AUTOMATIC CONTROL AND SYSTEM THEORY

NON LINEAR SYSTEMS: ANALYSYS AND CONTROL

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Non linear systems

Analysis and design methods based on the assumption of linear and time-invariant systems are very common and allow to use efficient and relatively simple tools.

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & Ax(t)+Bu(t)\\ y(t) & = & Cx(t)\end{array}\right. \longrightarrow \quad G(S)$$

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & f(x,u,t)\\ y(t) & = & h(x,t)\end{array}\right.$$

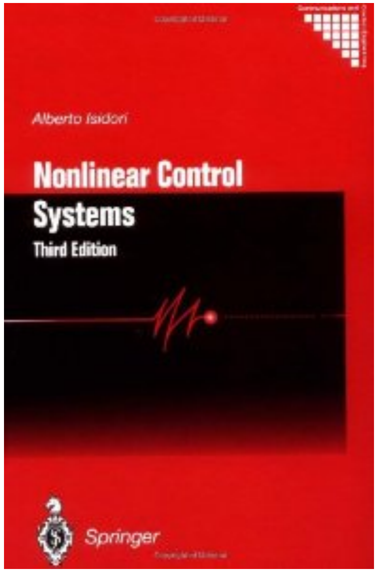
On the other hand, the linearity assumption is justified only if the considered signals and variables of the system vary in relatively small ranges. As a matter of fact, all physical systems are non linear, and behave approximately as linear only for “small signals”.

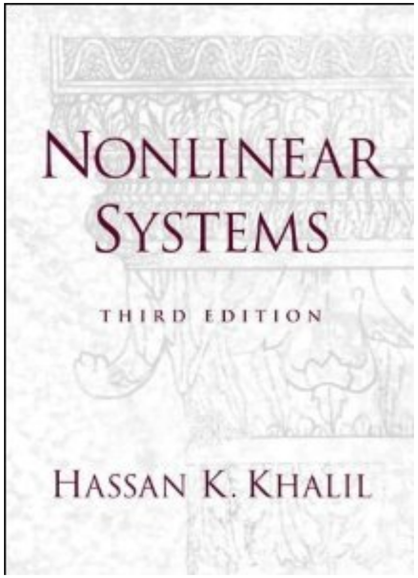
However, there are systems that do not show a linear behavior even for “small signals”. Note that a non linear behavior does not necessarily mean something to be avoided: there are systems in which non linearities are introduced on purpose in order to obtain specific behaviors.

Non linear systems

Many good textbook available in the literature:

- A. Isidori, “Nonlinear Control Systems”, Springer, New York 1995
- H.K. Khalil, “Non linear Systems”, Prentice-Hall, 1996





Non linear systems

Problems with non linear systems:

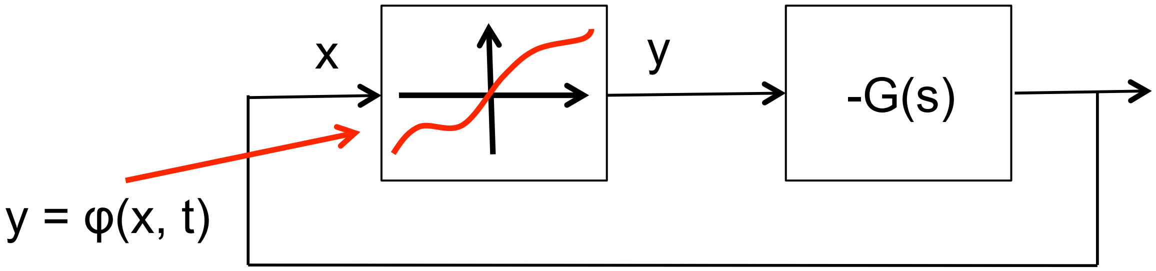
1) Analysis of the stability properties
   a. *Linearization (first Lyapunov method)*
   b. *Lyapunov (second Lyapunov method)*
   c. *Phase-plane plot (equilibrium points, limit cycles)*
   d. Circle criterion
   e. Popov criterion

2) Design of control systems
   a. Based on Lyapunov techniques
   b. Feedback linearization
   c. …

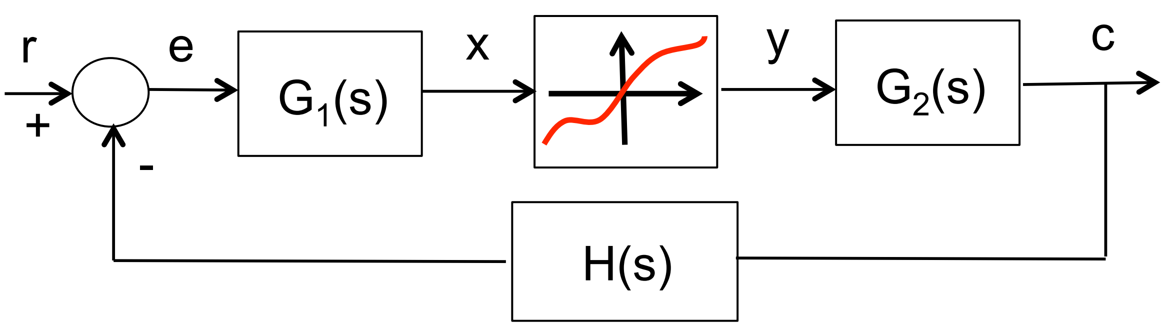
Circle Criterion – Popov Criterion

Goal: study the global asymptotic stability of a non linear system.

The “Circle criterion” and the “Popov criterion” give two ***sufficient conditions*** for the GAS of feedback autonomous, non linear systems in which a non linear algebraic part and a linear (stable) dynamics can be identified:



A scheme of this type is obtained from the more general scheme when r = cost.



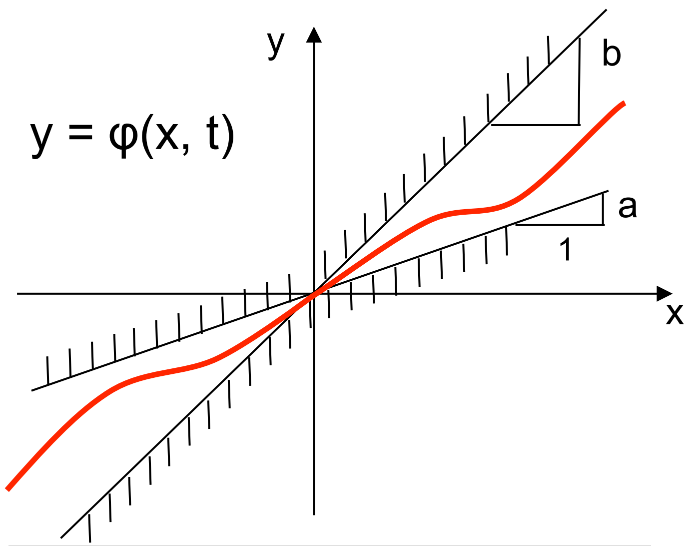
Circle Criterion

Sector condition

A non linear memoryless function φ(x, t) is said to satisfy a sector condition if some constants α, β, a, b (with b > a, α < 0 < β,) exist such that

$$ay^2 \leq y\ \varphi(y,t) \leq by^2, \qquad \forall t \geq 0,\ \forall y \in [\alpha, \beta]$$

If this condition hold ∀ y ∈(-∞, +∞), the sector condition holds globally and φ(x, t) belongs to the sector [a, b]



Circle Criterion

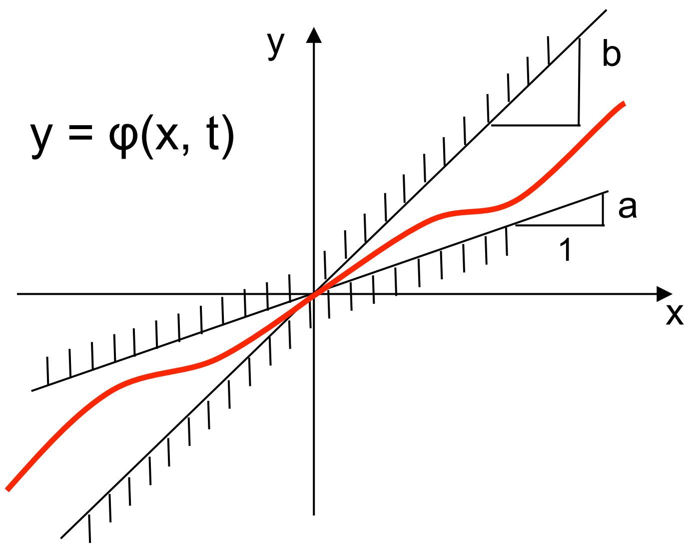
The circle criterion may be considered as an extension of the Nyquist criterion for stability of SISO linear systems.

