ENGG2440 Discrete mathematics for engineers

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Abstract

This is a note for ENGG2440 - Discrete mathematics for engineers for self-revision and concept understanding ONLY. Some contents are taken from lecture notes as well as only reference book.
Mistakes might be found. So please feel free to point out any mistakes.

Template of this note is adapted from https://github.com/sleepymalc.

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Chapter 1

Mathematical Induction

1.1 Introduction

In mathematics, there are some basic proof techniques that we can apply, including direct proof, proof by induction, proof by contradiction, and proof by contraposition. For most of these proving methods, you won't be learning their reasons or applications, but you will still use them in some simple proving questions. In this chapter, we will mainly discuss mathematical induction.

Definition 1.1.1 (Proposition). A **proposition** is a statement that is either true or false.

Definition 1.1.2 (Predicate). A **predicate** is a proposition whose truth depends on one or more variables.

1.2 Mathematical Induction

An analogy of the principle of mathematical induction is the game of dominoes. Suppose the dominoes are lined up properly, so that when one falls, the successive one will also fall. Now by pushing the first domino, the second will fall; when the second falls, the third will fall; and so on. We can see that all dominoes will ultimately fall.

The key point is only two steps:

1. the first domino falls;
2. when a domino falls, the next domino falls.

We use the above principle of Mathematical Induction to prove.

Process:

1. Let $P(n)$ be a predicate.
2. (Base Case) Show that $P(1)$ is true.
3. (Inductive Steps) Show that for $n = 1, 2, \ldots$, if $P(n)$ is true, then $P(n+1)$ is true.

Example.

$$P(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

1. Base Case

We need to show that $P(1)$ is true.

$$1 = \frac{(1)(1+1)}{2},$$

which is obviously true.

2. Inductive Step

For inductive hypothesis, we can assume

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Now, to show that $P(n+1)$ is true,

$$\begin{aligned} L.H.S. &= 1 + 2 + \cdots + n + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= R.H.S \end{aligned},$$

which shows that $P(n+1)$ is also true.

Hence, by the principle of MI, we can conclude that $P(n)$ is true for all integers $n \geq 1$.

Exercise. Show that for any integer $n \geq 1$, $n^3 - n$ is divisible by 3.

> **Note.** In inductive step, consider putting a constant q as $3q$ is divisible by 3.

Exercise. Prove that $n^3 < 2^n$ for all integers $n \geq 10$.

> **Note.** Consider bonding the lower order terms in terms of n^3.

1.3 Strong Mathematical Induction

As the name suggests, the method of induction used in this section is "stronger". This is because assuming only that $P(n)$ is true may be too restrictive, i.e., insufficient to prove the predicate. Thus, in the inductive step, you may show that $P(1), P(2), \cdots, P(n)$ are true, and then prove that $P(n+1)$ is true.

Example. The Fibonacci sequence is a sequence of number defined via the following recursion:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2$$
$$F_0 = 0; F_1 = 1$$

Prove that

$$P(n) : F_n \leq \phi^{n-1}, \text{where } \phi = \frac{1+\sqrt{5}}{2}$$

1. Base Case

$$F_1 = 1 \leq \phi^0 = 1$$
$$F_2 = 1 \leq \phi^1 \approx 1.618$$

Thus, $P(1)$ and $P(2)$ hold true, which means $P(3)$ also holds true.

2. Inductive Step

For inductive hypothesis, we assume

$$F_k \leq \phi^{k-1} \text{ for } k = 1, 2, \dots, n$$

Given the Fibonacci sequence

$$F_{n+1} = F_n + F_{n-1}$$

By the strong inductive hypothesis, we have

$$F_n \leq \phi^{n-1}, \quad F_{n-1} \leq \phi^{n-2}$$

From the definition of ϕ we have $\phi^2 = 1 + \phi$

Hence, we obtain

$$F_{n+1} \leq \phi^{n-1} + \phi^{n-2} = \phi^{n-2}(1 + \phi) = \phi^n$$

Chapter 2

Summation Techniques

2.1 Summation

When summing numbers with certain patterns, we can use summation notation. For example,

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

2.1.1 Distributive Law

Let c be a constant. Then, we can take c out of the summation:

$$\sum_{k \in \mathcal{K}} c a_k = c \sum_{k \in \mathcal{K}} a_k$$

Example.

$$\sum_{k=1}^{n} 2k = 2(1) + 2(2) + 2(3) + \cdots + 2(n) = 2(1 + 2 + 3 + \cdots + n) = 2 \sum_{k=1}^{n} k$$

