Meadows or Malls?

Three-Variable Equations, Three-Dimensional Coordinates, and Matrix Algebra

Teacher's Guide

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It's About Time
333 North Bedford Road
Mount Kisco, NY 10549
www.IAT.com

First Edition Authors
Dan Fendel, Diane Resek, Lynne Alper, and Sherry Fraser

Contributors to the Second Edition
Sherry Fraser, Jean Klanica, Brian Lawler, Eric Robinson, Lew Romagnano, Rick Marks, Dan Brutlag, Alan Olds, Mike Bryant, Jeri P. Philbrick, Lori Green, Matt Bremer, Margaret DeArmond

Interactive Mathematics Program
Year 4 Publishing Team

Project Editor
Josephine Noah

IMP Second Edition Publishing Team

Project Editor
Sharon Taylor

Consulting Editor
Mali Apple

Project Administrator
Juliana Tringali

Professional Reviewer
Rick Marks, Sonoma State University

Calculator Materials Editor
Josephine Noah

Math Checker
Carrie Gongaware

Production Editor
Andrew Jones

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Meadows or Malls? Unit Overview

Intent

Students solve a linear programming problem in six variables, using matrices to solve systems of linear equations.

Mathematics

The main concepts and skills that students will encounter and practice during this unit are:

General Linear Programming

- Seeing that for two-variable problems, the optimal value always occurs at a corner point of the feasible region
- Generalizing the corner-point principle to more than two variables
- Recognizing that for two-variable problems, corner points can be found as the intersections of lines corresponding to constraint equations or inequalities
- Generalizing the method of finding corner points to more than two variables

Solving Linear Equations

- Using substitution, graphing, and guess-and-check methods to solve systems of linear equations in two variables
- Developing and using the elimination method to solve systems of linear equations in two or more variables
- Using the concepts of **inconsistent, dependent,** and **independent** systems of equations

Geometry in the Plane and in 3-Space

- Extending the concept of coordinates to three variables by introducing a third axis perpendicular to the first two
- Graphing linear equations in three variables and recognizing that these graphs are planes in 3-space
- Seeing that two distinct points always determine a unique line and that two distinct lines in the plane determine a unique point unless the lines are parallel
- Examining the possible intersections of planes in 3-space

- Relating the possible intersections of lines and planes to the algebra of solving linear systems in two or three variables

Matrix Algebra

- Using matrices to represent information
- Using problem situations to motivate and develop the definitions of matrix addition and multiplication
- Examining whether matrix operations have certain properties, such as associativity and commutativity

Matrices and Systems of Linear Equations

- Seeing that systems of linear equations are equivalent to certain types of matrix equations
- Recognizing the role of identity and inverse elements in solving certain types of matrix equations
- Finding matrix inverses by hand by solving systems of linear equations
- Understanding the relationship between a system of linear equations having a unique solution and the coefficient matrix being invertible

Technology

- Entering matrices and doing matrix operations on a graphing calculator
- Using matrix inversion on a graphing calculator to solve systems of linear equations

Progression

In *Recreation Versus Development: A Complex Problem,* the unit opens with a city planning dilemma that is a linear programming problem in six variables. After a brief look at that problem, students spend several days reviewing the use of graphing to solve linear programming problems involving two variables, in *A Strategy for Linear Programming*. They see the need for a more general approach that does not require graphing when working with six variables in the central unit problem.

Based on work with several sample problems, students create a strategy for solving linear programming problems involving two variables without referring to a graph. In particular, they see that the optimal solution occurs at a corner point of the feasible region and that they can identify corner points by finding the common solution to pairs of linear equations involved in the problem. They test their algorithm (and correct or improve it if necessary) by working on other two-variable problems.

Equations, Points, Lines and Planes involves generalizing these concepts to three dimensions. Students are introduced to the coordinate system for three variables and see that the graph of a linear equation in three variables is a plane. They then investigate the possibilities for intersections of planes in 3-space. They see that "usually" three such planes determine a single point but that "exceptional" cases can occur, just as two lines in the plane do not necessarily intersect in a single point.

With these geometric insights, students apply their linear programming strategy to a three-variable problem in *Cookies, Cookies, Cookies*. They see that the feasible region for a three-variable linear programming problem is probably some sort of polyhedron and that the desired corner points are common solutions to sets of three linear equations involved in the problem. Through other problems, they see that if a constraint is an equation rather than an inequality, they should include it in all the combinations of constraints they use in their search for corner points.

As part of their work early in the unit, students review substitution methods for solving pairs of linear equations. Through the development of a general strategy for solving linear programming problems, they see that solving such systems is an essential part of the process, and in *Equations, Equations, Equations,* they develop the elimination method, which they extend beyond two variables. *Equations and More Variables in Programming* continues further development of writing constraints (including equations) and refining the process of solving linear programming problems.

Students also see, however, that as the number of variables increases, solving each system becomes more difficult. So they are highly motivated to find a tool that will do some of the work for them. In *Saved by the Matrices!,* looking at the elimination method leads them to focus on the matrix of coefficients. They develop the operations of addition and multiplication of matrices through problem contexts and learn how to use graphing calculators to do these operations on matrices.

Next, students investigate the process of representing systems of linear equations by matrices. Through an analogy with numerical equations, they develop the concepts of an identity element and an inverse, and they see that inverses of matrices are the key to solving linear systems. They learn that graphing calculators will also find inverses of matrices (when they exist), up to a certain size.

The use of the graphing calculators and matrix methods greatly simplifies the solving of linear programming problems. In *Solving Meadows or Malls?,* Students end the unit by solving the original six-variable problem.

Recreation Versus Development: A Complex Problem
A Strategy for Linear Programming
Equations, Points, Lines and Planes
Cookies, Cookies, Cookies Equations, Equations, Equations
Equations and More Variables in Programming
Saved by the Matrices!
Solving Meadows or Malls?

Meadows or Malls? and the Common Core State Standards for Mathematics

IMP is written to address the Common Core State Standards for Mathematics (CCSSM) High School standards.

Standards for Mathematical Practice

The eight Standards for Mathematical Practice are addressed exceptionally well throughout the *IMP* curriculum.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Standards for Mathematical Content

These specific content standards are addressed in the *Meadows or Malls?* unit. Additional content is covered that reinforces standards from earlier grades and courses.

N-VMQ.6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VMQ.7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

N-VMQ.8. (+) Add, subtract, and multiply matrices of appropriate dimensions.

