Hidden Markov Models

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Chapter 1

Introduction

The goal of this project is to determine if HMMs are suitable as rain generators.

The first task will be to extend on the work done by Grando. In her testing, she has concluded that HMMs are suitable as rain generators; however, she has used the same data for testing as she used for training. Doing so may, quite likely, have to lead to bias, and thus, we will extend her work by conducting out-of-sample tests.

If possible, we will build the software, so it is user friendly and efficient. With this, we can test data for multiple locations. This will allow us to understand if the result is genuinely significant, at least more than just one location.

Chapter 2

Preliminaries

In this section, we will briefly visit the foundations on which we will build throughout this paper. For most, this will be a simple refresher.

2.1 Mathematical Foundations

We start with a few key mathematical concepts.

2.1.1 Probability Theory

To discuss any probabilistic ideas, we must first understand general probability theory. We will start by defining a probability space.

Definition 2.1. Probability Space
A probability space is defined by $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is the non-empty set of all possible outcomes, such that all events $\omega \in \Omega$. $\mathbb{P}$ is a probability measure, a function $\mathbb{P}$(A) that maps event A to a number within [0,1] based on the likelihood of the event. $\mathcal{F}$ is a σ-algebra on Ω if

1. $\Omega \in \mathcal{F}$
2. A $\in \mathcal{F}$ implies $A^c \in \mathcal{F}$
3. if A_1, A_2, A_3,... are in $\mathcal{F}$ then so is $A_1 \cup A_2 \cup A_3$...

2.1.2 Conditional Probability

Sometimes we require the probability of an event assuming another event has occurred. In such situations, we require a conditional probability. Given two events A and B, the probability of event A occurring conditioned on event B:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \ \forall A \in \mathcal{F} \tag{2.1}$$

From 2.1 and the fact that for dependednt events $\mathbb{P}$(A $\cap$ B) = $\mathbb{P}$(B $\cap$ A) we can see that:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A),\ \forall A, B \in \mathcal{F} \tag{2.2}$$

Subsituting 2.2 into 2.1 we get the famous Bayes Theorem.

Theorem 2.2. *Bayes' Theorem*
For dependent events A and B with probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *, where* $\mathbb{P}$ *(B)* $\neq$ *0,*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)},\ \forall A \in \mathcal{F} \tag{2.3}$$

2.1.3 Stochastic Process

To be able to define a Markov model, of any kind, we must first define a stochastic process.

Definition 2.3. Stochastic Process
Given an ordered set T and probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a stochastic process is a collection of random variables $\mathrm{X} = \{X_t;\ \mathrm{t} \in \mathrm{T}\}$. Based on $\mathrm{t} \in \mathrm{T}$ and $\omega \in \Omega$ we get a numerical realization of the process. For simplicity, this may be viewed as a function; $X_t(\omega)$.

2.2 Applied Foundations

Chapter 3

Standard Markov and Markov Property

3.1 History

Andrei Markov discovered the Markov model while analyzing the relationship between consecutive letters from the text in the Russian novel "Eugene Onegin". A translation can be found here Markov, n.d. With a two-state model (states Vowel and Consonant) he proved that the probability of letters being in a particular state is not independent. Given the current state, he could probabilistically predict the next. With various probabilities to and from each state, this chain of states formed the foundation of the Markov Chain.

3.2 Markov Chain

A Markov chain is a network of connected states. At any given time the model is said to be in a particular state. At a regular discrete interval, the model can change states depending on the probabilities. To define a Markov chain, we must first address the Markov property GRIMMETT et al., 2020.

Definition 3.1. Markov Property
Let $\{X_t \ ; \ t \in \mathbb{N}_0\}$ denote a stochastic proccess 2.3, where t represents time. The process has the Markov property if and only if,

$$\mathbb{P}\{X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, ..., X_0 = i_0\} = \mathbb{P}\{X_{n+1} = i_{n+1} | X_n = i_n\} \tag{3.1}$$

We can now define a Markov chain.

Definition 3.2. Markov Chain
A stochastic process $\{X_t \ ; \ t > 0\}$ is a Markov Chain if and only if it satisfies the Markov property 3.1.

To store the sequence of states a Markov chain has been through, we use the set $Q = \{q_t; t \in \mathbb{N}_0\}$, where q_t represents the state at time t. We will use this notation throughout the paper.

Example 3.3. Given a Markov Model with states S = $\{S_1, S_2, S_3\}$, if the model starts at S_2 and then goes to S_3 and then back to S_2 the state sequence Q will be $Q = \{q_1 = S_2, q_2 = S_3, q_3 = S_2\}$.

From the Markov property, we can see that the only thing that influences q_t is q_{t-1}. We can use this to make predictions for q_{t+1} based on the outward transition probabilities from state q_t. By calculating all outward transition probabilities from the state at q_t, we can construct a probability measure that we may use to predict future states.
i.e.
Given a Markov chain with N states including i and j and discrete time $t \in \mathbb{N}_0$, we can use

$$\mathbb{P}(q_t = S_j | q_{t-1} = S_i)_{1 \leq i,j \leq N} \tag{3.2}$$

as a probility measure to help predict future states.

These probabilities can vary with time, but this can make the model quite complicated. Thus, we usually assume the probabilities are constant. These unique Markov models are called time-homogenous.

Definition 3.4. Time homogenous
Let $\{X_t$; t $\in \mathbb{N}_0\}$ denote a stochastic proccess 2.3, where t represents discrete time, and $p(i, j)$ represent the transition probability from state i to state j. If

$$\mathbb{P}\{X_n = j | X_{n-1} = i\} = p(i, j), \forall n \in \mathbb{N}_0 \tag{3.3}$$

then the process is time-homogenous.

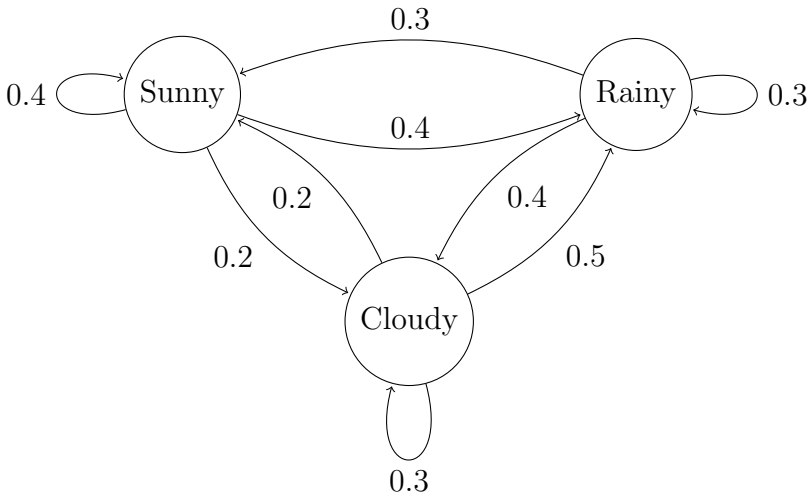
For a discrete Markov model with N states, there are N^2 total transitions, where $p(i, j) = 0$ represents an impossible transition. We must store each of these probabilities. Given a time-homogenous Markov chain, we can create a 2-dimensional N x N matrix of transition probabilities p. Unique Markov chains have unique transition matrices. These matrices can be defined as below:

$$p = \{p(i, j) = \mathbb{P}\{X_n = j | X_{n-1} = i\}\}_{1 \leq i,j \leq N} \tag{3.4}$$

All p matrices have some essential properties. The first is that all values within p must be within $[0, 1]$ which comes naturally as all values are probabilities and thus by definition, it must lie within $[0, 1]$. The second is that all rows, columns or both form stochastic vectors. If it is the rows, then the matrix is defined as a right-stochastic matrix; if it is the columns, it is called a left-stochastic matrix.

To demonstrate, we will present an example where the states represent the weather.

Example 3.5. Let $\{X_t; t \in \mathbb{N}_0\}$ denote a Markov Chain, with state-space S = {rainy, sunny, cloudy}, where t represents the number of days from the start. Since all states can eventually reach all other states, we say this model is ergodic. When all states within a model connect to all others, the model is always ergodic.



Arrows indicated the transition between states and adjacent values correspond to the probability of this transition.

Using 3.5 we can create a matrix p. To make this clear, we first create a table with our states labelled for rows and columns, where the $p(i, j)$ is the cell corresponding to row i and column j.

x	Sunny	Rainy	Cloudy
Sunny	0.4	0.4	0.2
Rainy	0.3	0.3	0.4
Cloudy	0.2	0.5	0.3

This content of this table forms the matrix p.

$$p = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \tag{3.5}$$

Now that we have a Markov Model with its p, we must reflect on its potential uses. Some natural questions one may ask are:

1. Given at time t the state was S_i, what is the most likely state time $t + 1$?
2. What is the probability of getting a particular state sequence O?
3. What is the probability of staying within a state for d time steps?

This first problem we will address using the matrix p. The motivation for creating the matrix p was to build a matrix where p_{ij} contains the probability of moving from state i to state j. Thus, we look at the current row and find the maximum probability and its corresponding j.

Example 3.6. Let us refer back to 3.5 and assume the current state is Cloudy. We can see that $0.2 < 0.3 < 0.5$ and that 0.5 corresponds to Rainy. Thus our the most likely next state would be Rainy.

Note: If the current state were sunny, we would have two maximums of 0.4. In such a case, the process is equally likely to go to either.

