

Bihar Board Class 12 Mathematics Question Paper 2018

INTERMEDIATE EXAMINATION - 2018

(ANNUAL)

MATHEMATICS

Time- $3\frac{1}{4}$ Hours

Full Marks: 100

Instruction for the candidates:

- 1) Candidates are required to give their answers in their own words as far as practicable.
- 2) Figures in the right hand margin indicate full marks.
- 3) 15 Minutes of extra time has been allotted for the candidates to read the questions carefully.
- 4) This question paper is divided into two **section A** and **section B**.
- 5) In **section-A**, there are **50 objective type questions** which are compulsory, each carry **1 mark**. Darken the circle with blue/black ball pen against the correct option on **OMR answer Sheet** provided to you. **Do not use whitener/Liquid/Blade/Nail etc. on OMR Sheet; otherwise the result will be invalid.**
- 6) In **Section-B**, there are **25 short answer** type question (each carrying **2 marks**), out of which **any 15** questions are to be answered. Apart this, there are **8 Long Answer Type** question (Each Carrying **5 Marks**), Out of which **any 4** questions to be answered.
- 7) Use of any electronic appliances is strictly prohibited.

Section-I : (Objective Type)

For the following Question Nos. 1 to 50 there is only one correct answer against each question. Mark the correct option on the answer sheet. [50 X 1=50]

1.let $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$ How many binary operation can be defined on this set ?

- | | |
|--------|-------|
| (a)8 | (b)10 |
| (c)-16 | (d)20 |

Sol:

Correct option is C

2.let $\{1, 2, 3\}$ which of the following function $f : A \rightarrow A$ does not an inverse function ?

(A) $\{(1,1), (2,2), (3,3)\}$

(b) $\{(1,2), (2,1), (3,1)\}$

(c) $\{(1,3), (3,2), (2,1)\}$

(d) $\{(1,2), (2,3), (3,1)\}$

Sol:

Correct option is B

3. If $A = \{1, 2, 3\}$, $B = \{6, 7, 8\}$ and $f : A \rightarrow B$ is a function such that $f(x) = x + 5$ then what type of a function is f ?

(a) into (b) one-one onto

(c) many-one onto (d) Constant function

SOL:

Correct option is B

4. What type of a relation is “Less than” in the set of real numbers?

(a) Only symmetric (b) Only transitive

(c) Only reflexive (d) equivalence relation

SOL:

Correct option is B

5. $\cos^{-1} \left(\cos \frac{8\pi}{5} \right) =$

(a) $\frac{8\pi}{5}$ (b) $\frac{12\pi}{5}$

(c) $\frac{2\pi}{5}$ (d) $\frac{4\pi}{5}$

SOL:

Correct option is D

6. $\cos^{-1}(2x - 1) =$

(a) $2\cos^{-1} x$ (b) $\cos^{-1} \sqrt{x}$

- (c) $2\cos^{-1}\sqrt{x}$ (d) None of these

SOL:

Correct option is D

7. $2\cot^{-1}3 + \cos^{-1}7 =$

- (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) π (d) $\frac{\pi}{6}$

SOL:

Correct option is B

8. $\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) =$

- (a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$
(c) $-\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

SOL:

Correct option is B

9. If $\lambda \in R$ and $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then $\lambda\Delta =$

- (a) $\begin{vmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{vmatrix}$ (b) $\begin{vmatrix} \lambda a & b \\ c & d \end{vmatrix}$
(c) $\begin{vmatrix} \lambda a & b \\ \lambda c & d \end{vmatrix}$ (d) None of these

SOL:

Correct option is A

10. If a,b and c are in A.P then $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} =$

SOL:

Correct option is C

11. If 7 and 2 are two roots of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ then the third root is:

- (a) -9
 - (b) 14
 - (c) $\frac{1}{2}$
 - (d) None of these

SOL:

Correct option is A

12. If $\omega \neq 1, \omega^3 = 1$ and $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$ then $x =$

SOL:

Correct option is D

13. If $\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ and $A + A' = I_2$ then $\alpha =$

- (a) π (b) $\frac{\pi}{3}$

- (c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{6}$

SOL:

Correct option is B

14. If A be a square matrix. Then $A + A'$ will be.....

- (a) Symmetric matrix
 - (b) Skew symmetric matrix
 - (c) Null matrix
 - (d) Unit matrix

SOL:

Correct option is A

15. If A is a matrix of order $A + A'$ such that $A + A'$ then $A + A'$ is equal to?

- (a) I_3 (b)A
 (c)3A (d) $I_3 - A$

SOL:

Correct option is A

16. Let A be a non-singular matrix of the order 2×2 then $|adj A| = \dots \dots \dots$

- (a) $2|A|$
 - (b) $|A|$
 - (c) $|A|^2$
 - (d) $|A|^3$

SOL:

Correct option is C

$$17. \frac{d}{dx} [\log(\sec x + \tan x)] =$$

- (a) $\frac{1}{\sec x + \tan x}$ (b) $\sec x$
 (c) $\tan x$ (d) $\sec x + \tan x$

SOL:

Correct option is A

18. If $x^2y^3 = (x+y)^5$ then, $\frac{dy}{dx}$

$$(a) \frac{x}{y} \quad (b) \frac{y}{x}$$

(c) $-\frac{y}{x}$ (d) $-\frac{x}{y}$

SOL:

Correct option is B

$$19. \frac{d}{dx} \left[\tan^{-1} \sqrt{1+x^2} - \cot^{-1} \left(-\sqrt{1+x^2} \right) \right] =$$

(a) π (b) 1

SOL:

Correct option is C

$$20. \frac{d(2^x)}{d(3^x)} =$$

$$\begin{array}{ll} \text{(a)} \left(\frac{2}{3} \right)^x & \text{(b)} \frac{2^{x-1}}{3^{x-1}} \end{array}$$

(c) $\left(\frac{2}{3}\right)^x \log_3 2$ (d) $\left(\frac{2}{3}\right)^x \log_2 3$

SOL:

Correct option is C

21. $f(x) = \sqrt{3} \sin x + \cos x$ is maximum then value of $x = \dots$

$$(a) \frac{\pi}{6} \qquad (b) \frac{\pi}{2}$$

- (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

SOL:

Correct option is C

22. If $y = \log \cos x^2$ then $\frac{dy}{dx}$ at $x = \sqrt{x}$ has the value

- (a) 1
 - (b) $\frac{\pi}{4}$
 - (c) 0
 - (d) $\sqrt{\pi}$

SOL:

Correct option is C

23. Equation of the tangent to the curve $x^2 + y^2 = a^2$ at (x_1, y_1) is

- (a) $xx_1 - yy_1 = 0$ (b) $xx_1 + yy_1 = 0$
 (c) $xx_1 - yy_1 = a^2$ (d) $xx_1 + yy_1 = a^2$

SOL:

Correct option is D

$$24. \frac{d}{dx} \left[\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} \right] =$$

- (a) $5a^4$
 - (b) $5x^4$
 - (c) 1
 - (d) 0

SOL:

Correct option is D

$$25. \int \sqrt{1+\cos 2x} dx =$$

- (a) $\sqrt{2} \cos x + c$ (b) $\sqrt{2} \sin x + c$

$$(c) -\cos x - \sin x + c$$

$$(d) \sqrt{2} \sin \frac{x}{2} + c$$

SOL:

