n^2 \rightarrow g^2$ in $L^1(\mathbb{R})$. Thus by letting $n \rightarrow \infty$ in the above inequality we get the claim as $f, g \in L^2(\mathbb{R})$. \square

August 2021

Problem 1. Let $n \in \mathbb{N}$ and $p_1, p_2, \dots, p_n \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n} = 1$. If $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \dots, f_n \in L^{p_n}(\mathbb{R})$, then show that $f_1 f_2 \cdots f_n \in L^1(\mathbb{R})$.

Proof. We prove by induction on n . The base case (when $n = 2$) follows from the Holder's inequality. Let $k \in \mathbb{N}_{\geq 2}$ and suppose that the claim is true for all $n = 2, \dots, k$. That is; if $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} = 1$ and $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \dots, f_k \in L^{p_k}(\mathbb{R})$, then $f_1 f_2 \cdots f_k \in L^1(\mathbb{R})$.

Need to show if $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} + \frac{1}{p_{k+1}} = 1$ and $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \dots, f_k \in L^{p_k}(\mathbb{R}), f_{k+1} \in L^{p_{k+1}}(\mathbb{R})$, then $f_1 f_2 \cdots f_k f_{k+1} \in L^1(\mathbb{R})$. Let $q = \frac{p_k p_{k+1}}{p_k + p_{k+1}}$. Then $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{q} = 1$. This implies $q > 1$. Thus if we can show $f_k f_{k+1} \in L^q(\mathbb{R})$, then we are done by the induction hypothesis as it implies $f_1 f_2 \cdots (f_k f_{k+1}) \in L^1(\mathbb{R})$. Notice that

$$f_k \in L^{p_k}(\mathbb{R}) \Rightarrow f_k^{p_k} \in L^1(\mathbb{R}) \Rightarrow f_k^q \in L^{\frac{p_k}{q}}(\mathbb{R}).$$

Similarly

$$f_{k+1} \in L^{p_{k+1}}(\mathbb{R}) \Rightarrow f_{k+1}^{p_{k+1}} \in L^1(\mathbb{R}) \Rightarrow f_{k+1}^q \in L^{\frac{p_{k+1}}{q}}(\mathbb{R}).$$

Since $\frac{q}{p_k} + \frac{q}{p_{k+1}} = 1$, by the Holder's inequality, $\|f_k^q f_{k+1}^q\|_1 \leq \|f_k^q\|_{\frac{p_k}{q}} + \|f_{k+1}^q\|_{\frac{p_{k+1}}{q}} < \infty$. Hence $f_k f_{k+1} \in L^q(\mathbb{R})$. \square

Problem 2. Let A and B be Lebesgue measurable subsets of \mathbb{R} , with $A \subset B$. Show that $m(A) = m(B)$ if and only if every subset C satisfying $A \subset C \subset B$ is Lebesgue measurable.

Proof. (\Rightarrow) Suppose $m(A) = m(B)$. Then for any $C \subset \mathbb{R}$ with $A \subset C \subset B$, there is a set $E \subset \mathbb{R}$, say, of Lebesgue measure zero such that $C = A \sqcup E$. Hence C is measurable.

(\Leftarrow) For contra-positive, assume $m(A) < m(B)$. Then by Vitali's theorem there is a non-measurable set $F \subset B \setminus A$. Then $C := A \sqcup F$ is non-measurable with $A \subset C \subset B$. \square

Problem 3.

- (a). Prove that for every $\epsilon > 0$ there is an open subset of $[0, 1]$, whose closure is $[0, 1]$, and whose Lebesgue measure is less than ϵ .
- (b). Prove that for every $a \in (0, 1]$ there is an open subset of $[0, 1]$, whose closure is $[0, 1]$, and whose Lebesgue measure is equal to a .

Proof. (a). If $\epsilon > 1$, then take $(0, 1)$ to be the open set. If $\epsilon \in (0, 1]$, then take the set below in part (b) when, for example, $a = \frac{\epsilon}{2}$.

- (b). Let $a \in (0, 1]$. Consider the standard middle third Cantor set (let's denote it by $C_{\frac{1}{3}}$). Instead of removing middle thirds (the ratio of "length of a removed interval in stage $n + 1$ " to "the length of the interval where it belonged as the middle third interval in stage n " is $1 : 3$), let's remove the middle $(\frac{1+2a}{a})^{th}$. That is; the ratio of "length of a removed interval in stage $n + 1$ " to "the length of the interval where it belonged as the middle $(\frac{1+2a}{a})^{th}$ interval in stage n " is $1 : \frac{1+2a}{a}$. Let U_a be the disjoint union of those

removed open intervals in this process. Hence U_a is open and clearly $\overline{U_a} = [0, 1]$ as $[0, 1] \setminus U_a$ does not contain any open interval. And

$$m(U_a) = \sum_{n=0}^{\infty} 2^n \left(\frac{a}{1+2a} \right)^{n+1} = a.$$

□

Problem 4. Let $f, g : [0, 1] \rightarrow \mathbb{R}$. Prove that

$$TV(fg) \leq TV(f) \sup_{x \in [0,1]} |g(x)| + TV(g) \sup_{x \in [0,1]} |f(x)|,$$

where TV is the total variation. Give examples where the inequality is strict, and where the equality holds although none of the functions f, g is constant.

Proof. Let $\mathcal{P} = \{0 = a_0 < a_1 < \dots < a_{n-1} < a_n = 1\}$ be a partition of $[0, 1]$. Then

$$\begin{aligned} V(fg, \mathcal{P}) &= \sum_{k=1}^n |f(a_k)g(a_k) - f(a_{k-1})g(a_{k-1})| \\ &\leq \sum_{k=1}^n |f(a_k) - f(a_{k-1})| |g(a_k)| + |g(a_k) - g(a_{k-1})| |f(a_{k-1})| \\ &\leq \sup_{x \in [0,1]} |g(x)| \sum_{k=1}^n |f(a_k) - f(a_{k-1})| + \sup_{x \in [0,1]} |f(x)| \sum_{k=1}^n |g(a_k) - g(a_{k-1})| \\ &= V(f, \mathcal{P}) \sup_{x \in [0,1]} |g(x)| + V(g, \mathcal{P}) \sup_{x \in [0,1]} |f(x)|. \end{aligned}$$

Then by taking the supremum over all the partitions of $[0, 1]$, we get the claim. □

For the strict inequality: Let $f(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$ and $g(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$.

Then $fg = 0$.

Thus $TV(fg) = 0$ and $TV(f) \sup_{x \in [0,1]} |g(x)| + TV(g) \sup_{x \in [0,1]} |f(x)| = 1 \cdot 1 + 1 \cdot 1 = 2$.

For the equality: let $f(x)$ be as above and let g be any function with $TV(g|_{[\frac{1}{2}, 1]}) = \infty$. Then the both sides of the inequality becomes ∞ .

If need a finite equality: Let $f(x)$ be as above and $g(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$. Then

$fg(x) := \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{else} \end{cases}$. Thus $TV(fg) = 2$ (for the partition, for example, $\{0 < \frac{1}{2} < 1\}$)

and $TV(f) \sup_{x \in [0,1]} |g(x)| + TV(g) \sup_{x \in [0,1]} |f(x)| = 1 \cdot 1 + 1 \cdot 1 = 2$.

