ENGG2760 Probability for Engineers

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Abstract

This is a note for **ENGG2760 - Probability for Engineers** for self-revision purpose ONLY. Some contents are taken from lecture notes and reference book.
Mistakes might be found. So please feel free to point out any mistakes.

Contents

Chapter 1

Probability and Counting

1.1 Introduction

We will start with some basic definitions.

Definition 1.1.1 (Sample Space). The sample space Ω is the set of all possible outcomes

For example, when flipping three coins, we have $2^3 = 8$ outcomes:

$$\Omega = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}$$

Definition 1.1.2 (Event). An event is a subset of the sample space.

Following the above example, if A is the event that at least two heads occur, we have:

$$A = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\}$$

Definition 1.1.3. The probability of an event is the sum of the probability of its outcomes.

- Probabilities are non-negative.
- Probabilities add up to one.

Again from the above example, we see that the probability of each event is equal to $\frac{1}{8}$, and they can be summed up to 1.

Definition 1.1.4. The probability of an event is the sum of the probabilities of its outcomes.

For event A, the probability would be

$$\mathbb{P}(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Proposition 1.1.1 (Uniform Probability Law). If the outcomes in Ω are equally likely, then the probability of event A will be

$$\mathbb{P}(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \Omega} = \frac{|A|}{|\Omega|}$$

Remark. It can only be used when every outcome is equally likely.

For event A, the probability would be

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$$

Example. We roll two dice. Which of the following outcome is more likely for the sum of the two dice?

1. 11
2. 12
3. equally likely

Solution: For the sum to be 11, we can have (5, 6) and (6, 5). However, for the sum to be 12, we can only have (6, 6). Therefore, for $|\Omega| = 6^2 = 36$,

$$\mathbb{P}(11) = \frac{2}{36}, \quad \mathbb{P}(12) = \frac{1}{36}$$

Therefore, the sum of 11 would be more likely to occur.

1.2 Permutation and Combination

1.2.1 Counting via Product Rule

Proposition 1.2.1 (Product Rule)**.** Suppose there are n possible outcomes for Experiment 1 and m possible outcomes for Experiment 2, where the two experiments are independent. Then, there are $m \times n$ possible outcomes for the two experiments.

For example, when flipping three coins, each of them has two possible outcomes. Therefore, there are in total $2 \times 2 \times 2 = 8$ possible outcomes.

We can then generalize this rule for cases that the outcomes of experiment 1 may affect the outcomes of experiment 2.

Proposition 1.2.2 (Generalized Product Rule)**.** Suppose that

- There are n possible outcomes for Experiment 1.
- For every outcome of Experiment 1, there are m possible outcomes for Experiment 2.

Then, there are $m \times n$ possible outcomes for the two experiments.

For example, when finding all possible outcomes for rolling two dice with different values, the outcomes of the first experiment, i.e. rolling the first die, would be 6. The outcomes of the second experiment, i.e. rolling the second die, would be 5 (since we need to exclude the outcome of the first die). Then, there are in total $6 \times 5 = 30$ possible outcomes.

Example. We roll two dice. What is the probability that they come out with different values?

Solution: Let A be the desired event. Then we have

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{6 \times 5}{6 \times 6} = \frac{5}{6}$$

Example. We roll two dice. What is the probability that the sum of dice equals 7? What is the probability that the sum of dice is an odd number?

Solution: Let A be the event that the sum of dice equals 7. Then we have

$$A = \{(1,6),(2,5),\cdots,(6,1)\}$$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{6}{6^2} = \frac{1}{6}$$

Let B be the event that the sum of dice is an odd number. Then we have

$$B = \{(1,2),(1,4),\cdots,(6,5)\}, \quad |B| = 6 \times 3,$$

where for each number in the first die, there will be exactly three numbers in the second die that can be added up to an odd number. Thus,

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{6 \times 3}{6^2} = \frac{1}{2}$$

Example. We again roll two dice. What is the probability that the first die is bigger than the second die?