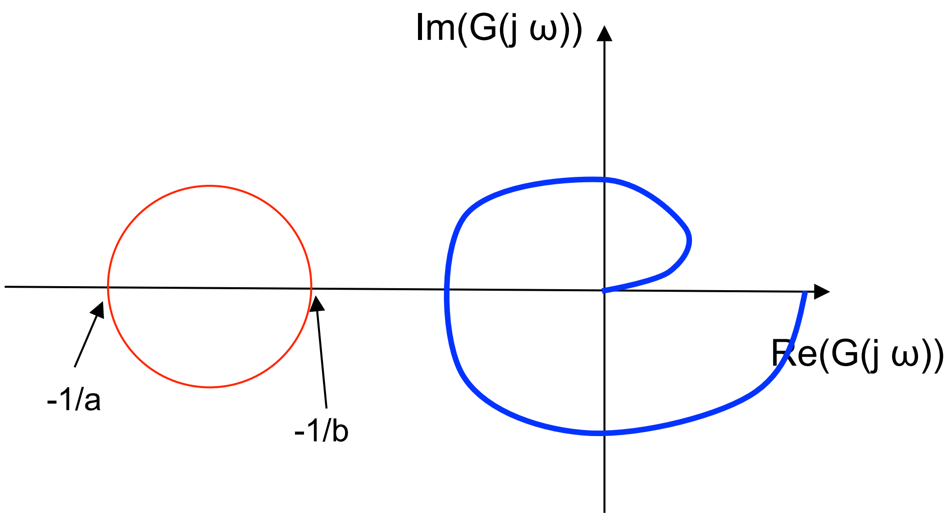
THEOREM Consider a linear system subject to non-linear feedback, i.e. a non linear element φ(x, t) is present in the feedback loop. Assume that φ(x, t) satisfies globally a sector condition [a, b].

Then the closed loop system is globally asymptotically stable if one of the following three conditions is satisfied:

1) If 0 < a < b, the Nyquist plot of G(j ω) does not penetrate the circle having as diameter the segment [-1/a,-1/b] (the *"critical circle" D(a, b)*) located on the x-axis and it surrounds the circle D(a,b) counterclockwise *m* times, being *m* the number of unstable poles of G(s)
2) If 0 = a < b, G(s) is stable and the Nyquist plot of G(j ω) is located on the right of the vertical line Re(s) = -1/b
3) If a < 0 < b, G(s) is stable and the Nyquist plot of G(j ω) is within the circle D(a,b)

Circle Criterion

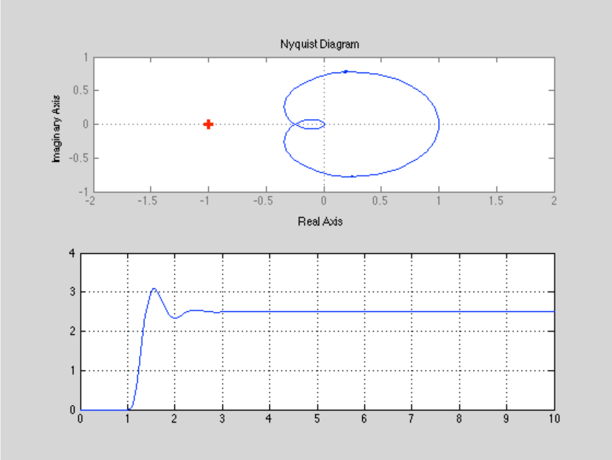




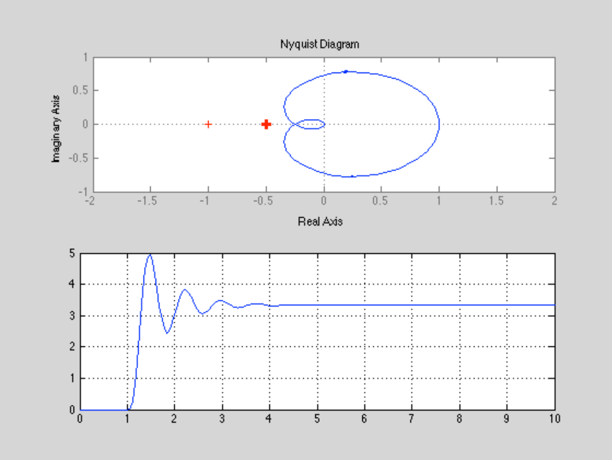
The circle criterion is very simple to be verified. It may be considered as an extension to the non linear case of the Nyquist criterion.

The Nyquist criterion is obtained in case a = b = m (linear case): the point -1/m (limit case of the circle when a = b = m) must not be encircled by the Nyquist plot.

