

Designing Robust Controllers With Optimal H_∞ Performance Using Frequency-Domain Data

Achille Nicoletti and Alireza Karimi

Abstract—In this paper, a new frequency-domain method for designing robust controllers is proposed that provides optimal \mathcal{H}_∞ performance. A nonlinear optimization problem is formulated which guarantees the stability of the closed-loop system whilst ensuring robust performance. It is shown that the optimal solution to the \mathcal{H}_∞ problem is attained for a fixed-structure controller without any conservatism. A convexified version of the \mathcal{H}_∞ criterion has been established in many works and have utilized a similar frequency gridding process; therefore, it is of interest to determine the definiteness of the solution produced by this convex optimization method with respect to the solution of the optimal \mathcal{H}_∞ problem proposed in this work.

I. INTRODUCTION

The uncertainties associated with many of today's complex systems pose challenging tasks for engineers and researchers within the control systems community. These uncertainties can cause performance degradations and stability problems that may lead to catastrophic events. Robust controller design methods seek to rectify this issue by designing controllers that ensure proper stability margins whilst maintaining suitable performance.

The robust controller design methodology can be realized in two manners: using time-domain or frequency-domain data. In this paper, the frequency-domain approach will be utilized for the controller design scheme. In a data-driven setting, this frequency domain approach can be used to avoid unmodeled dynamics associated with parametric models. A survey on the differences associated with model-based control and data-driven control has been addressed in [1] and [2]. The authors in [3] use the frequency response data of a stable system to design an optimal controller through a symmetric root locus technique. A robust frequency-domain control design method has been established in [4] that requires a solution to a non-linear optimization problem. A frequency-domain loop-shaping approach to design fixed-structure controllers is presented in [5]. However, in this method, the closed-loop stability is not guaranteed and should be verified a posteriori.

Robust controller design methods belonging to the \mathcal{H}_∞ control framework minimizes the \mathcal{H}_∞ norm of a weighted closed-loop sensitivity function. Solving the \mathcal{H}_∞ problem can be accomplished in a variety of manners. A convex approximation of the \mathcal{H}_∞ criterion has been discussed in [6] and [7] where convex constraints are imposed on the

Nyquist diagram. The problem is convexified by constructing a linearly parameterized controller and devising linear constraints with respect to a desired open-loop transfer function. A frequency-domain approach for computing low-order multivariable linearly parameterized controllers is presented in [8]. The \mathcal{H}_∞ constraints are convexified around an initial stabilizing controller. An iterative algorithm is used that converges to a local optimal solution of the non-convex problem. However, in [9], a convex-concave approximation of the \mathcal{H}_∞ constraint is used which is based on an initial stabilizing controller. The extension of this method to design multivariable PID controllers for stable systems is presented in [10]. This method uses the same convex approximation as in [8] but is limited to stable systems and PID structure. More recent works that implement an iterative loop-shaping method that ensures \mathcal{H}_∞ performance have been devised in [11] and [12]. These methods use the concepts from [10] to linearize non-convex constraints around an operating point.

In some of the works asserted above, it has been proved that a convex approximation of the \mathcal{H}_∞ problem yields favorable results; however, these approximations do not yield the true global optimal solution to the \mathcal{H}_∞ problem. This paper presents an extension of the work in [13], and its purpose is to devise a new frequency-domain approach for obtaining the optimal solution to the \mathcal{H}_∞ problem. In [13], it was shown that the optimal solution is obtained when the order of the controller increases to infinity; in the proposed method, the optimal solution can be obtained for a fixed low-order fully-parameterized controller (which has desirable practical implications). A nonlinear optimization problem is formulated to guarantee stability of the closed-loop system while ensuring robust performance (without any approximation). Additionally, it will be shown that the proposed method produces a solution that converges to the global optimal solution of the \mathcal{H}_∞ problem under certain conditions. A preset frequency grid will be used to satisfy the robust performance condition and solve for a finite number of constraints. It is desired to compare the solutions produced from the convex optimization algorithms in [13] with the proposed algorithm and correlate the results.

This paper is organized as follows: In section II, the class of models and controllers are defined. Section III will address the control objectives and the conditions required for robust performance; an optimization problem will be formulated to obtain a solution to the \mathcal{H}_∞ problem. Section IV will demonstrate the effectiveness of the proposed method and establish the efficacy for the convex approximation of the \mathcal{H}_∞ criterion. Finally the concluding remarks are given in

The authors A. Nicoletti and A. Karimi are with the Automatic Control Laboratory at Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland. Corresponding author: Alireza Karimi: alireza.karimi@epfl.ch

Section V.

Notation: In order to avoid the risk of any confusion, the notation for the symbols employed in this paper will be defined here. Bolded symbols will represent vectors.

\mathbb{R} : the set of all real numbers.

\mathbb{R}_+ : the set of all real numbers greater than zero.

$\Re\{\cdot\}$: the real part of a complex variable.

$\Im\{\cdot\}$: the imaginary part of a complex variable.

\mathbf{v} : column vector \mathbf{v} with elements $[v_1, \dots, v_n]^\top$.

s : complex frequency variable used to represent continuous-time systems.

z : complex frequency variable used to represent discrete-time systems.

II. PRELIMINARIES

A. Class of Models

The set of all linear time-invariant (LTI) single-input-signal-output (SISO) strictly proper models belonging to the family of perturbed plants with multiplicative uncertainty can be defined as follows:

$$\mathcal{G} = \{G_p(s)[1 + \Delta_p(s)W_{2p}(s)]; \quad p = 1, \dots, q\} \quad (1)$$

where $G_p(s)$ is the nominal model of the process, $W_{2p}(s)$ is an uncertainty weight with bounded infinity norm, and $\Delta_p(s)$ is an unknown stable transfer function satisfying $\|\Delta_p\|_\infty < 1$. For simplicity, one model from the set \mathcal{G} will be analyzed, and the subscript p will be omitted.

