ENGG2440 Discrete mathematics for engineers

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Abstract

This is a note for ENGG2440 - Discrete mathematics for engineers for self-revision and concept understanding ONLY. Some contents are taken from lecture notes as well as only reference book.
Mistakes might be found. So please feel free to point out any mistakes.

Template of this note is adapted from https://github.com/sleepymalc.

Contents

Chapter 1

Mathematical Induction

1.1 Introduction

In mathematics, there are some basic proof techniques that we can apply, including direct proof, proof by induction, proof by contradiction, and proof by contraposition. For most of these proving methods, you won't be learning their reasons or applications, but you will still use them in some simple proving questions. In this chapter, we will mainly discuss mathematical induction.

> **Definition 1.1.1** (Proposition). A **proposition** is a statement that is either true or false.

> **Definition 1.1.2** (Predicate). A **predicate** is a proposition whose truth depends on one or more variables.

1.2 Mathematical Induction

An analogy of the principle of mathematical induction is the game of dominoes. Suppose the dominoes are lined up properly, so that when one falls, the successive one will also fall. Now by pushing the first domino, the second will fall; when the second falls, the third will fall; and so on. We can see that all dominoes will ultimately fall.

The key point is only two steps:

1. the first domino falls;
2. when a domino falls, the next domino falls.

We use the above principle of Mathematical Induction to prove.

Process:

1. Let $P(n)$ be a predicate.
2. (Base Case) Show that $P(1)$ is true.
3. (Inductive Steps) Show that for $n = 1, 2, \ldots$, if $P(n)$ is true, then $P(n+1)$ is true.

> **Example.**
>
> $$P(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
>
> 1. Base Case
>
> We need to show that $P(1)$ is true.
>
> $$1 = \frac{(1)(1+1)}{2}$$

, which is obvioulsy true.

2. Inductive Step

For inducitve hypothesis, we can assume

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Now, to show that $P(n+1)$ is true,

$$\begin{aligned} L.H.S. &= 1 + 2 + \cdots + n + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= R.H.S \end{aligned}$$

which shows that $P(n+1)$ is also true.

Hence, by the principle of MI, we can conclude that $P(n)$ is true for all integers $n \geq 1$.

Exercise. Show that for any integer $n \geq 1$, $n^3 - n$ is divisible by 3.

Note. In inductive step, consider putting a constant q as $3q$ is divisible by 3.

Exercise. Prove that $n^3 < 2^n$ for all integers $n \geq 10$.

Note. Consider bonding the lower order terms in terms of n^3.

1.3 Strong Mathematical Induction

As the name suggests, the method of induction used in this section is "stronger". This is because assuming only that $P(n)$ is true may be too restrictive, i.e., insufficient to prove the predicate. Thus, in the inductive step, you may show that $P(1), P(2), \cdots, P(n)$ are true, and then prove that $P(n+1)$ is true.

Example. The Fibonacci sequence is a sequence of number defined via the following recursion:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2$$
$$F_0 = 0; F_1 = 1$$

Prove that

$$P(n) : F_n \leq \phi^{n-1}, \text{where } \phi = \frac{1+\sqrt{5}}{2}$$

1. Base Case

$$F_1 = 1 \leq \phi^0 = 1$$
$$F_2 = 1 \leq \phi^1 \approx 1.618$$

Thus, $P(1)$ and $P(2)$ hold true, which means $P(3)$ also holds true.

2. Inductive Step

For inductive hypothesis, we assume

$$F_k \leq \phi^{k-1} \text{ for } k = 1, 2, \ldots, n$$

Given the Fibonacci sequence

$$F_{n+1} = F_n + F_{n-1}$$

By the strong inductive hypothesis, we have

$$F_n \leq \phi^{n-1}, \quad F_{n-1} \leq \phi^{n-2}$$

From the definition of ϕ we have $\phi^2 = 1 + \phi$

Hence, we obtain

$$F_{n+1} \leq \phi^{n-1} + \phi^{n-2} = \phi^{n-2}(1 + \phi) = \phi^n$$

Chapter 2

Summation Techniques

2.1 Summation

When summing numbers with certain patterns, we can use summation notation. For example,