2.1.2 Associative Law

We can split the summand as follows:

$$\sum_{k \in \mathcal{K}} (a_k + b_K) = \sum_{k \in \mathcal{K}} a_k + \sum_{k \in \mathcal{K}} b_k$$

Example.

$$\begin{aligned}
\sum_{k=1}^{n} (k + k^2) &= (1 + 1^2) + (2 + 2^2) + \cdots + (n + n^2) \\
&= (1 + 2 + \cdots + n) + (1^2 + 2^2 + \cdots + n^2) \\
&= \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2
\end{aligned}$$

2.2 Close Form Formula

Close form formula is the formula that does not have the summation index k for a sum by simply writing it out explicitly. For example,

$$\sum_{k=1}^{n}(a_k - a_{k-1})$$

By expanding the sum, we have

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n - a_0$$

By cancelling the terms, we get $a_n - a_0$, which is the close form formula for the summation $\sum_{k=1}^{n}(a_k - a_{k-1})$.

2.3 Perturbation Method

It could be difficult to derive the close form formula for some summation. Therefore, we can use the perturbation method.

For summation

$$S_n = \sum_{k=1}^{n} a_k,$$

we can split off the first term and the last term, then rewrite it as

$$a_1 + \sum_{k=2}^{n+1} a_k = S_{n+1} = \sum_{k=1}^{n} a_k + a_{n+1}$$

Example (Geometric Sum). Let x be any number. Consider the sum

$$S_n = \sum_{k=1}^{n} x^k$$

$$x + \sum_{k=2}^{n+1} x^k = S_{n+1} = \sum_{k=1}^{n} x^k + x^{n+1}$$

Observe that

$$\sum_{k=2}^{n+1} x^k = x^2 + x^3 + \cdots + x^{n+1} = x(x + x^2 + \cdots + x^n) = xS_n$$

By substitution, we have

$$x + xS_n = S_n + x^{n+1}$$

If $x \neq 1$, then we can solve for S_n to get

$$S_n = \frac{x(1 - x^n)}{1 - x}$$

This summation is also called geometric sum.

By applying the perturbation method, we can find the close form formula for some common summation. Another example is Quadratic Series.

Example (Quadratic Series)**.** By applying perturbation method to the sum

$$S_n = \sum_{k=1}^{n} k^2,$$

we have

$$1 + \sum_{k=2}^{n+1} k^2 = S_{n+1} = \sum_{k=1}^{n} k^2 + (n+1)^2$$

Let $j = k - 1$,

$$\sum_{k=2}^{n+1} k^2 = \sum_{j=1}^{n} (j+1)^2$$

$$\begin{aligned}\sum_{j=1}^{n}(j+1)^2 &= \sum_{j=1}^{n}(j^2+2j+1) \\ &= \sum_{j=1}^{n} j^2 + 2\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\ &= S_n + 2\sum_{j=1}^{n} j + n\end{aligned}$$

Then, we have

$$\begin{aligned}1 + \sum_{k=2}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ 1 + S_n + 2\sum_{j=1}^{n} j + n &= S_n + (n+1)^2 \\ \sum_{j=1}^{n} j &= \frac{n(n+1)}{2}\end{aligned}$$

However, we obtain the Euler′s trick here instead. Thus, we may apply the perturbation method to another sum.

$$C_n = \sum_{k=1}^{n} k^3 \Rightarrow 1 + \sum_{k=2}^{n+1} k^3 = C_{n+1} = \sum_{k=1}^{n} k^3 + (n+1)^3$$

$$\begin{aligned}\sum_{k=2}^{n+1} k^3 &= \sum_{j=1}^{n}(j+1)^3 \quad \text{(By applying } j = k-1\text{)} \\ &= \sum_{j=1}^{n} j^3 + 3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\ &= C_n + 3S_n + \frac{3n(n+1)}{2} + n\end{aligned}$$

By substitution, we get

$$\begin{aligned}1 + C_n + 3S_n + \frac{3n(n+1)}{2} + n &= C_n + (n+1)^3 \\ 2 + 6S_n + 3n(n+1) + 2n &= 2(n+1)^3 \\ S_n &= \frac{2(n+1)^3 - 2n - 2 - 3n(n+1)}{6} \\ S_n &= \frac{(n+1)(2(n+1)^2 - 2 - 3n)}{6} \\ S_n &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

There are some shortcut expression that might be used without finding the closed form formula on your own.

