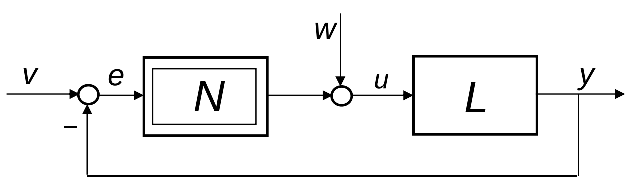
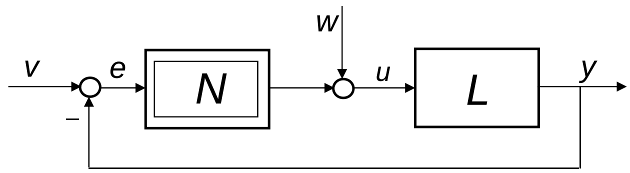
LUR'E PROBLEM:
ABSOLUTE STABILITY

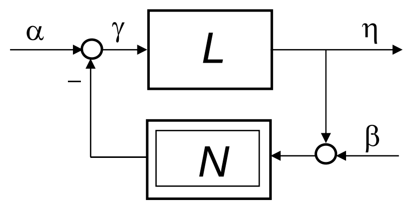
LUR'E SYSTEM



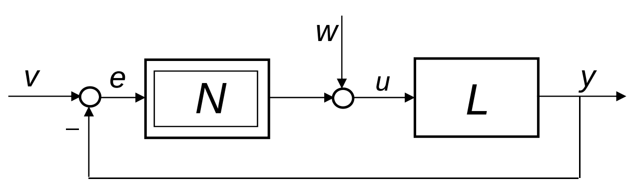
- L: time-invariant dynamic system
- N: nonlinear static system

LUR'E SYSTEM



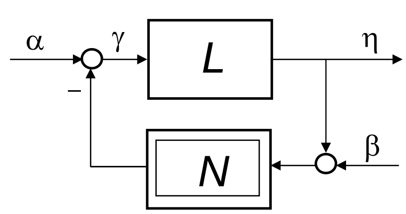


LUR'E SYSTEM

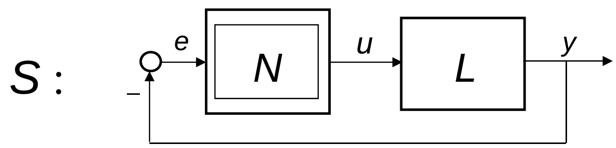


Equivalent forms

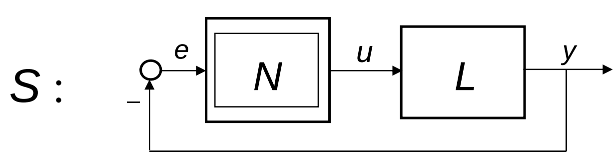
$\alpha = -w$
$\gamma = -u$
$\eta = -y$
$\beta = v$



AUTONOMOUS LUR'E SYSTEM



AUTONOMOUS LUR'E SYSTEM

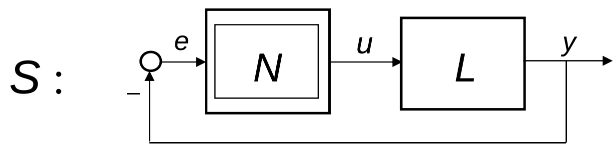


$$L: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$ (A,B,C) stabilizable

Assumption: (A,B) reachable & (A,C) observable

$$G(s) = C(sI - A)^{-1}B$$

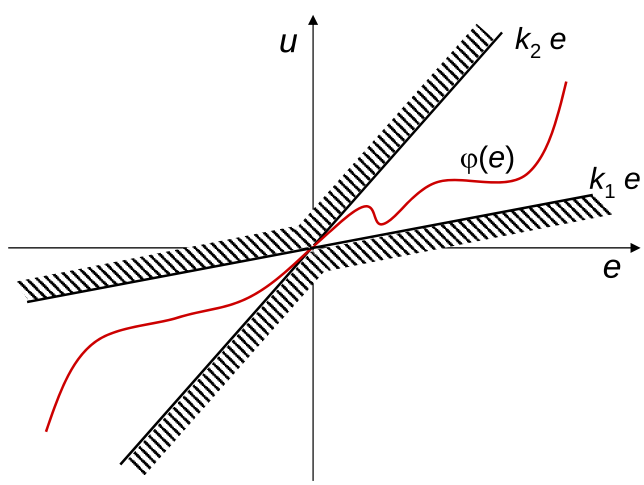
AUTONOMOUS LUR'E SYSTEM



$$N: \; u(t) = \varphi(e(t))$$

- $\varphi : \Re \to \Re$ piecewise continuous function
- $\varphi(\cdot) \in \Phi_{[k_1,k_2]} = \{\phi(\cdot): \; k_1 e \leq \phi(e) \leq k_2 e, \, \forall e \in \Re\}$

SECTOR NONLINEARITY



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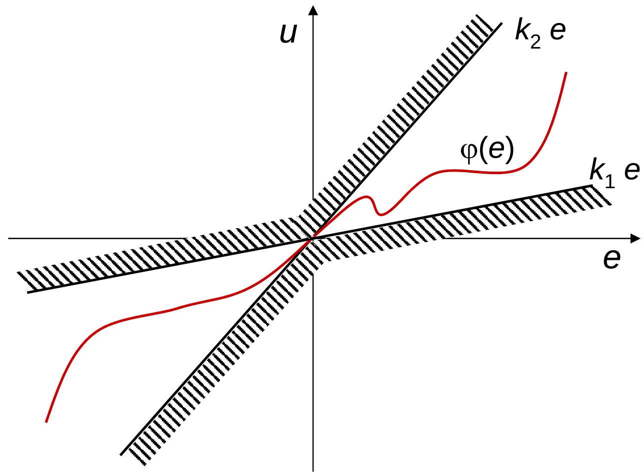