Correct option is B

$$26. \int x^2 \cdot e^{x^3} dx =$$

$$(a) e^{x^3} + c$$

$$(b) \frac{1}{3} e^{x^3} + c$$

$$(c) e^{x^2} + c$$

$$(d) \frac{1}{3} e^{x^2} + c$$

SOL:

Correct option is B

$$27. \int \frac{xe^x}{(x+1)^2} dx =$$

$$(a) \frac{e^x}{(x+1)^2} + c$$

$$(b) \frac{-e^x}{x+1} + c$$

$$(c) \frac{e^x}{x+1} + c$$

$$(d) \frac{-e^x}{(x+1)^2} + c$$

SOL:

Correct option is C

$$28. \int \frac{dx}{a^2 + x^2} =$$

$$(a) \frac{1}{a} \tan^{-1} \frac{a}{x} + c$$

$$(b) \tan^{-1} \frac{x}{a} + c$$

$$(c) \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$(d) \frac{1}{a} \tan^{-1} x + c$$

SOL:

Correct option is C

29. If $f(-x) = -f(x)$ then $\int_{-a}^a f(x)dx =$

- (a) $2 \int_0^a f(x)dx$ (b) 0
(c) 1 (d) -1

SOL:

Correct option is B

$$30. \int_{\alpha}^{\beta} \varphi(x) dx + \int_{\beta}^{\alpha} \varphi(x) dx =$$

SOL:

Correct option is D

31. Area between the x-axis and the curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$:

- (a) 2
 - (b) -1
 - (c) 1
 - (d) None of these

SOL:

Correct option is C

$$32. \int_0^1 x dx = \dots$$

- (a) 0
 - (b) 1
 - (c) 2
 - (d) $\frac{1}{2}$

SOL:

Correct option is D

33. The differential equation $1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^3$ is of order = and degree =

- (a) Order = 2, degree = 3
- (b) Order = , degree = 2
- (c) Order = 2, degree = 2
- (d) None of these

SOL:

Correct option is A

34. solution of the differential equation $ydx - xdy = xydx$ is

- (a) $\frac{y^2}{2} - \frac{x^2}{2} = xy + c$
- (b) $x = kye^x$
- (c) $x = kye^y$
- (d) None of these

SOL:

Correct option is B

35 Integrating factor (I.F) of differential equation $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ is

- (a) $\log x$
- (b) x
- (c) $\frac{1}{x}$
- (d) None of these

SOL:

Correct option is B

36. solution of $\frac{xdy - ydx}{x^2 + y^2} = 0$ is:

- (a) $\frac{x^2}{2} + \tan^{-1} \frac{x}{y} = k$
- (b) $\frac{x^2}{2} + \tan^{-1} \frac{y}{x} = k$
- (c) $\frac{x^2}{2} - \tan^{-1} \frac{x}{y} = k$
- (d) $\frac{x^2}{2} - \tan^{-1} \frac{y}{x} = k$

SOL:

Correct option is B

37. If $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$, then the corresponding unit vector \hat{a} in the direction of \vec{a} =

- (a) $\frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{6}}$ (b) $\frac{\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{6}}$
(c) $\frac{\vec{i} + \vec{j} + 2\vec{k}}{6}$ (d) None of these

SOL:

Correct option is B

38. The direction cosines of the vector $3\vec{i} - 4\vec{j} + 12\vec{k}$ is

- (a) $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$ (b) $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$
(c) $\frac{3}{\sqrt{13}}, \frac{4}{\sqrt{13}}, \frac{12}{\sqrt{13}}$ (d) $\frac{3}{\sqrt{13}}, \frac{-4}{\sqrt{13}}, \frac{12}{\sqrt{13}}$

SOL:

Correct option is B

39. $x\vec{i} - 3\vec{j} + 5\vec{k}, -x\vec{i} + x\vec{j} + 2\vec{k}$ are perpendicular to each other then value of x =

- (a) -2, 5 (b) 2, 5
(c) -2, -5 (d) 2, -5

SOL:

Correct option is D

$$40 \vec{i} \times (\vec{i} \times \vec{j}) + \vec{j} \times (\vec{j} \times \vec{k}) + \vec{k} (\vec{k} \times \vec{i})$$

- (a) $\vec{i} + \vec{j} + \vec{k}$ (b) 0
(c) 1 (d) $-(\vec{i} + \vec{j} + \vec{k})$

SOL:

Correct option is C

41. The direction Cosines of y axis are:

- (a) $(1, 0, 1)$

(b) $(0, 1, 0)$

(c) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(d) None of these

SQL ·

Correct option is B

42. The equation of the xy-plane is:

SOL_i

Correct option is B

43. If two planes $-2x - 4y + 3z = 5$ and $x + 2y + \lambda z = 12$ are perpendicular to each other, then $\lambda =$

- (a)-2
 - (b) 2
 - (c)3
 - (d) None of these

SOL:

Correct option is B

44. The distance between (4,3,7) and (1,-1,-5) is:

- (a) 13
 - (b) 15
 - (c) 12
 - (d) 5

SOL:

Correct option is A

45. If A' and B' are independent events then:

- $$(a) P(A' B') = P(A).P(B) \quad (b) P(A' B') = P(A') + P(B')$$

(c) $P(A' \cap B') = P(A') \cdot P(B')$ (d) $P(A' \cap B') = P(A') - P(B')$

SOL:

Correct option is C

46. If events A and B are mutually exclusive then:

(a) $P(A \cap B) = P(A) \cdot P(B)$ (b) $P(A \cap B) = 0$

(c) $P(A \cap B) = 1$ (d) $P(A \cup B) = 0$

SOL:

Correct option is B

47. If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$ then $P\left(\frac{A}{B}\right) =$

(a) 2 (b) $\frac{1}{2}$

(c) $\frac{2}{3}$ (d) $\frac{3}{2}$

SOL:

Correct option is C

48. If A and B are two events such that $P(A) \neq 0$ and $P\left(\frac{B}{A}\right) = 1$

(a) $B \subset A$ (b) $A \subset B$

(c) $B = \emptyset$ (d) $A \cap B = \emptyset$

SOL:

Correct option is B

49. $\int \frac{dx}{x + \sqrt{x}}$

(a) $\log x + \log(1 + \sqrt{x}) + C$ (b) $2 \log(1 + \sqrt{x}) + C$

- (c) $\log(1 + \sqrt{x}) + C$ (d) $\log(\sqrt{x}) + C$

SOL:

Correct option is B

50. solution of $\lim_{n \rightarrow \infty} \left[\frac{e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \dots + e^{\frac{n}{n}}}{n} \right]$ is

- | | |
|-------------|-------------|
| (a) $1 - e$ | (b) $e - 1$ |
| (c) e | (d) 1 |

SOL:

Correct option is C

SECTION-B (Non-Objective Type Question)

Question NO.1 to 22 are short answer type. Answer any 15 question. Each question carries 2 marks.

