ENGG2440 Discrete mathematics for engineers

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October 22, 2024

Abstract

This is a note for ENGG2440 - Discrete mathematics for engineers for self-revision and concept understanding ONLY. Some contents are taken from lecture notes as well as only reference book.
Mistakes might be found. So please feel free to point out any mistakes.

Template of this note is adapted from https://github.com/sleepymalc.

Contents

Chapter 1

Mathematical Induction

1.1 Introduction

In mathematics, there are some basic proof techniques that we can apply, including direct proof, proof by induction, proof by contradiction, and proof by contraposition. For most of these proving methods, you won't be learning their reasons or applications, but you will still use them in some simple proving questions. In this chapter, we will mainly discuss mathematical induction.

> **Definition 1.1.1** (Proposition). A **proposition** is a statement that is either true or false.

> **Definition 1.1.2** (Predicate). A **predicate** is a proposition whose truth depends on one or more variables.

1.2 Mathematical Induction

An analogy of the principle of mathematical induction is the game of dominoes. Suppose the dominoes are lined up properly, so that when one falls, the successive one will also fall. Now by pushing the first domino, the second will fall; when the second falls, the third will fall; and so on. We can see that all dominoes will ultimately fall.

The key point is only two steps:

1. the first domino falls;

2. when a domino falls, the next domino falls.

We use the above principle of Mathematical Induction to prove.

Let $P(n)$ be a predicate.

(Base Case) Show that $P(1)$ is true.

(Inductive Steps) Show that for $n = 1, 2, \ldots$, if $P(n)$ is true, then $P(n + 1)$ is true.

> **Lemma 1.2.1** (Useful lemma). Given the axioms of natural numbers $\mathbb{N}$, we have
> $$0 \neq 1.$$
> **An obvious proof.** Obvious. ∎

> **Proposition 1.2.1** (Useful proposition). From Lemma 5.1.1, we have
> $$0 < 1.$$

Exercise. Prove that $1 < 2$.

Answer. We note the following.

> **Note.** We have Proposition 5.1.1! We can use it iteratively!

With the help of Lemma 5.1.1, this holds trivially. ⊛

Example. We now can have $a < b$ for $a < b$!

Proof. Iteratively apply the exercise we did above. ⊛

Remark. We see that Proposition 5.1.1 is really powerful. We now give an immediate application of it.

Theorem 1.2.1 (Mass-energy equivalence)**.** Given Proposition 5.1.1, we then have

$$E = mc^2.$$

Proof. The blank left for me is too small,[a] hence we put the proof in Appendix A.1. ∎

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[a] https://en.wikipedia.org/wiki/Richard_Feynman

From Theorem 5.1.1, we then have the following.

Corollary 1.2.1 (Riemann hypothesis)**.** The real part of every nontrivial zero of the Riemann zeta function is $\frac{1}{2}$, where the Riemann zeta function is just

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots.$$

Proof. The proof should be trivial, we left it to you. ∎ ⟶ DIY

> **As previously seen.** We see that Lemma 5.1.1 is really helpful in the proof!

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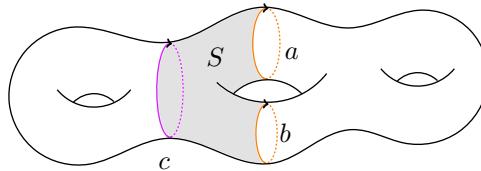
1.3 Figures
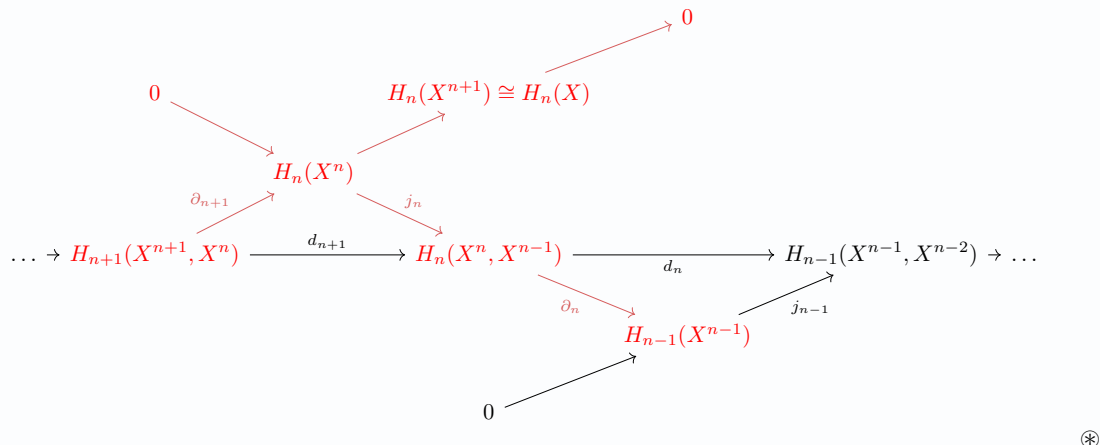
A simple demo for drawing:

Figure 1.1: A 3-torus.[1]

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1.4 Commutative Diagram

We can use the package `tikz-cd` to draw some commutative diagram.

