# Part MATH-UA 252 — Numerical Analysis

Based on lectures by Jason Kaye
Homework done by Aaron Ma

Fall 2024

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

# 1 Problem 1

## 1.1 a

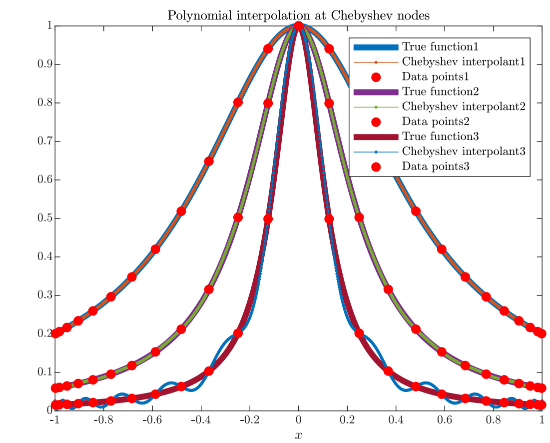


Figure 1: prob1a Plot of Polynomials with Interpolation

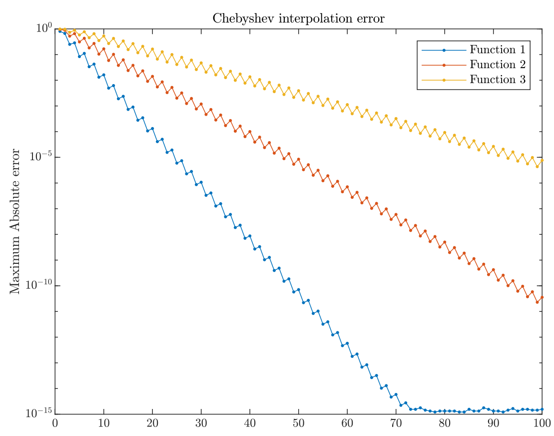


Figure 2: Plot of Maximum absolute error

As the graph suggests, the error converges exponentially. As $a$ increases, the convergence rate decreases. This is because the function increases when $a$ is larger, making the function peak around $x = 0$

## 1.2 b

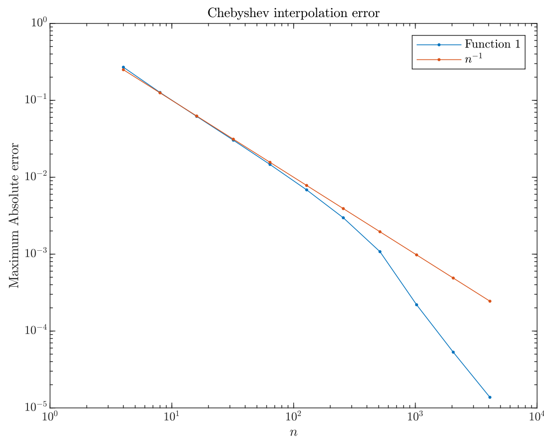


Figure 3: prob1b Plot of Maximum absolute error

The function is not so smooth: $|x|$ is not differentiable at $x = 0$, the middle of the function, in which Chebyshev nodes are sparse, and not 3 times differentiable. As a result, it converges slowly

## 1.3 c

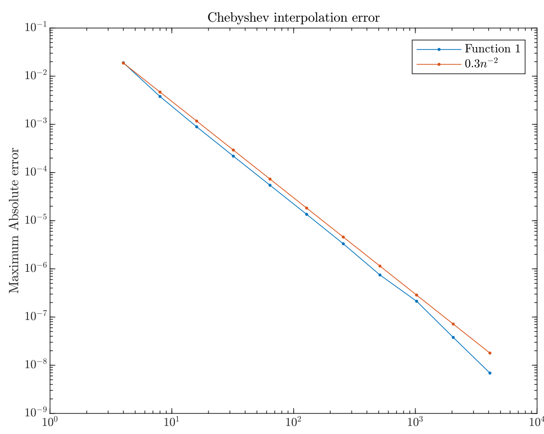


Figure 4: Maximum absolute error

## 1.4 d

[H]