Proposition 2.3.1 (Close form formula). (You can try the perturbation method to find the close form formula by yourself.)

- Geometric series
$$\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1}-1)}{r-1}, r \neq 1$$
- Euler′s trick
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
- Quadratic series
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
- Cubic series
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

2.4 Guess-and-Verify method

As the name suggests, we can often guess the closed-form formula. But how can we be certain it's correct? This is where mathematical induction comes in handy.

Example.
$$S_n = \sum_{k=1}^{n} k^2$$
Observe that S_n behaves like the sum of terms in a polynomial, allowing us to form an n-term polynomial. Since n^2 is the largest term in S_n, we conclude that $S_n \leq n^3$
$$S_n = a + bn + cn^2 + dn^3$$

$$\begin{aligned} 1 = S_1 &= a + b + c + d, \\ 5 = S_2 &= a + 2b + 4c + 8d, \\ 14 = S_3 &= a + 3b + 9c + 27d, \\ 30 = S_4 &= a + 4b + 16c + 64d \end{aligned}$$

Solving for above, we have $a = 0,\ b = \frac{1}{6},\ c = \frac{1}{2},\ d = \frac{1}{3}$.
$$S_n = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$$
After hypothesizing the closed-form formula, we need to use induction to verify its correctness. The steps are straightforward, and you can try proving it yourself. This process confirms that the formula is indeed the closed form for the summation.

2.5 Multiple Summation

All the summations above use only a single index. For example,

$$S_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2.$$

In this section, however, we introduce a summation with multiple indices. You can think of a single-index summation as summing over a 1D array. Extending this idea, a summation with two indices corresponds to summing over a 2D array. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

To sum up all the terms, we can use

$$r_j = a_{j1} + a_{j2} + \cdots + a_{jn} = \sum_{k=1}^{n} a_{jk}$$

Then, we can rewrite it as

$$S = \sum_{j=1}^{m} r_j = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}$$

We can also interchange the order of summation, which can be very useful for finding the closed-form formula. For the above summation, it is rather simple, we can simply do the interchange by

$$\sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} = \sum_{k=1}^{n} \sum_{j=1}^{m} a_{jk}$$

However, this does not work for all the summation.

Example. Considering

$$S = \sum_{j=1}^{n} \sum_{k=j}^{n} a_{jk}$$

To visualize that, we can again use matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$$

Then, we have

$$\begin{aligned} S = a_{11} + a_{12} + a_{13} + \cdots + a_{1n} \\ + a_{22} + a_{23} + \cdots + a_{2n} \\ \ddots \\ + a_{nn} \end{aligned}$$

Let

$$c_k = a_{1k} + a_{2k} + \cdots + a_{nk} = \sum_{j=1}^{n} a_{jk},$$

we get

$$S = \sum_{k=1}^{n} c_k = \sum_{k=1}^{n} \sum_{j=1}^{k} a_{jk}$$

Remark. Informally, form

$$S = \sum_{j=1}^{n} \sum_{k=j}^{n} a_{jk}$$

we have $1 \le j \le k \le n$, then we can simply interchange the order by

$$S = \sum_{k=1}^{n} \sum_{j=1}^{k} a_{jk}$$

Let's see another example, n-th harmonic number (H_n).

Example.

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

Again we can visualize it by using matrix

$$\begin{bmatrix} 1 & & & & \\ 1 & 1/2 & & & \\ 1 & 1/2 & 1/3 & & \\ & & & \ddots & \\ 1 & 1/2 & 1/3 & \cdots & 1/n \end{bmatrix}$$

$$S = \sum_{j=1}^{n} H_j = \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{1}{k}$$

Since $1 \le k \le j \le n$

$$S = \sum_{j=1}^{n} H_j = \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{1}{k} = \sum_{k=1}^{n} \sum_{j=k}^{n} \frac{1}{k}$$

Then we get

$$\begin{aligned} S &= \sum_{k=1}^{n} \sum_{j=k}^{n} \frac{1}{k} \\ &= \sum_{k=1}^{n} \frac{n-k+1}{k} \\ &= \sum_{k=1}^{n} \frac{n}{k} - \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \frac{1}{k} \\ &= (n+1)H_n - n \end{aligned}$$

Exercise. For any real number x, $\lfloor x \rfloor \le x \le \lfloor x \rfloor + 1$.

Find the closed form formula for

$$S_n = \sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor$$

where, for simplicity, we assume that n is a perfect square, i.e., $n = a^2$ for some integer $a \ge 1$.