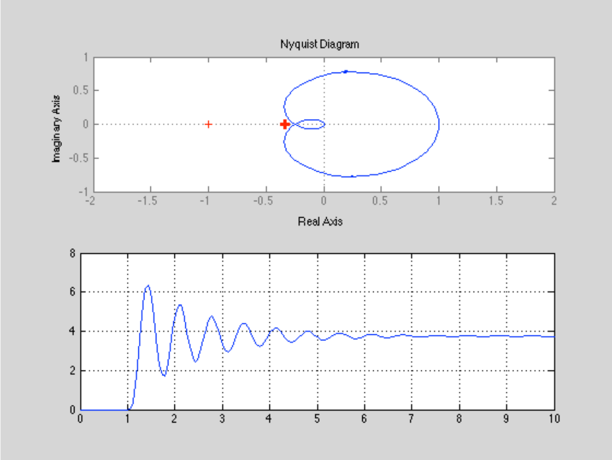
Circle Criterion - Example



m = 1

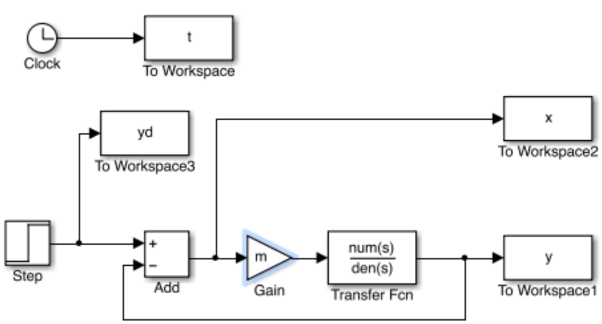


m = 2

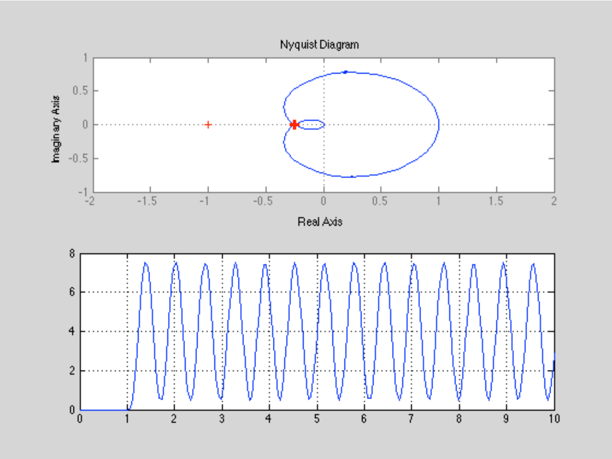


m = 3

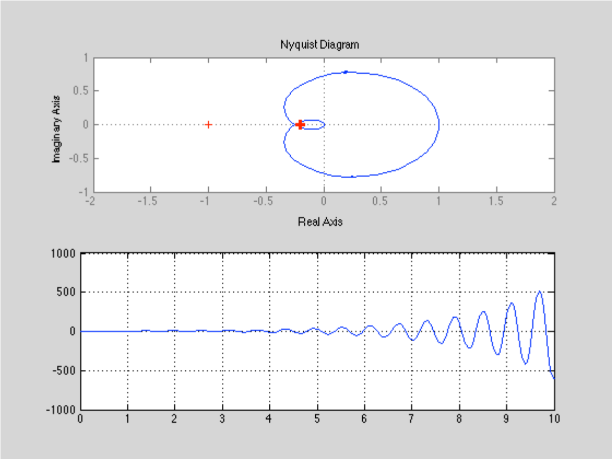
$$G(s) = \frac{10000}{(s+10)^4}$$



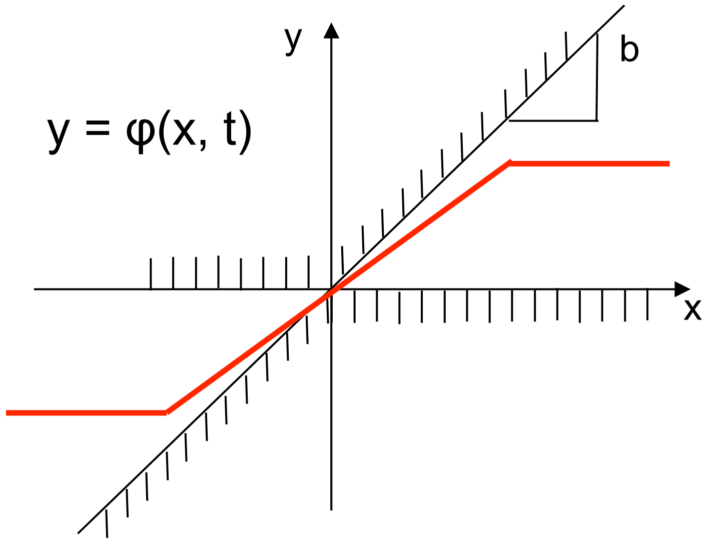
m = 4

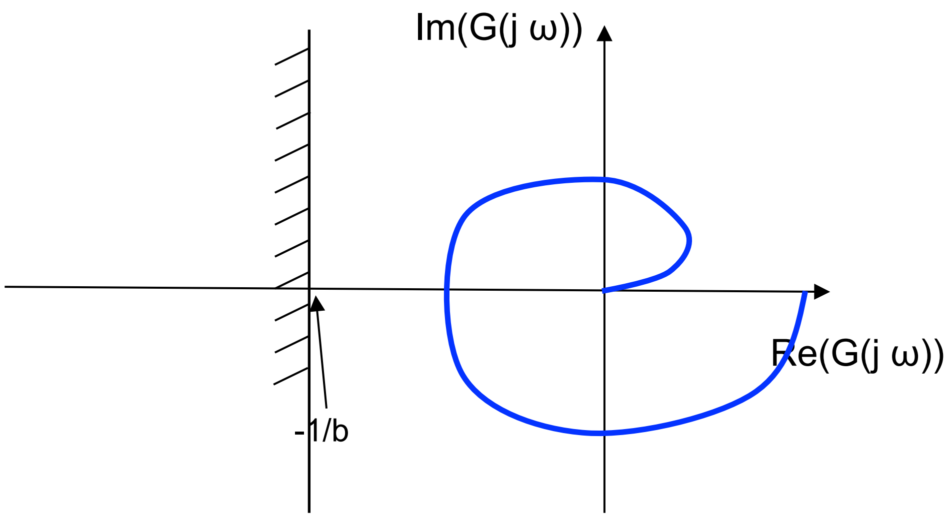


m = 5



Circle Criterion





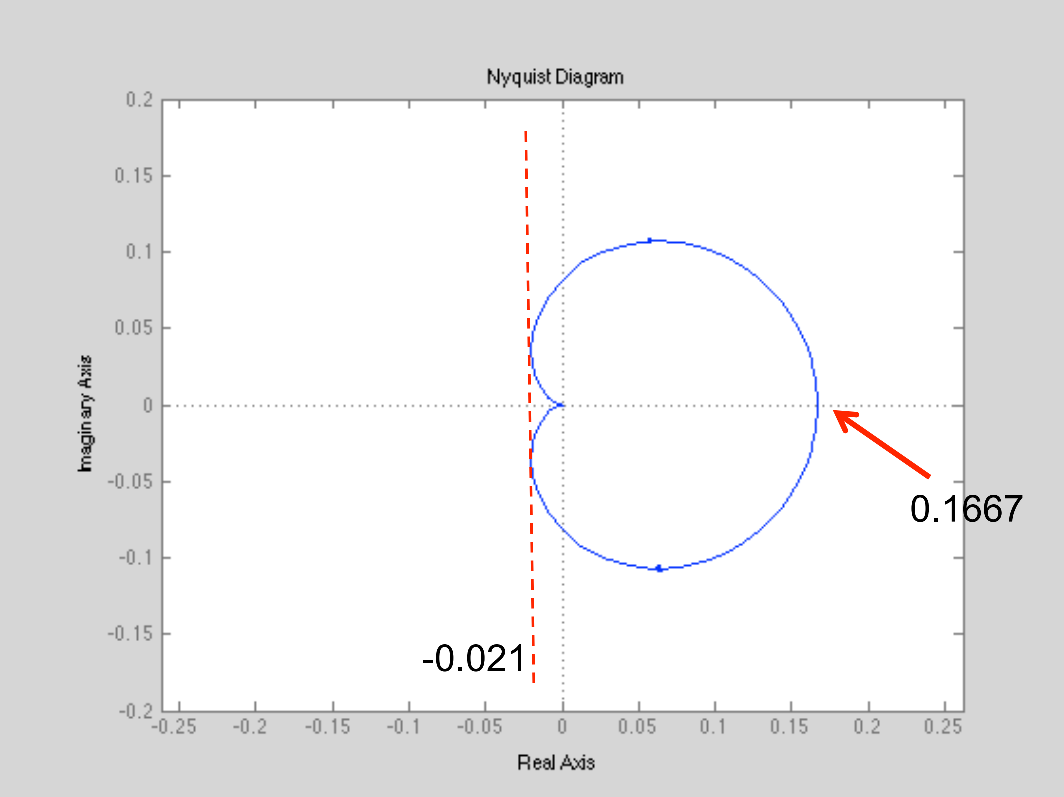
A quite frequent case is when a = 0 (sector [0, b]) (e.g. saturation).

In this case the circle degenerates to the half plane to the left of point -1/b.