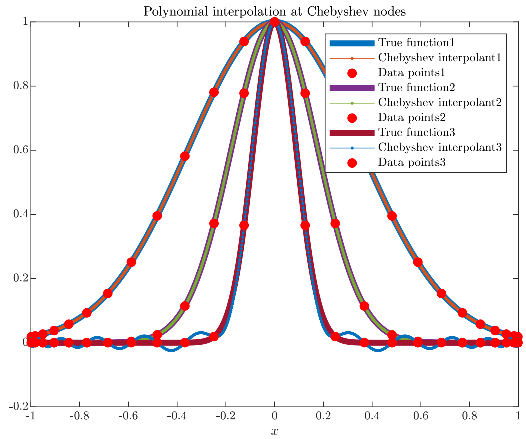


Figure 5: Plot of Polynomials with Interpolation

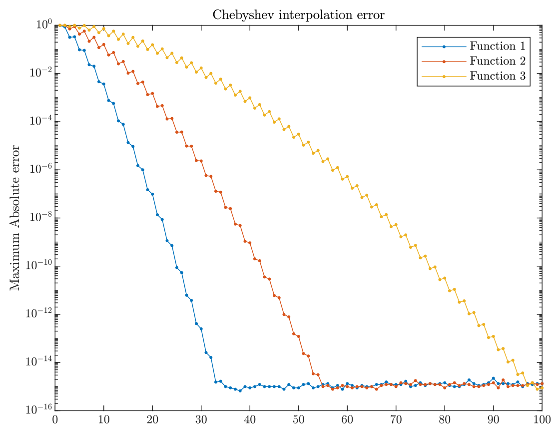


Figure 6: prob1d Plot of Maximum absolute error

From the graph, we can The convergence rate decreases as a increases. It is super-exponential convergence. This is because exponential function is infinitely-differentiable

## 1.5 e

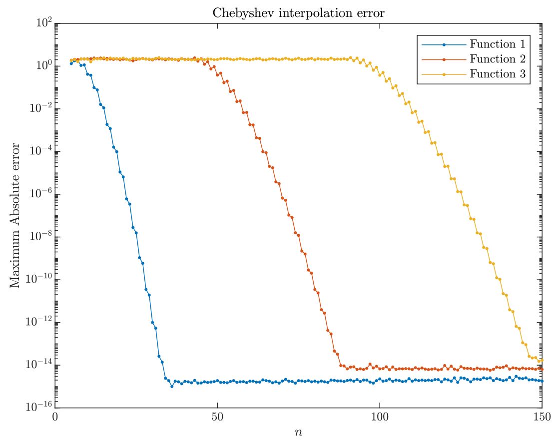


Figure 7: Plot of Maximum absolute error

The error first change algebraically and changes exponentially near $n \approx a$ until it is close to $\epsilon$
If $n < a$, the interpolation fails to resolve the oscillations and has a higher error, if$n > a$, the interpolation can capture the function and converges quickly.

# 2 Problem 2

## 2.1 a

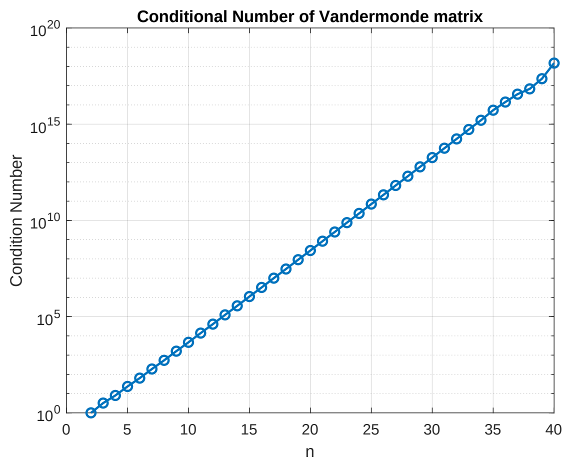


Figure 8: Number of Vandermonde matrix

This indicates an exponential growth of conditional number

## 2.2 b

The condition numbers for 10, 500, and 1000 nodes are: 1.41421356, 1.41421356, 1.41421356
They are the same