Let the set \mathcal{P} represent the family of all stable, proper, real-rational transfer functions. It is imperative to note that \mathcal{P} is closed under multiplication and addition; in other words, if $P_1(s), P_2(s) \in \mathcal{P}$, then

$$\{P_1(s) + P_2(s), P_1(s)P_2(s)\} \in \mathcal{P} \quad (2)$$

This definition will also apply to both continuous-time and discrete-time systems. Suppose that a SISO feedback control system structure is used where the plant is represented as $G(s) = N(s)M^{-1}(s)$ such that $\{N(s), M(s)\} \in \mathcal{P}$. As asserted in [14] and [15], if $\{N(s), M(s)\} \in \mathcal{P}$, then $G(s) = N(s)M^{-1}(s)$ is called a *coprime factorization* of $G(s)$ over \mathcal{P} . The controller $K(s) = X(s)Y^{-1}(s)$ can also be described in coprime form, where $\{X(s), Y(s)\} \in \mathcal{P}$.

B. Class of Controllers

The controllers $X(s)$ and $Y(s)$ for this control scheme will be parameterized in the decision vector $\boldsymbol{\rho}$, and can be expressed as

$$\begin{aligned} X(\boldsymbol{\rho}, s) &= \boldsymbol{\rho}_x^\top \boldsymbol{\phi}_x(\boldsymbol{\rho}, s) \\ Y(\boldsymbol{\rho}, s) &= \boldsymbol{\rho}_y^\top \boldsymbol{\phi}_y(\boldsymbol{\rho}, s) \end{aligned} \quad (3)$$

where $\boldsymbol{\rho}_x \in \mathbb{R}^{n_x}$, $\boldsymbol{\rho}_y \in \mathbb{R}^{n_y}$, and $\{\boldsymbol{\phi}_x(\boldsymbol{\rho}, s), \boldsymbol{\phi}_y(\boldsymbol{\rho}, s)\}$ are vectors of stable transfer functions chosen from a set of orthogonal basis functions. An example of such a function is the Laguerre basis function [16]:

$$\phi_{\{x,y\}1}(s) = 1, \quad \phi_{\{x,y\}q}(\boldsymbol{\rho}, s) = \frac{\sqrt{2\rho_c}(s - \rho_c)^{q-2}}{(s + \rho_c)^{q-1}} \quad (4)$$

for $q = 2, \dots, \{n_x, n_y\}$, where $\rho_c \in \mathbb{R}_+$. For discrete-time systems, the Laguerre basis function can be expressed as follows:

$$\phi_{\{x,y\}1}(z) = 1, \quad \phi_{\{x,y\}q}(\boldsymbol{\rho}, z) = \frac{\rho_d^*(1 - \rho_d z)^{q-2}}{(z - \rho_d)^{q-1}} \quad (5)$$

where $\rho_d^* = \sqrt{1 - \rho_d^2}$ and $\{\rho_d \in \mathbb{R} : -1 < \rho_d < 1\}$. Note that when $\rho_d = 0$, the discrete-time Laguerre basis function becomes a simple finite-impulse response (FIR) filter.

A PID controller can also be represented in coprime form. Suppose that the desired controller structure is given as:

$$K(\boldsymbol{\rho}, s) = \rho_1 + \rho_2 \frac{1}{s} + \rho_3 \frac{s}{T_f s + 1} \quad (6)$$

Then the coprime controllers can be expressed as follows:

$$\begin{aligned} X(\boldsymbol{\rho}, s) &= \frac{(\rho_1 T_f + \rho_3)s^2 + (\rho_1 + \rho_2 T_f)s + \rho_2}{(s + \rho'_c)^2} \\ Y(\boldsymbol{\rho}, s) &= \frac{s(T_f s + 1)}{(s + \rho'_c)^2} \end{aligned} \quad (7)$$

where $\rho'_c \in \mathbb{R}_+$. With the PID, note that if ρ'_c is fixed, then the coprime X is linearly parameterized while Y is a TF independent of the decision variables. This will always be the case when the controller K is linearly parameterized.

III. PROBLEM FORMULATION

The methods described in this paper are associated with minimizing the \mathcal{H}_∞ norm of a desired weighted sensitivity function. The sensitivity function $S(s)$ and complementary sensitivity function $T(s)$ associated with this control scheme can be formulated as follows:

$$\begin{aligned} S(s) &:= e/r = [1 + L(\boldsymbol{\rho}, s)]^{-1} \\ &= M(s)Y(\boldsymbol{\rho}, s)[N(s)X(\boldsymbol{\rho}, s) + M(s)Y(\boldsymbol{\rho}, s)]^{-1} \end{aligned} \quad (8)$$

$$\begin{aligned} T(s) &:= y/r = L(\boldsymbol{\rho}, s)[1 + L(\boldsymbol{\rho}, s)]^{-1} \\ &= N(s)X(\boldsymbol{\rho}, s)[N(s)X(\boldsymbol{\rho}, s) + M(s)Y(\boldsymbol{\rho}, s)]^{-1} \end{aligned} \quad (9)$$

where $L(\boldsymbol{\rho}, s) = G(s)K(\boldsymbol{\rho}, s)$; r is the reference input; y is the system output; e is the tracking error signal (i.e., $e = r - y$). From the above definitions of the sensitivity functions, it can be observed that $S(s) + T(s) = 1$.

