The Prime Counting Function

Before we dive into an examination of the prime number theorem, we first define the prime counting function, $\pi(x)$, to be equal to the number of prime numbers less than or equal to x, where $x \in \mathbb{Z}$ (although when $x \leq 1$, $\pi(x) = 0$).

An efficient algorithm for computing $\pi(x)$ when x is relatively small was known to the ancient Greeks, and involves marking the multiples of known primes as composite integers. This is known today as the Sieve of Eratosthenes and can be used to compute $\pi(x)$ for the following values of x. It also may be of interest to compute $\frac{\pi(x)}{x}$, which gives us the proportion of prime numbers to positive integers, to begin to conjecture how the primes might be distributed, so let us compute this as well,

x	$\pi(x)$	$\frac{\pi(x)}{x}$
10	4	0.4
100	25	0.25
1,000	168	0.168
10,000	1,229	0.1229
100,000	9,592	0.09592
1,000,000	78,498	0.07849

Immediately, we can observe that prime numbers become less frequent as we look at larger values of x, since the proportion of prime numbers (i.e. $\frac{\pi(x)}{x}$) is decreasing. For example, in the first 100 positive integers, one-quarter of them are prime, while only about one-sixth of the first 1000 positive integers are prime.

At the age of 15, the great mathematician Carl Freidrich Gauss computed similar tables, and began to notice a pattern in the frequency of prime numbers. He observed that as x becomes large, the proportion of prime numbers to positive integers up to x (i.e. $\frac{\pi(x)}{x}$) is approximately $\frac{1}{\log x}$ (where $\log x$ is the natural logarithm).

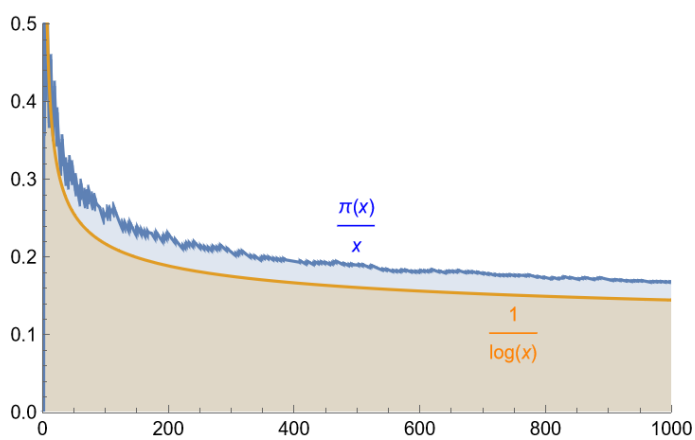

Figure 1: Plot of $\frac{\pi(x)}{x}$, $\frac{1}{\log x}$ (via Mathematica)

Therefore, to find an approximation for $\pi(x)$, or the number of primes less than or equal to x, it would make sense to compute a sum of these logarithms up to x,

$$\sum_{i=2}^{x} \frac{1}{\log i} = \frac{1}{\log 2} + \frac{1}{\log 3} + \cdots + \frac{1}{\log x}$$

This *logarithmic sum*, which can be denoted by $\mathrm{L_s}(x)$, has the property that,

$$\mathrm{L_s}(x) - 1.5 < \mathrm{Li}(x) < \mathrm{L_s}(x)$$

Where $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt$, is the *logarithmic integral*, and so we can conclude that the difference between $\mathrm{L_s}(x)$ and $\mathrm{Li}(x)$ is bounded.[1]

Another way to think of this, is to view $\frac{1}{\log x}$ as a sort of *probability density function* for an integer being prime. For example, if we assume that an integer being prime is a random event, then computing the approximate probability that a randomly chosen integer between 2 and x is prime would be being equivalent to finding the area under the curve of $\frac{1}{\log x}$.

Therefore, it is clear why Gauss made the following conjecture that, [2]

$$\mathrm{Li}(x) \approx \pi(x)$$

Around the same time, French mathematician Adrien-Marie Legendre independently conjectured that,

$$\frac{x}{\log x - B} \approx \pi(x)$$

Where $B \approx 1.08366$, and is known as Legendre's constant.[3] The origin of the exact value of the constant term B is unknown.[4] It has since been shown that if we let $B = 1$ instead, we get the most accurate approximation of $\pi(x)$ using the form $\frac{x}{\log x - B}$. [5]

