CS109/Stat121/AC209/E-109
Data Science
Regression Continued

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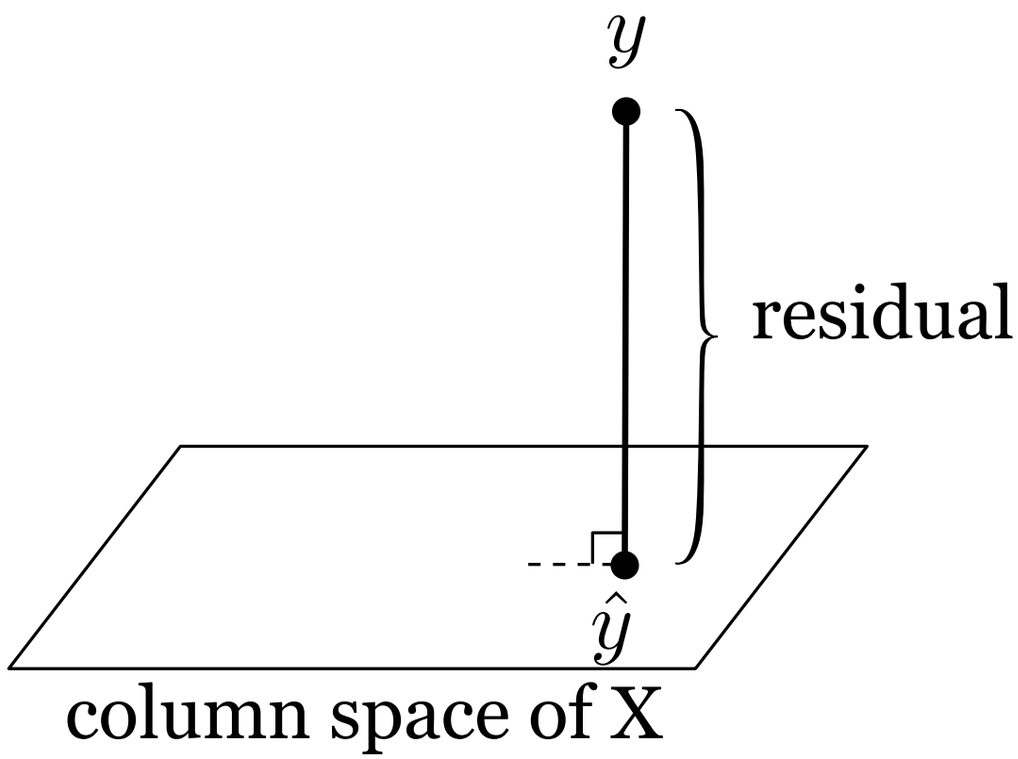
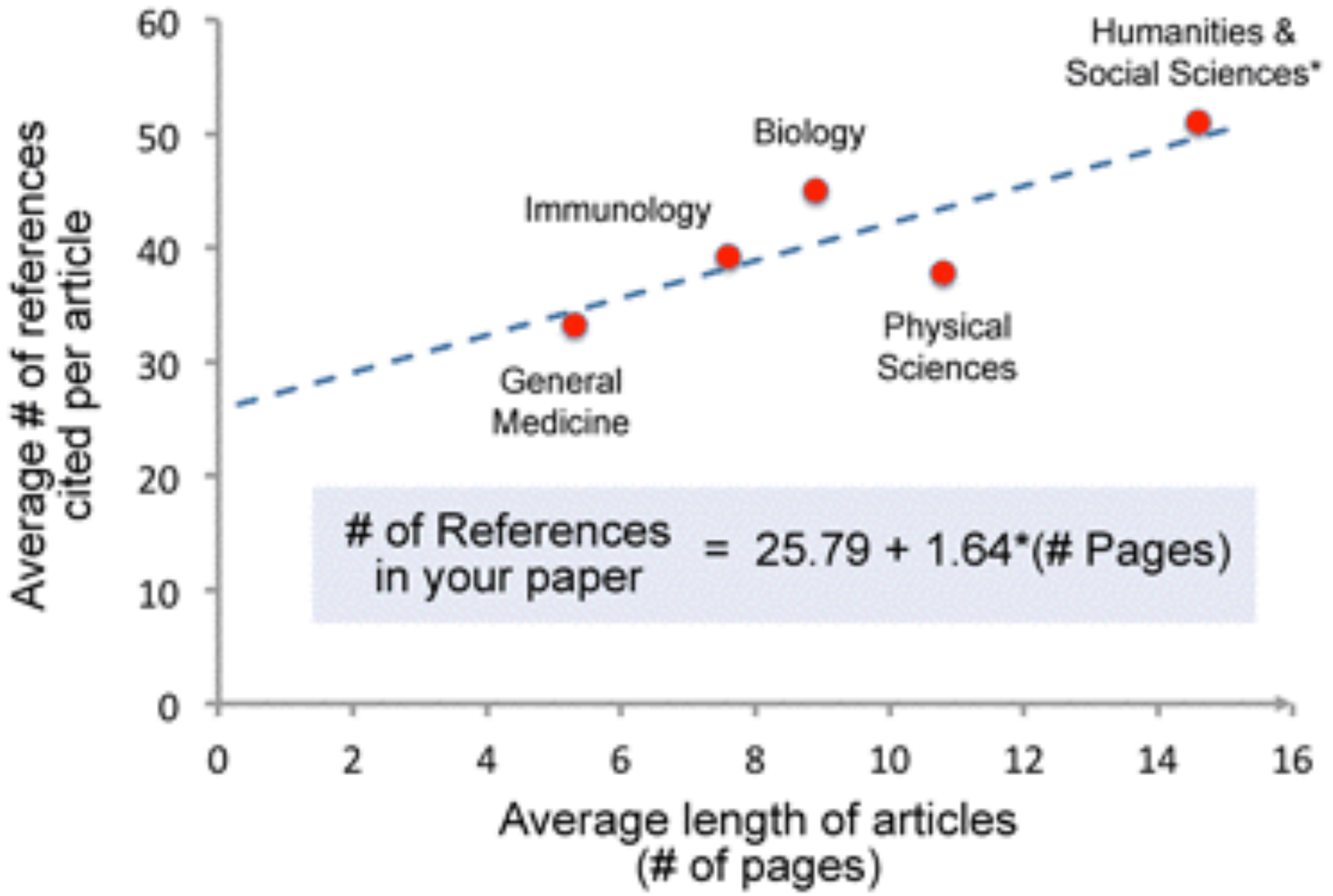


This Week

- HW2 due next Thursday (Oct 8) at 11:59 pm (Eastern Time)
- See updated Piazza posting guidelines (pinned note) and follow the format described there

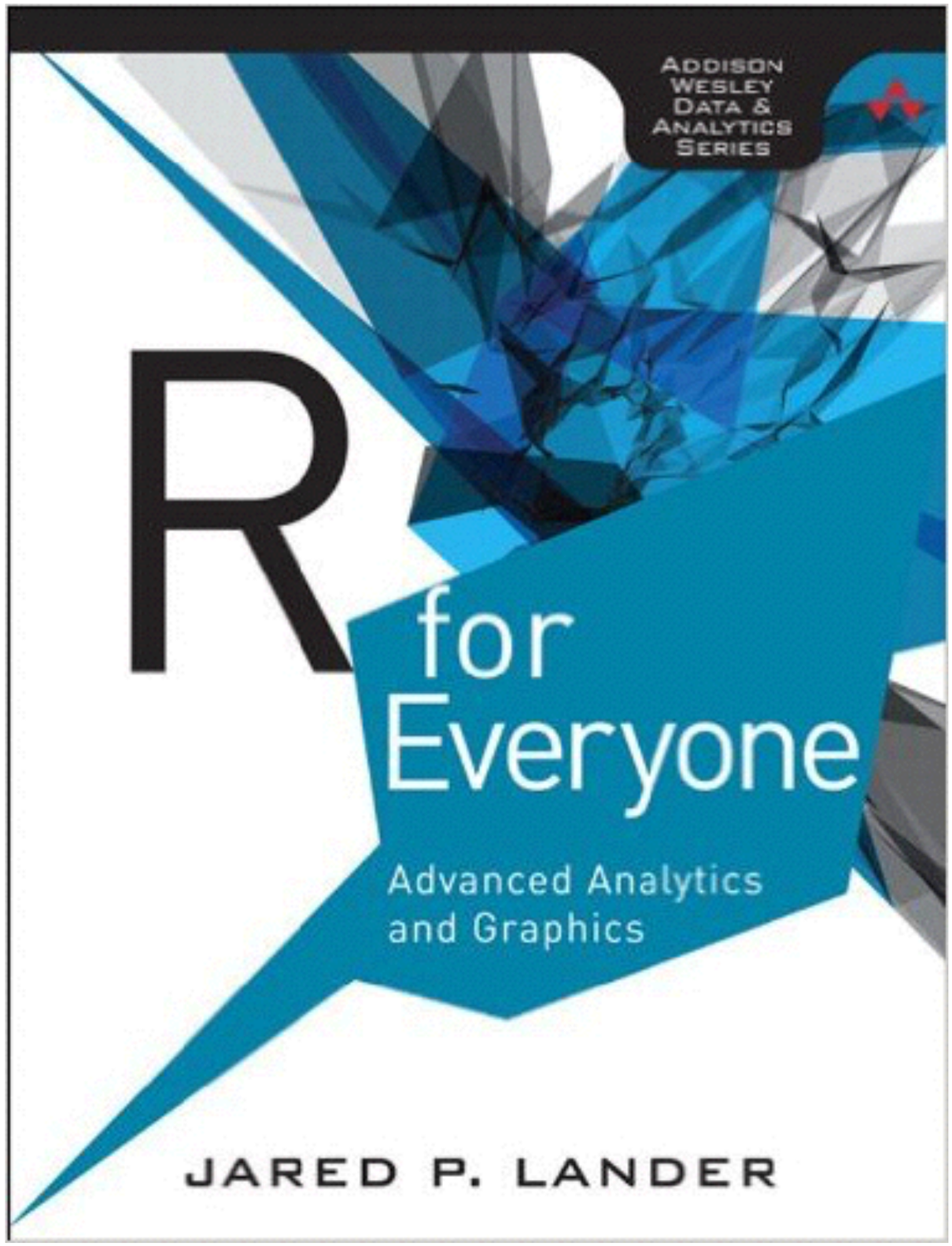
Need more References?