Solution: In this case, we cannot use generalized product rule since for every outcome in the first experiment, there will be a different outcome in the second experiment. Let A be the desired event. Then we have

$$A = \{(2,1),(3,1),\cdots,(6,5)\}$$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{15}{6^2} = \frac{5}{12}$$

1.2.2 Permutation

Definition 1.2.1 (Permutation)**.** A permutation of n different objects is an arrangement of the objects into an ordered sequence (order matters).

Proposition 1.2.3. For n different objects, there exists $n!$ different permutations:

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

Example. We roll six dice. How many ways are there for the six dice to have different values? What is the probability of that event?

Solution: Let A be the desired event. Then we have

$$|A| = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720, \quad \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{6!}{6^6}$$

Example (Birthday Paradox)**.** Suppose there are n people in a room. We assume that a year only has 365 days, and that every day is equally likely to be the birthday of a person. What is the probability that at least two people have the same birthday? Here we assume that $n < 365$.

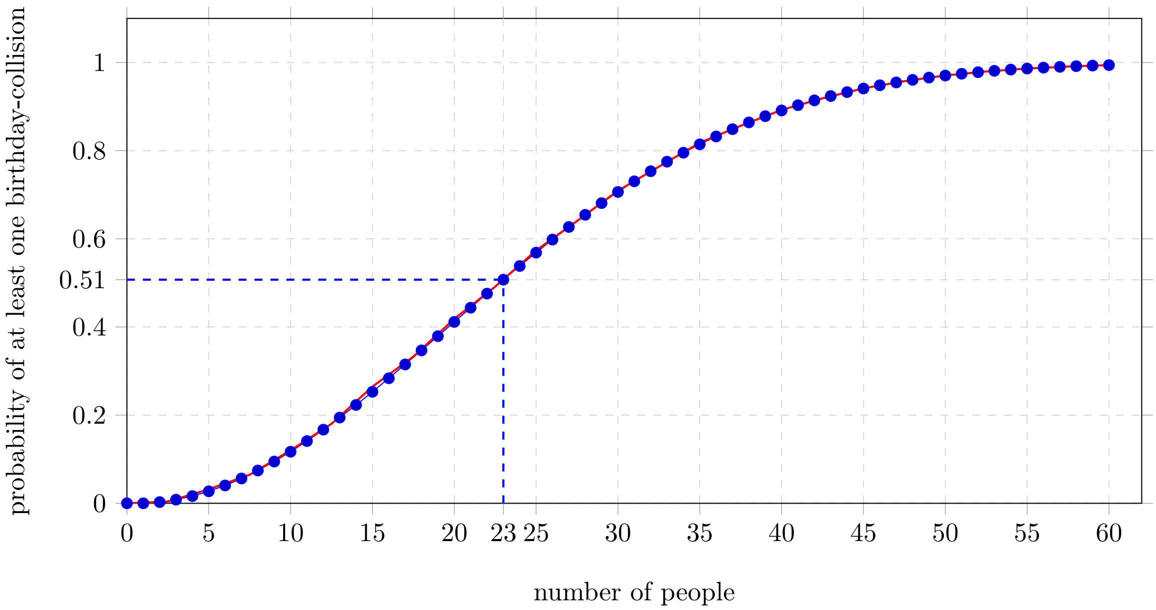
For sample space S we have the set of all possible sequences of n birthday, the $|S| = 365^n$.

Let T be the event in which at least two birthdays are the same. Then we have

$$\mathbb{P}(T) = 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n},$$

where the term $(365 \times 364 \times \cdots \times (365 - n + 1))$ is to count the possible outcomes for the event that all birthdays are distinct.

Birthday paradox could be visualized as below:



Adapted from MartinThoma

1.2.3 Binomial Coefficient

Proposition 1.2.4 (Binomial Coefficient or "n-Choose-k"). Given a set S of size n, the number of subsets of size k will be

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It can also be understood as the number of possible arrangements of k objects of Type A and $n-k$ objects of Type B into an ordered sequence.

Example. A box contains 8 red balls and 2 blue balls. You draw 2 balls at random(without replacement). What is the probability that the two balls have different colors?

Solution: Let A be the desired event.

