SUPPLEMENTAL MATERIAL

A Full proof of Theorem 6.1

To show this theorem, we first define the stochastic process obtained by the proposed method SPUR.

Definition A.1 (SPUR process): *Let $p^r_0 = 0, R_1 = \emptyset, \delta_1 = \alpha, H_1 = H$, a stochastic process $\{X_t\}_{t=1}^{T_{\text{SPUR}}}$ with $X_t = (\delta_t, R_t, p^r_t, h^r_t, \sigma_t, \tau_t)$ is said to be a SPUR process with stopping time T_{SPUR} if:*

$$
\begin{aligned}
for\ t > 1: \quad & H_t = \{h \in H | \forall_{h^r \in R_{t-1}}, h > h^r\} && (1)\\
& \delta_t = \delta_{t-1} - \tau_{t-1}(p^r_{t-1} - p^r_{t-1}) + p^r_{t-1} && (2)\\
for\ t \geq 1: \quad & p^r_t = \min_{h \in H_t} p_h,\ h^r_t = argmin_{h \in H_t} p_h && (3)\\
& \sigma_t = \max\{\sigma : (\sigma - p^r_{t-1})|\kappa_t(\sigma)| \leq \delta_t\} && (4)\\
& \tau_t = \delta_t / (\sigma_t - p^r_{t-1}) && (5)\\
& R_t = \begin{cases} R_{t-1} \cap \{h^r_{t-1}\} & (p^r_t \leq \sigma_t) \\ R_{t-1} & (otherwise) \end{cases} && (6)
\end{aligned}
$$

and $T_{\text{SPUR}} = \min\{t \in [|H|] : p^r_t > \sigma_t\}$

We remark that since we can obtain $H_t = \{h \in H | \forall_{h^r \in R_{t-1}}, h > h^r\}$, we do not need to include H_t in X_t. Using the stochastic process $\{X_t\}_{t=1}^{T_{\text{SPUR}}}$, we next rewrite the event of rejecting at least one true hypotheses, i.e., the event of occuring a Type-I error.

Lemma A.1: *Consider a SPUR process $\{X_t\}_{t=1}^{T_{\text{SPUR}}}$, let $E_t = \{h^r_t \in T, p^r_t \leq \sigma_t\}$ and $T_E = \min\{t \in [|H|] : t \leq T_{\text{SPUR}} \wedge E_t\}$, the following holds:*

$$\{R \cap T \neq \emptyset\} = \{T_E \leq |H|\}.$$

Proof.

$$
\begin{aligned}
& \{T_E = t_0\} && (7)\\
=& \{t_0 \leq T_{SPUR}\} \cap (\cap_{t=1}^{t_0-1} \bar{E}_t) \cap E_{t_0} && (8)\\
=& \{\forall_{t \leq t_0}, p^r_t \leq \sigma_t\} \cap (\cap_{t=1}^{t_0-1} \bar{E}_t) \cap E_{t_0} && (9)\\
=& \{\forall_{t \leq t_0}, p^r_t \leq \sigma_t\} \cap \{\forall_{t < t_0}, h^r_t \in F \vee p^r_t > \sigma_t\} && (10)\\
& \cap \{h^r_{t_0} \in T \wedge p^r_{t_0} \leq \sigma_{t_0}\} && (11)
\end{aligned}
$$

Since,

$$
\begin{aligned}
& \{\forall_{t \leq t_0}, p^r_t \leq \sigma_t\} && (12)\\
\implies & \{\forall_{t < t_0}, h^r_t \in F \vee p^r_t > \sigma_t\} = \{\forall_{t < t_0}, h^r_t \in F\} && (13)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \{T_E = t_0\} && (14)\\
=& \{\forall_{t < t_0}, h^r_t \in F \wedge p^r_t \leq \sigma_t\} \cap \{h^r_{t_0} \in T \wedge p^r_{t_0} \leq \sigma_{t_0}\} && (15)
\end{aligned}
$$