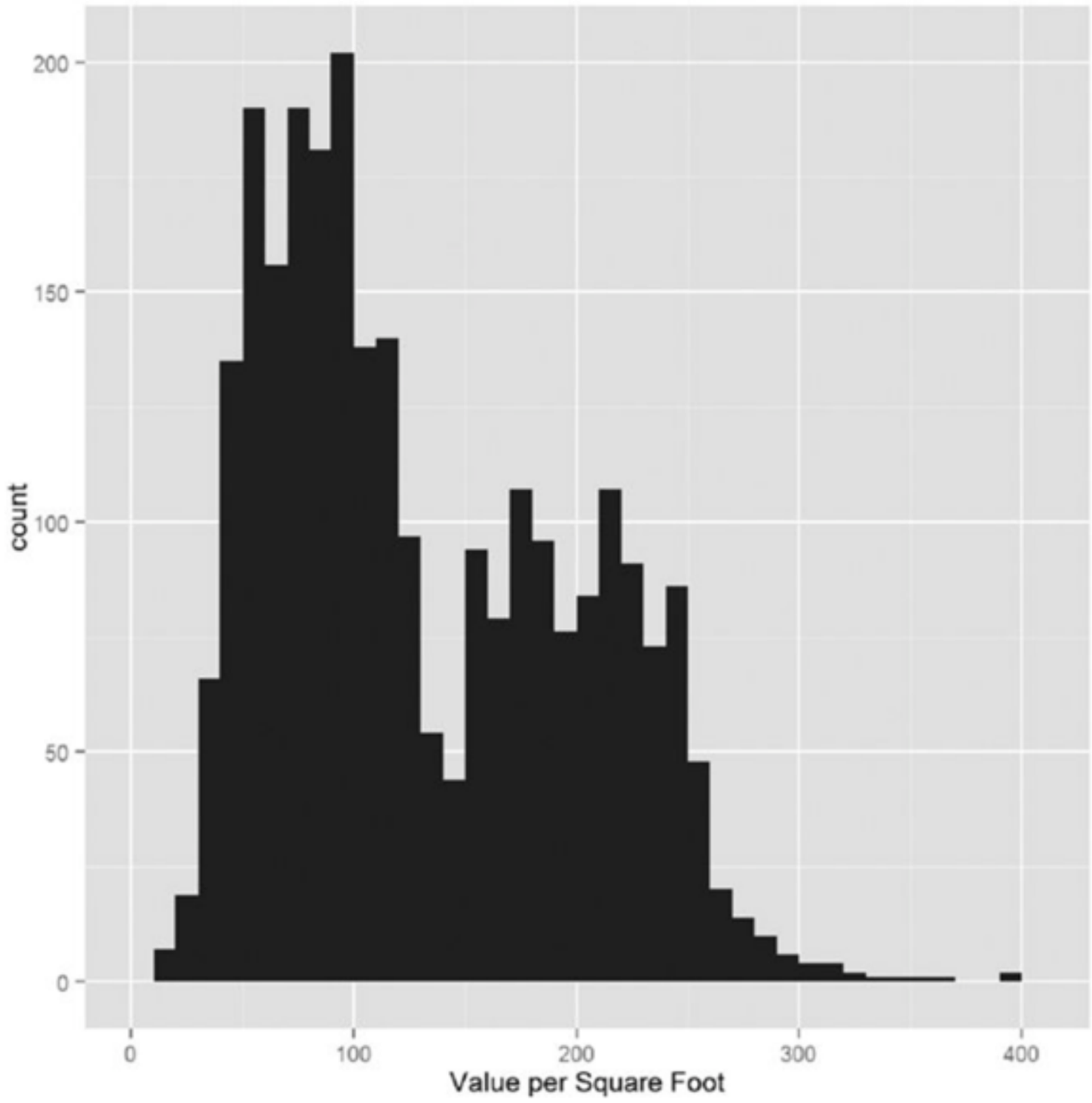
Sources: Abt, H. A. and Garfield, E. J. Am. Soc. for Info. Science & Tech. 53(13):1106-1112, Nov. 2002; Halevi, G. Res. Trends (32), March 2013; Beck, M., beckmw.wordpress.com July 2014. Humanties data estimated. Based on 1000-word pages.

http://www.phdcomics.com/comics.php?f=1823

NYC Housing Example

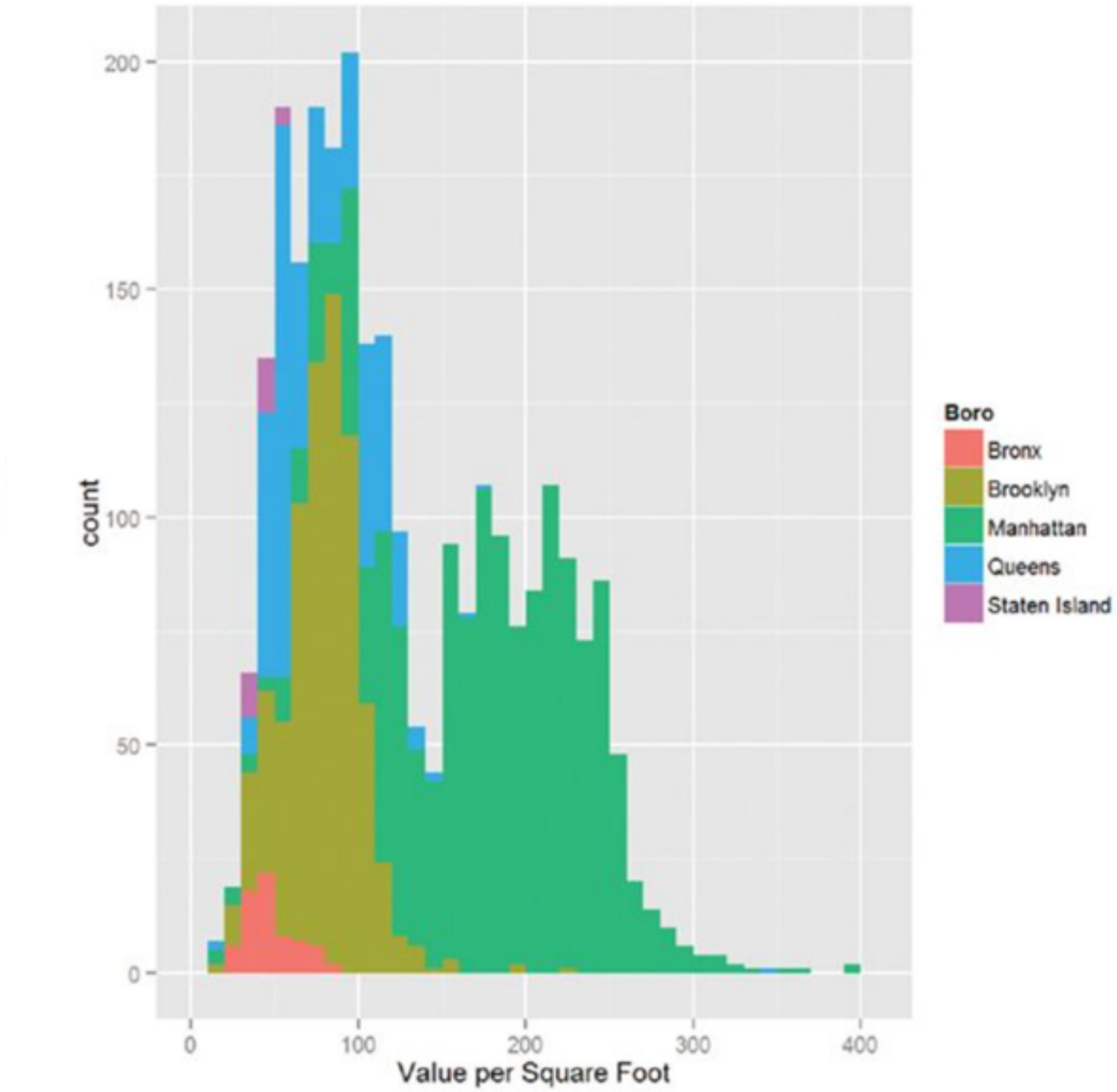


NYC Housing Example



Lander, R for Everyone; NYC Open Data

NYC Housing Example



Lander, R for Everyone; NYC Open Data

NYC Housing Example

- Response variable (y): price per square foot
- Predictor variables (x's): number of units in complex, number of square feet, borough indicators
- Try linear regression; it may help to take logs of some of the continuous variables first

It's perfectly Ok for the predictor variables X to be discreet variables, it's important to make sure the response variable Y is continuous non-binary.