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{8}{1}\binom{2}{1}}{\binom{10}{2}} = \frac{16}{45}$$

Proposition 1.2.5 (Multinomial Coefficient). For a set S of size n, the number of partitioning of the set to partitions of size $k_1, k_2, \cdots, k_t$ (noted that $n = k_1 + k_2 + \cdots + k_t$) will be

$$\binom{n}{k_1, k_2, \cdots, k_t} = \frac{n!}{k_1! k_2! \cdots k_t!}$$

It can also be understood as the number of possible permutations of k_1 objects of Type 1, k_2 objects of Type 2, ..., and k_t objects of Type t.

Chapter 2

Probability Models and Axioms

2.1 Basic Definitions

We will introduce some definitions here as well.

Definition 2.1.1 (Complement)**.** The complement of event A (denoted by A^c) is the opposite event of A. In other words, A^c happens if and only if A does not happen.

Again, when flipping three coins, we have the following sample space:

$$\Omega = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}.$$

Let A be the event that at least two heads occur. Then for A^c, we have:

$$A^c = \{\text{TTT}, \text{HTT}, \text{THT}, \text{TTH}\}$$

Definition 2.1.2 (Intersection of Events)**.** The intersection of events happens when all the events occur. We denote this intersection of event A and B with $A \cap B$.

Let B be the event that no consecutive heads occurs. Then, for $A \cap B$, we have the event that at least two heads and no consecutive heads occur.

$$A \cap B = \{\text{HTH}\}$$

Definition 2.1.3 (Union of Events)**.** The union of events happens when at least one of the events occur. We denote the union of events A and B with $A \cup B$.

For example, for $A \cup B$ in the above example, we have

$$A \cup B = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}$$

Definition 2.1.4 (Disjoint Events)**.** We call event $A_1, A_2, \cdots$ disjoint events (or mutually exclusive events) if the intersection of every two events $A_i, A_j (i \neq j)$ is the null event:

$$\forall i \neq j : \quad A_i \cap A_j = \varnothing$$

Let C be the event that at least three heads occur. Then

$$B \cap C = \varnothing.$$

2.2 Probability Axioms

Definition 2.2.1 (Axioms of Probability). A probability assignment $\mathcal{P}$ to sample space Ω should satisfy the following three axioms:

1. For every event A, $0 \leq \mathbb{P}(A)$;
2. $\mathbb{P}(\Omega) = 1$;
3. If event $A_1, A_2, \cdots$ are disjoint, $\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$

Follow these axioms, and we can prove most of the rules for probability calculation.

2.3 Rules for Probability Calculation

Proposition 2.3.1 (Complement Rule). For every event E and its complement E^c:

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$$

Proposition 2.3.2 (Difference Rule). If event E, F satisfy $E \subseteq F$, then:

$$\mathbb{P}(F \cap E^c) = \mathbb{P}(F) - \mathbb{P}(E)$$

Remark. As a result, if $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$

Proof.

$$\begin{aligned} \mathbb{P}(F \cap E^c) &= \mathbb{P}(F) - \mathbb{P}(E) \\ \mathbb{P}(F) &= \mathbb{P}(F \cap E^c) + \mathbb{P}(E) \end{aligned}$$

Since $(F \cap E^c) \cap E = F \cap (E^c \cap E) = F \cap \varnothing = \varnothing$, $\mathbb{P}(F \cap E^c) + \mathbb{P}(E) \Rightarrow (F \cap E^c) \cup E$

$$\begin{aligned} (F \cap E^c) \cup E &= (F \cup E) \cap (E^c \cup E) \\ (F \cap E^c) \cup E &= (F \cup E) \cap \Omega \\ (F \cap E^c) \cup E &= F \cup E \\ (F \cap E^c) \cup E &= F \text{ (for } E \subseteq F) \end{aligned}$$

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Proposition 2.3.3 (Inclusion-Exclusion Principle). For events E, F:

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$

Remark. We can generalize the principle to more than two events. For example,

$$\mathbb{P}(E \cup F \cup G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G) - \mathbb{P}(E \cap F) - \mathbb{P}(E \cap G) - \mathbb{P}(F \cap G) + \mathbb{P}(E \cap F \cap G)$$

Example. In a city, 10% of the people are rich, 5% are famous, and 3% are both rich and famous. For a randomly-selected person in the city, find the probability for the following Events.