# 3 Problem 3

## 3.1 a

See the program

## 3.2 b

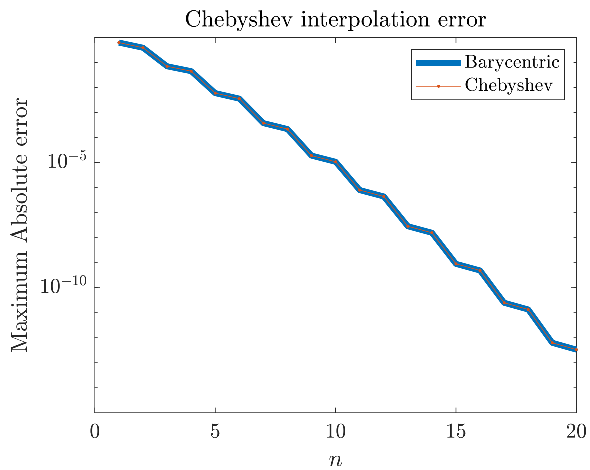


Figure 9: Plot of Maximum absolute error

They are almost the same

# 4 4

## 4.1 a

See the code

## 4.2 b

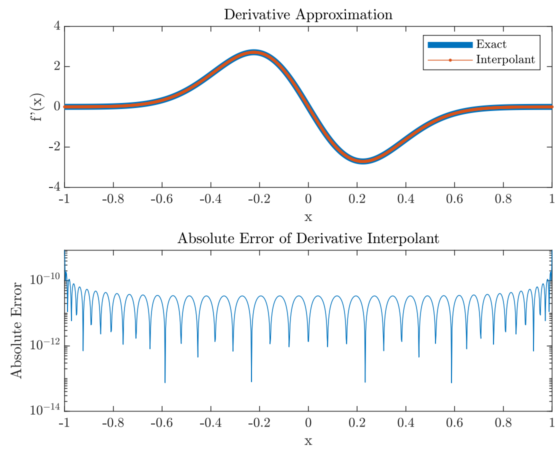


Figure 10: prob4b

## 4.3 c

$$
\begin{aligned}
f_k(\theta) &= T_k(cos(\theta)) \\
&= cos(k_\theta) \\
\therefore f'_{k+1}(\theta) &= -(k+1)sin((k+1))\theta) \\
f'_{k-1}(\theta) &= -(k-1)sin((k-1))\theta) \\
\because f'_k(\theta) &= -sin(\theta)T'_k(cos\theta) \\
\therefore T'_k(cos\theta) &= -\frac{1}{sin\theta}f'_k(\theta) \\
\therefore 2T_k(cos\theta) &= 2f_k(\theta) \\
&= -\frac{1}{sin\theta}(\frac{1}{k+1}f'_{k+1}(\theta) - \frac{1}{k-1}f'_{k-1}(\theta)) \\
\therefore 2T_k(x) &= \frac{1}{k+1}T'_{k+1}(x) - \frac{1}{k-1}T'_{k-1}(x)
\end{aligned}
$$

## 4.4 d

$$
\begin{aligned}
\int_{-1}^{x} \frac{T'_{k+1}(t)}{k+1} dt &= \frac{T_{k+1}(x)}{k+1} - \frac{T_{k+1}(-1)}{k+1} \\
\int_{-1}^{x} \frac{T'_{k-1}(t)}{k-1} dt &= \frac{T_{k-1}(x)}{k-1} - \frac{T_{k-1}(-1)}{k-1} \\
\therefore \int_{-1}^{x} 2T_k(t) dt &= \int_{-1}^{x} \frac{T'_{k+1}(t)}{k+1} dt + \int_{-1}^{x} \frac{T'_{k-1}(t)}{k-1} dt \\
&= \frac{T_{k+1}(x)}{k+1} - \frac{T_{k+1}(-1)}{k+1} - \frac{T_{k-1}(x)}{k-1} + \frac{T_{k-1}(-1)}{k-1} \\
\int_{-1}^{x} T_k(t) dt &= \frac{1}{2}(\frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1}) + \frac{(-1)^k}{1-k^2} \\
\therefore \int_{-1}^{x} T_0(t) dt &= \int_{-1}^{x} 1 dt \\
&= x + 1 \\
&= T_0(x) + T_1(x) \\
\int_{-1}^{x} T_1(t) dt &= \int_{-1}^{x} t dt \\
&= \frac{x^2+1}{2} \\
&= -\frac{1}{4} T_0(x) + \frac{1}{4} T_2(x) \\
\text{For } k > 1 & \\
S &= \\
\int_{-1}^{x} T_k(t) dt &= \frac{1}{2}(\frac{T_{k+1}(x_j}{k+1} - \frac{T_{k-1}(x_j}{k-1}) + \frac{(-1)^k}{1-k^2}
\end{aligned}
$$

## 4.5 e

By calculating $S^{cv}_{j+1,k+1}$ using what we have in (d), we can find $\int_{-1}^{x} T_k(t)dt$, then we subtract the integral from -1 to 0 and can get the result of the indefinite integration

## 4.6 f

See code

## 4.7 g

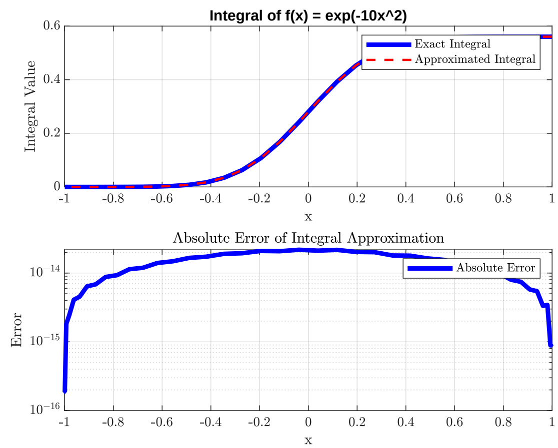


Figure 11: Enter Caption