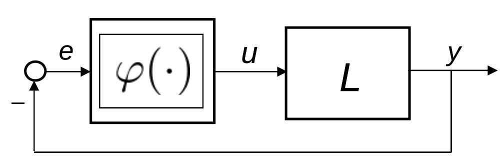
$$\Phi_{[k_1,k_2]} = \{\phi(\cdot) : \ k_1 e \le \phi(e) \le k_2 e \ \forall e \in \Re\}$$

$$\Updownarrow$$

$$\Phi_{[k_1,k_2]} = \{\phi(\cdot) : (k_2 e - u)(u - k_1 e) \ge 0, \ u = \phi(e), \ \forall e \in \Re\}$$

AUTONOMOUS LUR'E SYSTEM

$S:$



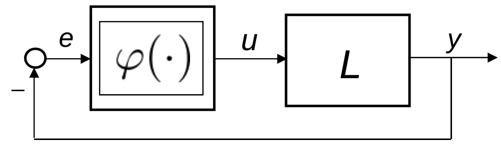
$$S: \begin{cases} \dot{x} = Ax + B\varphi(-Cx) \\ y = Cx \end{cases}$$

$$f(x) := Ax + B\varphi(-Cx)$$

$\varphi(0) = 0 \to f(0) = 0 \to \bar{x} = 0$ is an equilibrium for S, for any sector nonlinearity $\varphi(\cdot) \in \Phi_{[k_1,k_2]}$

ABSOLUTE STABILITY IN THE SECTOR $[k_1, k_2]$

S :



Definition

System S is absolutely stable in the sector $[k_1, k_2]$ is x = 0 is a a globally asymptotically stable equilibrium, for every sector nonlinearity $\varphi(\cdot) \in \Phi_{[k_1,k_2]}$

STABILITY OF AN EQUILIBRIUM

$$\dot{x}(t) = f(x(t))$$

Definition (equilibrium):

$x_e \in \mathbb{R}^n$ such that $f(x_e)=0$

Definition (stable equilibrium):

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \Rightarrow \|\boxed{x(t)} - x_e\| < \varepsilon, \forall t \geq 0$$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

execution starting from $x(0)=x_0$

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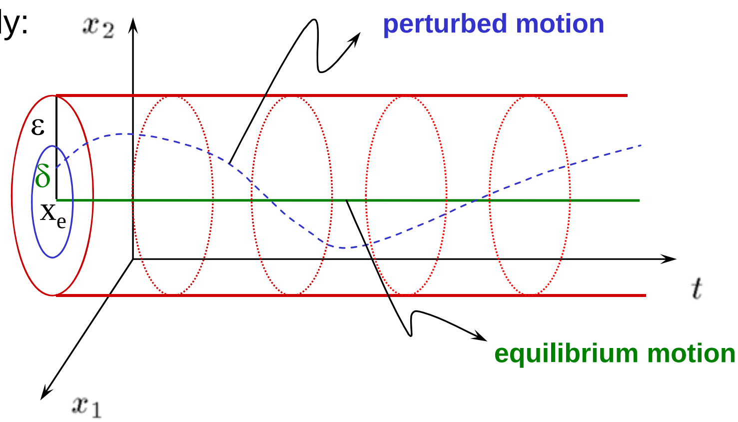
Graphically:



small perturbations lead to small changes in behavior

Definition (asymptotically stable equilibrium):

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x_0 - x_e\| < \delta \rightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq 0$$

and δ can be chosen so that $\lim_{t\rightarrow\infty}(x(t) - x_e) = 0$

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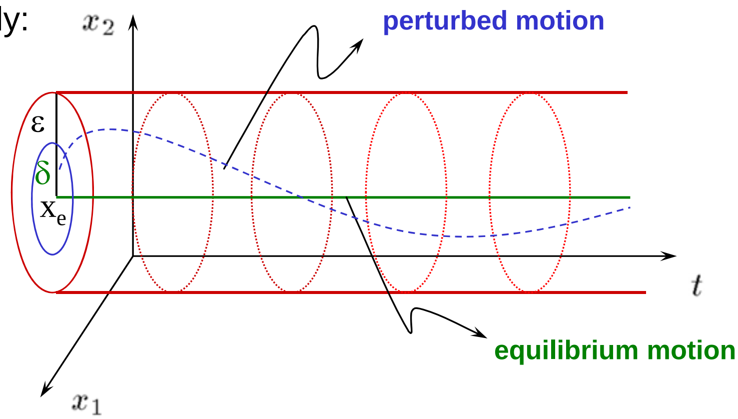

small perturbations lead to small changes in behavior and are re-absorbed, in the long run

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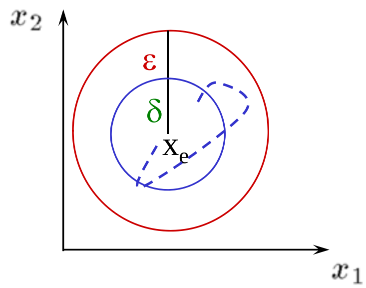


small perturbations lead to small changes in behavior and are re-absorbed, in the long run

Let x_e be asymptotically stable.

Definition (domain of attraction):

The domain of attraction of x_e is the set of x_0 such that

$$\lim_{t\rightarrow\infty}(\boxed{x(t)} - x_e) = 0$$

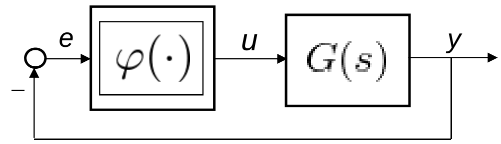
execution starting from $x(0)=x_0$

Definition (globally asymptotically stable equilibrium):

x_e is globally asymptotically stable (GAS) if its domain of attraction is the whole state space ?^{n}

LUR'E PROBLEM

S :

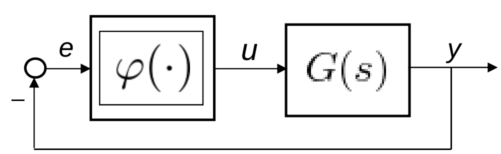


<u>Lur'e problem</u>

Given the transfer function *G(s)* of the linear system *L*, determine necessary and/or sufficient conditions for the absolute stability of S in the sector $[k_1, k_2]$.

