Part MATH-UA 252 — Numerical Analysis

Based on lectures by Jason Kaye

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

1 Problem 1

1.1 a

$$
\begin{aligned}
2\log(x+3) &= x \\
\log(x+3) &= \frac{x}{2} \\
x+3 &= e^{\frac{x}{2}} \\
x^2+6x+9 &= e^x \\
\therefore e^x - x^2 - 6x - 9 &= 0
\end{aligned}
$$

As a result, $f(x) = 0 \Leftrightarrow g(x) = x$ and the assertion is true

1.2 b

As the plot shows, there is a fixed point.

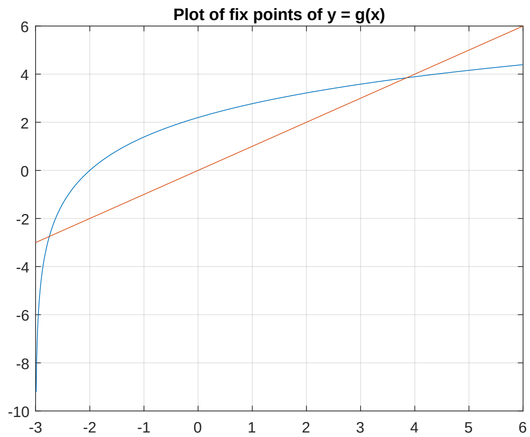

Figure 1: Plot of fixed points using y =g(x)

1.3 c

$$
\begin{aligned}
f(3) &= e^3 - 3^2 - 6\cdot 3 - 9 \\
&= e^3 - 36 < 0 \\
f(4) &= e^4 - 16 - 24 - 9 \\
&\approx 5.6 > 0
\end{aligned}
$$

As $f(x)$ is continuous and differentiable on $[3, 4]$, there must be a solution for $f(x)$ in that interval. That is to say, $g(x)$ has an interval in [3,4]
As

$$\forall x \in [3, 4], |g'(x)| = |\frac{2}{x+3}| < 1$$

Therefore, any initial guess in the interval will converge to a fixed point.

1.4 d

According to the program,
The solution is: 3.8478775007
The number of iteration used is: 21

2 Problem 2

2.1 a

For the normal Newton's method, what we do is use the tangent line to find the next iteration.
That is to say, we want to find the x-intercept of a line that crosses $(x_k, f(x_k)$ and the slope is $f'(x_k)$. As a result, we can find the new x as

$$h'(x_k) \cdot (x_{k+1} - x_k = h(x_{k+1}) - h(x_k)$$
$$x_{k+1} = x_k - \frac{h(x_k}{h'(x_k)}$$

where $h(x)$ is the tangent line of $f(x)$ at $x = x_k$
Similarly, for the multivariate case

$$f(x) \approx f(x_0) + J(x_0)(x - x_0)$$

As we want to find the x-intercept of the tangent phase, the iteration will lead to:

$$0 - f(x_k) = J(x_k)(x_{k+1} - x_k)$$
$$x_{k+1} = x_k - J^{-1}(x_k) f(x_k)$$

2.2 b

The solution is at the intercept point of the three phases as the graph shows, the point$(0.103448, 0.241379)$ is an approximation of the solution

2.3 c

According to the code, the solution is: (0.1025853042,0.2685767063) The number of iterations needed is: 6 The residual error is: 0.000000000007.

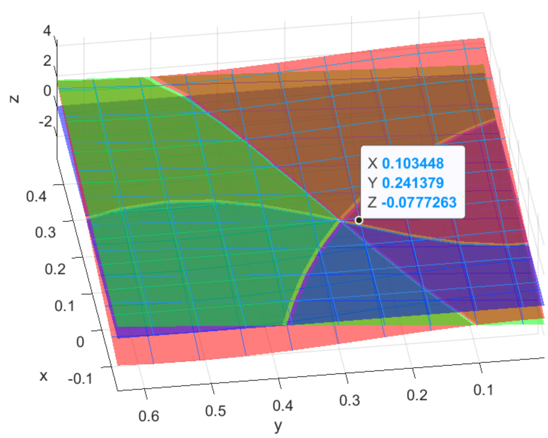

Figure 2: Figure of the three phases

3 Problem 3

3.1 a

$$
\begin{aligned}
f(x+h) &= f(x) + f'(x)h + \frac{f''(x)h^2}{2} + h.o.t \\
\therefore \frac{f(x+h) - f(x)}{h} &= \frac{f'(x)h + \frac{f''(x)h^2}{2} + h.o.t}{h} \\
\therefore e &= \frac{\frac{f''(x)h^2}{2} + h.o.t}{h} \\
&\approx \frac{f''(x)h}{2}
\end{aligned}
$$

