Geometry by Design

Transformations, Construction, and Proof

Teacher's Guide

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Geometry by Design Unit Overview

Intent

Geometry By Design is intended to provide students with historical knowledge about how humans created mathematics, and in particular, geometry. Students will use the ancient tools of straightedge and compass to do constructions, and ruler and protractor to make accurate drawings. The classical deductive system consisting of Euclid's postulates and theorems will be introduced to prove theorems about triangles and quadrilaterals.

Geometry By Design also introduces the modern geometric idea of transformations: reflection, rotation, translation and dilation. Students explore these transformations with constructions and drawings, and also on a coordinate system.

Mathematics

The concepts of construction, proof, and transformation are central to this unit. Students explore the following important ideas.

- The history of how geometry was created and developed by ancient peoples
- Basic geometric vocabulary and notation
- Basic straightedge and compass constructions: the perpendicular bisector of a segment, the angle bisector, and the perpendicular from a point to a line. Students use these as the foundation for more complex constructions, such as constructing a square and hexagon.
- Use of compass, ruler and protractor to draw triangles given SAS, ASA, and SSS.
- The idea of a deductive system, including postulates, theorems, and proofs
- Use of postulates to prove theorems involving congruent triangles, including the Isosceles Triangle Theorem and other theorems about constructions, triangles, and quadrilaterals
- Introduce transformations including the isometric transformations reflection, rotation, translation, and the non-isometric transformation, dilation
- Use transformations to solve geometric and algebraic problems

Progression

In *Do It Like The Ancients*, students practice making accurate geometric drawings using the tools of the ancients: compass, straightedge, ruler, and protractor. In *Construction and Deduction*, students experience how the ancient Greeks organized geometric ideas into a deductive system, so that new truths could be found not only by observation, but also by abstract thinking. *Isometric Transformations* and *Dilation* introduce modern tools of geometry: reflection, rotation, translation and dilation. These tools are used in computer software to produce geometric designs for manufacturing, for building, and for creating art. Students explore the ideas behind these new tools. Finally, in *Put the Pieces Together*, students revisit and combine application of ideas from earlier in the unit. They use what they've learned to design and draw original geometric figures.

Do It Like the Ancients

Construction and Deduction

Isometric Transformations

Dilation

Put the Pieces Together

Geometry by Design and the Common Core State Standards for Mathematics

Meaningful Math—Geometry is written to address the Common Core State Standards for Mathematics (CCSSM), and particularly the High School standards that Appendix A of the CCSSM recommends for inclusion in a Geometry course.

Standards for Mathematical Practice

The eight Standards for Mathematical Practice are addressed exceptionally well throughout the *Meaningful Math* curriculum.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Standards for Mathematical Content

These specific content standards are addressed in the *Geometry by Design* unit. Additional content is covered that reinforces standards from earlier grades and courses.

G-CO.1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

G-CO.2. Model transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus stretch in a specific direction).

G-CO.3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

G-CO.4. Develop definitions of rotations, reflections and translations in terms of angles, circles, perpendicular lines, parallel lines and line segments.

G-CO.5. Given a specified rotation, reflection or translation and a geometric figure, construct the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Construct a sequence of transformations that will carry a given figure onto another.

G-CO.6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a rigid motion on a figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

G-CO.8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence.

G-CO.9. Prove theorems about lines and angles.

G-CO.10. Prove theorems about triangles.

G-CO.11. Prove theorems about parallelograms.

G-CO.12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc).

G-CO.13. Construct an equilateral triangle, a square and a regular hexagon inscribed in a circle.

G-SRT.1a. Verify experimentally the properties of dilations: A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

G-SRT.1b. Verify experimentally the properties of dilations: The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

G-SRT.5. Use triangle congruence and similarity criteria to solve problems and to prove relationships in geometric figures.

G-SRT.10. Prove the Laws of Sines and Cosines and use them to solve problems.

G-SRT.11. Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

G-C.1. Prove that all circles are similar.

G-GPE.5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

G-MC.3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy constraints or minimize cost; working with typographic grid systems based on ratios).

Pacing Guides

50-Minute Pacing Guide (28–31 days)

Day	Activity	In-Class Time Estimate
Do It Like the Ancients		
1	*Reference: Who Invented the Rectangle?*	5
	Three Points in a Line—Introducing the Straightedge	10
	What Do We Already Know?	25
	Homework: *Circle and Arc Designs—Introducing the Compass*	10
2	*Postulates, Axioms, and a Theorem*	30
	Construct an Equilateral Triangle	10
	Homework: *Circle Designs*	10
3	*Reference: Measuring Tools—Introducing the Ruler and Protractor*	10
	Homework: *Drawing Triangles with Ruler and Protractor—SAS and ASA*	40
4	*Discussion: Drawing Triangles with Ruler and Protractor*	10
	What Else Do We Already Know?	35
	Homework: *Triangle Puzzles*	5
5	Discussion: *Triangle Puzzles*	5
	Drawing Specific Triangles—SSS	30
	Homework: *A Trough and a Tent*	15
6	*Discussion: A Trough and a Tent*	10
	POW 5: Draw Your Bike	10
	Bicycle Vocabulary—Geometry Notation	15
	Homework: *What Makes a Triangle?* and *Triangle Congruence Applications*	15

7	*Discussion: What Makes a Triangle?* and *Triangle Congruence Applications*	10
	Reference: Our Postulates and Theorems So Far?	10
	But Why?	10
	Proving the Isosceles Triangle Theorem	15
	Homework: *Using the Isosceles Triangle Theorem*	5
8	Discussion: *Using the Isosceles Triangle Theorem*	10
	Deduce Those Angles	20
	Homework: *Proofs for You!*	20
Construction and Deduction		
9	*Discussion: Proofs for You!*	10
	Walking Sets of Equidistant Points	40
	Homework: Finish POW 5 *Draw Your Bike*	0
10	Presentations: *POW 5: Draw Your Bike*	10
	Perpendicular Bisector Explorations	20
	Why it Works—The Perpendicular Bisector Construction	15
	Homework: *Construction Challenges*	5
11	*POW 6: How Did Humans Create Mathematics?*	20
	Angle Bisector Explorations	25
	Homework: *Properties of Special Quadrilaterals*	5
12	Discussion: *Properties of Special Quadrilaterals*	10
	POW 6: Assign Pairs and Topics	10
	Parallel Lines and Transversals	15
	Reference: Parallelograms and Trapezoids	5
	Homework: *Parallelogram and Trapezoid Proofs*	10
Isometric Transformations		
13	Discussion: *Parallelogram and Trapezoid Proofs*	20
	Isometric Transformation 1: Reflection	20
	Homework: *Reflection Challenges*	5
14	Discussion: *Reflection Challenges*	5
	Reflecting Lines	35
	Homework: *Reflection Designs*	10

15	*Discussion: Reflection Designs*	10
	Isometric Transformation 2: Rotation	40
	Homework: *POW 6: How Did Humans Create Mathematics?*	0
16	POW 6: Collect all presentation slides in a file on teacher's computer	10
	How Many Ways from A to B?	40
17	Discussion: *How Many Ways from A to B?*	10
	Begin Presentations: *POW 6: How Did Humans Create Mathematics?* *(see note at the bottom of this schedule)	40
	Homework: *Rotation with Coordinates*	5
18	Discussion: *Rotation with Coordinates*	5
	Continue Presentations: *POW 6: How Did Humans Create Mathematics?**	50
	Homework: none	
19	*Sloping Sides*	20
	Continue Presentations: *POW 6: How Did Humans Create Mathematics?**	25
	Homework: *Perpendicular Rotations Questions 1–4*	5
20	*Discussion: Perpendicular Rotations Questions 1–4*	5
	Continue Presentations: *POW 6: How Did Humans Create Mathematics?**	30
	Homework: *Perpendicular Rotations Questions 5–10*	10
21	*Discussion: Perpendicular Rotations Questions 5–10*	10
	Finish Presentations: *POW 6: How Did Humans Create Mathematics?**	25
	Isometric Transformation 3: Translation	10
	Homework: *Frieze Frame*	5
22	*Translation Investigations*	30
	Homework: *Translation Designs*	20
23	*Transforming One Shape Into Another* (optional depending upon time constraints)	50
Dilation		
24	*A Non-Isometric Transformation: Dilation*	35
	Homework: *Dilating a Right Triangle*	15

25	*Discussion: Dilating a Right Triangle*	5
	Enlarging on a Copy Machine	30
	Homework: *Dilation Investigations*	15
Put the Pieces Together		
26	Discussion: *Dilation Investigations*	10
	Think About It	30
	Homework: *Drawing Conclusions*	20
27	Discussion: *Drawing Conclusions*	20
	Homework: *Combinations of Transformations*	30
28	*Discussion: Combinations of Transformations*	15
	Digging Into Transformations (optional depending on time constraints)	15
	Homework: *Geometry by Your Design* (optional depending on time constraints)	10
	Homework: *Coordinate Transformations* (optional depending on time constraints)	10
29	*Tricky Transformations*	30
	Homework: *Geometry by Design Portfolio*	20
30	*In-Class Assessment*	40
	Homework: *Take-Home Assessment*	10
31	Assessment Discussion	30
	Unit Reflection	20

**Note about POW 6:* Presentations for POW 6 are best done over several days. The schedule presented for deadlines and presentations is only a suggestion. You may require more or less time depending on the size of your class and the availability of technology.

90-Minute Pacing Guide (18-21 days)

Day	Activity	In-Class Time Estimate
Do It Like the Ancients		
1	*Reference: Who Invented the Rectangle?*	5
	Three Points in a Line—Introducing the Straightedge	10
	What Do We Already Know?	25
	Circle and Arc Designs—Introducing the Compass	10
	Postulates, Axioms, and a Theorem	25
	Construct an Equilateral Triangle	10
	Homework: *Circle Designs*	5
2	*Reference: Measuring Tools—Introducing the Ruler and Protractor*	10
	Homework: *Drawing Triangles with Ruler and Protractor—SAS and ASA*	45
	What Else Do We Already Know?	30
	Homework: *Triangle Puzzles*	5
3	Discussion: *Triangle Puzzles*	5
	Drawing Specific Triangles—SSS	35
	A Trough and a Tent	25
	POW 5: Draw Your Bike	10
	Bicycle Vocabulary—Geometry Notation	15
	Homework: *POW 5*	0
4	*What Makes a Triangle?* and *Triangle Congruence Applications*	40
	Reference: Our Postulates and Theorems So Far?	10
	But Why?	15
	Proving the Isosceles Triangle Theorem	15
	Homework: *Using the Isosceles Triangle Theorem*	5

5	Discussion: *Using the Isosceles Triangle Theorem*	10
	Deduce Those Angles	15
	Proofs for You!	35
Construction and Deduction		
	Walking Sets of Equidistant Points	30
	Homework: Finish POW 5: *Draw Your Bike*	0
6	Presentations: *POW 5: Draw Your Bike*	10
	Perpendicular Bisector Explorations	20
	Why it Works—The Perpendicular Bisector Construction	15
	Construction Challenges	20
	POW 6: How Did Humans Create Mathematics?	20
	Homework: Angle Bisector Explorations	25
7	Discussion: *Angle Bisector Explorations*	10
	Properties of Special Quadrilaterals	30
	POW 6: Assign Pairs and Topics	15
	Parallel Lines and Transversals	20
	Reference: Parallelograms and Trapezoids	5
	Homework: *Parallelogram and Trapezoid Proofs*	10
Isometric Transformations		
8	Discussion: *Parallelogram and Trapezoid Proofs*	20
	Isometric Transformation 1: Reflection	20
	Reflection Challenges	20
	Homework: *Reflecting Lines*	30
	Homework: *Reflection Designs*	0
9	*Discussion: Reflecting Lines* and *Reflection Designs*	10
	Isometric Transformation 2: Rotation	50
	How Many Ways from A to B?	30
	Homework: *POW 6: How Did Humans Create Mathematics?*	0
10	*How Many Ways from A to B?* (continued)	30
	Rotation with Coordinates	20
	Homework: *Sloping Sides* and *POW 6*	40

11	Discussion: *Sloping Sides*	15
	POW 6: Collect all presentation slides in a file on teacher's computer	15
	Perpendicular Rotations Questions 1–4	15
	Begin Presentations: *POW 6: How Did Humans Create Mathematics?* *(see note at the bottom of this schedule)	40
	Homework: *Perpendicular Rotations Questions 5–10*	5
12	Discussion: *Perpendicular Rotations Questions 5–10*	10
	Presentations: *POW 6: How Did Humans Create Mathematics?**	50
	Isometric Transformation 3: Translation	20
	Homework: *Frieze Frame*	10
13	*Translation Investigations*	30
	Continue Presentations: *POW 6: How Did Humans Create Mathematics?**	40
	Homework: *Translation Designs*	20
14	Finish Presentations: *POW 6: How Did Humans Create Mathematics?**	50
	Transforming One Shape Into Another (optional depending upon time constraints)	40
Dilation		
15	*A Non-Isometric Transformation: Dilation*	35
	Homework: *Dilating a Right Triangle*	20
	Enlarging on a Copy Machine	30
	Homework: *Dilation Investigations*	5
Put the Pieces Together		
16	Discussion: *Dilation Investigations*	10
	Think About It	30
	Drawing Conclusions	20
	Homework: *Combinations of Transformations Question 1*	30

17	*Discussion: Combinations of Transformations Question 1*	10
	Combinations of Transformations Questions 2–3	30
	Digging Into Transformations (optional depending on time constraints)	25
	Homework: *Geometry by Your Design* (optional depending on time constraints)	10
	Homework: *Coordinate Transformations* (optional depending on time constraints)	0
18	*Tricky Transformations*	30
	Homework: *Geometry by Design Portfolio*	60
19	*In-Class Assessment*	50
	Homework: *Take-Home Assessment*	40
20	Assessment Discussion	30
	Unit Reflection	20

**Note about POW 6:* Presentations for POW 6 are best done over several days. The schedule presented for deadlines and presentations is only a suggestion. You may require more or less time depending on the size of your class and the availability of technology. See POW 6 notes in this teacher guide and blackline masters at the end for help.

Materials and Supplies

All *Meaningful Math* classrooms should have a set of standard supplies and equipment, and students are expected to have materials available for working at home on assignments and at school for classroom work. Lists of these standard supplies are included in the section “Materials and Supplies for the *Meaningful Math* Classroom” in *A Guide to Meaningful Math.* There is also a comprehensive list of materials for all units in Geometry.

Listed below are the supplies needed for this unit.

Geometry by Design

- Compass
- Straightedge/centimeter ruler
- Protractor
- 6–10 feet of string (per group)
- Computers with presentation software
- Colored paper (1 sheet per pair)
- Poster paper (1 sheet per pair)
- Scissors
- Glue
- Marker
- Meter sticks (1 per pair, optional)
- Centimeter dot paper
- Rubber bands
- Isometric dot paper

More About Supplies

- Graph paper is a standard supply for *Meaningful Math* classrooms. Blackline masters of 1-Centimeter Graph Paper, ¼-Inch Graph Paper, and 1-Inch Graph Paper are provided so you can make copies and transparencies for your classroom.

Assessing Progress

Geometry by Design concludes with two formal unit assessments. In addition, there are many opportunities for more informal, ongoing assessment throughout the unit. For more information about assessment and grading, including general information about the end-of-unit assessments and how to use them, see “Assessment and Grading” in *A Guide to Meaningful Math.*

End-of-Unit Assessments

Each unit concludes with in-class and take-home assessments. The in-class assessment is intentionally short so that time pressures will not affect student performance. Students may use graphing calculators and their notes from previous work when they take the assessments. You can download unit assessments from the *Shadows* Unit Resources.

Ongoing Assessment

Assessment is a component in providing the best possible ongoing instructional program for students. Ongoing assessment includes the daily work of determining how well students understand key ideas and what level of achievement they have attained in acquiring key skills.

Students’ written and oral work provides many opportunities for teachers to gather this information. Here are some recommendations of written assignments and oral presentations to monitor especially carefully that will offer insight into student progress.

- *Drawing Triangles with Ruler and Protractor—SAS and ASA*
- *POW 5: Draw Your Bike*
- *What Makes a Triangle?* (Tricky Triangles section)
- *POW 6: How Did Humans Create Mathematics?* (presentations)
- *Reflecting Lines*
- *Drawing Conclusions* (Drawing Transformations on Dot Paper section)
- *Combinations of Transformations*

Supplemental Activities Overview

Geometry by Design contains a variety of activities at the end of the student pages that you can use to supplement the regular unit material. These activities fall roughly into two categories.

- **Reinforcements** increase students' understanding of and comfort with concepts, techniques, and methods that are discussed in class and are central to the unit.
- **Extensions** allow students to explore ideas beyond those presented in the unit, including generalizations and abstractions of ideas.

The supplemental activities are presented in the teacher's guide and the student book in the approximate sequence in which you might use them. Below are specific recommendations about how each activity might work within the unit. You may wish to use some of these activities, especially the later ones, after the unit is completed.

***Hexagon Designs* (reinforcement)** These two designs can be assigned anytime after *Construct an Equilateral Triangle.*

***Pentagon Designs* (reinforcement)** These designs, based on a pentagon, are best done after the protractor is introduced, to help with drawing an accurate pentagon. Although a pentagon may be constructed using only compass and straightedge, the resulting pentagon is usually not accurate due to the physical limitations of the compass.