Example. The cellular homology agrees with singular homology.

Proof. The following commutative diagram shows everything.

$$
\begin{array}{c}
\\
0 \qquad\qquad H_n(X^{n+1}) \cong H_n(X) \qquad 0 \\
\\
H_n(X^n) \\
{\scriptstyle \partial_{n+1}} \qquad {\scriptstyle j_n} \\
\ldots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{\;d_{n+1}\;} H_n(X^n, X^{n-1}) \xrightarrow{\;d_n\;} H_{n-1}(X^{n-1}, X^{n-2}) \to \ldots \\
{\scriptstyle \partial_n} \qquad\qquad {\scriptstyle j_{n-1}} \\
H_{n-1}(X^{n-1}) \\
0
\end{array}
$$

⊛

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Definition. We have the following number system.

> **Definition 1.5.1** (Rational number)**.** The set of *rational number*, denote as $\mathbb{Q}$.

> **Definition 1.5.2** (Real number)**.** The set of *real number*, denote as $\mathbb{R}$.

> **Definition 1.5.3** (Complex number)**.** The set of *complex number*, denote as $\mathbb{C}$.

Note. And indeed, we can still reference them correctly. For instance, we can use rational numbers to define real numbers and then further use it to define complex numbers.

Furthermore, we can completely control the name of our environments. We already saw we can name definition, lemma, proposition, corollary and theorem environment. In fact, we can also name remark, note, example and proof as follows.

Example (Interesting Example)**.** We note that $1 \neq 2$!

> **Note** (Important note)**.** As a consequence, $2 \neq 3$ also.

> **Remark** (Easy observation)**.** We see that from here, we easily have the following theorem.
>
> > **Theorem 1.5.1** (Lebesgue Differentiation Theorem)**.** Let $f \in L^1$, then
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> > $$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0$$
> >
> > for a.e. x.
> >
> > **An obvious proof of Theorem 5.4.1.** Obvious. ∎

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Chapter 2

Summation Techniques

2.1 Introduction

Nothing is bugs-free. There are some known bugs which I don't have incentive to solve, or it is hard to solve whatsoever. Let me list some of them.

2.1.1 Footnote Environment

It's easy to let you fall into a situation that you want to keep using `footnote` to add a bunch of unrelated stuffs. However, with our environment there is a known strange behavior, which is following.

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>
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> [b]The final footnote which is ok!

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> **Proof of Theorem 5.1.1.** We need to prove the followings.
>
> > **Claim.** $E = mc^2$.

Proof. Nonsense.

Nonsense,

Nonsense,

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Nonsense.   ⊛

∎

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Chapter 3

Recurrences

3.1 Introduction

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Chapter 4

Asymptotics

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Definition 4.1.1 (Natural number)**.** We denote the set of *natural numbers* as $\mathbb{N}$.

Lemma 4.1.1 (Useful lemma)**.** Given the axioms of natural numbers $\mathbb{N}$, we have

$$0 \neq 1.$$

An obvious proof. Obvious. ∎

Proposition 4.1.1 (Useful proposition)**.** From Lemma 5.1.1, we have

$$0 < 1.$$

Exercise. Prove that $1 < 2$.

Answer. We note the following.

> **Note.** We have Proposition 5.1.1! We can use it iteratively!

With the help of Lemma 5.1.1, this holds trivially. ⊛

Example. We now can have $a < b$ for $a < b$!

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Remark. We see that Proposition 5.1.1 is really powerful. We now give an immediate application of it.

Theorem 4.1.1 (Mass-energy equivalence)**.** Given Proposition 5.1.1, we then have

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Proof. The proof should be trivial, we left it to you. ∎ DIY