A. Robust Performance via Convex Optimization

Consider a process from the multiplicative uncertainty set in (1). Given a performance weighting filter $W_1(s)$ with bounded infinity norm, a necessary and sufficient condition for achieving robust performance is given by [15]:

$$\|W_1 S\| + \|W_2 T\|_\infty < \gamma \quad (10)$$

where $\gamma = 1$. However, the problem of minimizing the upper bound γ will be considered in this paper, where $\gamma \in \mathbb{R}_+$. The condition in (10) can also be expressed as

$$|W_1(j\omega)S(\boldsymbol{\rho}, j\omega)| + |W_2(j\omega)T(\boldsymbol{\rho}, j\omega)| < \gamma, \quad \forall \omega \in \Omega \quad (11)$$

where $\Omega \in [0, \infty)$. For notation purposes, the dependence in $j\omega$ will be omitted and will only be reiterated when

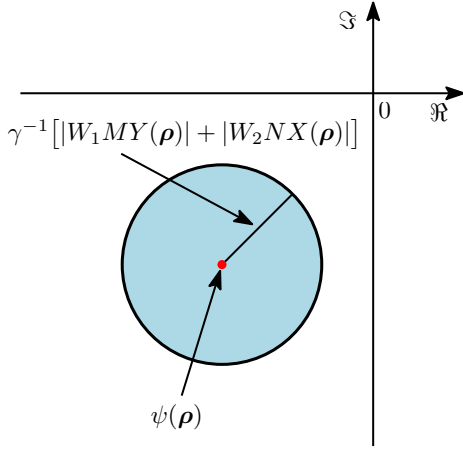


Fig. 1. Graphical interpretation of the \mathcal{H}_∞ robust performance constraint in the complex plane.

deemed necessary. The dependence on ρ will continue to be highlighted. By substituting (8) and (9) into (11), the condition for robust performance can be expressed as

$$\gamma^{-1} [|W_1MY(\rho)| + |W_2NX(\rho)|] < |\psi(\rho)|, \quad \forall \omega \in \Omega \quad (12)$$

where $\psi(\rho) = NX(\rho) + MY(\rho)$. Consider a circle in the complex plane at a specific frequency in Ω which is centered at $\psi(\rho)$ and has radius $\gamma^{-1} [|W_1MY(\rho)| + |W_2NX(\rho)|]$ (as shown in Fig. 1). The constraint in (12) ensures that for any frequency point in Ω , the circle associated with this frequency point will not encircle the origin. In [13], the authors show that there exists a function $F(s)$ that can rotate this circle such that it lies on the right-hand side of the $j\omega$ axis of the complex plane (i.e., all values on and within the circle have positive real parts). This condition is recalled with the following Lemma:

Lemma 1. Suppose that

$$\begin{aligned} H_1(\rho) &= W_1MY(\rho)\psi^{-1}(\rho) \\ H_2(\rho) &= W_2NX(\rho)\psi^{-1}(\rho) \end{aligned}$$

are frequency responses of bounded analytic functions in the right-half plane. Then, the following constraint is met

$$\sup_{\omega \in \Omega} (|H_1(\rho)| + |H_2(\rho)|) < \gamma \quad (13)$$

if and only if there exists a stable transfer function $F(s)$ that satisfies

$$\Re\{\psi(\rho)F\} > \gamma^{-1} [|W_1MY(\rho)F| + |W_2NX(\rho)F|]$$

for all $\omega \in \Omega$

Proof: The proof of a similar condition has been formally documented in [13]. Here, the condition has been extended for attaining robust performance instead of nominal performance. ■

With the above Lemma, a necessary and sufficient condition can be derived for attaining robust performance. In [13], it is shown that if $X(\rho)$ and $Y(\rho)$ are linearly parameterized,

then a convex optimization problem can be formulated as follows:

$$\begin{aligned} &\text{minimize}_{\rho} \quad \gamma \\ &\text{subject to:} \quad \gamma^{-1} [|W_1MY(\rho)| + |W_2NX(\rho)|] < \Re\{\psi(\rho)\} \\ &\quad \quad \quad \forall \omega \in \Omega \end{aligned} \quad (14)$$

Notice that (14) is a semi-infinite programming (SIP) optimization problem since there are a finite number of optimization variables and an infinite number of constraints. This problem can be solved by presetting a frequency grid ω and solving a finite number of constraints. This frequency grid can be predefined in a variety of manners (see [17], [18], and [19]).

B. Optimal \mathcal{H}_∞ Performance

In [13], it is shown that when the orders of $X(\rho)$ and $Y(\rho)$ approach infinity, then γ from (14) approaches the optimal solution to the \mathcal{H}_∞ problem. However, it is impractical and sometimes impossible to implement large order controllers in a real system.

