Asian and lookback Option Pricing using Monte Carlo simulations

1. Numerical methods

1.1 Monte Carlo simulations

In option pricing through Monte Carlo simulation, we generate random price movements for the underlying asset utilising a geometric Brownian motion. These simulated price paths play a crucial role in defining the option's payoff at its expiration. Through repeated simulations and the averaging of resulting payoffs, analysts can gauge the expected value of the option and assess its associated risk factors, including the probability of it expiring in or out of the money. This method entails simulating potential price trajectories for the underlying asset, computing the option's payoff for each trajectory, and then averaging these payoffs while discounting them back to present value. Ultimately, this Monte Carlo approach provides a means to determine the option's price effectively.

1.2 Euler-Maruyama Scheme

The Euler-Maruyama scheme is utilized to simulate the underlying stock price, which adheres to geometric Brownian motion (GBM). This motion is governed by the equation:

$$dS_t = rS_t dt + \sigma S_t dW_t \qquad (1) \tag{1}$$

where S_t is the price of the underlying at time t, σ is the constant volatility, r is the constant risk-free interest rate and W is a brownian motion.

The Euler-Maruyama method provides a convenient discrete time solution (2) to the continous time equation (1):

$$S_{t+\delta t} = S_t \left(1 + r\delta t + \sigma\phi\sqrt{\delta t}\right) \qquad (2)$$

where δt represents a discrete time-step and ϕ is a random variable of a standard normal distribution (mean = 0 and standard deviation = 1).

1.3 Stock paths generation

The first step is to simulate paths using the Euler-Maruyama Scheme with the following parameters:

Today's stock price S_t = S_0 = 100

Strike price = 100

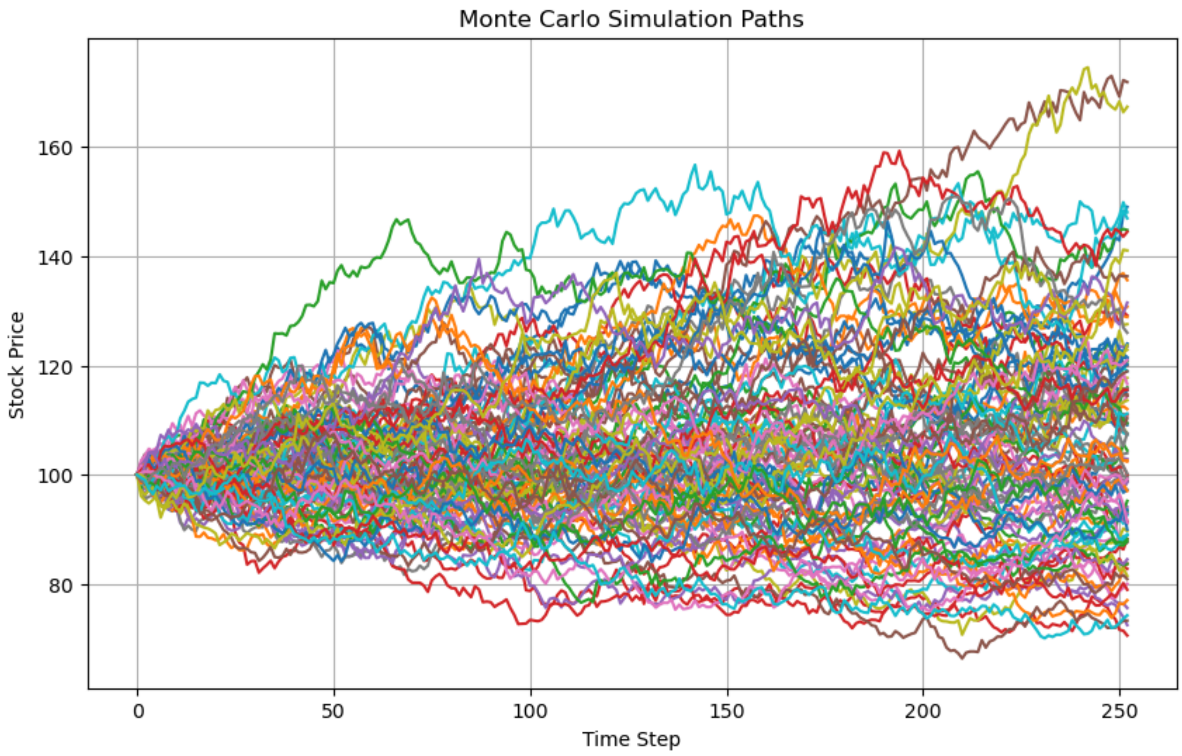
Time to expiry = (T-t)