N-VMQ.9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

N-VMQ.10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

A-CE.3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

A-REI.8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A-REI.9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

Pacing Guides

50-Minute Pacing Guide (40 days)

Day	Activity	In-Class Time Estimate
1	*Recreation Versus Development: A Complex Problem*	0
	Meadows or Malls?	45
	Homework: *Meadows, Malls, and Variables*	5
2	Discussion: *Meadows or Malls?*	10
	Discussion: *Meadows, Malls, and Variables*	25
	Introduce: *POW 1: That's Entertainment!*	15
	Homework: *Heavy Flying*	0
3	Discussion: *Heavy Flying*	50
	A Strategy for Linear Programming	0
	Homework: *Programming and Algebra Reflections*	0
4	Discussion: *Programming and Algebra Reflections*	15
	Ideas for Solving Systems	5
	Programming Puzzles	30
	Homework: *Donovan Meets the Beatles*	0
5	Discussion: *Programming Puzzles*	25
	Discussion: *Donovan Meet the Beatles*	10
	Homework: *Finding Corners Without the Graph*	15
6	Discussion: *Finding Corners Without the Graph*	15
	What Wood Would Woody Want?	35
	Homework: *Widening Woody's Woodwork*	0

7	Discussion: *Widening Woody's Woodwork*	30
	Homework: *More Equations*	20
8	Discussion: *More Equations*	25
	Equations, Points, Lines, and Planes	0
	Being Determined	25
	Homework: *How Much After How Long?*	0
9	Discussion: *How Much After How Long?*	30–50
	POW 1: That's Entertainment! (optional)	0–20
	Homework: *The Points and the Equations*	0
10	Discussion: *The Points and the Equations*	5
	The Three-Variable Coordinate System	45
	Homework: *What Do They Have in Common?*	0
11	Discussion: *What Do They Have in Common?*	50
	Homework: *Trying Out Triples*	0
12	Discussion: *Trying Out Triples*	45
	Homework: *More Cookies*	5
13	Presentations: *POW 1: That's Entertainment!*	20
	Discussion: *More Cookies*	10
	Just the Plane Facts	20
	Homework: *Solving with Systems*	0
14	Discussion: *Solving with Systems*	10
	Just the Plane Facts (continued)	30
	Homework: *Fitting a Line*	10
15	Presentations: *Just the Plane Facts*	25
	Discussion: *Fitting a Line*	15
	Cookies, Cookies, Cookies	0
	Introduce: *POW 2: SubDivvy*	10
	Homework: *The "More Cookies" Region and Strategy*	0
16	Discussion: *The "More Cookies" Region and Strategy*	30
	Homework: *Finishing Off the Cookies*	20

17	Discussion: *Finishing Off the Cookies*	50
	Equations, Equations, Equations	0
	Homework: *Easy Does It!*	0
18	Discussion: *Easy Does It!*	15
	Get Rid of Those Variables!	35
	Homework: *Eliminating More Variables*	0
19	Discussion: *Eliminating More Variables*	20
	Additional practice with the elimination method	30
	Homework: *Gardener's Dilemma*	0
20	Discussion: *Gardener's Dilemma*	10
	Elimination in Three Variables	40
	Homework: *More Equation Elimination*	0
21	Discussion: *More Equation Elimination*	15
	Equations and More Variables in Programming	0
	Ming's New Maneuver	35
	Homework: *Let Me Count the Ways*	0
22	Discussion: *Let Me Count the Ways*	5
	Ming's New Maneuver (complete and discuss)	45
	Homework: *Three Variables, Continued*	0
23	Discussion: *Three Variables, Continued*	25
	Homework: *Grind It Out*	25
24	Discussion: *Grind It Out*	15
	Presentations: *POW 2: SubDivvy*	20
	Introduce: *POW 3: Crack the Code*	15
	Homework: *Constraints Without a Context*	0
25	Discussion: *Constraints Without a Context*	15
	Eastside Westside Story	35
	Homework: *Fitting More Lines*	0
26	*Eastside Westside Story* (complete and discuss)	40
	Discussion: *Fitting More Lines*	10

	Homework: *Ages, Coins, and Fund-Raising*	0
27	Discussion: *Ages, Coins, and Fund-Raising*	5
	Saved by the Matrices!	0
	Matrix Basics	5
	Inventing an Algebra	35
	Homework: *Fitting Quadratics*	5
28	Discussion: *Fitting Quadratics*	10
	Flying Matrices	40
	Homework: *Matrices in the Oven*	0
29	Discussion: *Matrices in the Oven*	50
	Homework: *Fresh Ingredients*	0
30	Discussion: *Fresh Ingredients*	25
	Calculators to the Rescue	25
	Homework: *Make It Simple*	0
31	Discussion: *Make It Simple*	5
	Back and Forth	45
	Matrices and Linear Systems	0
	Homework: *Solving the Simplest*	0
32	Discussion: *Back and Forth*	10
	Discussion: *Solving the Simplest*	40
	Homework: *Things We Take for Granted*	0
33	Discussion: *Things We Take for Granted*	10
	Finding an Inverse	40
	Homework: *Inverses and Equations*	0
34	Discussion: *Inverses and Equations*	20
	Calculators Again	30
	Homework: *Fitting Mia's Bird Houses*	0
35	Discussion: *Fitting Mia's Bird Houses*	15
	Presentations: *POW 3: Crack the Code*	20
	Discussion: *Calculators Again*	15
	Solving "Meadows or Malls?"	0

	Homework: *Getting Ready for "Meadows or Malls?"*	0
36	Discussion: *Getting Ready for "Meadows or Malls?"*	15
	"Meadows or Malls?" Revisited	35
	Homework: *Beginning Portfolios–Part 1*	0
37	Discussion: *Beginning Portfolios–Part 1*	10
	"Meadows or Malls?" Revisited (continued)	40
	Homework: *Beginning Portfolios–Part II*	0
38	Discussion: *Beginning Portfolios–Part II*	10
	Presentations: *"Meadows or Malls?" Revisited*	40
	Homework: *"Meadows or Malls?" Portfolio*	0
39	*In-Class Assessment*	45
	Homework: *Take-Home Assessment*	5
40	Exam Discussion	35
	Unit Reflection	15

90-Minute Pacing Guide (27 days)

Day	Activity	In-Class Time Estimate
1	*Recreation Versus Development: A Complex Problem*	0
	Meadows or Malls?	55
	Introduce: *POW 1: That's Entertainment!*	20
	Homework: *Meadows, Malls, and Variables*	15
2	Discussion: *Meadows, Malls, and Variables*	20
	Heavy Flying	70
	A Strategy for Linear Programming	0
	Homework: *Programming and Algebra Reflections*	0
3	Discussion: *Programming and Algebra Reflections*	20
	Ideas for Solving Systems	5
	Programming Puzzles	65
	Homework: *Donovan Meets the Beatles*	0
4	Discussion: *Donovan Meets the Beatles*	10
	Finding Corners Without the Graph	60
	POW 1: That's Entertainment! (optional)	20
	Homework: *What Wood Would Woody Want?*	0
5	Discussion: *What Wood Would Woody Want?*	15
	Widening Woody's Woodwork	60
	Homework: *More Equations*	15
6	Discussion: *More Equations*	20
	Equations, Points, Lines, and Planes	0
	Being Determined	15
	How Much After How Long?	55

