ENGG2440 Discrete mathematics for engineers

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Abstract

This is a note for ENGG2440 - Discrete mathematics for engineers for self-revision and concept understanding ONLY. Some contents are taken from lecture notes as well as only reference book.
Mistakes might be found. So please feel free to point out any mistakes.

Template of this note is adapted from https://github.com/sleepymalc.

Contents

Chapter 1

Mathematical Induction

1.1 Introduction

In mathematics, there are some basic proof techniques that we can apply, including direct proof, proof by induction, proof by contradiction, and proof by contraposition. For most of these proving methods, you won't be learning their reasons or applications, but you will still use them in some simple proving questions. In this chapter, we will mainly discuss mathematical induction.

Definition 1.1.1 (Proposition). A **proposition** is a statement that is either true or false.

Definition 1.1.2 (Predicate). A **predicate** is a proposition whose truth depends on one or more variables.

1.2 Mathematical Induction

An analogy of the principle of mathematical induction is the game of dominoes. Suppose the dominoes are lined up properly, so that when one falls, the successive one will also fall. Now by pushing the first domino, the second will fall; when the second falls, the third will fall; and so on. We can see that all dominoes will ultimately fall.

The key point is only two steps:

1. the first domino falls;
2. when a domino falls, the next domino falls.

We use the above principle of Mathematical Induction to prove.

Process:

1. Let $P(n)$ be a predicate.
2. (Base Case) Show that $P(1)$ is true.
3. (Inductive Steps) Show that for $n = 1, 2, \ldots$, if $P(n)$ is true, then $P(n+1)$ is true.

Example.

$$P(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

1. Base Case

We need to show that $P(1)$ is true.

$$1 = \frac{(1)(1+1)}{2}$$

, which is obviouly true.

2. Inductive Step

For inducitve hypothesis, we can assume

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Now, to show that $P(n+1)$ is true,

$$\begin{aligned} L.H.S. &= 1 + 2 + \cdots + n + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= R.H.S \end{aligned}$$

which shows that $P(n+1)$ is also true.

Hence, by the principle of MI, we can conclude that $P(n)$ is true for all integers $n \geq 1$.

Exercise. Show that for any integer $n \geq 1$, $n^3 - n$ is divisible by 3.

Note. In inductive step, consider putting a constant q as $3q$ is divisible by 3.

Exercise. Prove that $n^3 < 2^n$ for all integers $n \geq 10$.

Note. Consider bonding the lower order terms in terms of n^3.

1.3 Strong Mathematical Induction

As the name suggests, the method of induction used in this section is "stronger". This is because assuming only that $P(n)$ is true may be too restrictive, i.e., insufficient to prove the predicate. Thus, in the inductive step, you may show that $P(1), P(2), \cdots, P(n)$ are true, and then prove that $P(n+1)$ is true.

Example. The Fibonacci sequence is a sequence of number defined via the following recursion:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2$$
$$F_0 = 0; F_1 = 1$$

Prove that

$$P(n) : F_n \leq \phi^{n-1}, \text{where } \phi = \frac{1+\sqrt{5}}{2}$$

1. Base Case

$$F_1 = 1 \leq \phi^0 = 1$$
$$F_2 = 1 \leq \phi^1 \approx 1.618$$

Thus, $P(1)$ and $P(2)$ hold true, which means $P(3)$ also holds true.

2. Inductive Step

For inductive hypothesis, we assume

$$F_k \leq \phi^{k-1} \text{ for } k = 1, 2, \ldots, n$$

Given the Fibonacci sequence

$$F_{n+1} = F_n + F_{n-1}$$

By the strong inductive hypothesis, we have

$$F_n \leq \phi^{n-1}, \quad F_{n-1} \leq \phi^{n-2}$$

From the definition of ϕ we have $\phi^2 = 1 + \phi$

Hence, we obtain

$$F_{n+1} \leq \phi^{n-1} + \phi^{n-2} = \phi^{n-2}(1 + \phi) = \phi^n$$

Chapter 2

Summation Techniques

2.1 Summation

When summing numbers with certain patterns, we can use summation notation. For example,

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

2.1.1 Distributive Law

Let c be a constant. Then, we can take c out of the summation:

$$\sum_{k \in \mathcal{K}} c a_k = c \sum_{k \in \mathcal{K}} a_k$$

Example.

$$\sum_{k=1}^{n} 2k = 2(1) + 2(2) + 2(3) + \cdots + 2(n) = 2(1 + 2 + 3 + \cdots + n) = 2 \sum_{k=1}^{n} k$$

2.1.2 Associative Law

We can split the summands as follows:

$$\sum_{k \in \mathcal{K}} (a_k + b_K) = \sum_{k \in \mathcal{K}} a_k + \sum_{k \in \mathcal{K}} b_k$$

Example.

$$\begin{aligned}
\sum_{k=1}^{n} (k + k^2) &= (1 + 1^2) + (2 + 2^2) + \cdots + (n + n^2) \\
&= (1 + 2 + \cdots + n) + (1^2 + 2^2 + \cdots + n^2) \\
&= \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2
\end{aligned}$$

2.2 Close Form Formula

Close form formula is the formula that does not have the summation index k for a sum by simply writing it out explicitly. For example,

$$\sum_{k=1}^{n}(a_k - a_{k-1})$$

By expanding the sum, we have

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n - a_0$$

By cancelling the terms, we get $a_n - a_0$, which is the close form formula for the summation $\sum_{k=1}^{n}(a_k - a_{k-1})$.

2.3 Perturbation Method

It could be difficult to derive the close form formula for some summation. Therefore, we can use the perturbation method.

