Discussion of Nonlinear Control Problem

July 18, 2016

Date: 9/05/2016

Discussion 1: Using describing function as an uncertainty

Suppose that the nominal plant transfer function is G with the perturbed transfer function of the form $\hat{G} = G(1 + \Delta W_2)$ (W_2 is fixed, stable transfer function). If $\|\Delta\|_\infty < 1$, then as described in [1], W_2 can be chosen such that

$$\left| \frac{\hat{G}(j\omega)}{G(j\omega)} - 1 \right| \leq |W_2(j\omega)|, \quad \forall \omega \tag{1}$$

so that $W_2(j\omega)$ provides the uncertainty profile.

Let us consider the saturation describing function (with a slope of 1) given as follows:

$$N(A) = \begin{cases} 1, & \text{if } A \leq \delta \\ \frac{2}{\pi}\left[\sin^{-1}\frac{\delta}{A} + \frac{\delta}{A}\sqrt{1 - \left(\frac{\delta}{A}\right)^2}\right], & \text{if } A > \delta \end{cases} \tag{2}$$

where A is the amplitude of the sine signal that enters the saturation, δ is the value at which saturation occurs. Notice that $N(A) \in [0, 1]$; this means that $N(A)$ can be modeled as an uncertainty. If the saturation block is in series with the plant (as is usually the case), then the perturbed model can be described as $\hat{G} = GN(A)$, where G is the nominal model. In other words, $N(A)$ can be treated as a multiplicative perturbation of the nominal plant. By using the condition in (1), we will get the following constraint:

$$|N(A) - 1| \leq |W_2(j\omega)|, \quad \forall \omega \tag{3}$$

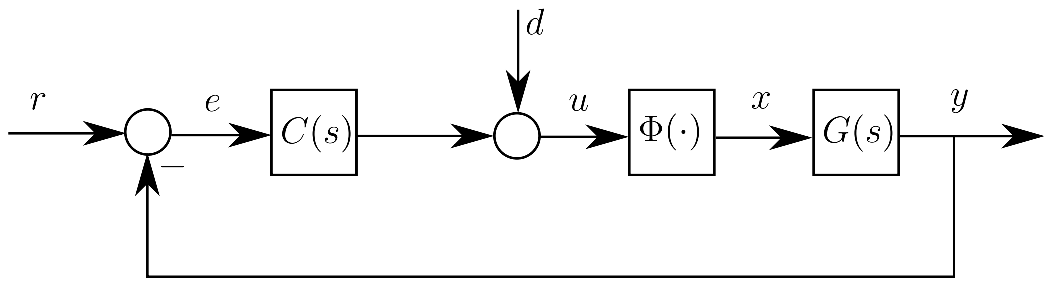

Figure 1: Block diagram with nonlinear component.

However, since $N(A) \in [0, 1]$, then we can consider $W_2(j\omega) \geq 1$ for all ω. Therefore, to reduce the conservatism, we can simply choose $W_2(j\omega) = 1$.

Discussion 2: Stability using circle criterion

I will now be deriving the conditions to achieve stability given a time-variant memoryless nonlinearity (in the frequency domain). As we had already spoken about, I will use the the circle criterion for this.

Suppose we have a closed-loop system with a nonlinearity Φ, a plant $G(s)$ and controller $C(s)$, as depicted in Fig. 1.

The type of nonlinearity that will be addressed here is of the sector type. The sector condition can be expressed as follows:

$$k_1 u \leq \Phi(t, u) \leq k_2 u, \quad \forall t \geq 0, \forall u \in [a, b] \tag{4}$$

where k_1, k_2, a and b are constants (with $k_2 > k_1$ and $a < 0 < b$). This condition can be interpreted as a nonlinearity which is bounded by two straight lines with slopes k_1 and k_2 that pass through the origin. Fig. 2 depicts this sector nonlinearity when (4) holds globally.

In order to analyze the stability, let us consider the autonomous form of the block diagram in Fig. 1, which is shown in Fig. 3. Let us define the open-loop system as $L(s) = C(s)G(s)$. The circle criterion states that given the sector nonlinearity in (4), then the closed-loop system is globally asymptotically stable at the origin if $L(j\omega)$ satisfies the Nyquist criterion and

$$\Re\left[\frac{1 + k_2 L(j\omega)}{1 + k_1 L(j\omega)}\right] > 0, \quad \forall\omega \tag{5}$$

For $L(j\omega)$ to satisfy the Nyquist criterion, it must encircle the point $-1/k_1 + j0$ m times in the counterclockwise direction, where m is the number poles