Lander, R for Everyone; NYC Open Data

NYC Housing Example

What matters more is what're the interpretation of the parameters and how large are the effects

```
Coefficients:
                   Estimate   Std. Error t value  Pr(>|t|)
(Intercept)        4.430e+01     5.342e+00    8.293     <2e-16
***
Units             -1.532e-01     2.421e-02   -6.330   2.88e-10
***
SqFt               2.070e-04     2.129e-05    9.723    < 2e-16
***
BoroBrooklyn       3.258e+01     5.561e+00    5.858   5.28e-09
***
BoroManhattan      1.274e+02     5.459e+00   23.343    < 2e-16
***
BoroQueens         3.011e+01     5.711e+00    5.272   1.46e-07
***
BoroStaten Island -7.114e+00    1.001e+01   -0.711      0.477
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```
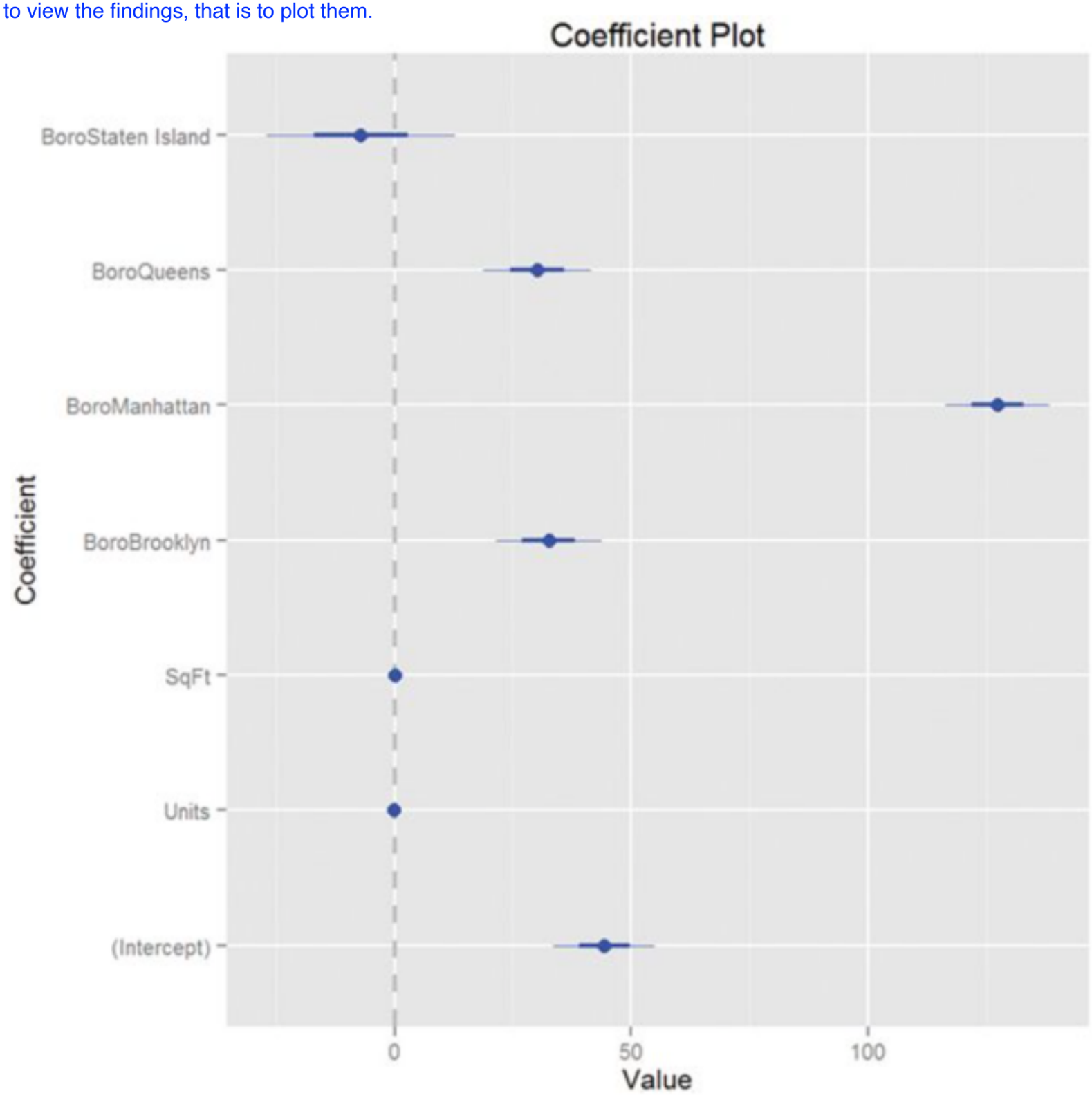

Lander, R for Everyone; NYC Open Data

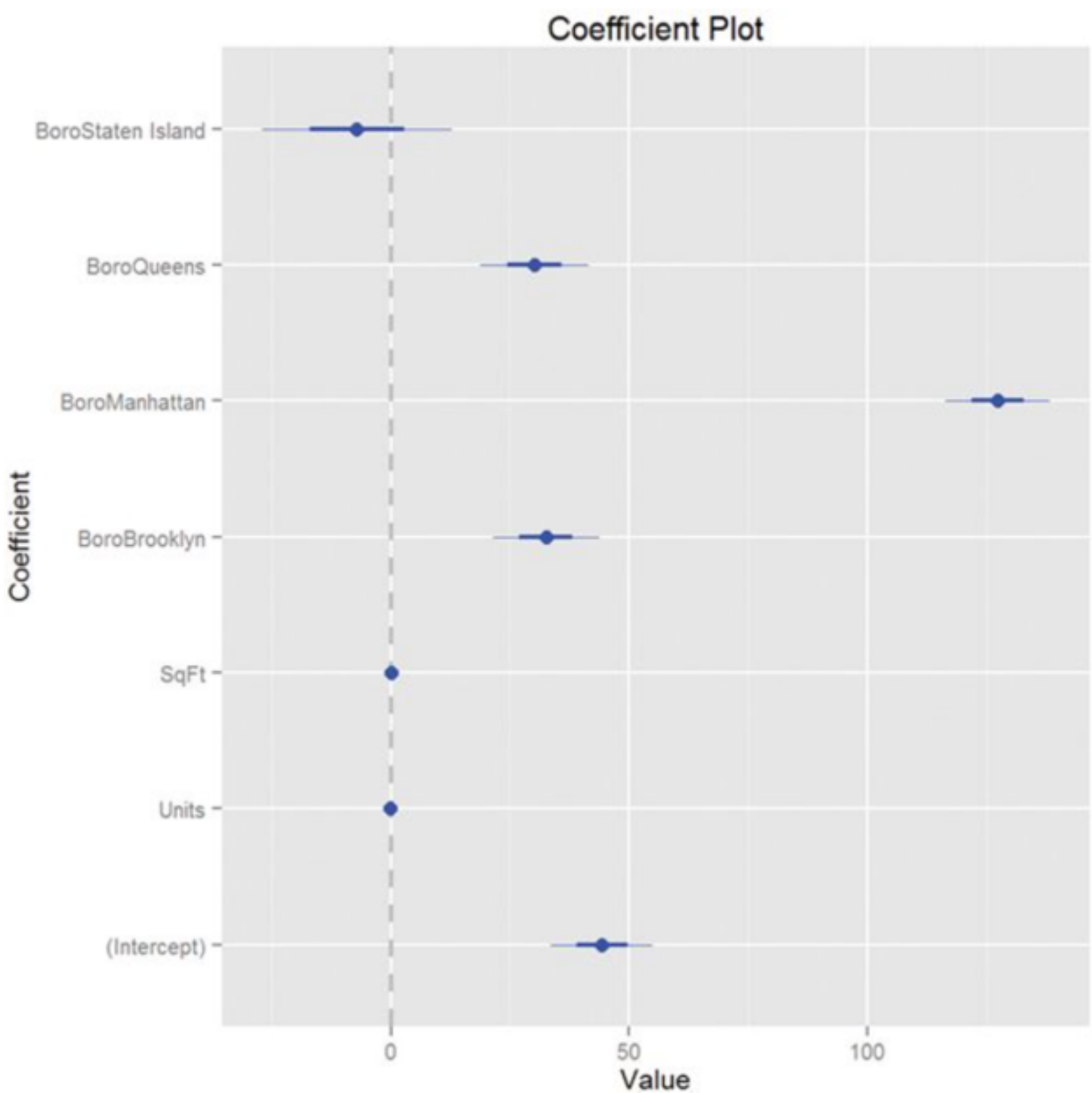
NYC Housing Example

This is a much better way to view the findings, that is to plot them.



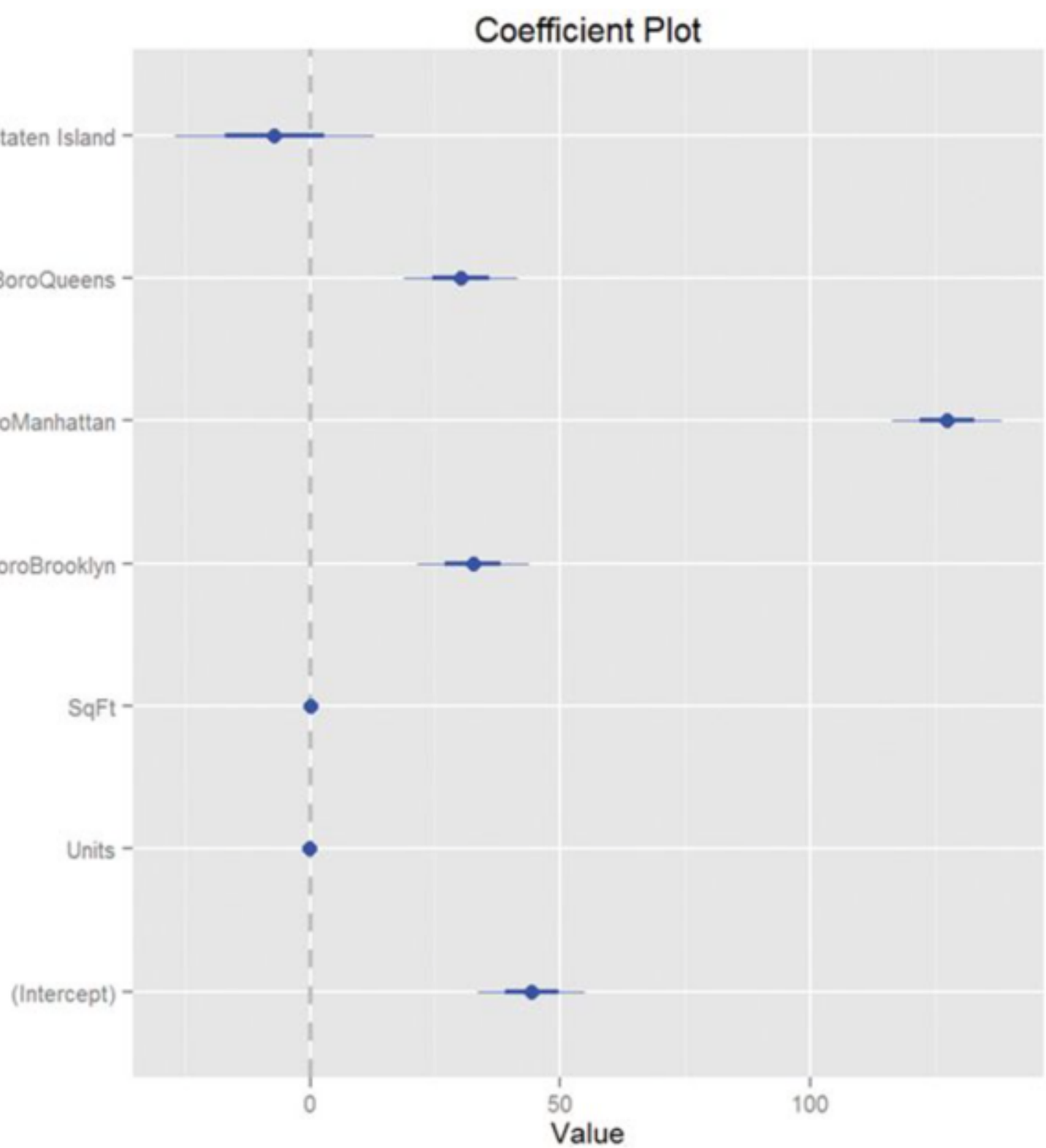
Point estimate surrounded by an error bar = estimations of probability.

Lander, R for Everyone; NYC Open Data



Where did the Bronx go?
Why are SqFT and Units so close to 0?

Lander, R for Everyone; NYC Open Data



Standardization would've been another route where you rescale/ center the variables where they have a mean of 0 and a standard eviation of 1.

This is useful if you want to put the variables on equal footing, but that would reduce interpretability because then you're talking about standardized prices vs. real housing prices.

Why are SqFT and Units so close to 0? Where did the Bronx go?

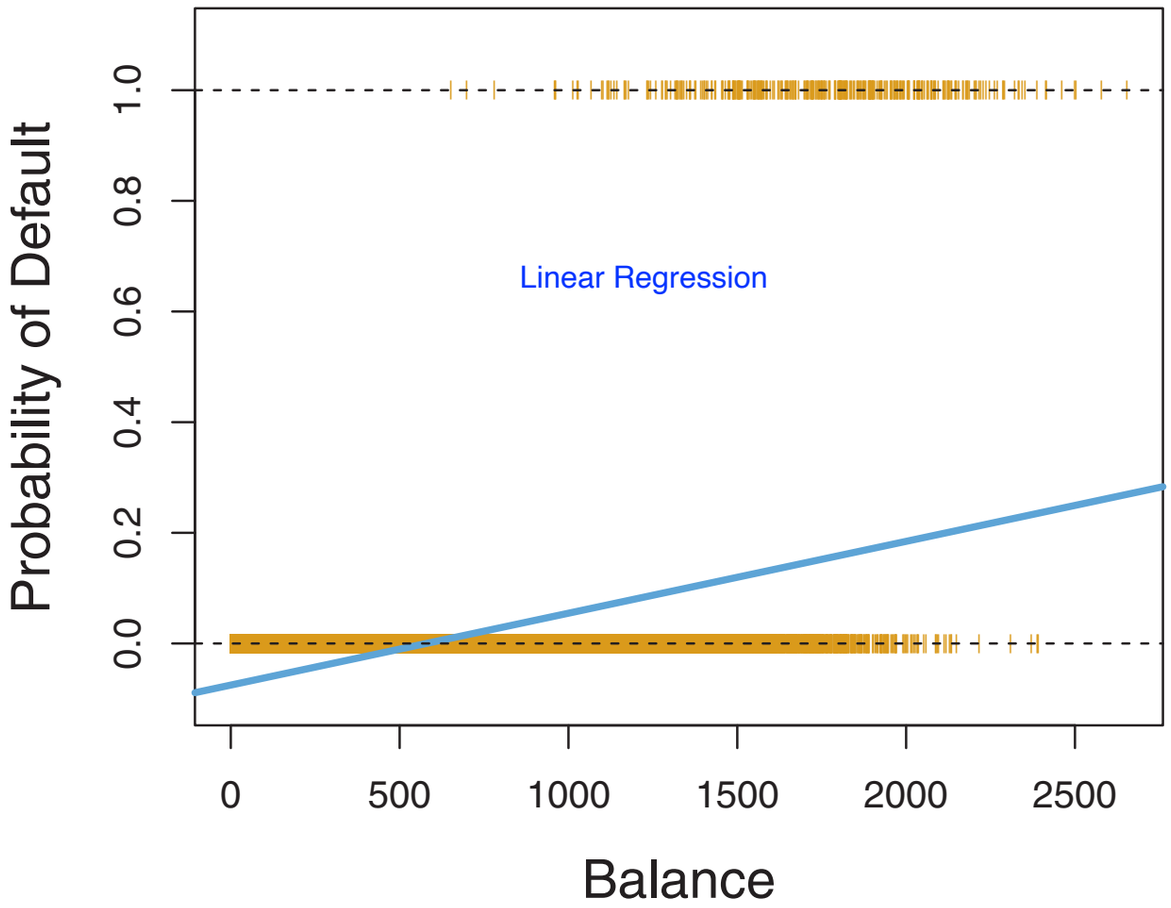
Lander, R for Everyone; NYC Open Data

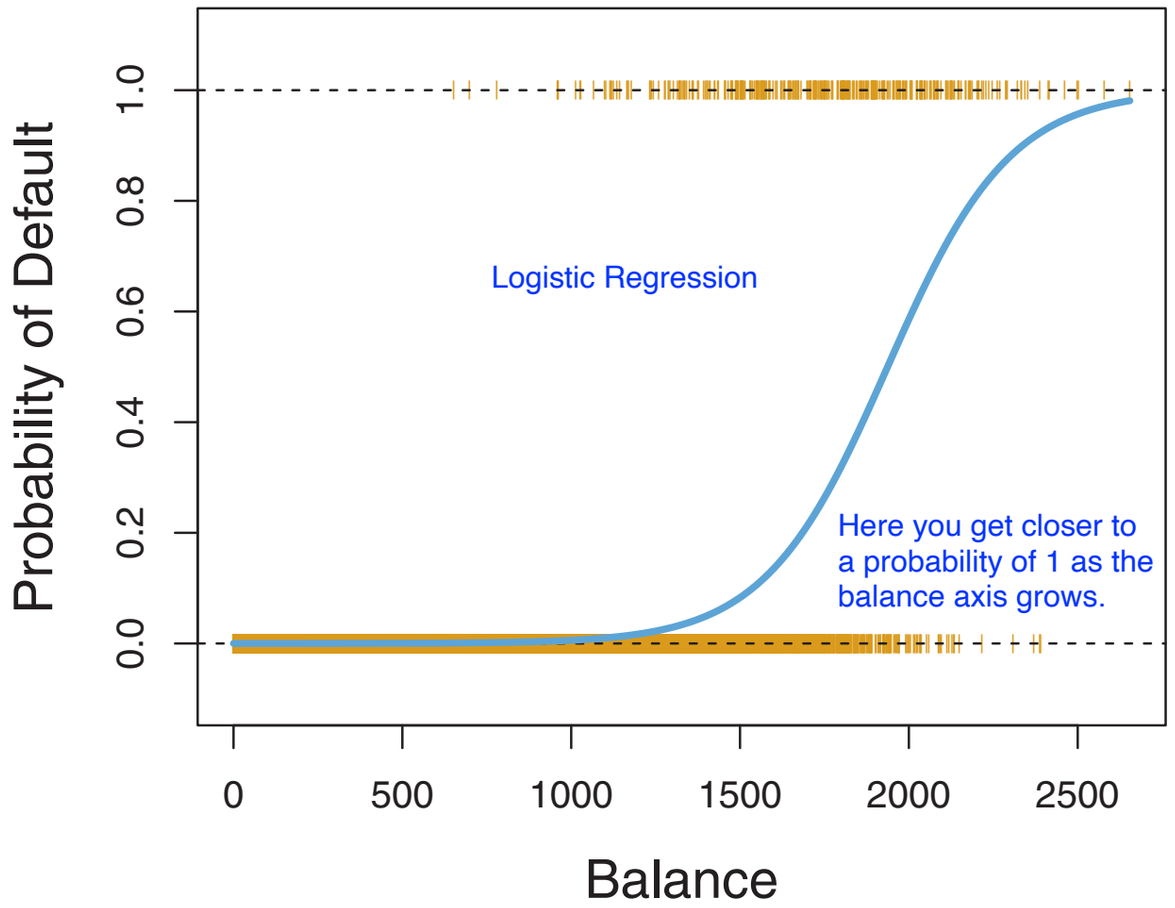
There is an effect for each square foot but because of the scale, the movement in price is small per each scare foot rather than the scale of the plot showing 50sq+ increments.

Collinearity

- Should avoid having predictor variables that are highly correlated with each other (collinearity results in instability, high variances in estimates, and worse interpretability)
- An extreme case of collinearity would be also including a Bronx indicator in the NYC Housing example. Instead, use one borough as a baseline.

Predicting a Binary Response





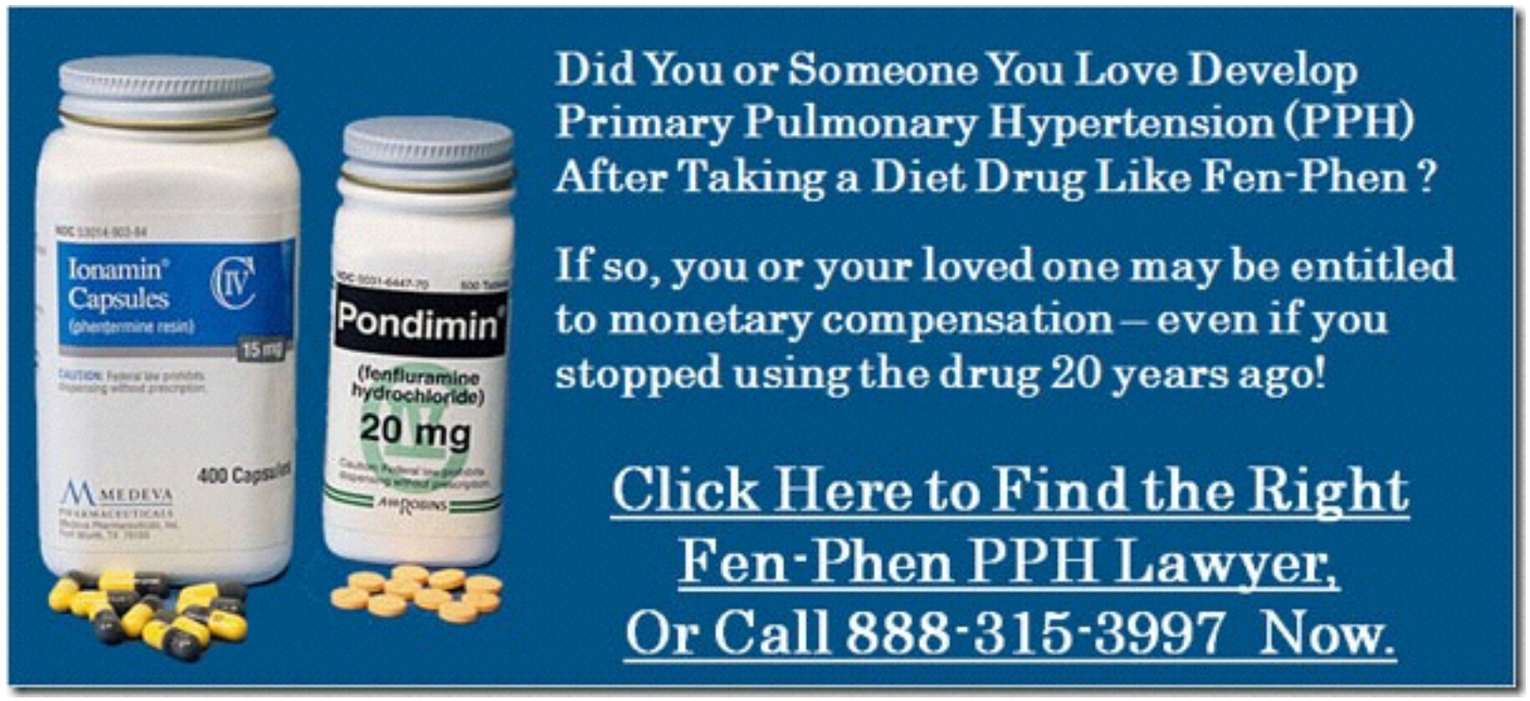
source: Introduction to Statistical Learning, James, Witten, Hastie, Tibshirani, http://www-bcf.usc.edu/~gareth/ISL/

