Discussion of H_2 Control Problem

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Discussion 1: Defining the optimality for the H_2 problem

You define the H_2^2 optimization problem (in page 3 of your note) to be the following:

$$
\begin{aligned}
\underset{\{X,Y,\gamma\}}{\text{minimize}} \quad & \sum_{k=1}^{Q} \gamma(\omega_k) \\
\text{subject to:} \quad & |W_1 M Y(\omega_k)|^2 < \gamma(\omega_k) |N X(\omega_k) + M Y(\omega_k)|^2 \\
& k = 1, \ldots, N
\end{aligned} \tag{1}
$$

Suppose that there exist optimal controllers X_0, $Y_0 \in \mathbf{RH}_\infty$ which satisfy the H_2^2 control problem. Then is it correct to state that the optimal solution $\gamma_0(\omega)$ is

$$\gamma_0(\omega) = \left| \frac{W_1 M Y_0(\omega)}{N X_0(\omega) + M Y_0(\omega)} \right|^2 \tag{2}$$

Answer by AK: Yes

Answer by AN:

In this case, I think that the proof for convergence is very similar to the one you presented in the journal paper sent to International Journal of Robust and Nonlinear Control (for the H_∞ criterion).

Let X_n^* and Y_n^* be the projections of X_0 and Y_0 into the subspace spanned by an n dimensional orthogonal basis functions and define γ_n^* as:

$$\gamma_n^*(\omega) = \left| \frac{W_1 M Y_n^*(\omega)}{N X_n^*(\omega) + M Y_n^*(\omega)} \right|^2 \tag{3}$$

Assume that $\gamma_n^*(\omega)$ is bounded and that $NX_n^* + MY_n^*$ has no zeros on the imaginary axis (which you proved in the journal paper). Since $\gamma_0(\omega_k) > 0$ and $\gamma_n^*(\omega_k) > 0$ for every k, then it is convenient to investigate the convergence of $|\sqrt{\gamma_n^*(\omega_k)} - \sqrt{\gamma_0(\omega_k)}|$ instead of $|\gamma_n^*(\omega_k) - \gamma_0(\omega_k)|$ as $n \to \infty$. By using the reverse triangle inequality, we will have the following condition:

$$\begin{aligned} \left|\sqrt{\gamma_n^*(\omega_k)} - \sqrt{\gamma_0(\omega_k)}\right| &\leq \left|\frac{W_1 M Y_n^*(\omega_k)}{NX_n^*(\omega_k) + MY_n^*(\omega_k)} - \frac{W_1 M Y_0(\omega_k)}{NX_0(\omega_k) + MY_0(\omega_k)}\right| \\ &\leq \left|\frac{W_1 MN[X_n^*(Y_0 - Y_n^*)(\omega_k) - Y_n^*(X_0 - X_n^*)(\omega_k)]}{[NX_n^*(\omega_k) + MY_n^*(\omega_k)][NX_0(\omega_k) + MY_0(\omega_k)]}\right| \\ &\qquad \text{for } k = 1, \ldots, Q \end{aligned} \tag{4}$$

X_n^* and Y_n^* are the projections of X_0 and Y_0, respectively, into the subspace spanned by orthogonal basis functions; therefore, according to [1],

$$\begin{aligned} \lim_{n\to\infty} \|X_0 - X_n^*\|_p = 0 \\ \lim_{n\to\infty} \|Y_0 - Y_n^*\|_p = 0 \end{aligned} \tag{5}$$

where $p \in (1, \infty)$. Therefore, since the frequency functions in (4) are bounded for every k, and the denominator has no zero on the imaginary axis, then

$$\begin{aligned} \lim_{n\to\infty} \left|\sqrt{\gamma_n^*(\omega_k)} - \sqrt{\gamma_0(\omega_k)}\right| = 0 \Longleftrightarrow \lim_{n\to\infty} \sqrt{\gamma_n^*(\omega_k)} = \sqrt{\gamma_0(\omega_k)} \\ \text{for } k = 1, \ldots, Q \end{aligned} \tag{6}$$

Since $\sqrt{\gamma_n^*(\omega_k)}$ is a convergent sequence, then so is $\gamma_n^*(\omega_k)$; therefore, it is evident that

$$\begin{aligned} \lim_{n\to\infty} \gamma_n^*(\omega_k) = \lim_{n\to\infty} \left[\sqrt{\gamma_n^*(\omega_k)}\right]^2 = \left[\lim_{n\to\infty} \sqrt{\gamma_n^*(\omega_k)}\right]^2 = \left[\sqrt{\gamma_0(\omega_k)}\right]^2 = \gamma_0(\omega_k) \\ \text{for } k = 1, \ldots, Q \end{aligned} \tag{7}$$

The above condition shows that as $n \to \infty$, the solution obtained with projections X_n^* and Y_n^* converges to the optimal solution.

Answer by AK: The problem is that the optimization in (1) is not convex, while for the H_∞ case we could make the optimization quasi-convex and solve it by bisection algorithm. In fact we should have two properties: first the convexity of the optimization problem and second the convergence to the optimal solution by increasing the order. What you did was to show the convergence for a non convex problem. This result is not useful because in a non-convex optimization you cannot show that the local optimal solution is monotonically non-increasing.