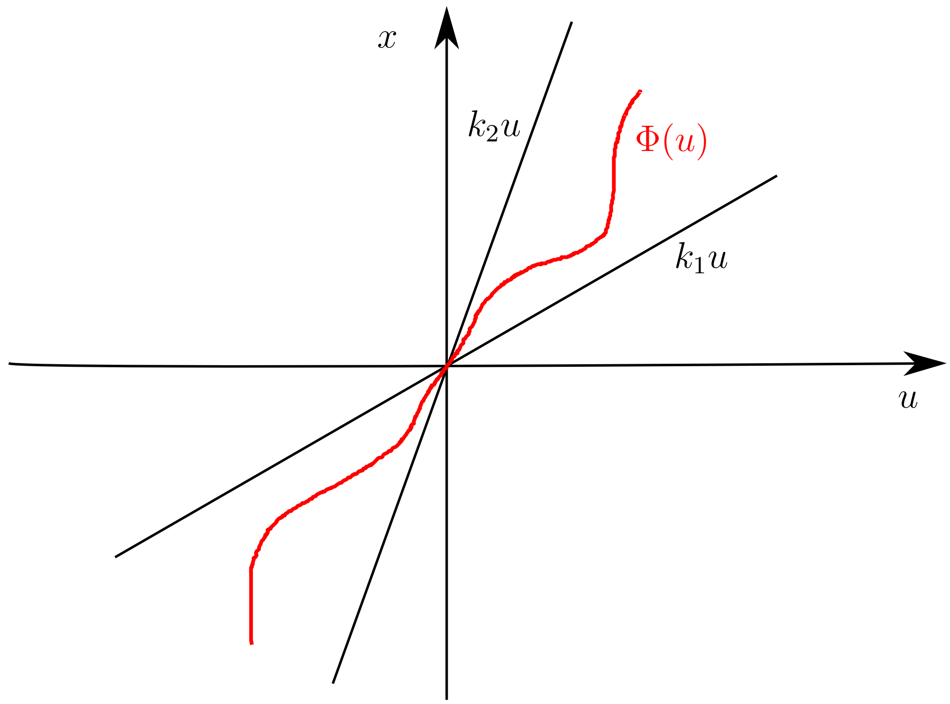

Figure 2: Nonlinear sector

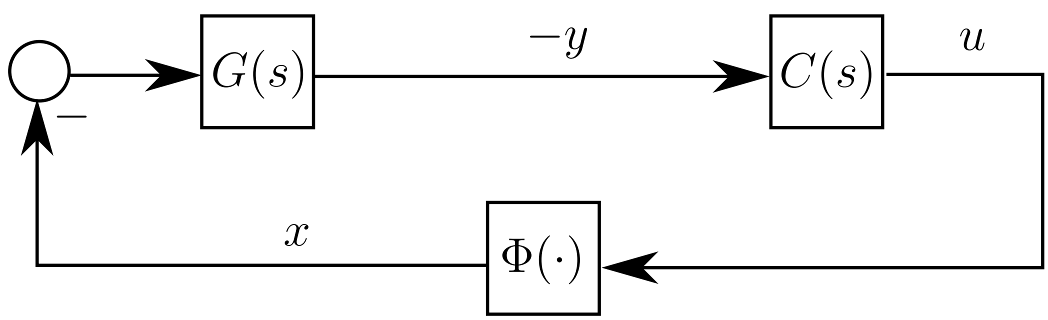

Figure 3: Equivalent block diagram in autonomous form

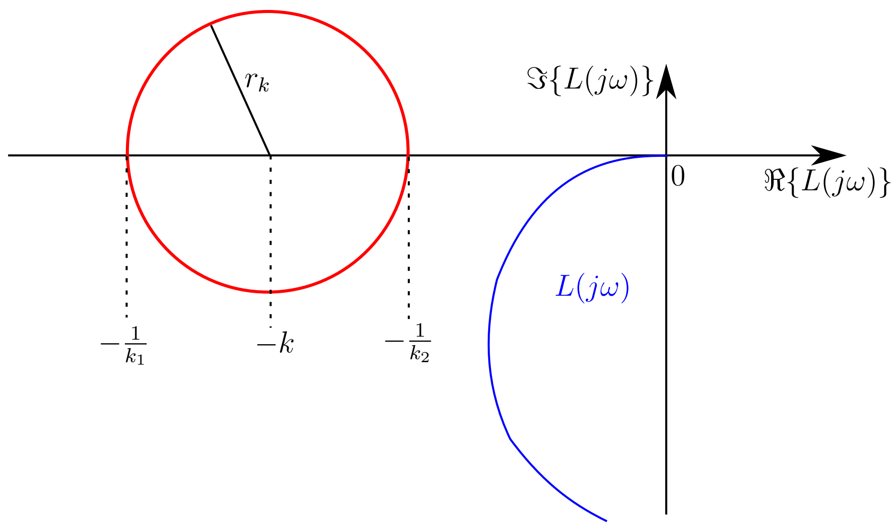

Figure 4: Absolute stability condition for sector nonlinearity in (4) (for $0 < k_1 < k_2$)

of $L(s)$ in the RHP. Let us first consider the case when $0 < k_1 < k_2$; then the condition in (5) can be interpreted as a disk $D(k_1, k_2)$ in the complex plane that is centered at $k = (k_1 + k_2)(2k_1k_2)^{-1}$ with radius $r_k = k - 1/k_2$ that does not include or intersect with $L(j\omega)$. In other words, the Nyquist plot of $L(j\omega)$ must be strictly outside the disk $D(k_1, k_2)$. Fig. 4 displays a graphical interpretation of this condition.