Problem 5. Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $L^1([0, 1])$, satisfying

$$\int_{[0,1]} |f_n| \leq \frac{1}{n^2}$$

for every $n \in \mathbb{N}$. Show that f_n converges to 0 pointwise a.e.

Proof. Let $\epsilon > 0$ be given. Then by the Chebychev's inequality,

$$m(\{x \in [0, 1] : |f_n(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_{[0,1]} |f_n| \leq \frac{\epsilon}{n^2}.$$

Thus $\sum_{n=1}^\infty m(\{x \in [0, 1] : |f_n(x)| \geq \epsilon\}) \leq \sum_{n=1}^\infty \frac{\epsilon}{n^2} < \infty$. Therefore by the Borel-Cantelli lemma, for almost all $x \in [0, 1]$ there is $N \in \mathbb{N}$ such that for each $n > N$, $x \notin \{x \in [0, 1] : |f_n(x)| \geq \epsilon\}$. In other words: for almost all $x \in [0, 1]$ there is $N \in \mathbb{N}$ such that for each $n > N$, $|f_n(x)| < \epsilon$. Hence the claim. \square

Problem 6. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Evaluate

$$\lim_{n \rightarrow \infty} \int_0^1 nx^{n-1} f(x) dx.$$

Answer. For a fixed $a \in \mathbb{R}$ and $k \in \mathbb{N}$, we get $\lim_{n \rightarrow \infty} \int_0^1 nx^{n-1} ax^k dx = a$. Thus by the linearity of the Lebesgue integral, $\lim_{n \rightarrow \infty} \int_0^1 nx^{n-1} P(x) dx = P(1)$ for any polynomial $P(x)$ on $[0, 1]$. Let f be a continuous function on $[0, 1]$. By Weierstrass approximation theorem, there is a sequence $(P_m(x))_{m=1}^\infty$ of polynomial on $[0, 1]$ such that $P_m \rightrightarrows f$. Because one of the limits is uniform, we can interchange the two limits. Thus

$$\begin{aligned} f(1) &= \lim_{m \rightarrow \infty} P_m(1) = \lim_{m \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \int_0^1 nx^{n-1} P_m(x) dx \right) \\ &= \lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} \int_0^1 nx^{n-1} P_m(x) dx \right) \\ &= \lim_{n \rightarrow \infty} \int_0^1 \lim_{m \rightarrow \infty} nx^{n-1} P_m(x) dx \\ &= \lim_{n \rightarrow \infty} \int_0^1 nx^{n-1} f(x) dx. \end{aligned}$$

Note: We can not start with $\lim_{n \rightarrow \infty} \int_0^1 nx^{n-1} f(x) dx$ as we still don't know if the limit exists.

Winter 2021

1. Give an example of a measurable set $E \subset \mathbb{R}$ of finite measure such that $m(E \cap (E+r)) > 0$ for every $r \in \mathbb{R}$ (where $E + r = \{x + r : x \in E\}$).

Example 1.

$$\bigsqcup_{n=1}^{\infty} [-n - \frac{1}{2^n}, -n] \sqcup [-1, 1] \sqcup \bigsqcup_{n=1}^{\infty} [n, n + \frac{1}{2^n}].$$

Example 2. Let $(q_n)_{n \in \mathbb{N}}$ an enumeration of \mathbb{Q} . Then

$$\bigcup_{n=1}^{\infty} [q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}].$$

2. Let $f_n : [0, 1] \rightarrow [0, \infty)$ ($n = 1, 2, 3, \dots$) be measurable functions. Prove that there exist numbers $a_n > 0$ ($n = 1, 2, 3, \dots$) such that the series $\sum_{n=1}^{\infty} a_n f_n(x)$ converges for almost every $x \in [0, 1]$.

Proof. Because f_n is finite, $\lim_{k \rightarrow \infty} f_n^{-1}((k, \infty)) = \emptyset$ for each $n \in \mathbb{N}$. Therefore for each $n \in \mathbb{N}$, there is $\delta(n) > 0$, say, such that $m(E_n) \leq 1/n^2$. Here $E_n := f_n^{-1}((\delta(n), \infty))$. Let $a_n = \frac{1}{\delta(n) \cdot 2^n}$. Then the series $\sum_{n=1}^{\infty} a_n f_n(x)$ diverges if and only if x belongs to infinitely many E_n 's. That is $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. Borel-Cantelli lemma says the set consists with such points has measure zero as $\sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} 1/n^2 < \infty$. \square

3. Let $f \in L^1(\mathbb{R})$. Prove that for every measurable set $E \subset \mathbb{R}$ of finite measure we have

$$\lim_{t \rightarrow \infty} \int_E f(x+t) dx = 0.$$

Proof. Let $\epsilon > 0$ be given. Because $f \in L^1(\mathbb{R})$ there is $\delta > 0$ such that for any $F \subset \mathbb{R}$ with $m(F) < \delta$ we get $\int_F |f| < \frac{\epsilon}{2}$. Because $m(E) < \infty$, we can find an interval $[-a, a]$, say, such that $\int_{\mathbb{R} \setminus [-a, a]} |f| < \frac{\epsilon}{2}$ and $m(E \setminus [-a, a]) < \min\{\delta, \frac{\epsilon}{2}\}$. Because Lebesgue measure is invariant under translation, for all $t > 2a$

$$\left| \int_E f(x+t) \right| \leq \int_E |f(x+t)| \leq \int_{E \cap [-a, a]} |f(x+t)| + \frac{\epsilon}{2} = \int_{(E \cap [-a, a]) + t} |f(x)| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

\square

4. Assume that for a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a constant $C \in \mathbb{R}$, the total variation of f on every compact interval is smaller than or equal to C . Prove that

$$\int_{\mathbb{R}} |f(x+h) - f(x)| dx \leq C|h|.$$

Proof. Let $h \in \mathbb{R}$ be given. Define $g : \mathbb{R} \rightarrow [0, \infty)$ by $g(x) := |f(x+h) - f(x)|$. For $i \in \mathbb{N}$, let $a_i := \sup\{i|h|, (i+1)|h|\}$. Then $\int_{i|h|}^{(i+1)|h|} g(x) dx \leq |h| \cdot g(a_i)$. Thus for any $n \in \mathbb{N}$,

$$\begin{aligned} \int_{[-n|h|, n|h|]} g(x) dx &= \sum_{i=-n}^{n-1} \int_{[i|h|, (i+1)|h|]} g(x) dx \leq |h| \sum_{i=-n}^{n-1} g(a_i) = |h| \sum_{i=-n}^{n-1} |f(a_i+h) - f(a_i)| \\ &\leq |h|V(f, \mathcal{P}) \leq |h|C. \end{aligned}$$

Here \mathcal{P} is a partition of $[-n|h|, n|h|]$ which includes the points $\{a_i, a_i+h\}_{i=-n}^{n-1}$. Because this is true for any $n \in \mathbb{N}$, we get the claim by the continuity of the Lebesgue integral. \square