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

2.1.1 Distributive Law

Let c be a constant. Then, we can take c out of the summation:

$$\sum_{k \in \mathcal{K}} c a_k = c \sum_{k \in \mathcal{K}} a_k$$

Example.

$$\sum_{k=1}^{n} 2k = 2(1) + 2(2) + 2(3) + \cdots + 2(n) = 2(1 + 2 + 3 + \cdots + n) = 2 \sum_{k=1}^{n} k$$

2.1.2 Associative Law

We can split the summands as follows:

$$\sum_{k \in \mathcal{K}} (a_k + b_K) = \sum_{k \in \mathcal{K}} a_k + \sum_{k \in \mathcal{K}} b_k$$

Example.

$$\begin{aligned}
\sum_{k=1}^{n} (k + k^2) &= (1 + 1^2) + (2 + 2^2) + \cdots + (n + n^2) \\
&= (1 + 2 + \cdots + n) + (1^2 + 2^2 + \cdots + n^2) \\
&= \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2
\end{aligned}$$

2.2 Close Form Formula

Close form formula is the formula that does not have the summation index k for a sum by simply writing it out explicitly. For example,

$$\sum_{k=1}^{n}(a_k - a_{k-1})$$

By expanding the sum, we have

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n - a_0$$

By cancelling the terms, we get $a_n - a_0$, which is the close form formula for the summation $\sum_{k=1}^{n}(a_k - a_{k-1})$.

2.3 Perturbation Method

It could be difficult to derive the close form formula for some summation. Therefore, we can use the perturbation method.

For summation

$$S_n = \sum_{k=1}^{n} a_k,$$

we can split off the first term and the last term, then rewrite it as

$$a_1 + \sum_{k=2}^{n+1} a_k = S_{n+1} = \sum_{k=1}^{n} a_k + a_{n+1}$$

Example (Geometric Sum)**.** Let x be any number. Consider the sum

$$S_n = \sum_{k=1}^{n} x^k$$

$$x + \sum_{k=2}^{n+1} x^k = S_{n+1} = \sum_{k=1}^{n} x^k + x^{n+1}$$

Observe that

$$\sum_{k=2}^{n+1} x^k = x^2 + x^3 + \cdots + x^{n+1} = x(x + x^2 + \cdots + x^n) = xS_n$$

By substitution, we have

$$x + xS_n = S_n + x^{n+1}$$

If $x \neq 1$, then we can solve for S_n to get

$$S_n = \frac{x(1 - x^n)}{1 - x}$$

This summation is also called geometric sum.

By applying the perturbation method, we can find the close form formula for some common summation. Another example is Quadratic Series.

Example (Quadratic Series). By applying perturbation method to the sum

$$S_n = \sum_{k=1}^{n} k^2,$$

we have

$$1 + \sum_{k=2}^{n+1} k^2 = S_{n+1} = \sum_{k=1}^{n} k^2 + (n+1)^2$$

Let $j = k - 1$,

$$\sum_{k=2}^{n+1} k^2 = \sum_{j=1}^{n} (j+1)^2$$

$$\begin{aligned}\sum_{j=1}^{n} (j+1)^2 &= \sum_{j=1}^{n} (j^2 + 2j + 1) \\ &= \sum_{j=1}^{n} j^2 + 2\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\ &= S_n + 2\sum_{j=1}^{n} j + n\end{aligned}$$

Then, we have

$$\begin{aligned}1 + \sum_{k=2}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ 1 + S_n + 2\sum_{j=1}^{n} j + n &= S_n + (n+1)^2 \\ \sum_{j=1}^{n} j &= \frac{n(n+1)}{2}\end{aligned}$$

However, we obtain the Euler′s trick here instead. Thus, we may apply the perturbation method to another sum.

$$C_n = \sum_{k=1}^{n} k^3 \Rightarrow 1 + \sum_{k=2}^{n+1} k^3 = C_{n+1} = \sum_{k=1}^{n} k^3 + (n+1)^3$$

$$\begin{aligned}\sum_{k=2}^{n+1} k^3 &= \sum_{j=1}^{n} (j+1)^3 \quad \text{(By applying } j = k - 1\text{)} \\ &= \sum_{j=1}^{n} j^3 + 3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\ &= C_n + 3S_n + \frac{3n(n+1)}{2} + n\end{aligned}$$