The second problem provides a state sequence $Q = \{q_t, q_{t+1}, q_{t+2}...\}$ and asks what the probability of this occuring is, i.e. $\mathbb{P}(Q|model)$. This can be simplified quite easily as shown below.

$$\begin{aligned} \mathbb{P}(Q|model) &= \mathbb{P}(\{q_t, q_{t+1}, q_{t+2}...\}|model) && (3.6)\\ &= \mathbb{P}(q_t)\mathbb{P}(q_{t+1}|q_t)\mathbb{P}(q_{t+2}|q_{t+1})... && (3.7)\\ &= \mathbb{P}(q_t)p(q_t, q_{t+1})p(q_{t+1}q_{t+2})... && (3.8) \end{aligned}$$

Example 3.7. We can now use 3.6 to demonstrate the probability of a state squence from 3.5. Let $Q = \{\text{Sunny, Sunny, Cloudy, Rainy}\}$. Given that we start from Sunny:

$$\begin{aligned} \mathbb{P}(Q|model) &= \mathbb{P}(\{Sunny, Sunny, Cloudy, Rainy\}|model) && (3.9)\\ &= \mathbb{P}(Sunny)\mathbb{P}(Sunny|Sunny)\mathbb{P}(Cloudy|Sunny)\mathbb{P}(Rainy|Cloudy) && (3.10)\\ &= 1 * 0.4 * 0.2 * 0.5 && (3.11)\\ &= 0.04 && (3.12) \end{aligned}$$

The third problem asks how long is the model likely to stay in any given state. Assume that the model stays in state S_i for d days. We can create a state sequence for this, $Q = \{q_t = S_i, q_{t+1} = S_i, ..., q_{t+d-1} = S_i, q_{t+d} \neq S_i\}$. Using the state sequence probabililty calculation from before we can compute the following equation:

$$\mathbb{P}(Q|model) = p(i,i)^{d-1}(1 - p(i,i)) = p_i(d) \tag{3.13}$$

We label this $p_i(d)$ to represent the discrete probability density function of the duration d in state i. From this we can calculate the expected stay in any particular state. This is done using the following formula:

$$\begin{aligned} \bar{d_i} &= \sum_{d=1}^{\infty} dp_i(d) && (3.14)\\ &= \frac{1}{1 - p(i,i)} && (3.15) \end{aligned}$$

Example 3.8. Reffering back to 3.5, suppose we would like to see how many days in a row we expect it to be sunny. We see that $p(Sunny, Sunny) = 0.4$. Now we can use 3.14 to find,

$$\begin{aligned} \bar{d_{Sunny}} &= \sum_{d=1}^{\infty} dp_{Sunny}(d) && (3.16)\\ &= \frac{1}{1 - p(Sunny, Sunny)} && (3.17)\\ &= \frac{1}{1 - 0.4} && (3.18)\\ &= 1.67 && (3.19) \end{aligned}$$

Thus we expect it to remain sunny for 1.67 days. Since we are dealing with discrete data, it is more appropriate to round up to 2 days.

The Markov model we have been discussing so far is observable as we can observe its events. Not all Markov models, however, are observable.

3.3 Motivating the Hidden Markov Model

Sometimes we do not observe our states but only see the effect of a state change. To help develop this idea, we borrow an example from Rabiner and Juang, 1986.

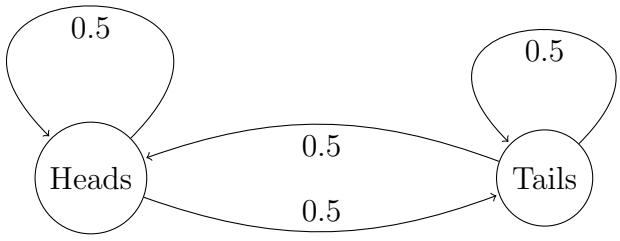
Example 3.9. Suppose someone is hiding behind a barrier or curtain, where we cannot see what they are doing. This person is doing some experiment with flipping coins and shouting heads or tails at regular intervals. We do not know:

i How many coins there are.

ii If the coin/s is/are fair.

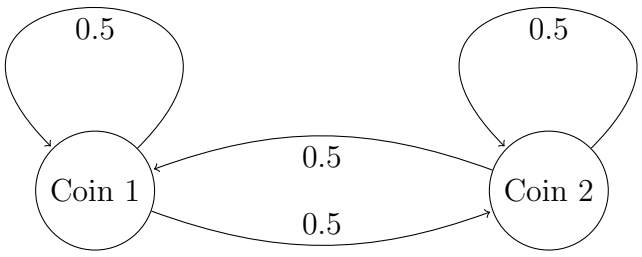
Since the problem is quite vague, we must experiment with various models and see which fits best the data. Since the only thing we can observe is the outcome, heads or tails, we will refer to this as our observation. Let us start with the simplest.

1. 1-Fair Coin
   In this model we have two states, heads and tails. There is a 0.5 probability for the model to change state and equally for staying in the same. The simplicity of this model comes at the cost of many assumptions that may not necessarily hold.



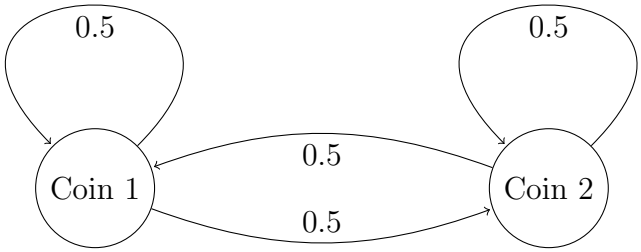
$$\begin{array}{ll} \mathbb{P}(Heads) & 0.5 \\ \mathbb{P}(Tails) & 0.5 \end{array}$$

2. 2-Fair Coins
   This model also has two states, but this time they represent two different coins. Both can produce heads and tails. Thus we do not know which produced the observation. The fact that they are both fair coins means that as an external observer, we will not see a difference between this model and the 1-Fair Coin. However, it is clear that the observations are now independent of the hidden states transitions, as they are equally likely regardless of the state. In the 1-Fair Coin model, we can determine perfectly which state the model is in from the observation, but here this is no longer possible.



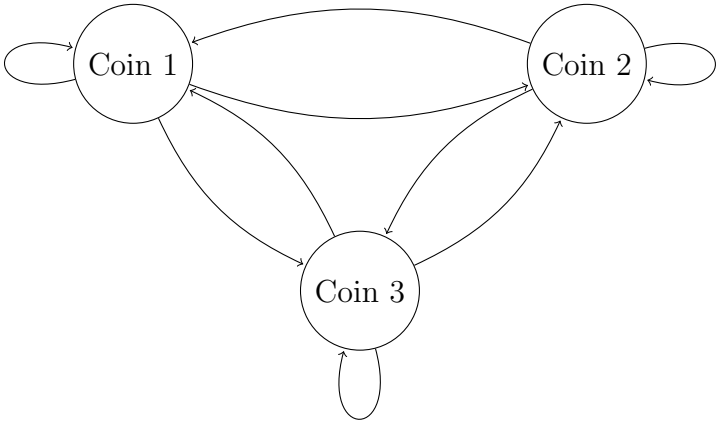
	$\mathbb{P}(Coin1)$	$\mathbb{P}(Coin2)$
$\mathbb{P}(Heads)$	0.5	0.5
$\mathbb{P}(Tails)$	0.5	0.5

3. 2-Biased Coins
   Although this model is similar to 2-Fair Coins, the change in $\mathbb{P}(Heads)$ and $\mathbb{P}(Tails)$ for each state has a large impact on the observation probabilities. If the model is in state one heads is more likely and if the state is in state two tails is more likely. As an observer we can now say if we see tails, it was most likely from state two and if we see heads, it was most likely from state one. The change between these two states must be an unrelated probability, such as a third coin or another source of randomness.



	$\mathbb{P}(Coin1)$	$\mathbb{P}(Coin2)$
$\mathbb{P}(Heads)$	0.75	0.25
$\mathbb{P}(Tails)$	0.25	0.75

4. 3-Biased Coins
   Similar to the 2-Biased Coins model, the probabilities for heads and tails vary with the three states.



	$\mathbb{P}(Coin1)$	$\mathbb{P}(Coin2)$	$\mathbb{P}(Coin3)$
$\mathbb{P}(Heads)$	0.6	0.25	0.45
$\mathbb{P}(Tails)$	0.4	0.75	0.55

After contemplating which model is best, an important conclusion one may make, is that it is quite challenging to decide on the number of states without a priori information. We must ensure that there are enough states, such that the model does not overgeneralize but also not so many that it requires too much data to train. Naturally, one would assume

choosing the larger number of states is more suitable as they can take the shape of a model with fewer states, while it is not for the opposite. However, larger models require much more data to be statistically reliable, as there are too many unknown variables. Below we show this for our models and where the unknowns come from (-s being from the state transition probabilities and -O being from the observation probabilities).

Model	Number of Unknowns	From
1-Fair Coin	0	-
2-Fair Coins	1	1-S
2-Biased Coins	4	2-S 2-O
3-Biased Coins	9	6-S 3-O

Hence, the best approach is to base the model state size on the amount of available data, which is not always guaranteed to give reliable results but can sometimes be the only option. For example, when limited by data.

One may notice that the first model has 0 unknowns. It has this because each state links with only one observation. Thus, as an observer, we can assert what state the model is in from an observation. Thus, the states are no longer hidden, and the model is simply a standard Markov model.

Another essential detail the avid reader may have noticed is in the case of 2-Biased Coins and 3-Biased Coins it is possible to make a reasonable guess of what state the model is in, i.e. which coin is flipped, based on the observation. Furthermore, the statistical properties of predictions generated using 1-Fair Coin and 2-Fair Coins should be identical but for 2-Biased Coins and 3-Biased Coins be somewhat unique. Due to this, unless the underlying system is entirely fair, like in one and two, it should be possible to determine which model best fits the data and determine the model.

We now arrive at the core idea behind hidden Markov models. Although we cannot directly observe the model or its properties, we can create a model that fits the data the best through sufficient observation data.

Chapter 4

Hidden Markov Model

4.1 Definition

A hidden Markov model is a doubly stochastic process, where the underlying process is markovian and hidden Rabiner and Juang, 1986, which comes from the fact that there are two stochastic processes, one determining the transition between states and one determining the output observation. Braum and his colleagues developed this in Leonard E Baum, Eagon, et al., 1967 and Leonard E. Baum and Petrie, 1966.

To define a hidden Markov model, we need five things.

1. N
   - i N is the number of hidden states. Usually based on something in the real world but sometimes can be unknown, as in 3.9.
   - ii The states are usually ergodic, i.e. from any given state, we can eventually reach another, but this is unnecessary.
   - iii The states are from the state space $S = \{S_1, S_2, ..., S_N\}$.
2. M
   - i M is the number of observable outputs.
   - ii These make a discrete alphabet of observations called $V = \{v_1, v_2, ..., v_M\}$.
3. A
   - i A is the state transition matrix.
   - ii This is the same as the p matrix 3.4.
   - iii A = $\{a_{ij}\}$
   - iv $a_{ij} = \{p(i,j) = \mathbb{P}\{X_n = j | X_{n-1}\}$
4. B
   - i B is the observation probability matrix.
   - ii B = $\{b_j(k)\}$
   - iii $b_j(k) = \mathbb{P}(V_k \ at \ t | q_t = S_j)_{1 \leq j \leq N, 1 \leq k \leq M}$

5. π

   i π is a vector containing all inital state probabilities.

   ii $\pi = \{\pi_i\}$

   iii $\pi_i = \mathbb{P}(q_1 = S_i) for 1 \leq i \leq N$

We can now combine the above to provide a formal definition of hidden markov models.