5. Let $f \in L^\infty([0, 1])$. Prove that

$$\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

Proof. Because $\|f\|_p \leq \|f\|_q$ for any $p \leq q$, the limit $L := \lim_{p \rightarrow \infty} \|f\|_p$ exists in $[0, \infty]$. But since $f \in L^\infty([0, 1])$, $L \in [0, \infty)$. Notice that $\|f\|_p = \left(\int_{[0,1]} |f|^p \right)^{1/p} \leq \left(\|f\|_\infty^p \right)^{1/p} = \|f\|_\infty$. So, $L \leq \|f\|_\infty$. To show the reverse inequality: for $n \in \mathbb{N}$, define

$$E_n := \{x \in [0, 1] : \|f\|_\infty(1 - \frac{1}{n}) \leq |f(x)| \leq \|f\|_\infty\}.$$

Then $m(E_n) > 0$. Thus

$$\|f\|_p^p = \int_{[0,1]} |f|^p = \int_{E_n} |f|^p + \int_{[0,1] \setminus E_n} |f|^p \geq \|f\|_\infty^p \cdot \left(1 - \frac{1}{n}\right)^p \cdot m(E_n)$$

Hence $L = \lim_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty \left(1 - \frac{1}{n}\right)$. Since this is true for any $n \in \mathbb{N}$, we get $L \geq \|f\|_\infty$ and therefore the claim. \square

6. Give an example of a function $f : [0, 1] \rightarrow \mathbb{R}$ such that $f \in L^p([0, 1])$ for every $p \in [1, \infty)$, but $f \notin L^\infty([0, 1])$.

Consider the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} n \chi_{\left(2^{-\frac{1}{n+1}}, 2^{-\frac{1}{n}}\right]}(x) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly it is unbounded and for any $p \geq 1$,

$$\|f\|_p^p = \sum_{n=1}^{\infty} \frac{n^p}{2^{n+1}} < \infty$$

by, for example, root test. Hence $f \in L^p([0, 1])$.

Summer 2020

Pavel Bleher

Problem 1. Let E be the set of all real numbers x on the interval $[0, 1]$ such that in the decimal form of $x = 0.i_1i_2i_3\cdots$ the number 5 appears infinitely many times. Prove that $mE = 1$.

Proof. Let $F := \{x \in [0, 1] : \text{the decimal expansion of } x \text{ has no } 5\}$. Then

$$F^c := [0, 1] \setminus F = \frac{1}{10}(5, 6) \sqcup \frac{1}{10^2} \bigsqcup_{\substack{i=0 \\ i \neq 5}}^9 (i5, i6) \sqcup \frac{1}{10^3} \bigsqcup_{\substack{i,j=0 \\ i,j \neq 5}}^9 (ij5, ij6) \sqcup \cdots$$

(Here we write 5 as $4.\bar{9}$ to make F compact)

Thus $m(F^c) = \frac{1}{9} \sum_{i=1}^{\infty} (\frac{9}{10})^i = 1$. This says $m(F) = 0$. For $n \in \mathbb{N} \cup \{0\}$, define

$$F_n := \{x \in [0, 1] : \text{the decimal expansion of } x \text{ has no } 5 \text{ after the } n^{\text{th}} \text{ place}\}.$$

Then $F_n \subset \frac{1}{10^n} \bigcup_{i=0}^{10^n-1} (i + F)$. Because the Lebesgue measure is invariant under translation, we then get $m(F_n) = 0$ and therefore $m(\bigcup_{n=1}^{\infty} F_n) = 0$. Since $E = [0, 1] \setminus (\bigcup_{n=1}^{\infty} F_n)$, we have the claim. \square

Problem 2. Prove that if functions f_1, f_2 are absolutely continuous on $[0, 1]$, then the function $f(x) = \max\{e^{f_1(x)}, |f_2(x)|\}$ is absolutely continuous on $[0, 1]$ as well.

Proof. Notice that e^x is Lipschitz on $[0, 1]$ (with a Lipschitz constant, for an example, e . To see: the line $y = x + e$ lies above the curve $y = e^x$ over $[0, 1]$. Or, the line $y = ex$ lies above the curve $y = e^x - 1$ over $[0, 1]$.) Hence $e^{f_1(x)} \in AC[0, 1]$. By the reverse triangle inequality, $|f_2| \in AC[0, 1]$. This together with the fact $|f(a) - f(b)| \leq |e^{f_1(b)} - e^{f_1(a)}| + ||f_2(b)| - |f_2(a)||$ for any $a, b \in [0, 1]$, we have $f \in AC[0, 1]$. \square

Problem 3. Prove that for any integrable function f on the interval $[a, b]$,

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \sin(\sin(nx)) \, dx = 0.$$

Proof. Because the collection of all step functions on $[a, b]$ with rational values is dense (under $\|\cdot\|_1$ norm) in $L^1([a, b])$, and the linearity of the Lebesgue integral, it is enough to show

$$\lim_{n \rightarrow \infty} \int_p^q \sin(\sin(nx)) \, dx = 0 \text{ for any interval } [p, q] \subseteq [a, b].$$

By changing the variable, we get

$$\lim_{n \rightarrow \infty} \left| \int_p^q \sin(\sin(nx)) \, dx \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \left| \int_{np}^{nq} \sin(\sin(x)) \, dx \right| \leq \lim_{n \rightarrow \infty} \frac{2}{n} = 0.$$

Note: Over any interval, the area under the curve $y = \sin x$ is in between -2 and 2 . \square

Problem 4. Prove that for any $1 \leq p < \infty$ there is a constant $C_p > 0$ such that for any integrable function f on $[0, 1]$,

$$\|\ln(1 + |f|)\|_p \leq C_p(1 + \|f\|_1).$$

Proof. For a fixed $p \geq 1$, defined the function $\varphi_p : [1, \infty) \rightarrow \infty$ by $\varphi_p(x) := (p + \ln x)^p$. Since $\varphi_p''(x) = -\frac{p}{x^2}(p + \ln x)^{p-2}(1 + \ln x) < 0$ on $[1, \infty)$, φ_p is concave. Let $f \in L^1([0, 1])$. Since $1 + |f| \geq 1$, by the Jensen's inequality for concave functions we get

$$\int_{[0,1]} \varphi_p \circ (1 + |f|) \leq \varphi_p\left(\int_{[0,1]} 1 + |f|\right).$$

Because

$$\|\ln(1 + |f|)\|_p^p \leq \int_{[0,1]} (p + \ln(1 + |f|))^p = \int_{[0,1]} \varphi_p \circ (1 + |f|)$$

and

$$\begin{aligned} \varphi_p\left(\int_{[0,1]} 1 + |f|\right) &= \left(p + \ln\left(\int_{[0,1]} 1 + |f|\right)\right)^p = \left(p + \ln(1 + \|f\|_1)\right)^p \\ &= \left(\ln e^p(1 + \|f\|_1)\right)^p \\ &\leq [e^p(1 + \|f\|_1)]^p, \end{aligned}$$

we get the claim by letting $C_p = e^p$. □

Problem 5. Let $f \in L^2[0, 1]$ and $m = \int_{[0,1]} f$. Prove that

$$m^2 + \left(\int_0^1 |f(x) - m| dx\right)^2 \leq \int_0^1 f^2(x) dx.$$

Proof. Because $f \in L^2[0, 1]$, $f \in L^1[0, 1]$. Hence $|m| < \infty$ and

$$m^2 + \left(\int_0^1 |f(x) - m| dx\right)^2 \leq m^2 + \int_0^1 (f - m)^2 = m^2 + \int_0^1 f^2 - 2m^2 + m^2 = \int_0^1 f^2.$$

The inequality is due to Jensen. □

Problem 6. Let $f \in L^1(\mathbb{R})$ be such that

$$\int_a^{\frac{a+b}{2}} f = \int_{\frac{a+b}{2}}^b f, \forall -\infty < a < b < \infty.$$

Prove that then $f(x) = 0$ almost everywhere on $(-\infty, \infty)$.