	Homework: *The Points and the Equations*	0
7	Discussion: *The Points and the Equations*	15
	The Three-Variable Coordinate System	45
	What Do They Have in Common?	30
	Homework: *POW 1: That's Entertainment!*	0
8	Presentations: *POW 1: That's Entertainment!*	15
	Discussion: *What Do They Have in Common?*	50
	Trying Out Triples	25
	Homework: *More Cookies*	0
9	Discussion: *Trying Out Triples*	45
	Discussion: *More Cookies*	10
	Introduce: *POW 2: SubDivvy*	10
	Just the Plane Facts	25
	Homework: *Solving with Systems*	0
10	Discussion: *Solving with Systems*	10
	Just the Plane Facts (continued)	25
	Presentations: *Just the Plane Facts*	25
	Fitting a Line	30
	Cookies, Cookies, Cookies	0
	Homework: *The "More Cookies" Region and Strategy*	0
11	Discussion: *Fitting a Line*	15
	Discussion: *The "More Cookies" Region and Strategy*	30
	Finishing Off the Cookies	45
	Equations, Equations, Equations	0
	Homework: *Easy Does It!*	0
12	Discussion: *Finishing Off the Cookies*	45
	Discussion: *Easy Does It!*	15
	Get Rid of Those Variables!	30
	Homework: *Eliminating More Variables*	0

13	Discussion: *Eliminating More Variables*	25
	Gardener's Dilemma	40
	Additional practice with the elimination method	25
	Homework: *POW 2: SubDivvy*	0
14	Presentations: *POW 2: SubDivvy*	15
	Elimination in Three Variables	45
	More Equation Elimination	30
	Homework: *Let Me Count the Ways*	0
15	Discussion: *Let Me Count the Ways*	5
	Discussion: *More Equation Elimination*	15
	Equations and More Variables in Programming	0
	Ming's New Maneuver	70
	Homework: *Three Variables, Continued*	0
16	Discussion: *Ming's New Maneuver*	10
	Discussion: *Three Variables, Continued*	20
	Introduce: *POW 3: Crack the Code*	15
	Grind It Out	45
	Homework: *Constraints Without a Context*	0
17	Discussion: *Constraints Without a Context*	15
	Eastside Westside Story	75
	Homework: *Fitting More Lines*	0
18	Discussion: *Fitting More Lines*	10
	Ages, Coins, and Fund-Raising	30
	Saved by the Matrices!	0
	Matrix Basics	5
	Inventing an Algebra	40
	Homework: *Fitting Quadratics*	5
19	Discussion: *Fitting Quadratics*	10
	Flying Matrices	45
	POW 3: Crack the Code	35
	Homework: *Matrices in the Oven*	0

20	Discussion: *Matrices in the Oven*	40
	Fresh Ingredients	50
	Homework: *Make It Simple*	0
21	Discussion: *Make It Simple*	5
	Calculators to the Rescue	30
	Back and Forth	55
	Matrices and Linear Systems	0
	Homework: *Solving the Simplest*	0
22	Discussion: *Solving the Simplest*	35
	Things We Take for Granted	35
	Homework: *Finding an Inverse*	20
23	Presentations: *POW 3: Crack the Code*	20
	Discussion: *Finding an Inverse*	25
	Inverses and Equations	45
	Homework: *Fitting Mia's Bird Houses*	0
24	Discussion: *Fitting Mia's Bird Houses*	15
	Calculators Again	55
	Solving "Meadows or Malls?"	0
	Homework: *Getting Ready for "Meadows or Malls?"*	20
25	Discussion: *Getting Ready for "Meadows or Malls?"*	15
	"Meadows or Malls?" Revisited	70
	Homework: *Take-Home Assessment*	5
26	Presentations: *"Meadows or Malls?" Revisited*	40
	In-Class Assessment	50
	Homework: *Beginning Portfolios–Part 1*	0
	Homework: *Beginning Portfolios–Part II*	0
27	Discussion: *Beginning Portfolios–Part 1*	10
	Discussion: *Beginning Portfolios–Part II*	10
	Unit Reflection	20
	Exam Discussion	45

	Homework: *"Meadows or Malls?" Portfolio*	5

Materials and Supplies

All IMP classrooms should have a set of standard supplies, described in the section "Materials and Supplies for the IMP Classroom" in *A Guide to IMP.*

Listed here are the supplies needed for this unit. Also available are general and activity-specific blackline masters, for transparencies or for student worksheets, in the "Blackline Masters" section.

Meadows or Malls? **Materials**

- A standard deck of cards (1 per group)
- Sentence strips
- Grid poster paper or transparencies (1 for each group)
- Small model of three-coordinate system, perhaps with the x- and y-axes displayed on a piece of cardboard, with a piece of spaghetti poked through a small hole at the origin to represent the z-axis
- Tape
- Three pieces of string, long enough to stretch across the classroom in all three directions
- Large towel, piece of cardboard, or flat surface to represent a plane (optional)
- String or yarn for each group
- Tape for each group
- A corner of a box (with three sides removed so the three remaining sides form a corner) for each group
- Corners of cardboard boxes (three sides removed so the three remaining sides form a corner): one per group
- One sheet of tagboard or a similar stiff, flat material
- Tagboard or index cards
- Pipe cleaners, skewers, or pick-up sticks
- An assortment of polyhedron blocks (optional)
- 3-D graphing software, such as *Grapher*, which is installed in the Applications: Utilities folder of many Apple computers (optional)

More About Supplies

Graph paper is a standard supply for IMP classrooms. Blackline masters of 1-Centimeter Graph Paper, ¼-Inch Graph Paper, and 1-Inch Graph Paper are provided, for you to make copies and transparencies.

Assessing Progress

Meadows or Malls? concludes with two formal unit assessments. In addition, there are many opportunities for more informal, ongoing assessments throughout the unit. For more information about assessment and grading, including general information about the end-of-unit assessments and how to use them, consult *A Guide to IMP*.

End-of-Unit Assessments

This unit concludes with in-class and take-home assessments. The in-class assessment is intentionally short so that time pressures will not affect student performance. Students may use graphing calculators and their notes from previous work when they take the assessments. You can download unit assessments from the *Meadows or Malls?* Unit Resources.

Ongoing Assessment

One of the primary tasks of the classroom teacher is to assess student learning. Although the assigning of course grades may be part of this process, assessment more broadly includes the daily work of determining how well students understand key ideas and what level of achievement they have attained on key skills, in order to provide the best possible ongoing instructional program for them.