Definition 4.1. Hidden Markov Model
A Hidden Markov Model is a 5-tuple $\{N, M, A, B, \pi\}$ that is used to represent a doubly stochastic process where the hidden process is markovian.

Before we continue, we will also provide some notation used for the rest of the paper.

i We will continue to use Q 3.3 to represent the Markov model's state sequence, i.e. the hidden process.

ii We will use $O = \{O_1, O_2, ..., O_T\},\ O_i \in V$ to represent the observation sequence.

iii we will occasionally represent the hidden markov model as $\{N, M, \lambda\}$ where $\lambda = \{A, B, \pi\}$

4.2 Using HMM

As with any mathematical model, we can use HMMs to predict future observations. As with simple Markov models 3.2, we can create probability measures, but this time we must use conditioning.

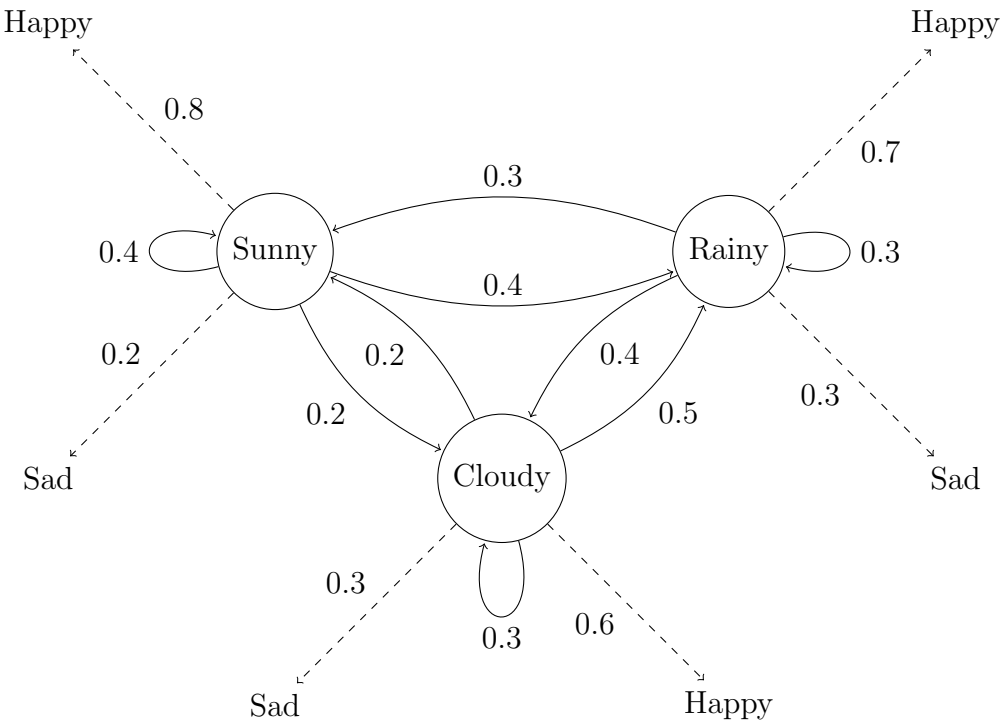
4.2.1 Predictive Model

We can use HMM to generate a sequence of potential observations. Given a a hidden markov model, lets call it H = $\{N, M, A, B, \pi\}$, to generate a sequence of T observations O we do the following steps:

1. Using π as a probability measure, set $t = 1$ and randomly select a state as the first $q_1 = s_i$.
2. Using $b_i(k)$ as a probability measure, randomly select the observation $O_t = v_k$.
3. Using a_{ij} as a probability measures, set $t = t + 1$ and randomly select a the next state $q_{t+1} = s_j$.
4. Repeat steps 2 and 3 until $t = T$

To demonstrate this method, we will use an adapted version of 3.5. This version will have the Markov chain from before as the underlying hidden process and another process with state-space {Happy, Sad}, which we can observe.

Example 4.2. Suppose Alice is hidden away from the world and has no access to information regarding the weather. She meets Bob every day and knows how weather affects his mood. For simplicity, assume Bob only has two moods, happy and sad. Given matrix A, B and vector π can she predict his mood for three consecutive days?
From context we can deduce the following: enumerate



The solid lines represent hidden probabilities, and the dashed represent observable.

$$\text{N} = 3, \text{ number of hidden states}$$

$$\text{M} = 2, \text{ number of possible observations}$$

$$A = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \tag{4.1}$$

$$B = \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \tag{4.2}$$

$$\pi = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix} \tag{4.3}$$

enumerate

We can now use 4.2.1 to create an observation sequence $O = \{o_1, o_2, o_3\}$. We will require multiple random variables generated using various probability measures. We will these through python using the "rand.py" file. We will use a uniform random variable. We will label this r.v. We select the state that corresponds to the region on the cumulative probability distribution that the random variable lies on.

i We generate a r.v. $= 0.0058$. Using Pi as the probaility distribution, we select Sunny. We can now set $q_1 = s_1$.

ii We generate a r.v. $= 0.1947$. Using $b_1(k)$ as the probability distribution we select Happy as the observation. We can now set $o_1 = v_1$.

iii we generate a r.v. $= 0.7168$. Using a_{1j} as the probability distribution we select Rainy as the next state. We now set $q_2 = s_2$.

iv We generate a r.v. $= 0.1060$. Using $b_2(k)$ as the probability distribution we select Happy as the observation. We can now set $o_2 = v_1$.

v we generate a r.v. $= 0.8977$. Using a_{2j} as the probability distribution we select Cloudy as the next state. We now set $q_3 = s_3$.

vi We generate a r.v. $= 0.1369$. Using $b_3(k)$ as the probability distribution we select Happy as the observation. We can now set $o_2 = v_1$.

Finally, we can look back on our prediction O and see that it is equal to $\{v_1, v_1, v_1\}$, i.e. we predict three consecutive Happy days.

4.2.2 Three Key Problems

There are many interesting questions one may pose regarding the HMM but there are three famous ones, highlighted in Rabiner and Juang, 1986 which we will focus on.

1. Evaluation
   Given model H $= \{N, M, A, B, \pi\}$ what is the probability that it generated the sequence of observations $O = \{o_1, o_2, ..., o_T\}$? i.e. $\mathbb{P}(O \mid H)$

2. Decoding
   What sequence of states $Q = \{q_1, q_2, ..., q_3\}$ best explains a sequence of observations $O = \{o_1, o_2, ..., o_T\}$?

3. Learning
   Given a set of observation $O = \{o_1, o_2, ..., o_T\}$, how can we learn the model H $= \{N, M, A, B, \pi\}$ that would generate them?

In the coming sections, we will be motivating uses and developing solutions for each problem.

4.3 Problem 1: Evaluation

Lets start by addressing question 1. Informally, we are looking for the probability that a given model generated a sequence of observations, i.e. $\mathbb{P}(O|\lambda)$.

This probability has many useful applications. For example, we may have multiple potential models λ_i and cannot decide which one is the most suitable. In this case, we can now calculate this probability for each λ_i and the largest.

To find this probability, we must consider the hidden internal states of the model. Since our probability of observations $\{b_j(k)\}$ is conditioned on the hidden state, we can start by

calculating this probability conditioned on these states. Lets assume we know what the state sequence $Q = \{q_1, q_2, ..., q_t\}$ is. To $\mathbb{P}(O|Q, \lambda)$ we can find the product of all the probabilities of an observation given the models state at all times t. In essence, this breaks the $\mathbb{P}(O|Q, \lambda)$ into T parts.

$$\mathbb{P}(O|Q, \lambda) = \prod_{t=1}^{T} \mathbb{P}(O_t|q_t, \lambda) \tag{4.4}$$

An observation one may make is that these probabilities are simply taken from the matrix B.

$$\mathbb{P}(O_t|q_t, \lambda) = b_{q_t}(O_t), \quad \forall t \in [0, T] \tag{4.5}$$

Thus we can rewrite 4.4 as:

$$\mathbb{P}(O|Q, \lambda) = b_{q_1}(O_1)b_{q_2}(O_2)...b_{q_T}(O_T) \tag{4.6}$$

Our next objective is to remove Q from the conditioned part of the probability. To do this, we must first calculate $\mathbb{P}(Q|\lambda)$. This is simply the probability of transitioning from q_1 to q_2, q_2 to q_3 etc. More formally we can use matrix A to find the probability of each of these transitions, and since we are finding the total for the entire sequence, we multiply them all together. We start with π_{q_1} as we also need the probability of starting at q_1.

$$\mathbb{P}(Q|\lambda) = \pi_{q_1} a_{q_1 q_2} a_{q_2 q_3} ... a_{q_{T-1} q_T} \tag{4.7}$$

We can now successfully remove Q from the condition using 4.6 and 4.7:

$$\begin{aligned} \mathbb{P}(O, Q|\lambda) &= \mathbb{P}(O|Q, \lambda)\mathbb{P}(Q|\lambda) && (4.8) \\ &= \pi_{q_1} b_{q_1}(O_1) a_{q_1 q_2} b_{q_2}(O_2) a_{q_2 q_3} ... a_{q_{T-1} q_T} b_{q_T}(O_T) && (4.9) \end{aligned}$$

This gives us the joint probability of observations and the internal states. In other words, it provides the probabilty that given observations O and internal state sequence Q was generated by model λ. To achieve our desired probablity all we need to do is get rid of the Q. Since it is another input, all we must do is sum each value of 4.8. As we have accounted for every possible Q, we no longer need to worry about its particular value. This leaves us with:

$$\begin{aligned} \mathbb{P}(O|\lambda) &= \sum_{allQ} \mathbb{P}(O|Q, \lambda) P(Q|\lambda) && (4.10) \\ &= \sum_{q_1, q_2, ..., q_T} \pi_{q_1} b_{q_1}(O_1) a_{q_1 q_2} b_{q_2}(O_2) a_{q_2 q_3} ... a_{q_{T-1} q_T} b_{q_T}(O_T) && (4.11) \end{aligned}$$

Although this solution is correct, calculating this is infeasible because it requires too many computations. For T timesteps and N states, to find every possible Q, we must sum over N^T state sequences. For each timestep we require a multiplication to $a_{q_{i-1} q_i}$ and $b_{q_i}(O_i)$, except the last where there are no transitions, which leads to $2T - 1$ multiplications for each state sequence. Lastly, we require N^T addition operations to sum the result for each state sequence. Our final total number of operations of $(2T - 1)N^T + (N^T - 1)$.

