Algorithms and Software for Computation in n-dimensional Thompson groups

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Introduction

Thompson groups F, T, and V are interesting objects first studied by Richard Thompson. They are finitely represented infinite groups with some interesting properties. The definition of V naturally generalizes to n dimensions.

Carrying out even basic computations in these groups (finding compositions of maps) by hand is a very tedious process, which necessitated writing a program to do them.

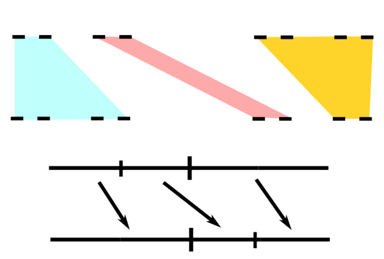
The result is nvTrees - a calculator which can be a very useful tool for anyone who wants to experiment with Thompson groups.

In particular, the software solves the **word problem**.

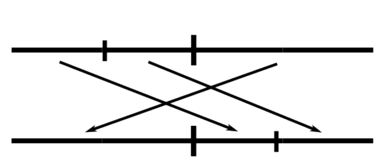
Thompson Groups

Let $\mathbf{C}$ denote the middle-thirds cantor set.

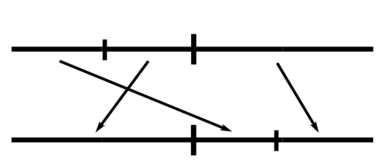
Definition 1 *F is the group of piecewise-linear, order-preserving homeomorphisms of* $\mathbf{C}$.



T is the group of piecewise-linear cyclic order-preserving homeomorphisms of $\mathbf{C}$.



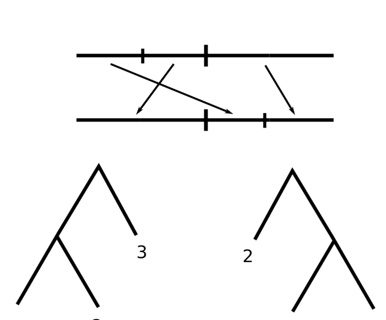
V is the group of piecewise-linear homeomorphisms of $\mathbf{C}$



We extend the definition of V to several dimensions:

Definition 2 *nV is the group of piecewise-linear homeomorphisms of $\mathbf{C^n}$ to itself.*

A useful way to represent maps in V is writing them as labeled tree pairs and a permutation on the leaves:



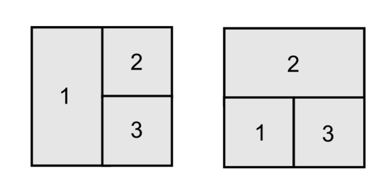
(we don't write the permutation when it is trivial).

Writing elements of nV

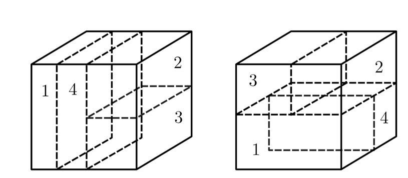
A pattern is a dyadic subdivision of an n-cube.

Maps in nV can be described as labeled pattern pairs.

Example in $2V$:

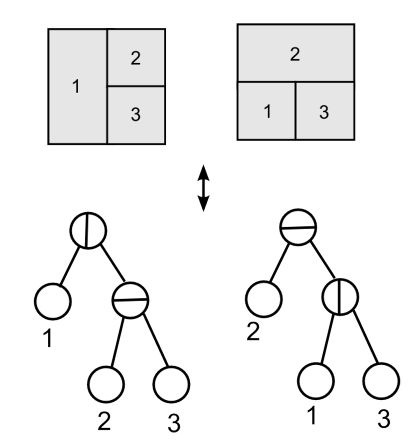


Example in $3V$:



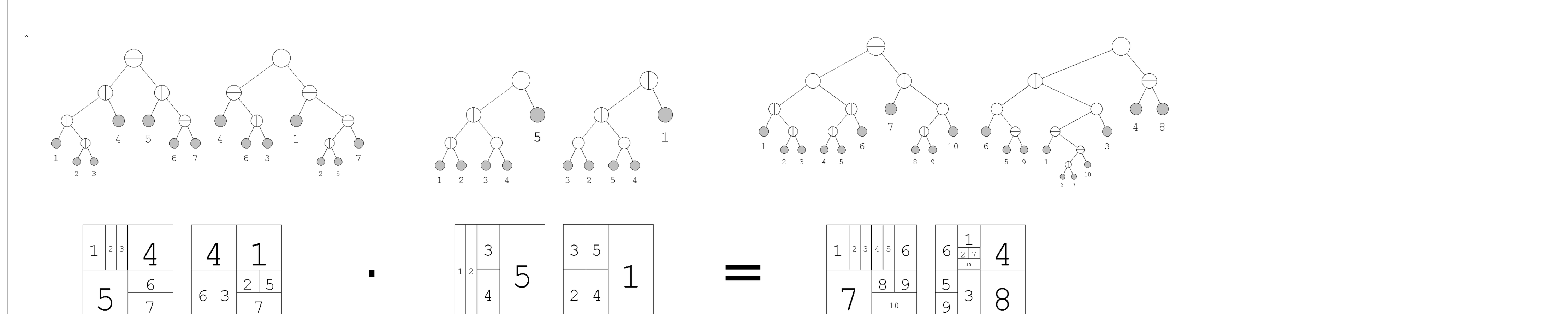
From pattern pairs to labeled trees

Maps in nV can be also written as pairs of labeled trees with permutations:



Color (or label) on the nodes identify the cut axis. To get from trees to patterns, think of the tree as guide on how to cut the square.

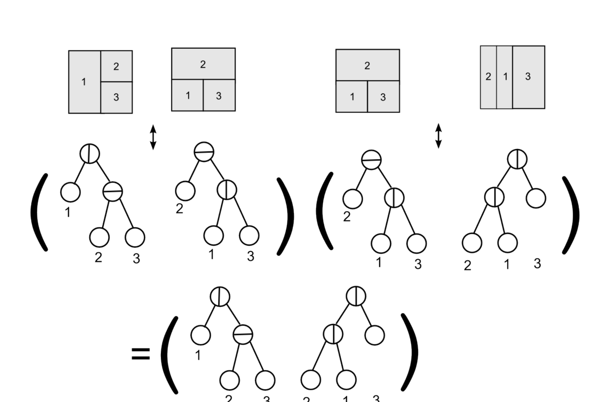
An example of computation with nvTrees



{2110100010200,1201002021000,4 6 3 1 2 5 7 } · {111002000,112002000,3 2 5 4 1 } = {2110100110001021000,1120200220210000200,6 5 9 1 2 7 10 3 4 8 }

Composition of maps

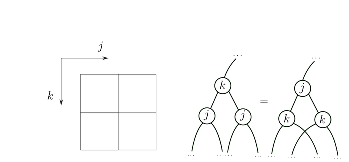
..is easy to compute if the trees in the middle are the same:



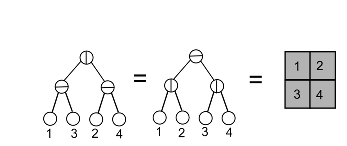
If the trees are different, we need to expand the trees until they match. In terms of patterns, that means finding a common refinement.

Implementation

The middle trees are expanded to full binary trees so that the number of nodes of each color on each root-leaf path match. The middle trees now represent the same pattern, as expressed by the relation below:



E.g.:



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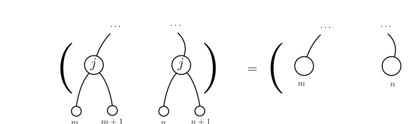
The software was written during 2008 REU at Cornell University. The poster was made using the template by Michael Gastpar and Ron Kumon.

Implementation Details

A map is stored as a pair of colored trees (or a collection of pattern blocks) and a permutation. The tree is entered by the user as a string which is a depth-first search traversal of the tree, where the symbol `0` indicates a leaf node, and symbols `1..9` indicate a non-leaf node having a corresponding label. An inverse of a map is obtained by swapping the trees in the pair.

To multiply two tree pairs, one counts the number of times each color occurs on each root-leaf path in the middle trees. The trees are then expanded to full binary trees. The outer trees need to be correspondingly modified by attaching subtrees to the leaves in order for the pairs to represent the same element.

A **standard reduced form** of a tree pair is obtained by expanding the right tree in the pair to full binary tree and then greedily reducing the pair using a fixed ordering of labels:



Theorem 1 *Each element of nV has a unique standard reduced form.*

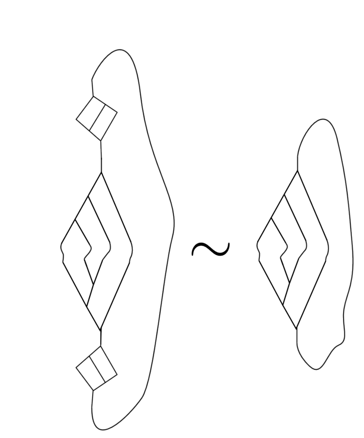
Word and conjugacy problems

The world problem is solved algorithmically by reducing to standard form.

A **further research goal** is finding an algorithm to determine if two elements in the group are conjugate and implementing it.

Such an algorithm exists for Thompson groups ($n = 1$), with an elegant version by Matucci and Belk[1], but does not immediately generalize to higher dimensions.

The difficulty arises in finding or proving the existence of a reduced form of a generalized strand diagram that is used in their proof:



The same algorithm cannot be applied to colored tree diagrams, since the size of the diagram might increase before a reduction can be applied.

References

James Belk, Francesco Matucci. *Conjugacy in Thompson's Groups.* Preprint at http://arxiv.org/abs/0708.4250