LUR'E PROBLEM

S :

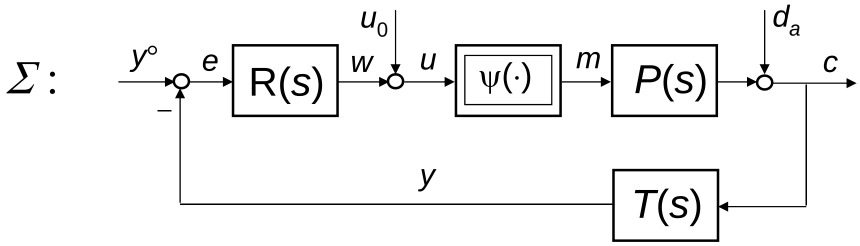


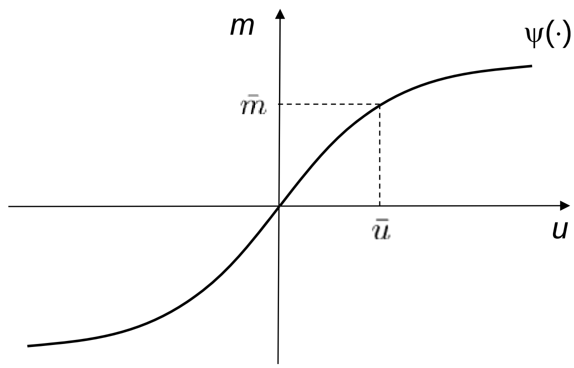
<u>Lur'e problem</u>

Given the transfer function *G(s)* of the linear system *L*, determine necessary and/or sufficient conditions for the absolute stability of S in the sector $[k_1, k_2]$.

Why is this problem meaningful?

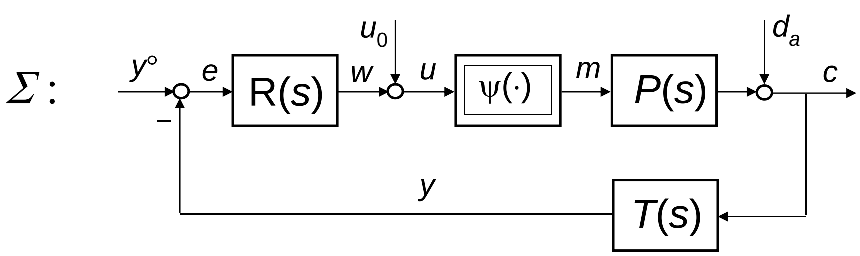
A SIGNIFICANT EXAMPLE





nonlinear
characteristic
of the actuator

A SIGNIFICANT EXAMPLE

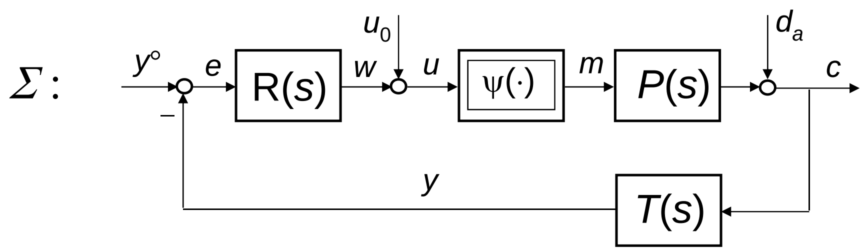


Assumption: $g_P = g_T = 0$ and $g_R = 1$

Let $\bar{y}^{\circ},\ \bar{d}_a,\ u_0$ be constant,

and denote with $\bar{x} = (\bar{x}'_P\ \bar{x}'_T\ \bar{x}'_R)'$ the corresponding equilibrium.

A SIGNIFICANT EXAMPLE



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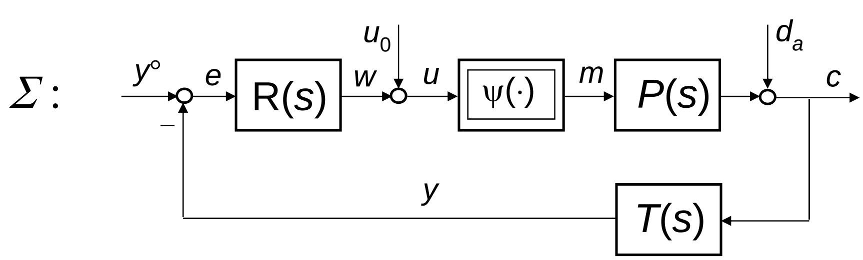
$$g_R = 1 \rightarrow \quad \bar{e} = 0 \rightarrow \bar{y} = \bar{y}^\circ \rightarrow \bar{c} = \frac{\bar{y}^\circ}{\mu_T}$$

$$\bar{c} = \bar{m}\mu_P + \bar{d}_a \rightarrow \bar{m} = \frac{1}{\mu_P}(\bar{c} - \bar{d}_a) = \frac{1}{\mu_P}\left(\frac{\bar{y}^\circ}{\mu_T} - \bar{d}_a\right)$$

$$\bar{u} = \psi^{-1}(\bar{m}) \qquad\qquad \bar{w} = \bar{u} - u_0$$

A SIGNIFICANT EXAMPLE

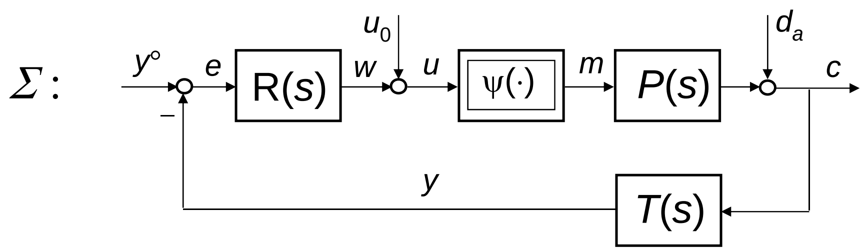


Typical control design approach:

‘linear’ design + nonlinear analysis

(for instance, by simulation)