1. Examine whether the function $f : R \rightarrow R$ is one-one(injective) if $f(x) = x^3, x \in R$

Sol:

Given that function $f : R \rightarrow R$ is denote as $f(x) = x^3, x \in R$

Let $x_1, x_2 \in R$

$$\begin{aligned}\Rightarrow f(x_1) &= f(x_2) \\ \Rightarrow x_1^3 &= x_2^3 \\ \Rightarrow x_1 &= x_2\end{aligned}$$

Thus the given function is one-one or injective.

$$2.\text{Prove that } \tan\left[\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}\right] = \frac{2x}{1-x^2}$$

Sol:

Given that,

$$\begin{aligned} 2\tan^{-1}x &= \sin^{-1}\frac{2x}{1+x^2} \\ &= \cos^{-1}\frac{1-x^2}{1+x^2} \\ &= \tan^{-1}\frac{2x}{1+x^2} \end{aligned}$$

Then,

$$\begin{aligned} L.H.S &= \tan\left[\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}\right] \\ &= \tan\left[\frac{1}{2}.2\tan^{-1}x + \frac{1}{2}.2\tan^{-1}x\right] \\ &= \tan\left[\tan^{-1}x + \tan^{-1}x\right] \\ &= \tan\left[2\tan^{-1}x\right] = \tan\left[\tan^{-1}\frac{2x}{1-x^2}\right] \\ &= \frac{2x}{1-x^2}\left[\tan\left[\tan^{-1}x\right]\right] = x \end{aligned}$$

Hence, LHS = RHS proved.

$$3.\text{Prove that } \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$$

Sol:

Let,

$$\alpha = \sin^{-1} \frac{3}{5}$$

$$\Rightarrow \sin \alpha = \frac{3}{5}$$

$$\therefore \cos \alpha = \sqrt{1 - \sin^2 \alpha}$$

$$\Rightarrow \cos \alpha = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad \text{---(1)}$$

and,

$$\beta = \cos^{-1} \frac{12}{13} \Rightarrow \cos \beta = \frac{12}{13}$$

$$\beta = \sqrt{1 - \cos^2 \beta} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13} \quad \text{---(2)}$$

Now, we know that: $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

Putting the value of (1) and (2), we get

$$\sin(\alpha - \beta) = \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13} = \frac{36}{65} - \frac{20}{65} = \frac{16}{65}$$

$$\Rightarrow \sin(\alpha - \beta) = \frac{16}{65}$$

$$\therefore \alpha + \beta = \sin^{-1} \frac{16}{65}$$

$$\Rightarrow \sin^{-1} \frac{3}{5} - \cos^{-1} \frac{12}{13} = \sin^{-1} \frac{16}{65}$$

4. Find the values of x and y when

$$\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} \text{ and } \begin{vmatrix} x & y \\ 2 & 1 \end{vmatrix} = \frac{7}{2}$$

Sol:

Given that,

$$\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} \text{ and } \begin{vmatrix} x & y \\ 2 & 1 \end{vmatrix} = \frac{7}{2}$$

Now,

$$\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} = 4 \Rightarrow 2x - 3y = 4$$

And,

$$\begin{vmatrix} x & y \\ 2 & 1 \end{vmatrix} = \frac{7}{2}$$
$$\Rightarrow x - 2y = \frac{7}{2}$$

Multiplying equation (2) by 2 and subtract from equation (1), we get

$$2x - 3y - 2x + 4y = 4 - \frac{14}{2} = \frac{8 - 14}{2}$$
$$\Rightarrow y = \frac{-6}{2} = -3$$

Putting the value of y in (i), we get

$$2x - 3y = 4 \Rightarrow 2x = 4 + 3y = 4 - 9 = -5$$
$$\Rightarrow 2x = -5$$
$$\Rightarrow x = \frac{-5}{2}$$

$$x = \frac{-5}{2} \quad y = -3$$

Thus, and

5. Prove that: $\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix} = x^2(x+a+b+c)$

Sol:

Let,

$$LHS = \begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$

$$\begin{aligned}
 & \begin{bmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{bmatrix} \\
 & = (x+a+b+c) \begin{bmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{bmatrix}
 \end{aligned}$$

Applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$ we get

$$\begin{aligned}
 & = (x+a+b+c) \begin{bmatrix} 1-1 & b-x-b & c-c \\ 1-1 & x+b & c-x-c \\ 1 & b & x+c \end{bmatrix} \\
 & = (x+a+b+c) \begin{bmatrix} 0 & -x & 0 \\ 0 & x+b & -x \\ 1 & b & x+c \end{bmatrix} \\
 & = (x+a+b+c) \begin{bmatrix} -x & 0 \\ x & -x \end{bmatrix} = (x+a+b+c)(x^2 - 0) \\
 & = x^2(x+a+b+c)
 \end{aligned}$$

6. If $2 \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then prove that:

$$f(x+y) = f(x), f(y)$$

Sol:

Given that,

$$\begin{aligned}
f(x) &= \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\Rightarrow f(y) &= \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos x \cos y - \sin x \sin y & -\cos x \sin y & 0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= f(x+y) = RHS
\end{aligned}$$

7. Find the value of x , such that:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}_{3 \times 3} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}_{1 \times 3} (x-5-1)=0$$

Sol:

Given that,

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}_{3 \times 3} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}_{1 \times 3} (x-5-1)=0$$

Arranged the given matrices as law of multiplication of matrix. We get

$$\begin{aligned}
 & [x - 5 - 1]_{1 \times 3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}_{3 \times 3} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}_{3 \times 1} = 0 \\
 \Rightarrow & [x \times 1 - 5 \times 0 - 1 \times 2 \quad x \times 0 - 5 \times 2 - 1 \times 0 \quad x \times 2 - 5 \times 1 - 1 \times 3] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} \\
 \Rightarrow & [x - 2 - 10 \quad 2x - 8] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0 \\
 \Rightarrow & [x(x - 2) - 10(4) + (2x - 8) \times 1]_{1 \times 1} = [0]_{1 \times 1} \\
 \Rightarrow & [x^2 - 48]_{[x]} = [0]_{[x]} \\
 \Rightarrow & x^2 - 48 = 0 \\
 \Rightarrow & x^2 = 48 \\
 \Rightarrow & x^2 = 16 \times 3 \\
 x = & 4 \pm \sqrt{3}
 \end{aligned}$$

8. Find $\frac{dy}{dx}$ if $x = y \log(xy)$

Sol:

Given that,

$$y = y \log(xy)$$

Differentiate w.r.to x in both side,

$$\begin{aligned}
\frac{dx}{dx} &= y \frac{d}{dy} [\log(xy)] + \log(xy) \frac{dy}{dx} \\
\Rightarrow 1 &= y \frac{d[\log(xy)]}{d(xy)} \times \frac{d(xy)}{dx} + \log(xy) \frac{dy}{dx} \\
\Rightarrow 1 &= y \times \frac{1}{xy} \left(x \frac{dy}{dx} + y \times 1 \right) + \log(xy) \frac{dy}{dx} \\
\Rightarrow 1 &= \frac{dy}{dx} + \frac{y}{x} + \frac{x}{y} \frac{dy}{dx} \quad [from(1)] \\
\Rightarrow 1 &= \frac{y}{x} \frac{dy}{dx} \left(1 + \frac{x}{y} \right) \\
\Rightarrow \frac{x-y}{x} &= \frac{dy}{dx} \left(\frac{x+y}{y} \right) \\
\Rightarrow \frac{dy}{dx} &= \frac{y(x-y)}{x(x+y)}
\end{aligned}$$

9. if $y = \tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right)$ then find $\frac{dy}{dx}$

Sol:

Given that,

$$y = \tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right)$$

$$y = \tan^{-1} \left[\frac{\sin \left(\frac{\pi}{2} - x \right)}{1 + \cos \left(\frac{\pi}{2} - x \right)} \right]$$

$$y = \tan^{-1} \left[\frac{2 \sin \left(\frac{\pi}{4} - \frac{x}{2} \right) \cos \left(\frac{\pi}{4} - \frac{x}{2} \right)}{2 \cos^2 \left(\frac{\pi}{4} - \frac{x}{2} \right)} \right]$$

$$\left[\because \sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \text{ and } 1 + \cos A = 2 \cos^2 \frac{A}{2} \right]$$

$$y = \tan^{-1} \left[\frac{\sin \left(\frac{\pi}{4} - \frac{x}{2} \right)}{\cos \left(\frac{\pi}{4} - \frac{x}{2} \right)} \right]$$

$$y = \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{x}{2} \right) \right]$$

$$y = \frac{\pi}{4} - \frac{x}{2}$$

$$[\because \tan^{-1}(\tan x) = x]$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left[\frac{\pi}{4} - \frac{x}{2} \right] = \frac{-1}{2}$$

10. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \dots \dots \text{to } \infty}}}$ then find $\frac{dy}{dx}$

Sol:

Given that,

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \dots \dots \text{to } \infty}}}$$

Let,

$$\sqrt{x + \sqrt{x + \dots \dots \dots}} = y$$

Then,

$$\begin{aligned}
 y &= \sqrt{x+y} \\
 \Rightarrow y^2 &= x+y \\
 \Rightarrow 2y \frac{dy}{dx} &= 1 + \frac{dy}{dx} \\
 \Rightarrow (2y-1) \frac{dy}{dx} &= 1 \\
 \therefore \frac{dy}{dx} &= \frac{1}{2y-1}
 \end{aligned}$$

11. Integrate $\int \sec^n \theta \tan \theta d\theta$

sol:

We know that

$$\int \tan^m x \sec^n x$$

Here $m=1$ i.e power or degree of $\sec x$ is odd positive integer

Then let $\sec x \Rightarrow dz = \sec x \tan x dx$

Now,

$$\begin{aligned}
 &\int \sec^n \theta \tan \theta d\theta \\
 &= \int \sec^{n-1} \theta \sec \theta \tan \theta d\theta \\
 &= \int z^{n-1} dz = \frac{z^{n-1+1}}{n+1+1} + c \\
 &= \frac{z^n}{n} + c \\
 &= \frac{\sec^n \theta}{n} + c \\
 \therefore \int \sec^n \theta \tan \theta d\theta &= \frac{\sec^n \theta}{n} + c
 \end{aligned}$$

12. prove that $\int_0^{2\pi} |\cos x| dx = 4$

Given that,

$$L.H.S = \int_0^{2\pi} |\cos x| dx$$

Here $|\cos x|$ line between $0 \leq x \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$

i.e., $0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$

$$\begin{aligned}\therefore \int_0^{2\pi} |\cos x| dx &= \int_0^{\frac{\pi}{2}} |\cos x| dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\cos x| dx + \int_{\frac{3\pi}{2}}^{2\pi} |\cos x| dx \\ \int_0^{\frac{\pi}{2}} \cos x dx &= \int_0^{\frac{\pi}{2}} |\cos x| dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx + \int_{\frac{3\pi}{2}}^{2\pi} |\cos x| dx \\ &= [\sin]_0^{\frac{\pi}{2}} - [\sin]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + [\sin]_{\frac{3\pi}{2}}^{2\pi} \\ &= \left[\sin \frac{\pi}{2} - \sin 0 \right] - \left[\sin \frac{3\pi}{2} - \sin \frac{\pi}{2} \right] + \left[\sin 2\pi - \sin \frac{3\pi}{2} \right] \\ &= [1 - 0] - [-1 - 0] + [1 + 1] \\ &= 1 + 1 + 2 = 4\end{aligned}$$

13. Evaluate $\therefore \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n+r}{n^2 + r^2}$

Sol:

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n+r}{n^2 + r^2} = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n \left(1 + \frac{r}{n}\right)}{n^2 \left(1 + \frac{r^2}{n^2}\right)}$$

Now, putting $\frac{r}{n} = x \Rightarrow \frac{r^2}{n^2} = x^2$ and $\frac{1}{n} = dx$

Also taking limits between 0 to 1, we get

$$\begin{aligned}
\lim &= \int_0^1 \frac{1+x}{1+x^2} dx = \int_0^1 \frac{dx}{1+x^2} + \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx \\
\Rightarrow \lim &= \left[\tan^{-1} x \right]_0^1 + \frac{1}{2} \left[\log(1+x^2) \right]_0^1 \\
\Rightarrow \lim &= \left[\tan^{-1}(1) - \tan^{-1}(0) \right] + \frac{1}{2} [\log 2 - \log 1] \\
\Rightarrow \lim &= \left[\tan^{-1}\left(\tan \frac{\pi}{4}\right) - 0 \right] + \frac{1}{2} [\log 2 - 0] \\
\Rightarrow \lim &= \frac{\pi}{4} + \frac{1}{2} \log 2 \\
\therefore \lim &= \sum_{r=1}^n \frac{n+r}{n^2+r^2} = \frac{\pi}{4} + \frac{1}{2} \log 2
\end{aligned}$$

14. solve: $(x^2 - y^2) \frac{dy}{dx} = 2xy$

Given that,

$$\begin{aligned}
(x^2 - y^2) \frac{dy}{dx} &= 2xy \\
\Rightarrow \frac{dy}{dx} &= \frac{2xy}{x^2 - y^2}
\end{aligned}$$

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

Putting the value of y and $\frac{dy}{dx}$ in (i) we get

$$\begin{aligned}
v + x \frac{dv}{dx} &= \frac{2x.vx}{x^2 - v^2 x^2} = \frac{2v.x^2}{(1-v^2)x^2} \\
&= \frac{2v}{1-v^2} \\
\Rightarrow x \frac{dv}{dx} &= \frac{2v}{1-v^2} - v = \frac{2v-v+v^3}{1-v^2} = \frac{v+v^3}{1-v^2} = \frac{v(1+v^2)}{1-v^2} \\
\Rightarrow x \frac{dv}{dx} &= \frac{v(1+v^2)}{1-v^2} \Rightarrow \frac{1-v^2}{v(1+v^2)} dv = \frac{dx}{x} \\
\Rightarrow \frac{dx}{x} &= \frac{1+v^2-2v^2}{v(1+v^2)} dv = \frac{1+v^2}{v(1+v^2)} dv - \frac{2v^2 dv}{v(1+v^2)} \\
\Rightarrow \frac{dx}{x} &= \frac{dv}{v} - \frac{2vdv}{1+v^2} \\
\Rightarrow \int \frac{dx}{x} &= \int \frac{dv}{v} - \int \frac{2vdv}{1+v^2} \\
\Rightarrow \log x &= \log v - \log(1+v^2) + \log k \\
\Rightarrow \log v k &- \log(1+v^2) \\
\Rightarrow x &= \frac{\frac{ky}{x}}{\frac{x^2+y^2}{x^2}} = \frac{kxy}{x^2+y^2} \\
\Rightarrow x^2 + y^2 &= ky
\end{aligned}$$