volatility σ = 20%

risk-free interest rate = 5%

In [85]:
```
# importing our own code to simulate and visualise paths
from MonteCarlo import MonteCarloSimulations

mc_sim = MonteCarloSimulations(s0=100, expiry=1, nbr_sim=100, time_step=252, sigma=
paths = mc_sim.get_paths()
mc_sim.plot_paths(paths)
```



2. Asian and lookback options pricing

An option price can be defined as the expected value of the discounted payoff under the risk-neutral density $\mathbb{Q}$:

$$V(S,t) = \mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\mathbb{E}^{\mathbb{Q}}[\mathrm{Payoff}(\mathrm{S_T})] \tag{2}$$

European and exotic options have distinct payoff functions. In the case of a call Asian option, the payoff hinges on the disparity between the average value of the underlying stock and the strike price, but only if this average surpasses the strike price. This relationship can be expressed succinctly as follows:

$$\mathrm{Payoff} = \max\left(0, \left[\frac{1}{n}\sum_{i=1}^{n} S_i - K\right]\right)$$

The payoff function of a call lookback option entails calculating the difference between the maximum value attained by the underlying stock and the strike price, provided that the maximum stock value exceeds the strike price. This can be expressed as follows:

$$\text{Payoff} = \max(S_{\max} - K, 0)$$

where:

- S_i is the price of the underlying asset at time i.
- S_{max} is the maximum value of the underslying stock price.
- K is the strike price.
- n is the number of underlying stock observations over which the averaging is performed.

Both Asian and lookback options are path-dependent, meaning their payoff is significantly influenced by the trajectory of the underlying stock. This characteristic adds complexity to their pricing compared to classic European-style options, as it introduces additional variables into the option functions. Computing closed-form solutions for options with increased dimensions becomes more challenging. Therefore, Monte Carlo simulation emerges as a viable method for pricing these path-dependent options due to its ability to handle the intricate dynamics of their underlying assets.

2.1 Arithmetic vs geometric average

Asian options offer the flexibility to compute the average using either arithmetic or geometric methods. The arithmetic average is calculated by summing all underlying stock values and dividing the total by the number of observations.

Geometric average calculates the central tendency of a set of numbers by multiplying them together and then taking the nth root of the product, where n is the total number of values. It can be expressed as follows:

$$\text{Asian call option payoff with geometric average} = \max\left(\left(\sqrt[n]{X_1 \cdot X_2 \cdot \ldots \cdot X_n}\right) - K, 0\right)$$

2.2 Discrete vs continous sampling

Averages for Asian options and maximum values for lookback call options can be computed from discrete or continuous underlying stock values. Discrete values represent a subset of the total underlying stock price values, such as considering stock values every 5 days for instance. In contrast, continuous sampling encompasses all underlying stock price values.

2.3 Fixed vs floating strike

The strike can either be fixed or floating. In a fixed strike option, the strike price is predetermined at the time of contract signing between the buyer and seller. Conversely, in a floating strike option, the strike price is determined by the value of the underlying stock at maturity (S_T).