***Draw These Triangles* (reinforcement)** If your students need more practice drawing triangles given SAS, ASA, or SSS, you my want to assign these problems. You can add one or more of them to each homework assignment after *Drawing Specific Triangles.* Here are the correct measurements for each problem, with lengths rounded to the nearest tenth of a cm and angles rounded to the nearest degree:

1. $PA = 7.2$, $m\angle A = 79°$
2. $EC = 12.6$, $m\angle C = 31°$
3. $m\angle U = 93°$
4. $NW = 7.5$, $WO = 9.1$
5. $ED = 3.5$

***Draw These Quadrilaterals* (extension)** These problems extend the triangle drawing problems to include drawing of special quadrilaterals. These problems are not intended to be formal construction problems.

Students are probably familiar with the basic properties of square, rectangle, rhombus, and parallelogram. They can use these properties to figure out how to accurately draw each shape using protractor, ruler and compass. For example, here is one way to do Question 1, "draw a square with diagonals 9 cm." By making a sketch, a student can realize the diagonals of a square bisect and are perpendicular to each other. (There's no need for a formal proof of this here.) So the square can be drawn by using a protractor to make a 90° angle, and extending the angle 4.5 cm in each direction. To draw the rectangles in Question 2, students know that the angles in a rectangle measure 90°. So one way to draw the rectangle is to first draw a right angle and mark off 8 cm along one ray. Then set the compass to measure 10 cm and use the compass to mark off the diagonal on the other ray from the endpoint of the side. Finally use the compass to complete the rectangle with an SSS construction.

Congruent Triangle Proofs **(reinforcement)** If you decide to spend more time with your class doing formal geometric proofs, you may want to assign some of these problems. Each proof included here can be done using only the postulates listed in *Our Postulates and Theorems So Far.* Proving triangles congruent is necessary in every proof.

The Girl and The Mirrors **(extension)** May be assigned any time after doing *Isometric Transformation 1: Reflection.* This is a fun problem in a context many students encounter each morning. Expect lots of animated discussion!

Build a Model Roof **(extension)** This activity may be used as a POW anytime after *Parallelograms and Trapezoids*. The roofs of houses are interesting and challenging for students to model. They must draw shapes accurately so that edges on the roof that meet are the same length.

You can set your own parameters for this POW. Best to make sure you specify the parameters carefully so the work includes enough geometry. As a minimum, require the measurements of all lengths and angles for each panel, the scale, and a photo of the real building.

A Pencil Proof **(extension)** This problem demonstrates how translation and rotation can be used to prove that the sum of the angles of any triangle is 180°. Note that after rotation through all the angles, the pencil is pointing the exact opposite direction from the start. Therefore, the sum of the turns must be 180°. To paraphrase Rene Descartes: "A Proof is making obvious that which was not obvious." If you agree with Descartes, then "turn the pencil" is a proof.

The "turn the pencil" method also shows the sum of the angles of any quadrilateral to be 360°, and can be extended to other polygons. Sometimes in mathematics, we make things so complicated that we obscure the obvious.

The Law is on Your Side **(extension)** Students will not be able to prove the law of cosines until they explore the Pythagorean theorem in the unit *Do Bees Build It Best?* But they did learn about cosine in the *Shadows* unit, so they should be able to apply the law of cosines to do calculations. They might want to try to use the formula to calculate some of the lengths they found by drawing. The law of cosines works well to find the length of the missing side in SAS drawings.

Geometry Problems To Puzzle Your Brain **(extension)** Students enjoy problem solving. These problems may be assigned any time during the unit as extras to liven up students' brains. It's best not to give answers or explain how to do these.

If a student asks you "Am I right?" then suggest they think a bit more until they know for sure they are right and can explain why to you!

Question 4 is especially nice, as it helps students understand the meaning of fractions.

Exterior Angles and Polygon Sums **(extension)** Euclid included theorems about exterior angles in his books. You may want to assign this problem if you want to include exterior angles as part of your course. The proof essentially uses the same technique as "Turn the Pencil" in *A Pencil Proof.*

The Design of Culture **(extension)** This research activity provides an opportunity to add another multicultural dimension to the unit.

Do it Like the Ancients

Intent

In this first section, the goals are for students to experience how mathematics was created and communicated by humans, and to use the tools invented by ancient peoples to do geometry.

Mathematics

Humans invented geometry. This section exposes students to both ancient and modern tools of geometry, and to deductive reasoning and proof.

This unit is more traditional than many *Meaningful Math* units, and has a theme rather than a central unit problem. It is written to fully address the Common Core State Standards' emphasis on transformational geometry, through a hands-on and exploratory approach.

Materials

In class and at home throughout this unit, students will need a compass, centimeter ruler (which serves as a straightedge), and protractor. You may also want students to do their work for this unit in a composition book filled with lined paper—or even better, graph paper.

Progression

Students learn about the tools of geometry. They are introduced to the construction tools of straightedge and compass, and then the measuring tools of ruler and protractor. Students also are exposed to the history of geometry, and learn about and begin to write proofs.

Reference: Who Invented the Rectangle?

Three Points in a Line—Introducing the Straightedge

What Do We Already Know?

Circle and Arc Designs—Introducing the Compass

Postulates, Axioms, and a Theorem

Construct an Equilateral Triangle

Circle Designs

Reference: Measuring Tools—Introducing the Ruler and Protractor

Drawing Triangles with a Ruler and Protractor—SAS and ASA

What Else Do We Already Know?

Triangle Puzzles

Drawing Specific Triangles—SSS

A Trough and a Tent

POW 5: Draw Your Bike

Bicycle Vocabulary—Geometry Notation

What Makes a Triangle

Triangle Congruence Applications

Reference: Our Postulates and Theorems So Far

But Why?

Proving the Isosceles Triangle Theorem

Using the Isosceles Triangle Theorem

Deduce Those Angles

Proofs for You!

Reference: Who Invented the Rectangle?

Intent

The intent of this reference section is to introduce students through a modern news event to the idea that geometry is an invention of the human mind.

Mathematics

Circles are common in nature. For example, raindrops falling into a pond generate hundreds of nearly perfect, concentric circles. On the other hand, rectangles are unnatural. The rectangular shapes we see are almost exclusively the result of humans.

Progression

Read *Who Invented the Rectangle?* and discuss.

Approximate Time

5 minutes

Classroom Organization

Whole-class discussion

Doing the Activity

Have students individually read *Who Invented the Rectangle?,* or select students to read paragraphs aloud to the class.

Discussing and Debriefing the Activity

Ask if anyone has heard about this patent lawsuit between Apple and Samsung. What is the current status of the lawsuit?

Ask, **What shapes do you see in nature? Can you think of any rectangles that occur in nature? Why would humans invent rectangles?**

Key Questions

Where do circles occur in nature? Where do hexagons occur in nature? What about rectangles?

Who wrote the first set of books about geometry? When and where were these books written?

What is the meaning of "ubiquitous"? Are rectangles ubiquitous?

Three Points in a Line—Introducing the Straightedge

Intent

Students are introduced to their first ancient mathematical tool, the straightedge. Basic to Greek philosophy is the belief in ideal forms. To the Greeks, the ideal line was perfectly straight, infinitely long, and had no thickness. The Greeks used a straightedge to draw approximations of lines.

When students are doing the constructions and drawings for this unit, they should strive for the ideal by making their lines as thin and straight as possible. Results are best when pencils are sharp. Ballpoint pens often work better than pencils as they draw uniform lines and don't need sharpening.

Mathematics

The three-points-in-a-line construction uses only a straightedge. The fact that the construction always results in three collinear points is called Pappas' Theorem. Pappas was a Greek living in Alexandria around 300 C.E. His elegant theorem is basic to a branch of mathematics called *projective geometry.*

Progression

First have students complete Questions 1 and 2. Before starting the construction in Question 3, have a brief discussion about the fact that three randomly chosen points in a plane are highly unlikely to be collinear. Then have the students follow the instructions to complete the construction.

Approximate Time

10 minutes

Classroom Organization

Students work individually, but help each other to follow the directions correctly

Materials

Supply each student with a straightedge, or require each student to acquire one of their own. Note that centimeter rulers will be required later in the unit, so it may be most efficient for students to have centimeter rulers, which perhaps they can turn over to use as a straightedge without measurement marks. Straightedges (and centimeter rulers) are needed frequently in this unit, both inside and outside of class.

Doing the Activity

To start the construction, encourage students to randomly place the points *A, B, C* and *X, Y, Z* along each line. Once located, the points need to be labeled along the line in alphabetical order, e.g. *A*-*B*-*C*, not *A*-*C*-*B.*

Encourage students to draw ideal lines that pass perfectly through the given points.

If some students do not get three collinear points, have them check to see if they have followed directions and labeled points correctly. Then have them redo the construction.

After finishing the construction once, some students might like to test the theorem again by trying the construction another time starting with different random points.

Discussing and Debriefing the Activity

Lead a short discussion using the Key Questions below.

Review the geometric notation given at the end of the activity. Students saw this notation in passing in the *Shadows* unit, but it's worth reviewing the distinction between notation for line, line segment, and length. You can choose how particular you want to be about students using the notation precisely.

Key Questions

What does collinear mean?

Can a straightedge be used to draw a perfect line?

How is a straightedge different than a ruler?

Of course humans make errors, so hand-drawn lines won't be perfect. Can a computer be used to display a perfect line?

When doing a construction, what are some ways of minimizing "human error"?

What Do We Already Know?

Intent

Students start geometry courses with a wide range of exposure to formal geometric vocabulary and notation. This activity is designed to bring everyone in the class to the same foundation of knowledge. Student responses also enable the teacher to assess prior knowledge.

Mathematics

The first four statements in Part I: True or False? are the first four geometric postulates in Euclid's Elements. The geometric words that fill in the blanks in Questions 6 to 12 are usually introduced in grades prior to high school.

Progression

Students first work individually, then in their group, and finally as whole class.

Approximate Time

25 minutes

Classroom Organization

Individuals, then groups, then whole class.

Doing the Activity

Have each student number 1 to 12 in his or her notebook. Then allow 5 minutes for each student to work alone and write down as many answers as are known.

After the 5 minutes are up, tell each group to compare answers and come to an agreement on their answers. Finally, have a whole-class discussion.

Discussing and Debriefing the Activity

Run through the items and call on a different group to provide their answer for each item. Questions 1 to 4 are all "true" for Euclidean Geometry. Question 5 is true unless the three points happen to be collinear

Most students will know the answers to Questions 6 through 10. Questions 11 and 12 have more than one possible correct answer. Questions 11 can be "adjacent" or "supplementary," and Question 12 can be "vertical" or "congruent." If students find these multiple possible answers, congratulate them. In the upcoming activity *Postulates, Axioms, and a Theorem,* we will prove the theorem "Vertical angles are equal," proposed here in Question 12.

If students have varied background knowledge, you may wish to start a word wall or poster with new vocabulary, including: perpendicular, parallel, radius, diameter, adjacent angles, vertical angles.

Circle and Arc Designs—Introducing the Compass

Intent

Students are introduced to the two uses for the compass: drawing circles and arcs, and measuring out lengths. Some students need time to get competent at using a compass accurately.

Mathematics

In Euclidean geometry, diagrams made only with compass and straightedge are called *constructions*. When solving geometric problems, the ancient Greeks restricted themselves to only these two tools. Solving geometric problems using only compass and straightedge is mathematically equivalent to solving algebraic equations using only simple arithmetic operations (add, subtract, multiply, divide) and square root.

Progression

Students learn to use a compass for marking out lengths, and for drawing circles and arcs.

Approximate Time

5 minutes for introduction

20 minutes (at home or in class)

5 minutes for discussion

Classroom Organization

Individuals

Materials

Supply each student with a good-quality compass, or require each student to acquire one of their own. Compasses are needed frequently in this unit, both inside and outside of class.

Doing the Activity

For Question 1, students can set the distance between the points to measure 10 miles using the scale in the book, and then use the compass to measure out the distance from Albion to Bayonne.

In Question 2, the designs shown are *not intended to be copied at the same size.* The goal is to produce *larger images* of the designs (taking up about half a sheet of paper each) precisely and accurately. Students will get more comfortable with the use of a compass through building the designs. The design in Question 2a is easiest to make if students start by using a compass to mark off three equal segments on a line. The design in Question 2b can be started by marking off four equal segments along a line.

After producing clean, precise versions of the two given designs, some students may want to make up their own circle designs.

Discussing and Debriefing the Activity

Have a couple of students share how they approached making their designs.

Focus on quality and precision. Designs should be smoothly and accurately drawn.

If any students created their own designs using a compass and straightedge, have them share with the class if they wish.

Key Questions

What two tools did the ancient Greeks allow for making diagrams?

Suppose while drawing you make a mistake. What are your options?

What are some dos and don'ts for making accurate constructions with compass and straightedge?

Postulates, Axioms, and a Theorem

Intent

Students are introduced to the elements of a **deductive** system: **axioms**, **postulates**, and **theorems**.

Mathematics

Euclid's first five axioms are given and used to prove the Vertical Angles Theorem. Proofs are essential in doing mathematics beyond basic algebra. The vertical angles proof is straightforward; later proofs involving geometric figures will be more challenging.

Progression

Students are introduced to the idea of deductive arguments and related terminology. They work in groups to make sense of Euclid's axioms, and make some observations about vertical angles. Finally, the idea of formal proof is introduced, and the Vertical Angle Theorem is proven.

Approximate Time

30 minutes

Classroom Organization

Whole-class introduction, followed by group work, followed again by whole-class discussion

Doing the Activity

Read the introductory paragraphs and discuss as a whole class. Have groups work on *Apply the Axioms* and *The Vertical Angles Theorem*. Work through *A Proof of the Vertical Angles Theorem* as a class.

Discussing and Debriefing the Activity

The abstract proof in *A Proof of the Vertical Angles Theorem* will be easier for students to understand if they've tackled the concrete work for Questions 6 and 7 first.

Key Questions

What are the five postulates in the U.S. Declaration of independence?

Which of Euclid's five axioms are used in solving of algebraic equations?

What distinguishes a postulate from a theorem?

Construct an Equilateral Triangle

Intent

Students use straightedge and compass to construct an equilateral triangle.

Mathematics

Euclid's third postulate is used to justify the construction.

Progression

Students work individually to analyze and replicate a construction of an equilateral triangle.

Approximate Time

10 minutes (at home or in class)

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Emphasize precision and accuracy.

The equilateral triangle is easiest to construct if it is made about the size of a person's palm.

Discussing and Debriefing the Activity

Verify that all students have accurately constructed an equilateral triangle. Have a student explain why it is definitely equilateral.

Key Questions

What steps of the equilateral triangle construction guarantee that the three sides of the triangle are the same length?

What might be true about the angles of an equilateral triangle?

Circle Designs

Intent

This activity provides practice using compass and straightedge, and performing the equilateral triangle construction that students learned in *Construct an Equilateral Triangle.*

Mathematics

Before high school, most students have used pattern blocks to arrange six equilateral triangles to form a regular hexagon. This idea can be used to construct the two circle designs, based on the equilateral triangle construction, with compass and straightedge.

Progression

May be assigned for homework.

Approximate Time

5 minutes for introduction

20 minutes for activity (in class or at home)

Classroom Organization

Individuals

Doing the Activity

The endpoints of each line segment and the center and radius of each arc need to be precisely located in order to draw the designs successfully. You may want to demonstrate how to start the designs using a compass and straightedge on an overhead projector.

Rather than drawing and erasing, you may wish to suggest that students draw interim construction lines lightly and just leave them. Designs may be shaded or colored to make them more attractive.

Discussing and Debriefing the Activity

Several attempts might be needed before a student completes an acceptable construction.

In future classes, you may want to do a "Design of the Day" by showing students a simple design based on equilateral triangles or regular hexagons on the overhead or projector and have them try to reproduce the design in their notebooks using a compass and straightedge.

Key Questions

Where in each design is the equilateral triangle construction used?

What qualities make an attractive design?

Supplemental Activity

Hexagon Designs (reinforcement) provides practice with the equilateral triangle construction in the context of creating designs with hexagons.

Reference: Measuring Tools—Introducing the Ruler and Protractor

Intent

Introduce the ruler and protractor.

Mathematics

Many geometric shapes may be accurately constructed using only compass and straightedge, but specifying the size of the shape requires the use of measurement tools: the ruler for lengths, and the protractor for angles.

Progression

Students had some exposure to using protractors in the *Shadows* unit, and so may be competent with their use by now. However, some students may have misconceptions about how to read measurements with either the protractor or ruler. For example, I have had many students tell me that they didn't have a centimeter ruler because their ruler said "ml" on the end. With protractors, some students are confused by the two sets of numbers. Others place the end on the angle vertex, rather than the center. Walk around the classroom as students work and watch out for these and other errors.

Approximate Time

10 minutes

Classroom Organization

Whole class

Materials

Students will need centimeter rulers and protractors frequently in this unit, both in and out of class. Either provide these tools, or ask students to acquire their own.

Doing the Activity

Have students read the material.

Discussing and Debriefing the Activity

Lead a brief discussion using the Key Questions below.

Key Questions

What are the differences between a construction, a drawing, and a sketch?