Students' written and oral work provides many opportunities for teachers to gather this information. We make some recommendations here of written assignments and oral presentations to monitor especially carefully that will give you insight into student progress.

- Presentations of *Programming Puzzles*
- Presentations or write-ups of *Just the Plane Facts*
- *Three Variables, Continued*
- *Matrices in the Oven*
- *Inverses and Equations*
- Presentations of *Meadows or Malls? Revisited*

Discussion of Unit Assessments

Have students volunteer to explain their work on each of the problems. Encourage questions and alternate explanations from other students.

In-Class Assessment

Let volunteers pick a system and explain which method they used, what results they got, and why they used the particular method.

Presumably, students will have used matrices for the last system. The second equation of Question 2 simplifies to $w = 2z - 6$, so this system is particularly suitable for substitution, but there is no "right" or "wrong" in the choice of method.

The solutions are:

- Question 1: $f = 5$, $g = 10$
- Question 2: $w = 8$, $z = 7$
- Question 3: $a = 3$, $b = -1$
- Question 4: $x = 1$, $y = -2$, $z = -1$, $w = 3$

Take-Home Assessment
Question 1 is fairly straightforward. Students will probably represent this system of linear equations with the matrix equation

$$\begin{bmatrix} 13 & 12 & -15 \\ 24 & -3 & 36 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 20 \\ 71 \\ 36 \end{bmatrix}$$

Question 2 is more complicated, because there are various ways to set up the matrices. The important element of this problem is setting up the matrices in a compatible way and using the correct matrix product to find the answer.

One method involves using the matrix

$$[A] = \begin{bmatrix} 0.5 & 0.1 \\ 0.3 & 0.07 \end{bmatrix}$$

to display the information about making vases and cups (so that the first column represents amounts of clay needed and the second column represents amounts of glaze needed).

One can then use the matrix

$$[B] = \begin{bmatrix} 40 & 20 \\ 30 & 45 \end{bmatrix}$$

to display the information about how many of each type of product will be made each week (so that the first row represents the first week and the second row represents the second week).

With this setup, the product [B][A] will give the amount of clay and glaze needed each week. The product comes out to

$$\begin{bmatrix} 26 & 5.4 \\ 28.5 & 6.15 \end{bmatrix}$$

which means that Aki needs 26 pounds of clay and 5.4 pounds of glaze the first week and 28.5 pounds of clay and 6.15 pounds of glaze the second week.

As noted previously, there are other ways to do this correctly, but there are also many incorrect expressions. For instance, the product [A][B] is a well-defined matrix product, but its entries do not answer Aki's question.

Question 3

Students should be able to set up a system of linear equations to solve this problem. For instance, if they use *f* for the number of free throws, *w* for the number of two-point field goals, and *h* for the number of three-point field goals, then the system could look like this:

$$f + w + h = 13$$
$$f + 2w + 3h = 27$$
$$f = \frac{1}{2}w$$

The solution is $f = 3$, $w = 6$, $h = 4$.

Supplemental Activities

Each unit contains a variety of activities that you can use to supplement the regular unit material. These activities are included at the end of the student pages for the unit and fall roughly into two categories.

Reinforcements increase students' understanding of and comfort with concepts, techniques, and methods that are discussed in class and are central to the unit.

Extensions allow students to explore ideas beyond those presented in the unit, including generalizations and abstractions of ideas.

The supplemental activities are presented in the *Teacher's Guide* and the student book in the approximate sequence in which you might use them. Listed here are specific recommendations about how each activity might work within the unit. You may wish to use some of these activities, especially the later ones, after the unit is completed.

***How Many Regions?* (extension)** This problem can be used at any time in the unit. In this problem, students subdivide the coordinate plane using the graphs of linear equations. They are asked to find all the possibilities for the number of regions that would be created, to give specific examples using equations for the lines, and to use inequalities to describe the regions.

***The Eternal Triangle* (reinforcement)** This activity is similar to the problems in the activity *Programming Puzzles* and can be used with students who need more experience with such two-variable linear programming situations.

***The Jewelry Business* (reinforcement)** This linear programming problem involves a profit line that is parallel to one of the constraint lines, so it makes a good follow-up to *Widening Woody's Woodwork.*

***Special Planes* (extension)** This activity makes a good follow-up to *Trying Out Triples*, which worked with the three-variable coordinate system and the equations of planes.

***Embellishing the Plane Facts* (extension)** This activity can be used anytime after *Just the Plane Facts* is discussed. The problem asks students to connect the algebra they have been studying to the geometry of that activity.

***A Linear Medley* (reinforcement)** The situations in this activity involve linear equations in one or two variables, and might be used after *Eliminating More Variables*. Note that Question 4 cannot be solved with the information given; students should propose a way to change the problem so that there is a unique solution.

***The General Two-Variable System* (extension)** By the time they complete *Gardener's Dilemma*, many students should be fairly comfortable with systems of two linear equations in two variables. This activity asks them to find a general formula for the solution to such a system and to discuss the relationship between their formula and whether such a system has a unique solution.

***Playing Forever* (reinforcement)** This activity follows up on *Let Me Count the Ways,* providing more situations involving the systematic creation of lists.

***The Shortest Game* (extension)** This activity poses a challenging and complex question about the game described in *POW 2: SubDivvy* and makes a good follow-up to the POW.

***Producing Programming Problems—More Variables* (extension)** This activity asks students to invent and solve a linear programming problem involving four or more variables. You can assign this problem after the presentations of *Eastside Westside Story*.

***Your Own Three-Variable Problem* (extension)** This activity asks students to make up a problem that can be solved using three variables and three linear equations. It makes a good follow-up to *Ages, Coins, and Fund-Raising*.

***Fitting a Plane* (extension)** This activity can be used as a follow-up to *Fitting More Lines,* although you may prefer to wait until students have completed *Fitting Quadratics.*

***Surfer's Shirts* (reinforcement)** This activity describes two situations in which students have to set up matrices and matrix expressions that will provide certain information. It is a good follow-up to *Fresh Ingredients*.

***An Associative Proof* (extension)** This activity is a natural follow-up to *Things We Take for Granted*, and can be assigned after the discussion of that activity.

***When Can You Find an Inverse?* (extension)** In *Inverses and Equations,* students see that not every 2 x 2 matrix has an inverse. In this problem, they investigate more fully when matrix inverses exist, for both 2 x 2 and 3 x 3 matrices.

Note: One way to state the result is to say that a 2 x 2 matrix has an inverse unless one row is a multiple of another and that a 3 x 3 matrix has an inverse unless one row is the sum of multiples of the other two rows.

Students might not get a statement as complete or as clear as this.