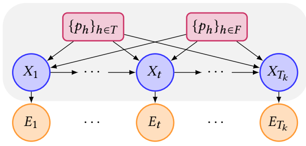

Figure 5: Relation graph of $\{X_t\}_{t=1}^{T_{\text{SPUR}}}$ (shaded) and event E_t (we let $k = T_{\text{False}}$ in the graph)

In the other hand, we consider the event of rejecting at least a true hypothesis

$$
\begin{aligned}
& \{R \cap T \neq \emptyset\} && (16)\\
=& \cup_{t_0=1}^{|H|}(\{T_{\text{SPUR}} = t_0\} \cap \{\exists_{t \leq t_0}, h^r_t \in T\}) && (17)\\
=& \cup_{t_0=1}^{|H|}(\{T_{\text{SPUR}} = t_0\} \cap (\cup_{t=1}^{t_0}\{\forall_{i<t_0} h^r_i \in F, h^r_t \in T\}) && (18)\\
=& \cup_{t_0=1}^{|H|}(\cup_{t=1}^{t_0}\{T_{\text{SPUR}} = t_0\}) \cap \{\forall_{i<t} h^r_i \in F, h^r_t \in T\} && (19)\\
=& \cup_{t_0=1}^{|H|}\{t_0 \leq T_{\text{SPUR}}\} \cap \{\forall_{i<t_0} h^r_i \in F, h^r_{t_0} \in T\} && (20)\\
=& \cup_{t_0=1}^{|H|}\{\forall_{i \leq t_0}, p^r_i \leq \sigma_t\} \cap \{\forall_{i<t_0}, h^r_i \in F, h^r_{t_0} \in T\} && (21)\\
=& \cup_{t_0=1}^{|H|}\{\forall_{t<t_0}, h^r_t \in F \wedge p^r_t \leq \sigma_t\} \cap \{h^r_{t_0} \in T \wedge p^r_{t_0} \leq \sigma_{t_0}\} && (22)\\
=& \cup_{t_0=1}^{|H|}\{T_E = t_0\} = \{T_E \leq |H|\} && (23)
\end{aligned}
$$

This concludes our proof. □

Here, $E_t = \{h^r_t \in T, p^r_t \leq \sigma_t\}$ is the event of rejecting a true hypothesis at the t-th iteration and $T_E = \min\{t \in [|H|] : t \leq T_{\text{SPUR}} \wedge E_t\}$ is the first iteration that a true hypotheses got rejected. Lemma A.1 claims that the FWER is actually the probability of the first Type-I error occurs at some step $t_0 \leq |H|$. To show the FWER controlling, we have to consider the relationship between the p-values of the true hypotheses, the false hypotheses, and the rejection threshold at each iteration. However, such the dependence are complicated in the SPUR process $\{X_t\}_{t=1}^{T_{\text{SPUR}}}$. Thus, we instead consider an alternative stochastic process that only depends on the false hypotheses and show the FWER controlling by analyzing this process. In addition, we illustrate the relation of entities in the SPUR process using a graph shown in Figure 5.

Definition A.2 (False hypotheses based process): *Let $p^*_0 = 0, H^*_1 = H, R^*_0 = \emptyset, \delta^*_1 = \alpha$, a stochastic process $\{Y_t\}_{t=1}^{T_{\text{FALSE}}}$ with $Y_t = (\delta^*_t, R^*_t, H^*_t, p^{r*}_t, h^{r*}_t, \sigma^*_t, \tau^*_t)$ is said to be a false-hypotheses based SPUR process with stopping time T_{FALSE} if:*