Example 4.3. Lets refer back to 4.2. Suppose we would like to find the probability of a string of 10 observations using model H. Here $N = 3$ and $T = 10$.

$$\begin{aligned} operations &= (2T - 1)N^T + (N^T - 1) && (4.12) \\ &= (20 - 1)3^10 + (3^10 - 1) && (4.13) \\ &= 1180979 && (4.14) \end{aligned}$$

We have a problem as even with a smaller model 4.3 we require a considerable number of calculations. For a moderate to large-sized model, we would require an infeasible amount of calculations. To overcome this problem, we look for a more efficient method, the Forward-Backward algorithm.

4.3.1 Forward-Backward Algorithm

The Forward-Backward algorithm (F-B) comprises two helper functions α and β. We will start by discussing the former.

$$\alpha_t(i) = \mathbb{P}(O_1, O_2, O_3, ..., O_t, q_t = S_i|\lambda) \tag{4.15}$$

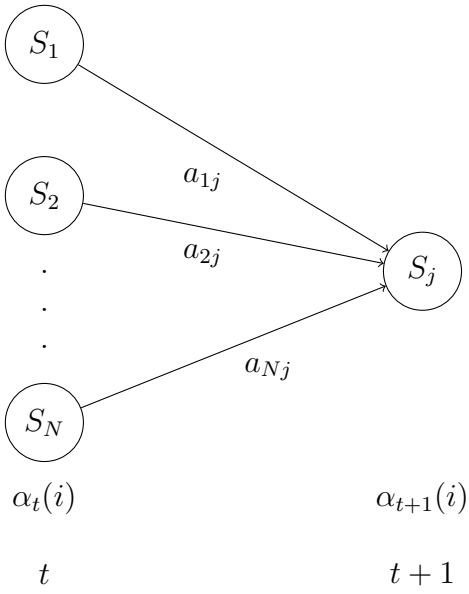
α is an extremly powerful tool in reducing the number of calculations. As given by 4.3.1, it providse the probability that at time t we have seen a sequence of observations and are currently at state $q_t = S_i$. This is not quite $\mathbb{P}(O|\lambda)$ but it represents a part of it. Instead of the probability of the whole sequence, it breaks it into a chunk of size t, commits to end at a particular state and then calculates the same proabbility for this.

We can combine with induction to produce an iterative process that can calculate **??**.

i Base case
For the base case we require the probability of the q_1 being equal to S_1 and giving us observation O_1. The former is addressed by π_i and the later by $b_i(O_1)$. This gives us:

$$\alpha_1(i) = \pi_i b_i(O_1), \quad i \in [1, N] \tag{4.16}$$

ii Inductive step:



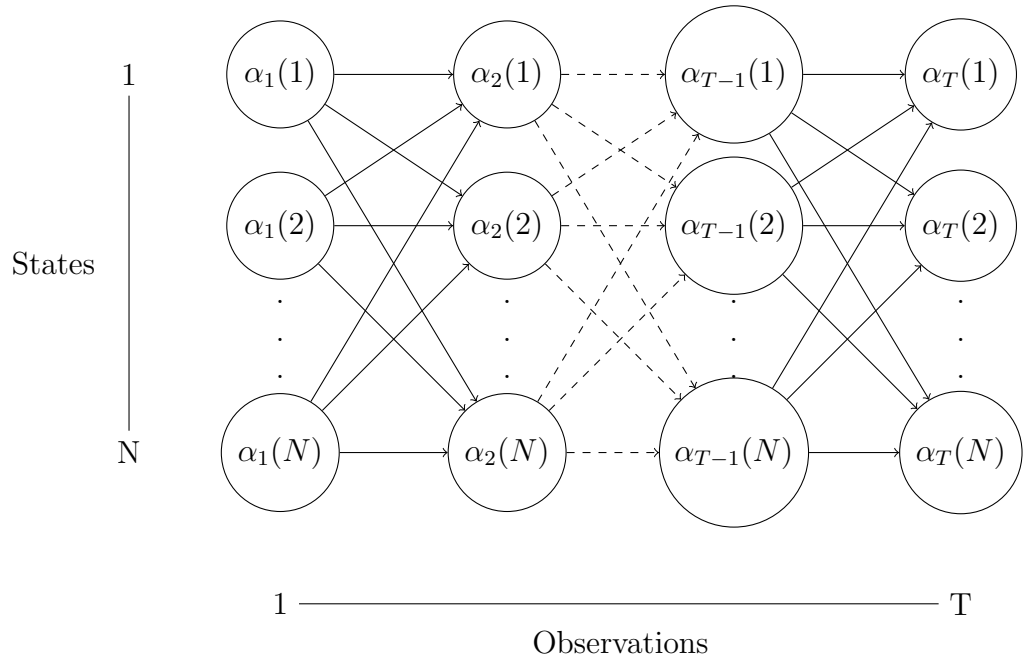
For each state j, for each inductive step, we follow the above method. Each$\alpha_t(i)$ is multiplied by each a_{ij} and then summed up. This resultmultipled by $b_j(O_{t+1}$ gives us $\alpha_{t+1}(j)$.

For the inductive step we must consider how to approach the next timestep. We will again be calculating for all $j \in [1, N]$ and as such must take into consideration, for each j, every possible i. This is again the same set of $[1, N]$. Therefore, to account for all possible previous states and their transition to the current state, we must sum over 1 to N the product of $\alpha_t(i)$ and a_{ij}. For the given observation, as before, we compute $b_j(O_{t+1})$. Additionally, we must stop before reaching the final step as there is no outward transition and thus this would not be applicable. This gives us:

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(O_{t+1}), \quad j \in [1, N], t \in [1, T-1] \tag{4.17}$$

iii Termination step:
For the termination step, we sum over all alpha at the final time T. Each alpha represents the probability of being in that given state at that particular time, through all possible sequences that it may have followed.

$$\mathbb{P}(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i) \tag{4.18}$$



For each α at time t, we require all α at time $t-1$. Thus werequire N^2 calculations for each timestep.

One can see that this method is far more efficient than 4.10. It requires N^2 multiplications for $\alpha_t(i)$ and a_{ij} for T time periods. At each T there are N addition operations for the summation and N multiplication for the $b_j(O_{t+1})$, which is true for all but the first and last timestep, where there are N multiplications and N additions respectively. This gives us $(T-2)(N^2+N+1)+2N$ calculations. Considering the order, we see that the previous method had order $O(TN^T)$ whereas F-B gives us an order of $O(TN^2)$. It is now feasible to compute $\mathbb{P}(O|\lambda)$.

Example 4.4. We attempt to solve the problem 4.3 again but this time using the F-B algorithm. Suppose we would like to find the probability of a string of 10 observations using model H. Here $N = 3$ and $T = 10$.

$$\begin{aligned} operations &= (T-2)(N^2+N+1)+2N && (4.19)\\ &= (10-2)(3^2+10+1)+6 && (4.20)\\ &= 118 && (4.21)\end{aligned}$$

118 is many orders of magnitude smaller than 1180979, which was the requirement without F-B algorithm. Thus it is a significant improvement. The improvement against the base method will be even more drastic for larger models, as F-B has a fixed N^2 instead of the N^T term.

For question 1 this is sufficient. However, we will later require the β, the backward component and as such we will describe it now. Logically it is the same principle as the forward component except this time instead of moving forward step by step we are moving backwards.

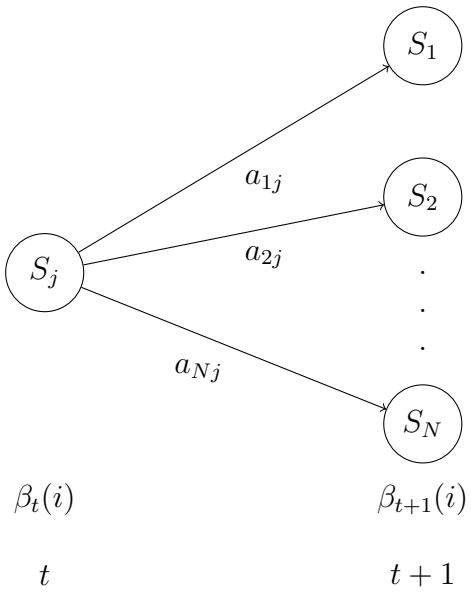
$$\beta_t(i) = \mathbb{P}(O_{t+1}, O_{t+2}, , ..., O_T | q_t = S_i, \lambda) \tag{4.22}$$

This represents the probability of seeing the observations O_{t+1} up to O_T given that at time t the model λ is at state S_i. We again calculate this inductively.

1. Base case
   For each state we must assume that it is the final state in order to iterate from it. This gives us:

$$\beta_T(i) = 1, \quad i \in [1, N] \tag{4.23}$$

2. Inductive step:



Similar to α, we use induction. However, this time we look into the future of the sequence and take steps backwards.