As $f''(x)$ is bounded, $\exists C$ such that $|f''(h)| < M$ where M is a constant and all the higher order terms are small enough to be neglected when h is chosen sufficiently small, therefore the error is bounded by Ch

3.2 b

$$
\begin{aligned}
value &= \frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h} \\
&= \frac{f(x+h) - f(x)}{h} + \frac{\delta_1 f(x+h) - \delta_2 f(x)}{h} \\
\therefore error &= \frac{\delta_1 f(x+h) - \delta_2 f(x)}{h} \\
&= \delta_1 \frac{f(x+h) - f(x)}{h} + \frac{\delta_1 - \delta_2}{h} f(x)
\end{aligned}
$$

As $|\delta| < \varepsilon, |\delta_1 - \delta 2| < 2\varepsilon$

$$
\begin{aligned}
\therefore |error| &\leq 2\varepsilon \frac{f(x+h) - f(x)}{h} + 2\varepsilon \frac{f(x)}{h} \\
&= 2\varepsilon \frac{f(x+h)}{h} \\
&\leq 2\varepsilon \frac{M}{h}
\end{aligned}
$$

3.3 c

When $h = \sqrt{\varepsilon}$, we have the minimum value for $h + \frac{\varepsilon}{h}$, which is $2\sqrt{\varepsilon}$

3.4 d

The figure shows that the error is bounded by the two lines, which are derived from the previous questions by setting $M = |f(1)|$ and $C = |f''(1)|$ as those are the whole range is a small interval starting from $x = 1$

3.5 e

Assume $g(x) = a + bx + cx^2 + dx^3$
Then it can be represented as the vector $v = (a, b, c, d)^T$
Therefore,

$$
\begin{aligned}
Dv &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
&= \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}
\end{aligned}
$$

Meaning the result is $b + 2cx + 3dx^2$, which is exactly the derivative of $g(x)$

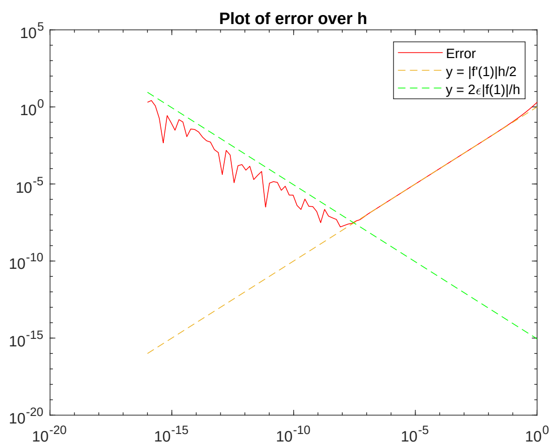

Figure 3: figure of error vs h

3.6 f

$$
\begin{aligned}
f(1) &= a_0 + a_1 + a_2 \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \\
&= e^T v
\end{aligned}
$$

3.7 g

$$
\begin{aligned}
e^T D &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \\
\therefore d &= \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}
$$

Assume $f(x) = a + bx + cx^2 + dx^3$

$$
\begin{aligned}
f'(x) &= b + 2cx + 3dx^2 \\
f'(1) &= b + 2c + 3d \\
&= \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} && = d^T v
\end{aligned}
$$

3.8 h

The result of f'(1) withou noise is : 2
The result of f'(1) with noise is: 2.000000000000001
There is no error for the answer with no noise, for the one with noise, there is little error so it is stable

4 Problem 4

4.1 a,b

See the program

4.2 c

The solution by LU factorization is $\begin{bmatrix} 1.000000000000000 \\ 1.000000000000000 \\ 1.000000000000000 \end{bmatrix}$

The error in the Euclidean norm is 1.110223024625157e-16

4.3 d

The solution by LU factorization is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The actual solution is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The error in the Euclidean norm is 1

4.4 e

The error in the Euclidean norm is 4.427186048556673e-10, which is still larger than ε but way closer to it compared to question d

5 5

For forward substitution, we need to solve $Ly = b$ where L is a lower triangular matrix.
As a result, we start with the first row, get the first result, and proceed to the next
The equation is

$$y_i = \frac{b_i - \sum_{j=1}^{i-1} L_{i,j} yj}{L_[I, i]}$$

As a result, the complexity for each y is $i - 1 + i - 2 + 1 + 1 = 2i - 1$
The total complexity is

$$\sum_{i=1}^{n} 2i - 1 = O(n^2)$$