Remark. Consider $\lfloor x \rfloor = \sum_{k=1}^{x} 1$, then how to do the order interchange such that we can get the following summation

$$S_n = \sum_{j=1}^{a} \sum_{k=j^2}^{a^2} 1$$

Let's look at the last example.

Example. Let $n \leq 1$ be an integer and $x \neq 1$ be a real number.

$$S = \sum_{j=1}^{n} \sum_{k=1}^{j} kx^j$$

By interchanging the summation orders yields

$$S = \sum_{k=1}^{n} \sum_{j=k}^{n} kx^j$$

Notice that for any integer k satisfying $1 \leq k \leq n$ it holds that

$$\sum_{j=k}^{n} x^j = \sum_{j=1}^{n} x^j - \sum_{j=1}^{k-1} x^j = \frac{x - x^{n+1}}{1-x} - \frac{x - x^k}{1-x} = \frac{x^k - x^{n+1}}{1-x}$$

Equivalently,

$$\sum_{j=k}^{n} x^j = \frac{x^k - x^{n+1}}{1-x} = x^{k-1} \cdot \frac{x - x^{n-k+2}}{1-x} = x^{k-1} \cdot \sum_{j=1}^{n-k+1} x^j$$

Then we have

$$\begin{aligned}
S &= \sum_{k=1}^{n} \sum_{j=k}^{n} kx^j \\
&= \sum_{k=1}^{n} k \sum_{j=k}^{n} x^j \\
&= \sum_{k=1}^{n} k \cdot \frac{x^k - x^{n+1}}{1-x} \\
&= \frac{1}{1-x} \cdot \sum_{k=1}^{n} kx^k - \frac{x^{n+1}}{1-x} \cdot \sum_{k=1}^{n} k \\
&= \frac{1}{1-x} \cdot \frac{x \cdot (nx^{n+1} - (n+1)x^n + 1)}{(x-1)^2} - \frac{x^{n+1}}{1-x} \cdot \frac{n(n+1)}{2}
\end{aligned}$$

Chapter 3

Recurrences

Sometimes recurrences are closely related to sums. Thus, if we are going to find the closed-form formula, we can express the sum as a recurrence, then the problem will be comparatively easier.

3.1 Homogeneous Recurrences

Definition 3.1.1 (Linear homogeneous recurrence)**.** A linear homogeneous recurrence relation of degree d with constant coefficients is a recurrence relation of the form

$$T(n) = a_1T(n-1) + a_2T(n-2) + \cdots + a_dT(n-d),$$

where $a_1, a_2, \cdots, a_k \in \mathbb{R}$ are given constants and $a_k \neq 0$

To solve for homogeneous recurrences with distinct root, we can use the following procedure:

1. Solve the characteristic equation to get root $r_1, \cdots, r_d$.
2. If the roots are all distinct, form, the candidate solution

$$T_0(n) = \theta_1 r_1^n + \theta_2 r_2^n + \cdots + \theta_d r_d^n$$

3. Use the initial conditions on $T(1), \ldots, T(d)$ to determine $\theta_1, \ldots, \theta_d$

Example.

$$\begin{cases} T(n) = T(n-1) + 2T(n-2) & \text{for } n \geq 2 \\ T(0) = 2, T(1) = 7 \end{cases}$$

Let $T(n) = x^n$. Then, for characteristic equation, we have

$$\begin{aligned} x^n &= x^{n-1} + 2x^{n-2} \\ x^2 &= x + 2 \quad \text{(characteristic equation)} \end{aligned}$$

The roots are $r_1 = 2, r_2 = -1$. Since they are distinct, we can form the candidate solution

$$T_0(n) = \theta_1 2^n + \theta_2(-1)^n.$$

By using the initial conditions, we have

$$\begin{aligned} 2 &= T_0(0) = \theta_1 + \theta_2 \\ 7 &= T_0(1) = 2\theta_1 - \theta_2 \end{aligned}$$

Solving above, we have $\theta_1 = 3, \theta_= -1$. Then, we have

$$T(n) = 3 \times 2^n - (-1)^n$$

However, if the root of the characteristic equation has a multiplicity $m \geq 1$, i.e., the root is repeated for m times, then we have $T(n) = n^{m-1}x^n$. Yet the procedures are the same as solving linear homogeneous recurrence with distinct roots.