15. Evaluate $x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1$

Given that,

$$\begin{aligned}
x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) &= 1 \\
\Rightarrow \frac{dy}{dx} + \frac{y(x \sin x + \cos x)}{x \cos x} &= \frac{1}{x \cos x} \\
\Rightarrow \frac{dy}{dx} \left(\frac{x \sin x}{x \cos x} + \frac{\cos x}{x \cos x} \right) y &= \frac{1}{x \cos x} \\
\Rightarrow \frac{dy}{dx} \left(\tan x + \frac{1}{x} \right) y &= \frac{1}{x \cos x}
\end{aligned}$$

It is linear differential equation in the form $\frac{dy}{dx} + Py = Q$

where $P = \tan x + \frac{1}{x}$ and $Q = \frac{\sec x}{x}$

$$\therefore (I.F) = e^{\int (\tan x + \frac{1}{x}) dx} = e^{\int \tan x dx + \int \frac{dx}{x}} = e^{\log \sec x + \log x} \\ = e^{\log(x \sec x)} = x \sec x$$

Now, multiplying (1) by I.F and integrating, we get

$$y \times I.F = \int Q \times I.F + C \Rightarrow yx \sec x = \int \frac{\sec x}{x} \times x \sec x = \tan x + c$$

16. Prove by vector method, that the angle inscribed in a semi-circle is right angle.

Sol:

Let O be the centre of semicircle with BOA as its diameter. Let P be any point. The $\angle BPA$ is an angle in a semicircle. Take O as the origin. Let the position vectors of A and P are \vec{a} and \vec{r} respectively. Then the position vector of B is $-\vec{a}$

Now,

$$\overrightarrow{AP} = (\text{Position vector of P}) - (\text{Position vector of A})$$

$$\Rightarrow AP = \overrightarrow{OP} - \overrightarrow{OA} = (\vec{r} - \vec{a})$$

$$\overrightarrow{BP} = (\text{Position vector of P}) - (\text{Position vector of B})$$

$$\Rightarrow BP = (OP - OB) = [\vec{r} - (-\vec{a})] = (\vec{r} + \vec{a})$$

$$\therefore \overrightarrow{AP} \cdot \overrightarrow{BP} = (\vec{r} - \vec{a})(\vec{r} + \vec{a})$$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = \vec{r} \cdot \vec{r} + \vec{r} \cdot \vec{a} - \vec{a} \cdot \vec{r} - \vec{a} \cdot \vec{a}$$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = |\vec{r}|^2 - |\vec{a}|^2 [\because \vec{a} \cdot \vec{r} = \vec{r} \cdot \vec{a}]$$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0 \quad [\because |\vec{r}| = |\vec{a}|]$$

$$\therefore BP \perp AP \Rightarrow \angle BPA = \frac{\pi}{2}$$

Thus the angle inscribed in a semicircle is a right angle.

17. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors and $\vec{a} + \vec{b} + \vec{c} = 0$ prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Sol:

Given that,

$$\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{b} = -\vec{a} - \vec{c}$$

Multiplying both sides by cross product \vec{a} , we get

$$\begin{aligned}
 \vec{b} &= \vec{a} - \vec{c} \Rightarrow \vec{a} \times \vec{b} = \vec{a} \times (-\vec{a} - \vec{c}) \\
 &\Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{a} + \vec{a}(-\vec{c}) \Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{a} - \vec{a} \times \vec{c} \\
 &\Rightarrow \vec{a} \times \vec{b} = \vec{a} \times \vec{a} + \vec{c} \times \vec{a} \Rightarrow \vec{a} \times \vec{b} = 0 + \vec{c} \times \vec{a} \\
 &\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad \text{-----(i)} \\
 [\because \vec{a} \times \vec{a} &= 0, -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}]
 \end{aligned}$$

Similarly $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{c} = -\vec{a} - \vec{b}$

Multiplying both sides by \vec{b} cross product, we get

$$\begin{aligned}
 \vec{c} &= -\vec{a} - \vec{b} \Rightarrow \vec{b} \times \vec{c} = \vec{b} \times (-\vec{a} - \vec{b}) \\
 &\Rightarrow \vec{b} \times \vec{c} = -\vec{b} \times \vec{a} + \vec{b} \times (-\vec{b}) \Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} - \vec{b} \times \vec{b} \\
 &\Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} - 0 \Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \quad \text{---(ii)} \\
 &[\because -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}, \vec{b} \times \vec{b} = 0]
 \end{aligned}$$

From equation (1)and (2),we get

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

18. Find the value of P, if:

$$\left(2\vec{i} + 6\vec{j} + 27\vec{k}\right) \times \left(\vec{i} + 3\vec{j} + p\vec{k}\right) = 0$$

sol:

Given that,

$$\left(2\vec{i} + 6\vec{j} + 27\vec{k}\right) \times \left(\vec{i} + 3\vec{j} + p\vec{k}\right) = 0$$

$$\text{let } \vec{a} = 2\vec{i} + 6\vec{j} + 27\vec{k} \quad \text{and} \quad \vec{b} = \vec{i} + 3\vec{j} + p\vec{k}$$

then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 6 & 27 \\ 1 & 3 & p \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = (6p - 81)\vec{i} - (2p - 27)\vec{j} + (6 - 6)\vec{k} = 0$$

$$6p - 81 = 0, -(2p - 27) = 0$$

$$6p - 81 = 0 \Rightarrow 6p = 81$$

$$p = \frac{81}{6}$$

$$p = \frac{27}{2}$$

and,

$$-2p + 27 = 0$$

$$\Rightarrow 2p = 27$$

$$p = \frac{27}{2}$$

Thus $\vec{a} \times \vec{b} = 0$ when $p = \frac{27}{2}$.

19. Prove by direction numbers, that the point $(1, -1, 3), (2, -4, 5)$ and $(5, -13, 11)$ are in a straight line.

Sol:

Let the three given points are $A(1, -1, 3), B(2, -4, 5)$ and $C(5, -13, 11)$. The direction ratios of the line AB joining the points A and B:

$$2-1, -4-(-1), 5-3 = 2-1, -4+1, 5-3 = 1, -3, 2 \dots (i)$$

Similarly, the direction ratios of the line BC joining the points B and C

$$5-2, -13-(-4), 11-5 = 3, -9, 6 \dots (ii)$$

It is clear from (1) and (2) that direction ratios of AB and BC are proportional i.e AB is parallel to BC. But the point B is common to both AB and BC. Therefore A, B and C are collinear points.