Circle Criterion

Example 1. Study the GAS of the following system

$$G(s) = \frac{1}{(s+2)(s+3)}$$



Circle Criterion

Example 1. Study the GAS of the following system

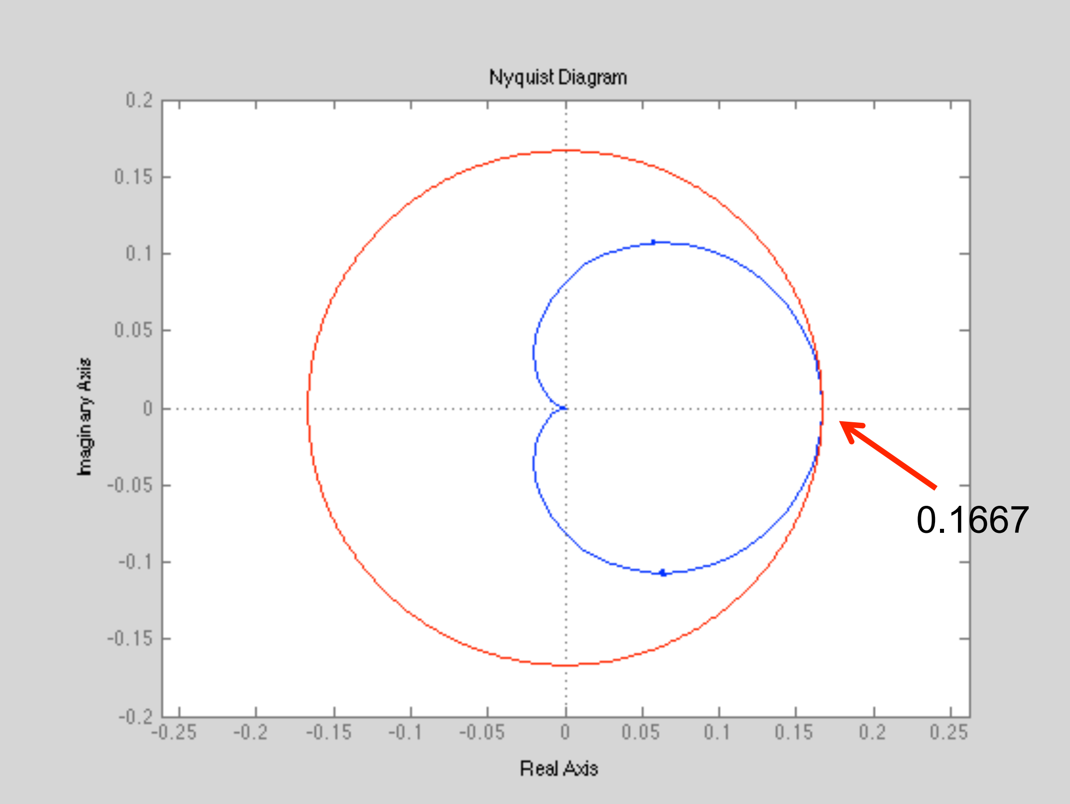
$$G(s) = \frac{1}{(s+2)(s+3)}$$

Since G(s) is stable, it is possible to consider a < 0 and apply the third case of the Theorem.

A circle D(a,b) encircling the Nyquist plot must be defined: several choices are possible!

a) If the center of D(a, b) is placed in the origin, then the circle D(-r, r) must be defined, being r the maximum value of |G(j ω)|.

From the plot, sup |G(j ω)| = 0.1667, then the system is GAS for all the nonlinearities in the sector [-1/0.1667, +1/0.1667] = [-6, 6].



Circle Criterion

Example 1. Study the GAS of the following system

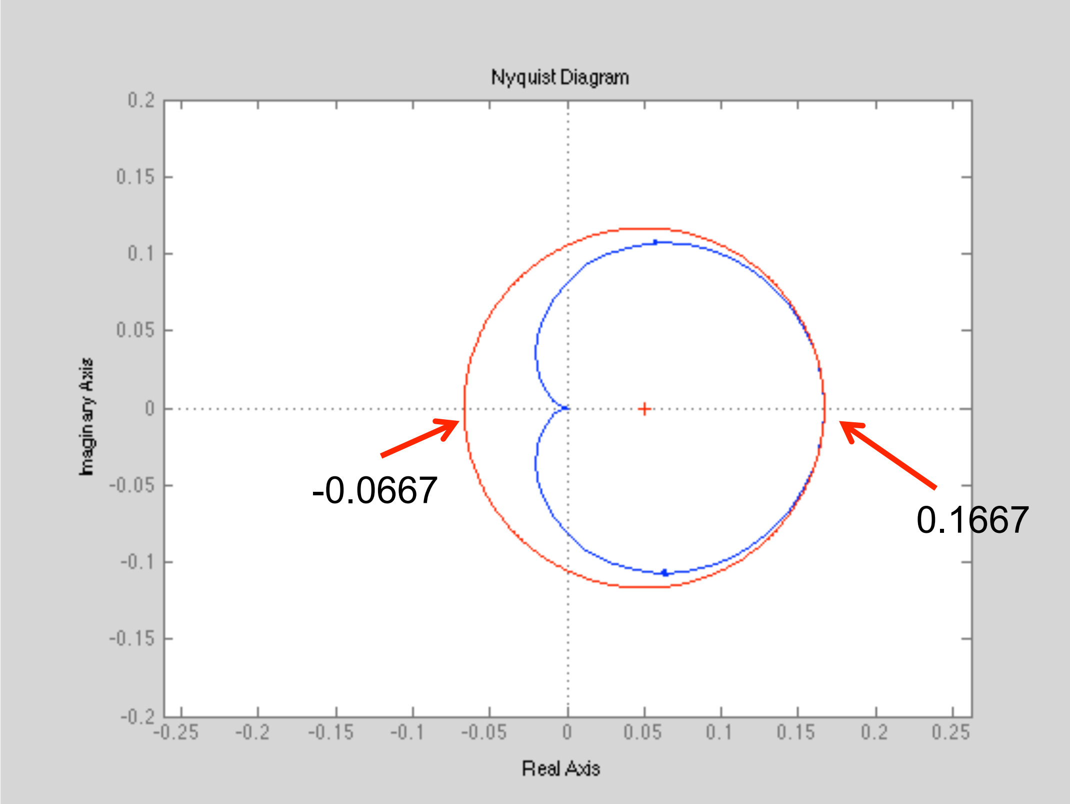
$$G(s) = \frac{1}{(s+2)(s+3)}$$

Since G(s) is stable, it is possible to consider a < 0 and apply the third case of the Theorem.

A circle D(a,b) encircling the Nyquist plot must be defined: several choices are possible!

b) If the center of D(a,b) is in (0.05, 0), then the maximum distance from this point to the Nyquist plot is 0.1167 (= r).

The intersections of the circle with the real axis are -0.0667 (= -(0.1167 - 0.05)) and 0.1167. The system is GAS for all the nonlinearities in the sector [-1/0.1167, 1/0.0667] = [-6, +15].



Circle Criterion

Example 1. Study the GAS of the following system

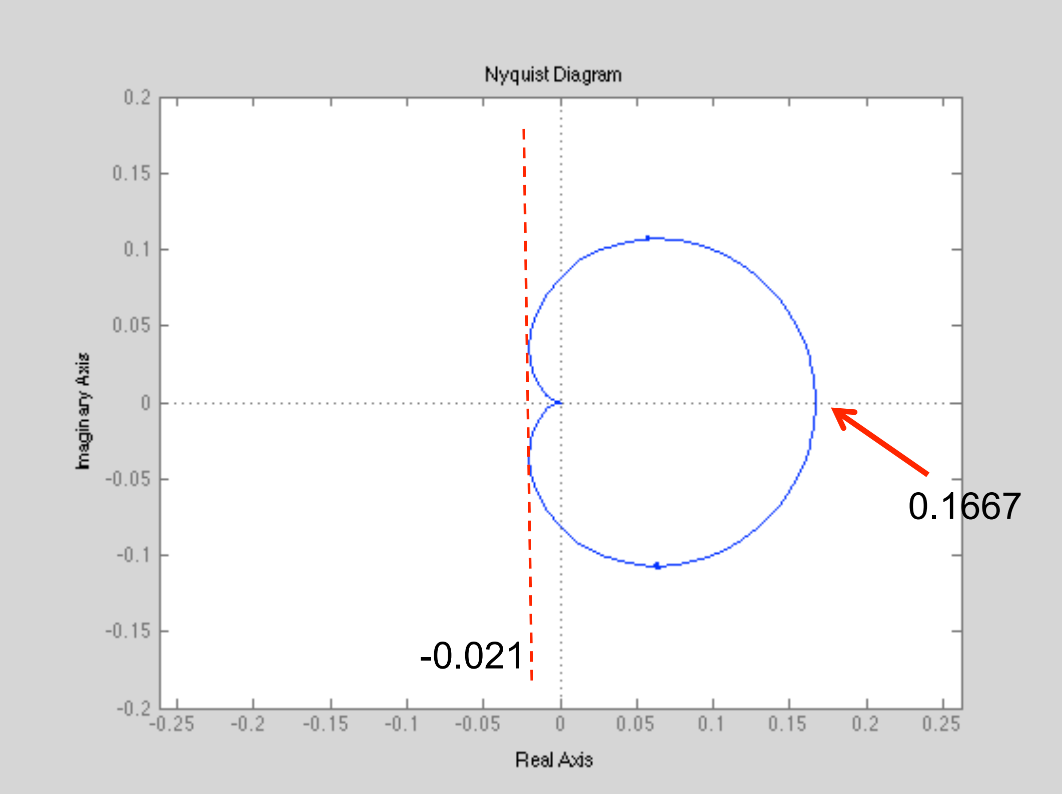
$$G(s) = \frac{1}{(s+2)(s+3)}$$

c) Let us assume a = 0 and apply the second case of the Theorem.