One solution to this problem is to find the optimal solution for a fixed low-order controller. According to Lemma 1, it is known that there exists a stable transfer function F such that the constraint to the \mathcal{H}_∞ problem is satisfied. Therefore, the optimal solution to this problem can be accomplished by fixing the orders of $X(\rho)$ and $Y(\rho)$ and implementing the following optimization problem:

$$\begin{aligned} &\text{minimize}_{\rho} \quad \gamma \\ &\text{subject to:} \quad \gamma^{-1} |F(\rho)| [|W_1MY(\rho)| + |W_2NX(\rho)|] \\ &\quad \quad \quad < \Re\{F(\rho)\psi(\rho)\} \\ &\quad \quad \quad \forall \omega \in \Omega \end{aligned} \quad (15)$$

where $F(\rho)$ can be chosen such that it incorporates the Laguerre basis functions defined in (4), i.e.,

$$F(\rho) = \rho_{f_1} + \sum_{p=2}^q \rho_{f_p} \frac{\sqrt{2\rho_o}(s - \rho_o)^{p-2}}{(s + \rho_o)^{p-1}} \quad (16)$$

where $\rho_o \in \mathbb{R}_+$. It should be noted that the optimal solution to the \mathcal{H}_∞ problem is achieved when $K(\rho) = X(\rho)Y^{-1}(\rho)$ has the same structure as the optimal controller. However, if the structure of the optimal controller is unknown a priori, then one can simply fix the order of the basis $F(\rho)$ and vary the orders of the controllers $X(\rho)$ and $Y(\rho)$ until γ converges to a constant value. The concept of this method will be shown in one of the examples in section IV.

It will now be shown that the solution to the optimization problem in (15) converges monotonically to the optimal solution of the \mathcal{H}_∞ problem as the order of $F(\rho)$ increases. The proof will omit the dependence on ρ since the result is independent on the parameterization of the basis functions.

Theorem 1. Suppose that the controller $K_o(s)$ achieves the optimal \mathcal{H}_∞ robust performance condition for the plant

model $G(s) = N(s)M^{-1}(s)$ such that

$$\gamma_o = \sup_{\omega \in \Omega} \left(|W_1(1 + L_o)^{-1}| + |W_2L_o(1 + L_o)^{-1}| \right) \quad (17)$$

where $L_o = GK_o$. Additionally, suppose that γ_n is the optimal solution of the problem in (15) when X and Y are constructed such that $K = XY^{-1}$ has the same structure as K_o and F is parameterized by an n dimensional orthogonal basis function. Then γ_n converges monotonically from above to γ_o when $n \rightarrow \infty$.

Proof : According to Lemma.1, there exists $\{X_o(s), Y_o(s)\} \in \mathcal{P}$ and stable transfer function $F_o(s)$ such that $K_o(s) = X_o(s)Y_o^{-1}(s)$ and

$$\gamma_o = \sup_{\omega \in \Omega} \left(\left| \frac{W_1MY_oF_o}{\Re\{\psi_oF_o\}} \right| + \left| \frac{W_2NX_oF_o}{\Re\{\psi_oF_o\}} \right| \right) \quad (18)$$

where $\psi_o = NX_o + MY_o$. Now take F_n^* as the projection of F_o into the subspace spanned by an n dimensional orthogonal basis function and define the following quantity:

$$\gamma_n^* = \sup_{\omega \in \Omega} \left(\left| \frac{W_1MY_oF_n^*}{\Re\{\psi_oF_n^*\}} \right| + \left| \frac{W_2NX_oF_n^*}{\Re\{\psi_oF_n^*\}} \right| \right) \quad (19)$$

By contradiction, it can be shown that γ_n^* is a bounded function (using the fact that $\Re\{\psi_oF_o\} > \epsilon > 0$); assume that $j\omega^*$ is a zero of $\Re\{\psi_oF_n^*\}$. Therefore, at $\omega = \omega^*$,

$$\Re\{\psi_oF_o\} = \Re\{\psi_o(F_o - F_n^*)\} > \epsilon \quad (20)$$

However, $\Re\{\psi_o(F_o - F_n^*)\}$ can be made arbitrarily small by increasing n ; therefore, for a large but finite n , $\Re\{\psi_oF_n^*\} \neq 0$.

By using the relations in (18) and (19), one can compute $|\gamma_o - \gamma_n^*|$ as follows:

$$|\gamma_o - \gamma_n^*| \leq \sup_{\omega \in \Omega} \left| \frac{|W_1MY_oF_o| + |W_2NX_oF_o|}{|\Re\{\psi_oF_o\}|} - \frac{|W_1MY_oF_n^*| + |W_2NX_oF_n^*|}{|\Re\{\psi_oF_n^*\}|} \right| \quad (21)$$

However, according to [20], it is known that

$$\lim_{n \rightarrow \infty} \|F_o - F_n^*\|_\infty = 0 \quad (22)$$

From this relation, it can be deduced that $\lim_{n \rightarrow \infty} |F_n^*| \rightarrow |F_o|$ and $\lim_{n \rightarrow \infty} \Re\{F_n^*\} \rightarrow \Re\{F_o\}$. Therefore, we obtain the following result:

$$\lim_{n \rightarrow \infty} |\gamma_o - \gamma_n^*| = 0 \quad (23)$$

However, γ_n (the solution to the optimization problem in (15)) is always less than or equal to γ_n^* and greater than the optimal solution γ_o if the structures of X and Y are the same as X_o and Y_o , respectively (which is the case, as postulated in the Theorem). Therefore, γ_n converges from above to γ_o , and this convergence is monotonic because the basis functions F_n of order n are a subset of those of order $n + 1$, which ensures that $\gamma_{n+1} \leq \gamma_n$. ■

Note that the constraint in the optimization problem (15) is not convex and a non-linear optimization algorithm will

need to be implemented in order to solve this problem. In this manner, the parameter ρ_o in $F(\rho)$ need not be fixed, and the optimal solution can be obtained for a lower-order $F(\rho)$ (as will be demonstrated in subsequent sections of this paper).