Here we let R be the event that the person is rich, F be the event that the person is famous,

1. The person is not rich.

$$\mathbb{P}(R^c) = 1 - \mathbb{P}(R) = 1 - 0.1 = 0.9$$

2. The person is not rich but is famous.

$$\mathbb{P}(R^c \cap F) = \mathbb{P}(F) - \mathbb{P}(F \cap R) = 0.05 - 0.03 = 0.02$$

3. The person is neither rich nor famous.

$$\mathbb{P}(F^c \cap R^c) = 1 - \mathbb{P}(F \cup R) = 1 - \mathbb{P}(F) - \mathbb{P}(R) + \mathbb{P}(F \cap R) = 1 - 0.05 - 0.1 + 0.03 = 0.88$$

Chapter 3

Conditional Probability and Bayes' Rule

3.1 Conditional Probability

Let's begin with an example

Example. We toss 3 coins. You win if at least two heads come out. What is the probability of winning?

Solution:

$$\Omega = \{\text{HHH}, \text{HHT}, \cdots, \text{TTT}\} \quad \Rightarrow \quad \mathbb{P}(W) = \frac{|W|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$$

However, if it is given that the first coin was tossed and came out head. How does this affect your chances of winning? Since the sample space now changes due to the condition that the first toss is heads, the way we find the probability differs.

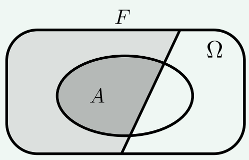
$$\Omega' = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}\}, \quad W' = \{\text{HHH}, \text{HHT}, \text{HTH}\}$$

This give the probability $\frac{3}{4}$.

3.1.1 Conditional Probability

Definition 3.1.1 (Conditional Probability).

The Conditional Probability $\mathbb{P}(A|F)$ represents the probability of event A assuming (or given) that event F happened.



Remark. All the outcomes of Ω and event A in F should be excluded in the calculation.

The conditional probability of A with respect to reduced sample space F is given by the formula:

$$\mathbb{P}(A|F) = \frac{\mathbb{P}(A \cap F)}{\mathbb{P}(F)}$$

Example. You roll two dice. You win if the sum of the outcomes is 8. If the first die toss is a 4, should you be happy?

Solution: Considering the initial case, we have:

$$\mathbb{P}(W) = \frac{|W|}{|\Omega|} = \frac{5}{36}$$

Let D be the event that the first toss is a 4. Given D, we have:

$$\mathbb{P}(W|D) = \frac{\mathbb{P}(W \cap D)}{\mathbb{P}(D)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

Therefore, the probability of winning increases.

Example. For the same game, you win if the sum of the outcomes is 7. The first toss is a 4. Should you be happy?

Solution: Considering the initial case, we have:

$$\mathbb{P}(W) = \frac{|W|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

Let D be the event that the first toss is a 4. Given D, we have:

$$\mathbb{P}(W|D) = \frac{\mathbb{P}(W \cap D)}{\mathbb{P}(D)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

Therefore, the probability does not change.

3.1.2 Properties of Conditional Probability

Proposition 3.1.1 (Properties of Conditional Probability). Conditional Probability $\mathbb{P}(\cdot|D)$ are probabilities over reduced sample space F and satisfy probability axioms:

1. For every A, $\mathbb{P}(A|F) \geq 0$
2. $\mathbb{P}(F|F) = 1$
3. For disjoint events A, B: $\mathbb{P}(A \cup B|F) = \mathbb{P}(A|F) + \mathbb{P}(B|F)$

Remark. There is no assumption about A or B are subsets of F.

We can then generalize the conditional probability using Uniform Probability Law. Under equally-likely outcomes in F:

$$\mathbb{P}(A|F) = \frac{\text{Number of outcomes in } A \cap F}{\text{Number of outcomes in } F} = \frac{|A \cap F|}{|F|}$$

Example.