Example.

$$\begin{cases} T(n) = 2T(n-1) - T(n-2) & \text{for} n \geq 2, \\ T(0) = 0, T(1) = 1 \end{cases}$$

Characteristic equation: $x^2 = 2x - 1$.

Solving above we have $r_1 = 1$ with multiplicity $m_1 = 2$. Then we have

$$T_0(n) = \theta_1(1)^n + n\theta_2(1)^n = \theta_1 + n\theta_2$$

By using initial conditions, we have

$$\begin{aligned} 0 &= T_0(0) = \theta \\ 1 &= T_0(1) = \theta_1 + \theta_2 \end{aligned}$$

Then, it follows that the solution to the recurrences is given by

$$T(n) = n$$

Let's see another example

Example.

$$\begin{cases} T(n) = 4T(n-1) - 5T(n-2) + 2T(n-3) & \text{for} n \geq 3, \\ T(0) = 0, T(1) = 1, T(2) = 3. \end{cases}$$

Characteristic equation: $x^3 = 4x^2 - 5x + 2$

This equation has two distinct roots, $r_1 = 1, r_2 = 2$. The multiplicity of r_1 is $m_1 = 2$. Hence,

$$T_0(n) = \theta_1(1)^n + n\theta_2(1)^n + \theta_3 2^n = \theta_1 + n\theta_2 + \theta_3 2^n$$

Using initial conditions, we have

$$\begin{aligned} 0 &= T_0(0) = \theta_1 + \theta_3 \\ 1 &= T_0(1) = \theta_1 + \theta_2 + 2\theta_3 \\ 3 &= T_0(2) = \theta_1 + 2\theta_2 + 4\theta_3 \end{aligned}$$

Solving for above, we have

$$T(n) = -1 + n \times 0 + 1 \times 2^n = -1 + 2^n$$

3.2 Non-homogeneous Recurrences

A recurrence relation of the form

$$T(n) = a_1T(n-1) + a_2T(n-2) + \cdots + a_dT(n-d) + f(n)$$

is called non-homogeneous recurrences.

To solve for non-homogeneous recurrences, we can use the following procedures:

1. Solve the associated linear homogeneous recurrence.

2. Find the particular solution $T_p(n)$ to the linear non-homogeneous recurrence by examining the function class of $f(n)$.

3. Form the candidate solution $T_0(n) = T_h(n) + T_p(n)$, and use the initial conditions to find the parameters in $T_h(n)$.

In general, to solve non-homogeneous recurrence, we can consider the following particular solutions:

$f(n)$	$T_p(n)$
s	x_0
n	$x_1 n + x_0$
n^2	$x_2 n^2 + x_1 n + x_0$
s^n	$x_0 s^n$
ns^n	$(x_1 n + x_0)s^n$

Let's see some examples

Example. Consider the recurrence

$$\begin{cases} T(n) = 2T(n-1) + 1 & \text{for } n \geq 1 \\ T(1) = 1 \end{cases}$$

Let $T_0(n) = T_h(n) + T_p(n)$, where $T_h(n) = 2T(n-1)$.

For $T_h(n)$, characteristic equation: $x = 2$, then we have $T_h(n) = \theta 2^n$.

Since $f(n) = 1$, let $T_p(n) = x$,

$$\begin{aligned} x &= 2x + 1 \\ x &= -1 \end{aligned}$$

Then, we have

$$T_0(n) = T_h(n) + T_p(n) = \theta 2^n - 1$$

Using the initial conditions, we have

$$1 = T_0(1) = 2\theta - 1$$

This gives $\theta = 1$. Hence, the solution to the recurrence is given by

$$T(n) = 2^n - 1$$

Example. Consider the recurrence

$$\begin{cases} T(n) = 5T(n-1) - 6T(n-2) + 7^n & \text{for} n \geq 2 \\ T(0) = 0, T(1) = 1. \end{cases}$$

Let $T_0(n) = T_h(n) + T_p(n)$, where $T_h(n) = 5T(n-1) - 6T(n-2)$.

For $T_h(n)$, characteristic equation: $x^2 = 5x - 6$, with $r_1 = 3, r_2 = 2$.

Hence, we have $T_h(n) = \theta_1 3^n + \theta_2 2^n$, which is the homogeneous solution.