Proof. We can assume $f \geq 0$. Suppose the set $E := f^{-1}((0, \infty))$ has a positive measure. Then there is $a > 0$ such that $M := \int_{-a}^a f > 0$. Then by the hypothesis $\int_{-a}^{3a} f = 2M$ and therefore by induction, for any $n \in \mathbb{N}$, $\int_{-a}^{(2n-1)a} f = nM$. Since $M > 0$, $\int_{\mathbb{R}} f = \infty$. This contradicts the fact f is integrable. Hence $m(E) = 0$. The general case follows by considering f_+ and f_- separately and $|f| = f_+ + f_-$. □

Winter 2020

Pavel Bleher

Problem 1. Let E be the set of real numbers x on the interval $[0, 1]$ such that in the decimal expansion form of $x = 0.i_1i_2i_3\cdots$, there is no 5, so that $i_k \neq 5$ for all k . Prove that

1. the set E is uncountable;
2. $m(E) = 0$, where $m(E)$ is the Lebesgue measure of E .

Proof. 1. The set $\{x \in [0, 1] : \text{the decimal expansion of } x \text{ consists with only 0 and 1}\}$ is uncountable (by Cantor's diagonal argument) and is a subset of E .

2. Notice that

$$[0, 1] \setminus E = \frac{1}{10}[5, 6) \sqcup \frac{1}{10^2} \bigsqcup_{\substack{i=0 \\ i \neq 5}}^9 [i5, i6) \sqcup \frac{1}{10^3} \bigsqcup_{\substack{i,j=0 \\ i,j \neq 5}}^9 [ij5, ij6) \sqcup \frac{1}{10^4} \bigsqcup_{\substack{i,j,k=0 \\ i,j,k \neq 5}}^9 [ijk5, ijk6) \sqcup \cdots$$

Thus $m([0, 1] \setminus E) = \frac{1}{9} \sum_{i=1}^{\infty} (\frac{9}{10})^i = 1$ and therefore $m(E) = 0$.

□

Problem 2. Let E be the same set as in Problem 1. Calculate the fractal dimension of E , defined as

$$d := \lim_{k \rightarrow \infty} \frac{\ln N(k)}{\ln k},$$

where $N(k)$ is the smallest number of intervals of length $\frac{1}{k}$ covering E . Prove rigorously the existence of the limit.

Answer. Since

$$[0, 1] \setminus E = \frac{1}{10}[5, 6) \sqcup \frac{1}{10^2} \bigsqcup_{\substack{i=0 \\ i \neq 5}}^9 [i5, i6) \sqcup \frac{1}{10^3} \bigsqcup_{\substack{i,j=0 \\ i,j \neq 5}}^9 [ij5, ij6) \sqcup \frac{1}{10^4} \bigsqcup_{\substack{i,j,k=0 \\ i,j,k \neq 5}}^9 [ijk5, ijk6) \sqcup \cdots,$$

for each $n \in \mathbb{N}$, E can be covered by 9^n number of intervals of length $1/10^n$. For $k \in \mathbb{N} \cup \{0\}$, let $n_k \in \mathbb{N}$ be such that $10^{n_k} \leq k < 10^{n_k+1}$. Then $9^{n_k} = N(10^{n_k}) \leq N(k) \leq N(10^{n_k+1}) = 9^{n_k+1}$. Hence

$$\frac{n_k \ln 9}{(n_k + 1) \ln 10} \leq \frac{\ln N(k)}{\ln k} \leq \frac{(n_k + 1) \ln 9}{n_k \ln 10}$$

and therefore by the squeeze theorem, $d = \frac{\ln 9}{\ln 10}$.

Problem 3. Let us enumerate all rational numbers on $[0, 1]$, so that $\mathbb{Q} \cap [0, 1] = \{r_1, r_2, r_3, \cdots\}$. Prove that for any $\alpha > -1$, the series

$$f(x) = \sum_{k=1}^{\infty} \frac{|x - r_k|^\alpha}{k^2}$$

is convergent for almost all x on $[0, 1]$.

Proof. For $n \in \mathbb{N}$, define $f_n : [0, 1] \rightarrow [0, \infty)$ by $f_n(x) := \sum_{k=1}^n \frac{|x-r_k|^\alpha}{k^2}$. Then

$$\begin{aligned} \int_{[0,1]} f_n(x) dx &= \int_{[0,1]} \left(\sum_{k=1}^n \frac{|x-r_k|^\alpha}{k^2} \right) dx = \sum_{k=1}^n \int_{[0,1]} \frac{|x-r_k|^\alpha}{k^2} dx \\ &= \sum_{k=1}^n \int_{[0,r_k]} \frac{(r_k-x)^\alpha}{k^2} dx + \int_{[r_k,1]} \frac{(x-r_k)^\alpha}{k^2} dx \\ &= \sum_{k=1}^n \frac{1}{k^2} \left(\frac{r_k^{\alpha+1}}{\alpha+1} + \frac{(1-r_k)^{\alpha+1}}{\alpha+1} \right) \\ &\leq \frac{1+2^{\alpha+1}}{\alpha+1} \sum_{k=1}^{\infty} \frac{1}{k^2} =: M < \infty. \end{aligned} \quad (0.1)$$

Since M does not depend on n , it is a uniform bound of the sequence $(\int_{[0,1]} f_n)_{n=1}^{\infty}$. Because $(f_n)_{n=1}^{\infty}$ is an increasing sequence, the pointwise limit $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ exists in $\overline{\mathbb{R}}$. Then by the Monotone Convergence Theorem and (0.1), it exists almost everywhere in \mathbb{R} . In other words, the given series converges for almost every $x \in [0, 1]$. \square

Problem 4. Let $f(x)$ be the function defined in Problem 3. Prove that for any $\alpha > 0$, $f(x)$ is absolutely continuous on $[0, 1]$.