The Nyquist plot is on the left of the vertical line Re(s) = -0.021.

The system is GAS for all nonlinearities in the sector [0, 1/0.021] = [0, 47.619].

Note that this case gives the best estimation of b, but only nonlinearities in the first and third sectors are allowed!



Circle Criterion

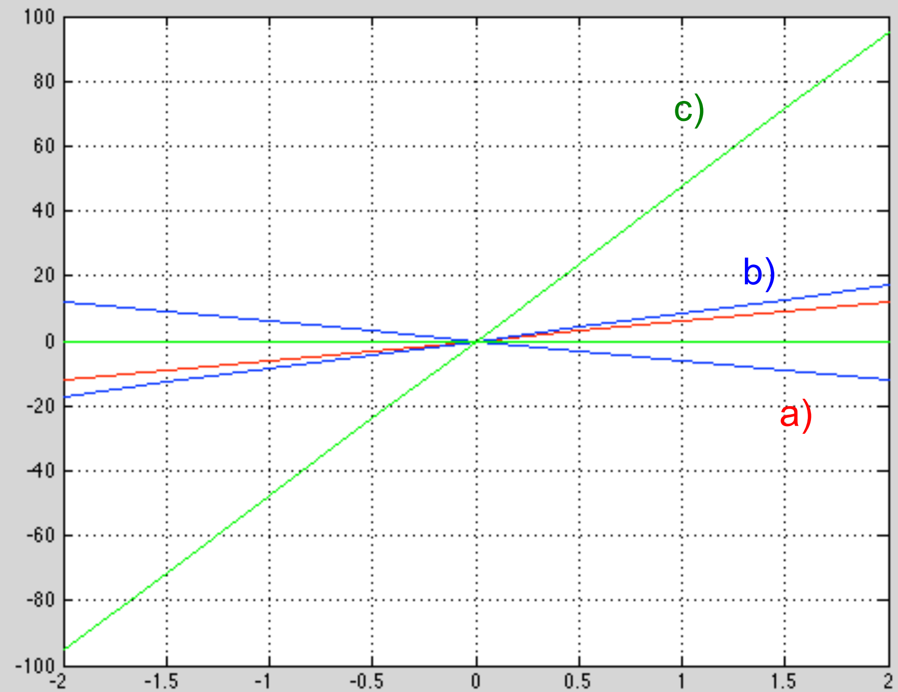
Example 1. Study the GAS of the following system

$$G(s) = \frac{1}{(s+2)(s+3)}$$

Three different stability bounds have been identified.

a) [-6, 6]
b) [-6, +15]
c) [0, 47.619]

Notice that from the previous analysis we can conclude that the system is GAS for nonlinearities satisfying the sector condition [-6, 47.619].



Circle Criterion

Example 2. Study the GAS of the following system

$$G(s) = \frac{s+1}{(s+2)^2(s-1)}$$

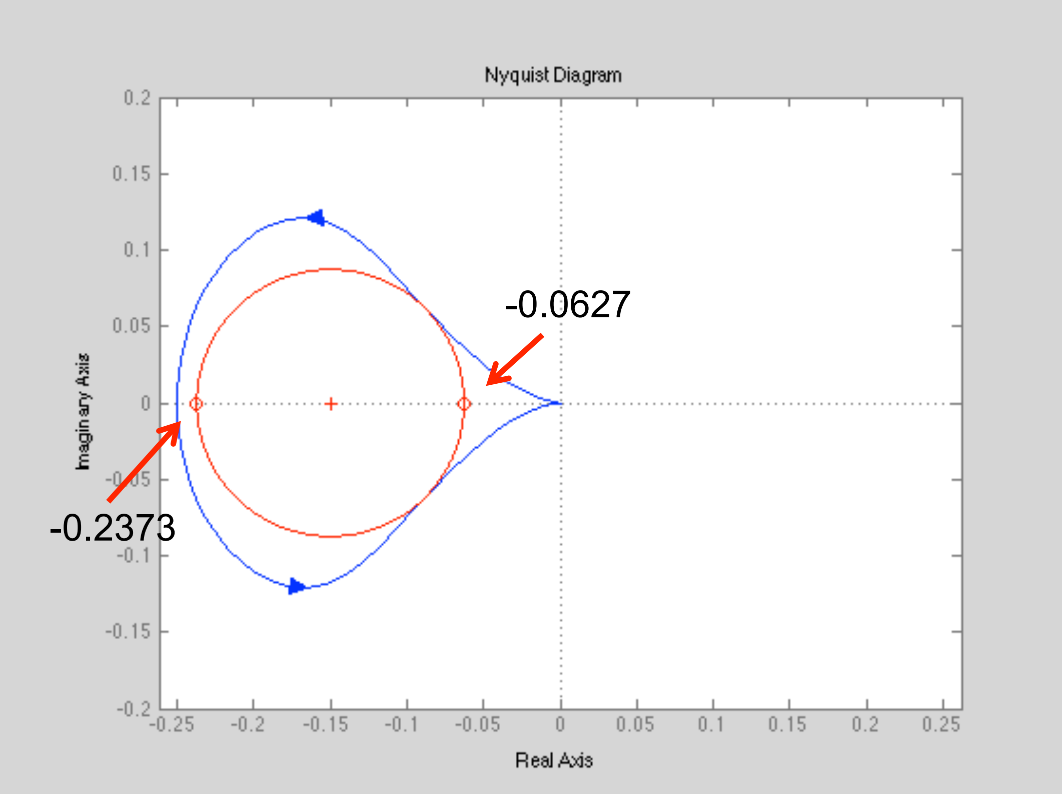
The system is non stable. Case 1) of the Theorem must then be considered (a > 0).

The Nyquist plot must encircle counterclockwise the circle D(a,b) (once).

Selecting the point c = [-0.15, 0] as center, the minimum distance between c and the Nyquist plot is 0.0873 (radius of the circle D(a,b)).

The points on the real axis are: -1/b = -0.2373, -1/a = -0.0627.

The system is GAS for all nonlinearities in the sector [1/0.2373, 1/0.0627] = [4.2141, 15.949].



Circle Criterion

Example 2. Study the GAS of the following system

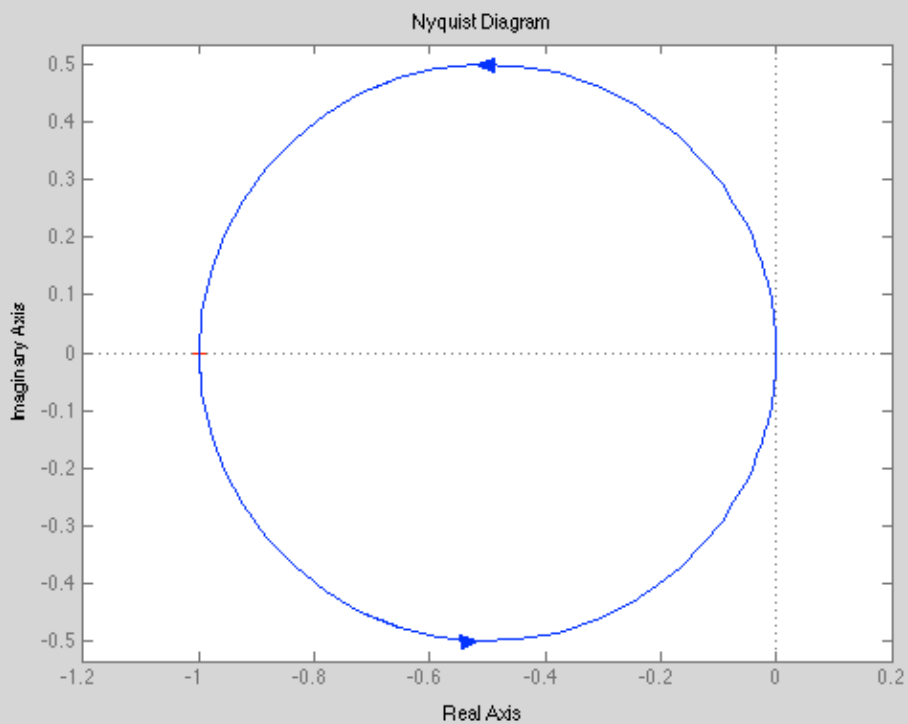
$$G(s) = \frac{s}{s^2 - s + 1} \qquad p_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j$$

The system is non stable: 2 poles with positive real part.