$$
\begin{aligned}
for\ t > 1: \quad & H^*_t = \{h \in H^*_{t-1} \mid h > h^{r*}_{t-1}\} && (24)\\
& \delta^*_t = \delta^*_{t-1} - \tau_{t-1}(p^{r*}_{t-1} - p^{r*}_{t-1}) + p^{r*}_{t-1} && (25)\\
for\ t \geq 1: \quad & p^{r*}_t = \min_{h \in H^*_t \cap F} p_h,\ h^{r*}_t = argmin_{h \in H^*_t \cap F} p_h && (26)\\
& \sigma^*_t = \max\{\sigma : (\sigma - p^*_{t-1})|\kappa^*_t(\sigma)| \leq \delta^*_t\} && (27)\\
& \tau^*_t = \delta^*_t / (\sigma^*_t - p^*_{t-1}) && (28)\\
& R^*_t = \begin{cases} R^*_{t-1} \cap \{h^*_{t-1}\} & (p^{r*}_t \leq \sigma^*_t) \\ R^*_{t-1} & (otherwise) \end{cases} && (29)
\end{aligned}
$$

and $T_{\text{FALSE}} = \min\{t \in [H] : p^*_t > \sigma^*_t\}$

Next, we define a sequence of true hypotheses' p-values $\{p_t^{T*}\}_{t=1}^{T_{\text{False}}}$ where each p_t^{T*} is obtained using the value Y_t of the false hypotheses based process.

Definition A.3 (Alternative true hypotheses sequence): *A set of r.v. $\{p_t^{T*}\}_{t=1}^{T_{\text{False}}}$ is said to be an alternative true hypotheses sequence obtained from the false hypotheses based process $\{Y_t\}_{t=1}^{T_{\text{FALSE}}}$ if for $t \le T_{\text{FALSE}}$:*

$$p_t^{T*} = f(Y_t, \{p_h\}_{h\in T}) = \min_{h\in H_t^*\cap T} p_h.$$

Since at each step, SPUR rejects the most significant hypothesis $h_t^r = min_{h\in H_t} p_h$, the event $\{h_t^r \in T\}$ (and $\{h_t^r \in F\}$) depends on the comparison of $min_{h\in H_t\cap F} p_h$ and $min_{h\in H_t\cap R} p_h$. We next consider this comparison via the false hypotheses based process and the alternative true hypotheses sequence, while claims its relation to Lemma A.1.

Lemma A.2: *Consider $\{Y_t\}_{t=1}^{T_{\text{False}}}$ and $\{p_t^{T*}\}_{t=1}^{T_{\text{False}}}$ as defined in Definition A.2 and A.3. Let $E_t^* = \{p_t^{T*} \le p_t^{r*}, p_t^{T*} \le \sigma_t^*\}$ and $T_E^* =$* $\min\{t \in [|H|] : t \le T_{\text{False}} \wedge E_t^*\}$ *and T_E as defined in Lemma A.1,*

$$T_E \ge T_E^* \text{ almost surely.}$$

Proof.

$$\{T_E^* = t_0\} \tag{30}$$

$$=\{t_0 \le T_{\text{False}}\} \cap (\cap_{t=1}^{t_0-1} \bar{E}_t^*) \cap E_{t_0}^* \tag{31}$$

$$=\{\forall_{t<t_0}, p_t^* \le \sigma_t^*\} \cap \{\forall_{t<t_0}, p_t^{T*} > p_t^{r*} \vee p_t^{T*} > \sigma_t^*\} \tag{32}$$

$$\cap \{p_{t_0}^{T*} \le p_{t_0}^{r*} \wedge p_{t_0}^{T*} \le \sigma_{t_0^*}\} \tag{33}$$

$$=\{\forall_{t<t_0}, p_t^{T*} > p_t^{r*} \wedge p_t^{r*} \le \sigma_t^*\} \cap \{p_{t_0}^{T*} \le p_{t_0}^{r*} \wedge p_{t_0}^{T*} \le \sigma_{t_0^*}\} \tag{34}$$

