Robust Controller Design By Convex Optimization

Alireza Karimi
Laboratoire d'Automatique, EPFL

References

1. Linear Matrix Inequalities in System and Control Theory, S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, SIAM 1994.

2. Convex Optimization, S. Boyd and L. Vandenberghe, Cambridge University Press 2004 (available on web).

3. LMIs in Control, C. Scherer and S. Weiland, Course notes Delft and Eindhoven Universities (available on web).

4. LMI Optimization with Applications in Control, D. Henrion and D. Arzelier, slides available on web

Introduction

Robust performance condition : $\||W_1 S| + |W_2 T|\|_\infty < 1$

Objective : Find Controller $C(s)$ by numerical optimization such that the robust performance condition is satisfied.

Optimization problem :

$$\min_{C(s) \in \text{SSC}} \gamma$$

$$\left| \frac{W_1(j\omega)}{1 + C(j\omega)P(j\omega)} \right| + \left| \frac{W_2(j\omega)C(j\omega)P(j\omega)}{1 + C(j\omega)P(j\omega)} \right| < \gamma \qquad \forall \omega \in [0 \quad \infty[$$

Difficulties :

1. The set of stabilizing controllers SSC is not defined.
2. There is an infinite number of non convex constraints.
3. The algorithm converges to a local minimum instead of a global one.

Laboratoire
d'Automatique

Convex Optimization

The main property of a convex optimization problem is that any local minimum is a global minimum.

Convex set: A set $\mathcal{C}$ in a vector space is said to be convex if the line segment between any two points of the set lies inside the set.

$$x_1, x_2 \in \mathcal{C} \Rightarrow \lambda x_1 + (1 - \lambda)x_2 \in \mathcal{C} \qquad \forall \lambda \in [0 \quad 1]$$

Convex set   Non-convex set
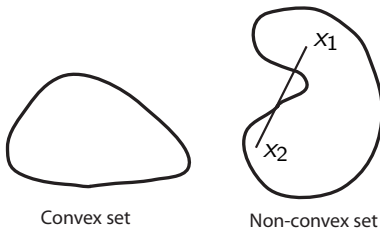
The intersection of convex sets is a convex set.

Convex Optimization

Convex Combination: Let $x_1, \ldots, x_n \in \mathcal{C}$ then :

$$x = \sum_{i=1}^{n} \lambda_i x_i \quad \text{with} \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1$$

is called a convex combination of $x_1, \ldots, x_n$.

Convex hull: For any set $\mathcal{C}$ in a vector space the convex hull consists of all convex combinations of the elements of $\mathcal{C}$ and is a convex set.

Convex function: A function $f : \mathcal{C} \rightarrow \mathcal{R}$ is convex if

1. $\mathcal{C}$ is a convex set and
2. for all $x_1, x_2 \in \mathcal{C}$ and $\lambda \in [0 \quad 1]$ there holds that

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

Convex Optimization

Example: $f(x) = x^2$ on $\mathcal{R}$, $f(x) = \sin x$ on $[\pi \quad 2\pi]$ and $f(x) = |x|$ on $\mathcal{R}$ are convex, but $f(x) = -x^2$ is not convex.

A twice differentiable function is convex if its domain is convex and its second derivative is non negative.

Properties:

1. Linear and affine functions are convex.
2. If $f(y)$ is convex and $y = g(x)$ is linear, $f(g(x))$ is convex.
3. Convex combination of convex functions is also a convex function.

$$g = \sum_{i=1}^{n} \lambda_i f_i \qquad \lambda_i \in [0 \quad 1] \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1$$

is convex if $f_1, \ldots, f_n$ are convex functions.

Convex Optimization

Links between convex sets and convex functions:

1. Epigraph of a convex function is a convex set.

2. If $f(x)$ is a convex function, then

$$D = \{x \mid f(x) \leq 0\} \text{ is a convex set}$$

3. If $f(x)$ is a linear function on $\mathcal{R}$ then

$$f(x) \leq 0, \quad f(x) \geq 0 \quad \text{and} \quad f(x) = 0 \quad \text{define the convex sets.}$$

4. Let $f_1, \ldots, f_n$ be convex functions then

$$D = \{x \mid f_i(x) \leq 0 \quad \text{for } i = 1, \ldots, n\}$$

is the intersection of convex sets and defines a convex set.

5. If $f(x)$ is a nonlinear convex function, neither $f(x) \geq 0$ nor $-f(x) \leq 0$ defines a convex set.

Laboratoire
d'Automatique

Convex Optimization

$$\min_x f_0(x)$$

subject to

$$f_i(x) \leq 0 \qquad i = 1, \ldots, n \qquad f_i \text{ Convex}$$

$$g_j(x) = 0 \qquad j = 1, \ldots, m \qquad g_j \text{ Linear}$$

Linear programming: $f_0(x)$ and $f_i(x)$ are linear.

Quadratic programming: $f_0(x)$ is quadratic, but $f_i(x)$ are linear.

Semidefinite programming: $f_0(x)$ is linear and the optimization variables are symmetric positive semidefinite matrices with linear constraints in matrices (Linear Matrix Inequalities).

Semi-infinite programming: The constraints are defined for a parameter $\theta \in \Theta$ (number of constraints goes to infinity). These type of problems are called robust optimization.

Laboratoire
d'Automatique

Robust Performance Control by Convex Optimization

Given P, W_1 and W_2 compute a controller C that minimizes $\||W_1 S| + |W_2 T|\|_\infty$

1. Parameterize all stabilizing controllers by coprime factorization,

$$P = NM^{-1}, \quad NX + MY = 1, \quad C = \frac{X + MQ}{Y - NQ}$$

2. Then $S(j\omega) = M(j\omega)(Y(j\omega) - N(j\omega)Q(j\omega))$ and $T(j\omega) = N(j\omega)(X(j\omega) + M(j\omega)Q(j\omega))$ are linear w.r.t Q and

$$|W_1 M(Y - NQ)| + |W_2 N(X + MQ)| < \gamma \qquad \forall \omega \in [0 \quad \infty[$$

is a set of convex constraints w.r.t Q.

3. If $Q(j\omega)$ is linear w.r.t its parameters $q_1, \ldots, q_n$, above constraints are convex w.r.t $q_1, \ldots, q_n$.

4. This leads to a convex optimization problem with infinite number of constraints. It can be solved approximately by gridding ω.

Laboratoire
d'Automatique

Robust Performance Control by Convex Optimization

Example: Given $P(s) = \dfrac{s-1}{(s-2)(s-3)}$ and $W_2(s) = \dfrac{s+1}{s+100}$,

compute a robust controller that minimize $\|W_2 T\|_\infty$.

Step I: Do a coprime factorization for $P(s)$

$$N = \frac{s-1}{(s+1)^2}, \quad M = \frac{(s-2)(s-3)}{(s+1)^2}$$

$$X = \frac{5s+17}{s+1}, \quad Y = \frac{s+3}{s+1}$$

Step II: Linearly parameterize $Q(s)$

$$Q(s) = \frac{x_1 s^{n_q} + \cdots + x_{n_q+1}}{(s+1)^{n_q}}$$

Step III: Solve the following convex optimization problem

$$\min_x \gamma$$

$$|W_2(j\omega)N(j\omega)(X(j\omega) + M(j\omega)Q(j\omega))| - \gamma < 0 \qquad \forall \omega \in [0 : 0.1 : \omega_{\max}]$$

Step IV: Compute $C = \dfrac{X + MQ}{Y - NQ}$

Laboratoire
d'Automatique

Robust Performance Control by Convex Optimization

MATLAB program:

```
function [c,ceq]=nonlconstr(x)
num=x(2:end)';den=1;
for i=1:length(x)-2;den=conv(den, [1 1]);end
Q=tf(num,den);
s=tf('s');W2=(s+1)/(s+100);
N=(s-1)/(s+1)*(s+1);M=(s-2)*(s-3)/(s+1)*(s+1);
X=(5*s+17)/(s+1);Y=(s+3)/(s+1);
W2T=W2*N*(X+M*Q);
omega=0:0.1:15;
[m,p]=bode(W2T,omega);
c(1:length(omega))=m(1,1,:)-x(1);c=c';
ceq=[];
x = fmincon(inline('x(1)'),zeros(4,1),
[],[],[],[],[],[],@nonlconstr, optimset('Largescale','off'))
C=(X+MQ)/(Y-NQ); C=minreal(C,0.01)
```

Robust Performance Control by Convex Optimization

Exercise: Consider the plant model

$$P(s) = \frac{s - 10}{(s + 1)(s + 10)}$$

when the nominal performance is to minimize $\|W_1 S\|_\infty$ with

$$W_1(s) = \frac{1}{s + 0.001}$$

and the weighting filter for additive uncertainty is given by :

$$W_2(s) = \frac{s + 2}{s + 10}$$

Compute a robust controller that minimizes:

$$\||W_1 S| + |W_2 C S|\|_\infty$$

Pole Placement with Sensitivity Function Shaping [1]

Pole placement: Given a discrete-time plant model

$G(q^{-1}) = \dfrac{q^{-d}B(q^{-1})}{A(q^{-1})}$ with

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}$$

$$B(q^{-1}) = b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}$$

and desired closed-loop polynomial :

$$P(q^{-1}) = P_D(q^{-1})P_F(q^{-1})$$

The polynomials $R(q^1)$ and $S(q^{-1})$ of the controller
$K(q^{-1}) = \dfrac{R(q^{-1})}{S(q^{-1})}$ are solutions of :

$$A(q^1)S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1}) = P_D(q^{-1})F_D(q^{-1})$$