What are some modern professions that involve drawing accurate geometric figures?

Supplemental Activity

Pentagon Designs (reinforcement) provides practice using a protractor in the context of creating designs.

Drawing Triangles with Ruler and Protractor—SAS and ASA

Intent

Students will accurately draw and measure triangles given either two sides and the included angle (SAS) or two angles and the included side (ASA).

Mathematics

By drawing many specific triangles given SAS or ASA, students will gain insight as to why these criteria are reasonable as postulates for proving triangles congruent.

Progression

Students may start work immediately drawing the triangles as you circulate around the room helping individuals. Expect wide variation in student measurement skills. Many have only looked at pictures in textbooks of lines and angles drawn for them and have never been required to drawn lines themselves.

Approximate Time

40 minutes in class; finish in class or at home

10 minutes for discussion of Questions 8 and 9

Classroom Organization

Students work individually but may get help from group members, and compare their work to classmates' in Questions 3 and 7

Doing the Activity

While students work, circulate and help students who have trouble. Use of the protractor may be difficult for some, especially drawing and measuring obtuse angles.

You may want to put up the answers for the measured lengths as students work so they can check their own drawings for accuracy.

The correct lengths and angles for each problem, obtained by calculation, are:

1. In $\triangle PEA$, $AP = 6.3$ cm and $m\angle A = 54°$
2. In $\triangle PAC$, $PC=10.9$ cm and $m\angle C = 26°$
4. In $\triangle NUT$, $NT = 10.1$ cm and $TU = 8.6$ cm
5. In $\triangle CAT$, $CT = 12.9$ cm, $AT = 7.2$ cm, and $m\angle T = 34°$
6. In $\triangle MAN$, $MA = 5.0$ cm and $NA = 8.5$ cm

If your students need more practice with drawing triangles to specifications you can easily make up problems yourself and use them as daily "warm ups" until everyone becomes competent with the use of ruler and protractor. Here's a way to make up more drawing problems using the technique "start with the answer." You might introduce this strategy to students and have some of them provide warm ups too!

Step 1: Use a straightedge to draw a triangle.

Step 2: Obtain the specifications you will give to students by measuring the triangle you drew. For ASA, measure any two angles and the included side. For SAS, measure any two sides and the included angle.

Step 3: Give the measurements to students. Have them draw the triangle and then measure an "unknown" side and/or angle.

Step 4: To find the correct values for the "unknown" sides and/or angles, you can measure the values on the original triangle you drew in Step 1 (or you could calculate the values using the Law of Sines or Law of Cosines).

Discussing and Debriefing the Activity

Students can compare their measurements with the calculated values above to confirm the accuracy of their drawings. They should be able to get length measurements correct to within 0.2 centimeter and angle measurements within 2 degrees.

Discuss Questions 8 and 9 as a class. Students are demonstrating the side-angle-side and angle-side-angle triangle congruence postulates, which will be established in the upcoming activity *What Makes a Triangle?*

Key Questions

Does specifying two sides and the included angle (SAS) always result in a unique triangle?

Does specifying two angles and the included side (ASA) always result in a unique triangle?

When drawing triangles to specifications, what are some things to watch out for in order to minimize "human error"?

Supplemental Activity

Draw These Triangles provides more practice with drawing triangles to specifications.

What Else Do We Already Know?

Intent

By working and discussing the questions in this activity, students will review what they know about angles, and possibly learn a few new concepts.

Mathematics

Angle vocabulary and notation are reviewed or introduced. Vocabulary words include: bisector, straight angle, adjacent, acute, obtuse, supplementary, and perpendicular.

Progression

Each student first attempts the 20 questions alone, then with their group, and finally in a whole-class discussion.

Approximate Time

35–40 minutes

Classroom Organization

Individuals, then groups, then whole-class discussion

Doing the Activity

Have each student set up an answer column by numbering 1 to 19 in his or her notebook. Instruct students to work individually for 8 minutes to come up with as many answers as they can. Then have students compare answers with group members for another 7 minutes. Circulate around the room as groups share answers and note which ideas students seem unfamiliar with.

Finally, have groups share answers with the whole class and discuss.

If you keep a word wall or vocabulary poster in your room (highly recommended), write words up as they are discussed.

Discussing and Debriefing the Activity

Tell students that they are responsible for knowing the meaning and spelling of each word.

Note that several words can fill the blank in Question 15: supplementary, adjacent, and right angles. Ask students to share their answers to this question.

Key Questions

Some geometry texts define angle as "the union of two rays sharing a common endpoint." Other texts define angle as an "amount of turn." How are these two definitions related?

Triangle Puzzles

Intent

These two challenges are a fun review of ideas.

Mathematics

The two puzzles review ideas about triangles and circles.

Progression

Students work on the puzzles independently.

Approximate Time

5 minutes for introduction

30 minutes for activity (at home or in class)

5 minutes for discussion

Classroom Organization

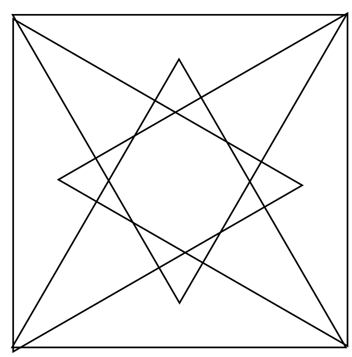
Individuals, followed by class discussion

Doing the Activity

This activity requires no introduction.

Discussing and Debriefing the Activity

In Puzzle 1, if the constructions were done carefully, a regular octagon is formed in the middle.



For Puzzle 2, students need to use the facts that the diagonals of a rectangle are equal and radii of a circle are equal. These principles will be discussed later in the unit.

Have students present their solutions to the two puzzles. In the unlikely event that a student points out that we haven't yet proven that the diagonals of a rectangle have equal length, congratulate them for thinking deductively.

Drawing Specific Triangles—SSS

Intent

Students will accurately draw and measure triangles given three sides.

Mathematics

By drawing specific triangles given three sides, students will gain insight as to why the SSS criterion is reasonable as a postulate for proving triangles congruent.

Progression

Students work individually, but compare their drawings to their classmates' in Question 3.

Approximate Time

20 minutes for activity

10 minutes for discussion

Classroom Organization

Individuals, with occasional group collaboration

Doing the Activity

Some students may attempt to draw triangles given three sides using the ruler alone. Let them try to do it this way. They will soon discover that just drawing a 7 cm. line from one endpoint of a segment doesn't guarantee a 6 cm segment from the other endpoint.

To do an SSS drawing, the compass must be used in addition to a ruler.

To draw a triangle given SSS, start by drawing one of the sides. The endpoints of this side are two of the vertices of the triangle. But how can we locate the third vertex? This is when the compass must be used. You may want to do the example described in the text and have students do it along with you.

Students should complete the drawings for Questions 1a, 1b, but will find that Question 1c describes an impossible triangle.

Students can use their measurements to check the accuracy of their drawings.

The correct angles for each triangle of Question 1, obtained by calculation and rounded to the nearest whole degree, are:

a. In $\triangle COW$, $m\angle C = 65°$, $m\angle O = 41°$, and $m\angle W = 74°$

b. In $\triangle HAT$, $m\angle H = 32°$, $m\angle A = 95°$, and $m\angle T = 53°$

c. $\triangle BAD$ is impossible to draw because $5 + 4.8 < 10$.

You may easily make up more problems like these using the "start with the answer" technique described in this Teacher's Guide in the section *Drawing Triangles with Ruler and Protractor—SAS and ASA.*

Circulate around the room as students work, helping those who are stuck, and encouraging accuracy, especially with angle measurement.

Discussing and Debriefing the Activity

Guide a brief discussion of Question 4.

Also ask a student to explain why the triangle in Question 1c is impossible to draw. They should be able to name the triangle inequality principle, which was learned in the *Shadows* unit in the activity *What's Possible?*

Key Questions

Given SSS, is a unique triangle always determined?

Given SSS, when might a unique triangle not be determined?

A Trough and a Tent

Intent

This activity provides practice with drawing triangles in applied situations.

Mathematics

Students are expected to do these two problems by making accurate scale drawings.

Progression

Students work independently on two similar problems.

Approximate Time

10 minutes for introduction

20 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals.

Doing the Activity

Tell students that these problems should be done by making accurate scale drawings. Careful measuring and drawing is required. Students will probably do SSS drawings for each question, though other approaches are possible. For example, students may realize (based on the *Shadows* unit) that all equilateral triangles are similar and relate answers to Questions 1–3.

Discussing and Debriefing the Activity

Correct values obtained by calculation, correct to the nearest whole degree and hundredth of a foot, are:

Trough:	47°
Tents:	1. 6.93 ft
	2. 4.33 ft
	3. 11.55 ft

You can easily make up more problems like these using the “start with the answer” technique as described in this Teacher’s Guide in the section *Drawing Triangles with Ruler and Protractor—SAS and ASA.*

Key Questions

Did anyone use a strategy that did not involve drawing an SSS triangle?

POW 5: Draw Your Bike

Intent

This POW provides students with practice measuring physical objects and drawing accurate geometric shapes. It is also an application of scale drawing.

Mathematics

The Bike Project reviews the basic skills of how to measure lengths with a ruler and draw angles using a protractor, and also serves as a familiar anchor in the student's mind for the geometric abstractions coming later in the course.

Students will find that when making a scale drawing, lengths must all be scaled down by the same scale factor, but angle measures must be kept the same in the drawing as they are on the bike.

Progression

Students work on the POW on their own and share their creations in class.

Approximate Time

10 minutes for introduction

1–3 hours (at home)

10 minutes for discussion and sharing

Classroom Organization

Individuals, followed by whole-class discussion and sharing

Doing the Activity

To introduce this POW, bring a bicycle into the classroom. If you can't provide a bike, ask for a student to bring one—perhaps a student who commutes to school via bicycle. If possible, set the bike up on a table or hang it up on hooks so everyone can see it. Whenever relevant, use the real bicycle to point out geometric terminology (such as in the next activity, *Bicycle Vocabulary—Geometry Notation*) and demonstrate how measurements can be taken.

This is not a partner activity. Each student should take their own measurements and make their own scale drawings. Students who don't have a bicycle can borrow one to measure, find an old one, or go to a bike shop and measure a new one. The bike may be any size or kind: mountain, motocross, road, commuter or other.

To fit the scale drawing on a page of paper, lengths must all be scaled down by the same scale factor, but angle measures must be kept the same on the paper as they are on the bike.

If you think students need more support, ask students to draw a rough sketch of the bike's frame and wheels and then write down as many geometric names as they can of the shapes they see on the bike. Then use the bike in the classroom to demonstrate how to take actual measurements (of both angles and lengths), how to scale the measurements for drawing the frame, and how to draw the wheels using a compass.

Tell your students when the project will be due (give at least a week) and how their work will be assessed.

Discussing and Debriefing the Activity

Have students post their bicycle scale drawings around the room, and conduct a gallery walk. Then bring the class together to share comments—mathematical or aesthetic—about the work.

Key Questions

What are the minimum essential measurements necessary in order to specify a triangle? A quadrilateral? A circle?

Bicycle Vocabulary—Geometry Notation

Intent

Recognize that abstract geometric shapes are used to build real objects.

Mathematics

Mathematical vocabulary and notation are routinely used in many professions, including the design and construction of bicycles.

Progression

Approximate Time

5 minutes for introduction

10 minutes for activity (at home or in class)

Classroom Organization

Individuals, or groups

Doing the Activity

This activity needs little introduction, and could be assigned as homework. It's likely that students don't know what a tangent line is, referenced in Question 9. This is a good opportunity to note that students should look up unfamiliar terms in the text's glossary, or elsewhere.

Discussing and Debriefing the Activity

Check that correct notation is used, for example that angles indicated with three letters are written in the correct order. For example, $\angle EDC$ (or $\angle CDE$) is an obtuse angle. But $\angle ECD$ or $\angle CED$ are not angles in the drawing.

Have students compare answers with their group mates and resolve any disputes. Students will likely have different answers, but many possibilities are correct.

What Makes a Triangle?

Intent

This activity introduces triangle congruence postulates, which will be used to justify statements made in proofs.

Mathematics

Prior work drawing triangles to specifications should convince students that ASA, SAS, and SSS are reasonable to use as postulates for congruence.

(However, in Euclid's original work, ASA and SAS were postulates, while SSS was proven.)

Progression

Have students read the activity and then write answers to the Questions 1 and 2. Discuss these answers. Then work on the questions in the Tricky Triangles section.

Approximate Time

35 minutes (at home or in class)

10 minutes for discussion

Classroom Organization

Individual, group, and whole-class discussion

Doing the Activity

Students can work on this activity independently, or you could work through the first section as a class discussion. Be sure to note that the activity includes the Tricky Triangles section on page 136.

Discussing and Debriefing the Activity

Have students share answers to the Tricky Triangles questions:

3. Impossible because angles sum to more than 180°
4. Many triangles are possible because only one side and one angle are specified.
5. Many triangles are possible but they are all similar. (This relies on recollection of AA from the *Shadows* unit.)
6. SAS, so one unique triangle is determined. $UT = 9.25$ cm.
7. This triangle is impossible because $4 + 2 < 7$. (The triangle inequality principle was learned in the *Shadows* unit.)
8. Many triangles are possible because only two sides are given.
9. SSS, so one unique triangle is determined. $\angle D = 159°$

Triangle Congruence Applications

Intent

This activity provides contexts to which students apply triangle congruence postulates.

Mathematics

Scale drawing and application of triangle congruence postulates are practiced.

Progression

Students work independently on the questions.

Approximate Time

5 minutes for introduction

15 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals

Doing the Activity

This activity may be assigned as homework.

Discussing and Debriefing the Activity

Use a projector to display student drawings for Question 2. . By scale drawing, a value of *DC* between 810 and 820 would be acceptable. The height of the hill, calculated using the law of sines, is 814.8 feet.

Reference: Our Postulates and Theorems So Far

Intent

In this reference section, students review the postulates we will use to justify statements made in the proofs that follow.

Mathematics

A deductive system is the foundation of most branches of modern mathematics. A deductive system consists of clearly stated postulates/axioms, and theorems that logically follow from those postulates. Geometry was the first branch of mathematics to use such a system. Beginning in the 1700's, number theory, algebra, calculus, probability, and statistics were also organized by mathematicians into deductive systems.

Mathematicians use intuition, experimentation, and observation to discover and develop new theorems. But ultimately, the gold standard for asserting the truth of a theorem requires a logical proof within a deductive system. Because of this, mathematics is often called "the exact science."

In many of the activities that follow, students will use a deductive system based on the postulates studied in this section to prove elementary geometric theorems.

A Note on Numbering: The postulates and theorems have been numbered in this text for convenience. For 2000 years, geometry courses used Euclid's original numbering scheme. But in recent years, numbers have been changed at various times by various authors. The numbering scheme used here is not Euclid's.

Progression

Have students read the list of fourteen postulates and three theorems, and reference them as the complete the next activity, *But Why?* That activity will reinforce understanding of the postulates and axioms.

Approximate Time

10 minutes to read and discuss

Classroom Organization

Individuals or whole-class

Doing the Activity

You can have students read these reference pages independently, or you can do so as a class.

Discussing and Debriefing the Activity

When doing proofs, students will be referring back to these pages of postulates often. You may want to have students flag this page with a sticky note or tab.

Key Questions

What is a deductive system?

When making a logical argument, what are the advantages of clearly stating the postulates on which the argument is based?

Many people assert that every high school student should take and pass geometry. What might be some of the postulates people are assuming to support this assertion?

But Why?

Intent

In this activity, students use postulates or theorems from *Our Postulates and Theorems So Far* to justify statements.

Mathematics

Students begin to practice justifying statements with established postulates or theorems—the foundation of a deductive system.

Progression

Students can work independently or in groups, referring back to *Our Postulates and Theorems So Far*.

Approximate Time

10–15 minutes

Classroom Organization

Individuals or groups

Doing the Activity

Students will reference *Our Postulates and Theorems So Far*.

Discussing and Debriefing the Activity

Have groups discuss their answers. Then have a whole class discussion by calling on a different group for the answer to each question.

Proving the Isosceles Triangle Theorem

Intent

This activity introduces the two common formats for writing a proof: paragraph and two-column.

Mathematics

In the Middle Ages, the Isosceles Triangle Theorem was called the *pons asinorum,* literally translated from Latin as Bridge of Asses/Fools. Euclid's proof involved adding lines that made the diagram look like a bridge.

According to Wikipedia, "There are two possible explanations for the name *pons asinorum,* the simplest being that the diagram used resembles an actual bridge. But the more popular explanation is that it is the first real test in *The Elements* of the intelligence of the reader and functions as a 'bridge' to the harder propositions that follow."

It has been the experience of this writer over 30 years of teaching that the aptitude for proof of students today follows the same normal distribution as those in the Middle Ages. You will likely find that a few of your students will understand the Isosceles Triangle Theorem proof immediately, while many others will struggle to understand what is being said.

Be patient with those who "don't get it." Encourage them to keep trying, read carefully, ask questions, draw diagrams, and hold the big picture in mind while working on the details.