Case 1) of the Theorem must be considered (a > 0): the Nyquist plot should encircle twice the circle D(a,b).

Unfortunately, this is not possible!

A sector for which the system is GAS does not exist!



Popov Criterion

The Popov criterion may be applied when:

1) The non linear function φ(x) is defined in the sector [0, b]
2) The non linear function satisfies φ(0) = 0
3) The linear transfer function G(s) has deg(p(s)) < deg(q(s))

$$G(s) = \frac{1}{s^h} \frac{p(s)}{q(s)}$$

4) The poles of G(s) are in the left hand side plane or on the imaginary axis
5) The system is marginally stable in the singular case

Popov Criterion

THEOREM

The closed loop system is globally asymptotically stable if $\varphi(x) \in [0, b]$, $0<b<\infty$, and there exists a constant q such that the following equation is satisfied

$$Re[G(j\omega)] - qj\omega Im[G(j\omega)] > -\frac{1}{b}, \qquad \forall \omega \in [-\infty, \infty]$$

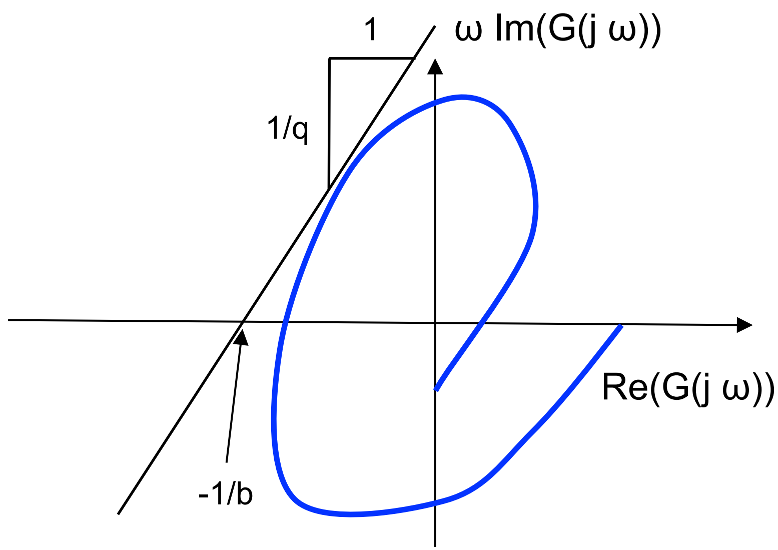
Graphical interpretation

Let us define the *Popov plot* P as

$$P(j\omega) = \{z = Re[G(j\omega)] + j\omega Im[G(j\omega)], \quad \omega > 0\}$$

Then, the closed loop system is GAS if P lies to the right of the line through the point (–1/b + j 0) with a slope 1/q.

Popov Criterion



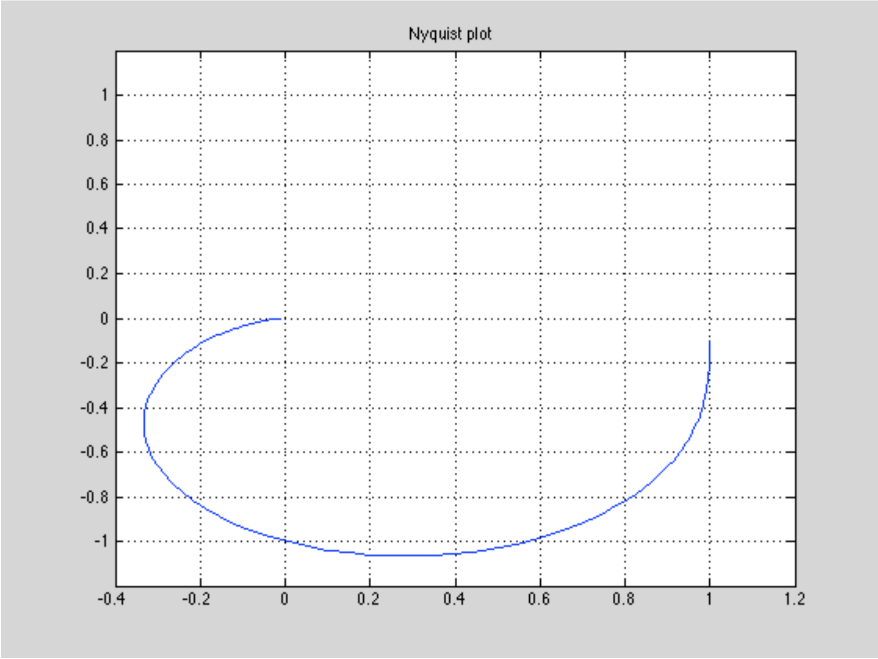
- The Popov plot is not the same as the Nyquist plot (increased “difficulty”)
- The Popov criterion is less conservative than the Circle criterion: both are sufficient conditions! (Popov may indicate GAS for systems for which the Circle criterion does not give indication)
- Popov applies only to time-invariant nonlinearities φ(x), while the Circle criterion can be applied also to time-variant functions φ(x, t).

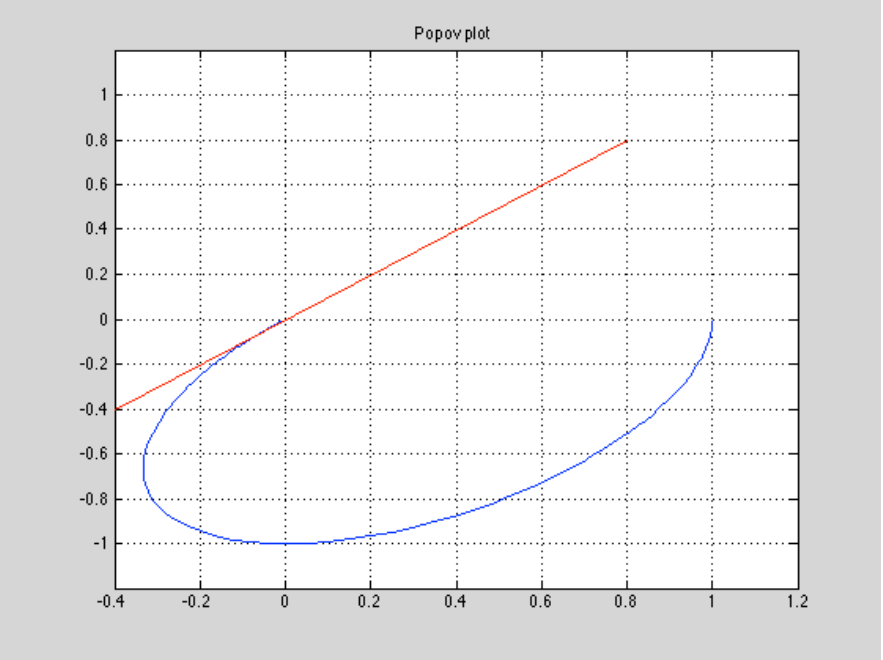
Popov Criterion

Example. Stability analysis of

$$G(s) = \frac{1}{s^2 + s + 1}$$

The line is through the origin (1/b = 0, b → ∞) with a unit slope (q = 1).
The stability sector is then [0, +∞)





Control of non linear systems

A wide and detailed literature is available for the design of controllers for non linear systems.