Definition 1: *Consider an open-loop system $L(s)$ with $\Phi(\cdot,\cdot)$ that satisfies the sector condition in (4) globally. Then the system is absolutely stable for $0 < k_1 < k_2$ if the Nyquist plot of $L(j\omega)$ does not enter the disk $D(k_1, k_2)$ and encircles it m times in the counterclockwise direction, where m is the number of poles of $L(s)$ with positive real parts.*

This condition can be expressed as follows:

$$\sup_{\omega} \left| \frac{r_k}{k + L(j\omega)} \right| < 1 \tag{6}$$

This inequality will ensure that the Nyquist plot of $L(j\omega)$ will never intersect the disk $D(k_1, k_2)$. Notice that this condition looks similar to the H_∞ norm of the weighted sensitivity function (i.e., when the weight $W = r_k$ and $k = 1$).

Theorem 1. *Suppose that $C(s)$ is linearly parameterized. A convex constraint which ensures the absolute stability of the autonomous system with* *the sector nonlinearity in (4) can be asserted as follows:*

$$r_k|k + L_d(j\omega)| - \Re\{[k + L_d(-j\omega)][k + L(j\omega)]\} < 0, \quad \forall\omega \tag{7}$$

where $L_d(j\omega)$ is the FRF of the desired open-loop transfer function.

Proof :
We know that

$$\Re\{[k + L_d(-j\omega)][k + L(j\omega)]\} \le |[k + L_d(-j\omega)][k + L(j\omega)]| \tag{8}$$

Therefore, noting that $|k + L_d(-j\omega)| = |k + L_d(j\omega)|$, we can conclude the following:

$$r_k - |k + L(j\omega)| < 0, \quad \forall\omega \tag{9}$$

which leads directly to (6). To ensure stability, we must show that the Nyquist criterion is met for $L(j\omega)$. Notice that (7) implies that

$$\Re\{[k + L_d(-j\omega)][k + L(j\omega)]\} > 0 \tag{10}$$

which in turn implies the following:

$$\mathrm{wno}\{[k + L_d(-j\omega)][k + L(j\omega)]\} = 0 \tag{11}$$

where "wno" stands for the winding number of the Nyquist plot around the origin. Therefore, we can conclude that

$$\mathrm{wno}[k + L_d(j\omega)] = \mathrm{wno}[k + L(j\omega)] \tag{12}$$

Thus if L_d meets the Nyquist stability requirement, then so will L. ■

H_∞ Performance for Sector Nonlinearity

The constraint in Theorem 1 guarantees that the system is globally asymptotically stable at the origin for the sector nonlinearity. We will now investigate how we can achieve H_∞ performance with the sector nonlinearity. In one of the papers with Gorka, you derived a sufficient condition to guarantee H_∞ performance by loop-shaping $L(j\omega)$ around a desired L_d:

$$\gamma^{-1}|W_1(j\omega)(1 + L_d(j\omega))| - \Re\{[1 + L_d(-j\omega)][1 + L(j\omega)]\} < 0, \quad \forall\omega \tag{13}$$

where γ is minimized. Let us replace the nonlinearity $\Phi(\cdot)$ in Fig. 1 with a simple gain $Q > 0$. We can define the modified plant as $G_2 = QG$. The output can then be expressed as $x = Qu$, which is a line in the $u - x$ plane in Fig. 2 that passes through the origin and has slope Q. Notice that (13) is convex in Q. Therefore, if (13) is satisfied for $Q = k_1$ AND $Q = k_2$, and because it is convex in Q, then (13) will be satisfied for all $Q \in [k_1, k_2]$. If this is the case, then H_∞ performance is guaranteed for the nonlinearity sector defined in (4) because the performance is ensured for all slopes between k_1 and k_2. In other words, you can always select a slope $Q \in [k_1, k_2]$ such that it intersects with the sector nonlinearity in (4). Therefore, one can impose the following optimization problem to guarantee the performance for the nonlinear sector in (4):

$$\begin{array}{ll} \underset{\{C,\gamma\}}{\text{minimize}} & \gamma \\ \text{subject to:} & \gamma^{-1}|W_1(j\omega)(1 + L_d(j\omega))| - \Re\{[1 + L_d(-j\omega)][1 + QL(j\omega)]\} < 0 \\ & Q \in \{k_1, k_2\} \\ & \forall\omega \end{array} \tag{14}$$

Comments by AK for the second discussion:

- Although the stability constraint in Theorem 1 seems to be fine the performance constraint in (14) is unlikely valid for time-varying sector non-linearilty. It should be fine with time-invariant sector non-linearity.
- The condition in Theorem 1 is a sufficient condition. You could develop a necessary and sufficient condition using your formulation instead of that of Gorka.

References

[1] C. J. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. New York: Mc Millan, 1992.