Remark. One problem with solving the nonlinear problem in (15) is defining the initial conditions. Since there can be many variables involved in this optimization problem, defining the initial conditions to achieve the global optimal solution to the \mathcal{H}_∞ problem may not be trivial. One solution to this problem is to first solve the convex optimization problem in (14) and use the solution to this problem as the initial conditions for the nonlinear problem in (15). In this manner, the likelihood of obtaining the global optimal solution to the nonlinear problem will significantly increase.

IV. SIMULATION EXAMPLES

Let us now consider several examples in order to determine the validity of the proposed method. For each example, the MATLAB software was used in conjunction with the `fmincon` function to solve the proposed optimization problem.

A. Example 1: Convergence to Optimal Solution

Consider the following discrete time SISO system (with a sampling time of $T_s = 1s$):

$$G(z) = \frac{z - 0.186}{z^3 - 1.116z^2 + 0.465z - 0.093} \quad (24)$$

Note that since this system is stable, then the coprimes can be selected as $N(z) = G(z)$ and $M(z) = 1$.

Remark. Discrete-time systems can also be considered since the proposed method uses the FRF of a system for the controller synthesis. The FRF between continuous-time and discrete-time systems are nearly identical when the Shannon sampling condition is met.

The control objective of this case study is to design a controller (with integral action) that minimizes the nominal performance condition $\|W_1S\|_\infty$ (i.e., minimize γ in $\|W_1S\|_\infty < \gamma$), where

$$W_1(z) = \frac{0.4902(z^2 - 1.0431z + 0.3263)}{(z - 1)(z - 0.282)} \quad (25)$$

For this controller synthesis, we can simply select the basis functions in (5) to be FIR filters (i.e., $\rho_d = 0$) for $X(\rho)$ and $Y(\rho)$. In order to avoid unboundedness of W_1 at $\omega = 0$ and have integral action, the basis functions for $Y(\rho)$ will be multiplied by $(z - 1)/z$. It will be shown that as the order of $F(\rho)$ increases, the optimal solution γ will decrease monotonically and converge to the optimal solution.

1) *Simulation Results:* The `mixsyn` command from MATLAB's Robust Control toolbox yields an optimal value of $\gamma^* = 0.5522$ with a 5th order controller. Therefore, according to Theorem.1, a 5th order controller was also implemented with the proposed method (i.e., to ensure convergence to the optimal solution). Additionally, two simulations were

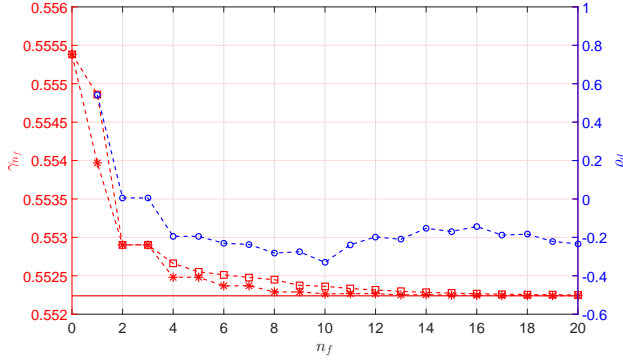


Fig. 2. Optimal solution obtained from `mixsyn` command (straight-red line); γ_{n_f} for $F(\rho)$ as an FIR filter (- \square -); γ_{n_f} for $F(\rho)$ with ρ_d as a decision variable (- \ast -); the value of ρ_d when used as a decision variable (- \circ -). $n_f = 0$ signifies the solution to the problem when $F(\rho) = 1$.

performed with the proposed method; one simulation with $F(\rho)$ selected such that the basis function is an FIR filter (i.e., Laguerre basis functions in (5) with $\rho_d = 0$) and one simulation with $F(\rho)$ selected such that ρ_d is a decision variable. Let γ_{n_f} denote the optimal solution of the nominal performance condition for a given n_f -th order of $F(\rho)$. Figure. 2 shows the convergence of the optimal solution γ_{n_f} as a function n_f ; it can be observed that with the proposed frequency-domain method, the optimal solution to the \mathcal{H}_∞ problem converges monotonically as the order of $F(\rho)$ increases. Additionally, when the parameter ρ_d is a decision variable, the convergence time is reduced.

Note that the controller order was fixed in this case study, since it was known (from the `mixsyn` command) that the optimal controller order was 5. However, suppose that the order and structure of the optimal controller $K_o = X_o Y_o^{-1}$ was unknown. In this case, one can simply fix the structure of the basis function of $F(\rho)$ with a large order and solve the optimization problem as the order of the controller is varied. The lowest order controller that achieves convergence can then be selected as the optimal controller. Let n_k denote the order of the controller $K(\rho)$ and γ_{n_k} the optimal solution to the \mathcal{H}_∞ problem for a given n_k . Figure. 3 shows the convergence to the optimal solution when the order of the basis function $F(\rho)$ is fixed to 20 (with $F(\rho)$ selected as an FIR filter) and the controller order n_k is varied; it can be observed that convergence to the optimal solution does indeed occur with a 5th order controller. In fact, according to Fig. 3, a 3rd order controller generates the performance similar to that of the 5th order optimal controller that MATLAB's Robust Control toolbox provides.