A SIGNIFICANT EXAMPLE

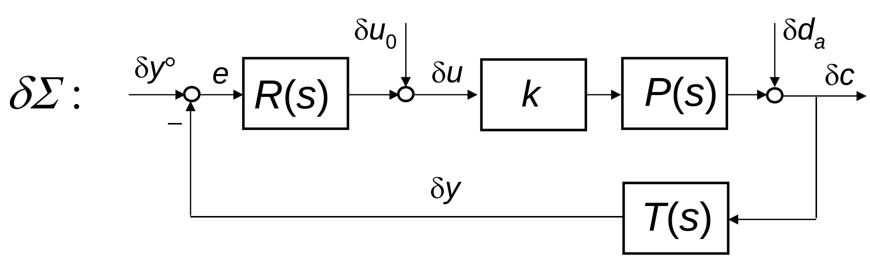


Linear design:

- build the system δΣ by linearizing Σ around the equilibrium associated with the constant inputs $\bar{y}^{\circ},\, \bar{d}_a,\, u_0$

$$k := \frac{\partial\psi}{\partial u}(\bar{u})$$

A SIGNIFICANT EXAMPLE



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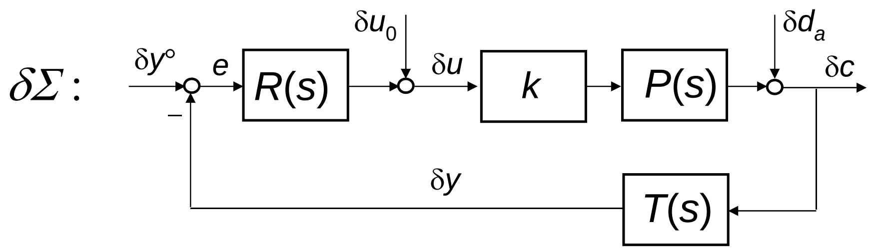
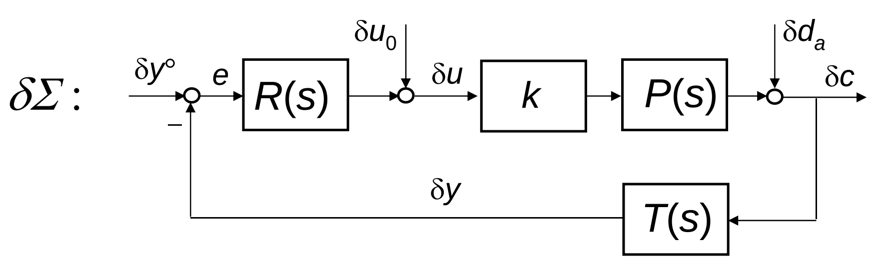


Linear design:

- build the system $\delta\Sigma$ by linearizing Σ around the equilibrium associated with the constant inputs $\bar{y}^{\circ},\, \bar{d}_a,\, u_0$
- choose *R(s)* that makes $\delta\Sigma$ asymptotically stable

A SIGNIFICANT EXAMPLE



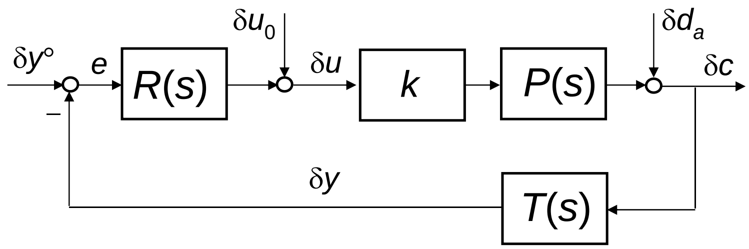
Different triples $\bar{y}^{\circ},\, \bar{d}_a,\, u_0$ map into different equilibria for Σ.

Hence, the linear gain *k* of the actuator is uncertain

$$k := \frac{\partial \psi}{\partial u}(\bar{u}) \in [k_{\min}, k_{\max}]$$

A SIGNIFICANT EXAMPLE

$\delta\Sigma$:



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Hence, the linear gain *k* of the actuator is uncertain

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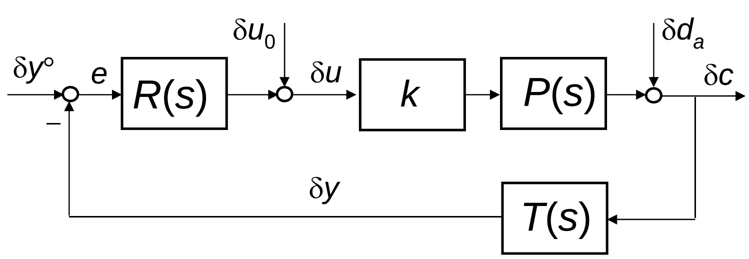
→ robust linear control design needed to guarantee that $\delta\Sigma$ is asymptotically stable for every k in the admissible range

A SIGNIFICANT EXAMPLE

$\delta\Sigma$:

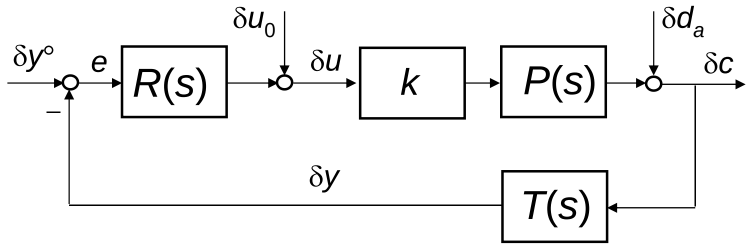


<u>Guarantees for $\delta\Sigma$:</u>

every equilibrium of $\delta\Sigma$ *associated* with constant inputs is *globally asymptotically stable* and the controlled variable will converge to the desired set-point after some suitable transient, irrespectively of the (constant) value of the disturbances

A SIGNIFICANT EXAMPLE

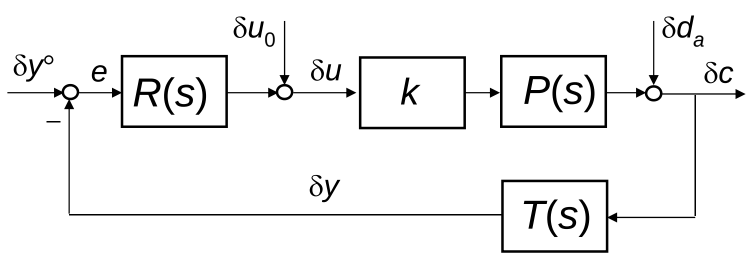
$\delta\Sigma$:



What about the nonlinear system Σ?