20. Find the distance of the point $(4, -5, 6)$ from the plane $\vec{r}(4\vec{i} - 4\vec{j} + 7\vec{k}) = -6$

Sol:

Here $\vec{a} = 4\vec{i} - 5\vec{j} + 6\vec{k}, \vec{N} = 4\vec{i} - 4\vec{j} + 7\vec{k}$ and $d = -6$

We know that the point \vec{a} plane \vec{N} and distance d. i.e, the plane in the form of $\vec{r} \cdot \vec{N} = d$ where \vec{N} is normal to the plane. Then the perpendicular distance

$$\begin{aligned}\frac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|} &= \frac{|(4\vec{i} - 5\vec{j} + 6\vec{k}) \cdot (4\vec{i} - 4\vec{j} + 7\vec{k}) + 6|}{|4\vec{i} - 4\vec{j} + 7\vec{k}|} \\ &= \frac{|16 + 20 + 42 + 6|}{\sqrt{4^2 + (-4)^2 + (7)^2}} = \frac{|84|}{\sqrt{81}} \\ &= \frac{84}{9} \\ &= \frac{2\sqrt{21}}{9} \text{ units.}\end{aligned}$$

21 If A and B are independent events then prove that:

$$P(A \cup B) = 1 - P(A')P(B')$$

Sol:

Since A and B are two independent events, therefore

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \Rightarrow P(A \cup B) &= P(A) + P(B) - P(A)P(B) \\ P(A \cup B) &= P(A) + P(B)[1 - P(A)] \\ P(A \cup B) &= P(A) + P(B)P(A') \\ P(A \cup B) &= 1 - P(A') + P(B)P(A') \\ P(A \cup B) &= 1 - P(A')[1 - P(B)] \\ P(A \cup B) &= 1 - P(A')P(B').\end{aligned}$$

22. Odds are 8:5 against a man, who is 55 years old, living till he is 75 and 4:3 against his wife who is now 48, living till she is 68. Find the probability that the Couple will be alive 20 years hence.

Sol:

Let A = the event of husband will be alive 20 years and B = the event of wife will be alive 20 years. Clearly A and B are independent events,

Let $P(A) = a$ and $P(A') = b$ then odds against of A = $\frac{P(A')}{P(A)} = \frac{b}{a}$

By question $b:a = 8:5 \Rightarrow b = \frac{8}{5}a$

Again, since $P(A) + P(A') = 1$ therefore $a+b=1 \Rightarrow b=1-a$

Putting this value of b in (i), we get

$$b = \frac{8}{5}a \Rightarrow 1-a = \frac{8}{5}a$$

$$\Rightarrow \frac{8}{5}a + a = 1$$

$$\Rightarrow \frac{8a+5a}{5} = 1$$

$$\Rightarrow 13a = 5$$

$$a = \frac{5}{13}$$

$$\Rightarrow P(A) = \frac{5}{13}, \text{ Similarly } P(B) = \frac{3}{7}$$

Then, the probability of the couple will be alive 20 years

$$= P(A \cap B) = P(A).P(B) = \frac{5}{13} \times \frac{3}{7} = \frac{15}{91}$$

Long Answer Type Question

Question No. 23 to 33 are long Answer Type Question. Each question carries 5 marks. Each question has an alternative as "or". You have to the answer each question or its alternative.

23. Find the maximum and minimum values $x^3 - 2x^2 + x + 6$

Sol:

Let,

$$y = x^3 - 2x^2 + x + 6$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 4x + 1 = 0$$

$$\frac{d^2y}{dx^2} = 6x - 4$$

Now, for maximum and minimum value of y , $\frac{dy}{dx} = 0$

$$3x^2 - 4x + 1 = 0$$

$$\Rightarrow 3x^2 - 3x - x + 1 = 0$$

$$\Rightarrow 3x(x-1) - 1(x-1) = 0$$

$$\Rightarrow (3x-1)(x-1) = 0$$

$$x = 1, \frac{1}{3}$$

Again $\frac{d^2y}{dx^2}$ at $x=1$

$$\frac{d^2y}{dx^2} = 6x - 4 = 6 \times 1 - 4 = 2 > 0 \text{ and } \frac{d^2y}{dx^2} \text{ at } x = \frac{1}{3}$$

$$\frac{d^2y}{dx^2} = 6x - 4 = 6 \times \frac{1}{3} - 4 = 2 - 4 = -2 < 0$$

It is clear that maximum and minimum value of y at $x = \frac{1}{3}$ and $x = 1$ respectively

\therefore putting $x = \frac{1}{3}$ in $y = x^3 - 2x^2 + x + 6$, we get

$$\begin{aligned} \Rightarrow y &= \left(\frac{1}{3}\right)^3 - 2 \times \left(\frac{1}{3}\right)^2 + \frac{1}{3} + 6 \\ &= \frac{1}{27} - \frac{2}{9} + \frac{1}{3} + 6 \\ &= \frac{1-6+9+162}{27} \\ &= \frac{172-6}{27} = \frac{166}{27} = 6.15 \end{aligned}$$

Again at $x = 1$, $y = 1^3 - 2 \cdot 1^2 + 1 + 6 = 1 - 2 + 1 + 6 = 8 - 2 = 6$

Thus maximum and minimum value of y is $= \frac{166}{27} = 6.15$ and 6 respectively.

OR,

If $x^m y^n = (x+y)^{m+n}$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

Sol:

Given that,

$$x^m y^n = (x+y)^{m+n}$$

Taking logarithm on the both sides, we get

$$\begin{aligned} \log(x^m \cdot y^n) &= \log(x+y)^{m+n} \\ \Rightarrow m \log x + n \log y &= (m+n) \log(x+y) \end{aligned}$$

Differentiating both sides, with respect to x, we get

$$\begin{aligned} \frac{m}{n} + \frac{n}{y} \frac{dy}{dx} &= \frac{(m+n)}{(x+y)} \left[1 + \frac{dy}{dx} \right] \\ \Rightarrow \frac{m}{n} + \frac{n}{y} \frac{dy}{dx} &= \frac{m+n}{x+y} + \frac{m+n}{x+y} \frac{dy}{dx} \\ \Rightarrow \left(\frac{n}{y} - \frac{m+n}{x+y} \right) \frac{dy}{dx} &= \frac{m+n}{x+y} - \frac{m}{x} \\ \Rightarrow \left[\frac{n(x+y) - y(m+n)}{y(x+y)} \right] \frac{dy}{dx} &= \frac{x(m+n) - m(x+y)}{x(x+y)} \\ \Rightarrow \left(\frac{nx+ny-my-ny}{y(x+y)} \right) \frac{dy}{dx} &= \frac{mx+nx-mx-my}{x(x+y)} \\ \Rightarrow \left(\frac{nx-my}{y(x+y)} \right) \frac{dy}{dx} &= \frac{nx-my}{x(x+y)} \\ \therefore \frac{dy}{dx} &= \frac{nx-my}{x(x+y)} \times \frac{y(x+y)}{nx-my} \\ \Rightarrow \frac{dy}{dx} &= \frac{y}{x} \end{aligned}$$

Hence, LHS = RHS proved.

24. Prove that: $\int_0^{\frac{\pi}{2}} \log(\tan \theta + \cot \theta) d\theta = \pi \log 2$

Sol:

$$\begin{aligned}
LHS &= I = \int_0^{\frac{\pi}{2}} \log \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right) d\theta \\
&= \int_0^{\frac{\pi}{2}} \log \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} d\theta \\
&= \int_0^{\frac{\pi}{2}} \log \frac{d\theta}{\sin \theta \cos \theta} = - \int_0^{\frac{\pi}{2}} \log(\sin \theta \cos \theta) d\theta \\
&= \left[\int_0^{\frac{\pi}{2}} \log \sin \theta d\theta + \int_0^{\frac{\pi}{2}} \log \cos \theta d\theta \right] \dots\dots\dots(A) \\
I &= -[I_1 - I_2] \dots\dots\dots(B)
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} \log \sin \theta d\theta \dots\dots\dots(1) \\
\Rightarrow I_1 &= \int_0^{\frac{\pi}{2}} \log \sin \left(\frac{\pi}{2} - \theta \right) d\theta = I_1 = \int_0^{\frac{\pi}{2}} \log \cos \theta d\theta \dots\dots\dots(2)
\end{aligned}$$