By substitution, we get

$$\begin{aligned}1 + C_n + 3S_n + \frac{3n(n+1)}{2} + n &= C_n + (n+1)^3 \\ 2 + 6S_n + 3n(n+1) + 2n &= 2(n+1)^3 \\ S_n &= \frac{2(n+1)^3 - 2n - 2 - 3n(n+1)}{6} \\ S_n &= \frac{(n+1)(2(n+1)^2 - 2 - 3n)}{6} \\ S_n &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

Chapter 3

Recurrences

3.1 Introduction

Nothing is bugs-free. There are some known bugs which I don't have incentive to solve, or it is hard to solve whatsoever. Let me list some of them.

3.1.1 Footnote Environment

It's easy to let you fall into a situation that you want to keep using `footnote` to add a bunch of unrelated stuffs. However, with our environment there is a known strange behavior, which is following.

> **Example.** Footnote![a]
>
> > **Remark.** Oops! footnote somehow shows up earlier than expect![a]
> >
> > [a]This is a footnote!
> > [a]This is another footnote!
>
> Bugs caught![b]
>
> [b]The final footnote which is ok!

As we saw, the footnote in the `Example` environment should show at the bottom of its own box, but it's caught by `Remark` which causes the unwanted behavior. Unfortunately, I haven't found a nice way to solve this. A potential way to solve this is by using `footnotemark` with `footnotetext` placing at the bottom of the environment, but this is tedious and needs lots of manual tweaking.

Furthermore, not sure whether you notice it or not, but the color box of `Remark` is not quite right! It extends to the right, another trick bug...

3.1.2 Mdframe Environment

Though `mdframe` package is nice and is the key theme throughout this template, but it has some kind of weird behavior. Let's see the demo.

> **Proof of Theorem 5.1.1.** We need to prove the followings.
>
> > **Claim.** $E = mc^2$.

Proof. Nonsense.

Nonsense,

Nonsense,

Nonsense,

Nonsense,

Nonsense. ⊛

■

I expect it should break much earlier, and this seems to be an algorithmic issue of `mdframe`. One potential solution is to use `tcolorbox` instead, but I haven't completely figure it out, hence I can't really say anything right now.

Chapter 4

Asymptotics

4.1

Definition 4.1.1 (Natural number). We denote the set of *natural numbers* as $\mathbb{N}$.

Lemma 4.1.1 (Useful lemma). Given the axioms of natural numbers $\mathbb{N}$, we have

$$0 \neq 1.$$

An obvious proof. Obvious. ■

Proposition 4.1.1 (Useful proposition). From Lemma 5.1.1, we have

$$0 < 1.$$

Exercise. Prove that $1 < 2$.

Answer. We note the following.

> **Note.** We have Proposition 5.1.1! We can use it iteratively!

With the help of Lemma 5.1.1, this holds trivially. ⊛

Example. We now can have $a < b$ for $a < b$!

Proof. Iteratively apply the exercise we did above. ⊛

Remark. We see that Proposition 5.1.1 is really powerful. We now give an immediate application of it.

Theorem 4.1.1 (Mass-energy equivalence). Given Proposition 5.1.1, we then have

$$E = mc^2.$$

Proof. The blank left for me is too small,[a] hence we put the proof in Appendix A.1. ■

[a] https://en.wikipedia.org/wiki/Richard_Feynman

From Theorem 5.1.1, we then have the following.

Corollary 4.1.1 (Riemann hypothesis). The real part of every nontrivial zero of the Riemann zeta function is $\frac{1}{2}$, where the Riemann zeta function is just

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

Proof. The proof should be trivial, we left it to you. ■

DIY

As previously seen. We see that Lemma 5.1.1 is really helpful in the proof!

Internal Link

You should see all the common usages of internal links. Additionally, we can use citations as [New26], which just link to the reference page!

4.2 Figures

A simple demo for drawing:

Figure 4.1: A 3-torus.[1]

4.3 Commutative Diagram

We can use the package `tikz-cd` to draw some commutative diagram.

Example. The cellular homology agrees with singular homology.