***Determining the Determinant* (extension)** This is a research assignment, in which students investigate the concept of a *determinant* and report on what they find. This activity is a good follow-up to the supplemental problem *The General Two-Variable System*. It provides a more general and more abstract approach to ideas that are also dealt with in the supplemental problem *When Can You Find an Inverse?* We suggest that you let interested students work on this following the discussion of *Inverses and Equations*.

***Cracking Another Code* (reinforcement)** This problem can be assigned anytime after *POW 3: Crack the Code* is discussed.

Recreation Versus Development: A Complex Problem

Intent

This sequence of activities introduces the central unit problem and helps students recall the basic steps in solving a linear programming problem.

Mathematics

Many of your students will likely have had prior exposure to solving a linear programming problem in two variables using a graphical approach. They will need to see that the six variables in the central unit problem render graphing unfeasible. Nevertheless, the major steps in solving a linear programming problem will still be pertinent: expressing the constraints and profit algebraically, and then finding the point(s) of intersection of constraints to determine an optimal profit. This involves the broader concept of solving a system of equations.

Progression

Meadows or Malls? introduces the central unit problem about three different pieces of land being given to the city, causing a city planning dilemma. In *Meadows, Malls, and Variables,* students express the constraints for the unit problem algebraically.

The activity *Heavy Flying* serves as the basis for a review of linear programming in two variables.

Meadows or Malls?

Meadows, Malls, and Variables

POW 1: That's Entertainment!

The Standard POW Write-Up

Heavy Flying

Meadows or Malls?

Intent

This activity introduces the unit problem.

Mathematics

Meadows or Malls? introduces the unit problem, and challenges students to find one way to allocate the land that satisfies the constraints, and to find the cost for that allocation. The discussion focuses on the difficulty of solving a problem of this nature and the need for developing a more efficient approach than guess-and-check.

Progression

Students work on the activity in groups, and share their results with the class.

Students will find it challenging to discover numbers that satisfy all of the constraints of this problem and will have little confidence that they are finding the best solution. This difficulty will help students to appreciate the usefulness of linear programming, which they will study in this unit.

Approximate Time

10 minutes for introduction
35 minutes for activity
10 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

The central unit problem has the same title as the unit itself. You might introduce the problem by having students read the description of the situation aloud.

Students who have had previous exposure to linear programming may initially try to tackle the problem using graphing, and you should have graph paper available for students who want to try this approach. It is likely that many students will recognize a similarity between this problem and the central problem of the Year 1 unit *Cookies.* They will soon find out that they have too many variables to graph, but it is preferable that they realize this themselves through trial and error.

If groups get too frustrated trying to graph, you can discuss with the class why this isn't appropriate: while graphing is a valuable tool in solving linear programming problems in two variables, a problem in more than two variables does not lend itself to this approach. If you like, tell them that there are ways to draw three-dimensional pictures to represent situations with three variables but that for more than three variables, as in the unit problem, graphing can't be used.

Students should spend the period investigating the problem, but don't expect them to solve it at this time. In fact, they may not come up with any allocation that meets the conditions. They will see an example in Question 3 of *Meadows, Malls, and Variables*, and if needed, you can give them time to find one of their own after discussion of that activity.

Discussing and Debriefing the Activity

Based on the results from this activity and the discussion of *Meadows, Malls, and Variables*, you may want to allow groups more time to come up with allocations that fit the constraints, following the discussion of the latter activity.

Question 1

Have groups report their allocations. Create a table like this, using poster paper:

Group	Goodfellow land for rec.	Army land for rec.	Mining land for rec.	Goodfellow land for dev.	Army land for dev.	Mining land for dev.	Cost
1	200	0	0	100	100	150	410,000
2	200	0	50	100	100	100	365,000

As each allocation is reported, have other groups check to see that the combination satisfies all the conditions of the problem.

Save this poster. At the end of the unit, it will be interesting for students to see just how close they came to discovering the optimal solution.

Question 2

The main goal of Question 2, about approaches to the unit problem that the group might like to try, is for students to realize how difficult the problem is and to recognize the need for more mathematical tools to solve it.

Ask each group to report on their approaches. (If the inadequacy of the graphing approach wasn't discussed previously, it may come up either now or in the next discussion, following *Meadows, Malls, and Variables.*) Some students might suggest writing a computer program that would test many different combinations of land allocation in a systematic way. Another idea would be to ignore some constraints, see if the problem can be solved without them, and then refine these solutions to fit the other constraints.

Meadows, Malls, and Variables

Intent

Students develop the constraints and cost expression for the unit problem.

Mathematics

In *Meadows, Malls, and Variables*, students express conditions from the unit problem algebraically, develop the cost function, and test two possible allocations. The discussion of this activity will review what a **linear programming** problem is and introduce the concept of **linear algebra.**

Progression

Students work on the activity independently. In the discussion, the class agrees upon constraints, and posts these constraints and the cost expression.

Approximate Time

30 minutes for activity (at home or in class)
20 to 25 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

Question 1: The Constraints

You may want to begin by posting the definitions of the variables from the activity for reference. For your convenience, we list them here as they were given in the activity:

- G_R is the number of acres of Mr. Goodfellow's land to be used for recreation.
- A_R is the number of acres of army land to be used for recreation.
- M_R is the number of acres of mining land to be used for recreation.
- G_D is the number of acres of Mr. Goodfellow's land to be used for development.
- A_D is the number of acres of army land to be used for development.
- M_D is the number of acres of mining land to be used for development.

Have students from different groups report on how they expressed the conditions in the problem. Once you get agreement from the class on the formulation of the constraints, post the results for reference during the rest of the unit. Students will probably come up with a list similar to this:

I $\quad G_R + G_D = 300$

II $\quad A_R + A_D = 100$

III $M_R + M_D = 150$

IV $G_D + A_D + M_D \geq 300$

V $A_R + M_R \leq 200$

VI $A_R + G_D = 100$

VII $G_R \geq 0$

VIII $A_R \geq 0$

IX $M_R \geq 0$

X $G_D \geq 0$

XI $A_D \geq 0$

XII $M_D \geq 0$

If students don't mention the last six constraints at this point, leave them off the list. The issue of requiring that the variables be nonnegative will come up in the discussion of the feasible region for the problem in *Heavy Flying*. At that point, you can add Constraints VII through XII to the list.

Important: The numbering of the constraints as shown here will be used throughout this guide, so you may want to post your list with this numbering. When students ultimately solve the problem, it will be helpful to have the constraints numbered for easy reference.

Question 2: The Cost to the City

Have a student give you the cost expression:

$$50G_R + 200A_R + 100M_R + 500G_D + 2000A_D + 1000M_D$$

Post this expression along with the definition of the variables and the constraints.

Question 3: Testing Allocations

The allocation in Question 3a violates Constraint IV. The allocation in Question 3b fits all the constraints and would cost the city $410,000.