Since $f(n) = 7^n$ and $s = 7$ is not a root of the characteristic equation. Let $T_p(n) = x_0 7^n$ (particular solution),

$$\begin{aligned} x_0 7^n &= 5x_0 7^{n-1} - 6x_0 7^{n-2} + 7^n \\ x_0 &= 5x_0 7^{-1} - 6x_0 7^{-2} + 1 \\ x_0 - \frac{5}{7}x_0 + \frac{6}{49}x_0 &= 1 \\ x_0 &= \frac{49}{20} \end{aligned}$$

Then, we have

$$T_0(n) = T_h(n) + T_p(n) = \theta_1 3^n + \theta_2 2^n + \frac{49}{20}7^n$$

Using the initial conditions, we have

$$\begin{aligned}0 &= T_0(0) = \theta_1 + \theta_2 + \frac{49}{20}\\1 &= T_0(1) = 3\theta_1 + 2\theta_2 + \frac{343}{20}\end{aligned}$$

This gives $\theta_1 = -\frac{225}{20}, \theta_2 = \frac{176}{20}$. Hence, the solution to the recurrence is given by

$$T(n) = -\frac{225}{20}3^n + \frac{176}{20}2^n + \frac{49}{20}7^n$$

Remark. Let $f(n) = s^n$, r_1 and r_2 be the roots of the characteristic equation. Then

- If $s \neq r_1$, $s \neq r_2$, then $T_p(n) = x_0 s^n$;
- If $s = r_1$, $r_1 \neq r_2$, then $T_p(n) = x_0 n s^n$;
- If $s = r_1 = r_2$, then $T_p(n) = x_0 n^2 s^n$;

where x_0 is constant to be determined in all cases.

Example. Consider the recurrence

$$\begin{cases}T(n) = 6T(n-1) - 9T(n-2) + 3^n & \text{for} n \geq 2\\T(0) = 0, T(1) = \frac{1}{2}.\end{cases}$$

Let $T_0(n) = T_h(n) + T_p(n)$, where $T_h(n) = 6T(n-1) - 9T(n-2)$.

For $T_h(n)$, characteristic equation: $x^2 = 6x - 9$, $r_1 = 3$ with multiplicity $m_1 = 2$.

Hence, we have $T_h(n) = \theta_1 3^n + n\theta_2 3^n$, which is the homogeneous solution.

Since $f(n) = 3^n$ and $s = 3$ is also a root of the characteristic equation. Let $T_p(n) = x_0 n^2 3^n$ (particular solution),

$$\begin{aligned}x_0 n^2 3^n &= 6x_0(n-1)^2 3^{n-1} - 9x_0(n-2)^2 3^{n-2} + 3^n\\9x_0 n^2 &= 18x_0(n-1)^2 - 9x_0(n-2)^2 + 9\\9x_0 n^2 &= 18x_0(n^2 - 2n + 1) - 9x_0(n^2 - 4n + 4) + 9\\9x_0 n^2 &= 9x_0 n^2 - 18x_0 + 9\\x_0 &= \frac{1}{2}\end{aligned}$$

Then, we have

$$T_0(n) = T_h(n) + T_p(n) = \theta_1 3^n + n\theta_2 3^n + \frac{1}{2}n^2 3^n$$

Using the initial conditions, we have

$$\begin{aligned}0 &= T_0(0) = \theta_1\\\frac{1}{2} &= T_0(1) = 3\theta_1 + 3\theta_2 + \frac{3}{2}\end{aligned}$$

This gives $\theta_1 = 0, \theta_2 = -\frac{1}{3}$. Hence, the solution to the recurrence is given by

$$T(n) = -\frac{1}{3}n3^n + \frac{1}{2}n^2 3^n$$

Chapter 4

Asymptotics

Asymptotic notation is a shorthand used to give a quick measure of the behavior of a function $f(n)$ as n grows large.

4.1 Big O

Big O is the most frequently used asymptotic notation. It is used to give an upper bound on the growth of a function, such as the running time of an algorithm.

Definition 4.1.1. We say that $f(x) = O(g(x))$ iff there exists a constant $c > 0$ and an $x_0 \geq 0$ such that

$$f(x) \leq cg(x) \quad \text{for all } x \geq x_0$$

Let's see an example

Example. Let $f(x) = x^2$ and $g(x) = x^3$.

By taking $c = 1$ and $x_0 = 1$, we can simply conclude that $x^2 = O(x^3)$.

However, to prove that **there is no constant c such that $x^2 \leq cx^3$ for all** $0 \leq x \leq 1$, we need to use proof by contradiction.