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Discussion 2: Proving the optimality for the H_∞ problem

So now we must re-prove the theorem for converging to the optimal solution for the H_∞ case. I have one mathematical question I would like to pose. Suppose we have 2 real numbers x and y. Then we know that

$$|x - y| = 0 \text{ if and only if } x = y \tag{8}$$

Now suppose that X_0 is an optimal controller for the H_∞ problem and that X_n is the controller with an n-dimensional orthogonal basis function (for the convex problem). Then we know that

$$\lim_{n\to\infty} \|X_0 - X_n\|_p = 0 \tag{9}$$

where $p \in (1, \infty)$. Can the above equation be equivalently stated as

$$\lim_{n\to\infty} \|X_0 - X_n\|_p = 0 \text{ if and only if } X_0 = X_{n\to\infty} \tag{10}$$

Answer by AK: X_0 cannot be equal to X_n because they do not have the same structure.

Answer by AN: Ah yes, sorry I forgot to put the norm in the statement:

$$\lim_{n\to\infty} \|X_0 - X_n\|_p = 0 \text{ if and only if } \|X_0\|_p = \|X_{n\to\infty}\|_p \tag{11}$$

X_0 and X_n in (9) are both functions of ω, right? So is structure important? I believe that in [1], they prove this convergence based on the Fourier series of the basis functions.

Answer by AK: The left hand side is much stronger. You may have $\|X_0\|_p = \|X_{n\to\infty}\|_p$ while X_0 and X_n are completely different. In other words $\|X_0\|_p = \|X_{n\to\infty}\|_p$ is a sufficient condition for $\lim_{n\to\infty} \|X_0 - X_n\|_p = 0$ but it is not necessary.

Answer by AN: Here is the analysis I performed for proving the convergence for the H_∞ problem. The optimal solution for the true H_∞ optimization problem is

$$\gamma_0 = \sup_{\omega\in\Omega}\left|\frac{W_1MY_0}{NX_0+MY_0}\right| \tag{12}$$

Now define

$$\gamma_n^* = \sup_{\omega\in\Omega}\left|\frac{W_1MY_n^*}{\Re\{NX_n^*+MY_n^*\}}\right| \tag{13}$$

where X_n^* and Y_n^* are the projections of X_0 and Y_0 into the subspace spanned by an n dimensional orthogonal basis function. We assume that γ_n^* is bounded (i.e., $NX_n^*+MY_n^*$ has no zero on the imaginary axis, which you have already proved in the journal paper). By using the fact that for two real functions f and g, $\sup(f)-\sup(g)\leq\sup(f-g)$, we get the following condition:

$$\begin{aligned}
\gamma_0-\gamma_n^* &= \sup_{\omega\in\Omega}\left|\frac{W_1MY_0}{NX_0+MY_0}\right| - \sup_{\omega\in\Omega}\left|\frac{W_1MY_n^*}{\Re\{NX_n^*+MY_n^*\}}\right| \\
&\leq \sup_{\omega\in\Omega}\left[\left|\frac{W_1MY_0}{NX_0+MY_0}\right| - \left|\frac{W_1MY_n^*}{\Re\{NX_n^*+MY_n^*\}}\right|\right] \\
&= \sup_{\omega\in\Omega}\left[\frac{|W_1MY_0\Re\{NX_n^*+MY_n^*\}|-|W_1MY_n^*(NX_0+MY_0)|}{|F|}\right] \\
&\leq \sup_{\omega\in\Omega}\left[\frac{|W_1MY_0(NX_n^*+MY_n^*)|-|W_1MY_n^*(NX_0+MY_0)|}{|F|}\right] \\
&\leq \sup_{\omega\in\Omega}\left|\frac{W_1MY_0(NX_n^*+MY_n^*)-W_1MY_n^*(NX_0+MY_0)}{F}\right| \\
&= \sup_{\omega\in\Omega}\left|\frac{W_1MNX_n^*(Y_0-Y_n^*)-W_1MNY_n^*(X_0-X_n^*)}{F}\right|
\end{aligned} \tag{14}$$

where $F=(NX_0+MY_0)\Re\{NX_n^*+MY_n^*\}$. But according to [1],

$$\begin{aligned}
\lim_{n\to\infty}\|X_0-X_n^*\|_p &= 0 \\
\lim_{n\to\infty}\|Y_0-Y_n^*\|_p &= 0
\end{aligned} \tag{15}$$

where $p\in(1,\infty)$. Therefore, since (14) is bounded, and the denominator has no zero in the imaginary axis,

$$\lim_{n\to\infty}\gamma_n^* = \gamma_0 \tag{16}$$

On the other hand, γ_n (i.e., the solution to the convex optimization problem) is always less than or equal to γ_n^* and greater than the optimal solution γ_0. Thus γ_n converges from above to γ_0 and this convergence is monotonic because the basis functions of order n are a subset of those of order $n+1$, which ensures that $\gamma_{n+1} \leq \gamma_n$.

Answer by AK: It's nice but it is not clear if X_n and Y_n are the projection of X_0 and Y_0 or they are the solutions of the convexified optimization problem in (13). In the paper we use two different notation for that leading to X_n^* and Y_n^*. If you suppose that these are the same you should prove it.

Answer by AN: Yes, I should have made it more clear. I modified the solution (see above).

Answer by AK: It's now perfect.

References

[1] H. Akcay and B. Ninness, "Orthonormal basis functions for continuous-time systems and l_p convergence," *Mathematics of Control, Signals, and Systems*, vol. 12, pp. 295–305, 1999.