Ask, **What problems have you seen before that were like *Meadows or Malls?* What did you call this type of problem?** If needed, remind students that problems of this type are called **linear programming** problems. Probably many students will recognize a similarity between this problem and the central problem of the Year 1 unit *Cookies.* Ask, **How does the *Meadows or Malls?* problem compare to the central unit problem from Cookies?**

Students may have solved linear programming problems in other courses with a graphical approach. Be sure students understand that such an approach will not work with this problem, because it has six variables.

Linear Programming and Linear Algebra

Ask, **What is a linear equation? Why are these equations called linear?** Ask if anyone can explain the use of the term *linear* in the phrase *linear programming*.

Students will probably be able to articulate that the word *linear* is used because the expressions involved in these problems are linear expressions. (You need not get into details now such as the distinction between a linear equation, a linear inequality, and a linear expression.)

Ask students for examples of linear equations using x and y. They should be able to give examples like $3x + 5y = 7$ or $y = 3x - 9$.

Then ask why such expressions are called *linear*. As needed, review that the name comes from the fact that the graphs of these equations are straight lines. Remind students, however, that in mathematics the word *linear* is used to refer to both the algebraic form of an equation and the geometry of its graph.

Follow this up by asking, **What's an example of a linear equation in only one variable?** Students might suggest equations like $3x + 5 = 11$ or $5a - 2 = 4(a + 7)$.

For clarification, ask for some examples of *nonlinear* equations in one or two variables. Students will probably give other types of polynomial equations, such as ones involving higher powers of the variables (for instance, $2x^2 + 5x - 9 = 0$) or products of variables (for instance, $xy = 10$).

Next ask, **What do you think a linear equation in three variables would look like?** For example, using variables x, y, and z, they should come up with equations like $6x - 5y + 8z = 12$ or $z = 2x - 4y + 9$.

Linear Equations in n Variables and Linear Algebra

Finally, ask **What do you think a linear equation in *n* variables would look like?** Students will probably be able to articulate that it cannot include higher powers of variables. Make sure they also state that it cannot include products of variables, like xy. You might elicit some examples of linear equations in five or six variables as illustrations of the general idea.

Inform students that formally, a *linear equation in n variables* $x_1, x_2, \ldots, x_n$ is any equation equivalent to one of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where the coefficients $a_1, a_2, \ldots, a_n$ and the value b are any specific numbers.

Tell students that the field of mathematics that deals with linear equations and related ideas is called **linear algebra,** and that in this unit they will learn ideas from linear algebra that can be used in many more situations than simply linear programming problems. In particular, they will learn some powerful tools for solving systems of linear equations, including systems involving more than two variables.

Key Questions

What problems have you seen before that were like *Meadows or Malls?*

What did you call this type of problem?

How does the *Meadows or Malls?* problem compare to the central unit problem from *Cookies*?

What is a linear equation?

Why are these equations called *linear*?

What's an example of a linear equation in only one variable?

What do you think a linear equation in three variables would look like?

What do you think a linear equation in *n* variables would look like?

POW 1: That's Entertainment!

Intent

Students solve a complex problem requiring investigation and careful communication of their efforts and results.

Mathematics

That's Entertainment! describes a card trick and challenges students to figure out how it is performed. In solving this POW, students will look for a pattern in data having two dependent variables controlling a single independent variable. The POW has no direct mathematical connection to the unit.

Progression

Students will work on the activity for about a week, followed by presentations from several students.

Approximate Time

15 minutes for introduction
1 to 3 hours for activity (at home)
15 minutes for presentations

Classroom Organization

Individuals, followed by several student presentations

Materials

A standard deck of cards (1 per group)

Doing the Activity

Take some time to introduce the POW and to accumulate some initial data for students to work with. Act out part of the trick (or have students act it out) to make sure they understand this POW.

Although the example in the POW has six piles of cards at the trick's end, students need to look at other scenarios with differing numbers of piles. You may want to emphasize that the entertainer gets two pieces of information—the number of cards in the incomplete pile and the number of complete piles—so students should look for a function based on two inputs.

Give students a week or more to work on the POW. On the day before the POW is due, choose three students to make POW presentations on the following day, and give them overhead transparencies and pens to take home to use for preparing.

Discussing and Debriefing the Activity

Have the three students make their presentations, and then ask other students to contribute things they discovered. If no student has found an algebraic answer to the problem and you have time available, you may want to let students work in their groups to find one. A question like, **How can you tell how many cards will be in a given pile by knowing the initial card?** should help to get them started. You may want to elicit this key observation from the class before having students work in their groups:

If the initial card is n, then there are 12 – n more cards added to the pile. Therefore, a pile whose initial card is n has (12 – n) + 1 = 13 – n cards.

Key Question

How can you tell how many cards will be in a given pile by knowing the initial card?

The Standard POW Write-Up

Intent

This reference page reviews Problems of the Week and the standard POW write-up. As needed, students may refer to this reference page throughout the year to aid with their POW write-ups.

Mathematics

Communicating about mathematical thinking is an important part of doing mathematics. This reference page is designed to support students' written communication about their findings when exploring large mathematical problems. Stating the problem, discussing one's methods, and concluding succinctly and with justification, so that a reader will understand what has been written, should be the goal of every student writer. By suggesting extensions to the problem, students will be saying that the mathematics has not been fully explored, given the time constraints. Their self-assessment will be an evaluation of the effort and quality of their work, what they take pride in, and what they wish they could have done better.

Approximate Time

None—for reference only

Heavy Flying

Intent

This activity sets the stage for a review of linear programming in two variables.

Mathematics

The activity describes a linear programming problem in two variables. Students are asked to find several combinations of numbers that fit the constraints and to figure the earnings from each. They are then asked to rewrite the constraints algebraically.

The discussion reviews the procedure for solving a linear programming problem graphically, using the **feasible region** and a family of parallel lines for optimization. Solving systems of equations is reviewed.

Progression

This activity reviews linear programming, feasible regions, profit lines, and solving systems of equations. These concepts were covered in the Year 1 unit *Cookies*, and will be used throughout this unit.

Approximate Time

25 minutes for activity (at home or in class)
45 to 50 minutes for small-group work and discussion

Classroom Organization

Individuals, followed by a combination of small-group work and whole-class discussion

Doing the Activity

It may take students some time to recall how to set up problems like this one. They are not expected to find the optimal solution for the problem, but merely to start investigating. If you are assigning this as class work, set a firm time limit on the activity and reserve 40 to 50 minutes for reviewing how to find the optimal solution.

Discussing and Debriefing the Activity

Have groups check their members' suggested loads for Lindsay to see if they fit the conditions of the problem.

