Problem 1.1

(1)

$$\sum_{i=0}^{\infty}\frac{1}{4^i}=\sum_{i=0}^{\infty}(\frac{1}{4})^i=\frac{(\frac{1}{4})^{n+1}-1}{\frac{1}{4}-1}=\frac{4-4^{-\infty}}{3}=\frac{4}{3}$$

(9)

$$\sum_{i=0}^{n}i^3=\frac{n^2(n^2+2n+1)}{4}$$

(13)

Assume that $2^{2n}=O(2^n)$, then we have $2^{2n}\leq c\cdot 2^n$ for $c>0$ and an $n_0\geq 0$ for all $n\geq n_0$.

$$2^{2n}\leq c\cdot 2^n$$
$$2^n\leq c$$

Since for $n_0\geq 0$, there does not exist a fixed c such that $2^n\leq c$, by contradiction, it can be proved that $2^{2n}\neq O(2^n)$.

Problem 1.3

(3)

Let $T_0(n)=T_h(n)+T_p(n)$, where $T_h(n)=aT(n-1)$.
For $T_h(n)$, characteristic equation $x=a$, then we have $T_h(n)=\theta a^n$.
Since $f(n)=bn^c, s=n$, let $T_p(n)=x_0n^c$.

$$x_0n^c=ax_0(n-1)^c+bn^c$$
$$x_0=\frac{bn^c}{n^c-a(n-1)^c}$$

Then, we have

$$T_0(n)=\theta a^n+\frac{bn^{2c}}{n^c-a(n-1)^c}$$

Since $T(1)=1$, we have

$$1=\theta\cdot a+b\Longrightarrow\theta=\frac{1-b}{a}$$

Hence, the solution to the recurrence is given by

$$T(n)=(1-b)a^{n-1}+\frac{bn^{2c}}{n^c-a(n-1)^c}$$

(8)

Let $T_0(n) = T_h(n) + T_p(n)$, where $T_h(n) = 3T(n-1)$.
For $T_h(n)$, characteristic equation $x = 3$, then we have $T_h(n) = \theta 3^n$.
Since $f(n) = 2$, let $T_p(n) = x$.

$$x = 3x + 2$$
$$x = -1$$

Then, we have

$$T_0(n) = \theta 3^n - 1$$

Since $T(1) = 1$, we have

$$1 = 3\theta - 1$$

This gives $\theta = \frac{2}{3}$. Hence, the solution to the recurrence is given by

$$T(n) = 2 \cdot 3^{n-1} - 1$$

Problem 1.4

(2)

Let $P(n)$ be the predicate

$$P(n) : \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

For $n = 0$, $L.H.S = 2^0 = 1, R.H.S = 2^{0+1} - 1 = 1$, which holds true.
Assume that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

For $P(n+1)$,

$$\begin{aligned}\sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} \\ &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2 \times 2^{n+1} - 1 \\ &= 2^{(n+1)+1} - 1 \\ &= R.H.S\end{aligned}$$

which shows that $P(n+1)$ is also true. Hence, by the principle of MI, we can conclude that $P(n)$ is true for all integers $n \geq 1$.

(4)

Let $f(x) = (n+1)^2, g(x) = n^2$.

$$f(x) = n^2 + 2n + 1 \leq n^2 + 2n^2 + n^2 = 4n^2$$

Thus, by taking $c = 4, n_0 = 1$, for $n > n_0$, $(n+1)^2 = O(n^2)$.

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