Adding (1) and (2), we get

$$\begin{aligned}
2I_1 &= \int_0^{\frac{\pi}{2}} \log (\sin \theta \cos \theta) d\theta = \int_0^{\frac{\pi}{2}} \log \left(\frac{2 \sin \theta \cos \theta}{2} \right) d\theta \\
2I_1 &= \int_0^{\frac{\pi}{2}} \log \frac{\sin 2\theta}{2} d\theta - \log \int_0^{\frac{\pi}{2}} \log \sin 2\theta d\theta - \int_0^{\frac{\pi}{2}} \log 2 d\theta \\
2I_1 &= \int_0^{\frac{\pi}{2}} \log \sin 2\theta d\theta - \log 2 \int_0^{\frac{\pi}{2}} d\theta \\
2I_1 &= I_3 - \log 2 [\theta]_0^{\frac{\pi}{2}} = I_3 - \frac{\pi}{2} \log 2 \\
2I_1 &= I_3 - \frac{\pi}{2} \log 2 \dots\dots\dots(3)
\end{aligned}$$

for I_3 let $2\theta = z \Rightarrow 2d\theta$

$$= dz \Rightarrow d\theta = \frac{dz}{2} \text{ also when } \theta = 0 \text{ Then } z = 0$$

and when $\theta = \frac{\pi}{2}$ Then $z = \pi$

$$\begin{aligned}\therefore I_3 &= \log \int_0^{\frac{\pi}{2}} \log \sin \theta d\theta - \int_0^{\pi} \log \sin z \frac{dz}{2} \\ \Rightarrow I_3 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \log \sin z dz = \frac{1}{2} \int_0^{2-\frac{\pi}{2}} \log \sin z dz \\ \Rightarrow I_3 &= \frac{2}{z} \int_0^{\frac{\pi}{2}} \log \sin z dz = \int_0^{\frac{\pi}{2}} \log \sin \theta d\theta\end{aligned}$$

$$\int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx \text{ if } f(2a-x) = f(x)$$

$$\text{and } \int_0^a f(x)dx = \int_0^a f(y)dy$$

\therefore from (3), we get

$$2I = I_1 - \frac{\pi}{2} \log 2$$

$$\Rightarrow I_1 = -\frac{\pi}{2} \log 2$$

Similarly,

$$I_2 = -\frac{\pi}{2} \log 2$$

Lastly from (A), we get

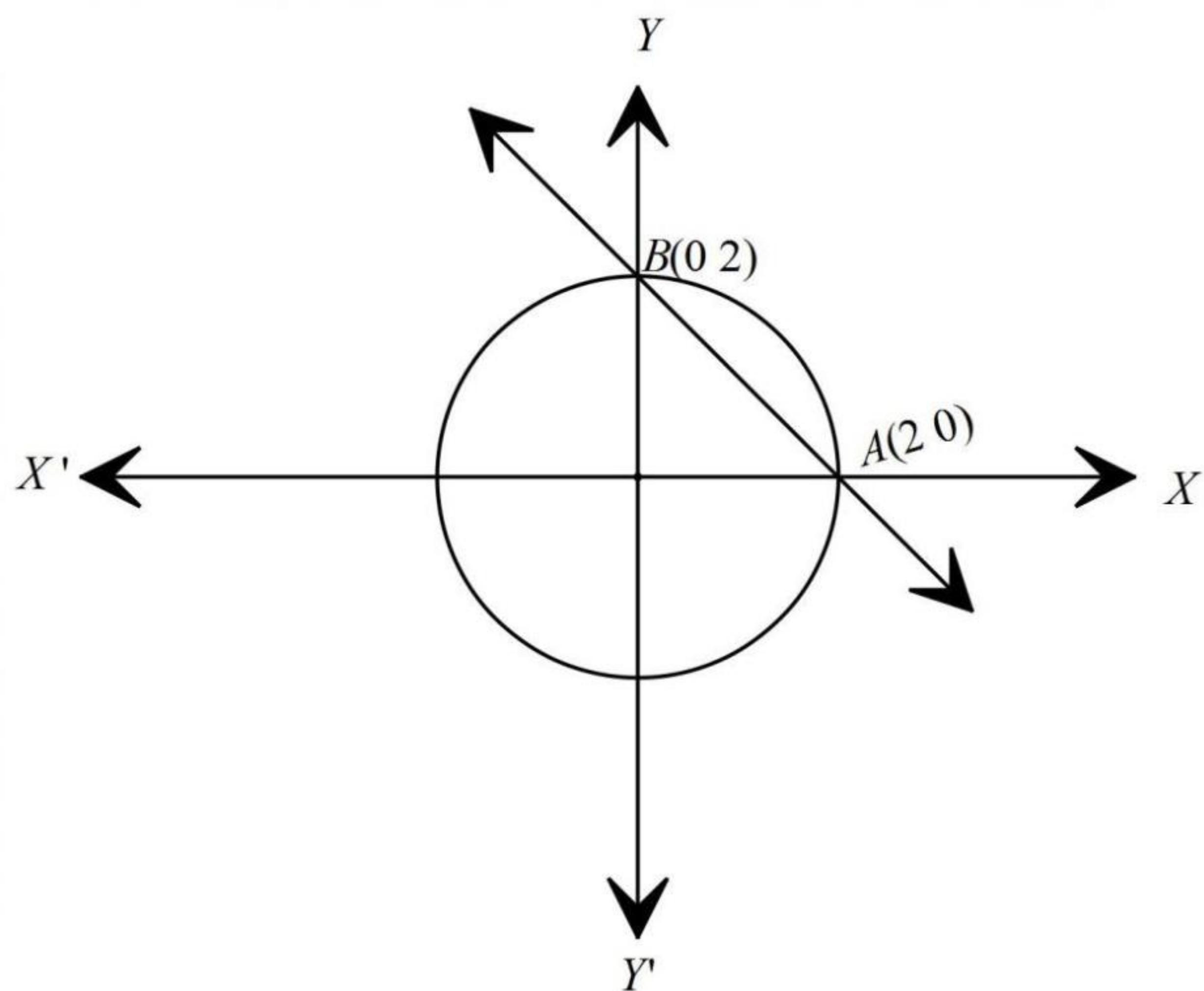
$$\begin{aligned}I_1 &= \left[-\frac{\pi}{2} \log 2 - \frac{\pi}{2} \log 2 \right] = \frac{\pi}{2} \log 2 + \frac{\pi}{2} \log 2 \\ &= \frac{2\pi}{2} \log 2 = \pi \log 2 = RHS\end{aligned}$$

OR,

Find the area of the Smaller portion of the circle $x^2 + y^2 = 4$ cut off by the line $x + y = 2$.