A SIGNIFICANT EXAMPLE

$\delta\Sigma$:



What about the nonlinear system Σ?

We need to verify that *all* equilibria associated with admissible constant inputs are *globally asymptotically stable*.

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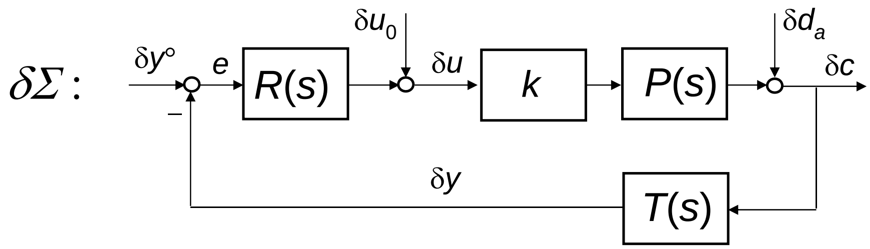

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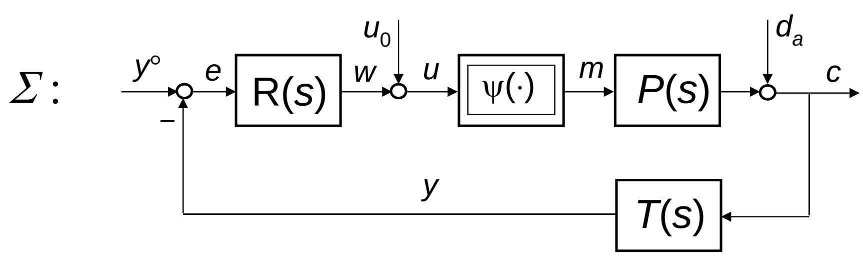
We need to verify that *all* equilibria associated with admissible constant inputs are *globally asymptotically stable*.

→ Lur'e problem

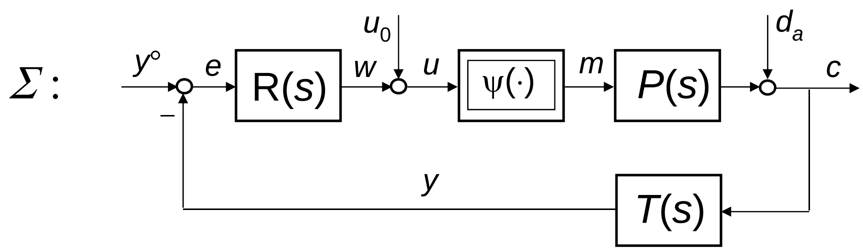
A SIGNIFICANT EXAMPLE



Consider the constant input values $\bar{y}^\circ$, $\bar{d}_a$, u_0 and the corresponding equilibrium. We can then adopt the following expressions:

$$x(t) = \bar{x} + \Delta x(t)$$
$$e(t) = \bar{e} + \Delta e(t)$$
$$w(t) = \bar{w} + \Delta w(t)$$
$$u(t) = \bar{u} + \Delta u(t)$$
$$m(t) = \bar{m} + \Delta m(t)$$
$$c(t) = \bar{c} + \Delta c(t)$$
$$y(t) = \bar{y} + \Delta y(t)$$

A SIGNIFICANT EXAMPLE



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$$
\begin{aligned}
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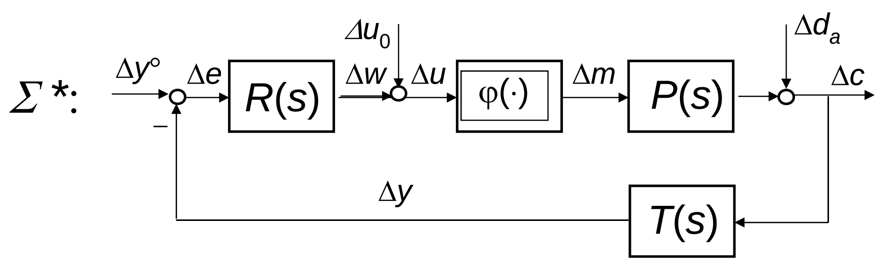
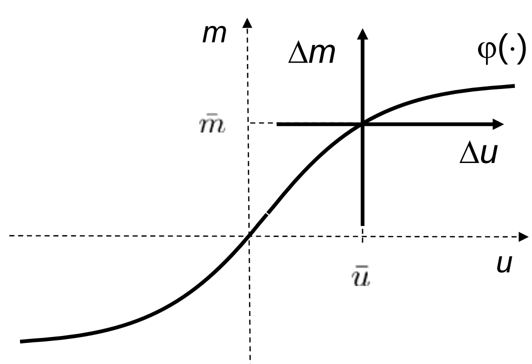
c(t) &= \bar{c} + \Delta c(t) \\
y(t) &= \bar{y} + \Delta y(t)
\end{aligned}
\quad \text{where} \quad
\begin{aligned}
\Delta x(t) &:= x(t) - \bar{x} \\
\Delta e(t) &:= e(t) - \bar{e} \\
\Delta w(t) &:= w(t) - \bar{w} \\
\Delta u(t) &:= u(t) - \bar{u} \\
\Delta m(t) &:= m(t) - \bar{m} \\
\Delta c(t) &:= c(t) - \bar{c} \\
\Delta y(t) &:= y(t) - \bar{y}
\end{aligned}
\quad \text{derive from } \Sigma^*
$$

