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# $H_2$ controller design using frequency-domain Data by convex optimization

Assumption: $G(j\omega) = NM^{-1}$ is given for a SISO-LTI system when $N(s)$ and $M(s)$ are in $R_L H_\infty$.

objective: Compute a controller $K(s) = X(s)Y^{-1}(s)$ that stabilizes the closed loop system and minimizes $\|H\|_2^2$ when $H$ is a weighted closed-loop sensitivity function.

Idea: suppose that at every $\omega$ we have $H = W_1 S$ and

$$|H| = \frac{|W_1 MY|}{|NX + MY|} < \gamma(\omega)$$

Then we have the following optimization problem:

$$\min_{X, Y, \gamma} \int \gamma^2(\omega)\, d\omega$$

subject to:

$$|W_1 MY| < \gamma(\omega)\, |NX + MY| \qquad \forall \omega$$

problem: the optimization is not convex!

For a finite set of frequencies we have:

$$\min_{X, Y, \gamma} \sum_{k=1}^{N} \gamma_k^2$$

subject to $|W_1 MY| < \gamma_k \, \mathrm{Re}\{NX + MY\} \qquad k = 1, \dots, N$

(2) The approximation of the constraint by replacing the absolute value of $NX+MY$ with $Re\{NX+MY\}$ has the interest that with increasing the order of ~~X~~ X and Y we can converge to the real constraint. However, the new constraint is not convex because of multiplication with $\gamma_k$. It is not quasi-convex because we have N variables ($\gamma_k$ $k=1,\dots N$).

Main research question: How we can convexify the constraint and keeping the property of convergence to the real constraint when order of X and Y increases

Note that if we have a stabilizing initial controller say $K_0 = X_0 Y_0^{-1}$ we can convexify the constraints and solve the problem but we ~~also~~ cannot show that the solution converges to the optimal one by increasing the order. In the next page we show how we can convexify the problem by linearizing the non convex part around the initial controller.

③ Let's define the following optimization problem

$$\min_{X,Y,\gamma} \sum_{K=1}^{N} \gamma_K$$

subject to $|W_1 MY|^2 < \gamma_K |NX+MY|^2$ $K=1,\dots,N$

In order to convexify the constraints around $K_0 = X_0 Y_0^{-1}$, take

$Z = NX+MY \Rightarrow |NX+MY|^2 = Z^* Z$

we have $(Z-Z_0)^*(Z-Z_0) \geq 0$ for any $Z_0 = NX_0 + MY_0$

$$\Rightarrow Z^* Z \geq Z^* Z_0 + Z_0^* Z - Z_0^* Z_0$$

$$\Rightarrow (W_1 MY)^* \gamma_K^{-1} (W_1 MY) < Z^* Z_0 + Z^* Z - Z_0^* Z_0$$

using the schur complement lemma the above constraints is equivalent to the following LMIs

$$\begin{bmatrix} Z^* Z_0 + Z^* Z - Z_0^* Z_0 & (W_1 MY)^* \\ W_1 MY & \gamma_K \end{bmatrix} > 0$$

So this LMI can be used as an inner convex approximation of the nonconvex constraint $|W_1 MY|^2 < \gamma_K |NX+MY|^2$.

The optimization problem can be improved iteratively by replacing $K_0$ with the solution of the previous optimization

This algorithme will converge to a local optimal solution.