[1]Adaptive Control by I.D. Landau, R. Lozano and M. M'Saad

Pole Placement with Sensitivity Function Shaping

Diophantine equation has a unique solution if B and A are coprime and :

$$n_p \leq n_a + n_b + d - 1, \qquad n_s = n_b + d - 1, \qquad n_r = n_a - 1$$

with
$$
\begin{aligned}
S(q^{-1}) &= 1 + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s} \\
R(q^{-1}) &= r_0 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}
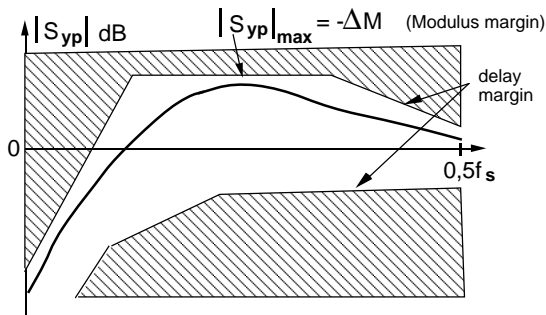\end{aligned}
$$

Sensitivity shaping: Pole placement does not necessarily lead to a good controller. The magnitude of the sensitivity functions should not be too large :

$$S_{yp}(q^{-1}) = \frac{1}{1 + K(q^{-1})G(q^{-1})} = \frac{A(q^{-1})S(q^{-1})}{P_D(q^{-1})P_F(q^{-1})}$$

$$S_{up}(q^{-1}) = \frac{K(q^{-1})}{1 + K(q^{-1})G(q^{-1})} = \frac{A(q^{-1})R(q^{-1})}{P_D(q^{-1})P_F(q^{-1})}$$

Laboratoire
d'Automatique

Pole Placement with Sensitivity Function Shaping

Constraints on output sensitivity function:

Constraint on input sensitivity function: In order to reduce the effect of noise and high frequency disturbance on $u(t)$ the magnitude of $S_{up}(e^{-j\omega})$ should be small in high frequencies:

$$|S_{up}(e^{-j\omega})| < U_{max} \qquad \text{for } \omega > \omega_h$$

Pole Placement with Sensitivity Function Shaping

Iterative method: Sensitivity functions can be shaped using fixed terms in R and S as follows:

$$R(q^{-1}) = H_R(q^{-1})R'(q^{-1}) \qquad S(q^{-1}) = H_S(q^{-1})S'(q^{-1})$$

Then the Diophantine equation becomes:

$$A'(q^{-1})S'(q^{-1}) + q^{-d}B'(q^{-1})R'(q^{-1}) = P_D(q^{-1})P_F(q^{-1})$$

with $\qquad A'(q^{-1}) = A(q^{-1})H_S(q^{-1}) \qquad B'(q^{-1}) = B(q^{-1})H_R(q^{-1})$

Effects of H_S, H_R and P_F on the sensitivity functions:

- H_S: should contain the internal model of the disturbance to be rejected e.g. $H_S = 1 - q^{-1}$ to reject step disturbance. It reduces the magnitude of S_{yp} in a given frequency range.

- H_R: can open the loop at frequencies where $H_R(e^{-j\omega})$ is small or zero. For example $H_R = 1 + q^{-1}$ will open the loop at Nyquist frequency and reduce the magnitude of S_{up}.

- P_F: high frequency aperiodic poles will generally reduce the magnitude of S_{yp} in the high frequency region.

Pole Placement with Sensitivity Function Shaping

Iterative procedure: Choose P_D, P_F, H_R and H_S for nominal performance.

Step I: Compute the controller from the Diophantine equation and check the shape of S_{yp}. If it is inside the template go to Step IV else if the maximum of $|S_{yp}|$ is in LF go to Step II else go to Step III.

Step II: Put a pair of complex zeros in H_S with a frequency near to that of maximum of $|S_{yp}|$ and a damping factor between 0.3 and 0.8. If there exists already a pair of zeros near to the maximum value, just decrease a bit its damping factor. Go to Step I.

Step III: Choose $P_F = (1 - p_1 q^{-1})^{n_f}$ with $0.05 \leq p_1 \leq 0.5$ and $n_f \leq n_p - n_d$. If p_1 has already been defined increase it. Go to Step I.

Step IV: Check the input sensitivity function. If it is large in high frequencies choose $H_R = 1 + \alpha q^{-1}$ with $0.85 \leq \alpha \leq 1$ and go to Step I else STOP.

Pole Placement with Sensitivity Function Shaping

Convex optimization method [1]**:** Choose P_D, P_F, H_R and H_S for nominal performance.

1. Compute the nominal controller:

$$A'(q^1)S_0'(q^{-1}) + q^{-d}B'(q^{-1})R_0'(q^{-1}) = P_D(q^{-1})P_F(q^{-1})$$

   If the constraints on S_{yp} and S_{up} are not satisfied, go to 2 else STOP.

2. Parameterize all controllers that put the closed-loop poles in the same locations:

$$R'(q^{-1}) = R_0'(q^{-1}) + A'(q^{-1})Q(q^{-1})$$

$$S'(q^{-1}) = S_0'(q^{-1}) - q^{-d}B'(q^{-1})Q(q^{-1})$$

   where $Q(q^{-1}) = x_0 + x_1q^{-1} + \cdots + x_{n_q}q^{-n_q}$ is linearly parameterized with respect to x. Consider N frequency points between 0 and Nyquist frequency : $\omega_k \quad k = 1, \dots, N$. Choose $n_q = 1$ and go to 3.

Pole Placement with Sensitivity Function Shaping

3. Find a feasible point for following convex feasibility problem:

$$\left| \frac{A'(e^{-j\omega_k})[S_0'(e^{-j\omega_k}) - q^{-d}B'(e^{-j\omega_k})Q(e^{-j\omega_k})]}{P_D(e^{-j\omega_k})P_F(e^{-j\omega_k})} \right| < S_{\max}(\omega_k) \quad \forall \omega_k$$

$$\left| \frac{A'(e^{-j\omega_k})[R_0'(e^{-j\omega_k}) + A'(e^{-j\omega_k})Q(e^{-j\omega_k})]}{P_D(e^{-j\omega_k})P_F(e^{-j\omega_k})} \right| < U_{\max}(\omega_k) \quad \forall \omega_k$$

4. If there is no feasible point then increase n_q.

5. A convex function can also be minimized in order to improve the performance. For example :

$$\min_x \left\| W_1 \frac{q^{-d}B'[S_0' - q^{-d}B'Q]}{P_D P_F} \right\|_2$$

will reduce the two-norm of the disturbance.

---

[1]Combined pole placement and sensitivity function shaping method using convex optimization criteria, Langer and Landau, Automatica 1999

Linear Matrix Inequalities with Applications in Control

A linear matrix inequality is an expression of the form:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \succ 0$$

- $x = [x_1, \ldots, x_m]$ is a vector of m decision variables,
- $F_i = F_i^T \in \mathcal{R}^{n \times n}, i = 0, \ldots, m,$
- The special inequality $\succ 0$, means positive definite.

Positive definite matrices: Matrix $F \succ 0$ if $u^T F u > 0$ for all $u \in \mathcal{R}^n$ and $u \neq 0$. The following statements are necessary for a real symmetric matrix F to be positive definite:

- All eigenvalues of F are positive.
- All diagonals of F are positive.
- The determinant of all principal minors of F are positive.
- All principal minors of F are positive.

Laboratoire
d'Automatique

Linear Matrix Inequalities

Further properties of real positive definite matrices:

1. Every positive definite matrix is invertible and the inverse is also positive definite.

2. If $F \succ 0$ and $\lambda > 0$ is a real number, then $\lambda F \succ 0$.

3. If $F \succ 0$ and $G \succ 0$ then $F + G \succ 0$ and $GFG \succ 0$ and $FGF \succ 0$ and $\text{tr}(FG) > 0$. The product FG is also positive definite if $FG = GF$

4. If $F \succ 0$ then there is $\delta > 0$ such that $F \succeq \delta I$ (it means that $F - \delta I \succeq 0$).

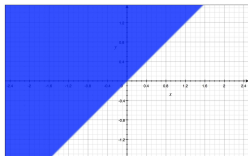**Main property:** $F(x) \succ 0$ defines a convex set on x.

That is the set $\mathcal{C} = \{x | F(x) \succ 0\}$ is convex. It means if $x_1, x_2 \in \mathcal{C}$ and $\lambda \in [0 \quad 1]$, then:

$$F(\lambda x_1 + (1 - \lambda)x_2) = \lambda F(x_1) + (1 - \lambda)F(x_2) \succ 0$$

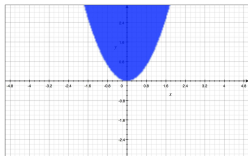$F(x) \succeq 0, F(x) \preceq 0$ and $F(x) \prec 0$ define the convex sets.