Proof. Let f_n be as in Problem 3. Notice that each f_n is differentiable almost everywhere on $[0, 1]$ (in fact, it is differentiable at all but finitely many points: r_1, \dots, r_n) and $f'_n(x) = \sum_{k=1}^n \delta(r_k) \frac{\alpha(x-r_k)^{\alpha-1}}{k^2}$. Here $\delta(r_k) = 1$ if $x > r_k$ and $\delta(r_k) = -1$ if $x < r_k$. Notice that $|f'_n(x)| \leq \alpha \sum_{k=1}^n \frac{|x-r_k|^{\alpha-1}}{k^2}$. Because $\alpha > 0$, $\alpha - 1 > -1$. So, by replacing α with $\alpha - 1$ in Problem 3 we see that the series $g(x) := \sum_{k=1}^{\infty} \delta(r_k) \frac{\alpha(x-r_k)^{\alpha-1}}{k^2} (= \lim_{n \rightarrow \infty} f'_n(x))$ converges absolutely for almost every $x \in [0, 1]$. Hence

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \left(f_n(0) + \int_0^x f'_n(t) dt \right) \\ &= \lim_{n \rightarrow \infty} f_n(0) + \lim_{n \rightarrow \infty} \int_0^x f'_n(t) dt \\ &= f(0) + \int_0^x \lim_{n \rightarrow \infty} f'_n(t) dt, \text{ by general LDCT} \\ &= f(0) + \int_0^x g(t) dt. \end{aligned}$$

By comparison test, $\lim_{n \rightarrow \infty} f_n(0) = f(0)$.

This says f is an indefinite integral over $[0, 1]$. Hence $f \in AC([0, 1])$. \square

Problem 5. Prove that for any integrable function f on $(-\infty, \infty)$,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \cos(nx^2) f(x) dx = 0.$$

Proof. By the definition of Lebesgue integral of a non-negative function (we can assume $f \geq 0$). The general case follows by considering the positive and negative parts of f separately, if we can show the claim for any non-negative bounded function of finite support, then we are done. But any such a function can be approximated by (in this case by $\|\cdot\|_1$) step functions. Therefore by the linearity of the Lebesgue integral, if we can show the claim for the case $f(x) = 1$ then we are done. In fact

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} \cos(nx^2) dx \right| &= \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} \frac{\cos(x^2)}{\sqrt{n}} dx \right|, \text{ by letting } x = \frac{x}{\sqrt{n}}. \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \int_{\mathbb{R}} \cos(x^2) dx \right| \\ &\leq \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0. \end{aligned}$$

This implies $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \cos(nx^2) dx = 0$, and therefore the claim. \square

Problem 6. Let $f(x)$ be an integrable function on $(-\infty, \infty)$ and

$$f_n(x) = n \int_x^{x+\frac{1}{n}} f(t) dt, \quad n = 1, 2, 3, \dots$$

Prove that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n(x) - f(x)| dx = 0.$$

Proof. • **Proof 1** Let $n \in \mathbf{N}$. Notice that

$$\begin{aligned} \int_{\mathbb{R}} |f_n(x)| dx &= \sum_{k=-\infty}^{\infty} \int_0^{\frac{1}{n}} |f_n(x + \frac{k}{n})| dx = \int_0^{\frac{1}{n}} \left(\sum_{k=-\infty}^{\infty} |f_n(x + \frac{k}{n})| \right) dx \\ &\leq \int_0^{\frac{1}{n}} \left(\sum_{k=-\infty}^{\infty} n \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} |f(t)| dt \right) dx \\ &= \int_0^{\frac{1}{n}} n \int_{\mathbb{R}} |f| dx = \int_{\mathbb{R}} |f| dx. \end{aligned}$$

Thus

$$\|f_n\|_1 \leq \|f\|_1, \text{ for all } n \in \mathbf{N}. \quad (0.2)$$

Let $s(x) := \sum_{i=1}^k a_i \chi_{I_i}(x)$ be a step function. Then

$$s_n(x) = n \int_x^{x+\frac{1}{n}} \sum_{i=1}^k a_i \chi_{I_i}(t) dt = n \sum_{i=1}^k a_i m(I_i \cap [x, x + \frac{1}{n}]).$$

Let x be a point in the interior of $\cup_{i=1}^k I_i$. WLOG assume $x \in \text{int}(I_j) = (p_j, q_j)$ for some $j \in \{1, \dots, k\}$ and $p_j, q_j \in \mathbb{R}$. Then there is $N \in \mathbf{N}$ (for example take N such that $\frac{1}{N} \leq \min\{x - p_j, q_j - x\}$) such that for each $n > N$, $[x, x + \frac{1}{n}] \subset I_j$. Hence for each $n > N$,

$s_n(x) = a_j = s(x)$. If $x \notin \bigcup_{i=1}^k I_j$, then clearly $s(x) = s_n(x) = 0$ for any $n \in \mathbb{N}$. And the set of end-points of the intervals I_i 's are finite. This says $s_n \xrightarrow{a.e.} s$. That is $(s_n - s) \xrightarrow{a.e.} 0$. Then by the Lebesgue Dominated Convergence Theorem we have the claim for any step function:

$$\lim_{n \rightarrow \infty} \|s_n - s\|_1 = 0. \quad (0.3)$$

Let $f \in L^1(\mathbb{R})$. Let $\epsilon > 0$ be given. Then we can find $s \in \text{step}(\mathbb{R})$ such that $\|f - s\|_1 < \frac{\epsilon}{2}$. Then by (0.2) we have $\|f_n - s_n\|_1 = \|(f_n - s_n)\|_1 \leq \|f - s\|_1 < \frac{\epsilon}{2}$. Thus

$$\|f_n - f\|_1 \leq \|f_n - s_n\|_1 + \|s_n - s\|_1 + \|s - f\|_1 < \|s_n - s\|_1 + \epsilon.$$

Then by taking the limit as $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$ as $\epsilon > 0$ is arbitrary.

• **Proof 2** Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by $F(x) := \int_0^x f(t)dt$. Because $f \in L^1(-\infty, \infty)$, this is well-defined and absolutely continuous. Thus $F'(x) = f(x)$ for almost every $x \in (-\infty, \infty)$. Notice that

$$\frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}} = f_n(x), \text{ for } n \in \mathbb{N}.$$

Thus $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for almost every $x \in \mathbb{R}$. By Minkowski's inequality and (0.2) we have $\|f_n - f\|_1 \leq \|f_n\|_1 + \|f\|_1 \leq 2\|f\|_1 < \infty$. Therefore by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n(x) - f(x)| dx = 0.$$

□

Summer 2019

Pavel Bleher

Problem 1. Let $E \subset \mathbb{R}$, $mE = 0$, and $P(x)$ a real polynomial of degree $d \geq 1$. Let

$$S = P^{-1}(E) = \{x \in \mathbb{R} : P(x) \in E\}.$$

Prove that $mS = 0$.