You might want to begin the whole-class discussion by having students choose variables and use them to express the conditions of the problem algebraically. We will use F for the number of containers of chicken feed and C for the number of boxes of calculators. So students should have these conditions (or equivalents):

$$40F + 50C \leq 37{,}000$$

$$2F + 3C \leq 2000$$

$$F \geq 0$$

$C \geq 0$

(If they don't include the conditions $F \geq 0$ and $C \geq 0$, you can bring up this issue when students discuss the feasible region later in the activity.)

Have a student give you an expression for computing Lindsay's earnings for a given load. (Using our variables, the expression is 2.20F + 3.00C.) Then have several volunteers give their suggested loads and the associated earnings. Have the class verify that each load fits the two constraints and that the earnings have been computed correctly.

***Continued Work on* Heavy Flying**

Ask, **In general, how might you solve the problem?** Be sure students use the terms **feasible region** and *family of profit lines* (or more accurately in this problem, *earnings lines*).

If students don't use these terms, ask, **What do you call the set of points that fit the constraints? What do you call the points with a given profit? How does this set change if you change the profit?**

Note: We will use the term *profit line* when discussing linear programming problems generically even though in specific problems the relevant linear function might be something else, such as a cost function. Similarly, we may refer to *maximizing the profit function* even though a particular problem might involve minimizing a cost function.

Let students work in groups to complete the problem. Pass out overhead transparencies and pens to the first groups to finish, for preparing presentations. For the discussion, it will be good if at least a few groups have drawn in some earnings lines to show why they are sure that the maximum lies at the intersection. While some groups are preparing their presentations, you can help those that were unable to finish or disagree on how to proceed.

Presentations

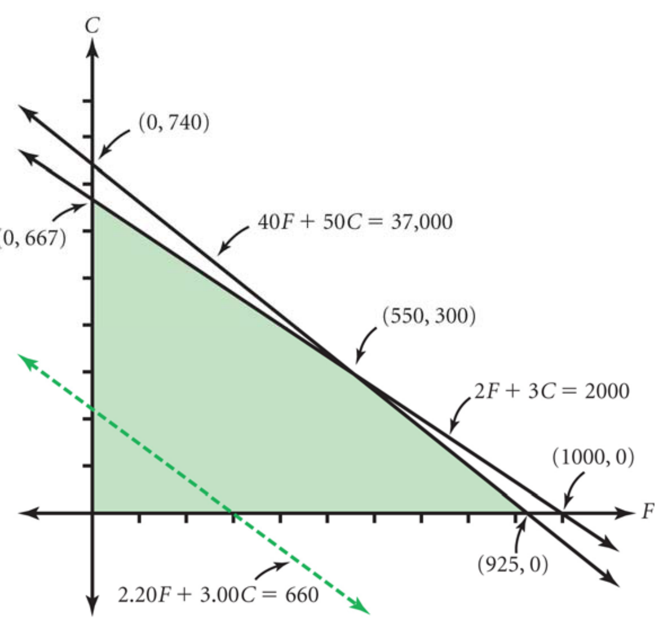
When groups are done and overheads have been prepared, ask the presenters to present their analyses of the *Heavy Flying* problem.

Students should begin with a graph that shows the feasible region, bounded by the coordinate axes and by the graphs of the two equations:

$$40F + 50C = 37{,}000$$

$$2F + 3C = 2000$$

This is the shaded area in this diagram.



(The dashed line is an earnings line, as discussed later.)

Constraints Stating That Variables Are Nonnegative

If students did not mention the constraints $F \geq 0$ and $C \geq 0$, this is a good time to introduce them. For instance, ask why the feasible region doesn't include points in the other quadrants.

Whenever this issue is first raised, have students go back to their list of constraints for the central unit problem, *Meadows or Malls?,* and add appropriate constraints .

The Family of Earnings Lines

The main focus in the discussion should be on the use of the family of earnings lines to explain where the maximum income is found.

Students should be able to explain that the points that give a particular income form a straight line and that changing the given income yields a different but parallel line. (We suggest that you temporarily postpone a discussion of how students know such lines are parallel—see the next subsection.)

For example, the dashed line in the preceding graph shows those points where Lindsay's earnings are $660; that is, the dashed line is the graph of the equation $2.20F + 3.00C = 660$. [The number 660 was chosen because it gives two easy solutions: (300, 0) and (0, 220).]

Students should be able to articulate that they are looking for the line in this parallel family that intersects the feasible region at the "last" possible place and that the point where the family of lines leaves the feasible region is their desired optimal solution.

Some students may have located their optimal point by drawing their graph carefully, paying attention to the "slant" of the income line as compared to the "slants" of the boundary lines for the feasible region.

A careful drawing shows that the family of parallel lines leaves the region at the point where the equations $40F + 50C = 37{,}000$ and $2F + 3C = 2000$ intersect: (550, 300). Thus, Lindsay's optimal load is 550 containers of chicken feed and 300 boxes of calculators. (See the following subsection, "Finding the Point of Intersection," for a discussion of how to handle the issue of identifying the point where the lines intersect.)

Why Are the Lines Parallel?

Ask, **How do you know that the different earnings lines are parallel?**
Students may have various explanations. One approach is to realize, for example, that the equations $2.20F + 3.00C = 660$ and $2.20F + 3.00C = 330$ cannot have any solutions in common, because the expression $2.20F + 3.00C$ cannot be equal to both 660 and 330. Because the equations have no common solutions, the graphs have no common points, which means the lines are parallel.

Finding the Point of Intersection

Ask, **How did you find the point of intersection of the lines?** If students did not use algebra, ask, **How could you find the common solution algebraically?**

Students may have found the point of intersection of the lines $40F + 50C = 37{,}000$ and $2F + 3C = 2000$ using graphing or guess-and-check, but you should ask them to review the algebraic methods they could use to find the common solution to these two equations.

Most will probably have used substitution or a variant of it, such as solving the second equation for $2F$ or solving both equations for F and setting the results equal. For instance, if students use the second equation to write $2F$ as $2000 - 3C$, then they can write $40F$ as $20 \cdot 2F$. The first equation becomes $20(2000 - 3C) + 50C = 37{,}000$, which they can solve to get $C = 300$, and then substitute back to get $2F = 1100$, and $F = 550$. Or they can get $F = 1000 - 1.5C$ from the second equation and $F = 925 - 1.25C$ from the first, set $1000 - 1.5C$ equal to $925 - 1.25C$, and solve for C.

Let students share ideas now, and tell them that they will be learning more about solving sets of linear equations later in the unit.

Note: If students find this system of equations too difficult to solve algebraically, you can give them a simpler pair of equations, such as $2x + y = 17$ and $y - x = 8$. Have them work in groups to solve the system algebraically, and then let volunteers share their results.

Key Questions

In general, how might you solve the problem?