Linear regression would be bad at this becuase it's trying to find the relationship along a continuous response variable verses a binary outcome.

Logistic Regression is a 'bread-and-butter' modeling technique when you're trying to predict a binary outcome

Fen-Phen Case Study

On July 8, 1997 Mayo Clinic investigators described 24 cases of valvular heart disease in patients taking the recently released appetite suppressant combination fen/phen (fenfluramine plus phentermine). The FDA issued an advisory to encourage reporting of similar cases.



Example from Stat S105 with Xiao-Li Meng, Yves Chretien

Strong Association?

Recall we obtained the following sample of patients in a follow-up study:

	Heart disease	No heart disease	Total
Fen/phen	53	180	233
Control	3	230	233

- So do you think there is strong association between heart disease and fen/phen usage?

- How would you defend your assertion scientifically? Can you just say "Well, 53/3=17.7 is very large to me"?

How about this one then?

Now suppose that instead of heart disease, you wanted to test whether fen/phen increased the risk of a rare type of cancer. Using the same patients, you observe that:

	cancer	No cancer	Total
Fen/phen	1	232	233
Control	0	233	233

Is that the strongest evidence of association one can ever get, since 1/0 is infinite?

You need a more principled way of determining how worried you should be.

Measures of Association: Odds Ratio

If someone's probability of experiencing an outcome is p , then that person's *odds* of the outcome are $p/(1-p)$

The *odds ratio* is the ratio of two different people's odds of some outcome. If people in group A have probability p_A of disease, and people in group B have probability p_B, then the odds ratio of group A vs. group B is

$$\text{Odds Ratio} = \frac{p_A}{1-p_A} \Big/ \frac{p_B}{1-p_B} = \frac{(1-p_B)p_A}{(1-p_A)p_B}$$

Crude Odds Ratio Estimate

The data in one study were as follows:

		Fen/phen		
		+	-	
Aortic Regurgitation	+	6	162	168
	-	13	2343	2356
		19	2505	2524

Also known as the cross product formula in epidemiology.

A crude estimate of the odds ratio is

$$\frac{6 \times 2343}{13 \times 162} = 6.7$$

This means the odds of heart disease are approximately 6.7 times higher for Fen-Phen users than non Fen-Phen users.

Palmieri V, Arnett DK, Roman MJ, Liu JE, Bella JN, Oberman A, Kitzman DW, Hopkins PN, Morgan D, de Simone G, Devereux RB. Appetite suppressants and valvular heart disease in a population-based sample: the HyperGEN study. Am J Med. 2002 Jun 15;112(9):710-5.