A Warning About Proofs:

Although proofs are considered essential in higher mathematics, whether to emphasize proof in a high school geometry course is controversial. In 1957, the seminal work done by the Van Heiles established five *Van Heile levels*: Visualization, Analysis, Abstraction, Deduction, and Rigor. The Van Heiles postulated that all people must progress, in order, through the sequence of levels. Their research showed that the main reason many high school students are unsuccessful doing proofs (the Deduction level) is because they have not had enough experience with the prior levels: Visualization, Analysis, and Abstraction. In *Geometry by Design*, work so far has been at these first three Van Heile levels.

An excellent article on the Van Heile levels, including questions you can use to assess your students' readiness for undertaking proofs, is *Spadework Prior to Deduction in Geometry,* by Mike Shaughnessy and Bill Burger (*Mathematics Teacher,* vol 78, September 1985).

You will need to decide whether your class is ready to tackle proofs. If your students have done well so far in this unit, they may be ready to undertake proofs. However, if your students have found the work so far in this unit to be challenging, then proofs will likely be difficult, if not impossible for many of them. You will have two options: 1) Spend more time constructing, drawing, and discussing the properties of basic figures, such as triangles and quadrilaterals, before moving into proofs; or, 2) De-emphasize proofs in this section and instead focus on basic constructions and POW 6. Later work in this unit on transformations does not require proofs as a prerequisite, nor does any other unit in this *Geometry* course.

Progression

Tell students they will be undertaking some classic geometric proofs which students in schools have been challenged to understand since ancient times. Not only will they be trying to understand the proof, but they also will be learning the format for proofs they will be writing on their own.

First present and discuss the paragraph version, then the two-column version. You may prefer to require students to use the two-column format when they write their own proofs.

Approximate Time

15–20 minutes

Classroom Organization

Class discussion, then groups

Doing the Activity

Work through the two versions of the proof with the whole class. Each individual should write the proofs (including diagrams) in their own notebook as the proofs are discussed.

First have the class read through the paragraph proof aloud, copying each sentence into their notebooks as it is read. Point out:

1. Each assertion is justified by one of our agreed-upon postulates.
2. The assertions are ordered logically so they make sense.
3. At the beginning of the proof the “givens” are restated, then intermediate steps build the argument, and finally the conclusion is the theorem itself.

Use the same approach to lead the class through the two-column version of the proof.

After working through the original versions of the proof, students can work in their groups on the section Another Way to Prove the Isosceles Triangle Theorem.

Discussing and Debriefing the Activity

Have volunteers share the midpoint version of the proof.

Key Questions

Focus on the two-column format. Is the order of the steps in the proof important? Are the steps in a logical order? Which steps could

be ordered in a different way and still be in a logical order? Which steps must only come after others?

How is the format of a proof like the steps in a trial in a courtroom?

In the U.S. Declaration of Independence, what theorem was Thomas Jefferson trying to prove? What is significant about the order of the sentences that Jefferson used?

Using the Isosceles Triangle Theorem

Intent

Provide students with a context to explore the meaning of the Isosceles Triangle Theorem.

Mathematics

Students practice accurate drawing, and complete a proof that uses the Isosceles Triangle Theorem as a “reason.”

Progression

May be assigned in class or as homework.

Approximate Time

5 minutes for introduction

25 minutes (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals

Doing the Activity

In Question 1a, students need to first figure out that the base angles are each 35°, then draw the triangle using ASA. When accurately drawn, the equal sides will measure 6.1 cm.

In Question 1b, students need to calculate that the vertex angle is 72°, then draw the triangle using SAS. When accurately drawn, the base will measure about 8.8 cm.

You may easily make up more problems like these to give to students using the “start with the answer” technique as described in this Teacher’s Guide in the activity *Drawing Triangles with Ruler and Protractor—SAS and ASA*.

Question 2 involves the Isosceles Triangle Theorem and the Vertical Angles Theorem. Note that $\triangle CLO$ is not congruent to $\triangle NEO$.

Discussing and Debriefing the Activity

Have one or more volunteers present their proofs for Question 2.

Deduce Those Angles

Intent

This activity applies several of the postulates and theorems established thus far.

Mathematics

Students will use T2: Sum of the angles of a triangle is 180°, and T4: Isosceles Triangle Theorem.

Progression

Students work on this activity independently.

Approximate Time

15–20 minutes (at home or in class)

5 minutes for discussion

Classroom Organization

Individuals

Materials

Deduce Those Angles blackline master

Doing the Activity

This activity needs no introduction. It may be assigned for homework.

Discussing and Debriefing the Activity

Have a student present his or her process for finding the angles.

Selected answers are: $m\angle a = 37°$, $m\angle d = 58°$, $m\angle h = 84°$, $m\angle m = 26°$, and $m\angle s = 58°$

Proofs for You!

Intent

This activity provides experience completing proofs that use the prior postulates and theorems as "reasons."

Mathematics

Given information about geometric figures, new properties of the figures are proven.

Progression

Students copy each diagram, and the "given" and "prove" statements. Then they write out the proof in either paragraph or two-column format.

Approximate Time

Note: If you have decided to deemphasize proofs you may omit this activity.

20 minutes to complete Question 1

35 minutes to complete Questions 2 and 3 (at home or in class)

Classroom Organization

Individuals

Doing the Activity

Circulate around the room as students work. Some may need help getting started. Talking through the steps of the proof before writing sometimes helps.

You may want to pick one of the problems, perhaps Question 2, and do it with the whole class, and then have students work on the other two independently.

Discussing and Debriefing the Activity

Assign each group one proof to share with the class. You may assign two or three groups to present the same proof and then compare for similarities and differences. Class will move along more quickly if you have students write their proof up on the board, overhead transparency, or document camera before the presentations begin.

Supplemental Activity

Congruent Triangle Proofs (reinforcement) includes six more proofs which may be assigned any time after this lesson.

Construction and Deduction

Intent

This section introduces some basic ancient straightedge and compass constructions, and allows students to develop their proof skills.

Mathematics

The construction techniques introduced in this section were known to many ancient peoples and used not only in drawing designs but also in laying out buildings and defining boundaries for land parcels. The first people to rigorously prove that these constructions work were the Greeks, around 500 B.C.E. In the Greek version of geometry, only these constructions were allowed when producing geometric figures and diagrams.

Progression

Students are introduced to some foundational constructions in this section—the perpendicular bisector and angle bisector. They prove why these constructions work, and continue their work with proof to explore and prove properties of various quadrilaterals.

This section includes the second POW of the unit, *How Did Humans Create Mathematics?* By completing this POW and listening to the presentations of others, students will learn about the historical significance of the math they are studying and help answer the perennial student question: "Why are we learning this?"

Walking Sets of Equidistant Points

Perpendicular Bisector Explorations

Why It Works—The Perpendicular Bisector Construction

Construction Challenges

POW 6: How Did Humans Create Mathematics?

Angle Bisector Explorations

Properties of Special Quadrilaterals

Parallel Lines and Transversals

Reference: Parallelograms and Trapezoids

Parallelogram and Trapezoid Proofs

Walking Sets of Equidistant Points

Intent

Introduce the idea of **equidistant.**

Equidistance from a point, from two points, from a line, from two lines, and from a line and a point, are the premises underlying all the classic compass and straightedge constructions.

Mathematics

The basic Greek constructions are all done only using compass (or string) and straightedge. When a person produces a diagram using only compass and straightedge, he or she is actually drawing a set of points satisfying particular conditions, called a *locus*. The simple, yet elegant, Greek constructions all involve conditions with one or more given points, one or more given lines, and equidistance. For example, the locus of points equidistant from a given point is a circle. The compass is the tool the Greeks used to mark off sets of equidistant points.

In this activity, the circle, perpendicular bisector, angle bisector, and parabola are each defined as a locus of points. (Note: Although we will not be constructing a parabola with compass and straightedge, students should be familiar with this shape from the *Fireworks* unit in Algebra 1.)

Progression

After a brief discussion of the meaning of *equidistant,* students work in groups on the activity. Then groups demonstrate their modeling of the sets of points for the class.

Approximate Time

15 minutes for group work

25–30 minutes for kinesthetic group presentations

Classroom Organization

Groups and then whole class

Materials

Optional: About 6 to 10 feet of string per group

Doing the Activity

Write the word *equidistant* on the board and discuss its meaning. Then direct students to begin working in groups on the activity. Tell them that after 15 minutes, each group will be randomly assigned one or more of the sets to act out (model) using group members and, if needed, string.

Circulate as groups work, giving minimal hints as needed.

Note which groups are successful with which sets of points, and use this information to choose groups to make presentations. For example, you may want to assign a group that is working more slowly to present the circle, while a group flying through the problems might be assigned the parabola.

Discussing and Debriefing the Activity

After the 15 minutes of group preparation time is up, begin the presentations.

Before presentations begin, have students write the title DEFINITIONS on a page in their notebooks.

When each group presentation is complete and students agree on which geometric object is formed, have students write out the relevant sentence and its completion. For example, "The set of points equidistant from <u>a given point</u> is a <u>circle</u>."

In subsequent lessons students will prove that the definitions for perpendicular bisector and angle bisector do in fact give the result named in the description.

Note: The statement "The set of points equidistant from <u>two parallel lines</u> is a <u>parallel line</u>" is the only one that is not a definition.

Key Questions

Compare these two definitions:

> **<u>Dictionary</u>: A circle is a round, plane figure.**
> **<u>Geometry class</u>: A circle is the set of points in a plane equidistant from a given point.**

List all the possible conditions that involve the words "set of points equidistant from" and the words "one point" or "two points", and/or "one line" or "two lines." Which of the possibilities are included in the list we studied? Of those not included, which are possible, and which are impossible?

Perpendicular Bisector Explorations

Intent

Introduce and use the perpendicular bisector construction.

Mathematics

Students use compass and straightedge to draw the set of points equidistant from the endpoints of a segment. In the next lesson, students will prove that this is the perpendicular bisector of the segment.

Progression

In Questions 1 and 2 students follow written instructions for the construction. In Questions 3 and 4 students construct the perpendicular bisectors of the sides of a triangle, and find that they meet at a single point. In classical geometry this point is called the **circumcenter** because it is the center of a circle that passes through the three vertices of the triangle.

Approximate Time

5 minutes for introduction

25 minutes for activity (at home or in class)

20 minutes for discussion

Classroom Organization

Individuals

Doing the Activity

Students have encountered perpendicular bisectors several times now, in the *Shadows* unit and in *Walking Sets of Equidistant Points.* This activity needs no or minimal introduction. However, students who have trouble reading or following the steps in Question 1 may need a demonstration of the construction.

Discussing and Debriefing the Activity

Check student work to be sure each student can do this construction accurately.

Ask students whether they noticed anything interesting about the perpendicular bisectors of the sides of a triangle in Questions 3 and 4. If they did their constructions well, they should have found that they meet in a point. If they're curious, let them know this point is called the **circumcenter** of a triangle. However, they need not be required to know this vocabulary term.

Try to get students to explain *why* they meet in a point—asked in Question 3c and 4c. It has to do with the fact that every point on the perpendicular bisector of a segment is equidistant from the endpoints. So, suppose you have triangle ABC. The perpendicular bisectors of sides $\overline{AB}$ and $\overline{AC}$ will intersect at a point—let's call it P. The distance from P to A is the same as the distance from P to B by the definition of perpendicular bisector (of $\overline{AB}$). The distance from P to A is also the same as the distance from P to C by the definition of perpendicular bisector (of $\overline{AC}$). So because $PB = PC$, the perpendicular bisector of $\overline{BC}$ will also pass through point P. Thus P is equidistant from all three vertices of the triangle.

Selected answers, rounded to nearest degree and nearest tenth of a cm are:

Question 3a: $m\angle C = 26°$, $m\angle P = 49°$

Question 4a: The third side measures 12.9 cm.

Key Questions

Must the three perpendicular bisectors of the sides of a triangle meet in a single point? Why?

Why It Works—The Perpendicular Bisector Construction

Intent

In this activity, students use postulates and theorems to prove that the perpendicular bisector construction indeed produces a line that is both perpendicular to and bisects the given segment.

Mathematics

The Greeks proved that each of their constructions done with compass and straightedge actually did produce the desired result.

Progression

Students fill in missing statements and reasons in a two-column proof.

Approximate Time

15–20 minutes

Classroom Organization

Whole class or groups

Doing the Activity

You might talk through this proof as a class, or have students work in groups and then share results in a whole-class discussion.

Construction Challenges

Intent

Students learn that the distance from a point to a line is defined as the shortest path, which is a perpendicular. They also apply their skill at constructing perpendicular bisectors in several contexts.

Mathematics

The shortest distance from a point to a line lies along the perpendicular to the line. Euclid proved this assertion using a technique known as proof by contradiction. We will not prove this assertion. Instead, an example is provided so students can see it is reasonable. Then we focus on altitudes of triangles, which is used in the upcoming *Do Bees Build It Best?* unit to find areas of triangles.

Progression

Read through What's the Shortest Path and briefly discuss. Students work through the remainder of the activity on their own.

Approximate Time

5 minutes for introduction and to begin activity

30 minutes to complete activity (at home or in class)

Classroom Organization

Whole-class, then individuals

Doing the Activity

Start in class and assign the remainder as homework.

Discussing and Debriefing the Activity

Check student notebooks to see that each student can construct the perpendicular from a point to line accurately.

Selected answers correct to one decimal place are:

Question 3: h = 4.3 cm

Question 4: h = 2.0 cm

It is important that students are able to construct a perpendicular to a line through a given point. Have a student demonstrate, or do so yourself. Then ask, **Can you prove that this construction does in fact produce a line perpendicular to the given line?** Give students a few minutes to work on this in their groups, or do it as a class, or propose that motivated students work on it on their own time.

If your students need more practice, you can easily make up additional problems like these using the "start with the answer" technique described on pages 14–15 of this Teacher's Guide. The additional problems can be assigned as homework or warm ups and used on quizzes.

Students already know what a square looks like and constructing one using only compass and straightedge should not be difficult. Later in this unit, a square will be defined to be a quadrilateral that is both equilateral and equiangular.

Key Questions

Can you prove that this construction does in fact produce a line perpendicular to the given line?

When will an altitude of a triangle lie outside the triangle?

POW 6: How Did Humans Create Mathematics?

Intent

Involve students in learning about the historical roots of mathematics and the significance of mathematics in the evolution of human culture. Work on this POW will build students' research skills, and takes advantage of the Internet as a primary source of knowledge and modern technology in making presentations.

Mathematics

Much of the mathematics students study in schools today is the same as the mathematics their parents studied, and the parents of their parents before that. Therefore, students, and even many parents, think that mathematics is an unchanging, static subject that has always been the way it is and will always remain the same. But the mathematics we study and use was all created gradually over the centuries by the human mind and continues to evolve today.

Progression

After an introduction, students work independently or in pairs on POW 6. Several days will be needed for presentations.

This is an extremely interesting and valuable POW. I've been amazed at the quality of student presentations and I strongly suggest you undertake it with your class. However, if you don't have time for it in your schedule, consider assigning *POW 7: Build a Model Roof,* found in the Supplementary Activities.

Approximate Time

20 minutes for introduction and brief discussion of topics

20 minutes to choose partners and assign topics

8–10 hours to prepare presentation (at home or library with internet access)

10 minutes of class time per presentation (After *Isometric Transformation 2: Rotation* might be a good time in the unit to begin *POW 6* presentations.

Classroom Organization

Individuals, followed by whole-class presentations

Materials

POW 6: How Did Humans Create Mathematics? blackline masters (found at the end of this *Teacher Guide*)

Students will need access to computers and presentation software (such as PowerPoint® or Prezi (www.prezi.com) The classroom will need a computer and projector for the presentations.

Doing the Activity

The presentations to the class are the most fascinating part of this POW. By preparing their own presentations, students will know intimately one aspect of mathematical history. By listening to other students present, they will broaden and deepen their image of math and its place in our culture.

<u>Getting Started</u>

The blackline master *POW 6: How Did Humans Create Mathematics?* (at the end of this *Teacher's Guide*) can help your students prepare this POW.

Assign pairs randomly or let students choose a partner or work individually.

You may assign topics randomly or let students choose topics, first-come, first-served.

Students will need several days to prepare their presentation. To get them started, you may want to spend one or two class periods in the school library or computer lab to let then research and set up their presentations. Beyond these periods, they work outside of class to complete their presentations.

The four lessons in the textbook that follow POW 6 are designed to be started in class with little introduction and then can be assigned as homework. By doing this, you can allow some class time for working in the computer lab or library on POW 6.

<u>Making the Presentations</u>

You can schedule the presentations over several weeks, three or four per week, or you might want to allocate several consecutive days for the presentations. Students can make better sense of the "big picture" if presentations are given in the order in which they are listed in the book.

However, *on the deadline you've set* (the day on which the first presentations will occur), save the slides from *all* students onto the computer that will be used for presentations. If you have multiple classes, make one folder per class. After this deadline, allow no more work on presentation slides—this ensures that everyone gets the same amount of time to prepare their presentation and also ensures that everyone pays attention during the presentations (instead of feverishly working last-minute on their own).

To help the process of the presentations move along, the two-page blackline master at the end of this Teacher's Guide may be used or modified. The form is based on a 120-point scale. You might want to make changes in these forms to meet your needs and then print then out for your students.

Discussing and Debriefing the Activity

After the presentations are complete, you may want to have each presenter produce a poster about the contents of their presentation. The posters will serve as a review and reminder when displayed around the classroom.