For summation

$$S_n = \sum_{k=1}^{n} a_k,$$

we can split off the first term and the last term, then rewrite it as

$$a_1 + \sum_{k=2}^{n+1} a_k = S_{n+1} = \sum_{k=1}^{n} a_k + a_{n+1}$$

Example (Geometric Sum). Let x be any number. Consider the sum

$$S_n = \sum_{k=1}^{n} x^k$$

$$x + \sum_{k=2}^{n+1} x^k = S_{n+1} = \sum_{k=1}^{n} x^k + x^{n+1}$$

Observe that

$$\sum_{k=2}^{n+1} x^k = x^2 + x^3 + \cdots + x^{n+1} = x(x + x^2 + \cdots + x^n) = xS_n$$

By substitution, we have

$$x + xS_n = S_n + x^{n+1}$$

If $x \neq 1$, then we can solve for S_n to get

$$S_n = \frac{x(1 - x^n)}{1 - x}$$

This summation is also called geometric sum.

By applying the perturbation method, we can find the close form formula for some common summation. Another example is Quadratic Series.

Example (Quadratic Series). By applying perturbation method to the sum

$$S_n = \sum_{k=1}^{n} k^2,$$

we have

$$1 + \sum_{k=2}^{n+1} k^2 = S_{n+1} = \sum_{k=1}^{n} k^2 + (n+1)^2$$

Let $j = k - 1$,

$$\sum_{k=2}^{n+1} k^2 = \sum_{j=1}^{n} (j+1)^2$$

$$\begin{aligned}
\sum_{j=1}^{n} (j+1)^2 &= \sum_{j=1}^{n} (j^2 + 2j + 1) \\
&= \sum_{j=1}^{n} j^2 + 2\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\
&= S_n + 2\sum_{j=1}^{n} j + n
\end{aligned}$$

Then, we have

$$\begin{aligned}
1 + \sum_{k=2}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\
1 + S_n + 2\sum_{j=1}^{n} j + n &= S_n + (n+1)^2 \\
\sum_{j=1}^{n} j &= \frac{n(n+1)}{2}
\end{aligned}$$

However, we obtain the Euler's trick here instead. Thus, we may apply the perturbation method to another sum.

$$C_n = \sum_{k=1}^{n} k^3 \Rightarrow 1 + \sum_{k=2}^{n+1} k^3 = C_{n+1} = \sum_{k=1}^{n} k^3 + (n+1)^3$$

$$\begin{aligned}
\sum_{k=2}^{n+1} k^3 &= \sum_{j=1}^{n} (j+1)^3 \quad \text{(By applying } j = k - 1\text{)} \\
&= \sum_{j=1}^{n} j^3 + 3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\
&= C_n + 3S_n + \frac{3n(n+1)}{2} + n
\end{aligned}$$

By substitution, we get

$$\begin{aligned}
1 + C_n + 3S_n + \frac{3n(n+1)}{2} + n &= C_n + (n+1)^3 \\
2 + 6S_n + 3n(n+1) + 2n &= 2(n+1)^3 \\
S_n &= \frac{2(n+1)^3 - 2n - 2 - 3n(n+1)}{6} \\
S_n &= \frac{(n+1)(2(n+1)^2 - 2 - 3n)}{6} \\
S_n &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Chapter 3

Recurrences

Chapter 4

Asymptotics

Chapter 5

Set Theory and Counting Principle

Chapter 6

Binomial Coefficients

6.1 Introduction

In this section, we introduce Binomial Coefficients.

6.1.1 Combinations

As it is introduced before,

Definition 6.1.1. An r-combination of the n-element ground set S_0 is an **unordered selection** of r elements from S_0.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

6.1.2 Permutation

Again, as it is introduced before,

Definition 6.1.2. An r-permutation of the n-element ground set S_0 is an **ordered selection** of r elements from S_0.

$$P(n,r) = \frac{n!}{(n-r)!}$$

6.1.3 Binomial Identities

Proposition 6.1.1. For any integers $m, r \geq 0$ with $0 \leq r \leq n$,

$$\binom{n}{r} = \binom{n}{n-r}$$

We have two ways to prove this proposition, namely Algebraic Proof and Combinatorial Proof.

Algebraic Proof.

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

■

Combinatorial Proof. Both side of the identity are supposed to be two different ways of solving a counting problem. We can define the counting problem as counting the number of different unordered selections of r elements from an n-element ground set. Then, we can define the RHS as the selecting number to be excluded. Then this identity holds. ■

6.1.4 Pascal's Identity

Theorem 6.1.1. For any integers $n,\ r \geq 0$ with $1 \leq r \leq n-1$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ OR } \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Again, we can prove this theorem by two ways.

Algebraic Proof.

$$\begin{aligned} RHS &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!} \\ &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{1}{r} + \frac{1}{n-r}\right) \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{n}{r(n-r)}\right) \\ &= \frac{n!}{r!(n-r)!} \\ &= \binom{n}{r} \end{aligned}$$

■

Combinatorial Proof. Recall that $\binom{n}{k}$ equals the number of subsets with k elements from a set with n elements. Suppose one particular element is uniquely labeled X in a set with n elements.

To construct a subset of k elements containing X, include X and choose $k-1$ elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k-1}$ such subsets.

To construct a subset of k elements **not** containing X, choose k elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k}$ such subsets.

Every subset of k elements either contains X or not. The total number of subsets with k elements in a set of n elements is the sum of the number of subsets containing X and the number of subsets that do not contain X, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

This equals $\binom{n}{k}$. ■

Appendix

Appendix A

Additional Proofs