Each inductive step we move back by one timestep. As before, we will be calculating for all N states except this time i will be varying instead of j. This makes sense as we are stepping backward and we want to see which previous state is the most likely previous state. At each step, we the transaction probability of having coming from i, a_{ij} to the probability of seeing the given observation at state S_j, $b_j(O_{t+1})$ and the liklihood of being at that state based on future states $\beta_{t+1}(i)$. This gives us:

$$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(O_{t+1}) \beta_{t+1}(i), \quad i \in [1, N], t \in [1, T-1] \tag{4.24}$$

4.4 Problem 2: Decoding

We now turn our attention to 2. This problem is somewhat more difficult, as the definition of 'best' is quite vague and open to interpretations. An obvious appraoch would be to look for the states that are individually most likely for each state for a set time t given all observations and the model. Lets define this probaility as γ.

$$\gamma_t(i) = \mathbb{P}(q_t = S_i | O, \lambda) \tag{4.25}$$

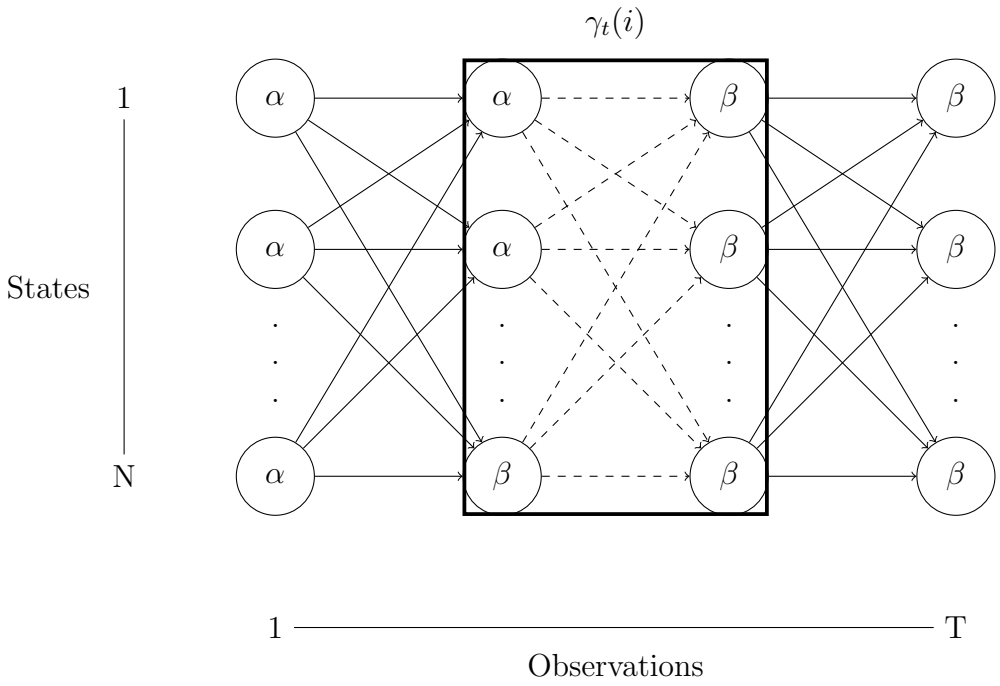
Using 4.3.1 and 4.22 we can solve for γ quite quickly. If we recall, $\alpha_t(i)$ provides us with the probability of being in state i after seeing all the previous observations and $\beta_t(i)$ gives us the probability of being in state i see all the remaining observations in the future. This gives us $\alpha_t(i)\beta_t(i)$. We now normalise this to get a probability, which is possible by dividing by the probability of getting this particular observation given all possible observation sequences $\mathbb{P}(O|\lambda)$. This is equivalent to dividing by the sum of $\alpha_t(j)\beta_t(j)$ over all possible states j.

$$\gamma_t(i) = \frac{\alpha_t(i)\beta_t(i)}{\mathbb{P}(O|\lambda)} \tag{4.26}$$

$$= \frac{\alpha_t(i)\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)} \tag{4.27}$$

This ensures that:

$$\sum_{i=1}^{N} \gamma_t(i) = 1 \tag{4.28}$$



Each dashed line represents a calculation for $\gamma_t(i)$. For each time step, we calculate these and then find the maximum. We select the state corresponding to the maximum and add it to the state sequence.

As such this ensures all $\gamma_t(i)$ are probabilities. We can now go through each timestep t calculating $\gamma_t(i)$ and selecting state i corresponding to the largest $\gamma_t(i)$.

This method will maximise the number of correct states and give the correct answer if the model is completely connected, i.e. each state can reach every other state. If this is not the case, we may get a state sequence that is not possible. e.g. At time t the model is in state i and at time $t+1$ it is in j, but $a_{ij} = 0$, thus is not possible.

In the scenario given, we must redefine 'best' to the path that maximises $\mathbb{P}(Q|O, \lambda)$. Since we are maximising this, we can equivalently maximise $\mathbb{P}(Q, O|\lambda)$. To solve this problem, we require the Viterbi Algorithm.

4.4.1 Viterbi Algorithm

The Viterbi Algorithm Viterbi, 1967, explored in depth in Forney, 1973 follows a similar lattice structure of the F-B algorithm. Similar to how we broke 1 into smaller pieces, we do the same to $\mathbb{P}(Q|O, \lambda)$.

$$\delta_t(i) = max_{q_1,q_2,...,q_t-1}\mathbb{P}(\{q_1, q_2, ...q_t = i\}, O_1, O_2, ..., O_t|\lambda) \tag{4.29}$$

An important point to note is $\delta_t(i)$ does not store the sequence. Thus we must introduce a new variable that will be responsible for doing so. We can call this $\psi_t(i)$, where it is equal to the state we have come from given we are at time t and state i. We will once again use induction. The process is as follows.

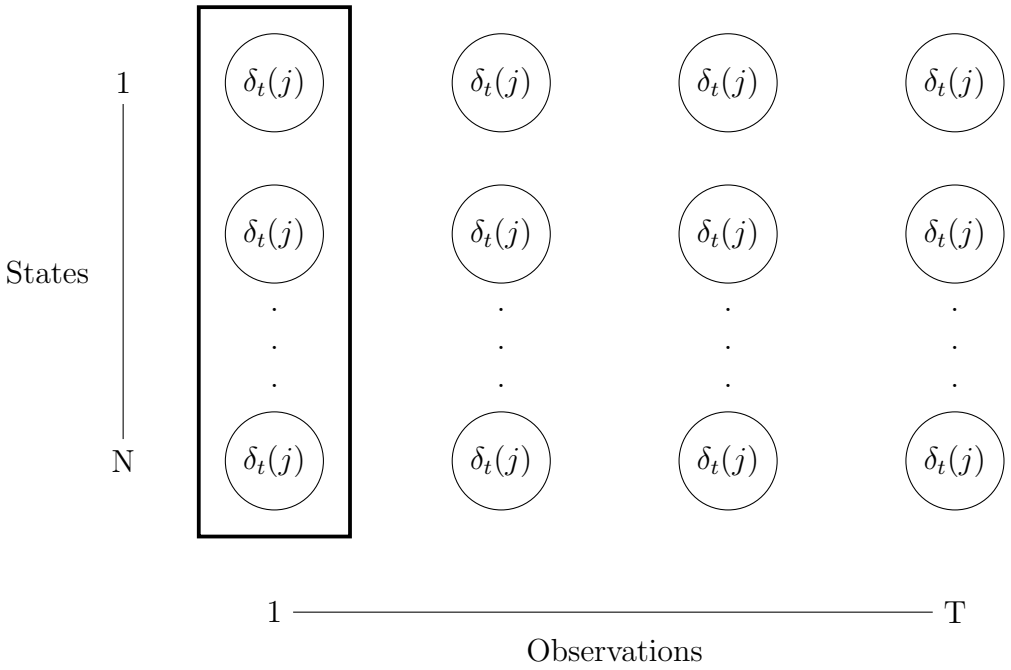
1. Base case
   For each state we must calculate $\pi_i b_i(O_1)$ to determine which intial state is most likely.

We also must set $\psi_1(i) = 0$ as there have not been any states until now.

$$\delta_1(i) = \pi_i b_i(O_1) \tag{4.30}$$
$$\psi_1(i) = 0 \tag{4.31}$$

2. Inductive step:
For each state we calculate the $\delta_{t-1}(i)$ times a_{ij} to find the most likely next state based on previous state liklihood and the transition probability. We maximize this term and the multiply by $b_j(O_t)$ to include a bias based on the observation probabilities. As before we store the argument of the maximum in ψ. This gives us:

$$\delta_t(j) = max_{1\leq i\leq N}[\delta_{t-1}(i)a_{ij}]b_j(O_t), t \in [2, T] \tag{4.32}$$
$$\psi_t(j) = argmax_{1\leq i\leq N}[\delta_{t-1}(i)a_{ij}], j \in [1, N] \tag{4.33}$$



Once $\delta_t(j)$ is calculated for a particular t, we find the maximum, which is the state we add to our state sequence. Since $\delta_t(j)$ accounts for transition probability a_{ij}, impossible transitions will always be equal to 0. Thus unless all transitions are equal to 0, they cannot be the maximum.

3. Termination step:
To terminate this recursion, we need to find the maximum $\delta_T(i)$. Here we maximise, as the values we already calculated the values in the induction. We will store this probability in $\mathbb{P}*$. Similarly, we need to find the argument corresponding to this max for the final state, and we set this to the final state q_T^*.

$$\mathbb{P}^* = max_{1\leq i\leq N}[\delta_T(i)] \tag{4.34}$$
$$q_T^* = argmax_{1\leq i\leq N}[\delta_T(i)] \tag{4.35}$$

To find any particular states that we may be interseted in q_t^*, we use $q_t^* = \psi_{t+1}(q_{t+1}^*)$.

4.5 Problem 3: Learning

The third problem 3 addresses learning. How can we learn a model from given data of observations? This problem is critical as we often do not have the model, and calculating it is not simple. In these cases, we must derive the model from the data we have, a set of observations. Given the set of observations $O = \{O_1, O_2, ..., O_T\}$ we want to find the most suitable $\lambda = (A, B, \pi)$. To do this, we use the Baum-Welch algorithm.