Next, we show that

$$\forall_{t<t_0}, p_t^{T*} > p_t^{r*} \implies \forall_{t<t_0}, X_t = Y_t \tag{35}$$

First, we have

$$p_t^{T*} > p_t^{r*} \tag{36}$$

$$\iff \min_{h\in H_t^*\cap T} > \min_{h\in H_t^*\cap F} \tag{37}$$

$$\iff \text{argmin}_{h\in H_t^*\cap F} = \text{argmin}_{h\in H_t^*} \tag{38}$$

In the other hand,

$$h_t^r \in F \iff \min_{h\in H_t} = \min_{h\in H_t\cap F} \tag{39}$$

$$h_t^r \in T \iff \min_{h\in H_t} = \min_{h\in H_t\cap T} \tag{40}$$

Hence,

$$p_t^{T*} > p_t^{r*} \wedge H_t = H_t^* \tag{41}$$

$$\implies \text{argmin}_{h\in H_t^*\cap F} = \text{argmin}_{h\in H_t} \tag{42}$$

$$\implies h_t^{r*} = h_1^r \tag{43}$$

$$\implies X_t = Y_t \tag{44}$$

Moreover, we have

$$p_t^{T*} > p_t^{r*} \wedge (H_t, \delta_t, \sigma_t, \tau_t) = (H_t^*, \delta_t^*, \sigma_t^*, \tau_t^*) \tag{45}$$

$$\implies X_t = Y_t \tag{46}$$

$$\implies (\delta_{t+1}, H_{t+1}, \sigma_{t+1}, \tau_{t+1}) = (\delta_{t+1}^*, H_{t+1}^*, \sigma_{t+1}^*, \tau_{t+1}^*) \tag{47}$$

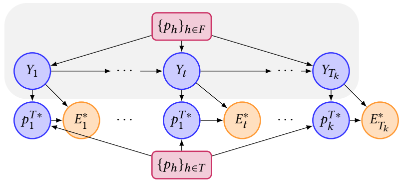

Figure 6: Relation graph of $\{Y_t\}_{t=1}^{T_{\text{False}}}$ (shaded), $\{p_t^{T*}\}_{t=1}^{T_{\text{False}}}$ and event E_t^* (we let $k = T_{\text{False}}$ in the graph)

The last line follows from the definition of two procedure. Moreover, since $(H_1, \delta_1, \sigma_1, \tau_1) = (H_1^*, \delta_1^*, \sigma_1^*, \tau_1^*)$,

$$p_1^{T*} > p_1^{r*} \implies X_t = Y_t \tag{48}$$

The claim (35) can now be shown using induction and we rewrite the event $\{T_E = t_0\}$ as

$$\{T_E = t_0\} \tag{49}$$

$$=\{\forall_{t<t_0}, h_t^r \in F \wedge p_t^r \le \sigma_t\} \cap \{h_{t_0}^r \in T \wedge p_{t_0}^r \le \sigma_{t_0}\} \tag{50}$$

$$=\{\forall_{t<t_0}, h_t^{r*} < h_t^{T*} \wedge p_t^{r*} \le \sigma_t^*\} \cap \{h_t^{T*} < h_{t_0}^{r*} \in T \wedge p_{t_0}^{r*} \le \sigma_{t_0}^*\} \tag{51}$$

$$\subseteq\{\forall_{t<t_0}, h_t^{r*} < h_t^{T*} \wedge p_t^{r*} \le \sigma_t^*\} \cap \{h_t^{T*} \le h_{t_0}^{r*} \in T \wedge p_{t_0}^{r*} \le \sigma_{t_0}^*\} \tag{52}$$

Thus,

$$\{T_E^* = t_0\} \implies \{T_E = t_0\} \tag{53}$$

This concludes our proof. □

We also give an illustration on the events E_t^* and their relationship with other entities in Figure 6. We have that $\Pr[R \cap T \neq \emptyset] \le \Pr[T_E \le |H|] \le \Pr[T_E^* \le |H|]$ from Lemma A.2. Moreover, since

$$\Pr[T_E^* \le |H|] = \mathrm{E}_{\{p_h\}_{h\in F}}\left[\Pr[T_E^* \le |H| \mid \{p_h\}_{h\in F}]\right].$$

we next find the upper bound of $\Pr[T_E^* \le |H| \mid \{p_h\}_{h\in F}]$.