Laboratoire
d'Automatique

Linear Matrix Inequalities

Many convex sets can be represented by LMI:

$$x_2 > x_1 \qquad\qquad x_2 > x_1^2 \qquad\qquad x_1^2 + x_2^2 < 1$$

$$F(x) = [x_2 - x_1] > 0 \qquad F(x) = \begin{bmatrix} 1 & x_1 \\ x_1 & x_2 \end{bmatrix} \succ 0 \qquad F(x) = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \succ 0$$
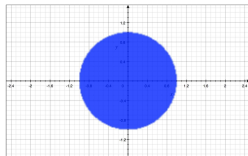
$F(x) \succ 0$ can always be represented as an LMI if its elements are affine w.r.t x. Moreover, any matrix inequality which is affine w.r.t $F(x)$ is also an LMI (i.e. $AF(x) + B$)

$$F(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
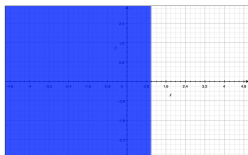
Linear Matrix Inequalities

Geometry of LMIs: An LMI is the intersection of constraints on some polynomial functions (determinants of the principal minors).

$$F(x) = \begin{bmatrix} 1 - x_1 & x_1 + x_2 & x_1 \\ x_1 + x_2 & 2 - x_1 & 0 \\ x_1 & 0 & 1 + x_2 \end{bmatrix} \succ 0$$
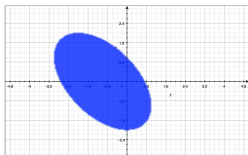
$m_1 :\quad 1 - x_1 > 0$
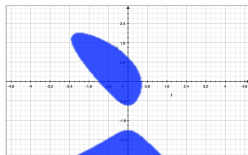
$m_2 :\quad (1 - x_1)(2 - x_1) - (x_1 + x_2)^2 > 0$

$m_3 :\quad x_1^2(x_2 - 2) + (1 + x_2)[(1 - x_1)(2 - x_2) - (x_1 + x_2)^2] > 0$

$m_1 > 0$ $m_2 > 0$ $m_3 > 0$

LMIs in Control

Stability analysis: A continuous-time LTI autonomous system $\dot{x}(t) = Ax(t)$ is asymptotically stable ($\lim_{t\to\infty} x(t) = 0, \quad \forall x_0 \neq 0$) iff there exists a quadratic Lyapunov function $V(x) = x^T P x$ such that:

$$V(x) > 0 \quad \text{and} \quad \dot{V}(x) < 0$$

These two conditions are verified iff there exists a symmetric matrix $P \succ 0$ such that

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x < 0$$

This is equivalent to feasibility of the LMI:

$$\begin{bmatrix} P & 0 \\ 0 & -(A^T P + PA) \end{bmatrix} \succ 0$$

or the existence of $P \succ 0$ and such that :

$$A^T P + PA + Q = 0 \qquad \text{where} \quad Q \succ 0$$

LMIs in Control

Stability of polytopic systems: Consider the LTI system $\dot{x}(t) = Ax(t)$ where $A \in \mathrm{co}\{A_1, \ldots, A_N\}$.

This system is quadratically stable iff there exists $P \succ 0$ such that :

$$A_i^T P + P A_i \prec 0 \qquad \forall i \in [1 \quad N]$$

Proof: We know that :

$$\mathrm{co}\{A_1, \ldots, A_N\} = \left\{ A \,\middle|\, A(\lambda) = \sum_{i=1}^{N} \lambda_i A_i, \quad \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i = 1 \right\}$$

On the other hand : $\sum_{i=1}^{N} \lambda_i (A_i^T P + P A_i) = A^T(\lambda) P + P A(\lambda) \prec 0$

- ▶ Stability of a polytopic system is ensured by stability of its vertices.
- ▶ Quadratic stability guarantees stability for fast parameter variations.
- ▶ Quadratic stability condition is too conservative for robust stability.

Laboratoire
d'Automatique

LMIs in Control

Bounded real lemma: Consider a stable strictly proper LTI system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ and $\gamma > 0$. The following statements are equivalent:

1. $\|G\|_\infty < \gamma$

2. The Hamiltonian matrix H has no eigenvalue on the imaginary axis.

$$H = \left[\begin{array}{cc} A & \gamma^{-2}BB^T \\ -C^T C & -A^T \end{array}\right]$$

3. There exists $P \succeq 0$ such that :

$$A^T P + PA + \gamma^{-2}PBB^T P + C^T C = 0$$

4. There exists $P \succ 0$ such that :

$$A^T P + PA + \gamma^{-2}PBB^T P + C^T C \prec 0$$

LMIs in Control

Schur lemma: If $A = A^T$ and $C = C^T$ then :

$$F = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \quad \begin{aligned} &\Longleftrightarrow \quad A \succ 0 \quad \text{and} \quad C - B^T A^{-1} B \succ 0 \\ &\Longleftrightarrow \quad C \succ 0 \quad \text{and} \quad A - BC^{-1}B^T \succ 0 \end{aligned}$$

$$F = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \prec 0 \quad \begin{aligned} &\Longleftrightarrow \quad A \prec 0 \quad \text{and} \quad C - B^T A^{-1} B \prec 0 \\ &\Longleftrightarrow \quad C \prec 0 \quad \text{and} \quad A - BC^{-1}B^T \prec 0 \end{aligned}$$

$C - B^T A^{-1} B$ and $A - BC^{-1}B^T$ are called Schur complements.

Proof:

$$\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}$$

Note that $M^T F M$ has the same sign of F if M is nonsingular.

LMIs in Control

Bounded real lemma: The matrix inequality

$$A^T P + PA + \gamma^{-2} PBB^T P + C^T C \prec 0$$

is not an LMI, but it can be converted to an LMI using the Schur lemma:

$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \prec 0$$

It can be further simplified using the Schur lemma again:
Multiplying by γ^{-1} and taking $P = \gamma^{-1} P$, we obtain:

$$A^T P + PA + \gamma^{-1} PBB^T P + \gamma^{-1} C^T C \prec 0$$

$$\begin{bmatrix} A^T P + PA + \gamma^{-1} C^T C & PB \\ B^T P & -\gamma I \end{bmatrix} = \begin{bmatrix} A^T P + PA & PB \\ B^T P & -\gamma I \end{bmatrix} + \begin{bmatrix} C^T \\ 0 \end{bmatrix} \gamma^{-1} I [\begin{array}{cc} C & 0 \end{array}]$$

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} \prec 0$$

LMIs in Control

Computing the infinity-norm: Given $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $\|G\|_\infty$ is the solution to the following convex optimization problems with LMI constraints:

$$\min \gamma$$

$$\left[\begin{array}{ccc} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{array}\right] \prec 0, \qquad P \succ 0$$

This problem can be solve by the following YALMIP code [1]:

```
gamma=sdpvar(1,1); P=sdpvar(n,n,'symmetric');
lmi=set([A'*P+P*A P*B C';B'*P -gamma D';C D
-gamma]<0); lmi=lmi+set(P>0);
options=sdpsettings('solver','lmilab');
solvesdp(lmi,gamma,options);
gamma=double(gamma)
```

---

[1] Free code developed by J. Löfberg accessible at
http://control.ee.ethz.ch/~joloef/yalmip.php

LMIs in Control

Model-matching problem: Given $T_1(s)$, $T_2(s) \in \mathcal{RH}_\infty$ find $Q(s) \in \mathcal{RH}_\infty$ such that $\|T_1 - T_2 Q\|_\infty$ is minimized. This problem is encountered in feedback controller design for nominal performance and robust stability.

Step I: Linearly parameterize $Q(s)$ using a set of stable orthonormal basis functions: $Q(s) = \sum_{i=1}^{n_q} x_i Q_i(s)$

Laguerre : $\quad Q_i(s) = \dfrac{\sqrt{2a}}{s+a}\left(\dfrac{s-a}{s+a}\right)^{i-1}$

Kautz : $\quad Q_i(s) = \dfrac{\sqrt{2a}s}{s^2+as+b}\left(\dfrac{s^2-as+b}{s^2+as+b}\right)^{i-1} \quad$ for $\quad i = 2k-1$

$\qquad\qquad Q_i(s) = \dfrac{\sqrt{2ab}}{s^2+as+b}\left(\dfrac{s^2-as+b}{s^2+as+b}\right)^{i-1} \quad$ for $\quad i = 2k$

General : $\quad Q_i(s) = \dfrac{\sqrt{2Re[a_i]}}{s+a_i}\prod_{k=1}^{i-1}\dfrac{s-\bar{a}_k}{s+a_k}$

LMIs in Control

Model-matching problem:

Step II: Find a controllable canonical state space realization of $T_1 - T_2Q$:

$$T_1(s) - T_2(s)Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

Note that the decision variables x_i appear linearly only in the numerator of $T_1 - T_2Q$, so they appear in C and may be in D if $T_1 - T_2Q$ is not strictly proper.