There are three cards with Black/Black, Red/Red and Red/Black sides. We randomly draw one card and observe that one of its sides is Black.
What is the probability that the card's other side is also Black?

Solution: Let B be the event that the first drawn color is black, and E be the event that the second drawn color is also black. Then we have:

$$\Omega = \{1F, 1B, 2F, 2B, 3F, 3B\}$$

$$\mathbb{P}(E|B) = \frac{|E \cap B|}{|B|} = \frac{2}{3}$$

3.1.3 The Multiplication Rule

Proposition 3.1.2 (The Multiplication Rule)**.** For events E_1, E_2 we can write the probability of their intersection $E_1 \cap E_2$ as

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2|E_1)$$

In general, for every $E_1, E_2, \cdots, E_n$, the multiplication rule says

$$\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2|E_1)\mathbb{P}(E_3|E_1 \cap E_2)\cdots\mathbb{P}(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1})$$

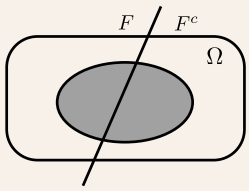
Example. A box contains 5 red balls and 15 blue balls. We randomly draw three balls from the box (without replacement). What is the probability that the balls are all red?

Solution: Let R_i be the event that the i-th ball being drawn is red.

Then we have

$$\mathbb{P}(R_1 \cap R_2 \cap R_3) = \mathbb{P}(R_1)\mathbb{P}(R_1|R_2)\mathbb{P}(R_3|R_1 \cap R_2) = \frac{5}{20} \times \frac{4}{19} \times \frac{3}{18} = \frac{1}{114}$$

Theorem 3.1.1 (Total Probability Theorem)**.**



For every event E and F and its complement F^c,

$$\begin{aligned}\mathbb{P}(E) &= \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)\\ &= \mathbb{P}(E|F)\mathbb{P}(F) + \mathbb{P}(E|F^c)\mathbb{P}(F^c)\end{aligned}$$

More generally, if events $F_1, F_2, \cdots, F_n$ partition Ω (disjoint events and $F_1 \cup F_2 \cup \cdots \cup F_n = \Omega$), then the total probability theorem says

$$\mathbb{P}(E) = \mathbb{P}(E|F_1)\mathbb{P}(F_1) + \mathbb{P}(E|F_2)\mathbb{P}(F_2) + \cdots + \mathbb{P}(E|F_n)\mathbb{P}(F_n)$$

Example. A box contains 5 red balls and 15 blue balls. We randomly draw two balls from the box (without replacement). What is the probability that the balls have different colors?

Solution: Let R be the event that the first ball is red, E be the event that the balls have different colors. Then we have

$$\mathbb{P}(E) = \mathbb{P}(E|R)\mathbb{P}(R) + \mathbb{P}(E|R^c)\mathbb{P}(R^c) = \frac{15}{19} \times \frac{5}{20} + \frac{5}{19} \times \frac{15}{20} = \frac{15}{38}$$

Example. For the situation that a group of students answered one multiple choice question where there are 4 options. What is the probability that a student knows the answer to the question?

Solution: Here we define event K as student knows the answer to the question, and event C as student correctly answers the question. Then, the total probability theorem says

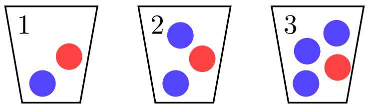
$$\begin{aligned}\mathbb{P}(C) &= \mathbb{P}(C|K)\mathbb{P}(K) + \mathbb{P}(C|K^c)\mathbb{P}(K^c)\\ &= 1 \times \mathbb{P}(K) + \frac{1}{4} \times (1 - \mathbb{P}(K))\\ &= \frac{3}{4}\mathbb{P}(K) + \frac{1}{4}\end{aligned}$$

This gives

$$\mathbb{P}(K) = \frac{4\mathbb{P}(C) - 1}{3}$$

3.2 Bayes' Rule

We choose a cup at random and then a random ball from that cup. The selected ball is red. Which cup do you guess the ball came from?



This is an example of cause and effect. Here, the cause is the cup number and the effect is the ball's color. Choosing different cups leads to different probabilities of choosing the color of balls.