3. Results

The following section displays the results of all the different combinations of Asian options with arithmetic and geometric averages, discrete continous sampling, fixed and floating strikes to give a total of $2^3 = 8$ types of call options and 8 types of put options. The keys for Asian option tables are the following:

CDFiA: Call option, Discrete sampling, Fixed strike, Arithmetic average

CDFiG: Call option, Discrete sampling, Fixed strike, Geometric average

CDFlA: Call option, Discrete sampling, Floating strike, Arithmetic average

CDFlG: Call option, Discrete sampling, Floating strike, Geometric average

CCFiA: Call option, Continous sampling, Fixed strike, Arithmetic average

CCFiG: Call option, Continous sampling, Fixed strike, Geometric average

CCFlA: Call option, Continous sampling, Floating strike, Arithmetic average

CCFlG: Call option, Continous sampling, Floating strike, Geometric average

PDFiA: Put option, Discrete sampling, Fixed strike, Arithmetic average

PDFiG: Put option, Discrete sampling, Fixed strike, Geometric average

PDFlA: Put option, Discrete sampling, Floating strike, Arithmetic average

PDFlG: Put option, Discrete sampling, Floating strike, Geometric average

PCFiA: Put option, Continous sampling, Fixed strike, Arithmetic average

PCFiG: Put option, Continous sampling, Fixed strike, Geometric average

PCFlA: Put option, Continous sampling, Floating strike, Arithmetic average

PCFlG: Put option, Continous sampling, Floating strike, Geometric average

We finally displays lookback options with fixed and floating strikes for call and put options. The key for lookback options are the following:

CFl: Call option with a floating strike

CFi: Call option with a fixed strike

PFl: Put option with a floating strike

PFi: Put option with a fixed strike

3.1 Asian Options

In [50]:
```
%run TablesDisplay.py
```

In [45]:
```
%run Standard_Error.py
```

3.1.1 Varying number of simulations

Table 3 demonstrates a notable trend: as the number of simulations increases, the standard error decreases. This phenomenon is explained by the central limit theorem (CLT), which asserts that with larger sample sizes, the distribution of sample means converges towards a narrower and more concentrated distribution around the population mean.

In [51]:
```
df_sim_call # call options TABLE 1
```

Out[51]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CDFiA	CDFiG	CDFIA	CDFIG	
0	100	0.05	0.2	100	100	5.706918	5.503210	3.789483	3.671611	5.7
1	1000	0.05	0.2	100	100	5.868848	5.629198	3.349311	3.222475	5.9
2	10000	0.05	0.2	100	100	5.758550	5.539273	3.572499	3.438622	5.8

In [52]:
```
df_sim_put # put options TABLE 2
```

Out[52]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PDFiA	PDFiG	PDFIA	PDFIG	
0	100	0.05	0.2	100	100	3.847766	3.973499	5.524167	5.720717	3.8
1	1000	0.05	0.2	100	100	3.520884	3.645179	5.893902	6.114764	3.5
2	10000	0.05	0.2	100	100	3.296719	3.416988	6.060448	6.280228	3.3

In [53]:
```
df_sim_call_err # call options standard errors TABLE 3
```

Out[53]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CCFiA	CCFiG
0	100	0.05	0.2	100	100	1.068840	1.057886
1	1000	0.05	0.2	100	100	0.374135	0.369789
2	10000	0.05	0.2	100	100	0.120128	0.118755

3.1.2 Varying strike price

As anticipated, we notice that reducing the strike price leads to higher option values for call options and lower values for put options. This outcome stems from the nature of the payoff function. With a lower strike price, the disparity between the average of the underlying stock and the strike price widens for call options and contracts for put options.