Lemma A.3: *Using the same definition of Lemma A.2 and let $k =$ T_{False}, under Assumption 3.1, the following holds:*

$$Pr[T_E^* \le |H| \mid \{p_h\}_{h\in F}]$$
$$\le \sum_{t_0<k} |\kappa_{t_0}^*(p_{t_0}^{r*}) \cap T|(p_{t_0}^{r*} - p_{t_0-1}^{r*}) + |\kappa_k^*(p_k^{r*}) \cap T|(\sigma_k^* - p_{k-1}^{r*}).$$

Proof. First, since the False-hypotheses based process $\{Y_t\}_{t=1}^{T_{\text{False}}}$ determined only by only the values of $\{p_h\}_{h\in F}$,

$$\Pr[T_E^* \le |H| \mid \{p_h\}_{h\in F}] \tag{54}$$

$$=\Pr[T_E^* \le |H| \mid \{p_h\}_{h\in F}, \{Y_t\}_{t=1}^{T_{\text{False}}}] \tag{55}$$

$$=\Pr[T_E^* \le |H| \mid p_F, Y_F, k] \tag{56}$$

In the last line, we let $k = T_{\text{False}}$, and compactly express the conditional term for the sake of space, where p_F and Y_F represent $\{p_h\}_{h\in F}$ and $\{Y_t\}_{t=1}^{T_{\text{False}}}$, respectively.

From the definition of T_E^*, we have:

$$T_E^* = \min\{t \in [|H|] : t \le T_{\text{False}} \wedge E_t^*\} \tag{57}$$

$$\implies T_E^* \le T_{\text{False}} \tag{58}$$

$$\implies \Pr[T_E^* \le |H| \mid p_F, Y_F, k] = \Pr[T_E^* \le k \mid p_F, Y_F, k] \tag{59}$$

Beside, using the result of T_E^* obtained before:

$$\{T_E^* \leq k\} \tag{60}$$
$$= \cup_{t_0 \leq k} \{\forall_{t<t_0}, p_t^{T*} > p_t^{r*} \wedge p_t^{r*} \leq \sigma_t^* \wedge p_{t_0}^{T*} \leq p_{t_0}^{r*} \wedge p_{t_0}^{T*} \leq \sigma_{t_0^*}\} \tag{61}$$
$$\subseteq \cup_{t_0 \leq k} \{p_{t_0-1}^{T*} > p_{t_0-1}^{r*} \wedge p_{t_0}^{T*} \leq p_{t_0}^{r*} \wedge p_{t_0}^{T*} \leq \sigma_{t_0^*}\} \tag{62}$$
$$\subseteq \cup_{t_0 \leq k} \{p_{t_0}^{T*} > p_{t_0-1}^{r*} \wedge p_{t_0}^{T*} \leq p_{t_0}^{r*} \wedge p_{t_0}^{T*} \leq \sigma_{t_0^*}\} \tag{63}$$