4.5.1 Baum-Welch Algorithm

For the Baum-Welch algorithm we will need all three parameters intrudced earlier; α 4.3.1, β 4.22 and γ 4.25. We will also need a new parameter $\xi_t(i, j)$. This will capture the probability of being in some state S_i at time t and then S_j at time $t + 1$ given the observations and the model.

$$\xi_t(i, j) = \mathbb{P}(q_t = S_i, q_{t+1} = S_j | O, \lambda) \tag{4.36}$$

To solve this problem we can refer back to 4.3.1 for the left hand side (including q_t) and 4.22 for the right hand side (indlucing q_{t+1}). To get from state i to j we use $a_{ij}b_j(O_{t+1})$. Putting this all togther we get a liklihood. To normalize this into a probabiltiy we once more use $\mathbb{P}(O|\lambda)$. This gives us:

$$\xi_t(i, j) = \frac{\alpha_t(i)a_{ij}b_j(O_{t+1})\beta_{t+1}(j)}{\mathbb{P}(O|\lambda)} \tag{4.37}$$

Since we are interested in the probability of going from one i to one j, for the denominator we use the sum of all i and all j.

$$\xi_t(i, j) = \frac{\alpha_t(i)a_{ij}b_j(O_{t+1})\beta_{t+1}(j)}{\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_t(i)a_{ij}b_j(O_{t+1})\beta_{t+1}(j)} \tag{4.38}$$

One may find it helpful to compare $\xi_t(i, j)$ with $\gamma_t(i)$. $\gamma_t(i)$ represents the probability of being at state i at timestep t. Summing over j for $\xi_t(i, j)$ leaves us with $\xi_t(i)$ which represents the probability of being at state i at timestep t. This is identical to $\gamma_t(i)$. We can now state:

$$\gamma_t(i) = \sum_{j=1}^{N}\xi_t(i, j) \tag{4.39}$$

For $\gamma_t(i)$, if we sum over t we can get a number that can be used as the expected number of times S_i is visited. Thus the expected number of transitions from S_i can be calculated by:

$$\sum_{t=1}^{T-1}\gamma_t(i) \tag{4.40}$$

Similarly for $\xi_t(i)$, if we sum over t we can get a number that can be used as the expected number of times S_i transitions to S_j.Thus the expected number of transitions from S_i to S_j can be calculated by:

$$\sum_{t=1}^{T-1}\xi_t(i, j) \tag{4.41}$$

Now that we have created a set of tools, we will need to tackle the learning problem itself. To build a improved λ, $\bar{\lambda}$, we need to find $\bar{\pi}$, $\bar{A}$ and $\bar{B}$.

i $\bar{\pi}$
Since $\gamma_t(i)$ represents the expected frequency in S_i at time t, if we let t=1 this now is equvialent to $\bar{\pi}$

$$\bar{\pi} = \gamma_1(i) \tag{4.42}$$

ii $\bar{a_{ij}}$
Here we can use the expected number of transitions from state i to j divided by expected number of transitions from i in total. This will provide the appropriate transition probability.

$$\bar{a_{ij}} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)} \tag{4.43}$$

iii $\bar{b_j}(k)$
For the B matrix, we will need to divide the expected number of times the model is in state j and observes v_k by the expected number of times it is j.

$$\bar{b_j}(k) = \frac{\sum_{t=1,s.t.O_t=v_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T-1} \gamma_t(j)} \tag{4.44}$$

We have used our λ and O to produce our parameters: $\alpha_t(i)$, $beta_t(i)$, $\gamma_t(i)$, $\xi_t(i,j)$. This is called parameter re-estimation. We have then used these parameters to produce our $\bar{\lambda}$ = $\{\bar{\pi}, \bar{a_{ij}}, \bar{b_j}(k)\}$. This is called expectation maximization. This method can be iterated and each iteration should provide an improvement for lambda or worst case senario be equvialent. Proof of this can be found in Leonard E Baum et al., n.d. Once lambda stabalizes and is no longer changing, we can assume a local optimumum has been reached.

4.6 Modified HMM

4.6.1 GMM

Chapter 5

Model Selection / Parameter Estimation

Two models can use exteremly different technqiues to predict the same thing. Thus being able to compare these models is vital. Even if the models are the same, for parameter estimation, we must compare the Performance of the model with multiple sets of paramters. To do so we need a metric that we can compare accross all models and parameter estimations. This metric is given by the likelihood function.

Definition 5.1. Likelihood Function
Given model θ and the observation $x_1, x_2, ..., x_n$, the likelihood function is given by

$$L(\theta|x_1, x_2, ..., x_n) = \mathbb{P}(x_1, x_2, ..., x_n|\theta) \tag{5.1}$$

In other words it is given by the probability that the given model produced the given observations.

This function allows us to directly compare which model has a higher probability of fitting the system, based on the data. To do so we use Maximum likelihood estiamtion.

5.1 Maximum Liklihood Estimators

Through maximizing the likelihood function 5.1, we can find the model that best fits the given observations. This process is called Maximum Likelihood estimation.

We often use $\log(L(\theta|x_1, x_2, ..., x_n))$ instead of $L(\theta|x_1, x_2, ..., x_n)$ as this often leads to simplier differenciation. This is possible since the log function is monotonically increassing thus they have their maximum at the same value of θ.

The method can be described as follows:

i Calculate the joint probability density of observeriving your data $X_1 = x_1, ..., X_n = x_n$ given your model θ, i.e.

$$\mathbb{P}(X_1 = x_1, ..., X_n = x_n|\theta) \tag{5.2}$$

ii Take the Natural logarithm of both sides.

$$\log(\mathbb{P}(X_1 = x_1, ..., X_n = x_n|\theta)) \tag{5.3}$$

iii Take the partial derivative with respect to each parameter.

iv For each paramter, set the derivative equal to 0 and rearrange for its value. The given value will be the maximum likelihood estimation of that parameter.

Unfortunately in real world applications the derivative of the log-Liklihood is analytically intractable. This is particurally true for Hidden Markov based models. Thus we often require alternative methods to compare models.

5.2 Approximate Bayesian Computation

Approximate Bayesian Computation (ABC) is an alternative method of parameter estiamtion and model Selection. Using Bayesian inference, we calcuate the posterior distribution, one containing information from botht the data and the prior distribtuion.

We uses Bayes original theorem applied to point probabilities and observations y to obtain the following:

$$\pi(\theta|y) \propto \pi(\theta)f(y|\theta) \tag{5.4}$$

where $\pi(\theta|y)$ is the probability distribtuion of the paramters given the data (posterior distribtuion), $\pi(\theta)$ is the probability distribtuion of the paramters before we take into account the evidence (prior distribtuion) and $f(y|\theta)$ is the likelihood.

Instead of calculating the probability directly, using some preset parameters ABC creates a simulation and then compares this to the observation dataset. The method can be described as follows: **********REFFERENCE**********

If N iterations are desired for i=1 to N:

1. Generate simulation θ' from the prior distribution π(data).
2. Generate Z (new value for parameter/s) from the likelihood $f(\text{data}|\theta')$
3. If distance between statistic η of y and z is less than or equal to some ϵ set $\theta_i = \theta'$ else repeat until condition is fullfilled.

Given a small enough tolerence (ϵ) and appropriate statistic η we have:

$$\pi_\epsilon(\theta|y) = \int \pi_{epsilon}(\theta, z|y)dz \approx \pi(\theta|y) \tag{5.5}$$

as the actual posterior distribtuion is the limit of 5.5 as ϵ goes to 0.

Chapter 6

Spacio-Temporal Rainfall Models

6.1 Coppertwait

Coppertwait's rainfall models have been used for decades with a good deal of sucess. Although we will not dive deep into his methodology the following papers may be refferenced if one wishes too **ref all coppertwait papers**.

Over time, Coppertwait has improved and adjusted his model for varying applications. For example, in one of his more recent papers **ref 2010 paper**, Coppertwait has turned the discrete variable describing the type of storm into a continous one. This changes and developments, while interesting, are not the focus of our paper. Instead, we will focus on **ref 1994 paper** to understand the model at its simplicity and one may apply these ideas to a Hidden Markov Model.

All of Coppertwaits models use a Neyman-Scott point processes to describe the arrival of Storms. The model itself can be described as follows:

- The origin of each storm is a Poisson process in space-time
- Each storm is a circular region in two dimensional space with radii given by independent exponential random variables.
- Each storm generates a cluster of two dimensional circular rain cells with random duration, poistion and radius.
- The duration of each cell is an exponential random variable.
- The intensity of each cell is a random exponential distributed but dependent on the type of cell. In **ref 1994** we have two cell types, light and heavy.
- Overall intensity at any point is given by the sum of overlaping intensities at the point in space and time.

The GNSRP Model, discussed in **ref paper 1994**, is a generalisation of the Neyman-Scott-Rectangular-Pulse (NSRP) Model with only one cell type. The key feature of the NSRP is that, in addition to those above, the distance from Cell origins to its Storm origin are independently exponentially distributed. Unlike other models that used and independent uniform distribution for this, NSRP accounted for the realistic effect that rainfall tends to be more intense near the center of the storm ** ref browning**.

6.2 Grando - HMM Based Model

6.2.1 Model

Grando extended on Coppertwait's ideas in **ref paper**. She completely removes the double disc structure and attempts to enclose the correlations within the larger Storm discs from Coppertwait in Hidden Markov States.

She defines her discrete time intensity process as follows: For $n \geq 1$ and $x \in A$, where $A \subset \mathbb{R}^2$

$$I_n(x) := \sum_{i=1}^{N^{(n)}} \sigma_i^{(n)} \mathbb{I}_{D(x_i^{(n)}, R_i^{(n)})}(x) \tag{6.1}$$

where:

- $I_n(x)$ is the rain intensity
- $N^{(n)} \sim \text{Pois}(\lambda)$ represents the number of rain discs generated
- $\sigma_i^{(n)} \sim \text{Exp}(\tau)$ is a random variable providing the rain intensity
- $D(x_i^{(n)}, R_i^{(n)})$ is a rain disc with centre $x_i^{(n)} \sim \text{Unif}(A)$ and radius $R_i^{(n)} \sim \text{Exp}(\xi)$.
- $\mathbb{I}_{D(x_i^{(n)}, R_i^{(n)})}(x)$ is an indicator function, equal to 1 if the current rain disc overlaps position x.

Parameters λ,ξ and τ are taken to be fixed for each type of storm. In other words, for each hidden state of the HMM there is a unique λ,ξ and τ. Informally, the model has hidden states which determine the type of storm. This then dictates the values of the three above parameters, which in turn influence the intensity of rain observed.

Thus we arrive at the learning problem. Due to the added complexity of additional parameters we are unable to directly use methods for HMMs discussed earlier in the paper. Direct likelihood calculation is complex for HMMs alone, after we introduce these additional parameters it is not feasible.

Given the model has m states, we will have $m^2 + 4m$ unknown parameters, m^2 for the transition probabilities, m for the intitial distribution and m for each of λ,ξ and τ. This is clearly a high-dimensional problem. To overcome this Grando fixes certain paramters in order to gain a better understanding of the remaining. She uses two methods; Nealder-Mead and Approximate Bayesian Computation. We will focus on the latter as it has provided superior results in her testing.

6.2.2 Model Fitting

We begin by trying to replicate Grando's results in order to establish a starting point. For ABC we require the prior distribution along with a summary statistic. Deriving the prior distribution is not a trivial problem. We not only have parameters for the HMM as well as parameters for the observations but have to extract both from only the intensity values.