Proof. Let $I \subset \mathbb{R}$ be a bounded interval. Then $P^{-1}(I)$ consists with at-most d number of disjoint bounded intervals. Notice that if $l(I)$ goes to 0 (that is when I contracts to a point), so does $m(P^{-1}(I))$. Otherwise there will be a point a such that $P^{-1}(\{a\})$ has an interval. But that contradicts $d \geq 1$. Thus for a given $\epsilon > 0$, there is $d(\epsilon) > 0$ such that if $l(I) < d(\epsilon)$, then $m(P^{-1}(I)) < \epsilon$ (basically because P is continuous and P' is bounded on compact sets). Because $m(E) = 0$, we can find a countable collection $\{I_{i,\epsilon}\}_{i \in \mathbb{N}}$ of disjoint open intervals which forms an open cover of E such that $l(I_{i,\epsilon}) \leq d(\frac{\epsilon}{2^i})$. Then $S = P^{-1}(E) \subset P^{-1}(\bigsqcup_{i=1}^{\infty} I_{i,\epsilon}) = \bigsqcup_{i=1}^{\infty} P^{-1}(I_{i,\epsilon})$ and therefore $m(S) \leq \sum_{i=1}^{\infty} m(P^{-1}(I_{i,\epsilon})) \leq \epsilon$ (S is measurable as E and f are measurable). Because this is true for any $\epsilon > 0$, $m(S) = 0$. \square

Problem 2. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|\tan(nx)|^{1/2}}{n^2}$$

is finite almost everywhere on the interval $[0, \pi]$.

Proof. If we can prove $f \in L^1([0, \pi])$, then we are done. For $k \in \mathbb{N}$, define

$$f_k(x) := \sum_{n=1}^k \frac{|\tan(nx)|^{1/2}}{n^2}.$$

Then

$$\begin{aligned} \int_0^{\pi} f_k(x) \, dx &= \sum_{n=1}^k \int_0^{\pi} \frac{|\tan(nx)|^{1/2}}{n^2} \, dx = \sum_{n=1}^k \int_0^{n\pi} \frac{|\tan(x)|^{1/2}}{n^3} \, dx \\ &\leq \sum_{n=1}^k \frac{1}{n^3} \sum_{i=0}^{n-1} \int_{i\pi}^{(i+1)\pi} \left| \left(i + \frac{1}{2}\right)\pi - x \right|^{-\frac{1}{2}} \, dx \\ &\quad \left(\because |\tan(nx)|^{1/2} \leq \left| \frac{\pi}{2} - x \right|^{-\frac{1}{2}} \text{ on } [0, \pi] \right) \\ &= \sum_{n=1}^k \frac{1}{n^3} \sum_{i=0}^{n-1} 2\sqrt{2\pi} = 2\sqrt{2\pi} \sum_{n=1}^k \frac{1}{n^2} \\ &\leq 2\sqrt{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} =: M < \infty. \end{aligned}$$

Since this is true for any $k \in \mathbb{N}$, by the Monotone Convergence Theorem, $\int_0^{\pi} f(x) \, dx \leq M < \infty$. \square

Problem 3. Prove that for any integrable function f on the line,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) [\sin(nx)]^n dx = 0.$$

Proof. • **Proof 1.** By the density of $step(\mathbb{R})$ in $L^1(\mathbb{R})$ under $\|\cdot\|_1$ and the linearity of the Lebesgue integral, it is enough to show the claim for the case when $f \in step(\mathbb{R})$. So let $f(x) \stackrel{\text{a.e.}}{=} \chi_{[a,b]}(x)$ for any interval $[a, b]$. By change of variables we get

$$\int_{-\infty}^{\infty} \chi_{[a,b]}(x) \cdot [\sin(nx)]^n dx = 1/n \int_{na}^{nb} (\sin x)^n dx.$$

Let $\alpha_n = 1 - 1/\ln(n+1)$. Then $\alpha_n \nearrow 1$ and $\lim_{n \rightarrow \infty} \alpha_n^n = 0$. For $n \in \mathbb{N}$ choose $E_n \subset [na, nb]$ so that $\alpha_n \geq \sup\{|\sin x| : x \in [na, nb] \setminus E_n\}$ and $\int_{E_n} |\sin x|^n dx \leq \alpha_n \leq 1$.

(Idea: Depending on $n \in \mathbb{N}$, choose a finite collection of disjoint intervals around the points $[na, nb] \cap \sin^{-1}(\{\pm 1\})$ with above given properties).

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \int_{na}^{nb} |\sin x|^n dx &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \int_{(na, nb) \setminus E_n} |\sin x|^n dx + \frac{1}{n} \int_{E_n} |\sin x|^n dx \right) \\ &\leq \lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \alpha_n^n \cdot (nb - na) + \frac{1}{n} \cdot 1 \right) \\ &= \lim_{n \rightarrow \infty} \alpha_n^n (b - a) \\ &= 0. \end{aligned}$$

• **Proof 2.** Instead of creating the sequence $(E_n)_{n=1}$ of sets as above, use the fact $m\left(\bigcup_{n=1}^{\infty} \sin^{-1}(\{\pm 1\})/n\right) = 0$ and the Lebesgue Dominated Convergence Theorem on $f_n(x) := \chi_{[a,b]}(x) \cdot \sin^n(nx)$. \square

Problem 4. Let $f \geq 0$ be a measurable function on $E \subset \mathbb{R}$ and $mE < \infty$. Let

$$E_n = \{x \in E : f(x) \geq n\}.$$

Prove that f is integrable if and only if

$$\sum_{n=1}^{\infty} mE_n < \infty.$$

Proof. Notice that $\sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} n \cdot m(E_n \setminus E_{n+1})$ and

$$\sum_{n=1}^{\infty} n \cdot m(E_n \setminus E_{n+1}) \leq \sum_{n=1}^{\infty} \int_{E_n \setminus E_{n+1}} f \leq \sum_{n=1}^{\infty} (n+1) \cdot m(E_n \setminus E_{n+1}). \quad (0.4)$$

Because $m(E) < \infty$,

$$f \text{ is integrable iff } \int_{E_1} f < \infty \text{ iff } \sum_{n=1}^{\infty} \int_{E_n \setminus E_{n+1}} f < \infty.$$

Then by the comparison tests (standard and limit comparison tests) and (0.4), we get the claim. \square

Problem 5. Let f be a measurable function on $[0, 1]$ and

$$A = \{x \in [0, 1] : f(x) \in \mathbb{Z}\}.$$

Prove that the set A is measurable and

$$\lim_{n \rightarrow \infty} \int_0^1 |\cos(\pi f(x))|^n dx = m(A).$$

Proof. Because $A = \bigsqcup_{n \in \mathbb{Z}} f^{-1}(\{n\})$, it is measurable as countable union of measurable sets is measurable.

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 |\cos(\pi f(x))|^n dx &= \lim_{n \rightarrow \infty} \int_{[0,1] \setminus A} |\cos(\pi f(x))|^n dx + \lim_{n \rightarrow \infty} \int_A |\cos(\pi f(x))|^n dx \\ &= 0 + \lim_{n \rightarrow \infty} \int_A 1 dx, \quad \because |\cos(\pi f(x)\chi_{[0,1] \setminus A}(x))|^n \rightarrow 0 \text{ and by LDCT.} \\ &= m(A). \end{aligned}$$

\square

Problem 6. Let $f_n(x) = \cos(nx)$ on $[0, 2\pi]$. Prove that there is no sub-sequence f_{n_k} converging almost everywhere in $[0, 2\pi]$.