The last line follows since $p_{t_0}^{T*} \geq p_{t_0-1}^{T*}$. We next consider the term $\{p_{t_0}^{T*} \leq p_{t_0}^{r*} \wedge p_{t_0}^{T*} \leq \sigma_{t_0}^*\}$ for different iteration t_0 regarding of the stopping time $k = T_{\text{FALSE}}$. Using the definition of T_{FALSE}:

$$k = \min\{t \in [H] : p_t^* > \sigma_t^*\} \tag{64}$$

Hence,

$$t_0 < k \Leftrightarrow p_{t_0}^{T*} \leq \sigma_{t_0}^* \tag{65}$$
$$\Rightarrow \{p_{t_0}^{T*} \leq p_{t_0}^{r*} \wedge p_{t_0}^{T*} \leq \sigma_{t_0}^*\} = \{p_{t_0}^{T*} \leq p_{t_0}^{r*}\} \tag{66}$$
$$t_0 = k \Leftrightarrow p_{t_0}^{T*} > \sigma_{t_0}^* \tag{67}$$
$$\Rightarrow \{p_{t_0}^{T*} \leq p_{t_0}^{r*} \wedge p_{t_0}^{T*} \leq \sigma_{t_0}^*\} = \{p_{t_0}^{T*} \leq \sigma_{t_0}^*\} \tag{68}$$

And then,

$$\{T_E^* \leq k\} \tag{69}$$
$$\subseteq \cup_{t_0<k} \{p_{t_0}^{T*} > p_{t_0-1}^{r*} \wedge p_{t_0}^{T*} \leq p_{t_0}^{r*} \wedge p_{t_0}^{T*} \leq \sigma_{t_0}^*\} \tag{70}$$
$$\cup \{p_k^{T*} > p_{k-1}^{r*} \wedge p_k^{T*} \leq p_k^{r*} \wedge p_k^{T*} \leq \sigma_{k^*}\} \tag{71}$$
$$\subseteq \cup_{t_0<k} \{p_{t_0-1}^{r*} < p_{t_0}^{T*} \leq p_{t_0}^{r*}\} \cup \{p_{k-1}^{r*} < p_k^{T*} \leq \sigma_k^*\} \tag{72}$$

The probability of this event is

$$\Pr\left[T_E^* \leq k \mid p_F, Y_F, k\right] \tag{73}$$
$$\leq \sum_{t_0<k} \Pr\left[p_{t_0-1}^{r*} < p_{t_0}^{T*} \leq p_{t_0}^{r*} \mid p_F, Y_F, k\right] \tag{74}$$
$$+ \Pr\left[p_{k-1}^{r*} < p_k^{T*} \leq \sigma_k^* \mid p_F, Y_F, k\right] \tag{75}$$
$$\leq \sum_{t_0<k} \cup_{h\in\kappa_{t_0}^*(p_{t_0}^{r*})\cap T} \Pr\left[p_{t_0-1}^{r*} < p_h \leq p_{t_0}^{r*} \mid p_F, Y_F, k\right] \tag{76}$$
$$+ \cup_{h\in\kappa_k^*(\sigma_k^*)\cap T} \Pr\left[p_{k-1}^{r*} < p_h \leq \sigma_k^* \mid p_F, Y_F, k\right] \tag{77}$$

Moreover, since $\{Y_t\}_{t=1}^k$ only relies on $\{p_h\}_{h\in F}$, by the Assumption 3.1:

$$\{p_h\}_{h\in F}, \{Y_t\}_{t=1}^k \perp \{p_h\}_{h\in F} \tag{78}$$

Then, since $p_h, h \in T$ follows the Uniform distribution,

$$\Pr\left[T_E^* \leq k \mid p_F, Y_F, k\right] \tag{79}$$
$$\leq \sum_{t_0<k} |\kappa_{t_0}^*(p_{t_0}^{r*}) \cap T|(p_{t_0}^{r*} - p_{t_0-1}^{r*}) + |\kappa_k^*(\sigma_k^*) \cap T|(\sigma_k^* - p_{k-1}^{r*}) \tag{80}$$

This concludes our proof. □

Actually, the proposed algorithm SPUR is designed to guarantee that the right side in the inequation of Lemma A.3 always less than α, as stated in Lemma A.4.