What about confounding factors?

The huge flaw in this calculation is that
there's no adjustment for confounding factors.

But what if there are confounding factors? For example, what if fen/phen users are more likely to be obese, and obesity increases the risk of heart disease?

We can set up a *logistic regression model* to predict a person's odds of heart disease, given the predictor variables.

Another famous model is pro-bit regression
and it's somewhat similar to logistic regression

We can also use this to *compare* fen/phen users vs. non-fen/phen users, controlling for the other predictors.

Then we can use the data to estimate the parameters, using Maximum Likelihood Estimation (MLE).

Variables in the model

$$Y = \begin{cases} 1, & \text{if cardiac valve abnormality} \\ 0, & \text{if not} \end{cases}$$

In logistic regression, it's not just predicting whether the result is yes or no, but what is the probability of one outcome vs. another, given all the X variables.

$$X_{fen} = \begin{cases} 1, & \text{if taking fen/phen} \\ 0, & \text{if not} \end{cases}$$

$$X_{age} = \text{subject's age}$$

$$X_{sex} = \begin{cases} 1, & \text{male} \\ 0, & \text{if female} \end{cases}$$

(plus other X-variables...)

$$p = P\left(Y = 1 \mid X_{fen}, X_{age}, X_{sex}, \ldots, X_k\right)$$

So, how is p related to all of these factors?

A logistic regression model

$$\text{logit}(p) = \ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_{fen}X_{fen} + \beta_{age}X_{age} + \beta_{sex}X_{sex} + \cdots + \beta_k X_k$$

Also said a ‘log odds’ called the logit

The parameters of the model (the β’s) are unknown, and are estimate from the data using MLE.

Trying to use a linear model, to predict the probability or logit or probability

This gave 1.84 as an estimate for the fen/phen parameter. How can that be interpreted?

Two patients, A and B, are the same age, same gender, and similarly identical on all other variables. Patient A has taken fen/phen and Patient B has not. The model predicts that

$$\text{logit}(p_A) = \ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_{age}X_{age} + \beta_{sex}X_{sex} + \cdots + \beta_k X_k + \beta_{fen}X_{fen}$$

$$\text{logit}(p_B) = \ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_{age}X_{age} + \beta_{sex}X_{sex} + \cdots + \beta_k X_k$$