Step III: Solve the following SDP problem :

$$\min \gamma$$

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0, \qquad P \succ 0$$

LMIs in Control

State feedback stabilization: Given $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, compute a state feedback controller K such that the closed-loop system is stable.

$$
\begin{array}{lll}
\dot{x} = Ax + Bu & & \dot{x} = (A - BK)x + Br \\
y = Cx & \Rightarrow & y = Cx \\
u = r - Kx & &
\end{array}
$$

Using the Lyapunov stability condition, we have :

$$(A - BK)^T P + P(A - BK) \prec 0, \qquad P \succ 0$$

which is not an LMI. Now, multiply it from left and right by $X = P^{-1}$:

$$X(A - BK)^T + (A - BK)X \prec 0, \qquad X \succ 0$$

Denoting $Y = KX$, we derive:

$$XA^T + AX - Y^T B^T - BY \prec 0, \qquad X \succ 0$$

LMIs in Control

Finsler lemma: The following statements are equivalent:

1. There exists $X \succ 0$ such that

$$XA^T + AX - Y^T B^T - BY \prec 0$$

2. There exists $X \succ 0$ such that

$$\tilde{B}^T(XA^T + AX)\tilde{B} \prec 0$$

   where $\tilde{B}$ is an orthogonal complement of B (i. e. $\tilde{B}^T B = 0$)

3. There exists $X \succ 0$ and a scalar $\sigma > 0$ such that

$$XA^T + AX - \sigma B^T B \prec 0$$

Without loss of generality we can take $\sigma = 1$ and compute X and then the controller by

$$K = \frac{1}{2} B^T X^{-1}$$

LMIs in Control

H_∞ **state feedback control:** Given $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, compute a state feedback controller K such that the infinity norm of the closed-loop system is minimized.

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \\ u = r - Kx \end{array} \quad \Rightarrow \quad \begin{array}{l} \dot{x} = (A - BK)x + Br \\ y = Cx \end{array}$$

Using the bounded real lemma we have :

$$(A - BK)^T P + P(A - BK) + \gamma^{-2} PBB^T P + C^T C \prec 0$$

Now, multiply it from left and right by $X = P^{-1}$:

$$X(A - BK)^T + (A - BK)X + \gamma^{-2} BB^T + XC^T CX \prec 0$$

Denoting $Y = KX$, we derive:

$$XA^T + AX - Y^T B^T - BY + \gamma^{-2} BB^T + XC^T CX \prec 0$$

LMIs in Control

H_∞ **state feedback control:** This matrix inequality can be converted to an LMI using the Schur lemma. First multiply the inequality by γ and then take $X = \gamma X$ and $Y = \gamma Y$:

$$XA^T + AX - Y^T B^T - BY + \gamma^{-1} BB^T + \gamma^{-1} X C^T C X \prec 0$$

Now, applying two times the Schur lemma we obtain:

$$\begin{bmatrix} XA^T + AX - Y^T B^T - BY & XC^T & B \\ CX & -\gamma I & 0 \\ B^T & 0 & -\gamma I \end{bmatrix} \prec 0, \quad X \succ 0$$

After minimizing γ subject to the above LMI constraints the controller is computed by $K = YX^{-1}$.

LMIs in Control

Computing the 2-norm: Given $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ we have:

$$\|G\|_2^2 = tr(CLC^T)$$

where L is the solution of $AL + LA^T + BB^T = 0$.

The two-norm can be computed by an equivalent convex optimization problem:

$$\min \gamma^2$$
$$AL + LA^T + BB^T \preceq 0 \qquad tr(CLC^T) < \gamma^2$$

Multiplying the first LMI by $X = L^{-1} \succ 0$ we obtain:

$$XA + A^T X + XBB^T X \prec 0 \qquad tr(CX^{-1}C^T) < \gamma^2$$

The second inequality is equivalent to $\exists Z$ such that $CX^{-1}C^T \prec Z$ and $tr(Z) < \gamma^2$.

Using the Schur lemma we have :

$$\begin{bmatrix} XA + A^T X & XB \\ B^T X & -I \end{bmatrix} \prec 0 \qquad \begin{bmatrix} Z & C \\ C^T & X \end{bmatrix} \succ 0$$

Laboratoire
d'Automatique

LMIs in Control

H_2 **state feedback control:** Given $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$

(controllable and observable), compute a state feedback controller K such that the two norm of the closed-loop system is minimized.

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \\ u = r - Kx \end{array} \quad \Rightarrow \quad \begin{array}{l} \dot{x} = (A - BK)x + Br \\ y = Cx \end{array}$$

we have :

$$\min \gamma^2$$

$$(A - BK)L + L(A - BK)^T + BB^T \preceq 0 \qquad tr(CLC^T) < \gamma^2$$

The first inequality can be converted to an LMI denoting $Y = KL$:

$$AL + LA^T - BY - Y^T B^T + BB^T \preceq 0$$

Positive Real Systems

Definition: The rational transfer function $H(s)$ is positive real (PR) if

1. $H(s) \in \mathcal{R} \qquad \forall s \in \mathcal{R}$ (The coefficients of s are real).
2. $H(s)$ is analytic for $\text{Re}[s] > 0$.
3. $\text{Re}\,[H(s)] \geq 0, \quad \forall s : \text{Re}[s] \geq 0$.

Strictly Positive Real Systems: $H(s)$ is SPR if $H(s - \epsilon)$ is PR for some $\epsilon > 0$.

Example: $H(s) = \frac{1}{s+\lambda}$ with $\lambda > 0$ is SPR.

$H(s) = \frac{1}{s}$ is PR but not SPR and $H(s) = \frac{-1}{s}$ is not PR.

Remark 1: $H(s)$ is PR if (1) and (2) hold and

- any pure imaginary pole of $H(s)$ is a simple pole with non negative residue,
- for all ω for which $j\omega$ is not a pole of $H(s)$, $\text{Re}[H(j\omega)] \geq 0$.

Remark 2: $H(s)$ is SPR if (1) holds and

- $H(s)$ is analytic for $\text{Re}[s] \geq 0$,
- $\text{Re}[H(j\omega)] > 0 \qquad \forall \omega$.

Laboratoire
d'Automatique

Positive Real Systems

Properties of SPR systems:

1. $H(s)$ has no pole in RHP and on the imaginary axis.
2. The Nyquist plot of $H(j\omega)$ lies strictly in right half complex plane.
3. The relative degree of $H(s)$ is 0, 1 or -1.
4. $H(s)$ is minimum phase ($H(s)$ has no zero with Re $[z] \geq 0$).
5. If $H(s)$ is SPR, then $1/H(s)$ is also SPR. Similarly, if $H(s)$ is PR, then $1/H(s)$ is PR.
6. If $H_1(s) \in$ SPR and $H_2(s) \in$ SPR then $H(s) = \alpha_1 H_1(s) + \alpha_2 H_2(s)$ is SPR for $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 > 0$.
7. If $H_1(s)$ and $H_2(s)$ are SPR then the feedback interconnection of two systems is also SPR.

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{1}{\frac{1}{H_1(s)} + H_2(s)} \in \text{SPR}$$

This property holds even if $H_1(s)$ is PR.

Positive Real Systems

Lemma (Kalman-Yakubovich-Popov): Consider $H(s)$ with following minimal realization:

(n: number of states, m number of inputs and outputs)

$$H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$$

▶ $H(s)$ is PR if and only if there exist $P \in \mathcal{R}^{n \times n} \succ 0$, $Q \in \mathcal{R}^{m \times n}$ and $W \in \mathcal{R}^{m \times m}$ such that:

$$PA + A^T P = -Q^T Q, \quad PB = C^T - Q^T W, \quad W^T W = D + D^T$$

$H(s)$ is SPR if in addition $\bar{H}(s) = W + Q(sI - A)^{-1}B$ has no zero on the imaginary axis.

▶ $H(s)$ is SPR, that is $H(j\omega) + H^*(j\omega) \succ 0 \quad \forall \omega$, if there exist $P \in \mathcal{R}^{n \times n} \succ 0$, $Q \in \mathcal{R}^{m \times n}$, $W \in \mathcal{R}^{m \times m}$ and $\epsilon > 0$ such that:

$$PA + A^T P = -\epsilon P - Q^T Q, \quad PB = C^T - Q^T W$$

$$W^T W = D + D^T$$

Laboratoire
d'Automatique

Positive Real Systems

Theorem (passivity): Consider the LTI system $H(s)$ with a minimal realization (A, B, C, D) interconnected with a sector nonlinearity $\phi(t, y)$ defined as:

$$\begin{cases} \phi(t, 0) = 0 & \forall\, t \geq 0 \\ y^T \phi(t, y) \geq 0 & \forall t \geq 0 \end{cases}$$

Then the closed-loop system is globally and exponentially stable if $H(s)$ is SPR.