For $0 \leq x \leq 1$, we have $x^2 \geq x^3$. Therefore, if such a constant $c > 0$ exists, then we must have $c \geq 1$ such that $x^2 \leq cx^3$. However, whenever $0 \leq x \leq \frac{1}{c}$, we have

$$x^3 \geq \frac{1}{c}x^2 > x^3.$$

Since we get $x^3 > x^3$, by contradiction, we know that there is no constant c such that $x^2 \leq cx^3$ for all $0 \leq x \leq 1$

We can also use differentiation to show the upper bound of a function.

Example. Let $f(x) = \ln x$ and $g(x) = x$.

Observe that $f(1) = \ln 1 = 0 < 1 = g(1)$. Moreover, $f'(x) = \frac{1}{x}$ and $g'(x) = 1$ for all $x > 0$, which implies that $g'(x) > f'(x)$ for all $x > 1$. It is the same as saying that $g(x)$ grows faster than the function $f(x)$ because the slope of the former is larger than that of the latter. It follows that

$$\ln x \leq x \text{ for all } x \geq 1.$$

Hence, by taking $c = 1$ and $x_0 = 1$ we have $\ln x = O(x)$.

Exercise. Let $f(x) = x^2$ and $g(x) = 2^x$. Show that $f(x) = O(g(x))$.

4.2 Big Omega

As we use Big O notation to express upper bound, for lower bound, we have the "Big Omega" notation.

Definition 4.2.1. We say that $f(x) = \Omega(g(x))$ iff there exists a constant $c > 0$ and an $x_0 \geq 0$ such that
$$f(x) \geq cg(x) \quad \text{for all } x \geq x_0$$

Since Big O and Big Omega are essentially "mirror image" of one another, we have

Theorem 4.2.1 (Big O vs. Big Omega)**.**
$$f(x) = O(g(x)) \iff g(x) = \Omega(f(x)).$$

Proof. By definition of big O, $f(x) = O(g(x))$ means
$$\exists c_1 > 0, x_1 > 0 \text{ such that } f(x) \leq c_1 g(x) \quad \forall x \geq x_1,$$
which means the same as
$$\exists c_1 > 0, x_1 > 0 \text{ such that } g(x) \geq \frac{1}{c_1} f(x) \quad \forall x \geq x_1,$$
Hence, by taking $c_0 = \frac{1}{c_1} > 0$ and $x_0 = x_1 \geq 0$, we have $g(x) = \Omega(f(x))$ ■

4.3 Theta

Theta can be understood as the approximation of the function.

Definition 4.3.1. We say that $f(x) = \Theta(g(x))$ iff $f(x) = O(g(x))$ and $g(x) = O(f(x))$.

> **Remark.** It is important to note that $f(x) = \Theta(g(x))$ does not mean that $f(x) = g(x)$, it just means that
> $$\exists c_{1,2} > 0 \quad \text{such that} \quad c_1 g(x) \leq f(x) \leq c_2 g(x) \quad \forall x \geq x_0$$

4.4 Little O

Little O notation can be understood as the strict upper bound on the growth of a function.

Definition 4.4.1 (Little O notation)**.** For functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say f is asymptotically smaller than g, in symbols,
$$f(x) = o(g(x)) \quad \text{iff} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Example. For example, let $f(x) = x, g(x) = e^x - 1$. Because
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{e^x - 1} = 0.$$
Hence, we have $f(x) = o(g(x))$

Example. Let

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Is $f(n) = o(1)$?

Let $g(n) = 1$. Upon noting $f(n) = 1 + (-1)^n$, we see that

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} f(n).$$

However, this limit does not exist, as $f(n)$ fluctuates between 0 and 2. Thus, $f(n) \neq o(1)$.

On the other hand, if $f(n) = o(n)$? Here, let $g(n) = n$. Note that $0 \leq f(n) \leq 2$. It follows that

$$0 \leq \frac{f(n)}{g(n)} \leq \frac{2}{n}$$

for $n \geq 1$. By the sandwich theorem, we then obtain

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Thus, $f(n) = o(n)$.

4.5 Little Omega

Little Omega notation can be understood as the strict lower bound on the growth of a function.