Lemma A.4: *Using the same definition of Lemma A.2 and let $k = T_{\text{False}}$, the following holds:*

$$\sum_{t_0<k} |\kappa_{t_0}^*(p_{t_0}^{r*}) \cap T|(p_{t_0}^{r*} - p_{t_0-1}^{r*}) + |\kappa_k^*(\sigma_t^*) \cap T|(\sigma_k^* - p_{k-1}^{r*}) \leq \alpha.$$

Proof. Let

$$\Delta_k = \sum_{t_0<k} |\kappa_{t_0}^*(p_{t_0}^{r*}) \cap T|(p_{t_0}^{r*} - p_{t_0-1}^{r*}) + |\kappa_k^*(\sigma_k^*) \cap T|(\sigma_k^* - p_{k-1}^{r*}) \tag{81}$$

By our definition of the False-hypotheses based process, we have

$$\forall_{t_0<k}, p_{t_0}^{r*} \leq \sigma_{t_0}^* \tag{82}$$
$$\implies \quad \forall_{t_0<k}, \kappa_{t_0}^*(p_{t_0}^{r*}) \subseteq \kappa_{t_0}^*(\sigma_{t_0}^*) \tag{83}$$

Thus,

$$\Delta_k \leq \sum_{t_0<k} |\kappa_{t_0}^*(\sigma_{t_0}^*) \cap T|(p_{t_0}^{r*} - p_{t_0-1}^{r*}) + |\kappa_k^*(\sigma_k^*) \cap T|(\sigma_k^* - p_{k-1}^{r*}) \tag{84}$$

Moreover,

$$T_{\text{False}} = k \implies \forall_{t_0<k}, |\kappa_{t_0}^*(\sigma_{t_0}^*) \cap F| \geq k - t_0 \tag{85}$$

And since,

$$\kappa_{t_0}^*(\sigma_{t_0}^*) = (\kappa_{t_0}^*(\sigma_{t_0}^*) \cap F) \cup (\kappa_{t_0}^*(\sigma_{t_0}^*) \cap T) \tag{86}$$
$$\text{and } F \cap T = \emptyset \tag{87}$$

The following holds

$$\forall_{t_0<k}, |\kappa_{t_0}^*(\sigma_{t_0}^*) \cap T| \leq |\kappa_{t_0}^*(\sigma_{t_0}^*)| - k + t \tag{88}$$

Hence,

$$\Delta_k \leq \sum_{t_0<k} (|\kappa_t^*(\sigma_{t_0}^*)| - k + t_0)(p_{t_0}^{r^*} - p_{t_0-1}^{r^*}) + |\kappa_k^*(\sigma_k^*)|(\sigma_k^* - p_{k-1}^{r^*}) \tag{89}$$
$$\leq \sum_{t_0<k} \left[|\kappa_t^*(\sigma_{t_0}^*)|(p_{t_0}^{r^*} - p_{t_0-1}^{r^*}) + p_{t_0}^{r^*}\right] + |\kappa_k^*(\sigma_k^*)|(\sigma_k^* - p_{k-1}^{r^*}) \tag{90}$$

Besides, since $|\kappa_{t_0}^*(\sigma)|(\sigma - p_{t_0-1}^{r^*}) \leq \delta_{t_0}^*$ for $t_0 \leq k$,

$$\forall_{t_0<k}, |\kappa_k^*(\sigma_{t_0}^*)| \leq \frac{\delta_{t_0}^*}{p_{t_0}^{r*} - p_{t_0-1}^{r^*}} = \tau_{t_0}^* \tag{91}$$
$$|\kappa_k^*(\sigma_k^*)|(\sigma_k^* - p_{k-1}^{r^*}) \leq \delta_k^* \tag{92}$$

Thus,

$$\Delta_k \leq \sum_{t_0<k} \left[\tau_t(p_t^{F^*} - p_{t-1}^{F^*}) + p_t^{F^*}\right] + \delta_k^* = \alpha \tag{93}$$

This concludes our proof. □

Theorem 6.1 is then shown by combining the above lemmas.