To review POW 6, you can use the set of multiple choice questions that each pair made up after their presentation (see the blackline master). Have each pair email you their questions, then cut and paste the questions to make up a quiz. Have students take the quiz as individuals, in a group, or as a whole class, with notes allowed.

Angle Bisector Explorations

Intent

Students learn the angle bisector construction, apply it, and prove that it works.

Mathematics

The angle bisector construction was first described and proven in Euclid's *Elements*. It, together with the perpendicular bisector construction, form the basis for many other constructions in geometry, such as constructing a regular pentagon and constructing an area equal to another area, but with a different shape.

Progression

Students learn the angle bisector construction, apply it to discover a property about the angle bisectors of the three angles in a triangle, then prove that the construction works.

This activity can be started in class and completed at home.

Approximate Time

20–25 minutes in class; finish at home if needed

5 minutes for discussion

Classroom Organization

Individuals

Doing the Activity

Have students read and follow directions to make drawings in their notebooks. They can work independently on this activity.

Discussing and Debriefing the Activity

Check that each student has completed the assignment. Look for clean constructions and accurate drawings indicating that students understand the construction.

In Questions 3 and 4. students construct the angle bisectors of the angles of a triangle, and find that they meet at a single point—the **incenter**. It is not important that students know this term.

Selected answers:

Question 3a: $MA = 8.5$ cm, $NA = 4.99$ cm

Question 4a, to nearest degree: $m\angle T=38°$, $m\angle A=31°$

To answer 3c will take some thinking. Ask students to offer their explanations for why the angle bisectors all meet in a point. Here is one possibility: Recall that a perpendicular bisector is the set of points equidistant from the endpoints of the segment. But in a triangle, each vertex is the endpoint for two segments. Therefore, the point common to the bisectors of all three endpoints must be equidistant from all three vertices.

Have one or more groups display their proof of *Why it Works—The Angle Bisector Construction* for the class to see and discuss.

Properties of Special Quadrilaterals

Intent

Introduce rectangle, rhombus, and square in the context of Euclidean geometry.

Mathematics

Most students come to high school with a clear image of what rectangles and squares are. But they may not be familiar with rhombi (the plural of rhombuses).

The goal in this lesson is to relate what students already think they know about rectangles, rhombuses, and squares to the Euclidean definitions and framework of postulates and theorems we have been building.

Progression

After a brief introduction, students conjecture about and prove properties of rectangles, rhombi, and squares.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

10 minutes for discussion

(You may want to use any available class time to take your students to the computer lab or library to work on POW 6.)

Classroom Organization

Individuals

Doing the Activity

After a brief introduction, students work independently.

If you have decided to deemphasize proofs in your class, you may make the proof portions of each problem optional.

Discussing and Debriefing the Activity

Have students offer their conjectures and proofs about rectangles, rhombi, and squares.

Supplemental Activity

Draw These Quadrilaterals (reinforcement) may be assigned any time after this lesson. This activity continues building students' skills of drawing to specifications, as well as reinforcing quadrilateral properties and terminology.

Parallel Lines and Transversals

Intent

Students will relate previous knowledge about parallel lines and transversals from the *Shadows* unit to the postulates for parallels as defined in a Euclidean deductive system.

Mathematics

One postulate and one theorem about parallels are introduced:

> P15. If two lines are cut by a transversal, the lines are parallel if and only if alternate interior angles are equal.
>
> T4. If parallel lines are cut by a transversal, then corresponding angles are equal.

Progression

Students learn a postulate and theorem related to parallel lines and transversals, and then apply these properties and previously-known ones to find angles in a diagram.

This activity may be started in class and completed as homework.

Approximate Time

5 minutes for introduction

15 minutes for activity (at home or in class)

5 minutes for discussion

(You may want to use any available class time to take your students to the computer lab or library to work on POW 6.)

Classroom Organization

Individuals

Materials

Parallel Lines and Transversals blackline master

Doing the Activity

The Postulate Puzzle helps students apply what they have learned.

Discussing and Debriefing the Activity

Ask students how many angles they found that are equal to *BEH.* Have the student who found the most to explain which ones are equal, and how they know.

Reference: Parallelograms and Trapezoids

Intent

Define parallelograms and trapezoids.

Mathematics

Definitions of parallelogram and trapezoid are given, as well as information on how to correctly name a polygon.

Progression

Review the information in this reference section, either as a class, or through independent reading.

Approximate Time

5 minutes

(You may want to use any available class time to take your students to the computer lab or library to work on POW 6.)

Classroom Organization

Individuals

Doing the Activity

Students may be familiar with the definitions of parallelograms and trapezoids already. Check that they understand how to name a polygon, with the vertices in order

Parallelogram and Trapezoid Proofs

Intent

Theorems about parallelograms and trapezoids are established, and students practice proof skills. If you have decided to deemphasize proofs for your class you may choose to omit this problem.

Mathematics

Proofs about properties of parallelogram and trapezoid are written and presented.

Progression

Students are shown a proof for

> Theorem T5: The opposite sides of a parallelogram are congruent.

They then prove four more theorems about parallelograms and trapezoids themselves.

Approximate Time

10 minutes to discuss T5

20 minutes for activity Questions 1–4 (at home or in class)

20 minutes for presentations

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students should work on these proofs individually to build their proof skills. If you wish, specify if students must use paragraph or two-column format. Some students will find these very challenging.

Discussing and Debriefing the Activity

You might have different groups or individuals present each proof. Be sure students realize that these theorems can now be used themselves as justifications in other proofs.

Isometric Transformations

Intent

This section introduces students to the isometric transformations: *reflection, rotation,* and *translation*.

The approach used supports the Common Core State Standards' focus on transformational geometry.

Mathematics

Many of our English words used in measurement and geometry have Greek origins. "Isometric" originates from the Greek words *isos* meaning "equal" and *metros* meaning "measure." Isometric transformations preserve both size and shape, and therefore all angle and length measurements. The image of a figure under an isometric transformation such as reflection, rotation, or translation is congruent to the original. In the next section students will explore a non-isometric transformation: dilation.

The three isometric transformations are available as operations in almost all computer drawing and drafting software. In this unit, students will be drawing images by hand using compass and straightedge or on coordinate axes. Students will gain essential understandings about these basic transformations in a kinesthetic way when they complete the problems by hand.

However, if you have access to software such as iDraw® or The Geometer's Sketchpad®, you may want to work on some activities using software. In particular, the activity *Frieze Frame* could be done using drawing software on a computer or tablet. This section and the next introduce coordinate geometry versions of constructions and transformations. All computer drawing programs are based on coordinate geometry, even if the software itself doesn't require coordinates as inputs.

Some students may find work in this section tedious, particularly the coordinate geometry work, because drawing accurate graphs is required. To those students you may want to point out a famous quote attributed to Euclid. Euclid did his work at the Library at Alexandria established by the first king of Egypt, Ptolemy I. King Ptolemy asked Euclid to teach him Geometry. The King was having a hard time understanding the material, so he asked Euclid (paraphrased), "Isn't there an easier way for me do this stuff? After all, I'm the King." Euclid replied, "Oh King, in the real world there are two kinds of roads, roads for the common people to travel upon and roads reserved for the King to travel upon. *In geometry, there is no royal road."*

Persistence and patience will be rewarded with understanding and correct answers.

Progression

Students learn reflection, then rotation, then transformation, both with and without a coordinate system. When coordinate systems are used, algebraic skills are practiced.

Isometric Transformation 1: Reflection

Reflection Challenges

Reflecting Lines

Reflection Designs

Isometric Transformation 2: Rotation

How Many Ways from A to B?

Rotation with Coordinates

Sloping Slides

Perpendicular Rotations

Isometric Transformation 3: Translation

Frieze Frame

Translation Investigations

Translation Designs

Transforming One Shape Into Another (optional)

Isometric Transformation 1: Reflection

Intent

Define reflection and learn how to reflect using a compass and straightedge.

Mathematics

A reflection is defined by specifying a line, called the *line of reflection.* After a reflection, a point and its **image** are equidistant from the line of reflection, but on opposite sides. Points lying *on* the line of reflection are their own image.

A Note About MIRAs: Transformational geometry was a popular geometric topic introduced in the 1970s, but then fell out of favor in the 1980s. If you look around at the old supplies in your math department, you might find some red plastic tools used back then for making reflections called MIRAs. You might want to supplement the reflection activities in this section with MIRA activities.

Progression

Students read and reflect on the two pages of the introduction. Then they learn how to use a compass to reflect a point.

Approximate Time

20 minutes

Classroom Organization

Whole class and individuals

Doing the Activity

After reading and discussing the introduction, students should try Reflect a Point With a Compass.

Writer's note: The woman in the picture is my mother. This is the photo that my dad carried with him while flying a B-29 bomber during World War II.

Discussing and Debriefing the Activity

Check that students have completed Reflect a Point With a Compass correctly. A common student error is to draw circles with center at point *P*—centers should be on line *m.*

Key Questions

How does reflection in geometry relate to reflection in a mirror?

Why does the construction produce the image of *P* under reflection? (Hint: If *P'* is the image of *P,* then the line of reflection must be the perpendicular bisector of *PP'*.)

Recall the construction of a perpendicular from a point to a line. How might this construction be used to reflect a point?

Supplementary Activity

The Girl and the Mirrors (extension) can be assigned any time after this lesson. It’s best if you first discuss the question, “How does reflection in geometry relate to reflection in a mirror?”

Reflection Challenges

Intent

Provide students with experiences reflecting figure using compass and straightedge, and on a coordinate system.

Mathematics

Reflection may be done within different contexts, yet the idea remains the same.

Progression

Students start by reflecting without coordinates using the ancient tools. Then they reflect images on a coordinate system.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

5 minutes for discussion

Classroom Organization

Individuals, with support from group members

Doing the Activity

Individuals each produce their own drawings in their notebooks. Provide graph paper if it's not already in their notebooks.

Discussing and Debriefing the Activity

The construction of the reflection described in Question 3 may be done by reflecting each vertex of the original triangle across line j and then connecting the image vertices with line segments. Another method depends on noticing that the points of intersection of the triangle and line of reflection are their own image. These two additional points can be used to help draw the image triangle.

After specific images under reflection are drawn, parts b of Questions 5–7 should be generalized:

Question 2b: Across the y-axis, image point of (p, q) is $(-p, q)$.

Question 3b: Across the x-axis, image point of (p, q) is $(p, -q)$

Question 4b: Across the line $y = x$, reflections are along lines at 90° to $y = x$. The image of point (p, q) is (q, p).

Key Questions

After reflection across the *y*-axis, why does the image point have the same *y*-coordinate as the original point?

After reflection across the *x*-axis, why do the original point and the image point have the same *x*-coordinate?

Consider the reflection of a point across the line $y = x$. Use a diagram to explain why the *y*-coordinate of the original point becomes the *x*-coordinate of the image point.

Reflecting Lines

Intent

Given an equation of a line, find the equation of its reflected image.

Mathematics

In a graphics computer program, images can be stored either as a set of pixel coordinates called bitmap images, or as a set of mathematical equations called vector images. Images stored as a set of pixels become grainy when enlarged or moved. But images held as equations can be enlarged and manipulated without changing the resolution. Most high-end graphics programs, and all video games, store images that are to be moved as a set of equations.

In this activity, students learn how an equation for a line is changed to get a new equation for the image of the line after reflection.

This activity provides a review of the graphs of linear equations.

Progression

Work in this activity parallels work in the prior activity. First the line $y = mx + b$ is reflected across the y-axis, then the x-axis, then the line $y = x$.

Approximate Time

35–45 minutes

Classroom Organization

Individuals, supported by group members and teacher as needed

Materials

Graph paper

Graphing calculators

Doing the Activity

Before doing this activity, you may want to briefly review with your students the slope intercept form of a line, $y = mx + b$ (also written as $y = ax + b$), and that m gives slope, while b is the y-intercept.

Students should carefully think about the reflections of $y = 2x - 6$ before attempting to generalize in Questions 4, 8, and 12.

Discussing and Debriefing the Activity

For Questions 1–4, the y-intercept of the image line is the same as the original line, and the slope is the negative of the original slope.

For Questions 5–8, the *x*-intercept of the image line is the same as the original line, and the slope is the negative of the original slope.

For Questions 9–12, the *y*-intercept value becomes the value of the *x*-intercept of the image line, and the slope of the image line is the reciprocal of the original slope.

Assign different groups to use a diagram to explain each pair of Key Questions below to the class

Key Questions

After reflection across the y-axis:

> **Why does the image of $y = mx + b$ have the same y-intercept?**
>
> **Why is the slope of the image of $y = mx + b$ equal to $-m$?**

After reflection across the x-axis:

> **Why does the image of $y = mx + b$ have the same x-intercept?**
>
> **Why is the slope of the image of $y = mx + b$ equal to $-m$?**

After reflection across the line $y = x$:

> **Why does the image of $y = mx + b$ have an x-intercept with the same value as the original line's y-intercept?**
>
> **Why is the slope of the image of $y = mx + b$ equal to $1 / m$?**

Reflection Designs

Intent

Images that can be reflected onto themselves are said to have *reflectional symmetry.* Reflectional symmetry is also called *line symmetry,* which students probably learned about in earlier grades.

Mathematics

Lines making up figures with reflectional symmetry about the *x*- or *y*-axis (or both) have equations related by the work in the prior lesson.

Progression

Students may use “guess and check” or algebra to write equations for the lines to produce the given graphing calculator images.

Approximate Time

20 minutes (at home or in class)

(Begin POW 6 presentations after this activity.)

Classroom Organization

Individuals

Doing the Activity

A graphing calculator (or graphing software) is needed for this activity, so be sure it’s available if you assign this for homework.

Discussing and Debriefing the Activity

Possible lines for Question 1 are: $y = x + 4$, $y = -x + 4$, $y = x + 8$, and $y = -x + 8$.

Possible lines for Question 2 are: $y = 4$, $y = -1.5x + 10$, $y = 1.5x + 10$, $y = 0.5x - 1$, and $y = -0.5x - 1$.

Isometric Transformation 2: Rotation

Intent

Introduce the isometric transformation of rotation.

Mathematics

A rotation is specified by a given point, called the *center of rotation,* and an *angle of rotation.* A positive rotation angle is counterclockwise. A negative rotation angle is clockwise.

The first page of this activity shows a mandala. The word *mandala* has its origins in Sanskrit and is used to describe a geometric figure representing the cosmos or universe. Many religions, including Hinduism, Buddhism, and Christianity, have mandala symbols with rotational symmetry symbolizing the circle of life. Note that the mandala shown has rotational symmetry and reflectional symmetry. This mandala has 8 identical sections, so the least number of degrees to map it onto itself is a turn of 360°/8, or 45°. Therefore, any integer multiple of 45° will map the mandala onto itself.

A Note About Tracing Paper: Tracing paper may be helpful when doing rotations. (A kind of tracing paper that works well is called “patty paper” because it is used to separate raw hamburger patties and wrap food. You might find some in your school cafeteria!) To use tracing paper in these activities, put the point of a compass through the tracing paper at the center of rotation, trace the figure to be rotated, and then spin the tracing paper through the angle of rotation to locate the image.

Progression

Students read through the introductory material and then begin questions that explore rotation.

Approximate Time

40–50 minutes

Classroom Organization

Groups can work on the activity, but each individual should write responses into their own notebooks.

Doing the Activity

Students will need rulers, compass, protractor, and graph paper.

Allow students time to complete this lesson, then assign groups to explain each part to the class.

Discussing and Debriefing the Activity

For Question 2, try to get students to give verbal descriptions. For reflections, have students describe the line of reflection; and for rotations, specify angle of rotation and center of rotation.

For example, the five lines of symmetry for the pentagon can be described as "the lines joining each vertex with the midpoint of the opposite side." A rotation that maps the pentagon onto itself can be described as, "a rotation of a multiple of 72° with center at the point of intersection of any two lines of symmetry."

Check each student's diagrams for accuracy and neatness.

For Question 3, rather than requiring a verbal description, you might want to require only an accurate sketch. Draw lines of symmetry for reflections and draw the center of rotation and specify the angle measure for the rotations.

Questions 4 and 5 will require more time. Students should be able to complete these problems with a compass, ruler, and protractor. Use of tracing paper is optional.

Key Questions

What two parameters are needed to specify a rotation?

When, if ever, might the image of a rotation be the same as the image of a reflection?

How Many Ways from *A* to *B*?

Intent

Provide experience relating a mapping by rotation to a mapping made by two reflections.

Mathematics

Every rotation is equivalent to two successive reflections.

Progression

Provide materials to pairs of students. Circulate around the room as students work.

Remind students that the two images to start the poster must be the same side up—not one turned over. If one is turned over, then NO rotation will ever map one onto the other!

Approximate Time

40–50 minutes

Classroom Organization

Pairs work together to make poster

Materials

For each pair:

- 1 sheet colored paper
- 1 sheet poster paper
- Scissors
- Glue
- Marker
- Ruler, compass, and protractor
- Meter sticks (optional)

Doing the Activity

Circulate as students work.