A similar function to the one found in Legendre's conjecture is $\frac{x}{\log x}$. This function is intimately related to both the logarithmic integral and the approximation of $\pi(x)$. We will cover the derivation of this function after a brief comparison of our approximations of $\pi(x)$ that we have found so far. The following table, where we have rounded $\frac{x}{\log x}$ and $\mathrm{Li}(x)$ to the nearest integer, will help illustrate the point, [6] [7]

[1] Zagier, Don. *The first 50 million prime numbers.* The Mathematical Intelligencer, 1977
[2] Klyve, Dominic. *The Origin of the Prime Number Theorem.* Mathematical Association of America, 2019
[3] Weisstein, Eric. *Legendre's Constant.* Wolfram MathWorld.
[4] Fong, Wallace. *Prime and Logs, Gauss's Childhood Discovery.* SublimeBlog
[5] Rosser, J. Barkley and Schoenfeld, Lowell. *Approximate Formulas for Some Functions of Prime Numbers.* Illinois Journal of Mathematics, 1962
[6] Online Encyclopedia of Integer Sequences - Integer Nearest to $\mathrm{Li}(10^n)$
[7] Wolfram Alpha - Integer Nearest to $\frac{x}{\log x}$

x	$\pi(x)$	$\mathrm{Li}(x)$	$\frac{x}{\log x}$
10	4	6	4
100	25	30	22
1,000	168	178	145
10,000	1,229	1,246	1,086
100,000	9,592	9,630	8,686
1,000,000	78,498	78,628	72,382

It appears, based on our computations, that $\mathrm{Li}(x)$ is an overestimate for $\pi(x)$ and that $\frac{x}{\log x}$ is an underestimate, but that $\mathrm{Li}(x)$ is a far better estimate overall for $\pi(x)$. This is illustrated in Figure 2 below.

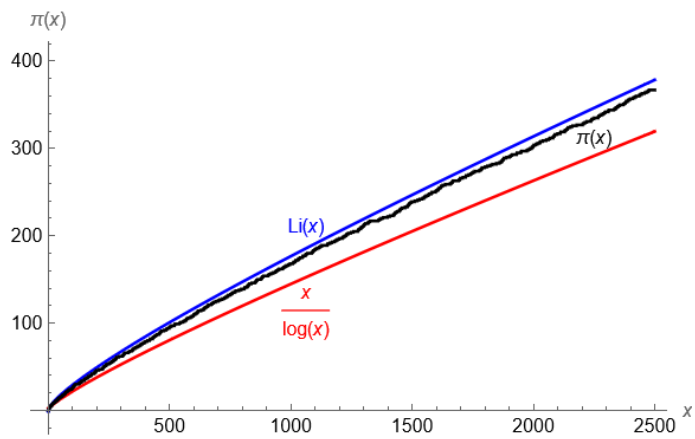

Figure 2: Plot of $\pi(x), \mathrm{Li}(x), \frac{x}{\log x}$ (via Mathematica)

To proceed, we define for functions $f(x), g(x)$, that $f(x) \sim g(x)$, if and only if,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

This relationship is known as *asymptotic equivalence*, and we say that f is *asymptotic* to g.

The concept of asymptotic equivalence will become crucial when we state the prime number theorem, the focus of this paper. A fact to keep in mind is that a careful distinction should be made when asserting the asymptotic equivalence of two functions. If $f(x) \sim g(x)$, this does not necessarily imply that $f(x) = g(x)$ eventually,[8] rather that as x becomes large, the two functions will have the same rate of change. This follows from a basic fact of calculus, because $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ which is finite and non-zero.[9]

To better understand the asymptotic equivalence of two functions, as will be stated in the

[8] Babai, Laszlo. *Asymptotic Equality and Inequality.* University of Chicago, 2021.

[9] Boelkins, Matt, Austin, David, and Schickler, Steve. *Active Calculus.* Grand Valley State University, 2014.

prime number theorem, we can explore the relationship between $\mathrm{Li}(x)$ and $\frac{x}{\log x}$ by looking at the end behavior of the quotient, $\frac{\left(\frac{x}{\log x}\right)}{\mathrm{Li}(x)}$. We will show that $\mathrm{Li}(x) \sim \frac{x}{\log x}$.