Sol:

Let $x^2 + y^2 = 4$ represent a circle whose centre $(0,0)$ and radius 2 units. AB : $x + y = 2$ be the line which passes through circle at the points $(2,0)$ and $(0,2)$



\therefore Area of region ACB = Area of quadrant OAB - Area of $\triangle OAB$ ----(1)

$$\text{Now } x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2 \Rightarrow y = \sqrt{4 - x^2}$$

\therefore Area of quadrant OAB

$$\begin{aligned} OAB &= \int_0^2 \sqrt{4 - x^2} dx \\ &= \left[\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2 \\ &\quad \left[\because \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right] \\ &= 0 + 2 \sin^{-1} \frac{2}{2} - 0 = 2 \sin^{-1} \frac{\pi}{2} \\ &= 2 \sin^{-1} \left(\sin \frac{\pi}{2} \right) = 2 \times \frac{\pi}{2} = \pi \text{ sq.units} \end{aligned}$$

Area of $\triangle OAB$ = Area of the region bounded by AB.

$$\therefore AB : x + y = 2 \Rightarrow y = 2 - x \text{ and } x = 0, y = 0$$

\therefore Area of $\triangle OAB$

$$\int_0^2 (2-x)dx = \left[2x - \frac{x^2}{2} \right]_0^2 = \left[4 - \frac{4}{2} \right] - 0 = 2$$

Area of region ACB = Area of quadrant OAB – Area of $\triangle OAB$

\therefore Area of region ACB = $(\pi - 2)$ sq. units.

25. Prove by Vector method, that in any $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Sol:

let ABC be a triangle, whose sides are $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$.

Then $\vec{a} + \vec{b} + \vec{c} = \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$

$$\vec{a} + \vec{b} + \vec{c} = 0$$

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times 0$$

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0 \quad [\because \vec{a} \times \vec{a} = 0]$$

$$\vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$$

Now, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ [$\because -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}$] ----- (1)
Similarly,

$$\vec{a} + \vec{b} + \vec{c} = 0$$

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times 0$$

$$\vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{b} \times \vec{c} = -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}$$

$$\vec{b} \times \vec{c} = \vec{a} \times \vec{b} \quad \text{----- (2)}$$

$$[\vec{b} \times \vec{b} = 0, -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}]$$

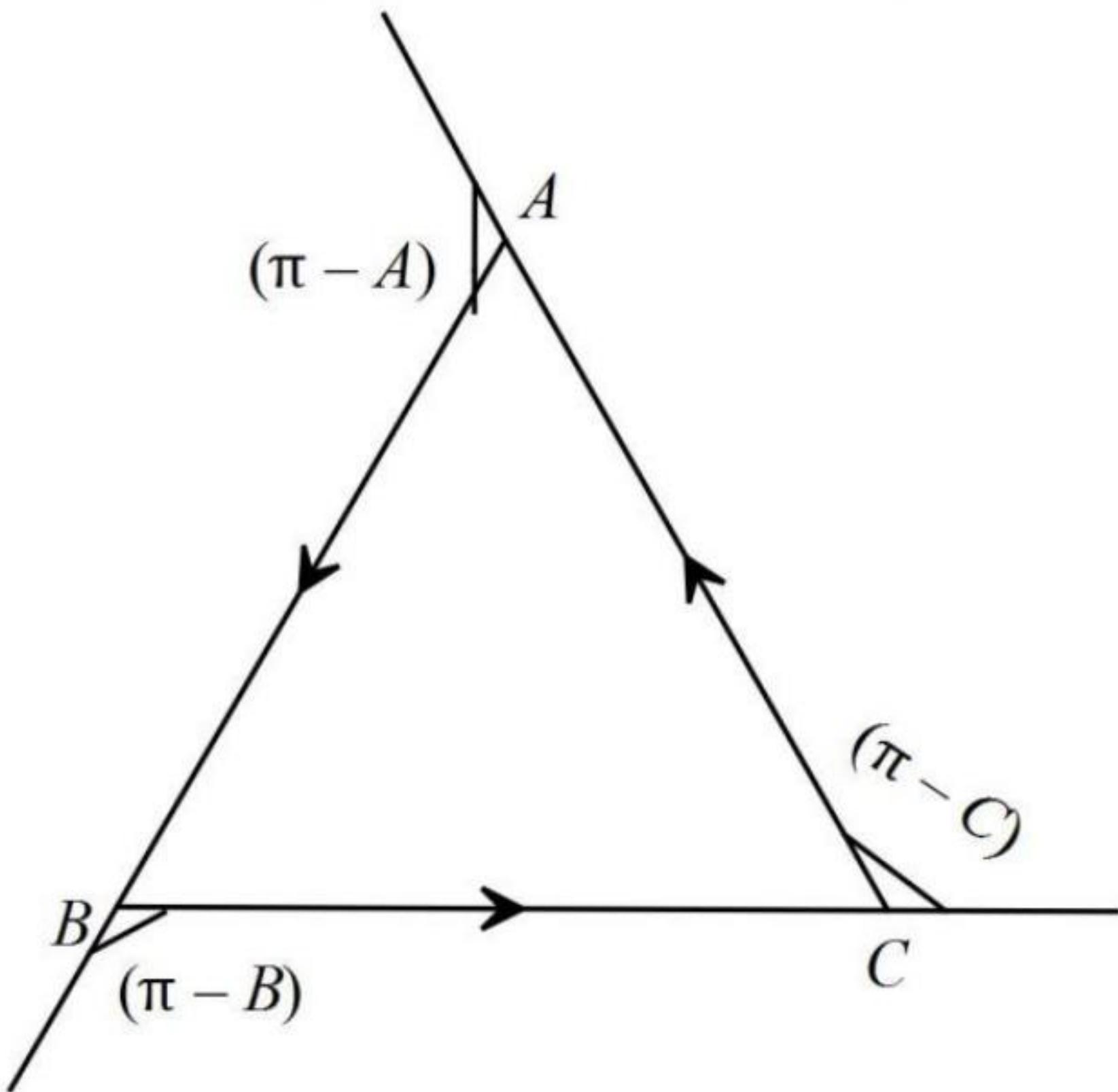
From (1) and (2), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$$\begin{aligned}
&\Rightarrow |\vec{a} \times \vec{b}| = |b \times c| = |c \times a| \\
&\Rightarrow ab \sin(\pi - C) = bc \sin(\pi - A) = ca \sin(\pi - B) \\
&\Rightarrow ab \sin C = bc \sin A = ca \sin B [\because \sin(\pi - \theta) = \sin \theta] \\
&\Rightarrow \frac{ab \sin C}{abc} = \frac{bc \sin A}{abc} = \frac{ca \sin B}{abc} \\
&\Rightarrow \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b} \\
&\Rightarrow \frac{c}{\sin C} = \frac{a}{\sin A} = \frac{b}{\sin B} \\
&\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\end{aligned}$$

OR,

Show that the line joining the point $(4, 7, 8), (2, 3, 4)$ is parallel to the line joining the points $(2, 4, 10), (-2, -4, 2)$

Sol:



Let the given points are $A(4, 7, 8), B(2, 3, 4)$ and $C(2, 4, 10), D(-2, -4, 2)$. Then direction ratios of the line AB:

$$2-4, 3-7, 4-8 = -2, -4, -4 \Rightarrow a_1 = -2, b_1 = -4, c_1 = -4$$

And the direction ratios of the line CD:

$$\begin{aligned}-2 - 2, -4 - 4, 2 - 10 &= -4, -8, -8 \\ \Rightarrow a_2 &= -4, b_2 = -8, c_2 = -8\end{aligned}$$

Now, two lines AB and CD are parallel if its direction ratios are in the form $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{-2}{-4} = \frac{-4}{-8} = \frac{-4}{-8} \Rightarrow \frac{1}{2} = \frac{1}{2} = \frac{1}{2}$$

Thus $AB \parallel CD$