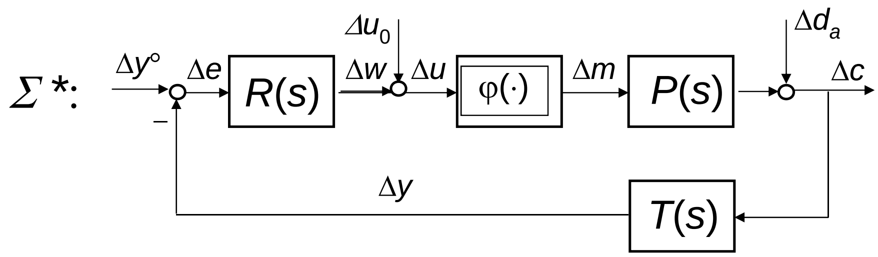
A SIGNIFICANT EXAMPLE

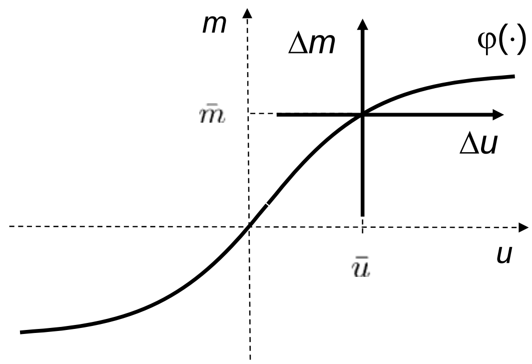




$$
\begin{aligned}
\Delta y^{\circ}(t) &:= y^{\circ}(t) - \bar{y}^{\circ} \\
\Delta u_0 &:= u_0(t) - u_0 \\
\Delta d_a(t) &:= d_a(t) - \bar{d}_a
\end{aligned}
$$

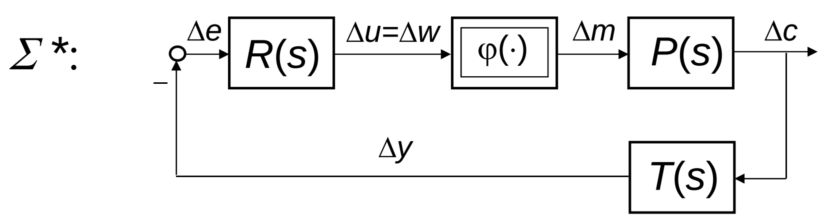
A SIGNIFICANT EXAMPLE





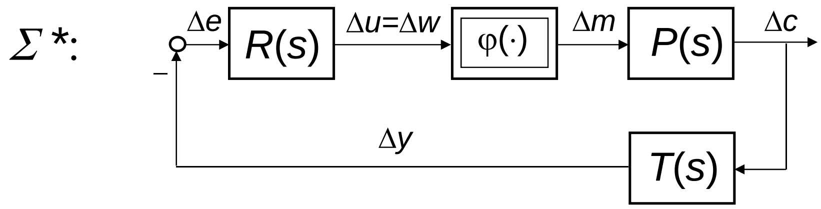
$$\underbrace{\begin{aligned} \Delta y^{\circ}(t) &:= y^{\circ}(t) - \bar{y}^{\circ} = 0 \\ \Delta u_0 &:= u_0(t) - u_0 = 0 \\ \Delta d_a(t) &:= d_a(t) - \bar{d}_a = 0 \end{aligned}}_{\text{(constant) inputs keep unchanged}}$$

A SIGNIFICANT EXAMPLE



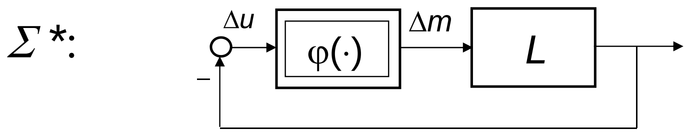
autonomous system

A SIGNIFICANT EXAMPLE



autonomous system

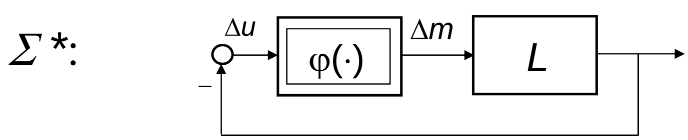
System Σ* in compact form:



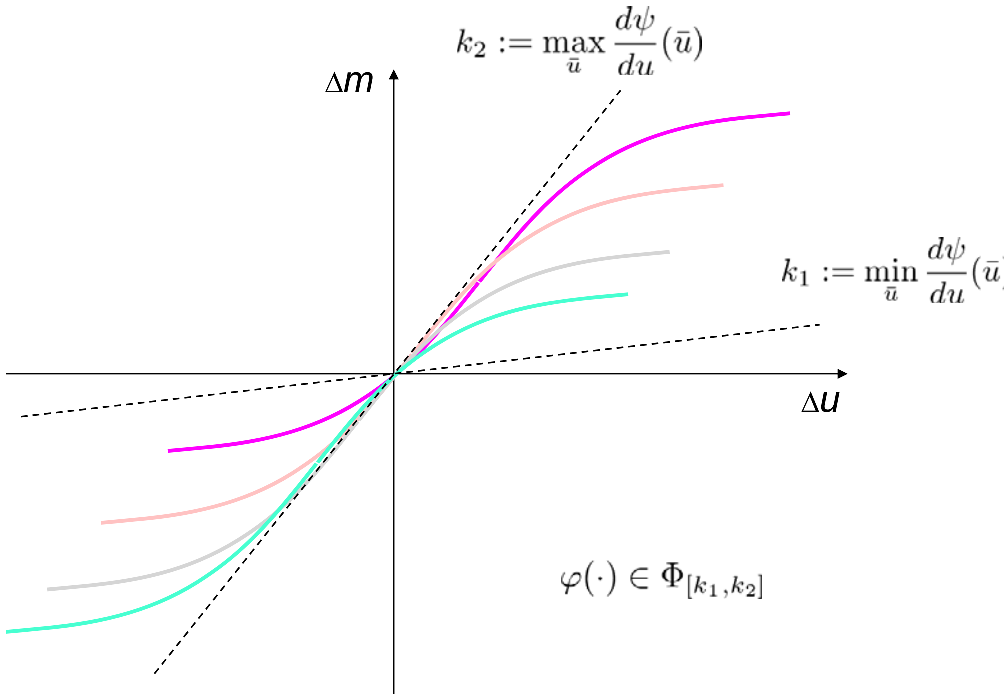
L: G(s) = P(s)T(s) R(s)