In [54]:
```
df_strike_call # call options TABLE 4
```

Out[54]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CDFiA	CDFiG	CDFIA	CDFIG	
0	10000	0.05	0.2	100	95	8.603593	8.356210	3.662577	3.525941	8.6
1	10000	0.05	0.2	100	100	5.604114	5.385347	3.562278	3.430316	5.6
2	10000	0.05	0.2	100	105	3.360457	3.182282	3.556949	3.422993	3.4

In [55]:
```
df_strike_put # put option TABLE 5
```

Out[55]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PDFiA	PDFiG	PDFIA	PDFIG	
0	10000	0.05	0.2	100	95	1.674112	1.765097	6.102207	6.324455	1.6
1	10000	0.05	0.2	100	100	3.252437	3.369501	6.103280	6.325050	3.2
2	10000	0.05	0.2	100	105	5.882572	6.037814	5.958625	6.176139	5.9

3.1.3 Varying volatility

Table 6 and Table 7 shows that when volatility rises, both call and put option prices also increase. This happens because volatility adds to the uncertainty or risk linked to how the underlying asset's price moves. Options get their value from the possibility of price changes in the underlying asset. Increased volatility means there's a higher chance of substantial price swings, boosting the likelihood of the option leading to a favorable outcome. As a result, buyers are ready to pay higher prices for these contracts, reflecting the heightened risk and potential for higher returns.

Table 8 demonstrates a similar trend with the standard error, which is expected. This occurs because an increase in volatility in the underlying stocks weakens the accuracy of the model's predictions.

In [61]:
```
df_vol_call # call options TABLE 6
```

Out[61]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CDFiA	CDFiG	CDFIA	CDFIG	
0	10000	0.05	0.2	100	100	5.646181	5.432524	3.516014	3.381845	
1	10000	0.05	0.3	100	100	7.914844	7.462766	5.826320	5.490431	
2	10000	0.05	0.4	100	100	10.018768	9.242379	8.249322	7.625878	1

In [62]:
```
df_vol_put # put options TABLE 7
```

Out[62]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PDFiA	PDFiG	PDFIA	PDFIG
0	10000	0.05	0.2	100	100	3.283243	3.403226	6.014188	6.235523
1	10000	0.05	0.3	100	100	5.352923	5.636825	8.490829	8.945482
2	10000	0.05	0.4	100	100	7.650315	8.184693	10.676160	11.431364

In [63]:
```
df_vol_call_err # standard error call options TABLE 8
```

Out[63]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CCFiA	CCFiG
0	10000	0.05	0.2	100	100	0.118202	0.116935
1	10000	0.05	0.3	100	100	0.180717	0.177687
2	10000	0.05	0.4	100	100	0.240845	0.234608

3.1.4 Varying risk-free rate

We notice that as we increase the risk-free rate, the option value rises when the strike is fixed. However, in the case of a floating strike, the option value declines with an increase in the risk-free rate.

When the risk-free rate increases and the strike is fixed, the option value increases because the present value of the payoff (the difference between the stock price average and the strike price) also increases. This is because a higher risk-free rate implies a higher discount factor, making future cash flows less valuable, which favors the buyer of the option.

On the other hand, when the strike is floating, the option value decreases with an increase in the risk-free rate. This is because a floating strike option effectively locks in the value of the underlying asset at maturity. As the risk-free rate increases, the expected future value of the underlying asset decreases (due to the higher discounting effect), leading to a lower payoff at maturity and hence a lower option value.

In [64]:
```
df_rfr_call # call options TABLE 9
```

Out[64]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CDFiA	CDFiG	CDFIA	CDFIG	
0	10000	0.03	0.2	100	100	5.298722	5.098498	3.892517	3.744767	5.3
1	10000	0.05	0.2	100	100	5.767261	5.546338	3.526949	3.393335	5.8
2	10000	0.07	0.2	100	100	6.270432	6.027878	3.141536	3.020236	6.3

In [65]:
```
df_rfr_put # put options TABLE 10
```

Out[65]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PDFiA	PDFiG	PDFIA	PDFIG	
0	10000	0.03	0.2	100	100	3.815377	3.953604	5.523201	5.726222	3.8
1	10000	0.05	0.2	100	100	3.323745	3.444342	6.130671	6.352272	3.3
2	10000	0.07	0.2	100	100	2.848182	2.951783	6.681856	6.928335	2.8

In [66]:
```
df_rfr_call_err # standard error call options TABLE 11
```

Out[66]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CCFiA	CCFiG
0	10000	0.03	0.2	100	100	0.116791	0.115730
1	10000	0.05	0.2	100	100	0.118772	0.117412
2	10000	0.07	0.2	100	100	0.119744	0.118081

In [88]:
```
%run lookback_option.py
```

3.1.5 Arithmetic vs Geometric means

Overall, Asian call options with arithmetic means are generally more expensive than those with geometric means due to the smoothing effect of the geometric mean. The arithmetic mean considers each observation equally, including extreme values, leading to potentially higher option prices. In contrast, the geometric mean mitigates the impact of extreme values by taking the nth root of the product of all observations, resulting in lower option prices.