Proof: We show that if $H(s)$ is SPR the closed-loop system is stable. From positive real lemma, there exist $P > 0, \epsilon > 0, Q, W$ such that:

$$PA + A^T P = -\epsilon P - Q^T Q, \quad PB = C^T - Q^T W, \quad W^T W = D + D^T$$

Now, we consider $V(x) = x^T P x$ as a Lyapunov function and we show that $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} < 0$. (Note that $u = -\phi(t, y)$)

$\dot{V}(x) = [Ax - B\phi]^T P x + x^T P[Ax - B\phi] = x^T(A^T P + PA)x - \phi^T B^T P x - x^T PB\phi$

$\dot{V}(x) = -\epsilon V(x) - x^T Q^T Q x - 2\phi^T y - \phi^T(D + D^T)\phi + 2\phi^T W^T Q x$

$\dot{V}(x) \leq -\epsilon V(x) - [Qx + W\phi]^T [Qx + W\phi] < 0$

Laboratoire d'Automatique

Positive Real Systems

LMI formulation:

Continuous-time: $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$ is SPR

iff there exists $P = P^T$ such that:

$$\left[\begin{array}{cc} A^TP + PA & PB - C^T \\ B^TP - C & -D - D^T \end{array}\right] \prec 0$$

Discrete-time: $H(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D$ is SPR iff

there exists $P = P^T$ such that:

$$\left[\begin{array}{cc} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - D - D^T \end{array}\right] \prec 0$$

Positive Real Systems

Relation between positive realness and infinity norm: It can be shown that

$$H \in \mathsf{SPR} \iff \left\| \frac{1 - H}{1 + H} \right\|_\infty < 1$$

Proof:

$$
\begin{aligned}
\left\| \frac{1-H}{1+H} \right\|_\infty < 1 \iff{}& |1 - H| < |1 + H| \quad \forall\, \omega \\
\iff{}& (1 - \mathsf{Re}\{H\})^2 + (\mathsf{Im}\{H\})^2 \\
& < (1 + \mathsf{Re}\{H\})^2 + (\mathsf{Im}\{H\})^2 \quad \forall\, \omega \\
\iff{}& 4\mathsf{Re}\{H\} > 0 \quad \forall\, \omega \\
\iff{}& H \in \mathsf{SPR}
\end{aligned}
$$

Note that if H is SPR then $\dfrac{1}{1 + H}$ and $\dfrac{H}{1 + H}$ are stable, so $\dfrac{1 - H}{1 + H}$ is stable.

Positive Real Systems

From bounded real lemma to positive real lemma and vice-verca: Show that the infinity norm of $H = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is less than γ, iff $\dfrac{\gamma - H}{\gamma + H}$ is SPR.

$\|H\|_\infty < \gamma : \exists P \succ 0,$ and $A^T P + PA + \gamma^{-2} PBB^T P + C^T C \prec 0$

$$\frac{\gamma - H}{\gamma + H} = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A - \gamma^{-1}BC & B \\ \hline -2\gamma^{-1}C & 1 \end{array}\right]$$

Multiply the matrix inequality by $2\gamma^{-2}$ and take $P = 2P\gamma^{-2}$:

$$A^T P + PA + \frac{1}{2}PBB^T P + 2\gamma^{-2}C^T C \pm \gamma^{-1}C^T B^T P \pm \gamma^{-1}PBC \prec 0$$

$$\Longleftrightarrow$$

$$(A - \gamma^{-1}BC)^T P + P(A - \gamma^{-1}BC) + (PB + 2\gamma^{-1}C^T)\frac{1}{2}(B^T P + 2\gamma^{-1}C) \prec 0$$

$$\Longleftrightarrow \quad \left[\begin{array}{cc} \bar{A}^T P + P\bar{A} & P\bar{B} - \bar{C}^T \\ \bar{B}^T P - \bar{C} & -\bar{D} - \bar{D}^T \end{array}\right] \prec 0, \qquad P \succ 0$$

Robust Control of Systems with Parameter Uncertainty

Parameter uncertainty: First principle modeling gives a model with physical parameter uncertainty.

$$G(s) = \frac{a}{s^2 + bs + c}$$

- Interval uncertainty :
  $$a \in [a_{\min}, a_{\max}], \quad b \in [b_{\min}, b_{\max}], \quad c \in [c_{\min}, c_{\max}]$$
- Linear parameter uncertainty :
  $$a = 1 + \theta_1 + 2\theta_2, \quad b = \theta_2 - \theta_1, \quad c = 3\theta_1 - \theta_2$$
- Multilinear parameter uncertainty :
  $$a = 1 + \theta_1\theta_2, \quad b = \theta_2 - 3\theta_1\theta_2, \quad c = 1 + \theta_1$$
- Nonlinear parameter uncertainty :
  $$a = 1 + \theta_1^2\theta_2, \quad b = \sqrt{\theta_1 + \theta_2}, \ldots$$
- Multimodel uncertainty :
  $$a \in \{a_1, a_2, a_3\}, \quad b \in \{b_1, b_2, b_3\}, \quad c \in \{c_1, c_2, c_3\}$$

Laboratoire
d'Automatique

Robust Control of Systems with Parameter Uncertainty

Polytopic System: is a polytope in the space of the coefficients of numerator and denominator of the model.

$$G(s) = \frac{a}{s^2 + bs + c}$$

$$a = F_1(\theta_1, \theta_2), \quad b = F_2(\theta_1, \theta_2), \quad c = F_3(\theta_1, \theta_2)$$

$$\theta_1 \in [\theta_1^{\min}, \theta_1^{\max}], \theta_2 \in [\theta_2^{\min}, \theta_2^{\max}]$$

In the three-dimensional space (a,b,c), four points (2^{n_θ}) can be defined :

$$p_1(\theta_1^{\min}, \theta_2^{\min}), p_2(\theta_1^{\min}, \theta_2^{\max}), p_3(\theta_1^{\max}, \theta_2^{\min}), p_4(\theta_1^{\max}, \theta_2^{\max})$$

The convex combination of $\{p_1, p_2, p_3, p_4\}$ makes a polytopic system.

Polytopic system a polytope in the space of system parameters $(a, b, c, \ldots)$ constructed by the convex combination of the mapping of the vertices of a hyper cube in the space of the independent parameters $(\theta_1, \theta_2, \ldots)$ under the functions $F_1, F_2, F_3, \ldots$

Laboratoire
d'Automatique

Robust Control of Systems with Parameter Uncertainty

Polytopic uncertainty:

- *covers interval and linear parameter uncertainty without any conservatism,*

- *represents the smallest polytope (convex set) outbounding the multilinear parameter uncertainty,*

- *can be used with some conservatism for multimodel uncertainty. However, the larger set covered by polytopic system is practically more realistic than the multimodel set.*

Robust Control of Systems with Parameter Uncertainty

Q-**Parameterization:** Given $G = NM^{-1}$ with $N, M \in \mathcal{RH}_\infty$ the coprime factors of G and $X, Y \in \mathcal{RH}_\infty$ satisfying $NX + MY = 1$, then the set of all stabilizing controllers is given by :

$$\mathcal{K} : \left\{ K = \frac{X + MQ}{Y - NQ} \ \middle| \ Q \in \mathcal{RH}_\infty \right\}$$

$S = M(Y - NQ)$ and $T = N(X + MQ)$ are convex in Q

Main problems of Q-parameterization :

▶ It cannot be applied to systems with parametric uncertainty, because the controller depends directly to the plant model.

▶ The controller order depends on the order of the plant model and the order of Q, so fixed-order controller (like PID) cannot be considered.

New Parameterization

Theorem : The set of all stabilizing controllers for $G = NM^{-1}$ is given by:
$$\mathcal{K} : \{K = XY^{-1} \,|\, MY + NX \in \mathcal{S}\}$$

where $X, Y \in \mathcal{RH}_\infty$ and $\mathcal{S}$ is the convex set of all Strictly Positive Real (SPR) transfer functions.

Elements of proof :

- ▶ SPR transfer functions are stable and inversely stable.

- ▶ Zeros of $MY + NX$ are the closed-loop poles.

- ▶ If $K_0 = X_0 Y_0^{-1}$ stabilizes the closed loop system then $F = (MY_0 + NX_0)^{-1} \in \mathcal{RH}_\infty$ and therefore $K_0 \in \mathcal{K}$ with $X = X_0 F$ and $Y = Y_0 F$ because $MY + NX = (MY_0 + NX_0)F = 1 \in \mathcal{S}$.

Laboratoire
d'Automatique

Robust Control of Systems with Parameter Uncertainty

Extension to Polytopic Systems: Consider a polytopic system with q vertices such that i-th vertex constitutes of the parameters of the model $G_i = N_i M_i^{-1}$. Assume that the model parameters appear linearly in the numerators of N_i and M_i. Then the set of all models in this polytopic system is given by : $(\lambda_i \geq 0, \sum_{i=1}^{q} \lambda_i = 1)$

$$\mathcal{G} : \{ G = NM^{-1} \mid N = \sum_{i=1}^{q} \lambda_i N_i, \ M = \sum_{i=1}^{q} \lambda_i M_i, \}$$

Example : $G(s) = \dfrac{s+a}{s+b}, \qquad a \in [a_{\min}, a_{\max}], \quad b \in [b_{\min}, b_{\max}]$

$$G_1 = N_1 M_1^{-1}, \qquad N_1 = \frac{s + a_{\min}}{s+1}. \quad M_1 = \frac{s + b_{\min}}{s+1}$$

$$G_2 = N_2 M_2^{-1}, \qquad N_2 = \frac{s + a_{\min}}{s+1}. \quad M_2 = \frac{s + b_{\max}}{s+1}$$

Extension to Polytopic Systems

Theorem : The set of all stabilizing controllers for $\mathcal{G}$ is given by:

$$\mathcal{K} : \{K = XY^{-1} \,|\, M_iY + N_iX \in \mathcal{S}, \quad i = 1, \ldots, q\}$$

where $X, Y \in \mathcal{RH}_\infty$. This set is convex in the pair (X, Y).

Elements of proof :

- If $M_iY + N_iX \in \mathcal{S}$ for $i = 1, \ldots q$ then their convex combinations is also in $\mathcal{S}$, hence $MY + NX$ is SPR.