Two techniques only are analysed as (important) examples, taking as case study the control of a robot manipulator:

- Control based on Lyapunov techniques
- Feedback linearization

The (Euler-Lagrange) dynamic model of a robot manipulator has the general expression:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$$

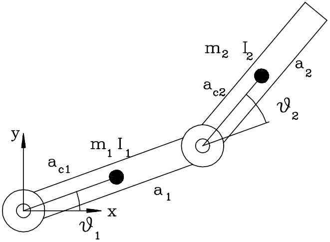
Where:

- $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \boldsymbol{\tau}$: joint pos., vel., acc. vectors, joint torques ($\in \mathbb{R}^n$)
- $\mathbf{M}(\mathbf{q}), \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}), \mathbf{D}, \mathbf{g}(\mathbf{q})$ Inertia matrix, Coriolis and centrifugal effects, Coulomb friction and gravity terms

Dynamic model of a 2 dof manipulator.

Dynamic model of a 2 dof manipulator. Consider:

- θ_i i-th joint variable;
- m_i i-th link mass ;
- $\tilde{I}_i$ i-th link inertia, about an axis through the CoM and parallel to **z**;
- a_i i-th link length;
- a_{Ci} distance between joint i and the CoM of the i-th link;
- τ_i torque on joint i;
- g gravity force along **y**;
- P_i, K_i potential and kinetic energy of the i-th link.



Dynamic model of a 2 dof manipulator.

From $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}$ we have

$$\begin{aligned} M_{11}\ddot{\theta}_1 + M_{12}\ddot{\theta}_2 + c_{121}\dot{\theta}_1\dot{\theta}_2 + c_{211}\dot{\theta}_2\dot{\theta}_1 + c_{221}\dot{\theta}_2^2 + g_1 &= \tau_1 \\ M_{21}\ddot{\theta}_1 + M_{22}\ddot{\theta}_2 + c_{112}\dot{\theta}_1^2 + g_2 &= \tau_2 \end{aligned}$$

or

$$\begin{aligned} [m_1 a_{C1}^2 + m_2(a_1^2 + a_{C2}^2 + 2a_1 a_{C2} C_2) + \tilde{I}_1 + \tilde{I}_2]\ddot{\theta}_1 + [m_2(a_{C2}^2 + a_1 a_{C2}^2 C_2) + \tilde{I}_2]\ddot{\theta}_2 & \\ -m_2 a_1 a_{C2} S_2 \dot{\theta}_2^2 - 2m_2 a_1 a_{C2} S_2 \dot{\theta}_1 \dot{\theta}_2 & \\ +(m_1 a_{C1} + m_2 a_1) g C_1 + m_2 g a_{C2} C_{12} &= \tau_1 \\ [m_2(a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{I}_2]\ddot{\theta}_1 + [m_2 a_{C2}^2 + \tilde{I}_2]\ddot{\theta}_2 & \\ +m_2 a_1 a_{C2} S_2 \dot{\theta}_1^2 & \\ +m_2 g a_{C2} C_{12} &= \tau_2 \end{aligned}$$

$S_i, C_i = \sin(\theta_i), \cos(\theta_i), \quad C_{ij} = \cos(\theta_i + \theta_j)$

Centralized control

Two important centralized control schemes are now introduced:

1. PD + gravity compensation
2. Inverse dynamics control

PD controller with gravity compensation

Given a desired reference configuration $\mathbf{q}_d$, the goal is to define a controller ensuring the global asymptotic stability of the nonlinear dynamical system (i.e the robot) described by:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}$$

For this purpose, let us define the error as

$$\tilde{\mathbf{q}} = \mathbf{q}_d - \mathbf{q}$$

and consider a dynamic system with state $\mathbf{x}$ given by

$$\mathbf{x} = \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\mathbf{q}} \end{bmatrix}$$

The *direct Lyapunov method* is exploited for the control law definition.

PD controller with gravity compensation

Let us consider the following candidate Lyapunov function

$$V(\mathbf{x}) = V(\tilde{\mathbf{q}}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_P \tilde{\mathbf{q}} > 0 \qquad \forall\ \dot{\mathbf{q}}, \tilde{\mathbf{q}} \neq \mathbf{0}$$

where $\mathbf{K}_P$ is a square $(n \times n)$ positive-definite matrix.

Function $V(\tilde{\mathbf{q}}, \dot{\mathbf{q}})$ is composed by two terms:

- $1/2\ \ \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$
  expressing the kinetic energy of the system;
- $1/2\ \ \tilde{\mathbf{q}}^T \mathbf{K}_P \tilde{\mathbf{q}}$
  that can be interpreted as elastic energy stored by springs with stiffness $\mathbf{K}_P$; these springs are a ‘physical interpretation’ of the position control loops.

PD controller with gravity compensation

The reference configuration $\mathbf{q}_d$ is constant. Then $\dot{\tilde{\mathbf{q}}} = -\dot{\mathbf{q}}$, and the time derivative of V is:

$$\dot{V} = \dot{\mathbf{q}}^T \mathbf{M}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^T \dot{\mathbf{M}}\dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{K}_P \tilde{\mathbf{q}}$$

Since the robot dynamics can be rewritten as $\mathbf{M}\ddot{\mathbf{q}} = \mathbf{u} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{D}\dot{\mathbf{q}} - \mathbf{g}$, then

$$\begin{aligned}
\dot{V} &= \dot{\mathbf{q}}^T \mathbf{M}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^T \dot{\mathbf{M}}\dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{K}_P \tilde{\mathbf{q}} \\
&= \dot{\mathbf{q}}^T (\mathbf{u} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{D}\dot{\mathbf{q}} - \mathbf{g}) + \frac{1}{2}\dot{\mathbf{q}}^T \dot{\mathbf{M}}\dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{K}_P \tilde{\mathbf{q}} \\
&= \frac{1}{2}\dot{\mathbf{q}}^T [\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})]\dot{\mathbf{q}} - \dot{\mathbf{q}}^T \mathbf{D}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T [\mathbf{u} - \mathbf{g}(\mathbf{q}) - \mathbf{K}_P \tilde{\mathbf{q}}]
\end{aligned}$$

In order to compute the control input $\mathbf{u}$, note that:

- $\dot{\mathbf{q}}^T [\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})]\dot{\mathbf{q}} = 0$ due to the structure of $\mathbf{C}$ (Christoffel symbols)
- $-\dot{\mathbf{q}}^T \mathbf{D}\dot{\mathbf{q}}$ is negative-definite

Thus, by setting

$$\mathbf{u} = \mathbf{g}(\mathbf{q}) + \mathbf{K}_P \tilde{\mathbf{q}}$$

it is possible to guarantee that $\dot{V}$ is negative-semidefinite. In fact:

$$\dot{V} = 0 \qquad \dot{\mathbf{q}} = 0, \qquad \forall \tilde{\mathbf{q}}$$

PD controller with gravity compensation

The same result can be achieved also by adding a second term to the control $\mathbf{u}$:

$$\mathbf{u} = \mathbf{g}(\mathbf{q}) + \mathbf{K}_P\tilde{\mathbf{q}} - \mathbf{K}_D\dot{\mathbf{q}}$$

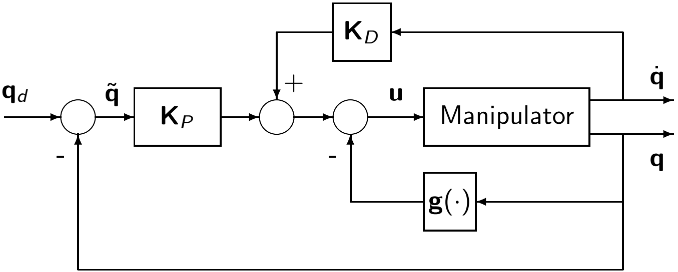
By defining $\mathbf{K}_D$ as a positive-definite matrix, it results

$$\dot{V} = -\dot{\mathbf{q}}^T(\mathbf{D} + \mathbf{K}_D)\dot{\mathbf{q}}$$

As a consequence, the convergence speed of the system to the equilibrium is increased.