Some students may use trial and error to locate the center of rotation, but a precise approach uses ideas from the Rotate a Triangle section of the prior lesson, *Isometric Transformation 2: Rotation.* One precise method uses the fact that the set of all points equidistant from the endpoints of a line segment is the perpendicular bisector of that segment. In a rotation, a point and its image lie along a circular arc equidistant from the center of rotation. So, choose a point on the original figure and its corresponding image point. Join these points with a line segment. Draw the perpendicular bisector of this segment. The center of rotation must lie on this line. Choose another point on the original figure and its image and do the same. The two perpendicular bisectors must intersect at the center of rotation!

An even simpler method that uses the perpendicular bisector idea involves folding the paper. Fold the paper over to match up corresponding points on the original figure and image and then crease the paper. Then match up two different corresponding points and crease the paper again. The two creases will intersect at the center of rotation!

Finding the lines of reflection can also be done by folding and creasing the paper. The first line of reflection can be almost anywhere. The second depends on the location of the first.

Note: Since the second reflection ends up on top of the original, only two copies of the original are actually needed. The third copy is extra.

Discussing and Debriefing the Activity

Put posters up as they are done. Have each pair briefly describe their poster and the process they used to make it to the class.

Key Questions

Based on what you observe looking at all the posters, do you think any image made by a rotation can be made also by two reflections?

For a particular rotation, are the equivalent two reflections unique?

Is the angle of rotation related to the angle between the lines of reflection? If so, how?

Rotation with Coordinates

Intent

Rotation on a coordinate system is introduced.

Mathematics

Students discover the resulting coordinates when a point is rotated 90°.

Progression

Students figure out coordinates for the new vertices of a triangle with a rotation of 90° with center of rotation at (0, 0). Then they use what they have learned to generalize for any point (p, q).

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

5 minutes for discussion

Classroom Organization

Individuals

Doing the Activity

Accurate drawings are crucial in enabling students to solve this activity. Note that a rotation of 90° is a counterclockwise rotation.

Some students may want to use tracing paper to find and/or check the image triangle.

Discussing and Debriefing the Activity

For Question 2, students should find that when (p, q) is rotated 90° about the origin, the result is the image ($-q$, p).

Key Questions

Which direction is a rotation of 90°, clockwise or counterclockwise?

What is the resulting image when a horizontal segment is rotated 90°?

Because rotation is an isometric transformation, what is true about the original triangle and its image?

Sloping Sides

Intent

Given a line with slope of –3/5, students figure out the slope of a perpendicular line.

Mathematics

In this activity and the subsequent one, *Perpendicular Rotations,* students generalize the relationship between the slope of a given line and the slope of a line perpendicular to it.

Progression

The context of Ella Pea designing the slides helps students to generalize.

Approximate Time

20 minutes

Classroom Organization

Groups

Doing the Activity

An effective problem solving strategy in this case is to make an accurate drawing of the situation on graph paper.

Discussing and Debriefing the Activity

Ask one or more groups to present and explain their result.

Point out that in slope triangles, for vertical movement, up is positive and down is negative, and for horizontal movement, to the right is positive and to the left is negative. The slide on the left drops 3 units for every 5 units to the right, hence $\frac{-3}{5}$. For the slide on the right to be perpendicular, it must have drop of 5 for every 3 units to the left, hence $\frac{-5}{-3}$, or $\frac{5}{3}$. The slope of one slide must be the negative reciprocal of the slope of the other slide.

Key Questions

How can congruent right triangles help solve Ella Pea's design problem?

What is Ella Pea's middle name? (Not a Key Question.)

Perpendicular Rotations

Intent

Work on this problem is designed to teach the Common Core Standard: "G-GPE.5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point)."

Mathematics

Find the equations of perpendicular line segments that intersect at a given point.

Progression

Students may use the generalization they make in Question 8 to do Questions 9 and 10.

Approximate Time

50 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals work within group.

Doing the Activity

Circulate as students work. Check to see if their graphs are accurate.

Discussing and Debriefing the Activity

Have one or more groups present each part of the problem.

Key Questions

Given the equation of a line, how can we decide if a point is on the line or not?

How is the slope of a line related to the slope of a line parallel to it?

How is the slope of a line related to the slope of a line perpendicular to it?

Isometric Transformation 3: Translation

Intent

This activity introduces the last, and simplest, isometric transformation: translation.

Mathematics

Translation slides all the points of a figure a specific direction and distance. A translation does not include any rotation of the figure. Translations are often described as *vectors* using the notation <*h, k*>, indicating a horizontal move of *h* units and a vertical move of *k* units.

Progression

Read the activity together. Have students answer Questions 1–2 in groups or in a whole-class.

Approximate Time

10 minutes

Classroom Organization

Whole-class and groups

Doing the Activity

Have one student read aloud while others read along silently. Work through the translations of the “Y” together. Introduce vector notation for translations. Note that a vector <*h, k*> specifies a movement and is not the same as a point (*h, k*), which specifies a location.

Key Questions

What two parameters must be specified for a translation?

How is <4, 7> different than (4,7)?

How is a vector different from a point?

Frieze Frame

Intent

Students use translation to make a decorative border.

Mathematics

This activity provides an application of translation. The ancients used translation to design decorative borders and friezes.

Friezes of animals at the base of Hoysaleswara temple, India. Credit: Wikimedia Commons.

Progression

Hand out square dot paper.

Set students working.

Approximate Time

5 minutes for introduction

40–60 minutes for activity (at home or in class)

Classroom Organization

Individuals

Materials

Two or more sheets of square dot paper per student (see blackline masters)

Doing the Activity

Encourage creativity, but advise student to keep their basic design simple.

Slicing the design at 45° to make the mitered corners is tricky. To make an elegant border, the edges of the shaded regions along the cuts should match along the 45° lines.

Discussing and Debriefing the Activity

Display the best designs—especially good examples of both mitered joints and butt joints—for the class to see.

Key Questions

What kind of corners are used in most picture frames, mitered joints or butt joints?

Translation Investigations

Intent

Introduce translations on a coordinate system. This activity practices algebra skills with equations.

Mathematics

Students have already experienced translations of parabolic graphs in the Algebra 1 *Fireworks* unit. They found that the parabola $y = ax^2$ translated by the vector $<h, k>$ results in the parabola with equation $y = a(x - h)^2 + k$.

Equations of lines translate in a similar manner: $y = mx + b$ translated by the vector $<h, k>$ results in the line $y = m(x - h) + k$.

Progression

Students work independently to find the result of translating a point and a line by $<h, k>$.

Approximate Time

30 minutes (at home or in class)

Classroom Organization

Individuals

Materials

Graphing calculators or graphing software

Doing the Activity

Circulate as students work. Some may have difficulty with the horizontal shift in Question 2.

Discussing and Debriefing the Activity

Remind students about the connection between modifying an equation to translate a line and modifying an equation to translate a parabola.

Translation Designs

Intent

Students practice applying translations on a coordinate system.

Mathematics

Students apply what they noted in *Translation Investigations* about horizontal and vertical translation effects on a linear equation to find the equations of lines that make a design. Students also refresh their knowledge about slopes of parallel lines.

Progression

Students work independently to write equations for translated lines.

Approximate Time

30 minutes (at home or in class)

Classroom Organization

Individuals

Materials

Graphing calculators or graphing software

Doing the Activity

Students can do this activity with little support.

Discussing and Debriefing the Activity

The designs in both Questions 1 and 2 can be achieved through either horizontal or vertical translation (or both). Have students share the variety of equations they may have found, and justify why they work. When rewritten in standard form $y = mx + b$, all equations for the same line will become the same.

You might want to point out a general relationship for translation by recalling how parabolas were translated in the *Fireworks* unit:

> The parabola $y = ax^2 + b$ translated by $<h, k>$ becomes $y = a(x - h)^2 + b + k$.
>
> The line $y = mx + b$ translated by $<h, k>$ becomes $y = m(x - k) + b + k$

Key Question

What can you say about the slopes of parallel lines?

Transforming One Shape Into Another

Intent

This activity is optional. If your class is still doing presentations for POW 6 and you need more class time, you may either assign this entire lesson for homework or omit it.

The isometric transformations reflection, rotation and translation preserve shape and size and therefore preserve area. So isometric transformations can be used in creative ways to transform one figure into another figure with the same area, suggesting new area formulas.

Mathematics

When deriving area formulas, shapes such as triangles, parallelograms, and trapezoids are cut up and transformed into rectangles with the same area.

Progression

Have students read and start work.

Approximate Time

30 minutes for activity (at home or in class)

20 minutes for presentations and discussion

(This activity is optional.)

Classroom Organization

Groups or individuals

Doing the Activity

In modern geometry, mathematicians use well-defined isometric transformations to prove theorems about the properties of figures.

A transformation must have the necessary parameters specified:

- For reflection: the location of the line of reflection
- For rotation: the center of rotation and angle of rotation
- For translation: the horizontal and vertical translation (e.g. as a vector)

Make sure students clearly define all the transformation parameters they use in doing these problems.

Discussing and Debriefing the Activity

Assign groups to present solutions.

Dilation

Intent

Students explore the final transformation, dilation. Unlike the previous transformations, this one is not isometric.

Mathematics

The parameters needed to specify a dilation are a point, called the *center of dilation*, and a scale factor which tells how much the shape is enlarged (or shrunk).

A *projection line* is a line from the center of dilation passing through a point on an original figure and extended to the corresponding point on the image.

Dilation is not an isometric transformation because although the image has the same shape as the original, it may not have the same size. Thus, dilation is the transformation that produces images *similar* to the original, while the three prior transformations produced images *congruent* to the original. Similarity was introduced to students in the *Shadows* unit.

Dilations provide another context for multiplication in the real number system. In particular, students see that for dilations, the result of a positive times a positive is a positive, a positive times a negative is a negative, and the result of a negative times a negative is a positive.

Progression

This section is brief.

A Non-Isometric Transformation: Dilation

Dilating a Right Triangle

Enlarging on a Copy Machine

Dilation Investigations

A Non-Isometric Transformation: Dilation

Intent

Introduce the final transformation, dilation, and establish that it creates images that are *similar,* but not necessarily *congruent.*

Mathematics

Dilation transforms a figure into a similar figure. In a dilation, lengths in the original are each multiplied by the scale factor to get the corresponding lengths in the image.

Progression

Students read about dilation, then perform an investigation with rubber bands to experience dilation.

Approximate Time

35 minutes (at home or in class)

Classroom Organization

Whole class, then individuals

Materials

Rubber bands, preferably all the same, small size, so the activity fits on a sheet of standard paper.

Doing the Activity

Read through the first page of this activity and look at the dilation diagram of the kite-like figures. From the diagram, notice that scale factors for a dilation are not necessarily whole numbers. The scale factor for a dilation can be any real number, including negative numbers. A negative dilation inverts the original shape.

Pass out rubber bands and have students follow the instructions to do the rubber band dilation. If you are nervous about passing out rubber bands to your students, or you don't have any rubber bands available, you may assign this activity for homework.

By doing the rubber band activity, students experience the relationship between center of dilation, projection lines, distance from the center and the resulting size of the image, and the effects of different scale factors.

Discussing and Debriefing the Activity

Check student notebooks to see if they have successfully drawn a dilation.

Key Questions

How could rubber bands be used to produce a dilation with scale factor 3?

If an object is held up to a light source and produces a shadow on a wall, what must be true in order for the shadow to be a dilation?

Dilating a Right Triangle

Intent

Use a compass and straightedge to produce a dilation of a right triangle.

Mathematics

This activity provides an example of a center of dilation inside a figure, and provides practice with using a compass to mark off equal segments.

Progression

A ruler and protractor are used to draw right triangle *ABC* (or you can require that students construct a right angle). A compass and straightedge are then used to perform the dilation.

Approximate Time

15 minutes (at home or in class)

Classroom Organization

Individuals

Doing the Activity

With the center of dilation inside the triangle, students draw projection lines from this center through each of the vertices of the triangle. Then they use the compass to measure from center to vertex, and then mark off that distance again along the projection line passing through the vertex. They do this for each vertex, then connect the points to get the image triangle. When completed, the diagram will look a bit like a spider web.

In Question 5, the ratio of the lengths of the hypotenuses should be the same as the scale factors. To compare this ratio with the scale factors, students will have to divide the ratio and write it as a decimal. The ratios will probably not match exactly due to measurement and rounding errors.

Key Questions

The scale factor of a dilation may be written as a fraction, as a decimal, or as a percent. What would be the effect of a dilation of 100%? What would be the effect of a dilation of 3/5? What would be the effect of a dilation of 3.14159?

You've seen what happens with when the center of dilation is outside the figure and when it is inside the figure. What would happen if the center of dilation lies *on* the figure? Sketch a triangle. Locate the center of dilation at one of the vertices. Draw projection lines and sketch what you think a dilation of 200% would look like.

Enlarging on a Copy Machine

Intent

This activity provides a context for students to discuss the effects of successive dilations.

Mathematics

Consecutive enlargements are multiplicative, not additive.

Progression

Have groups read and discuss the problem. Then ask groups to share their results with the class.

Approximate Time

20 minutes

Classroom Organization

Groups

Doing the Activity

Circulate around the room while students discuss this problem. Encourage each group to include everyone's opinions. Try to give out just facts about copy machines and avoid giving answers. You will gain a lot of insight into your student's understanding of ratios by just listening.

Discussing and Debriefing the Activity

Have a different group report on each of Arnold's friends' ideas.

If confusion ensues, you can ask, **Suppose you did each friend's suggestion with a 10 cm line? What would be the length of the image line?**

If no agreement is reached, you may have to try these on an actual copier!

The friends who made suggestions that actually result in a 300% enlargement are Jackson, Kayla, and Madelyn.

Key Questions

Suppose you did each friend's suggestion with a 10 cm line? What would be the length of the image line?

You have seen that there are several ways to get a 300% enlargement on a copy machine with a 200% upper limit. How many other different pairs of settings are possible? List 5 more pairs of settings.

On a copy machine that has an upper limit of 200%, what single setting could be used four times in succession to get approximately a 300% enlargement?

Suppose, instead of 300%, you want to make a shape only 20% larger. What scale factor should be used?

Dilation Investigations

Intent

This lesson consists of dilations in two contexts:

Repeated Dilations looks at dilations of images of dilations.

Billy Bear Grows Up looks at what happens to perimeter and area of figures under dilation.

Mathematics

Dilations provide another context for students to make sense of the multiplication of signed numbers. For example, it makes sense that the net result of a negative dilation of a negative dilation is a positive dilation.

In a dilation, lengths are multiplied by the scale factor, but areas are multiplied by the square of the scale factor.

Progression

Students think through the results of repeated dilations, then draw dilations of a figure on isometric dot paper and compare lengths and areas to the original.

Approximate Time

30 minutes (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals

Materials

Isometric dot paper, one half-sheet per student

Doing the Activity

Pass out isometric dot paper for Billy Bear Grows Up. Start students working in class and assign leftovers for homework.

Note that the unit of area for Billy Bear is an equilateral triangle 1 unit on a side.

Discussing and Debriefing the Activity

Have group members compare answers, then have a whole-class discussion.

You may want to have a copy of the flag dilations available for the whole class to see so presenters can explain their thinking using the diagram.

Be sure students understand the results of repeated dilations, and that dilation increases lengths but the scale factor, but increases areas by the square of the scale factor.

Key Questions

Suppose a dilation with scale factor *k* is used to produce an image. What scale factor will take the image back onto the original? Your friends give their suggestions. Who is correct, and why?

> **Arnold says –*k* will do it.**
>
> **Berea says you need to use 1 / *k*.**
>
> **Clarissa says that neither of these scale factors will work.**

When a polygon such as Billy Bear is dilated by a scale factor of *k*, the lengths are multiplied by *k*, but the area is multiplied by k^2. Is this true for the areas of parts of Billy Bear? For example, look at one of Billy Bear's feet. His original foot has an area of 1 square unit. What are the areas of the images of his foot as scale factors increase from 2 to 3 to 4, and so on?

Put the Pieces Together

Intent

Time to start the review and synthesis!

By looking back over the entire unit, students connect big ideas and strengthen neural pathways for easier recall in the future.

This last section of *Geometry by Design* reviews and summarizes the ideas in the unit. In addition, problems that combine several ideas are included.

Mathematics

This last section combines all the transformations previously learned, and has an emphasis on finding single transformations that are equivalent to a combination of given transformations.

Progression

Students practice combining transformations, then complete the unit with a portfolio.

Think About It

Drawing Conclusions

Combinations of Transformations

**Geometry by Your Design*

**Digging Into Transformations*

**Coordinate Transformations*

Tricky Transformations

Geometry by Design Portfolio

Note: The starred * tasks are optional and may be omitted in case of time constraints.

Think About It

Intent

The true or false statements in *Think About It* review some important ideas in *Geometry by Design.*

Mathematics

These statements help students to refine and solidify their ideas about shapes and transformations. The questions are similar to those seen on many standardized tests.

Progression

You may want to have students do Part I first and then have a discussion. Then start Part II and have students finish it for homework along with *Drawing Conclusions*.

Approximate Time

30 minutes

Classroom Organization

Individuals, then groups, then whole-class discussion

Doing the Activity

Have students work alone on Part I for 5 minutes, then spend 5 minutes comparing answers within their group. Finally, share answers in a whole-class discussion. If a statement is true, encourage students to explain why; if a statement is false, ask a student to change it to make it true. For a bonus, students could also provide a counterexample for the false statements.