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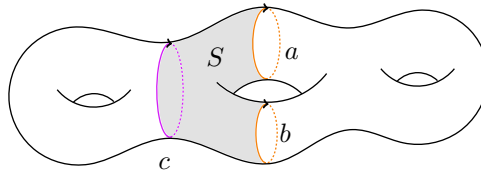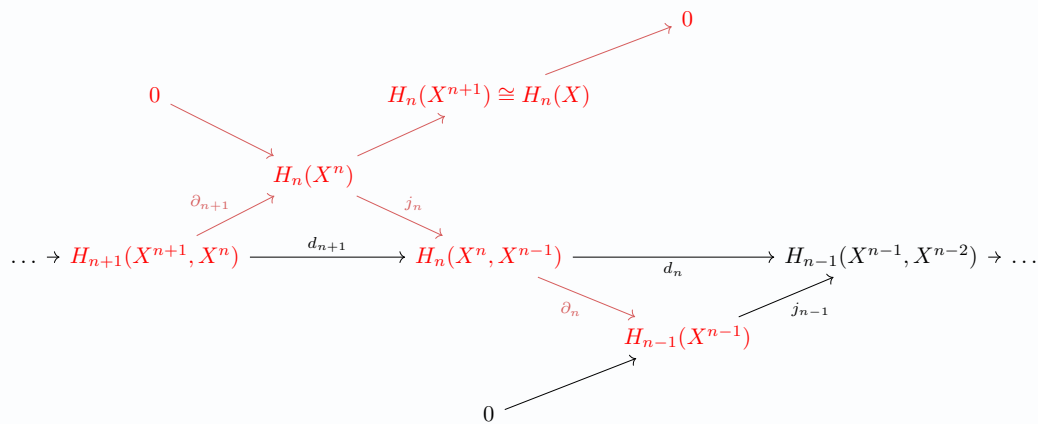
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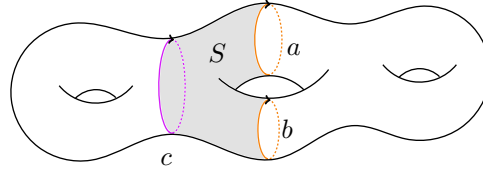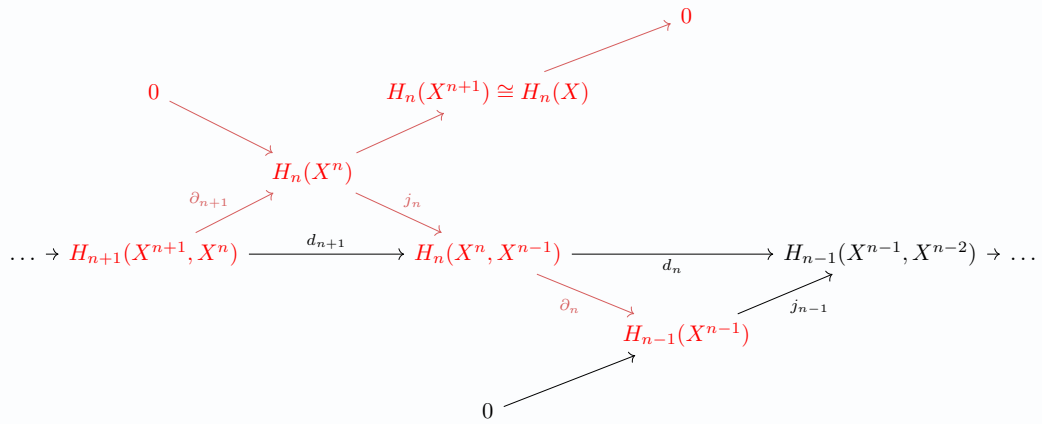
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Chapter 6

Binomial Coefficients

6.1 Introduction

In this section, we introduce Binomial Coefficients.

6.1.1 Combinations

As it is introduced before,

> **Definition 6.1.1.** An r-combination of the n-element ground set S_0 is an **unordered selection** of r elements from S_0.
> $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

6.1.2 Permutation

Again, as it is introduced before,

> **Definition 6.1.2.** An r-permutation of the n-element ground set S_0 is an **ordered selection** of r elements from S_0.
> $$P(n,r) = \frac{n!}{(n-r)!}$$

6.1.3 Binomial Identities

> **Proposition 6.1.1.** For any integers $m, r \geq 0$ with $0 \leq r \leq n$,
> $$\binom{n}{r} = \binom{n}{n-r}$$

We have two ways to prove this proposition, namely Algebraic Proof and Combinatorial Proof.

Algebraic Proof.
$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$
∎

Combinatorial Proof. Both side of the identity are supposed to be two different ways of solving a counting problem. We can define the counting problem as counting the number of different unordered selections of r elements from an n-element ground set. Then, we can define the RHS as the selecting number to be excluded. Then this identity holds. ∎

6.1.4 Pascal's Identity

Theorem 6.1.1. For any integers n, $r \geq 0$ with $1 \leq r \leq n-1$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ OR } \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Again, we can prove this theorem by two ways.

Algebraic Proof.

$$\begin{aligned}
RHS &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!} \\
&= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!} \left(\frac{1}{r} + \frac{1}{n-r} \right) \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!} \left(\frac{n}{r(n-r)} \right) \\
&= \frac{n!}{r!(n-r)!} \\
&= \binom{n}{r}
\end{aligned}$$

∎

Combinatorial Proof. Recall that $\binom{n}{k}$ equals the number of subsets with k elements from a set with n elements. Suppose one particular element is uniquely labeled X in a set with n elements.

To construct a subset of k elements containing X, include X and choose $k-1$ elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k-1}$ such subsets.

To construct a subset of k elements **not** containing X, choose k elements from the remaining $n-1$ elements in the set. There are $\binom{n-1}{k}$ such subsets.

Every subset of k elements either contains X or not. The total number of subsets with k elements in a set of n elements is the sum of the number of subsets containing X and the number of subsets that do not contain X, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

This equals $\binom{n}{k}$.

∎

Example. Footnote![a]

> **Remark.** Oops! footnote somehow shows up earlier than expect![a]
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Appendix

Appendix A

Additional Proofs

A.1 Proof of Theorem 5.1.1

We can now prove Theorem 5.1.1.

Proof of Theorem 5.1.1. See here. ∎

Bibliography

[New26]   I. Newton. *Philosophiae naturalis principia mathematica.* Innys, 1726. URL: https://books.google.com/books?id=WeZO9rjv-1kC.