Note that the terms $\mathbf{K}_D\dot{\mathbf{q}}$ is equivalent to a derivative action in the control loop (PD and gravity compensations).

PD controller with gravity compensation



Remarks:

- The control law is a linear PD controller with a nonlinear term (for gravity compensation). The system is globally asymptotically stable for any choice of $\mathbf{K}_P$, $\mathbf{K}_D$ (positive-definite);
- The derivative action is fundamental in systems with low friction effects **D**. Typical examples are manipulators equipped with Direct Drive motors: the low electrical dumping in this case is increased by the control action (derivative actions).

PD controller with gravity compensation

- The system evolves, and V decreases, as long as $\dot{\mathbf{q}} \neq 0$. Since $\dot{V}$ does not depend on $\mathbf{q}$ ($\dot{V} = -\dot{\mathbf{q}}^T(\mathbf{D} + \mathbf{K}_D)\dot{\mathbf{q}}$), it is not possible to guarantee that in steady state (when $\dot{\mathbf{q}} = 0$) also $\tilde{\mathbf{q}} = 0$.
- On the other hand, the steady state can be computed from the system equation

$$\underbrace{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q})}_{Robot\ \ dynamics} = \underbrace{\mathbf{g}(\mathbf{q}) + \mathbf{K}_P\tilde{\mathbf{q}} - \mathbf{K}_D\dot{\mathbf{q}}}_{PD+g(q)}$$

In fact, in steady state ($\dot{\mathbf{q}} = \ddot{\mathbf{q}} = \mathbf{0}$) it results

$$\mathbf{K}_P\tilde{\mathbf{q}} = \mathbf{0}$$

that is ($\mathbf{K}_P$ is positive definite):

$$\mathbf{q} = \mathbf{q}_d$$

- A perfect compensation of the gravity term $\mathbf{g}(\mathbf{q})$ is necessary, otherwise it is not possible to guarantee the stability of the system (*robust control* problem).

Inverse dynamics control

The manipulator is considered as a nonlinear MIMO system described by

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}$$

or, in short:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{u}$$

The goal is now to define a control input $\mathbf{u}$ such that the overall system can be regarded as a linear MIMO system.

This result (*global linearization*) can be achieved by using a nonlinear state feedback.
It can be shown that this is possible because:

- the model is linear in the control input $\mathbf{u}$;
- the matrix $\mathbf{M}(\mathbf{q})$ is invertible for any configuration of the manipulator.

Let us choose the control input $\mathbf{u}$ (based on the state feedback):

$$\mathbf{u} = \mathbf{M}(\mathbf{q})\mathbf{y} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

Inverse dynamics control

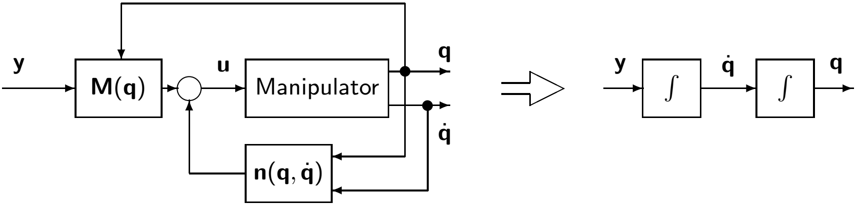
By using the control input $\mathbf{u}$ defined as

$$\mathbf{u} = \mathbf{M}(\mathbf{q})\mathbf{y} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})$$

it follows that

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{n} = \mathbf{M}\mathbf{y} + \mathbf{n} \quad \text{and thus (since } \mathbf{M} \text{ is invertible)} \quad \rightarrow \quad \ddot{\mathbf{q}} = \mathbf{y}$$

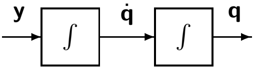
where $\mathbf{y}$ is the new input of the system.



Inverse dynamics control

This is called an inverse dynamics control scheme because the inverse dynamics of the manipulator must be calculated and compensated.

As long as y_i affects only q_i ($y_i = \ddot{q}_i$), the overall system is linear and decoupled with respect to $\mathbf{y}$.



Now, it is necessary to define a control law $\mathbf{y}$ that stabilizes the system. By choosing

$$\mathbf{y} = -\mathbf{K}_P\mathbf{q} - \mathbf{K}_D\dot{\mathbf{q}} + \mathbf{r}$$

from $\ddot{\mathbf{q}} = \mathbf{y}$ it follows

$$\ddot{\mathbf{q}} + \mathbf{K}_D\dot{\mathbf{q}} + \mathbf{K}_P\mathbf{q} = \mathbf{r}$$

that is asymptotically stable if the matrices $\mathbf{K}_P, \mathbf{K}_D$ are positive-definite.

Inverse dynamics control

If matrices $\mathbf{K}_P, \mathbf{K}_D$ are diagonal matrices defined by

$$\mathbf{K}_P = diag\{\omega_{ni}^2\} \qquad \mathbf{K}_D = diag\{2\delta_i\omega_{ni}\}$$

the dynamics of the i-th component is characterized by the natural frequency ω_{ni} and by the damping coefficient δ_i.

A predefined trajectory $\mathbf{q}_d(t)$ can be tracked by defining

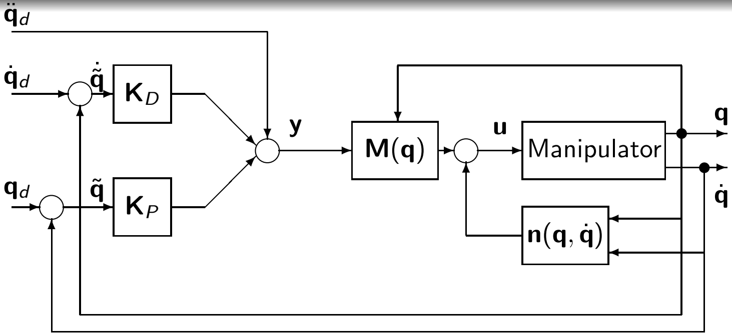
$$\mathbf{r} = \ddot{\mathbf{q}}_d + \mathbf{K}_D\dot{\mathbf{q}}_d + \mathbf{K}_P\mathbf{q}_d$$

Then, the dynamics of the tracking error is:

$$\ddot{\tilde{\mathbf{q}}} + \mathbf{K}_D\dot{\tilde{\mathbf{q}}} + \mathbf{K}_P\tilde{\mathbf{q}} = \mathbf{0}$$

The error is not null if and only if $\tilde{\mathbf{q}}(0) \neq \mathbf{0}, \dot{\tilde{\mathbf{q}}}(0) \neq \mathbf{0}$ and converges to zero with a dynamics defined by $\mathbf{K}_P, \mathbf{K}_D$.

Inverse dynamics control



Two control loops are present:

- the first loop is based on the nonlinear feedback of the state and it provides a linear and decoupled model between $\mathbf{y}$ and $\mathbf{q}$ (double integrator);
- the second loop is linear and is used to stabilize the whole system; the design of this outer loop is quite simple because it has to stabilize a linear system.

As the inverse dynamics controller is based on state feedback, all the terms in the manipulator dynamic model $(\mathbf{M}(\mathbf{q}), \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}), \mathbf{D}, \mathbf{g}(\mathbf{q}))$ must be known and computed in real-time.

Inverse dynamics control

This kind of controller has some implementation problems:

- it requires the *exact* knowledge of the manipulator model (including payload, non-modeled dynamics, mechanical and geometrical approximations, . . .);
- the real-time computation of all the dynamic terms involved in the control loop.

If, for computational reasons, only the principal terms are considered, then the control action cannot be precise due to the introduced approximations. It follows that control techniques able to compensate modeling errors are required:

- Robust control (sliding mode, ...)
- Adaptive control.

Inverse dynamics control - Example

Nonlinear system:

$$(2+\sin q)\ddot{q} + \dot{q}^3\sqrt{1-0.5\cos q} + \sqrt{1+q^2} = u$$

Desired trajectory: trapezoidal velocity profile. $k_p = 100, \quad k_d = 14$