$$\beta_{fen} = \text{logit}(p_A) - \text{logit}(p_B) = \ln\left(\frac{\frac{p_A}{1-p_A}}{\frac{p_B}{1-p_B}}\right)$$

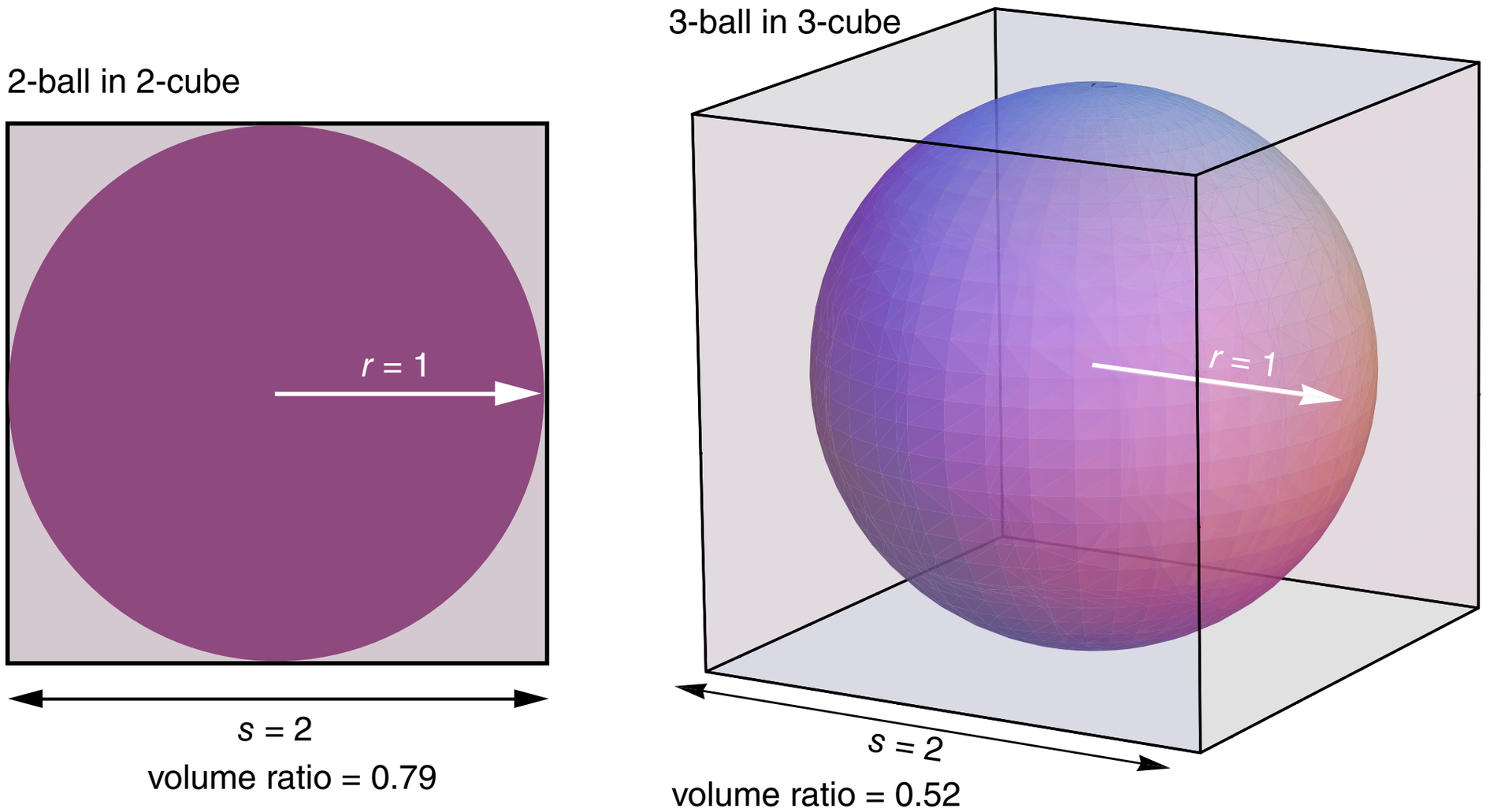
Using this model we can estimate an "adjusted" odds ratio that's the odds ratio for two people with all other known factors held constant:

$$e^{\hat{\beta}_{fen}} = e^{1.84} \approx 6.3$$

The 6.3 is lower than the crude amount calculated earlier which indicates some confounding but not a huge amount.

Curse of Dimensionality

For a uniformly random point in a box with side length 2, what is the probability that the point is in the unit ball?



source: An Adventure in the nth Dimension, Brian Hayes, American Scientist 2011

Curse of Dimensionality

For a uniformly random point in the box in d dimensions with length 2 in each dimension, what is the probability that the random vector is in the unit ball in d dimensions?

Most statistical methods were developed to try to understand problems that wee average or central. You're trying to undertand what happens in the middle.

d	probability
2	0.79
3	0.52
6	0.08
10	0.002
15	0.00001
100	$1.87 \cdot 10^{-70}$

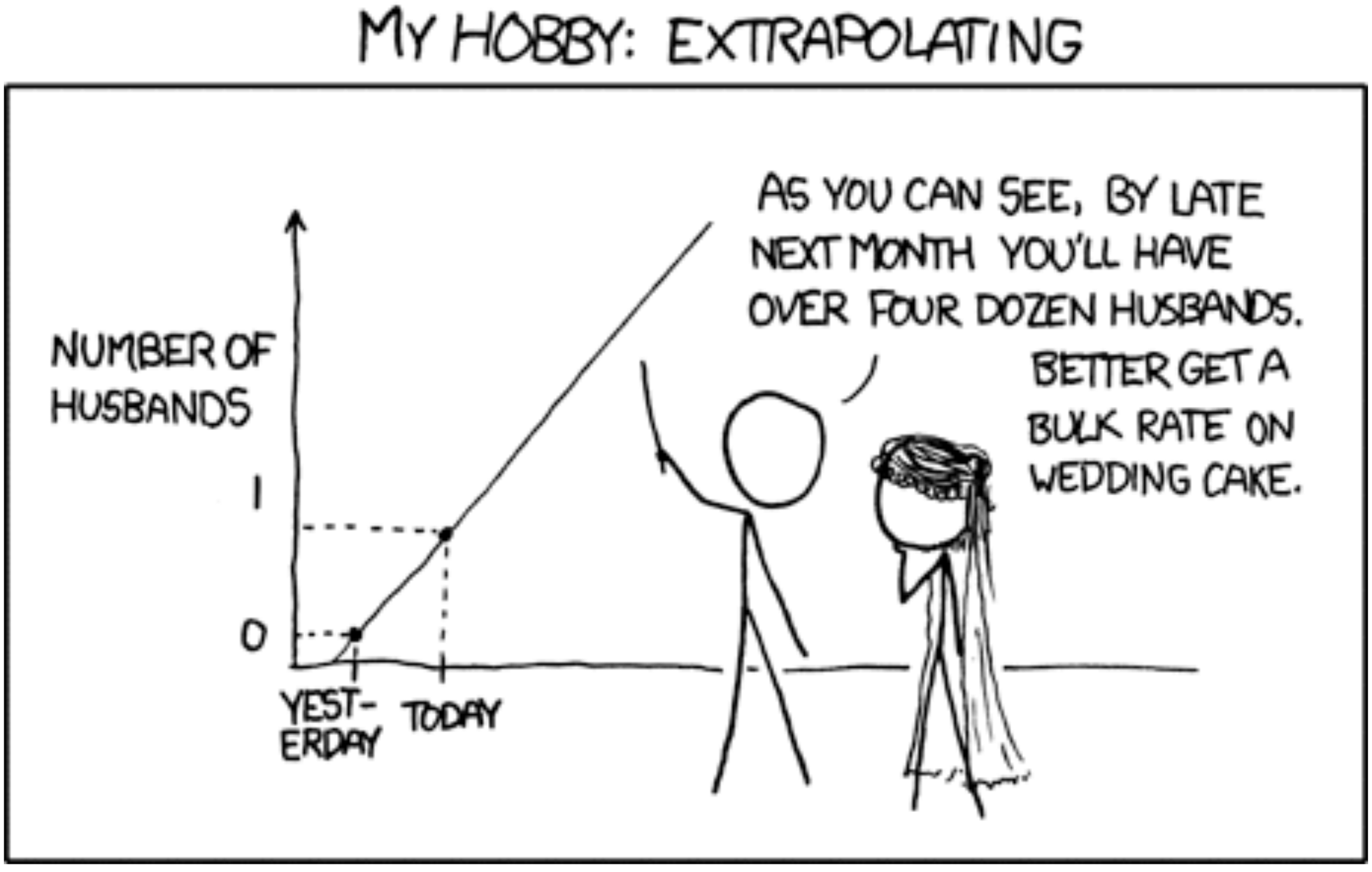
What this says is that by having 100 variables, almost all of that data exists out in the corniers, or rather not in the typical sphere or central area of the data.

This would be bad for many traditional statistical methods.

In many high-dimensional settings, the vast majority of data will be near the boundaries, not in the center.

Interpolation tries to figure out what happens in between data points vs. extrapolation takes current data point and predicts what's beyond them.

Interpolation vs. Extrapolation



In higher dimensions you may not have a choice but to extrapolate because the points are so far away from each other that you can't interpolate.

source: https://xkcd.com/605/

In high dimensions, nearest neighbor point tends to be very far away.
May be very hard to interpolate well, even with a lot of data points.

Blessing of Dimensionality

In statistics, “curse of dimensionality” is often used to refer to the difficulty of fitting a model when many possible predictors are available. But this expression bothers me, because more predictors is more data, and it should not be a “curse” to have more data....

