Part MATH-UA 252 — Numerical Analysis

Based on lectures by Jason Kaye

Homework done by Aaron Ma

Fall 2024

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

1 Problem 1

1.1 a

The maximum absolute error is: 1.110223024625157e-15

1.2 b

The power method will converge to the eigenvalue with the highest absolute value, which is 1, the speed of error decay depends on $\frac{\lambda_2}{\lambda_1} = 0.5$, every extra iteration cuts the error by almost half, i.e. $error \leq C \cdot (0.5^k)$

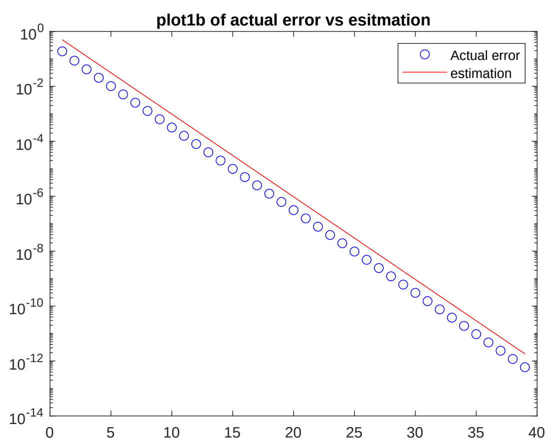

Figure 1: plot1b of actual error vs estimation.pdf

1.3 c

Similarly, $error \leq C \cdot (\frac{\lambda_2}{\lambda_1})^k = C(\frac{2}{3})^k$

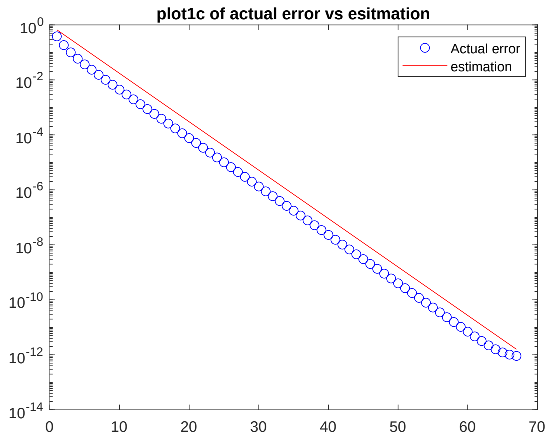

Figure 2: plot1c of actual error vs estimation

1.4 d

similarly, $error \leq C \cdot (\frac{\lambda_2}{\lambda_1})^k = C(\frac{|\frac{1}{20} - \frac{51}{1000}|}{|\frac{1}{21} - \frac{51}{1000}|})^k$

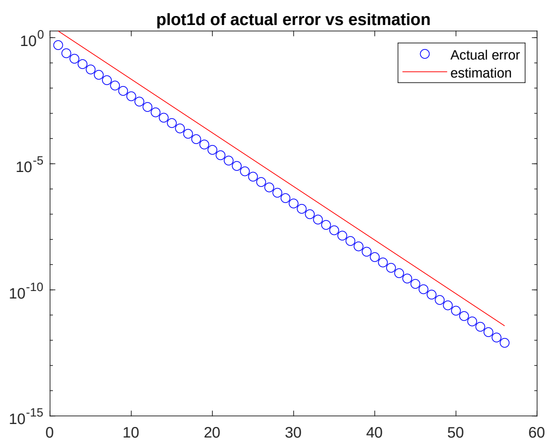

Figure 3: plot1d of actual error vs estimation

1.5 e

It takes 5 iterations for Rayleigh quotient, while it takes 57 iterations for inverse iteration

2 2

2.1 a

See the code

2.2 b

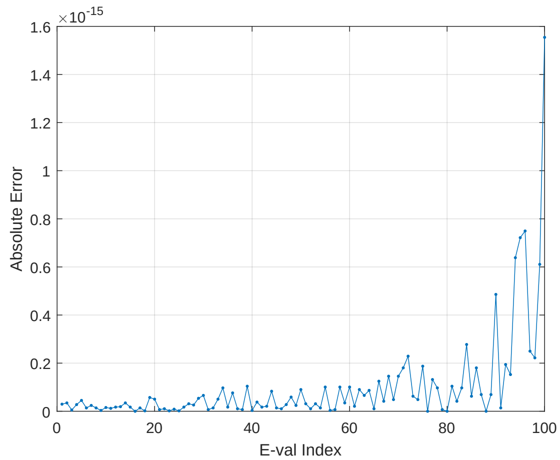

Figure 4: plot2b of error of e-val

3 3

3.1 a

$$
\begin{aligned}
u(x+h) &\approx u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u^{(3)}(x) + \frac{h^4}{24}u^{(4)}(x) \\
u(x-h) &\approx u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u^{(3)}(x) + \frac{h^4}{24}u^{(4)}(x) \\
\therefore |u''(x) - \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}| &\approx \frac{h^2}{12}u^{(4)}(x) \\
&= O(h^2)
\end{aligned}
$$