- If $K_0 = X_0 Y_0^{-1}$ stabilizes the polytopic system then the characteristic polynomials c_i of vertices make a polytope of stable polynomials. For such a polytope there exist d such that c_i/d is SPR. Therefore, choosing
  $F = (MY_0 + NX_0)^{-1}c_i/d \in \mathcal{RH}_\infty$ and $K_0 \in \mathcal{K}$ with $X = X_0F$ and $Y = Y_0F$ because
  $MY + NX = (MY_0 + NX_0)F = c_i/d \in \mathcal{S}$.

Robust Control of Systems with Parameter Uncertainty

Pole placement for a single model: Consider the following optimization problem : $\min\limits_{X,Y} \|MY + NX - 1\|$

$$\text{Subject to:} \quad MY + NX \in \mathcal{S}$$

This optimization problem is convex if :

$$X = \sum_{i=1}^{m} x_i \beta_i \qquad ; \qquad Y = \sum_{i=1}^{m} y_i \beta_i$$

where β_i are the basis functions:

$$\beta_i(s) = \frac{\sqrt{2a}}{s+a} \left(\frac{s-a}{s+a} \right)^{i-1} \quad \text{or} \quad \beta_i(s) = \frac{\sqrt{2Re[a_i]}}{s+a_i} \prod_{k=1}^{i-1} \frac{s-\bar{a}_k}{s+a_k}$$

This optimization problem places the closed-loop poles on the denominator of $MY + NX$ which can be chosen arbitrarily.

Robust Control of Systems with Parameter Uncertainty

Pole placement for polytopic systems: An approximate pole placement can be achieved by the following optimization problem :

$$\min \gamma \quad \text{Subject to:}$$

$$\|M_i Y + N_i X - 1\| < \gamma \quad \text{for } i = 1, \ldots, q$$

$$M_i Y + N_i X \in \mathcal{S} \qquad \text{for } i = 1, \ldots, q$$

- ▶ Pole placement alone, does not necessarily give good performance and is not robust with respect to unstructured uncertainty.
- ▶ Sensitivity function shaping is needed to obtain good control.
- ▶ **Sensitivity functions are not convex with respect to the new parameterization.**

$$S = (1 + KG)^{-1} = (1 + XY^{-1}NM^{-1})^{-1} = \frac{MY}{MY + NX}$$

Laboratoire
d'Automatique

Robust Control of Systems with Parameter Uncertainty

Sensitivity Function Shaping: Note that

$$
\begin{aligned}
S &= (1 + KG)^{-1} = MY(MY + NX)^{-1} \\
T &= KG(1 + KG)^{-1} = NX(MY + NX)^{-1}
\end{aligned}
$$

On the other hand:

$$\|MY + NX - 1\|_\infty < \gamma \Leftrightarrow |MY + NX - 1| < \gamma \quad \forall \omega$$

$$\Rightarrow 1 - \gamma < |MY + NX| < 1 + \gamma \quad \forall \omega$$

Therefore:
$$
\begin{aligned}
\|W_1 MY\| < 1 - \gamma &\quad \Rightarrow \quad \|W_1 S\| < 1 \\
\|W_1 NX\| < 1 - \gamma &\quad \Rightarrow \quad \|W_2 T\| < 1
\end{aligned}
$$

Robust pole placement with sensitivity function shaping

$$\text{Minimize } \max_i \gamma_i$$

$$
\begin{array}{lll}
\|M_i Y + N_i X - 1\|_\infty < \gamma_i & \quad \text{for} & \quad i = 1, \ldots, q \\
\|W_1 M_i Y\| < 1 - \gamma_i & \quad \text{for} & \quad i = 1, \ldots, q \\
\|W_2 N_i X\| < 1 - \gamma_i & \quad \text{for} & \quad i = 1, \ldots, q \\
M_i Y + N_i X \in \mathcal{S} & \quad \text{for} & \quad i = 1, \ldots, q
\end{array}
$$

Laboratoire
d'Automatique

Robust Control of Systems with Parameter Uncertainty

LMI formulation: We have 2-norm, infinity-norm and SPRness constraints on some transfer functions in which the denominators are known and the optimization variables appear only in the numerators. Using a controllable canonical state-space realization (A, B, C, D), the optimization variables appear only in C and D.

Infinity-norm:

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i & C_i^T \\ B_i^T P_i & -\gamma_i I & D_i^T \\ C_i & D_i & -\gamma_i I \end{bmatrix} \prec 0, \qquad P_i \succ 0$$

SPRness:

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i - C_i^T \\ B_i^T P_i - C_i & -D_i - D_i^T \end{bmatrix} \prec 0, \qquad P_i \succ 0$$
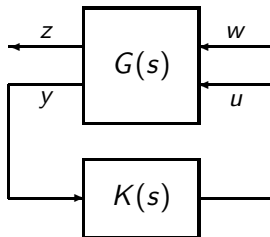
Linear Fractional Transformation

A feedback control system can be rearranged as an LFT:

w: all external inputs

u: control inputs

z: outputs or error signals

y: measured outputs

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \qquad u = Ky$$

$$T_{zw} = F_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

▶ LFT is a useful way to standardize block diagrams for robust control analysis and design

▶ $F_l(G, K)$ is the transfer function between the error signals and external inputs. In $\mathcal{H}_\infty$ control problems the objective is to minimize $\|F_l(G, K)\|_\infty$

Linear Fractional Transformation

Example 1: Show the nominal performance problem as an LFT (Find the system matrix G):
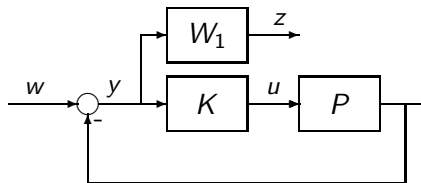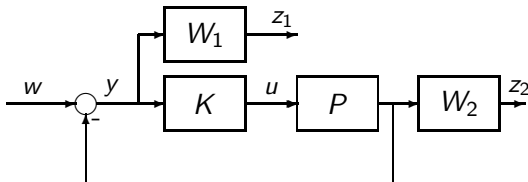

$$z = W_1(w - Pu)$$
$$y = w - Pu$$

$$\begin{pmatrix} z \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} W_1 & -W_1P \\ 1 & -P \end{pmatrix}}^{G} \begin{pmatrix} w \\ u \end{pmatrix}$$

$$
\begin{aligned}
F_l(G, K) &= G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \\
&= W_1 + (-W_1P)K(1 + PK)^{-1}1 \\
&= W_1\left(1 - \frac{PK}{1 + PK}\right) = \frac{W_1}{1 + PK} = W_1 S
\end{aligned}
$$

Linear Fractional Transformation

Example 2: Show the robust performance problem for multiplicative uncertainty as an LFT:

$$z_1 = W_1(w - Pu)$$
$$z_2 = W_2 Pu$$
$$y = w - Pu$$

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} W_1 & -W_1 P \\ 0 & W_2 P \\ 1 & -P \end{pmatrix}}^{G} \begin{pmatrix} w \\ u \end{pmatrix}$$
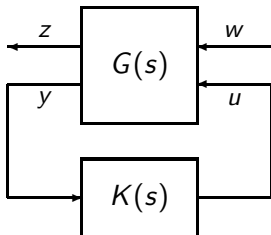
$$F_l(G, K) = \begin{pmatrix} W_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -W_1 P \\ W_2 P \end{pmatrix} K(1 + PK)^{-1} = \begin{pmatrix} W_1 S \\ W_2 P \end{pmatrix}$$

$$\|F_l(G, K)\|_\infty = \left\| \begin{pmatrix} W_1 S \\ W_2 T \end{pmatrix} \right\|_\infty$$

H_∞ Control

Consider the system described by:

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]$$

Optimal H_∞ Control: Find all admissible controllers $K(s)$ such that $||T_{zw}||_\infty$ is minimized.

Suboptimal H_∞ Control: Given $\gamma > 0$, find all admissible controllers $K(s)$, if there are any, such that $||T_{zw}||_\infty < \gamma$.

Assumptions:

(A1) (A, B_2, C_2) is stabilizable and detectable.

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank $\forall \omega$.

(A2) D_{12} has full column rank and D_{21} has full raw rank.

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full raw rank $\forall \omega$.

Laboratoire
d'Automatique

H_∞ Control

What can we do if the Assumptions are not satisfied?

(A2) If D_{12} has not full column rank, it means that some control inputs have no effect on the controlled outputs $z = C_1 x + D_{12} u$ (these control inputs can go to infinity). The solution is to add some weighting filters on these control inputs (even a very small gain to avoid the singularity in the computations).

If D_{21} has not full raw rank, it means that one of the measured outputs $y = C_2 x + D_{21} w$ is not affected by any of external inputs (the feedback from this output says anything about the variation in input variables). This makes the problem singular and cannot be solved. The solution is to add some external inputs (noise or disturbance) on all measured output (with a very small gain to avoid the singularity in the computations).

(A3) and (A4) These Assumptions are not satisfied if there is some poles of the plant model or weighting filters on the imaginary axis. This problem can be solved by a small perturbation of the poles on the imaginary axis.