With multilevel modeling, there is no curse of dimensionality. When many measurements are taken on each observation, these measurements can themselves be grouped. Having more measurements in a group gives us more data to estimate group-level parameters (such as the standard deviation of the group effects and also coefficients for group-level predictors, if available).

In all the realistic “curse of dimensionality” problems I’ve seen, the dimensions–the predictors–have a structure. The data don’t sit in an abstract K-dimensional space; they are units with K measurements that have names, orderings, etc.

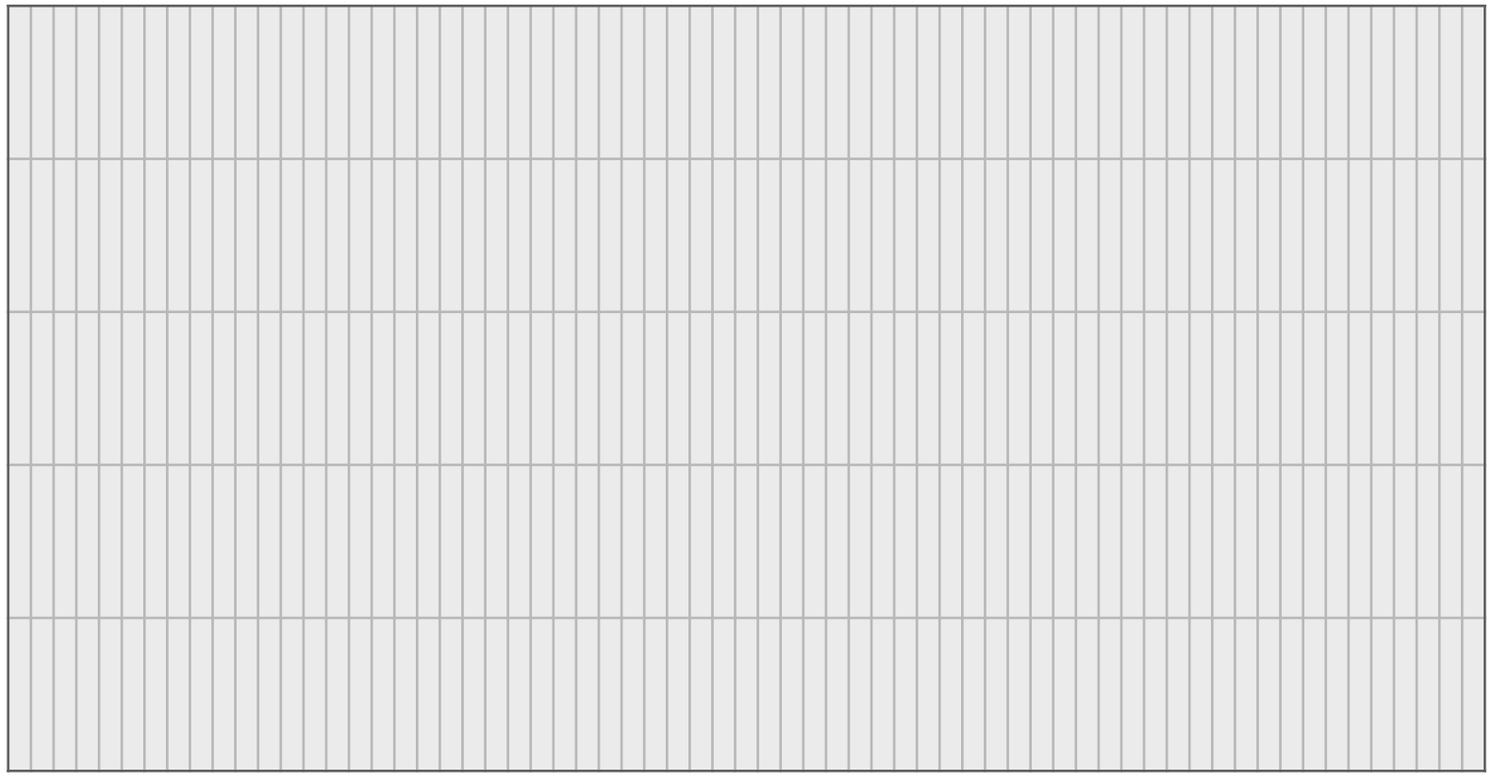
Andrew Gelman, http://andrewgelman.com/2004/10/27/the_blessing_of/

Hierachical Modeling is an approach towards dealing with high dimensions to your data.

Tall data vs. wide data

Traditional statistics prefers the example on the left where you don't have very many variables and you have a healthy number of partiicipants or observations.

vs.



n rows (individuals), p columns (variables)

p measurements

n people

Wide data are increasingly common in applications, e.g., neuroimaging, microarrays, MOOC data. But many traditional statistical methods assume n greater than p.

Ridge Regression and Shrinkage

In a linear regression model, in place of minimizing the sum of squared residuals, ridge regression says to minimize

$$\sum_{i=1}^{n}\left(y_i-\beta_0-\sum_{j=1}^{p}\beta_j x_{ij}\right)^2+\lambda\sum_{j=1}^{p}\beta_j^2=\mathrm{RSS}+\lambda\sum_{j=1}^{p}\beta_j^2,$$

The idea behind these regularization methods is that you create a penalty term. You don't want the sum of ridge betas to be too large.

Residual Sum of Squares

You start with OLS but you wan t to punish the model for being too large. YOu're still doing OLD, or minimizing the residuals, but subject to a constraint, that you don't want the sum of beta to be too large. We can't let the sum of squares grow too large but we're still trying to find the least sum of squares.

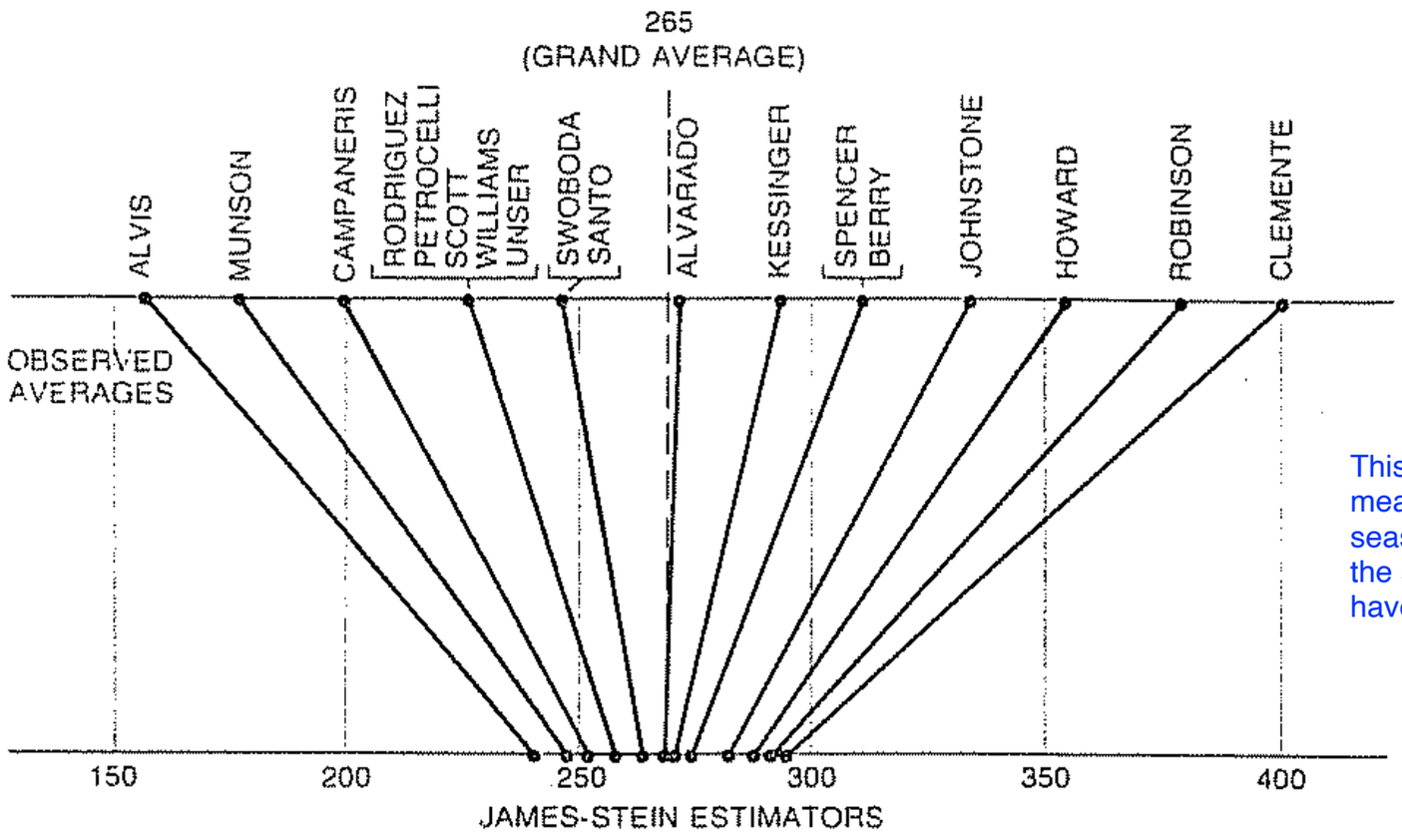
"Shrinkage" because it's shrinking back towards the origin

Stein's Paradox and Shrinkage Estimation

Let $y_1 \sim \mathcal{N}(\theta_1, 1), y_2 \sim \mathcal{N}(\theta_2, 1), \ldots, y_k \sim \mathcal{N}(\theta_k, 1)$ with $k \geq 3$. How should we estimate the vector θ, under sum of squared error loss?

Stein: the vector y is *inadmissible*; uniformly beaten by the James-Stein estimator

$$\hat{\theta}_j = \left(1 - \frac{k-2}{\sum_i y_i^2}\right) y_j.$$