3.2 b

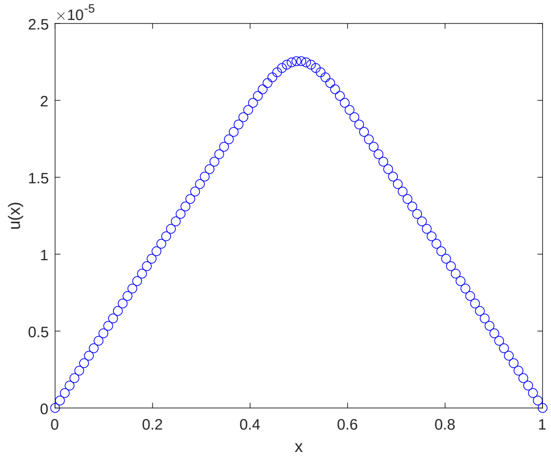

Figure 5: solution of u(x)

3.3 c

The first three eigenvalues are:
lambda1 = 9.86881
lambda2 = 39.46569
lambda3 = 88.76200
The eigenfunctions are:

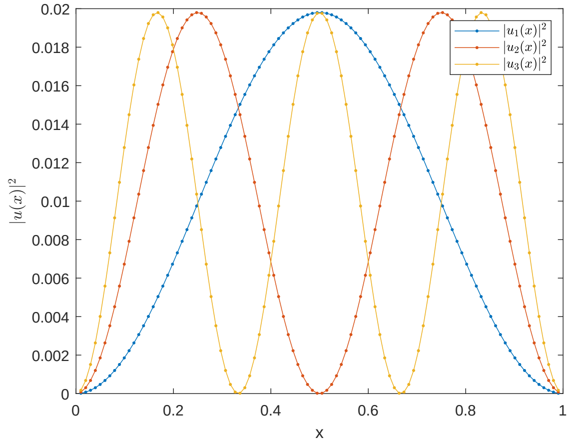

Figure 7: $u(x)^2$

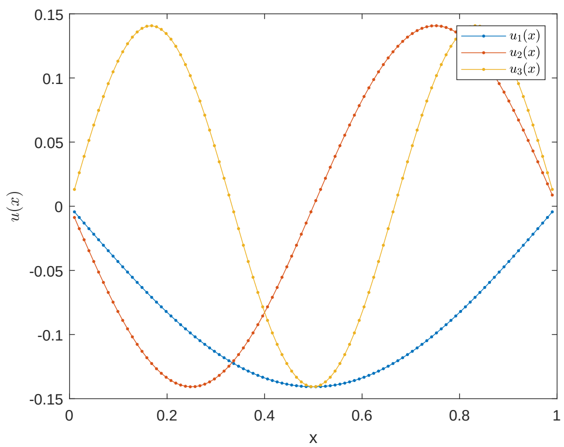

Figure 6: E-function of u(x)

4 4

4.1 a

See code

4.2 b

$$\frac{||e^{(k)}||_A}{||e^{(0)}||_A} \leq 2(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1})^k$$

where

$$\kappa = \frac{\lambda_{max}}{\lambda_{min}} < \frac{\text{upper bound of e-val}}{\text{lower bound of e-val}}$$

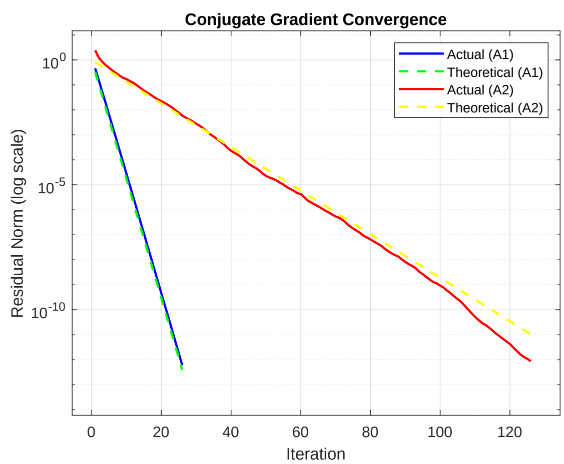

Figure 8: Error