B. Example 2: Robust Performance

Consider the unstable non-minimum phase system analyzed in [21] and [22] (which is subject to multiplicative uncertainty):

$$G(s) = \frac{s-1}{s^2 + 0.8s - 0.2} \quad (26)$$

The objective of this case study is to design a stabilizing PID controller such that the robust performance condition in

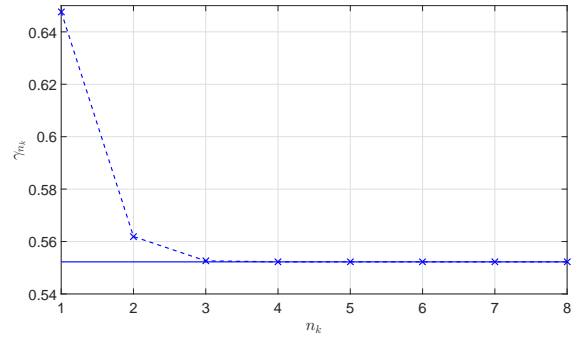


Fig. 3. Optimal solution obtained from `mixsyn` command (straight-blue line); γ_{n_k} for $F(\rho)$ selected as a 20th order FIR filter (- \times -).

(10) is satisfied. The weighting filters for this design will be chosen as those defined in [21]:

$$W_1(s) = \frac{10}{100s + 1}, \quad W_2(s) = \frac{s + 0.1}{s + 1} \quad (27)$$

Since $G(s)$ is unstable, the coprime functions can be selected as follows:

$$N(\rho, s) = \frac{s-1}{(s+\rho_g)^2}, \quad M(\rho, s) = \frac{(s^2 + 0.8s - 0.2)}{(s+\rho_g)^2} \quad (28)$$

where $\rho_g \in \mathbb{R}_+$. For a PID controller, the structure of the controllers $X(\rho)$ and $Y(\rho)$ can be selected as those defined in (7) with $T_f = 0.01$.

1) *Simulation Results:* The problem in (15) is solved by converting it to an SDP problem and gridding the frequency from 10^{-3} to 10^3 rad/s (using 500 logarithmically spaced points). A 20-th order basis function for $F(\rho)$ is selected (as defined in (16)). For comparative purposes, the MATLAB `mixsyn` function from the Robust Control Toolbox is used to compute a controller with the given weighting filters; the toolbox yields a 4th order controller.

Table.I shows the values of the optimal solutions obtained from various methods¹. The convex problem in (14) was solved with $\rho_g = \rho'_c = 1$. Since `mixsyn` minimizes $\|[W_1 S \ W_2 T]\|_\infty$, it can be observed that a smaller value of this criteria is obtained (with a 4-th order controller). However, with a simple PID controller, the proposed method yields the best performance for the *true* robust performance criteria in (10). The optimal value for the the decision variable of $F(\rho)$ was obtained as $\rho_o^* = 0.933$, while the optimal values of the remaining decision variables (for both the convex and proposed optimization methods) are shown in Table.II.

C. Example 3: Multimodel Uncertainty

For this example, a robust controller will be designed for a family of unstable systems. This example is taken from the Robust Control Toolbox of MATLAB. The nominal plant

¹The method proposed in [21] produced a set of PID controllers. A random solution from the set was used for comparative purposes.

TABLE I

COMPARISON OF OPTIMAL SOLUTIONS TO THE \mathcal{H}_∞ PROBLEM (ROBUST PERFORMANCE VS. MIXED SENSITIVITY)

	$\ W_1 S\ + \ W_2 T\ _\infty$	$\left\ \frac{W_1 S}{W_2 T} \right\ _\infty$
Convex method (14)	1.327	1.223
\mathcal{H}_∞ parametric approach in [21]	1.205	0.923
MATLAB [®] (mixsyn)	1.111	0.789
Proposed method (15)	1.019	0.966

TABLE II

OPTIMAL VALUES OF PID CONTROLLERS OBTAINED FROM THE CONVEX AND PROPOSED OPTIMIZATION PROBLEMS

	ρ_1^*	ρ_2^*	ρ_3^*	ρ_4^*	$(\rho_c')^*$
Convex Method (14)	-0.565	-0.013	-0.397	1	1
Proposed Method (15)	-0.486	-0.021	-0.486	1.647	1.647

model for this family of systems is given as follows:

$$G_0(s) = \frac{2}{s-2} \quad (29)$$

This model is perturbed through various types of uncertainties such as time delay, high frequency resonance, pole/gain migration, and extra lag; the family of perturbed models are given as follows:

$$\begin{aligned} G_1(s) &= G_0(s) \frac{1}{0.06s+1} & G_4(s) &= G_0(s) e^{-0.04s} \\ G_2(s) &= G_0(s) \frac{50^2}{s^2+10s+50^2} & G_5(s) &= \frac{2.4}{s-2.2} \\ G_3(s) &= G_0(s) \frac{70^2}{s^2+28s+70^2} & G_6(s) &= \frac{1.6}{s-1.8} \end{aligned} \quad (30)$$

The performance filter $W_1(s)$ and noise filter $W_2(s)$ are chosen to be equal to the filters asserted in [16], i.e.,:

$$\begin{aligned} W_1(s) &= \frac{0.33s+4.248}{s+0.008496} \\ W_2(s) &= \frac{0.1975s^2+0.6284s+1}{7.901 \cdot 10^{-5}s^2+0.2514s+400} \end{aligned} \quad (31)$$

For this problem, the control objective is to minimize γ and satisfy the following criteria for all seven models:

$$\|W_1 S_\ell\|_\infty < \gamma \quad \text{and} \quad \|W_2 T_\ell\|_\infty < \gamma \quad (32)$$

for $\ell = 0, \dots, 6$. Note that this control objective satisfies the mixed sensitivity robust condition, and not the robust performance condition in (10). The μ -synthesis approach in the MATLAB Robust Control Toolbox uses this criteria to design a controller. In this approach, a multiplicative uncertainty is considered and an appropriate weighting filter is designed to ensure that the condition in (32) is satisfied for all seven models. The controller produced by MATLAB with this μ -synthesis approach yields an 18th order controller and satisfies the control objective for all models.