Proof. Recall that $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt$,

Then notice that because $\lim_{x\to\infty} \mathrm{Li}(x) = \infty$ and $\lim_{x\to\infty} \frac{x}{\log x} = \infty$, that $\lim_{x\to\infty} \frac{\left(\frac{x}{\log x}\right)}{\mathrm{Li}(x)}$ has the indeterminate form $\frac{\infty}{\infty}$. Since both functions are differentiable, we can compute,

$$\begin{aligned}
\frac{d}{dx}\left(\mathrm{Li}(x)\right) &= \frac{d}{dx}\int_2^x \frac{1}{\log t} dt \\
&\overset{FTC}{=} \frac{1}{\log x} \\
\frac{d}{dx}\left(\frac{x}{\log x}\right) &= \frac{\log x - 1}{\log^2 x}
\end{aligned}$$

Then applying L'Hospital's Rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\left(\frac{x}{\log x}\right)}{\mathrm{Li}(x)} &= \lim_{x\to\infty} \frac{\left(\frac{\log x - 1}{\log^2 x}\right)}{\frac{1}{\log x}} \\
&= \lim_{x\to\infty} \frac{\log^2 x - \log x}{\log^2 x} \\
&= \lim_{x\to\infty} 1 - \frac{1}{\log x} \\
&= \lim_{x\to\infty} 1
\end{aligned}$$

Therefore, $\mathrm{Li}(x) \sim \frac{x}{\log x}$. □

This fact will become important when we state the prime number theorem, which can be written as either $\pi(x) \sim \mathrm{Li}(x)$ or $\pi(x) \sim \frac{x}{\log x}$, precisely because the end behavior of $\mathrm{Li}(x)$ and $\frac{x}{\log x}$ are so similar.

Another way to arrive at this conclusion is to use repeated integration by parts on $\mathrm{Li}(x)$ to find that the leading term in its *asymptotic expansion* is $\frac{x}{\log x}$ and thus dictates its end behavior. Observe,

$$\begin{aligned}
\mathrm{Li}(x) &= \int_2^x \frac{1}{\log t} dt \\
&= \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{\log^2 t} dt \\
&= \frac{x}{\log x} - \frac{2}{\log 2} + \frac{x}{\log^2 x} - \frac{2}{\log^2 2} + \int_2^x \frac{2}{\log^3 t} dt
\end{aligned}$$

Now that we have established the asymptotic equivalence of $\mathrm{Li}(x)$ and $\frac{x}{\log x}$, let's return to our discussion of the prime counting function. As mentioned previously, in our computations of $\pi(x)$, we noticed that $\mathrm{Li}(x)$ was an overestimate for $\pi(x)$ for x values up to 10^6. Were we to use a more efficient algorithm to compute $\pi(x)$, this would continue to be the

case for x values even an order of magnitude larger. It would be natural to ask whether there is any positive integer x where $\mathrm{Li}(x) < \pi(x)$?

The answer to this question was discovered in the early 20th century by mathematicians, John Edensor Littlewood and Stanley Skewes. Littlewood proved, using techniques in both complex analysis and number theory, that there are infinitely many points such that $\mathrm{Li}(x) < \pi(x)$ [10], although he was not able to construct a specific x for which this was true.

Skewes was able to provide two upper bounds, known as Skewes's numbers, [11] depending on whether the Riemann hypothesis, an outstanding conjecture about the zeroes of function of a complex variable, was assumed to be either true or not. If the Riemann hypothesis is assumed, then $\mathrm{Li}(x) < \pi(x)$ for some $x < 10^{10^{10^{34}}}$. Without assuming the Riemann hypothesis, Skewes found a larger upper bound of $10^{10^{10^{10^3}}}$. Now let us remember an important fact that Don Zagier mentions in his paper, *The first 50 million primes.* The size of Skewes's numbers should put an immediate pause on any conclusions drawn about prime numbers based on numerical data alone.

Here we see for the first time a link between the Riemann hypothesis, which appears to live solely in the domain of complex analysis, and the distribution of the prime numbers. On the surface, it appears that a complex function and the distribution of the prime numbers should exist in completely different areas of mathematics with minimal overlap. A further exploration into analytic number theory, however, shows that there is a deep relationship between these two concepts. We will explore the connection between the Riemann hypothesis and the prime numbers more extensively in a later section, but first let us state the prime number theorem.