Circulate among groups as students discuss answers. Note any misconceptions to bring up during the class discussion.

Discussing and Debriefing the Activity

As items are discussed, note any statements that the class cannot agree upon. Write them up on a wall and encourage students to continue further research and discussion until everyone is satisfied with the answer.

Drawing Conclusions

Intent

In earlier activities, we've established that to prove triangles are congruent we can use the SAS, ASA, and SSS Triangle Congruence Postulates. But what about other combinations of triangle parts?

The first three tasks in this activity examine possibilities for SSA and AAS:

The last task in this activity, Drawing Transformation on Dot Paper, provides students with a first experience with combinations of successive transformations, made easier by doing it on isometric dot paper.

Mathematics

In Arnold's Deduction, students note that it is not enough for just any two sides and one angle in one triangle to be congruent to two sides and an angle in another triangle. For the SAS postulate (or any of the triangle congruence postulates) to apply, the parts must be corresponding.

What About SSA? includes a diagram that shows why side-side-angle may not result in congruent triangles.

In What About AAS?, students learn that AAS is really just a special case of ASA, because the angles of a triangle sum to 180°.

Finally, in Drawing Transformations on Dot Paper, students figure out how the result of several successive transformations can be accomplished by a single transformation.

Progression

Depending on the timing of your class, these four activities may be split up and assigned on different days. Also, one or more may be made optional assignments, depending on your objectives and your emphasis on proof.

Approximate Time

20 minutes in class to start activity; finish at home

20 minutes for discussion

Classroom Organization

Individuals and groups

Materials

Isometric dot paper (blackline master available at the end of this *Teacher's Guide*)

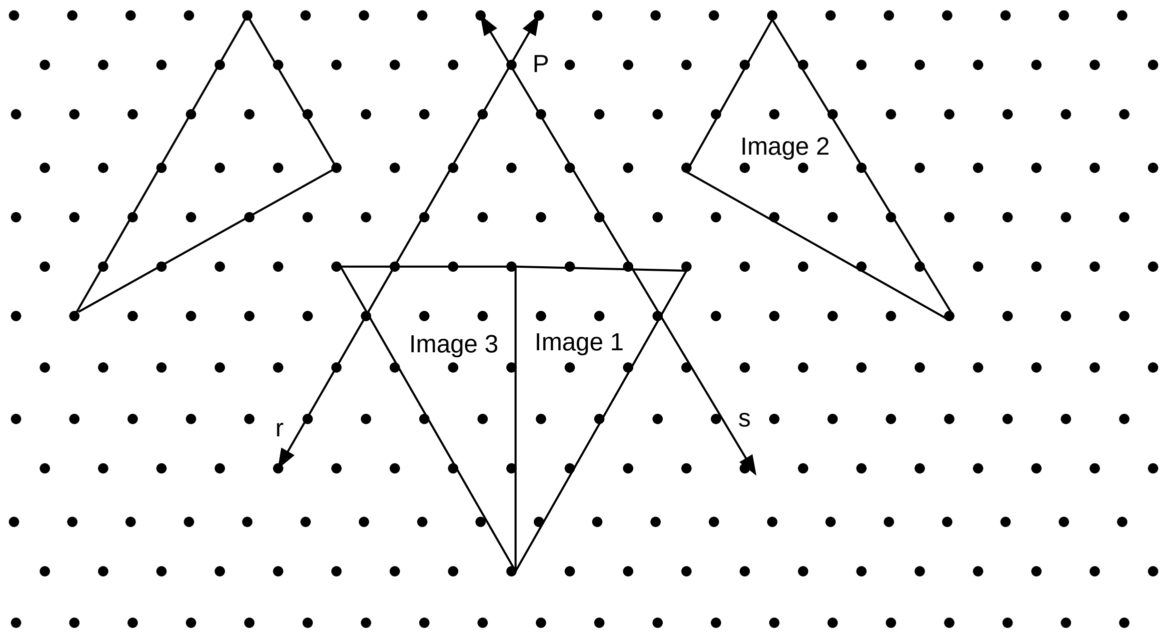
Doing the Activity

Tasks may be assigned in any order on different days. Students can work in groups or individually.

Discussing and Debriefing the Activity

You can assign the three tasks Arnold's Deduction, What about SSA?, and What about AAS to groups to present.

Drawing Transformations on Dot Paper will be a challenge for many students. Careful drawings will help. After completing the steps, students should get the diagram below. The single transformation mapping Image 3 back to the original would be a rotation of –60° about point P.



Combinations of Transformations

Intent

Each part of this activity involves drawing a shape on coordinate axes, moving the shape through a succession of transformations, and finally figuring out how to accomplish the moves using just one transformation.

Mathematics

In this activity, the focus is on transformations as mappings. Transformations play the same role in geometry as functions do with the real number system. A transformation exists independently from the shapes on which it may operate.

In fact, transformations may be used to map an entire plane, together with its coordinate system, onto a new plane.

Sometimes a set of successive transformations can be resolved into one single transformation; sometimes they cannot. In the three problems in this set, students will find that if the steps are followed accurately, all these problems come out “nice.”

Progression

Each of these three problems are independent and may be assigned together, or one at a time over several days.

Approximate Time

20 minutes for each question (at home or in class)

Classroom Organization

Students can work independently or in pairs to do these problems.

Materials

Graph paper

Doing the Activity

Have students set up axes with $-10 < x < 10$ and $-10 < y < 10$. All three questions of this activity will fit within these boundaries.

Advise students that care must be taken to get images right: Save time by double-checking each transformation as you do it! Compare what you get with others in your group. A mess-up on one transformation will make all the rest of the work that follows incorrect.

If all transformations are done correctly, the end result will be “nice”—meaning, you’ll be able to find a single equivalent transformation.

For students having a difficult time figuring out the transformations, suggest that they trace the original triangle on tracing paper and use the tracing to help find images.

Circulate around the room as students work. Suggest that they think of each problem as a puzzle to figure out. Remind them to be patient and double-check each translation.

Discussing and Debriefing the Activity

As you walk around the room, look for students who have completed each question correctly. For each question, choose one student with a correct answer to share their work with the class.

Selected answers for parts d:

Question 1d: Rotate –90° about point *W*.

Question 2d: Reflect across the line $x = -2$.

Question 3d: Rotate –90° about (–2.5, –3)

Key Questions

Suppose in Question 1 the three successive transformations were done on a different shape. Do you think the answer would be the same for part d? Why or why not?

When you drive in a car, the car's movement can be described by transformations. What car movement corresponds to a translation? What car movement corresponds to a rotation? Is there a car movement that corresponds with reflection?

In mathematics, sometimes transformations are used to move the entire coordinate system! For example, in astronomy a 3-D coordinate system is used. When measurements are taken by an observer on Earth, the locations usually need to be transformed by translation and possibly rotation into locations with the sun as center.

Suppose an entire *x*-*y* coordinate system was rotated 90° about the origin. What would be the image of the *y*-axis? Which points, if any, would remain fixed? What would be the image equation of the line $y = x$?

Geometry by Your Design

Intent

This is an optional activity that provides students an opportunity to use constructions and transformations to create their own original designs.

Mathematics

Transformations have been used since antiquity in art and architecture to create images pleasing to the human eye.

Progression

Provide students with guidelines that you choose (see Doing the Activity), and set them to work.

Approximate Time

10 minutes for introduction

20–60 minutes for activity (at home)

Note: This activity is optional.

Classroom Organization

Students work individually

Doing the Activity

This could be an optional or “extra credit’ assignment. You could also use it as an option for the end-of-unit assessment, particularly if you require detailed description of the transformations involved.

This is a fairly open-ended assignment for you as well as for students. You may want to restrict the tools that can be used to compass and straightedge; or compass, straightedge, and protractor. Or, if your students have access to a computer program such as iDraw or The Geometer’s Sketchpad, you can have them create designs on a computer.

To start, you might want to share images of friezes and mandalas and then brainstorm with the class, **What are the properties that make for a pleasing design?** Some possible responses are: line or rotational symmetry; repetition of a simple shape to make a complex design; interesting color combinations; accurate, clean lines; and so on.

You may want to allow students to choose a design from the Internet and then modify it to draw their own design.

Designs may be done on blank paper, graph paper, isomeric dot paper, or poster-sized paper. Remind students that this is not a free-hand art project where anything goes. Designs should involve constructions and/or transformations.

Discussing and Debriefing the Activity

Put student work up in the classroom as it is handed in. Some teachers who lack wall space fix a string across the room and use clothespins to hang up drawings.

Key Questions

What are the properties that make for a pleasing design?

Digging Into Transformations

Intent

Students investigate relationships between transformations.

Mathematics

Transformations can be generalized as operations independent from any particular shape or coordinate system.

Progression

You may either assign one or more of the seven statements to each group to investigate, or let each group choose two.

Approximate Time

20 minutes

Note: This activity is optional.

Classroom Organization

Groups

Doing the Activity

A simple way to test one of the statements is to cut out a shape and then use the shape to trace images using the transformations in the statement.

You might also suggest students look at prior work for ideas and examples. They can look at transformations on dot paper, with compass and straightedge, and with coordinates.

Do not attempt to prove results.

Discussing and Debriefing the Activity

Have each group report briefly on their results.

Answers:

1. True. This is because reflections across parallel lines will not produce any rotation. The original can be translated into the final image along a line perpendicular to the parallel lines
2. True. The lines of reflection are the perpendicular bisectors of corresponding points on each image. The perpendicular bisectors must pass through the center of rotation.
3. False. Rotations alone will never “flip over” a shape.
4. True. Just add the respective horizontal components and vertical components and you get the components of a single translation that will do the job.

5. False. A dilation of N followed by dilation of $-N$ gives a dilation of $-N^2$.

6. True.

7. True.

Key Questions

If a sequence of transformations includes just one reflection, could the final image be moved back to the original by a rotation or a translation? Why or why not?

True or false: The result of any sequence of transformations can always be replaced by just one or two reflections through different lines.

Coordinate Transformations

Intent

Students find image equations for a line that is reflected, rotated, or translated.

Mathematics

Students solve problems involving transformations where the answer is an equation.

Progression

This activity can be started in class, or assigned for homework.

Approximate Time

30 minutes (at home or in class)

Note: This activity is optional.

Classroom Organization

Individuals

Doing the Activity

Students can start by accurately drawing the given information on graph paper.

Discussing and Debriefing the Activity

Select one or more students to share diagram and answers with the class.

Answers:

Question 1: $y = -\frac{2}{3}x + 4\frac{2}{3}$

Question 2: $y = -4x + 37$

Question 3: $y = 3x - 5$

Tricky Transformations

Intent

Problems similar to these recently appeared on standardized tests.

Mathematics

These problems involve of identifying and analyzing transformations.

Progression

Allow time for students to work independently before sharing answers and strategies with other group members.

Approximate Time

30 minutes (at home or in class)

Classroom Organization

Individuals or groups

Doing the Activity

For Question 2, some students may find it helpful to trace the original triangle, cut it out, and move it to follow the sequence of transformations.

Each transformation should be specified using all required parameters.

Discussing and Debriefing the Activity

Selected answers:

> Question 1: a. A; b. B; c. C
>
> Question 2a: Reflect across $\overline{HZ}$; then rotate 180° about the midpoint of $\overline{OZ}$; then reflect across $\overline{WZ}$; finally rotate 180° about the midpoint of $\overline{WI}$.
>
> Question 2b: In one transformation: Rotate about the intersection of the perpendicular bisectors of $\overline{HI}$ and $\overline{WZ}$, through an angle equal to $\angle OWI$

Geometry by Design Portfolio

Intent

In this final activity, students review and consolidate the learning of the unit to compile their unit portfolios.

Mathematics

The portfolio gives students an important opportunity to reflect on the many topics in the unit and how they interrelate. These topics include constructions, deduction and proof, and transformations with compass and straightedge; on dot paper, and on a coordinate system.

Progression

Students work independently to review their work over the unit, select samples, reflect on the evidence of their learning the samples demonstrate, and write cover letters.

If students have done all work in a composition book, you may have them mark pages by writing on Post-It notes and sticking the note to the page with an edge protruding.

Approximate Time

5 minutes for introduction (at home or in class)

45 minutes for activity (at home)

Classroom Organization

Individuals

Doing the Activity

Provide some time in class for students to get started on their Cover Letter and Personal Growth write-up.

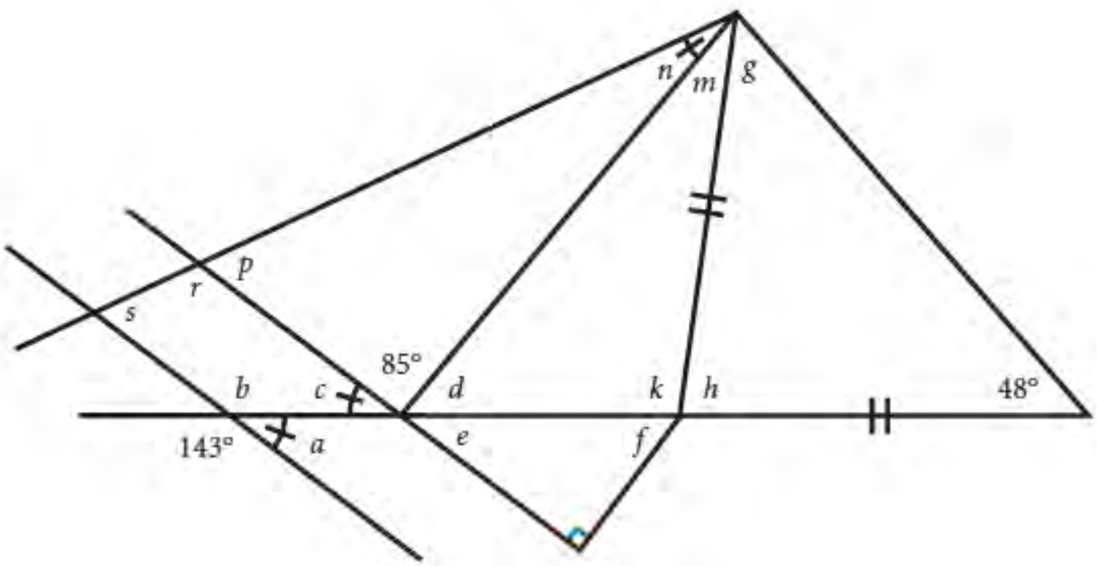
Discussing and Debriefing the Activity

Examine student portfolios and read their writing.

You might want to pair up students with a student from another group and have them share portfolios with each other.

Blackline Master

Deduce Those Angles



Blackline Master

POW 6: How Did Humans Create Mathematics?

Presentation Guidelines

Work with a partner or individually.
Topics will be given out first-come, first-served; only one pair/person per topic.
As soon as possible, check with your teacher to reserve your topic, but definitely before ________________.

Historical Mathematicians
Socrates Asks Questions
Plato's School
The Cult of Pythagoras
Euclid's Elements
Eratosthenes Measures Earth
Aristotle the Man and his Accomplishments
Aristotelian Logic: Syllogisms
Archimedes' Eureka! Moment
René Descartes Invents Coordinate Geometry

Geometric Topics
Babylonian Geometry
Egyptian Pyramid Designs
Ancient Chinese Geometry
Ancient Islamic Geometry
Trisecting An Angle
The Parallel Postulate
Non-Euclidean Geometries: Spherical Geometry
Riemann's Geometry and The Theory of Relativity

Geometry used at work by a parent or other professional (This topic may be chosen by more than one group if different sources are used.)

*You may select a math biography or topic of your own choosing, but if not from the above list, your idea must be cleared with the teacher before you begin work.

Your work on this task is worth 120 points:

- Your presentation is worth 100 points.
- After all presentations are done, you will take a multiple choice test on the presentations worth 20 points. You may use notes on the multiple choice test.

Your presentation must be 5 to 10 minutes long and include at least 10 but no more than 25 computer "slides." Slides may include photos or videos from the internet, photos or videos you make yourself, text pages, pages you hand-write and then scan/photograph, and so on.

Your slides must be selected/created before ________________.
You should be ready to make your presentation ANY TIME AFTER
________________.

Name(s)____

Presentation topic____

GRADING CRITERIA

ESSENTIAL COMPONENTS (70 POINTS POSSIBLE)	Points possible	Points earned
Progress check-ins with teacher ON OR BEFORE ____________: Select partner and topic. ON OR BEFORE ____________: Slides prepared	5 5	
Includes 10 but no more than 25 slides	5	
Length: minimum 5 minutes long, not including bonus math problem/activity	5	
Historical setting (What was happening then in that region of the world?)	10	
For Bios: Tell/show life events For Topics: Why important? What was the motivation?	15	
Describe the mathematics and its significance. Be specific. Clearly answer the question: If this person/topic never happened, how would our world be different?	15	
Hand in two multiple choice questions based on your presentation. (After presentation)	5	
Self-evaluation (Do after your presentation; see below.)	5	

QUALITY COMPONENTS (30 POINTS POSSIBLE)		
Style (clean, clear slides; clear explanations)	10	
Engagement	10	
Creativity	10	
Total Math problem/activity related to topic for class to try after the slide presentation	100 Bonus 10	

Self Evaluation (to be completed after your presentation)
Answer each question.