The same problem is now solved using the proposed approach. First, the coprime factors $N_\ell(s)$ and $M_\ell(s)$ for

TABLE III

COMPARISON OF OPTIMAL SOLUTIONS

	n	ρ_c^*	ρ_p^*	ρ_o^*	γ^*
MATLAB μ -Synthesis	18	—	—	—	1.024
Algorithm in [13] (Convex)	6	20	100	—	0.881
Algorithm in [13] $\{\rho_p, \rho_c\} \rightarrow \text{variable}$	6	23.89	73.87	—	0.870
Proposed method	6	16.72	127.1	63.68	0.814

$\ell = 0, \dots, 6$ must be established. Since each model is unstable, then each coprime factor must be selected such that $\{N_\ell(s), M_\ell(s)\} \in \mathcal{P}$ for all ℓ . A simple choice is to divide both the numerator and denominator of each model by a factor $(s + \rho_p)^{\lambda_\ell}$, where $\rho_p \in \mathbb{R}_+$ and λ_ℓ is the largest degree of the denominator of the ℓ -th respective plant model. For example, the coprime factors for the plant $G_2(s)$ can be formed as follows:

$$N_2(s) = \frac{2 \cdot 50^2}{(s + \rho_p)^3}, \quad M_2(s) = \frac{(s-2)(s^2+10s+50^2)}{(s + \rho_p)^3} \quad (33)$$

From these relations, it is evident that $G_2(s) = N_2(s)M_2^{-1}(s)$. To further simplify the design, the same ρ_p can be selected for each ℓ -th coprime. To solve the convex optimization problem proposed in [13], ρ_p must be fixed a priori; however, with the proposed method, ρ_p need not be fixed.

The optimization problem (with the proposed approach) for the mixed \mathcal{H}_∞ criteria in (32) can be formulated as follows:

$$\begin{aligned} &\underset{\rho}{\text{minimize}} && \gamma \\ &\text{subject to:} && \gamma^{-1} |W_1 M_\ell(\rho) Y(\rho) F(\rho)| < \Re\{\psi_\ell(\rho) F(\rho)\} \\ & && \gamma^{-1} |W_2 N_\ell(\rho) X(\rho) F(\rho)| < \Re\{\psi_\ell(\rho) F(\rho)\} \\ & && \omega \in \Omega, \ell = 0, \dots, 6 \end{aligned} \quad (34)$$

where $\psi_\ell = N_\ell(\rho)X(\rho) + M_\ell(\rho)Y(\rho)$. In solving the above optimization problem, the parameters in $\{\rho_c, \rho_p, \rho_o\}$ will not be fixed, and will be optimized by the nonlinear programming algorithm (as shown with the notation $N_\ell(\rho)$ and $M_\ell(\rho)$).

1) *Simulation Results:* The problem in (34) is converted to a SDP problem by considering a logarithmically spaced frequency grid with $Q = 200$ points from $\omega_1 = 10^{-3} \text{ rad/s}$ to $\omega_{200} = 10^4 \text{ rad/s}$. For the convex problem, a 6-th order controller (5-th order controller with one integrator) will be designed by using the Laguerre basis functions defined in (4) with $\rho_c = 20$ and $\rho_p = 100$ (as defined in [13]). With the proposed method (and for comparative purposes), a 6-th order controller will also be used while a 20-th order Laguerre basis function will be considered for $F(\rho)$ (as defined in (16)).

The optimal solutions from several optimization algorithms for satisfying the criteria in (32) are tabulated in Table.III (where n is the controller order of the respective method). The table also shows the solutions when the convex optimization problem in [13] is solved when the variables

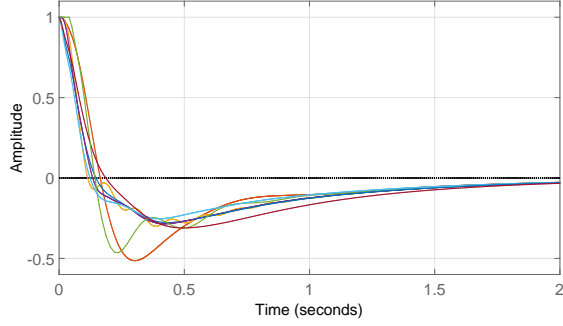


Fig. 4. Step response of $S(s)$ for all seven models using the convex optimization algorithm in [13].

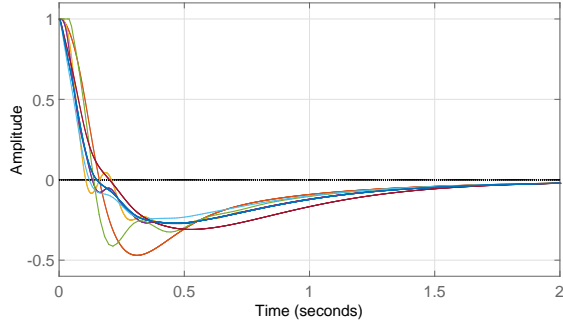


Fig. 5. Step response of $S(s)$ for all seven models using the proposed optimization algorithm.

ρ_c and ρ_p are decision variables (thus making the problem non-convex). It can be observed that the proposed method produces the best performance.

Figure. 4 shows the step response of $S(s)$ (disturbance response) for all seven models using the convex optimization algorithm, while Figure. 5 shows the step response of $S(s)$ using the proposed method. It can be observed that the proposed method produces shorter settling times and reduced overshoot.