→ Lur'e autonomous system

A SIGNIFICANT EXAMPLE

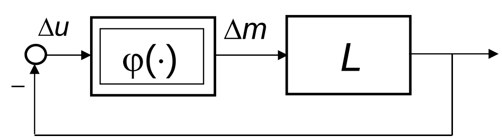


- Given that $x(t) = \bar{x} + \Delta x(t)$, then, the global asymptotic stability of the equilibrium $\bar{x}$ of Σ is equivalent to that of the equilibrium Δx = 0 of Σ*
- Function φ(·) depends on $\bar{x}$



A SIGNIFICANT EXAMPLE

Σ*:



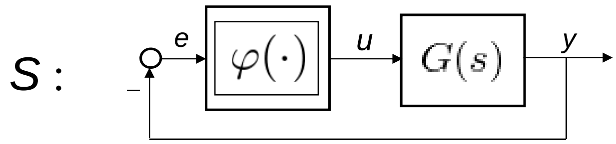
L: G(s) = P(s)T(s) R(s)

→ autonomous Lur'e system with $\varphi(\cdot) \in \Phi_{[k_1, k_2]}$

Conclusions:

If Σ* is absolutely stable in the sector [k_1, k_2], then, all equilibria of Σ are globally asymptotically stable

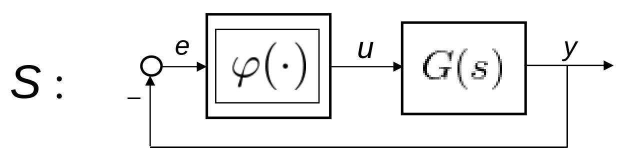
LUR'E PROBLEM



<u>Lur'e problem</u>

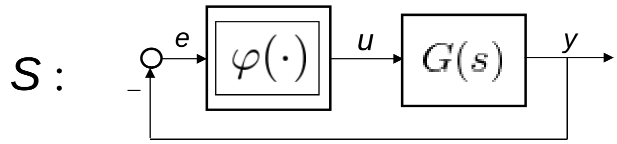
determine necessary and/or sufficient conditions for the absolute stability of S in some sector $[k_1, k_2]$.

A NECESSARY CONDITION



$$\varphi(\cdot) \in \Phi_{[k_1,k_2]}$$

A NECESSARY CONDITION



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Admissible sector functions can be linear:

$$\varphi(e) = ke \in \Phi_{[k_1,k_2]}$$

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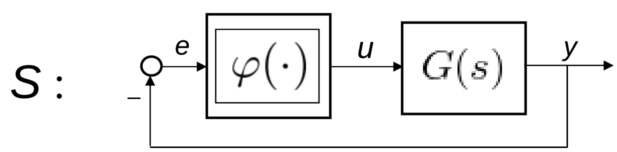
S :
e
u
y
–
$\varphi(\cdot)$
$G(s)$

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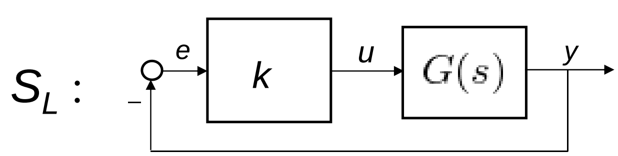
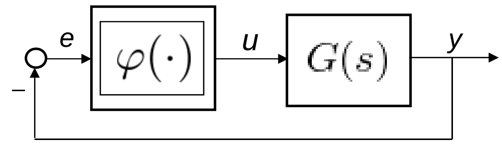


A NECESSARY CONDITION

$S:$



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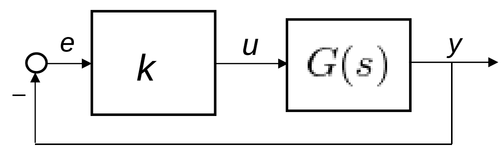
$S_L:$

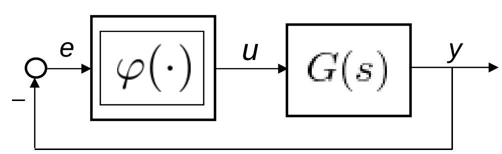


If S is absolutely stable in $[k_1, k_2]$,

then, S_L is (globally) asymptotically stable for any $k \in [k_1, k_2]$.

A NECESSARY CONDITION

$S:$



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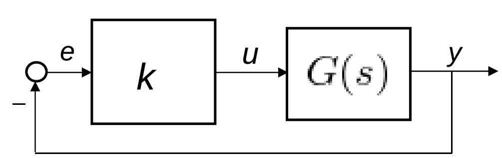
$S_L:$

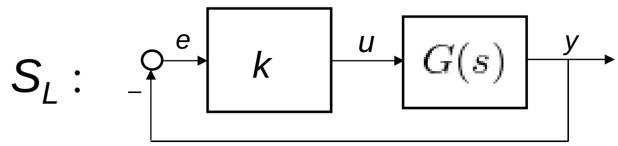


If S is absolutely stable in $[k_1, k_2]$,

then, S_L is (globally) asymptotically stable for any $k \in [k_1, k_2]$.