Definition 4.5.1 (Little O notation)**.** For functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say f is asymptotically smaller than g, in symbols,

$$f(x) = \omega(g(x)) \quad \text{iff} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

4.6 Properties for Asymptotic Analysis

4.6.1 Rules for Asymptotic Analysis

- Transitivity

  If $f(n) = \Pi(g(n))$ and $g(n) = \Pi(h(n))$, then $f(n) = \Pi(h(n))$, where $\Pi = O, o, \Omega, \omega, \Theta$

- Rule of sums

  $$f(n) + g(n) = \Pi(max\{f(n), g(n)\}), \text{ where } \Pi = O, \Omega, \text{ or } \Theta.$$

- Rule of products

  If $f_1(n) = \Pi(g_1(n)), f_2(n) = \Pi(g_2(n))$, then $f_1(n)f_2(n) = \Pi(g_1(n)g_2(n))$, where $\Pi = O, o, \Omega, \omega, \Theta$.

- Transpose symmetry

  $$f(n) = O(g(n)) \text{ iff } g(n) = \Omega(f(n)).$$

- Transpose symmetry

  $$f(n) = o(g(n)) \text{ iff } g(n) = \omega(f(n)).$$

- Reflexivity

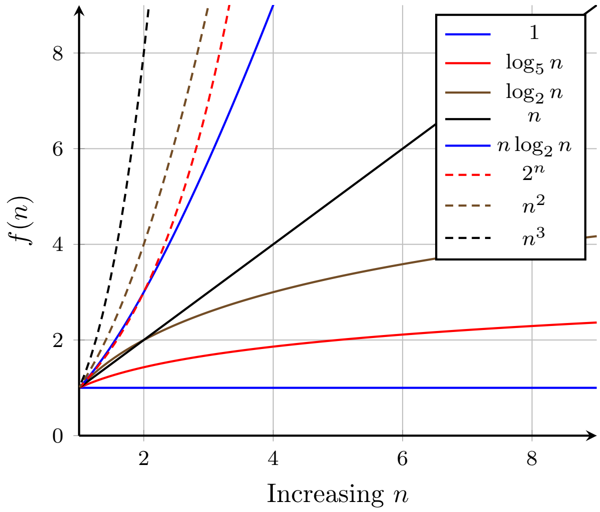
  $$f(n) = \Pi(f(n)), \text{ where } \Pi = O, \Omega, \Theta.$$

- Symmetry

  $$f(n) = \Theta(g(n)) \text{ iff } g(n) = \Theta(f(n)).$$

4.6.2 Graph for Functions

One can understand the growth of functions by the following graph.



Chapter 5

Set Theory and Counting Principle

Chapter 6

Binomial Coefficients

6.1 Introduction

In this section, we introduce Binomial Coefficients.

6.1.1 Combinations

As it is introduced before,

Definition 6.1.1. An r-combination of the n-element ground set S_0 is an **unordered selection** of r elements from S_0.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

6.1.2 Permutation

Again, as it is introduced before,

Definition 6.1.2. An r-permutation of the n-element ground set S_0 is an **ordered selection** of r elements from S_0.

$$P(n,r) = \frac{n!}{(n-r)!}$$

6.1.3 Binomial Identities

Proposition 6.1.1. For any integers $m, r \geq 0$ with $0 \leq r \leq n$,

$$\binom{n}{r} = \binom{n}{n-r}$$

We have two ways to prove this proposition, namely Algebraic Proof and Combinatorial Proof.

Algebraic Proof.

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

■

Combinatorial Proof. Both side of the identity are supposed to be two different ways of solving a counting problem. We can define the counting problem as counting the number of different unordered selections of r elements from an n-element ground set. Then, we can define the RHS as the selecting number to be excluded. Then this identity holds. ■

6.1.4 Pascal's Identity

Theorem 6.1.1. For any integers $n,\ r \geq 0$ with $1 \leq r \leq n-1$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ OR } \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Again, we can prove this theorem by two ways.

Algebraic Proof.

$$\begin{aligned} RHS &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!} \\ &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{1}{r} + \frac{1}{n-r}\right) \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{n}{r(n-r)}\right) \\ &= \frac{n!}{r!(n-r)!} \\ &= \binom{n}{r} \end{aligned}$$

■

Combinatorial Proof. Recall that $\binom{n}{k}$ equals the number of subsets with k elements from a set with n elements. Suppose one particular element is uniquely labeled X in a set with n elements.

To construct a subset of k elements containing X, include X and choose $k-1$ elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k-1}$ such subsets.

To construct a subset of k elements **not** containing X, choose k elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k}$ such subsets.

Every subset of k elements either contains X or not. The total number of subsets with k elements in a set of n elements is the sum of the number of subsets containing X and the number of subsets that do not contain X, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

This equals $\binom{n}{k}$. ■

Appendix

Appendix A

Additional Proofs