Grando determines that even without the HMM parameters, the problem is high dimensional ($3m$ paramters). Due to this she decides to fix all HMM paramters to uniform(0,1) for each test, whilst insuring the transition matrix is stochastic through normalization of rows.

She believes that this will allow her to retrieve close approximations of all λ,ξ and τ from the data, which she can then use to calculate the HMM paramters.

For each simulation she will generate a new value for all λ,ξ and τ. Then plot the summary statistic against the values for each and then find areas of high concentration to decide what a sutiable approximation for the true value of each is. Here she is setting her ABC tolerance to ∞ and then manually narrowing her range through the graph.

Grando picks values for each paramter using the uniform distribution between preset values. She says "This technique is purely heuristic and it is only aimed to explore the problem, as decisions made are completely arbitrary.". Theoretically we could pick any values for but for consistency we will match the values she has used.

She decides to use the mean intensity as her summary statistic for ABC. She calculates this for both the simulated and given data. She then calculates the normalised Euclidean distance between the two as her metric for model performance.

6.3 Numerical Results

We will be using the same algorithm, same data and same parameters as Grando to ensure consistency. Our data measures rainfall across 30 sites in Germany over the period from 1st of january 1931 to 31st December 2000. Co-ordinate data for each site is also given in an additional file. These are labeled as Easting and Northing.

Grando fixes m=3, allowing for a low, medium and high state for rainfall. This gives us 21 unknown parameters of which we are attempting to estimate 9.

All programs have been run on the personal desktop (AMD Ryzen™ 9 3900X with 12 Cores and 24 Threads running from 3.8GHz up to 4.6GHz) using C++20.

6.3.1 Code

To ensure validity of results, we wanted to run multiple tests. Grando's code was written in R and required multiple hours to produce a result. To improve on this we have rewritten the program in C++ and used Python on Jupyter Notebooks for analysis.

Data preperation took place in excel. For simplicity in importing into C++, the data needed to be configured in a particular way. This is described throughly in the notes given with the code.

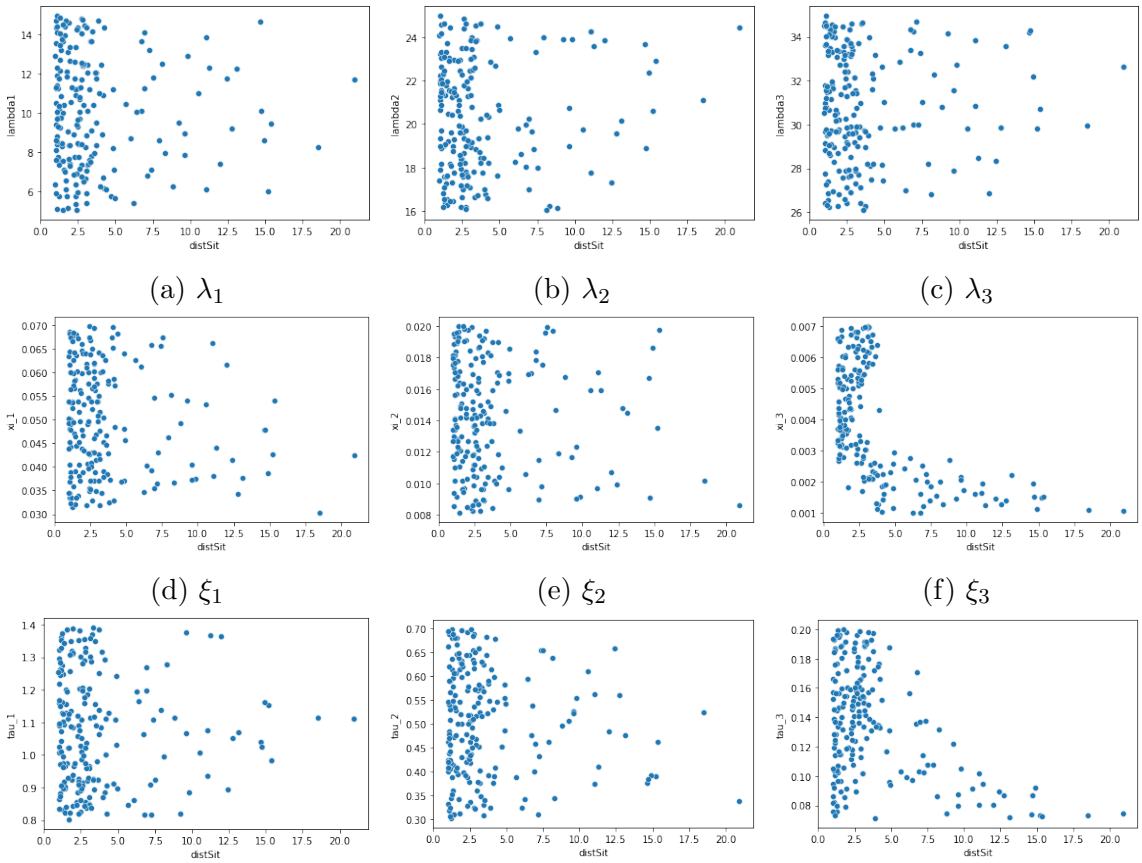
C++ was used for the actual simulation as this was the slowest part of Grandos algorithm. Many parameters have been hardcoded. This is to allow for efficiency and can later be altered if needed. Notes on mininpulating the code can be found **reference to later**. Multi-threading has been used for further reduction in run-time. Admittdely the code can be further parallised but in its current state it is acceptable; run times of under 5 minutes.

Analysis was done using Python in Jupyter Notebooks. The C++ program created multiple CSV files as output and these were taken as input for analysis.

6.3.2 Results - Nine Parameter Estimation

We begin with using ABC to estimate all parameters, found on pages 75-76 **Grando ref**. The parameters are picked as follows:

- $\lambda_1 \sim$ Unif(5,15), $\lambda_2 \sim$ Unif(16,25) and $\lambda_3 \sim$ Unif(26,35)



(a) λ_1 (b) λ_2 (c) λ_3

(d) ξ_1 (e) ξ_2 (f) ξ_3

(g) τ_1 (h) τ_2 (i) τ_3

Figure 6.1: Parameter values plotted against corresponding normalised Euclidean Distance between simulation and data means.

- $\xi_1 \sim$ Unif(0.03,0.07), $\xi_2 \sim$ Unif(0.008,0.02) and $\xi_3 \sim$ Unif(0.001,0.007)
- $\tau_1 \sim$ Unif(0.8,1.4), $\tau_2 \sim$ Unif(0.3,0.7) and $\tau_3 \sim$ Unif(0.07,0.2)

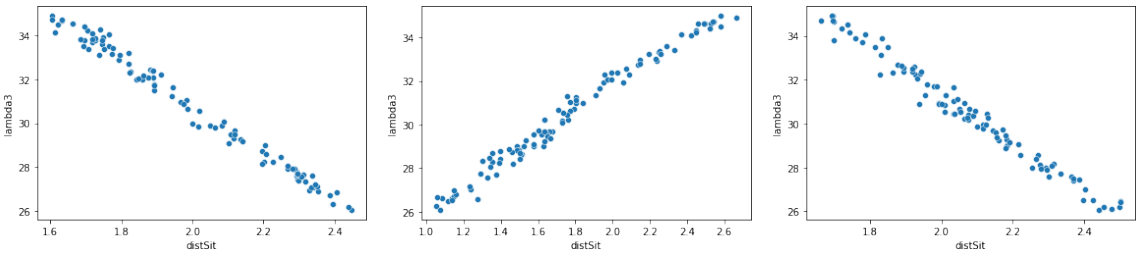
For each simulation we generate data for 365x5 days and calculate the normalised Euclidean distance between the mean of these and the mean of the whole given dataset. Due to random sampling we attempt to find estimates for all 9 parameters simultaneously. We completed 200 iterations on the personal desktop in 42.2841 seconds.

The results show ABC on nine parameters was not successful. Figure 6.1 shows for all parameters an extremly unifrom scatter suggesting the posterior distribution is still uniform in a neighbourhood of the true parameters. This is an exact match to Grando's results.

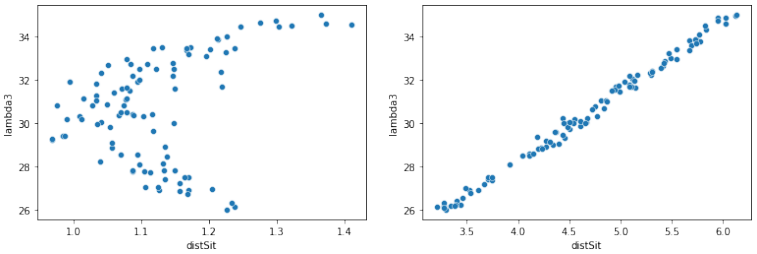
6.3.3 Results - Three Parameter Estimation

Grando then decides to perform ABC on a subset of our nine parameters. She claims choosing only λ_i,only ξ_i or only τ_i does not yield better outcomes. She then goes on to test for only λ_3,ξ_3 and τ_3, fixing all other parameters to abritary values. Once again, for consistency we choose values to match those of Grando.

- $\lambda_1 = 10$, $\lambda_2 = 20$ and $\lambda_3 = 30$



(a) Attempt 1 (b) Attempt 2 (c) Attempt 3



(d) Attempt 4 (e) Attempt 5

Figure 6.2: λ_3 values plotted against corresponding normalised Euclidean Distance between simulation and data means for 5 attempts.

- $\xi_1 = 0.05$, $\xi_2 = 0.01$ and $\lambda_3 = 0.005$
- $\tau_1 = 1.1$, $\tau_2 = 0.5$ and $\tau_3 = 0.1$

For each test we require three simulations. When computing for λ_3 we change its value from 30 to $\lambda_3 \sim$ Unif(26,35). Similarly, when computing for ξ_3, we change its value to $\xi_3 \sim$ Unif(0.001,0.007) and when computing for τ_3 we change its value to $\tau_3 \sim$ Unif(0.07,0.2). At any given time there is one random variable between our nine that we are testing. For efficiency we process this over multiple threads. We completed 100 iterations and ran 5 attempts on the personal desktop. Attempt time ranged between 152.725 in to 199.386 seconds.