Proof. Let $(f_{n_k})_{k \in \mathbb{N}}$ be a sub-sequence of $(f_n)_{n \in \mathbb{N}}$. For the sake of contradiction let's assume that it converges almost everywhere on $[0, 2\pi]$. Then the sequence $((f_{n_k} - f_{n_{k+1}})^2)_{k=0}^{\infty}$ converges to 0 almost everywhere on $[0, 2\pi]$. Therefore by LDCT,

$$0 = \lim_{k \rightarrow \infty} \int_0^{2\pi} (f_{n_k} - f_{n_{k+1}})^2 = \lim_{k \rightarrow \infty} \int_0^{2\pi} (\cos(n_k x) - \cos(n_{k+1} x))^2 dx = 2\pi.$$

Hence the contradiction. \square

Winter 2019

Pavel Bleher

Problem 1. Let F be a bounded, closed set on the line, and

$$F_\epsilon = \bigcup_{x \in F} [x - \epsilon, x + \epsilon], \quad \epsilon > 0.$$

Prove that $\lim_{\epsilon \rightarrow 0} mF_\epsilon = mF$, where m is the Lebesgue measure.

Proof. Because $F = \bigcap_{k=1}^\infty F_{1/k}$, we have the claim by the continuity of Lebesgue measure (and, if you want, the squeeze theorem). \square

Problem 2. Let $E_1 \subset E_2 \subset \dots$ be an increasing sequence of Lebesgue measurable sets on the line, such that the set $E = \bigcup_{n=1}^\infty E_n$ has a finite Lebesgue measure, $mE < \infty$. Prove that for any set $A \subset \mathbb{R}$ (not necessarily measurable),

$$\lim_{n \rightarrow \infty} m^*(A \cap E_n) = m^*(A \cap E).$$

Proof. Because $E_n \cap A \subset E \cap A$ for all $n \in \mathbb{N}$, we have

$$\limsup_{n \rightarrow \infty} m^*(A \cap E_n) \leq m^*(A \cap E).$$

On the other hand, because $m^*(A \cap E) < m(E \setminus E_n) + m^*(A \cap E_n)$, we have

$$m^*(A \cap E) \leq \liminf_{n \rightarrow \infty} m^*(A \cap E_n).$$

Thus

$$\lim_{n \rightarrow \infty} m^*(A \cap E_n) = m^*(A \cap E).$$

\square

Problem 3. Let $f \in L^1(-\infty, \infty)$. Find the limit,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \left(\cos x + \frac{\sin^2 x}{2} \right)^n dx,$$

and justify your answer.

Answer. Notice that $0 \leq (1 - \cos x)^2$ implies $\cos x + \sin^2 x/2 \leq 1$. Hence the maximum value the function $g(x) := |\cos x + \sin^2 x/2|$ can obtain is 1, and $g^{-1}(\{1\})$ is countable. Because $f \in L^1(-\infty, \infty)$, $f^{-1}(\{\infty\})$ has measure zero. Therefore the set $E := g^{-1}(\{1\}) \cup f^{-1}(\{\infty\})$ also has measure zero. Define the sequence $(f_n)_{n=1}$ of measurable functions on \mathbb{R} by $f_n(x) := f(x) \left(\cos x + \frac{\sin^2 x}{2} \right)^n$. Then it is integrable as $|f_n| \leq |f|$, and $f_n \rightarrow 0$ for all $x \in \mathbb{R} \setminus E$. Thus by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \left(\cos x + \frac{\sin^2 x}{2} \right)^n dx = 0.$$

Problem 4. Prove that for any integrable function f on the interval $[a, b]$,

$$\lim_{n \rightarrow \infty} \int_a^b \frac{\cos(nx)}{1 + \sin^2(nx)} f(x) dx = 0.$$

Proof. Due to the separability of $L^1[a, b]$ by $\mathcal{S}'[a, b]$ (the collection of all step functions on $[a, b]$ with rational values) under $L^1[a, b]$ norm, and the linearity of the Lebesgue integral, if we can show the claim when $f(x) = 1$ on any sub-interval $[p, q] \subseteq [a, b]$, then we are done. In fact

$$\lim_{n \rightarrow \infty} \int_p^q \frac{\cos(nx)}{1 + \sin^2(nx)} dx = \lim_{n \rightarrow \infty} \left[\frac{\tan^{-1}(\sin(nx))}{n} \right]_p^q \leq \lim_{n \rightarrow \infty} \frac{\pi}{n} = 0.$$

□

Problem 5. Let $E \subset [0, 1]$ be a measurable set such that there exists $\epsilon > 0$ such that

$$m(E \cap [a, b]) \geq \epsilon |b - a|$$

for all $[a, b] \subset [0, 1]$. Prove that $mE = 1$.

Proof. We show $m(E^c) = m([0, 1] \setminus E) = 0$. For $k > 1$, choose collection $\{I_n\}_{n=1}$ of open disjoint intervals such that $E^c \subset \bigsqcup_{n=1} I_n$ and $\sum_{n=1} m(I_n) < m(E^c) + \epsilon/k$. Then

$$\epsilon \sum_{n=1} m(I_n) \leq \sum_{n=1} m(E \cap I_n) < m\left(\left(\bigsqcup_{n=1} I_n\right) \setminus E^c\right) = \sum_{n=1} m(I_n) - m(E^c) < \frac{\epsilon}{k}.$$

This implies $\sum_{n=1} m(I_n) \leq 1/k$ for any $k > 1$. Thus $\sum_{n=1} m(I_n) = 0$. Because $E^c \subset \bigsqcup_{n=1} I_n$, $m(E^c) = 0$. □

Problem 6. Prove that for any function $f \in L^2[0, 1]$,

$$\|\ln(1 + |f|)\|_{L^1[0,1]} \leq \|f\|_{L^2[0,1]}.$$

Proof. By Jensen's inequality (on $\varphi(x) = x^2$),

$$\|\ln(1 + |f|)\|_{L^1[0,1]}^2 \leq \int_0^1 (\ln(1 + |f|))^2. \quad (0.5)$$

A fact:

$$\text{For any } x \geq 0, \ln(1 + x) \leq x. \quad (0.6)$$

□

By combining (0.5) and (0.6), we get the claim.

Summer 2018

Pavel Bleher

Problem 1. Let E be the set of real numbers x on the interval $[0, 1]$ such that in the decimal form of $x = 0.i_1i_2i_3\cdots$ there is no string of four consecutive digits 2018. Prove that

1. the set E is uncountable
2. $mE = 0$, where mE is the Lebesgue measure of E .

Proof. 1. The set $\{x \in [0, 1] : \text{the decimal expansion of } x \text{ consists with only 3 and 4}\}$ is uncountable (by Cantor's diagonal argument), and is a subset of E .

2. Notice that

$$[0, 1] \setminus E = \frac{1}{10^4} [2018, 2019) \sqcup \frac{1}{10^8} \bigsqcup_{\substack{i=0 \\ i \neq 2018}}^{10^4-1} [i2018, i2019) \sqcup \frac{1}{10^{12}} \bigsqcup_{\substack{i,j=0 \\ i,j \neq 2018}}^{10^4-1} [ij2018, ij2019) \sqcup \cdots$$

Thus $m([0, 1] \setminus E) = \frac{1}{10^4-1} \sum_{i=1}^{\infty} \left(\frac{10^4-1}{10^4}\right)^i = 1$, and therefore $m(E) = 0$.