If $0 \in [k_1, k_2]$, then, system L with t.f. G(s) is asymptotically stable

ASYMPTOTIC STABILITY OF S_L

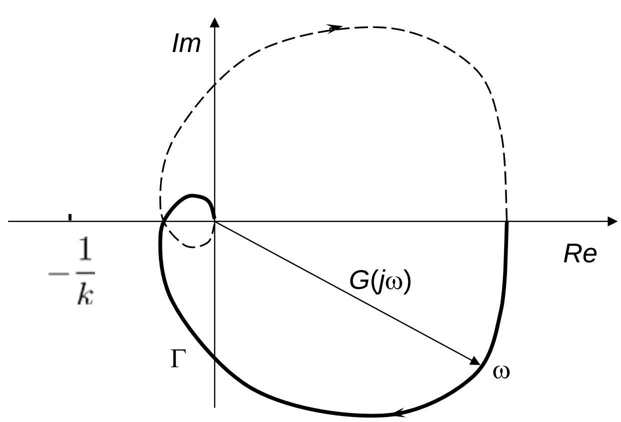


<u>For a given *k*</u> ,

System S_L is asymptotically stable if and only if the Nyquist plot of $G(s)$ encircles (anti-clockwise) the point in the complex plan corresponding to the real number $-1/k$ as many times as the number of poles of $G(s)$ with positive real part

(Nyquist criterion)

ASYMPTOTIC STABILITY OF S_L



ROBUST ASYMPTOTIC STABILITY OF S_L

If $k \in [k_1, k_2] \rightarrow$ robust stability of S_L

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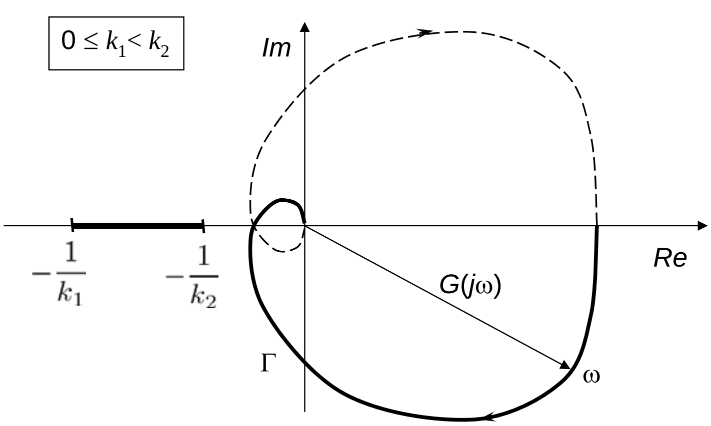
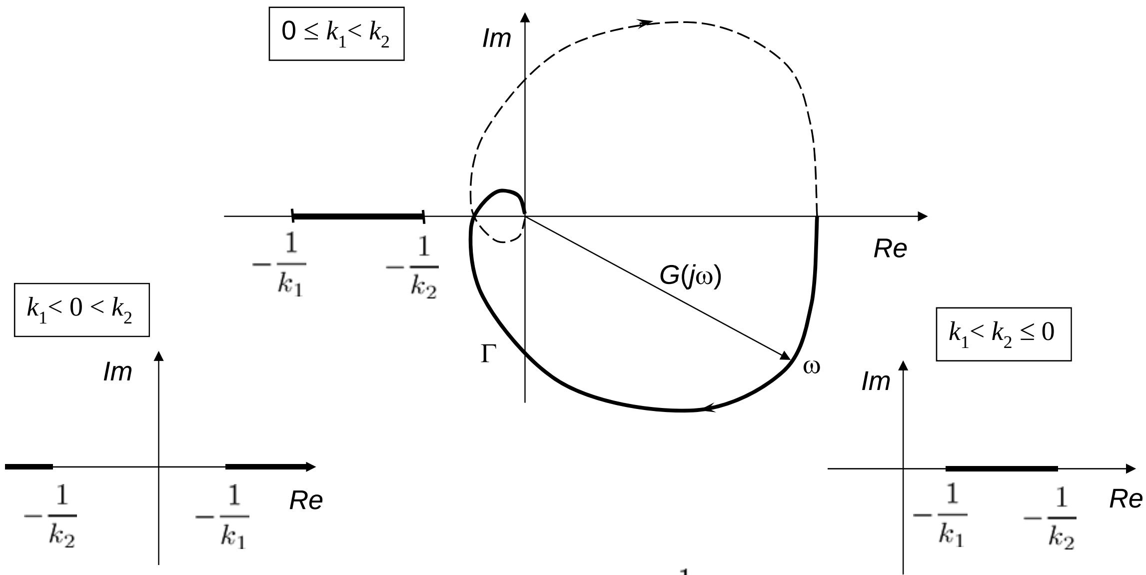


$$I(k_1, k_2) := \{\alpha \in \Re : \ \alpha = -\frac{1}{k}, \ k \in [k_1, k_2]\}$$

ROBUST ASYMPTOTIC STABILITY OF S_L

If k $\in$ $[k_1, k_2]$ → robust stability of S_L



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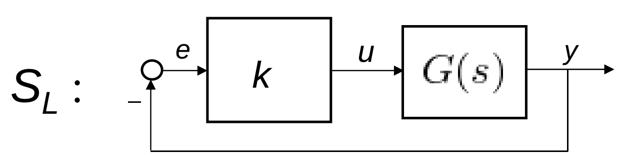
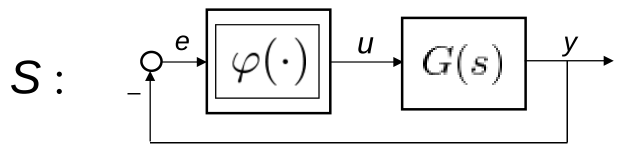
ROBUST ASYMPTOTIC STABILITY OF S_L



k uncertain, k $\in$ $[k_1, k_2]$,

System S_L is asymptotically stable *for any* $k \in [k_1, k_2]$ if and only if the Nyquist plot of $G(s)$ encircles (anti-clockwise) $I(k_1, k_2)$ as many times as the number of poles of $G(s)$ with positive real part.

A NECESSARY CONDITION



$$\varphi(\cdot) \in \Phi_{[k_1, k_2]}$$

Theorem (necessary condition)

If S is absolutely stable in the sector $[k_1, k_2]$,

then the Nyquist plot of $G(s)$ encircles (anti-clockwise) $I(k_1, k_2)$ as many times as the number of poles of $G(s)$ with positive real part.

In particular, if $0 \in [k_1, k_2]$, then system L with transfer function $G(s)$ is asymptotically stable.