$D_{22} \neq 0$: This problem occurs when we transform a discrete system to a continuous system. In this case we can solve the problem for $D_{22} = 0$ and compute the controller K_0 and then the final controller is:
$$K = K_0(I + D_{22}K_0)^{-1}$$

H_∞ Control

State-Space Solution There exists a stabilizing controller such that $||T_{zw}||_\infty < \gamma$ iff these conditions hold:

1. There exists $X \succ 0$ such that

$$XA + A^T X + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X + C_1^T C_1 = 0$$

2. There exists $Y \succ 0$ such that

$$AY + YA^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2)Y + B_1 B_1^T = 0$$

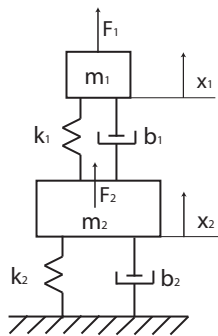3. $\rho(XY) < \gamma^2 \qquad \rho(\cdot) = |\lambda_{\max}(\cdot)|$: spectral radius

Moreover, one such controller is:

$$K_{\text{sub}}(s) = \left[\begin{array}{c|c} \hat{A} & (I - \gamma^{-2} YX)^{-1} Y C_2^T \\ \hline -B_2^T X & 0 \end{array} \right]$$

where:

$$\hat{A} = A + \gamma^{-2} B_1 B_1^T X - B_2 B_2^T X - (I - \gamma^{-2} YX)^{-1} Y C_2^T C_2$$

H_∞ Control

Example: Consider the following mass/spring/damper system

$F_1 = m_1\ddot{x}_1 + b_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2)$

$F_2 = m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2 x_2 + k_1(x_2 - x_1)$
$+ b_1(\dot{x}_2 - \dot{x}_1)$

States: $x_1, x_2, \dot{x}_1 = x_3, \dot{x}_2 = x_4$

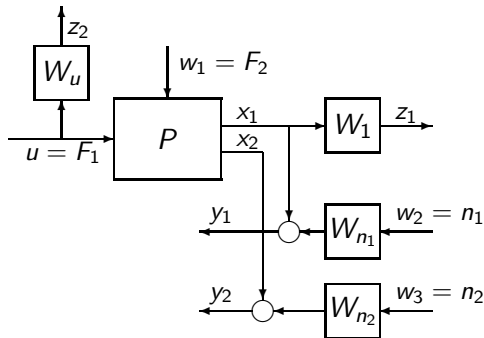**Inputs:** F_1 (control input) F_2 (disturbance)

Outputs: x_1, x_2 (measured)

Parameters: $m_1 = 1, m_2 = 2, k_1 = 1,$
$k_2 = 4, b_1 = 0.2, b_2 = 0.1$

Objective: Reduce the effect of disturbance force
(F_2) on x_1 in the frequency range $0 \leq \omega \leq 2$.

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{-k_1}{m_1} & \frac{k_1}{m_1} & \frac{-b_1}{m_1} & \frac{b_1}{m_1} \\
\frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\frac{1}{m_1} & 0 \\
0 & \frac{1}{m_2}
\end{bmatrix}
\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
$$

Laboratoire
d'Automatique

H_∞ Control : Example (mass/spring/damper)

$$W_u = \frac{s+5}{s+50}$$

$$W_1 = \frac{10}{s+2}$$

$$W_{n_1} = W_{n_2} = \frac{0.01s + 0.1}{s+100}$$

$$x_1(s) = P_{11}(s)F_1(s) + P_{12}(s)F_2(s) \qquad x_2(s) = P_{21}(s)F_1(s) + P_{22}(s)F_2(s)$$

Build the augmented plant:

$$\begin{bmatrix} z_1 \\ z_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} W_1 P_{12} & 0 & 0 & W_1 P_{11} \\ 0 & 0 & 0 & W_u \\ P_{12} & W_{n_1} & 0 & P_{11} \\ P_{22} & 0 & W_{n_2} & P_{21} \end{bmatrix} \begin{bmatrix} F_2 \\ n_1 \\ n_2 \\ F_1 \end{bmatrix}$$

H_∞ Control : Example (mass/spring/damper)

Matlab Codes:
```
Ap=[0 0 1 0;0 0 0 1;-1 1 -0.2 0.2;0.5 -2.5 0.1
-0.15];
Bp=[0 0;0 0;1 0;0 0.5];Cp=[1 0 0 0;0 1 0 0];
Dp=[0 0;0 0];
P=ss(Ap,Bp,Cp,Dp); % State-space model of the plant
Ptf=tf(P); % Transfer function model
% Weighting filters:
W1=tf(5,[0.5 1]);Wu=tf([1 5],[1 50]);
Wn=tf([0.01 0.1],[1 100]);
% Augmented plant:
G=[W1*Ptf(1,2) 0 0 W1*Ptf(1,1);0 0 0 Wu;Ptf(1,2) Wn 0
Ptf(1,1);Ptf(2,2) 0 Wn Ptf(2,1)]
% Convert state-space object to a system matrix
(appropriate for Robust Control toolbox)
G1=ss(G);[A,B,C,D]=ssdata(G1);Gsys=pck(A,B,C,D);
```

Laboratoire
d'Automatique

H_∞ Control : Example (mass/spring/damper)

H_∞ controller design

```
[K,T,gopt] = hinfsyn(Gsys,nmeas,ncon,gmin,gmax,tol);
```

- ▶ nmeas=2: Number of controller inputs,
- ▶ ncon=1: Number of controller outputs,
- ▶ gmin=0.1, gmax=10 (for the bisection algorithm)
- ▶ K Controller: 24 states, two inputs, one output
- ▶ gopt=0.2311
- ▶ T Closed-loop system (between w and z):

    48 states, 3 inputs, two outputs

Note that the order of the plant model was 4 !

H_2 controller design

```
[K2,T2] = h2syn(Gsys,nmeas,ncon);
```

Laboratoire
d'Automatique

H_∞ Control : Integral Control

How can we design an H_∞ controller with integral action?
Introduce an integral in the performance weight W_1, then the
transfer function between z_1 and w_1 is given by:

$$T_{zw} = W_1(1 + PK)^{-1}$$

Now if the resulting controller K stabilizes the plant P and make
$||T_{zw}||_\infty$ finite, then K must have a pole at $s = 0$.

Problem: H_∞ theory cannot be applied to systems with poles
on the imaginary axis.

Solution: Consider a pole very close to the origin in W_1 (i. e.
$W_1 = \frac{1}{s+\epsilon}$) and solve the H_∞ problem. The resulting
controller will have a pole very close to the origin
which can be replaced by an integrator.

Model & Controller Order Reduction

Introduction: Robust controller design based on the H_∞ method leads to very high-order controllers. There are different methods to design a low-order controller:

- ▶ Design directly a fixed-order controller for a high-order plant (open problem)
- ▶ Reduce the plant model order and design a low-order controller for the low-order model.
- ▶ Design a high-order controller and then reduce the controller order.

Model Reduction Techniques:

Classical Approaches: Zero-pole cancelation, omitting non-dominant poles and zeros, ...

Modern Approaches: Balanced model reduction, weighted balanced model reduction, Hankel norm model reduction, ...

Based on Identification: Time-domain approaches, frequency-domain approaches.

Laboratoire
d'Automatique

Model Order Reduction

Problem: Given a full-order model $G(s)$, find a lower-order model (r-th order model $G_r(s)$) such that G and G_r are close in some sense (e.g. infinity-norm).

Additive model reduction:

$$\inf_{\deg(G_r) \leq r} \| G - G_r \|_\infty$$

Multiplicative model reduction:

$$\inf_{\deg(G_r) \leq r} \| G^{-1}(G - G_r) \|_\infty$$

Frequency-weighted model reduction:

$$\inf_{\deg(G_r) \leq r} \| W_o(G - G_r)W_i \|_\infty$$

Example: In model reduction for control purpose, the objective is to find a reduced order model such that the closed-loop transfer functions are close to each other:

$$\| S - S_r \|_\infty = \| T - T_r \|_\infty = \| U(G - G_r)S_r \|_\infty$$

Model Order Reduction

Preliminaries: Consider the following LTI system

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = C(sI - A)^{-1}B + D$$

Theorem: The following are equivalent

1. (A, B) is controllable.

2. The controllability matrix $\mathcal{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$ has full-row rank.

3. The eigenvalues of $A + BF$ can be freely assigned by a suitable choice of F.

4. The solution P to the Lyapunov equation

$$AP + PA^T + BB^T = 0$$

is positive definite (assuming A stable).

Model Order Reduction

Theorem: The following are equivalent

1. (C, A) is observable.
2. The observability matrix $\mathcal{O} = [C \quad CA \quad CA^2 \quad \ldots \quad CA^{n-1}]^T$ has full-column rank.
3. The eigenvalues of $A + LC$ can be freely assigned by a suitable choice of L.
4. The solution Q to the Lyapunov equation

$$A^T Q + QA + C^T C = 0$$

   is positive definite (assuming A stable).

Theorem: A state-space realization of $G(s)$ is minimal (i.e. A has the smallest possible dimension) if and only if (A, B) is controllable and (C, A) is observable.

Example: For a SISO system, if there is a zero-pole cancellation the corresponding state-space realization is not minimal and either (A, B) is not controllable or (C, A) is not observable.