Remark. The μ -synthesis approach by MATLAB introduces some conservatism in determining the optimal controller. Additionally, with the μ -synthesis approach, the plant $G_4(s)$ was approximated with a first-order Pade function, since pure delay's cannot be used for synthesis. In the proposed approach, the true delay function can be considered in the design while eliminating the conservatism associated with modeling the multi-model uncertainties.

V. CONCLUSION

A frequency-domain approach has been implemented in order to design robust controllers that achieve optimal \mathcal{H}_∞ performance. With the proposed method, it has been shown that the solution to the \mathcal{H}_∞ problem converges monotonically to the optimal solution when the order of the basis function F approaches infinity. This solution is obtained by considering an optimization problem which guarantees the stability of the

closed-loop system. The optimization problem is converted from an SIP problem to an SDP problem by gridding the frequency vector over a finite interval. The simulation examples show that the proposed approach achieves the optimal performance for a given controller order without any conservatism. Although the proposed approach does yield the best solution for the robust performance condition, it should be emphasized that the convex problem proposed in [13] generates a solution that is comparable with the methods proposed in this paper. For future work, it will be desired to implement this method for multi-input-multi-output (MIMO) systems and systems with controllers that possess multiple degrees-of-freedom.

REFERENCES

- [1] Z.-S. Hou and Z. Wang, "From model-based control to data-driven control: Survey, classification and perspective," *Information Sciences*, vol. 235, pp. 3–35, 2013.
- [2] A. S. Bazanella, L. Campestri, and D. Eckhard, *Data-driven Controller Design: The H_2 Approach*. Springer, 2012.
- [3] R. Hoogendijk, A. J. Den Hamer, G. Angelis, R. van de Molengraft, and M. Steinbuch, "Frequency response data based optimal control using the data based symmetric root locus," in *IEEE Int. Conference on Control Applications*, Yokohama, Japan, 2010, pp. 257–262.
- [4] S. Khadraoui, H. Nounou, M. Nounou, A. Datta, and S. Bhattacharyya, "Robust control design method for uncertain system using a set of measurements," in *American Control Conference (ACC)*, 2013, 2013, pp. 4325–4330.
- [5] —, "A control design method for unknown systems using frequency domain data," in *Control Conference (ASCC)*, 2013 9th Asian. IEEE, 2013, pp. 1–6.
- [6] A. Karimi and G. Galdos, "Fixed-order H_∞ controller design for nonparametric models by convex optimization," *Automatica*, vol. 46, no. 8, pp. 1388–1394, 2010.
- [7] A. Karimi, G. Galdos, and R. Longchamp, "Robust fixed-order H_∞ controller design for spectral models by convex optimization," in *47th IEEE Conference on Decision and Control*, Cancun, MEX, 2008.
- [8] M. Saeki, M. Ogawa, and N. Wada, "Low-order H_∞ controller design on the frequency domain by partial optimization," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 3, pp. 323–333, 2010.
- [9] M. Hast, K. J. Åström, B. Bernhardsson, and S. Boyd, "PID design by convex-concave optimization," in *European Control Conference*, Zurich, Switzerland, 2013, pp. 4460–4465.
- [10] S. Boyd, M. Hast, and K. J. Åström, "MIMO PID tuning via iterated LMI restriction," *International Journal of Robust and Nonlinear Control*, vol. 26, no. 8, pp. 1718–1731, 2016.
- [11] T. Usami, K. Yubai, D. Yashiro, and S. Komada, "Multivariable fixed-structural controller design for H_∞ loop shaping method by iterative LMI optimization using frequency response data," in *International Conference on Advanced Mechatronic Systems (ICAMechS)*. IEEE, 2016, pp. 218–223.
- [12] C. Kammer and A. Karimi, "Robust distributed averaging frequency control of inverter-based microgrids," in *IEEE 55th Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 4973–4978.
- [13] A. Karimi, A. Nicoletti, and Y. Zhu, "Robust H_∞ controller design using frequency-domain data via convex optimization," available online in *International Journal of Robust and Nonlinear Control*, 2016.
- [14] K. Zhou and J. C. Doyle, *Essentials of robust control*. N.Y.: Prentice-Hall, 1998.
- [15] C. J. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. New York: Mc Millan, 1992.
- [16] A. Karimi, "Frequency-domain robust control toolbox," in *52nd IEEE Conference in Decision and Control*, 2013, pp. 3744 – 3749.
- [17] P. Seiler, B. Vanek, J. Bokor, and G. J. Balas, "Robust H_∞ filter design using frequency gridding," in *American Control Conference (ACC)*, 2011. IEEE, 2011, pp. 1801–1806.
- [18] G. Galdos, A. Karimi, and R. Longchamp, " H_∞ controller design for spectral MIMO models by convex optimization," *Journal of Process Control*, vol. 20, no. 10, pp. 1175 – 1182, 2010.

- [19] G. Ferreres and C. Roos, "Efficient convex design of robust feedforward controllers," in *44th IEEE Conference on Decision and Control*, Seville, Spain, December 2005, pp. 6460–6465.
- [20] H. Akcay and B. Ninness, "Orthonormal basis functions for continuous-time systems and l_p convergence," *Mathematics of Control, Signals, and Systems*, vol. 12, pp. 295–305, 1999.
- [21] Z. Zhen-ping, "Determination of the admissible parameter set in designing PID controllers with guaranteed robust performance," in *Electrical and Control Engineering (ICECE), 2010 International Conference on*. IEEE, 2010, pp. 3625–3628.
- [22] M.-T. Ho, "Synthesis of H_∞ PID controllers: A parametric approach," *Automatica*, vol. 39, no. 6, pp. 1069–1075, 2003.