Proof. The following commutative diagram shows everything.

$$\begin{tikzcd}
& & & & & 0 \\
& & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & \\
& & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[urr, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
& & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
& & & 0 \arrow[ur] & & &
\end{tikzcd}$$

$\circledast$

4.4 Fancy Stuffs

With this header, you can achieve some cool things. For example, we can have multiple definitions under a parent environment, while maintains the numbering of definition. This is achieved by `definition*` environment with `definition` inside. For example, we can have the following.

[1] For detailed information, please see `https://github.com/sleepymalc/VSCode-LaTeX-Inkscape`.

Definition. We have the following number system.

Definition 4.4.1 (Rational number)**.** The set of *rational number*, denote as $\mathbb{Q}$.

Definition 4.4.2 (Real number)**.** The set of *real number*, denote as $\mathbb{R}$.

Definition 4.4.3 (Complex number)**.** The set of *complex number*, denote as $\mathbb{C}$.

Note. And indeed, we can still reference them correctly. For instance, we can use rational numbers to define real numbers and then further use it to define complex numbers.

Furthermore, we can completely control the name of our environments. We already saw we can name definition, lemma, proposition, corollary and theorem environment. In fact, we can also name remark, note, example and proof as follows.

Example (Interesting Example)**.** We note that $1 \neq 2$!

Note (Important note)**.** As a consequence, $2 \neq 3$ also.

Remark (Easy observation)**.** We see that from here, we easily have the following theorem.

Theorem 4.4.1 (Lebesgue Differentiation Theorem)**.** Let $f \in L^1$, then

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0$$

for a.e. x.

An obvious proof of Theorem 5.4.1. Obvious. ■

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[2]This time I mean it!

Chapter 5

Set Theory and Counting Principle

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 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

⊛

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Chapter 6

Binomial Coefficients

6.1 Introduction

In this section, we introduce Binomial Coefficients.

6.1.1 Combinations

As it is introduced before,

Definition 6.1.1. An r-combination of the n-element ground set S_0 is an **unordered selection** of r elements from S_0.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

6.1.2 Permutation

Again, as it is introduced before,

Definition 6.1.2. An r-permutation of the n-element ground set S_0 is an **ordered selection** of r elements from S_0.

$$P(n,r) = \frac{n!}{(n-r)!}$$

6.1.3 Binomial Identities

Proposition 6.1.1. For any integers $m, r \geq 0$ with $0 \leq r \leq n$,

$$\binom{n}{r} = \binom{n}{n-r}$$

We have two ways to prove this proposition, namely Algebraic Proof and Combinatorial Proof.

Algebraic Proof.

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

■

Combinatorial Proof. Both side of the identity are supposed to be two different ways of solving a counting problem. We can define the counting problem as counting the number of different unordered selections of r elements from an n-element ground set. Then, we can define the RHS as the selecting number to be excluded. Then this identity holds. ■

6.1.4 Pascal's Identity

Theorem 6.1.1. For any integers $n,\ r \geq 0$ with $1 \leq r \leq n-1$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ OR } \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Again, we can prove this theorem by two ways.

Algebraic Proof.

$$\begin{aligned}
RHS &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!} \\
&= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{1}{r} + \frac{1}{n-r}\right) \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{n}{r(n-r)}\right) \\
&= \frac{n!}{r!(n-r)!} \\
&= \binom{n}{r}
\end{aligned}$$

■

Combinatorial Proof. Recall that $\binom{n}{k}$ equals the number of subsets with k elements from a set with n elements. Suppose one particular element is uniquely labeled X in a set with n elements.

To construct a subset of k elements containing X, include X and choose $k-1$ elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k-1}$ such subsets.

To construct a subset of k elements **not** containing X, choose k elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k}$ such subsets.

Every subset of k elements either contains X or not. The total number of subsets with k elements in a set of n elements is the sum of the number of subsets containing X and the number of subsets that do not contain X, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

This equals $\binom{n}{k}$. ■

Appendix

Appendix A

Additional Proofs

A.1 Proof of Theorem 5.1.1

We can now prove Theorem 5.1.1.

Proof of Theorem 5.1.1. See here. ■

Bibliography

[New26] I. Newton. *Philosophiae naturalis principia mathematica*. Innys, 1726. URL: https://books.google.com/books?id=WeZO9rjv-1kC.