3.2 Lookback Options

3.2.1 Varying number of simulations

In [89]:
```
df_lookback_call_sim
```

Out[89]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CFI	CFi
0	100	0.05	0.2	100	100	18.768684	15.823710
1	1000	0.05	0.2	100	100	16.294379	19.015955
2	10000	0.05	0.2	100	100	16.537951	18.464815

In [90]:
```
df_lookback_put_sim
```

Out[90]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PFI	PFi
0	100	0.05	0.2	100	100	13.376245	11.805237
1	1000	0.05	0.2	100	100	13.434485	11.716488
2	10000	0.05	0.2	100	100	13.579708	11.876114

3.2.2 Varying strike price

Increasing the strike price does not lead to a decrease in the value of the call option when the strike is floating. This is because, in a floating strike scenario, the strike price is not factored into the computation of the option price, only the value of the underslying asset at S_T is considered.

In [91]:
```
df_lookback_call_str
```

Out[91]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CFl	CFi
0	10000	0.05	0.2	100	95	16.672385	23.103873
1	10000	0.05	0.2	100	100	16.538888	18.507048
2	10000	0.05	0.2	100	105	16.364774	13.968419

In [92]:
```
df_lookback_put_str
```

Out[92]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PFl	PFi
0	10000	0.05	0.2	100	95	13.372956	7.647362
1	10000	0.05	0.2	100	100	13.599451	11.786629
2	10000	0.05	0.2	100	105	13.409327	16.630422

3.2.3 Varying volatility

In [93]:
```
df_lookback_call_vol
```

Out[93]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CFl	CFi
0	10000	0.05	0.2	100	100	16.530051	18.302311
1	10000	0.05	0.3	100	100	23.048524	26.632161
2	10000	0.05	0.4	100	100	28.567883	36.019118

In [94]:
```
df_lookback_put_vol
```

Out[94]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PFl	PFi
0	10000	0.05	0.2	100	100	13.533488	11.745024
1	10000	0.05	0.3	100	100	21.742454	18.269241
2	10000	0.05	0.4	100	100	30.648808	24.046717

3.2.4 Varying risk-free rate

In [95]:
```
df_lookback_call_rfr
```

Out[95]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	CFl	CFi
0	10000	0.03	0.2	100	100	15.606496	17.531348
1	10000	0.05	0.2	100	100	16.560225	18.660007
2	10000	0.07	0.2	100	100	17.610131	19.205493

In [96]:

```
df_lookback_put_rfr
```

Out[96]:

	Nbr_of_simulations	Risk-free_rate	Volatility	S0	Strike	PFl	PFi
0	10000	0.03	0.2	100	100	14.453362	12.907658
1	10000	0.05	0.2	100	100	13.382007	11.840767
2	10000	0.07	0.2	100	100	12.493984	10.740701

4. Conclusion

The price differential between Asian and lookback options is evident. This discrepancy can be attributed to the distinctive nature of their payoff mechanisms. Asian call options, reliant on averages, tend to be priced lower compared to lookback options, which utilize the maximum value in their calculations.

The standard error is significantly influenced by both the number of simulations and the volatility. Other parameters, when varied, do not exert as much impact on the standard error. Monte Carlo simulation is well-suited for pricing path-dependent options due to its ability to handle complex payoff structures. However, it can be computationally intensive, particularly when simulating a large number of paths or when pricing options with high-dimensional features.