□

Problem 2. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - n^{-1}|^{1/2}}{n^2}$$

is absolutely continuous on $[0, 1]$.

Proof. Because $|x - 1/n| \leq 1$ for any $x \in [0, 1]$, the series $\sum_{n=1}^{\infty} \frac{|x - n^{-1}|^{1/2}}{n^2}$ converges absolutely for all $x \in [0, 1]$. Thus for any $a, b \in [0, 1]$, $a < b$ we have

$$|f(b) - f(a)| = \left| \sum_{n=1}^{\infty} \frac{|b - n^{-1}|^{1/2}}{n^2} - \sum_{n=1}^{\infty} \frac{|a - n^{-1}|^{1/2}}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{b - a}{2n^{3/2}} \leq M(b - a).$$

Here $M = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. Thus f is Lipschitz and therefore absolutely continuous.

□

Problem 3. Prove that for any integrable function f on the interval $[a, b]$,

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \sin^4(nx) dx = \frac{3}{8} \int_a^b f(x) dx.$$

Proof. We have that

$$\sin^4(nx) = \frac{3 - 4 \cos(2nx) + \cos(4nx)}{8}.$$

Thus if we can show that

$$\lim_{n \rightarrow \infty} \int_a^b f(x) \cos(knx) dx = 0,$$

for any $f \in L^1[a, b]$ and any constant $k \in \mathbb{R}$, then we are done. Because $\mathcal{S}[a, b]$ - the collection of all step functions on $[a, b]$, is dense in $L^1[a, b]$ under $L^1[a, b]$ norm, and the Lebesgue integral is linear, if we can show that

$$\lim_{n \rightarrow \infty} \int_p^q \cos(nx) dx = 0 \text{ for any } p, q \in \mathbb{R}, p < q,$$

then we are done. In fact

$$\lim_{n \rightarrow \infty} \left| \int_p^q \cos(nx) dx \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \left| \int_{np}^{nq} \cos x dx \right| \leq \lim_{n \rightarrow \infty} \frac{2}{n} = 0.$$

□

Problem 4. Let $f \in L^2(-\infty, \infty)$. Prove that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x)f(x+n)dx = 0.$$

Proof. Because $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x+n)dx$ for any $n \in \mathbb{N}$, from Holder's inequality we have

$$\int_{\mathbb{R}} |f(x)f(x+n)|dx \leq \|f(x)\|_2 \cdot \|f(x+n)\|_2 = \|f(x)\|_2^2 < \infty.$$

Thus $f(x)f(x+n) \in L^1(-\infty, \infty)$ for all $n \in \mathbb{N}$. Therefore if we can show

$$\lim_{n \rightarrow \infty} \int_{-N}^N f(x)f(x+n)dx = 0 \text{ for any } N \in \mathbb{N},$$

then we are done.

And if we can show this claim for any simple function on $[-N, N]$, then we are done by the simple approximation theorem. Let $\varphi(x) = \sum_{i=1}^k a_i \chi_{E_i}(x)$ where $E_i \subset [-N, N]$ for all $i = 1, \dots, k$. Then $\varphi(x+n) = \sum_{i=1}^k a_i \chi_{E_i-n}(x)$. Hence

$$\int_{[-N, N]} \varphi(x)\varphi(x+n)dx = \sum_{i=1, j=i}^k a_i a_j m(E_i \cap (E_j - n)).$$

Because each E_j is bounded, for large enough n (say $n > 2N$) we have that $E_i \cap (E_j - n) = \emptyset$ for all $i, j \in \{1, \dots, k\}$. Thus

$$\lim_{n \rightarrow \infty} \int_{[-N, N]} \varphi(x)\varphi(x+n)dx = 0.$$

□

Problem 5. Let $f \in L^3[0, \pi]$ and

$$g(x) = \frac{f(x)}{|\sin x|^{0.1}}.$$

Prove that $\|g\|_2 \leq 2\|f\|_3$.

Proof. Let $\theta \in [0, \pi/2]$ be such that $\theta^3 = \sin^2 \theta$. Then simple calculations show that $\pi/4 < \theta < \pi/3$. So, $x^3 < \sin^2 x$ on $[0, \theta]$ and $\sin^2 x < x^3$ on $(\theta, \pi/2]$. Hence

$$\begin{aligned} \int_0^\pi \frac{dx}{\sin^{0.6} x} &= 2 \int_0^{\pi/2} \frac{dx}{\sin^{0.6} x} \quad \because y = 1/\sin^{0.6} x \text{ is symmetric around } x = \pi/2 \\ &\leq 2 \left(\int_0^\theta \frac{dx}{x^{0.9}} + \int_\theta^{\pi/2} \frac{dx}{\sin^{0.6} x} \right) \\ &\leq 2 \left([x^{0.1}]_0^\theta + \frac{\pi/2 - \theta}{\sin^{0.6} \theta} \right) \leq 2 \left(\theta^{0.1} + \frac{\pi/4}{\theta^{0.9}} \right) \leq 2((\pi/3)^{0.1} + (\pi/4)^{0.1}) < 2(2 + 2) \\ &< 8. \end{aligned} \tag{0.7}$$

This says that $\sin^{0.2} x \in L^3[0, \pi]$. Then by (generalized) Holder's inequality we have

$$\|g\|_2 = \left(\int_{[0, \pi]} \frac{f(x)^2}{\sin^{0.2}(x)} dx \right)^{1/2} \leq \|f\|_3 \left(\int_{[0, \pi]} \frac{dx}{\sin^{0.6} x} \right)^{1/6} \leq \|f\|_3 8^{1/6} < 2\|f\|_3.$$

□

Problem 6. Let $f(x)$ be an integrable function on $(-\infty, \infty)$ and $n \geq 1$ an integer. Define

$$f_n(x) = n \int_x^{x+\frac{1}{n}} f(t) dt.$$

Prove that $\|f_n\|_1 \leq \|f\|_1$.

Proof. • **Proof 1.** By Tonelli,

$$\|f\|_n \leq \int_{\mathbb{R}} n \int_{\mathbb{R}} |f(t)| \cdot \chi_{[x, x+\frac{1}{n}]} dt dx = \int_{\mathbb{R}} n \int_{\mathbb{R}} |f(t)| \cdot \chi_{[x, x+\frac{1}{n}]} dx dt = \int_{\mathbb{R}} n |f(t)| \frac{1}{n} = \|f\|_1.$$

• **Proof 2.**

$$\begin{aligned} \|f\|_n &= \int_{\mathbb{R}} |f_n(x)| dx = \sum_{k=-\infty}^{\infty} \int_0^{\frac{1}{n}} |f_n(x + \frac{k}{n})| dx = \int_0^{\frac{1}{n}} \left(\sum_{k=-\infty}^{\infty} |f_n(x + \frac{k}{n})| \right) dx \\ &\leq \int_0^{\frac{1}{n}} \left(\sum_{k=-\infty}^{\infty} n \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} |f(t)| dt \right) dx \\ &= \int_0^{\frac{1}{n}} n \|f\|_1 dx = \|f\|_1. \end{aligned}$$

□