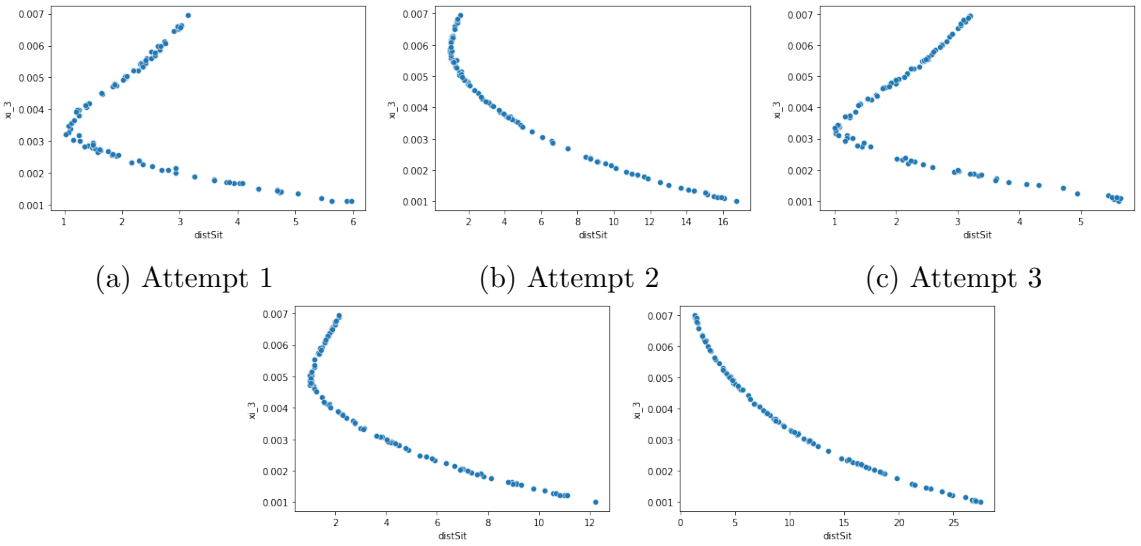
The graphs found in Figures 6.2, 6.3 and 6.4 correspond to the ones found in Grando's *** REF** paper pages 78-80. We have come accross unexpected results. Although the graphs show a similar pattern, there is an inconnsistency in values.

For λ_3 Grando's estimate was 80.436. This was after running the algorithm in a larger interval. Although we have no done this, our data is sufficient to draw conclusions. We are expecting a minimum distance, in other words we expect the graph be a parabola in y; the parameter values.

- The negative slope of 6.2a and 6.2c suggests the true value is larger than our upper limit of 35.
- The positive slope of 6.2b and 6.2e suggests the true value is lower than our lower bound of 26.
- 6.2d suggests the true value is somewhere within our interval of [26,35].

It is clear that not all three can be true as each contridicts the other. This suggests the value of λ_3 is changing through new attempts.

For ξ_3 Grando's estimate was 0.0027. From our results we can deduce the following:



(a) Attempt 1 (b) Attempt 2 (c) Attempt 3

(d) Attempt 4 (e) Attempt 5

Figure 6.3: ξ_3 values plotted against corresponding normalised Euclidean Distance between simulation and data means for 5 attempts.

- Both 6.3a and 6.3c suggest the true value is around 0.003.
- 6.3b and 6.2d suggest the true value is between 0.005 and 0.006.
- The negative gradient of 6.3e suggests the true value larger than our upper limit of 0.007.

Although it seems the value remains within the interval most of the time, it does not remain consistent. Again, as with λ_3, these changes in estimated value suggest the true value may be changing through new attempts.

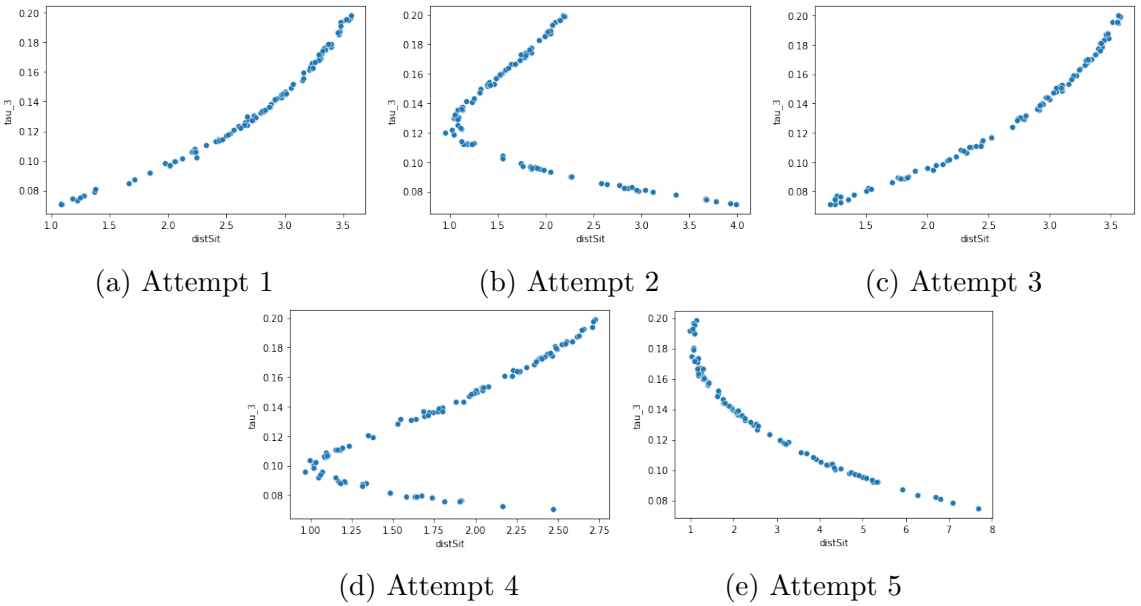
For τ_3 Grando's estimate was 0.0386. From our results we can deduce the following:

- The positive slope of 6.3a and 6.3c suggest the true value is below our lower bound of 0.07.
- 6.4b suggests the true value is around 0.13.
- 6.4d suggests the true value is around 0.10.
- The negative slope of 6.4e suggests the true value is above our upper limit of 0.2.

Once again, multiple attempts provide contradicting results. This suggests the true value is chaning with each attempt.

6.3.4 Adjusted ABC Algorithm

From all 6.2, 6.4 and 6.3 we have the same indication; the parameter values are changing for new attempts. To justify this we must isolate the cause of this change. Our model requires our nine paramters to be constants. Furthermore, we have controlled the values of each in our testing, thus any change in paramter value must be caused by external factors. The only other factors within the model are the hidden markov paramters.



(a) Attempt 1 (b) Attempt 2 (c) Attempt 3

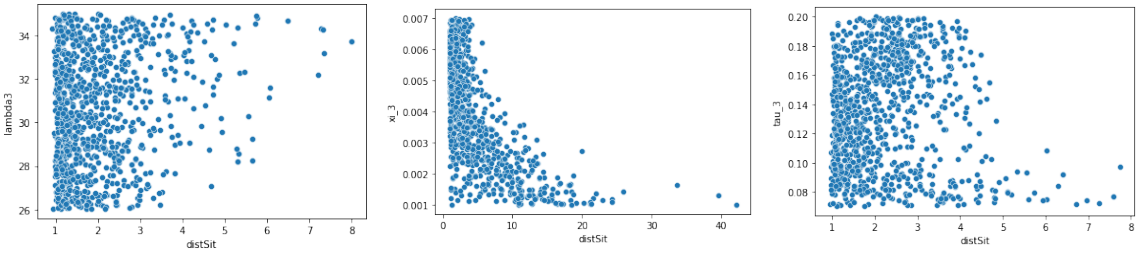
(d) Attempt 4 (e) Attempt 5

Figure 6.4: τ_3 values plotted against corresponding normalised Euclidean Distance between simulation and data means for 5 attempts.

Looking closely at our simulation algorithm, one may notice for each attempt the HMM parameters are generated by a uniform random variable and then kept the same for each simulation. In other words, they only change on new attempts. When we estiamte the parameters using ABC, we are conditioning on the paramters values we have preset; in particular the HMM paramters. Since these vary for each attempt, the condition our algorithm is based on varies for each attempt. Thus, we can confirm the model is different for each attempt and the parameters estimates are infact also different for each attempt. We can conclude ABC with the given algorithm has failed to provide parameter estiamtes. The increase in efficiency of the ABC algorithm has allowed for multiple attempts which has highlighted a flaw in the algorithm itself.

We can further highlight this issue through an adjustment to the ABC algorithm. We move the generation of HMM paramters to within the iterative process. This forces a new set of HMM paramters for each simulation. Whilst this would not be accurate to the model, this randomisation prevents the method providing parameter estiamtes that fit to a particular set of HMM paramters. We use this for three parameter estiamtion. We increase the number of iterations to 1000, to ensure we can find a pattern if there is one as there is an increase in randomisation. Using the same paramter values as 6.3.3, computation on the personal desktop took 652.842 seconds.

As we can see from Figures 6.5, we have a somewhat uniform scatter. This suggests we are unable to find an parameter estimate for our non-HMM paramters without fixing the HMM paramters. Thus we must search for an alternative method of parameter estimation. Our method must support high-dimensional paramter estimation in our current model we have 21 unkowns.



(a) λ_3 (b) ξ_3 (c) τ_3

Figure 6.5: Parameter values plotted against corresponding normalised Euclidean Distance between simulation and data means for simulation calculated using adjusted ABC algorithm

Chapter 7

HMM based Rainfall Models

In this chapter we will extend on Grando's ideas, adapt her model and try to decide if HMMs are useful in rainfall models. To awnser the question we must solve a 21-dimensional problem. Before we attempt to tackle this, we try a single state model to determine the performance before introducing HMM.

7.1 Extending on Grando's Model

In 6.2 we followed Grando's methodology for consistency in replication. However, we now aim to extend on her work and thus will begin by modifying her algorithm for improved efficiency.

Grando calculates the number of storm discs over the entire space (all sites),selects the discs within the current site and then repeats for all sites. This same process could be made more efficient by simulating for only one site at a time and removing the check to see if the disc covers a given site. Through this method we remove the generation, checking and discarding of a large number of poisson variables at each iteration.

7.2 Single State Model

7.3 Single State Model Numerical Results

Chapter 8

HMMs based Rainfall Models

8.1 Extending on Grando's Model

8.2 Single State Model

8.3 Single State Model Numerical Results

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