Model Order Reduction

Compute a Minimal Realization

Lemma: Consider $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$. Suppose that there exists a symmetric matrix

$$P = P^T = \left[\begin{array}{cc} P_1 & 0 \\ 0 & 0 \end{array} \right]$$

with P_1 nonsingular such that: $\qquad AP + PA^T + BB^T = 0$.

Now partition the realization (A, B, C, D) compatibly with P as

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array} \right]$$

Then $G(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array} \right]$. Moreover, (A_{11}, B_1) is controllable if A_{11} is stable.

Model Order Reduction

Compute a Minimal Realization

Procedure:

1. Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ be a stable realization.

2. Compute the controllability Gramian $P \geq 0$ from $AP + PA^T + BB^T = 0$.

3. Diagonalize P to get $P = [U_1 \quad U_2] \left[\begin{array}{cc} P_1 & 0 \\ 0 & 0 \end{array} \right] [U_1 \quad U_2]^T$ with $P_1 > 0$ and $[U_1 \quad U_2]$ unitary.

4. Then $G(s) = \left[\begin{array}{c|c} U_1^T A U_1 & U_1^T B \\ \hline C U_1 & D \end{array} \right]$ is a controllable realization.

Idea: Assume that $P_1 = \text{diag}(P_{11}, P_{12})$ such that $\lambda_{\max}(P_{12}) \ll \lambda_{\min}(P_{11})$; then the weekly controllable states corresponding to P_{12} can be truncated without causing much error.

Problem: The controllability (or observability) Gramian alone cannot give an accurate indication of the dominance of the system states.

Laboratoire
d'Automatique

Model Order Reduction

Balanced realization: A minimal realization of $G(s)$ for which the controllability and observability Gramians are equal is referred to as a balanced realization.

Lemma: The eigenvalues of the product of the Gramians are invariant under state transformation.

Proof: Consider that the state is transformed by a nonsingular T to $\hat{x} = Tx$. Then $\hat{A} = TAT^{-1}$, $\hat{B} = TB$ and $\hat{C} = CT^{-1}$. Now using the Lyapunov equations we find that $\hat{P} = TPT^*$ and $\hat{Q} = (T^{-1})^* QT^{-1}$. Note that $\hat{P}\hat{Q} = TPQT^{-1}$.

Remark: A transformation matrix T can always be chosen such that $\hat{P} = \hat{Q} = \Sigma$ where $\Sigma = \text{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_N I_{s_N})$. T and Σ are solutions to the following equations:

$$
\begin{aligned}
TAT^{-1}\Sigma + \Sigma(T^{-1})^T A^* T^T + TBB^T T^T &= 0 \\
(T^{-1})^T A^T T^T \Sigma + \Sigma TAT^{-1} + (T^{-1})^T C^T CT^{-1} &= 0
\end{aligned}
$$

Model Order Reduction

Balanced Realization

Procedure: If (A, B, C, D) is a minimal realization with Gramians $P \succ 0$ and $Q \succ 0$, then find a matrix R such that $P = R^T R$ and diagonalize $RQR^T = U\Sigma^2 U^T$. Now, let $T^{-1} = R^T U \Sigma^{-1/2}$ then

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array} \right] \text{ is balanced.}$$

Matlab command: `[Ab,Bb,Cb,sig,Tinv]=balreal(A,B,C);`

Balanced Truncation

Main Idea: Suppose $\sigma_r \gg \sigma_{r+1}$ for some r. Then the balanced realization implies that those states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_N$ are less controllable and less observable than those states corresponding to $\sigma_1, \ldots, \sigma_r$ and can be truncated.

Model Order Reduction

Theorem: Suppose $G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \in \mathcal{RH}_\infty$ is a

balanced realization with Gramian $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2)$: ($\Sigma_1$ and Σ_2 have no diagonal entries in common)

$$\begin{aligned}
\Sigma_1 &= \mathrm{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \ldots, \sigma_r I_{s_r}) \\
\Sigma_2 &= \mathrm{diag}(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \ldots, \sigma_N I_{s_N})
\end{aligned}$$

where σ_i has multiplicity s_i for $i = 1, \ldots, N$ and are decreasingly ordered. Then the truncated system $G_r \equiv (A_{11}, B_1, C_1, D)$ is balanced and asymptotically stable. Furthermore,

$$\|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^N \sigma_i \quad \text{and} \quad \|G - G(\infty)\|_\infty \leq 2 \sum_{i=1}^N \sigma_i$$

Laboratoire
d'Automatique

Model Order Reduction

Remarks on Balanced Truncation

Residualization: The truncated model has not the same static gain as the original high-order model. The residualization technique considers the DC contributions of the truncated states to modify the system matrix (instead of truncating the states their derivatives are forced to zero so at steady-state the original and reduced-order model have the same gain).

Unstable systems: Unstable systems can be factored as $G(s) = G_{st}(s)G_{unst}(s)$, then only the order of the stable part will be reduced.

Frequency weighted balanced reduction: In this technique the stability of the reduced-order model cannot be guaranteed. Moreover, an upper bound for the modeling error cannot be derived.

MATLAB Commands:

```
[Gb,sig]=sysbal(G); % find a balanced realization Gb
Gr=strunc(Gb,r); % truncate to the r-th order
Gres=sresid(Gb,r); % Truncation by residualization
[WGb,sig] = sfrwtbal(Gb,w1,w2);
WGr=strunc(WGb,r);
```
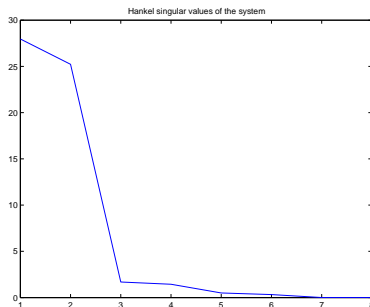
Model Order Reduction

Balanced Truncation

Example 1: Consider the mass/spring/damper example in H_∞ control. A minimal balanced realization of the augmented plant can be obtained by:

```
[Gb,sig] =sysbal(Gsys);
plot(sig)
```
Minimal realization is of 8th-order
(plant order+ filter orders)
The last two SVs are too small
so the corresponding states can be
truncated :
```
Gr=strunc(Gb,6)
```


Hankel singular values of the system

Example 2: Consider the transfer function between the control force F_1 and the filtered position z_1.
```
G41=pck(A,B(:,4),C(1,:),D(1,4)); [Gb,sig]=sysbal(G41);
plot(sig)
```

Model Order Reduction

Comparison of the frequency response of the original and reduced-order models

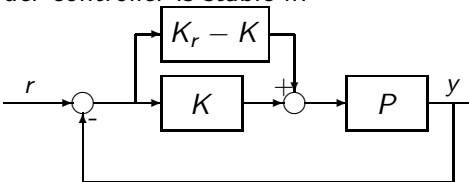The 3 last Hankel singular values are too small and can be truncated.
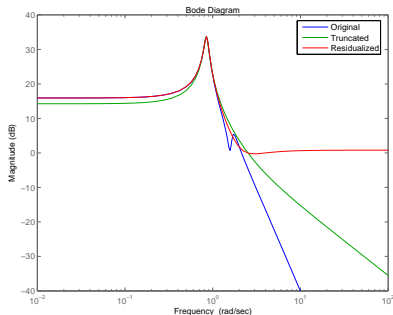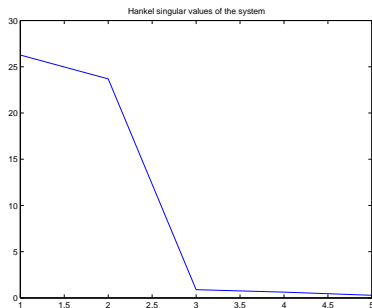
Modeling error: Note that $\sigma_4 + \sigma_5 + \sigma_6 = 1.797$

$$\|G - G_r\|_\infty = 1.71, \quad \|G - G_{res}\|_\infty = 1.99$$

Controller Order Reduction

In model order reduction the aim is to reduce $||G - G_r||_\infty$ whereas in controller order reduction the main issue is to preserve the stability and performance of the closed-loop system.

Stability: According to the small gain theorem, the closed-loop system with the reduced-order controller is stable if:

$$\left\| \frac{(K_r - K)P}{1 + KP} \right\|_\infty < 1$$

Performance: To preserve the closed-loop performance, the difference between the high-order and reduced-order closed-loop transfer function should be minimized.

$$\min \left\| \frac{1}{1 + KP} - \frac{1}{1 + K_r P} \right\|_\infty = \min \left\| \frac{(K - K_r)P}{(1 + KP)(1 + K_r P)} \right\|_\infty$$

$$\min \left\| \frac{KP}{1 + KP} - \frac{K_r P}{1 + K_r P} \right\|_\infty = \min \left\| \frac{(K - K_r)P}{(1 + KP)(1 + K_r P)} \right\|_\infty$$

Final project

1. Choose a plant with large uncertainty.

2. Design the set of models for robust control design (Unstructured: additive, multiplicative, etc. and/or Structured: multimodel, parametric uncertainty, etc.).

3. Define the closed-loop performance (iteratively improve the performance).

4. Design a robust controller with one of the methods explained in the course or a any other methods you propose.

5. Reduce the controller order if necessary.

6. Prepare a brief report containing the design steps and time-domain and frequency-domain results.

7. Present your project on July 16, 2007 (14h15-17h15). 30min presentation and 15min questions.