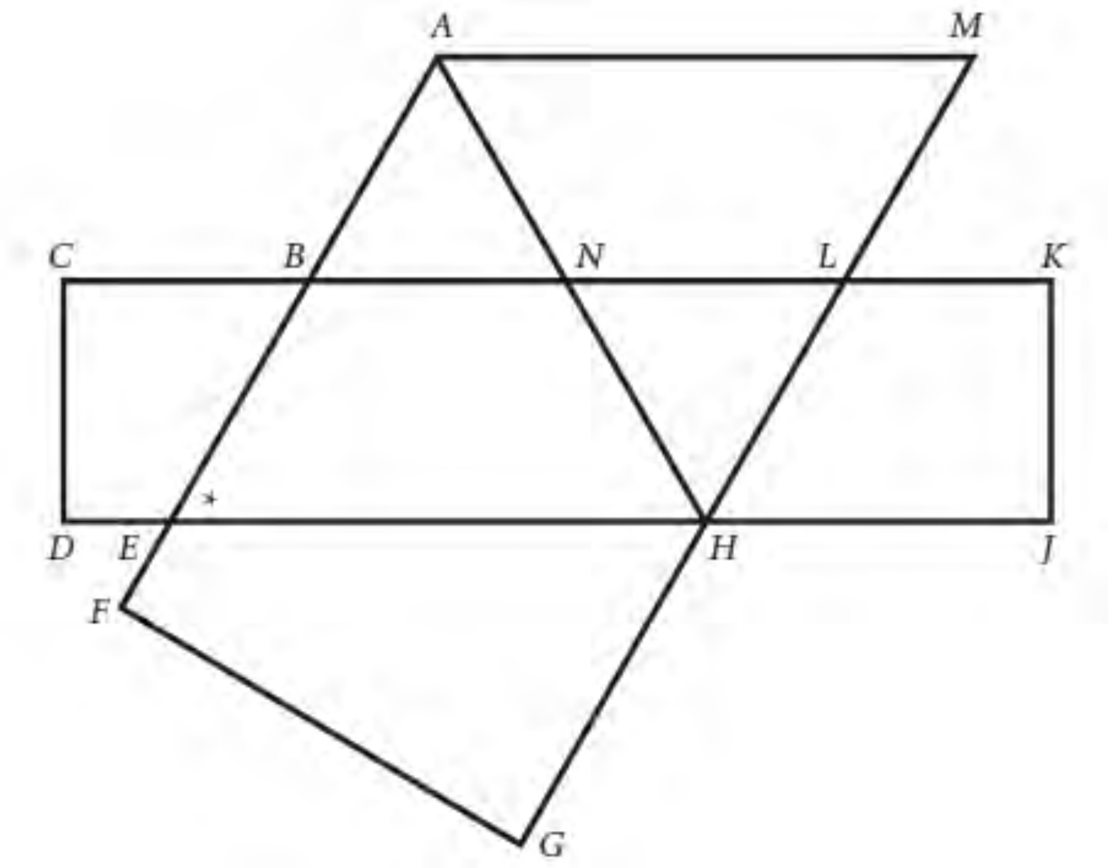
1. What was the best part of your presentation?

2. What could you have done better?

3. What grade out of 100 points do you think you deserve, and why?

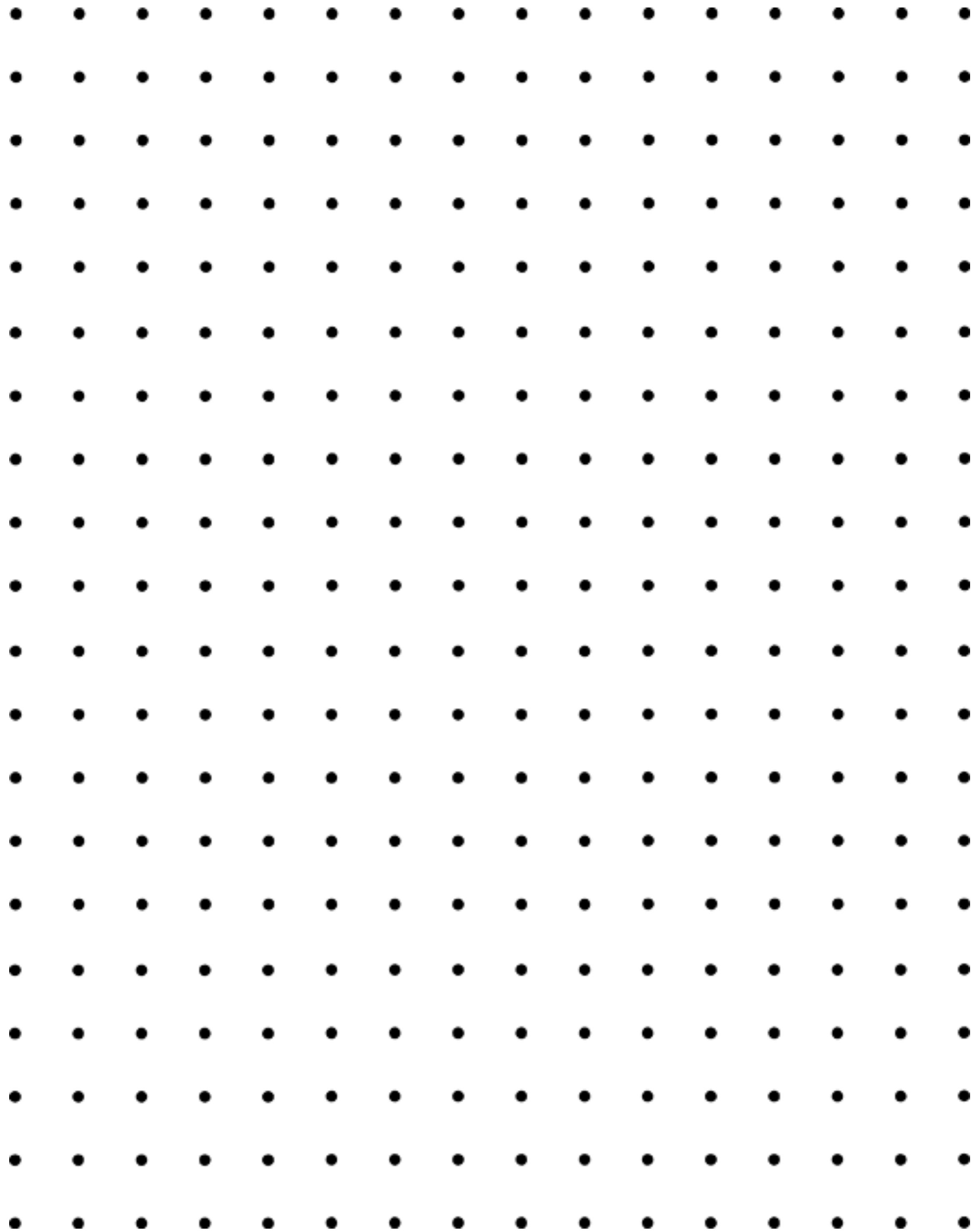
Blackline Master

Parallel Lines and Transversals



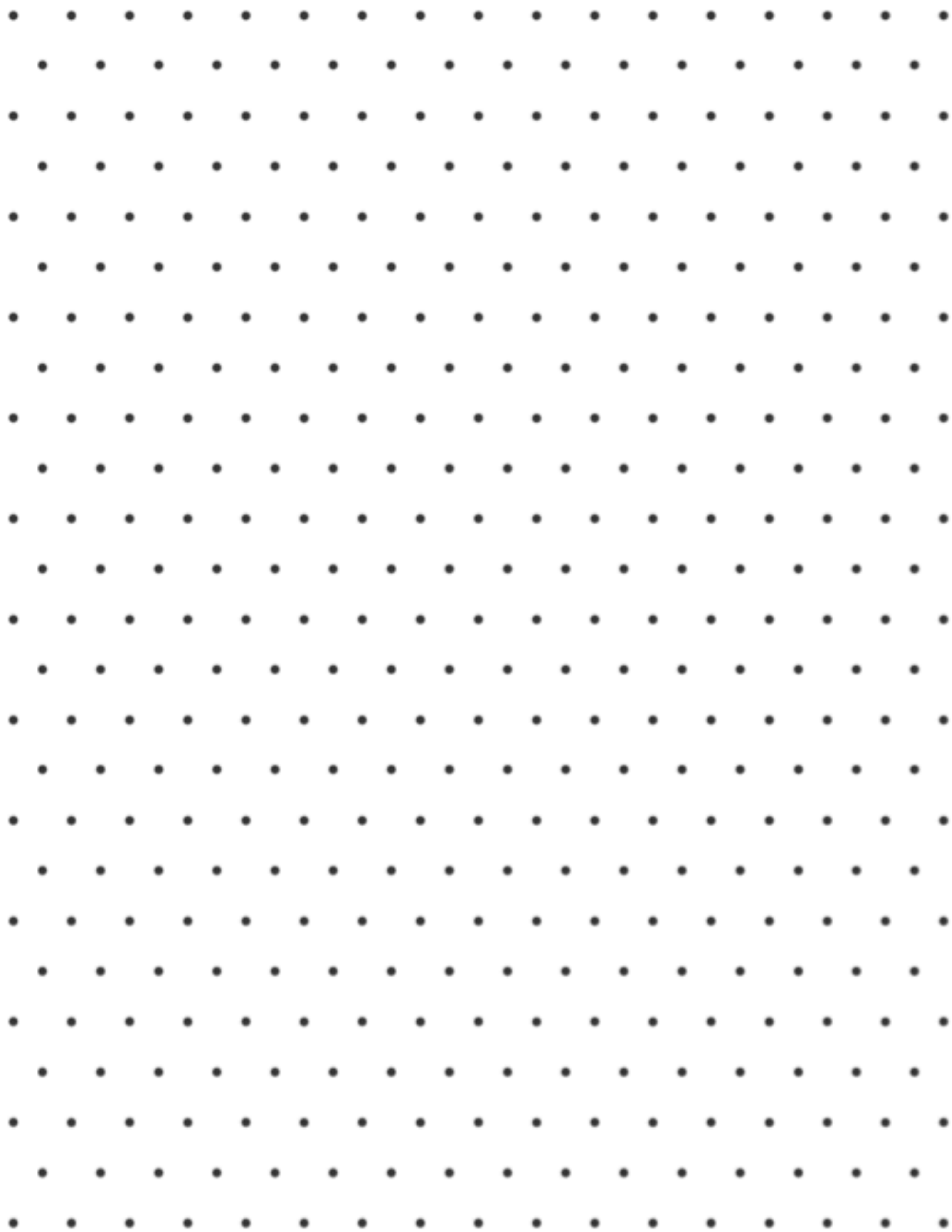
Blackline Master

Square Dot Paper



Blackline Master

Isometric Dot Paper



In-Class Assessment

Part I: Making Accurate Drawings and Constructions

1. a. Accurately draw $\triangle ONE$ with ON = 12.5 cm, $m\angle NOE$ = 38°, and OE = 5.2 cm.

b. Measure $\overline{EN}$ to the nearest tenth of a centimeter and $\angle OEN$ to the nearest degree.

c. Use a compass and straightedge only to construct the altitude from E to $\overline{ON}$. Show your construction marks!

2. a. Draw isosceles triangle TWO with $TO = WO$, TW = 6.5 cm, and $m\angle TOW$ = 32°.

   b. Measure $\overline{TO}$ and $\overline{WO}$ to the nearest tenth of a centimeter.

   c. Use compass and straightedge to construct the perpendicular bisector of $\overline{TW}$. Show construction marks.

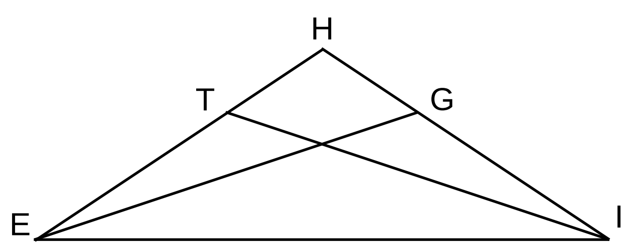
   d. Explain why the perpendicular bisector of $\overline{TW}$ must pass through O.

Part II: Understanding Vocabulary and Ideas

State whether each statement is true or false. If it is false, change it to make it true.

1. All squares are rhombuses.
2. The point (–2, 4) is on the line $y = 5x + 7$.
3. "All men are created equal" is an example of a postulate.
4. In a triangle if two sides and the angle between those sides are given, then all triangles with those specifications must be congruent.
5. In $\triangle ABC$, $AB = 3$ miles and $BC = 5$ miles. In $\triangle XYZ$, $XY = 3$ miles and $YZ = 5$ miles. Therefore, $\triangle ABC$ must be congruent to $\triangle XYZ$.
6. A rectangle is both equilateral and equiangular.
7. Rotation through 180° about (0, 0) is equivalent to reflection across the line $y = x$.
8. The image of the point (3, 6) rotated 90° about (0, 0) is the point (6, 3).
9. The image of the point (–2, 4) translated through <5, 4> is the point (3, 8).
10. When the line $y = 0.25x - 3$ is reflected across the line $y = x$, the image is the line $y = 4x + 12$.

Part III: Proof



Given: $\triangle EIH$ with $EH = HI$ and $m\angle GEI = m\angle TIE$

Prove: $TI = GE$

Take-Home Assessment

1. The original image below shows a resting head. Follow these instructions carefully, drawing each image in succession:

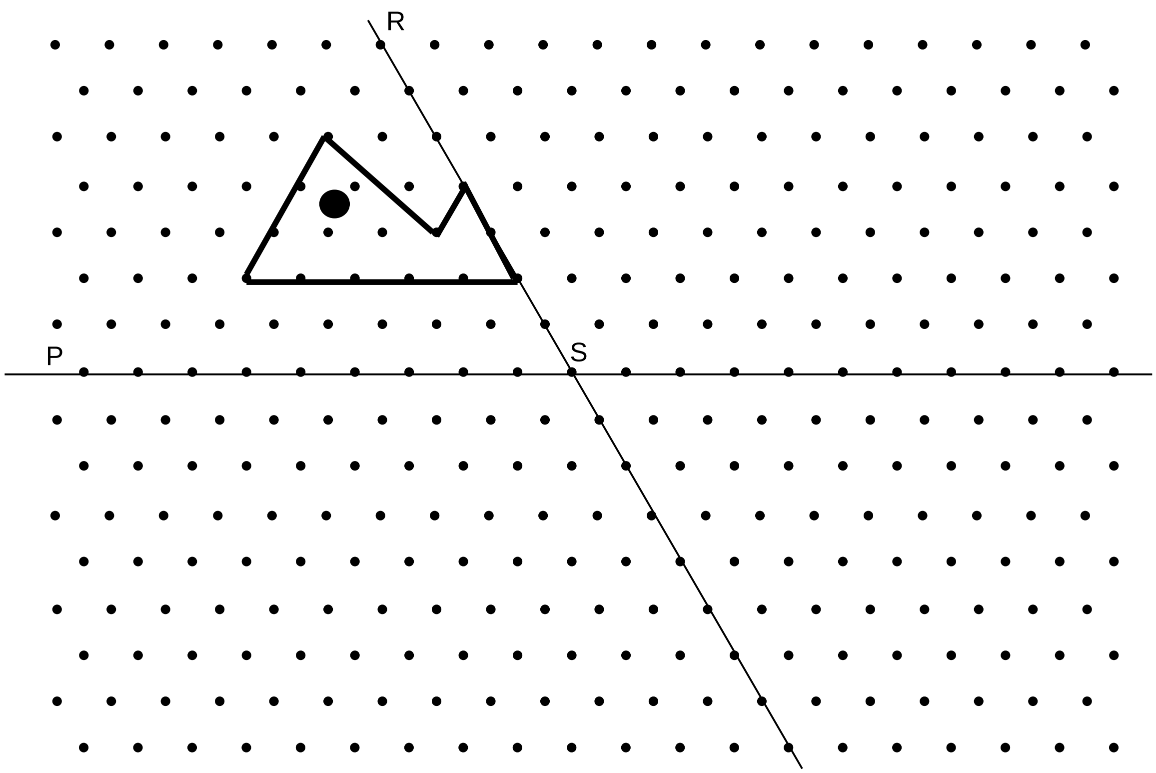
    a. Draw the image of the head reflected across line *PS*. Label it Image 1.

    b. Now rotate Image 1 by 60° with center of rotation *S.* Label it Image 2.

    c. Reflect Image 2 across line *RS*. Label the result Image 3.

    d. Finally reflect Image 3 across line *PS*. Label the result Image 4.

    e. Describe a transformation that will map Image 4 onto the original image. If you think it is a rotation, give the center and angle. If you decide it is a reflection, draw the line of reflection and label it.



2. Draw coordinate axes on graph paper.

   a. Plot $P(3, 9)$. Plot the image point that results when P is reflected across the y-axis. Give the coordinates of the image point and label it Q.

   b. Plot the image that results when P is reflected across the line $y = x$. Give the coordinates of this image point and label it R.

   c. Give the coordinates for a point S that makes a parallelogram with P, Q, and R. Draw parallelogram $PQRS$.

   d. Give the equation of line RS.

   e. Give the coordinates for a point T that makes an isosceles trapezoid with P, Q, and R. Draw isosceles trapezoid $PQRT$.

   f. Give the equation of line QT.