Theorem 3.2.1 (Bayes' Rule)**.** Consider events C and E. Then,

$$\mathbb{P}(C|E) = \frac{\mathbb{P}(E|C)\mathbb{P}(C)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|C)\mathbb{P}(C)}{\mathbb{P}(E|C)\mathbb{P}(C) + \mathbb{P}(E|C^c)\mathbb{P}(C^c)}$$

More generally, if $C_1, C_2, \cdots, C_n$ partition the set of possible causes S,

$$\mathbb{P}(C_1|E) = \frac{\mathbb{P}(E|C_1)\mathbb{P}(C_1)}{\mathbb{P}(E|C_1)\mathbb{P}(C_1) + \mathbb{P}(E|C_2)\mathbb{P}(C_2) + \cdots + \mathbb{P}(E|C_n)\mathbb{P}(C_n)}$$

Let us revisit the example above. Let R represent the event that a red ball is drawn, and let C_i denote the cup from which the red ball is drawn, where $i = 1, 2, 3$. We can then calculate the probability of each cup being the source of the red ball, given that the ball is red.

$$\mathbb{P}(C_i|R) = \frac{\mathbb{P}(R|C_i)\mathbb{P}(C_i)}{\mathbb{P}(R|C_1)\mathbb{P}(C_1) + \mathbb{P}(R|C_2)\mathbb{P}(C_2) + \mathbb{P}(R|C_3)\mathbb{P}(C_3)}$$

For sample space, we have:

$$\Omega = \{1B, 1R, 2B_1, 2B_2, 2R, 3B_1, 3B_2, 3B_3, 3R\} \quad \text{(not equally likely)}$$

	Cup 1	Cup 2	Cup 3
$\mathbb{P}(C_i)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbb{P}(R\|C_i)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

For $\mathbb{P}(R)$, by using total probability theorem, we have

$$\mathbb{P}(R) = \mathbb{P}(R|C_1)\mathbb{P}(C_1) + \mathbb{P}(R|C_2)\mathbb{P}(C_2) + \mathbb{P}(R|C_3)\mathbb{P}(C_3) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} = \frac{13}{36}$$

Then we have:

$$\mathbb{P}(C_1|R) = \frac{\mathbb{P}(R|C_1)\mathbb{P}(C_1)}{\mathbb{P}(R)} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{13}{36}} = \frac{6}{13}$$

$$\mathbb{P}(C_2|R) = \frac{\mathbb{P}(R|C_2)\mathbb{P}(C_2)}{\mathbb{P}(R)} = \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{13}{36}} = \frac{4}{13}$$

$$\mathbb{P}(C_3|R) = \frac{\mathbb{P}(R|C_3)\mathbb{P}(C_3)}{\mathbb{P}(R)} = \frac{\frac{1}{4} \times \frac{1}{3}}{\frac{13}{36}} = \frac{3}{13}$$

Example. Two classes take place in the same academic building. Class A has 100 students from whom 20% are female, Class B has 10 students from whom 80% are female.

Now we see a female student in this building. What is the probability that the student is from Class A?

Solution:

$$
\begin{aligned}
\mathbb{P}(A|F) &= \frac{\mathbb{P}(F|A)\mathbb{P}(A)}{\mathbb{P}(F|A)\mathbb{P}(A) + \mathbb{P}(F|B)\mathbb{P}(B)} \\
&= \frac{20\% \times \frac{10}{11}}{20\% \times \frac{10}{11} + 80\% \times \frac{1}{11}} \\
&= \frac{5}{7}
\end{aligned}
$$

This example demonstrates a counterintuitive result. Although the proportion of females in Class B is greater than that in Class A, the probability of a female student belonging to Class A is higher. This highlights the importance of using Bayes' rule to update our beliefs, as our initial hypotheses may not always align with the actual probabilities.

To summarize this chapter, we see that conditional probability will be a very powerful tool when

1. the studied environment includes causes and effect. This is especially useful for calculating the probability of a cause under an observation of the effect.

2. we want to calculate an ordinary probability and conditioning on the right event can simplify the description of the sample space.

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