Hidden Markov Models

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Preface

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Chapter 1

Introduction

The goal of this project is to determine if HMMs are suitable as rain generators.

The first task will be to extend on the work done by Grando. In her testing, she has come to the conclusion that HMMs are suitable as rain generators however she has used the same data for testing as she used for training. This, quite likely, has lead to bias and thus we will extend her work by conducting out-of-sample tests.

If possible, I will build the software so it is user friendly and efficient. With this, I can test data for multiple locations. This will allow me to understand if the result is truly signficant, at least more so than just one location.

Chapter 2

Preliminaries

In this section, we will briefly visit foundations on which we will build throughout this paper. For most, this will be a simple refresher.

2.1 Mathematical Foundations

We start with a few key mathematical concepts.

2.1.1 Probability Theory

To discuss any probablisitic ideas we must first understand general probability theory. This can be done through the definition of a probability space.

Definition 2.1. Probability Space
A probability space is defined by $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is the non-empty set of all possible outcomes, such that all events $\omega \in \Omega$. $\mathbb{P}$ is a probability measure, a function $\mathbb{P}$(A) that maps event A to a number within [0,1] based on the liklihood of the event. $\mathcal{F}$ is a σ-algebra on Ω if

1. $\Omega \in \mathcal{F}$
2. A $\in \mathcal{F}$ implies $A^c \in \mathcal{F}$
3. if A_1, A_2, A_3,... are in $\mathcal{F}$ then so is $A_1 \cup A_2 \cup A_3$...

2.1.2 Conditional Probability

Sometimes we require the probability of an event assuming another event has occured. In such situations we require conditional probability. Given two events A and B, the probability of event A occuring conditioned on the occurance of event B can be calculated as below.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)},\ \forall A \in \mathcal{F} \tag{2.1}$$

From 2.1 and the fact that for dependednt events $\mathbb{P}$(A $\cap$ B) = $\mathbb{P}$(B $\cap$ A) we can see that:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A),\ \forall A, B \in \mathcal{F} \tag{2.2}$$

Subsituting 2.2 into 2.1 we get the famous Bayes Theorem.

Theorem 2.2. *Bayes' Theorem*
For dependent events A and B with probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *, where* $\mathbb{P}$ *(B)* $\neq$ *0,*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)},\ \forall A \in \mathcal{F} \tag{2.3}$$

2.1.3 Stochastic Process

To be able to define a Markov model, of any kind, we must first define a stochastic process.

Definition 2.3. Stochastic Process
Given an ordered set T and probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a stochastic process is a collection of random variables X = $\{X_t;\ t \in T\}$. Based on t $\in$ T and $\omega \in \Omega$ we get a numerical realization of the process. For simplicity, this may be viewed as a function; $X_t(\omega)$.

2.2 Applied Foundations

Chapter 3

Standard Markov and Markov Property

3.1 History

Andrei Markov discovered the Markov model while analyzing the relationship between consecutive letters from text in the Russian novel ”Eugene Onegin”. With a two state model (states Vowel and Consonant) he proved that the probabililty of letters being in a particular state are not independent. Given the current state he could probabilistically predict the next. This chain of states, with various probabilities to and from each state, formed the foundation of the Markov Chain.

3.2 Markov Chain

To define a Markov chain we must first address the Markov property.

Definition 3.1. Markov Property
Let $\{X_t \ ; \ \mathrm{t} \in \mathbb{N}_0\}$ denote a stochastic proccess 2.3, where t represents discrete time. The process has the Markov property if and only if,

$$\mathbb{P}\{X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, ..., X_0 = i_0\} = \mathbb{P}\{X_{n+1} = i_{n+1} | X_n = i_n\} \tag{3.1}$$

Definition 3.2. Markov Chain
A stochastic process $\{X_t \ ; \ \mathrm{t} > 0\}$ is a Markov Chain if and only if it satisfies the Markov property 3.1.

To be able to use a Markov chain, we must first be able to define its transition probabilities. This becomes complex if the probabilities vary with time. Thus, we usually assume the probabilities are constant. These special Markov chains are called time-homogenous.

Definition 3.3. Time homogenous
Let $\{X_t \ ; \ \mathrm{t} \in \mathbb{N}_0\}$ denote a stochastic proccess 2.3, where t represents discrete time, and $p(i, j)$ represent the transition probability from state i to state j.

$$\mathbb{P}\{X_n = j | X_{n-1} = i\} = p(i, j), \forall n \in \mathbb{N}_0 \tag{3.2}$$

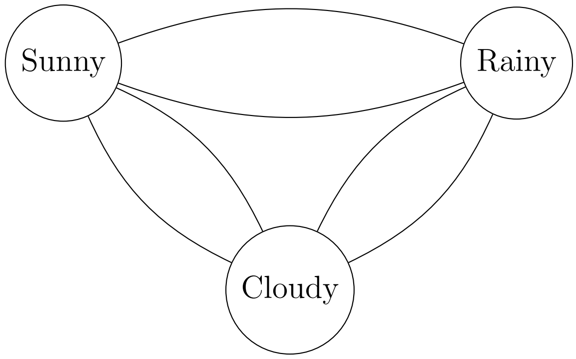
Given a time-homogenous Markov chain, we can create a 2-dimensional N x N matrix of transition probabilities, where N is the cardinality of the state space.Unique Markov chains have unique transition matrices. These matrices can be defined as below:

$$P = \{p(i,j) = \mathbb{P}\{X_n = j | X_{n-1} = i\}\}_{1 \leq i,j \leq N} \tag{3.3}$$

3.3 Applications of the Markov Model

Markov models have many applications. To demonstrate we will present an example where the weather is represented by the states.

Example 3.4. Let $\{X_t; t \in \mathbb{N}_0\}$ denote a Markov Chain, with state space S = {rainy, sunny, cloudy}, where t represents the number of days from start. Since any state can transition into any other state, we can say this model is ergodic. This can also be seen through fig() as each state is connected to all others.



3.4 Motivating the Hidden Markov Model

These examples should show the need for adaptations to simple markov models in particular cases.

Example 3.5.

Show how MM is a type of HMM.

Chapter 4

Hidden Markov Model

4.1 Derivation

Explain all 5 inputs for HMM using example from previous chapter

Definition 4.1. Define HMM

Provide notation that will be used for the rest of the paper.

4.2 Applications of HMM

As with any mathematical model, we can use HMMs for either verification or prediction. Through the output liklihood we can find the liklihood of paticular sequences and through the given probablities we can find the most likely states in the future.

4.2.1 Predicitive Model

4.2.2 Sequence Prediction

4.2.3 Three Key Problems

4.3 Problem 1: Evaluation

4.3.1 Forward-Backward Algorithm

4.4 Problem 2: Decoding

4.4.1 Viterbi Algorithm

4.5 Problem 3: Learning

4.5.1 Expectation Maximization

4.5.2 Baum-Welch Algorithm

4.6 Modified HMM

4.6.1 GMM

Bibliography

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