JAMES-STEIN ESTIMATORS for the 18 baseball players were calculated by "shrinking" the individual batting averages toward the overall "average of the averages." In this case the grand average is .265 and each of the averages is shrunk about 80 percent of the distance to this value. Thus the theorem on which Stein's method is based asserts that the true batting abilities are more tightly clustered than the preliminary batting averages would seem to suggest they are.

Source: Efron-Morris, Scientific American 1977

inadmissible means that there is another estimator that dominates it in that the expected loss is always lower. No matter what theta is this alternative estimator will always be lower in terms of expected loss.

It is inadmissible to solve those problems separately and it's better if you combine the data and solve them together… for independent events.

Here, take the grand average and shrink future estimations towards that value. It allowed for a 3-fold improvement to the statistical prediction.

This is using the Regression to the mean as better players earlier in the season may end up doing worse as the seasons progresses. Here you have one bundled loss function.

LASSO and Sparsity

In a linear regression model, in place of minimizing the sum of squared residuals, LASSO says to minimize

$$\sum_{i=1}^{n}\left(y_i-\beta_0-\sum_{j=1}^{p}\beta_j x_{ij}\right)^2+\lambda\sum_{j=1}^{p}|\beta_j| \;=\mathrm{RSS}+\lambda\sum_{j=1}^{p}|\beta_j|.$$

This helps induce *sparsity*, reducing the number of variables one has to deal with.

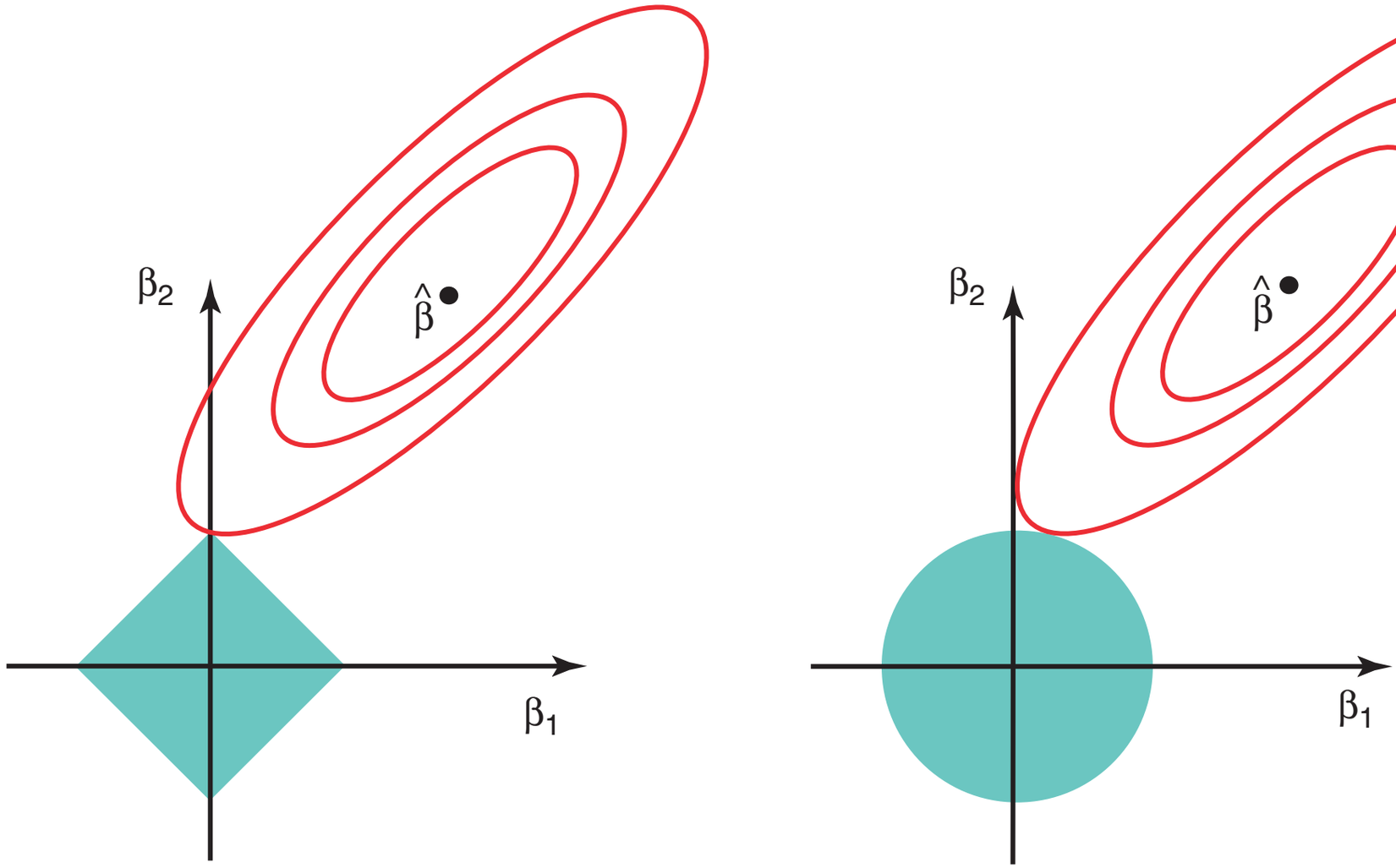
With Ridge Regression you're looking at the sum of squared betas. With LASSO you're doing sum of absolute values. So it's a different penalty term. This is penalizing the regression coefficient from growing too large.

Absolute value functions, if you plot them, are a V shape. A sharp corner means that a derivative doesn't exist.

LASSO vs. Ridge Constraints

Where Ridge will give you estimates towards 0, LASSO will give you actual 0 values for the estimates.

That makes it less overwhelming to have to deal with the model and it suggests which variables you should focus on, by shifting your attention to the ones that're not zero.



source: Introduction to Statistical Learning, James, Witten, Hastie, Tibshirani, http://www-bcf.usc.edu/~gareth/ISL/

This is better in situations where traiditional linear regressions break down.

If you're just doing OLS, then you don't get the gains of the high dimensional solutions like LASSO and Shrinkage.