Prime Number Theorem

Using the definitions above, we can state the prime number theorem as simply as

$$\pi(x) \sim \int_2^x \frac{dt}{\log(t)} \sim \frac{x}{\log(x)}.$$

We proved that $\int_2^x \frac{dt}{\log(t)} \sim \frac{x}{\log(x)}$ above, but actually proving that these two estimate the number of primes up to x is extremely difficult. Although renowned mathematicians such as Legendre and Gauss produced their conjecture in the late 1700s and early 1800s, it wasn't until 1886 that this fact was finally proven. [12]

These early proof involved complex analysis; however, some elementary proofs do exist. Pollack describes the search for these elementary proofs in detail in his book *Not Always Buried Deep.* He concludes that these proofs are often less useful than the analytical ones. "For example, no elementary proof of the prime number theorem is known which gives an estimate for the error term of the same quality as what was obtained by de la Vallée-Poussin

[10] Lee, Christine. *On the difference* $\pi(x) - \mathrm{Li}(x)$. University of Manchester, 2008.
[11] Weisstein, Eric. *Skewes's Number.* Wolfram MathWorld.
[12] Klyve. *The Origin of the Prime Number Theorem*

already in 1899" [13].

Chebyshev's Bias & Primes in Arithmetic Progressions

When computing $\pi(x)$ for various values of x, it may be of interest to observe which types of primes occur most frequently. We can look at primes that occur in arithmetic progressions, a sequence of numbers that take the form $a+nd$, where $n \in \mathbb{Z}$, so that we have $\{a, a+d, a+2d, a+3d, \dots\}$.[14].

Thm. *Dirichlet's Theorem on Arithmetic Progressions*

If $\gcd(a,d) = 1$*, then there are infinitely many primes of the form* $a + nd$*.*

Two of these sequences that are of particular interest to us are primes of the forms $4n+1$ and $4n+3$. Let's begin the exploration of this topic by first comparing,[15]

x	$\pi_{4n+1}(x)$	$\pi_{4n+3}(x)$	$\pi(x)$
100	11	13	25
500	44	50	95
1,000	80	87	168
5,000	329	339	669
10,000	609	619	1,229
50,000	2,549	2,583	5,133

We can immediately make two conjectures by looking at this table. The first being that $\pi(x) = \pi_{4n+1}(x) + \pi_{4n+3}(x) + 1$, with the second being $\pi_{4n+3}(x) \geq \pi_{4n+1}(x)$ for any real number x. Our first conjecture is quite easy to prove, so we will show this fact.

Proof. We will show that all odd primes have the form $4n+1$ or $4n+3$ for $n \in \mathbb{Z}_{\geq 0}$.

Recall that $\mathbb{Z}/m\mathbb{Z} = \{0, 1, 2, \dots, m-1\}$, so $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$. Equivalently this is the set of all possible remainders when dividing a positive integer by 4. Notice that for an odd prime, p, that $p \not\equiv 0, 2 \mod 4$, as this would imply that p is even. Therefore, $p \equiv 1, 3 \mod 4$ or $p = 4n+1$, $p = 4n+3$. $\square$

Turning our attention to the second conjecture, we see that 19th century Russian mathematician, Pafnuty Chebyshev noticed this same phenomenon, which became known *Chebyshev's Bias*[16]. Although the vast majority of numerical data collected shows that $\pi_{4n+3}(x) \geq \pi_{4n+1}(x)$, we can make use of an offshoot of both Dirichlet's theorem and the prime number theorem to show that this is not in fact the case.

[13] Pollack. *Not Always Buried Deep*

[14] Chebyshev's Bias - Michael Rubinstein and Peter Sarnak

[15] Granville, Andrew and Martin, Greg. *Prime Number Races.* The American Mathematical Monthly, 2006.

[16] Rubinstein, Michael and Sarnak Peter. *Chebyshev's Bias.* Experimental Mathematics, 1994.

Thm. *Prime Number Theorem for Arithmetic Progressions*

Let $\pi_{an+d}(x)$ *denote the prime of primes less than or equal to* x *that are congruent to* $a \bmod d$*, where* $a, d \in \mathbb{Z}_{>0}$ *and* $\gcd(a,d) = 1$*. Then,* $\pi_{an+d}(x) \sim \frac{1}{\phi(d)} \frac{x}{\log x}$*, where* $\phi(d)$ *is Euler's totient function.* [17]

So in the case of $4n+1$ and $4n+3$, we first notice that $\phi(4) = 2$ and so,

$$\pi_{4n+1}(x) \sim \frac{1}{2} \frac{x}{\log x}$$
$$\pi_{4n+3}(x) \sim \frac{1}{2} \frac{x}{\log x}$$

Therefore, the prime number theorem for arithmetic progressions tells us that we should expect half of the primes to fall into each of the congruence classes.

[17] Sopronouv, Ivan. *A Short Proof of the Prime Number Theorem for Arithmetic Progressions*. University of Toronto.