What do you call the set of points that fit the constraints? What do you call the points with a given profit? How does this set change if you change the profit?

How do you know that the different earnings lines are parallel?

How did you find the point of intersection of the lines?

How could you find the common solution algebraically?

A Strategy for Linear Programming

Intent

In this sequence of activities, students develop and implement a strategy for solving a two-variable linear programming problem without graphing.

Mathematics

Two-variable linear programming problems can be solved by graphing the system of constraint inequalities to display the feasible region, adding a family of parallel profit lines, and observing where the profit lines last leave the feasible region. This foundation enables students to better visualize what is happening in the solution of a more complex linear programming problem, such as the central unit problem, where there are too many variables to construct a graph.

In *A Strategy for Linear Programming,* students familiarize themselves once more with the graphical strategy for solving these problems. In doing so, they observe that the maximum profit (or minimum cost) always occurs at a corner of the feasible region. Based on this observation, they develop a strategy for solving two- variable linear programming problems without graphing, by finding the intersections of all possible pairs of equations associated with the constraint conditions, and then testing each intersection for feasibility and profit. Along the way, students practice procedures for solving systems of equations and encounter **dependent** and **inconsistent systems.**

Progression

Students begin by reviewing strategies for solving systems of two linear equations and for solving linear programming problems in two variables in *Programming and Algebra Reflections* and *Ideas for Solving Systems*. In *Programming Puzzles*, the class solves a sufficient number of linear programming problems to observe that the solution always falls at a corner of the feasibile region. This leads to development of a strategy that does not involve graphing in *Finding Corners Without the Graph.*

In the remaining activities in this sequence, students encounter a number of complications and special situations as they implement and practice their new strategy. These include an unbounded feasible region and the need to minimize rather than maximize the linear function (*Donovan Meets the Beatles*), parallel lines as constraint boundaries (*Widening Woody's Woodwork*), and dependent and inconsistent systems of equations (*More Equations*).

Programming and Algebra Reflections

Ideas for Solving Systems

Programming Puzzles

Donovan Meets the Beatles

Finding Corners Without the Graph

What Wood Would Woody Want?

Widening Woody's Woodwork

More Equations

Programming and Algebra Reflections

Intent

Students reflect on the processes for solving a system of linear equations and for solving a linear programming problem.

Mathematics

The first part of this activity asks students to describe what constitutes a linear programming problem, and the general steps for solving one. The second part gives two pairs of linear equations and asks students to solve each system and explain each step.

This activity may lead some students to realize (if they haven't already) that to solve a linear programming problem in two variables, they need only check the intersections of the constraining conditions (the corner points).

Progression

This activity reviews student's prior knowledge.

Approximate Time

30 minutes for activity (at home or in class)
15 to 20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Optional: Sentence strips for constructing poster during discussion

Doing the Activity

Students should be able to do this activity with no introduction.

Discussing and Debriefing the Activity

Begin the discussion with Part II, so that you can make a smooth transition from Part I to the next activity, *Programming Puzzles*.

Part II

Have volunteers share their work in solving each of the linear systems from the activity. After each presentation, ask if other students used different approaches. Take this opportunity to let students see a variety of methods.

For instance, in Question 3, students might solve the first equation for *y* in terms of *x* and substitute that expression into the second equation. Or they might solve each equation for *y* and then set the two expressions equal to each other. Use

Question 4 to bring out that they can just as well solve for *x* in terms of *y* (which is easily done in the second equation) as solve for y in terms of x. The solution to Question 3 is $x = 3$, $y = 1$; the solution to Question 4 is $x = 3$, $y = 2$.

If students seem rusty on the substitution method, you can make up another problem with two equations in two unknowns for them to solve. (For now, stick to examples involving pairs of independent equations. Dependent and inconsistent systems will arise in *More Equations* and the subsequent discussion.)

Part I

Have students share ideas about Part I in their groups. You might tell them that you will choose students at random to report on Questions 1 and 2.

After one student has answered Question 1, ask volunteers to add more information as needed. Here are some broad features of linear programming problems (in two variables) that the class should provide;

- The problem involves two quantities that need to be determined. Each of those two quantities is represented by a variable.
- The problem states conditions about those quantities that can be represented using linear inequalities (or equations) in those variables.
- The problem describes a particular linear function in those variables that is to be maximized (or minimized).
- The task is to find the values for the two variables that maximize (or minimize) the linear function while satisfying all the constraints.

You will find it useful to post this description (perhaps abbreviated) so students can refer to it when they try to generalize to more variables.

Question 2: A Strategy

Choose another student at random to answer Question 2. Again, you will probably want many students to contribute ideas.

After some discussion, students should come up with a general procedure for solving linear programming problems such as this:

1. Write the constraints as a set of linear inequalities, including inequalities stating that the variables are nonnegative.

2. Graph each of the inequalities and find where the graphs overlap; that is, draw the feasible region.

3. Draw one or more profit lines, each representing the set of points where the function being maximized or minimized has some specific value.

4. Determine where in the feasible region the profit function will have its maximum or minimum value by seeing where the family of parallel profit lines leaves the feasible region.

5. Find the coordinates of the point identified in the previous step.

Post this list alongside the description of two-variable linear programming problems.

Note: This list will be modified in subsequent discussions. Therefore, you may want to use sentence strips in posting the list so that you can replace portions of it more easily. Or you may find it more convenient to use chart paper for the initial strategy and then place sentence strips over it as needed to make changes. For instance, the preliminary problems use only inequalities as constraints, so this language is adequate for now, but the *Meadows or Malls?* constraints include equations as well.

A Generic Feasible Region

Ask, **In general, what might a feasible region look like?** For instance, they might say that it is a polygonal region in which parts of the coordinate axes are two of the sides. You need not get a definitive description here. In fact, in *Donovan Meets the Beatles* students will see that a feasible region need not fit this description, because it may be an unbounded region.

Key Question

In general, what might a feasible region look like?

Ideas for Solving Systems

Intent

These pages serve as reference material for students to review procedures for solving systems of equations.

Mathematics

This material describes two methods of solving a system of equations in two variables: solving both equations for the same variable and then setting the resulting expressions for that variable equal to each other, and using substitution.

Progression

No substantial discussion of the reference material is necessary.

Approximate Time

5 minutes for introduction

Doing the Activity

If students need help with solving pairs of linear equations, suggest that they look over this material.

Programming Puzzles

Intent

Students apply their linear programming strategy and discover that solutions occur at a corner of the feasible region.

Mathematics

Through looking at several variations of each problem in this activity, students examine how changes in the cost function in a linear programming problem affect where the solution is in the feasible region. In doing so, they observe that solutions to two-variable linear programming problems occur at corner points.

Progression

Students continue to solidify their understanding of how to solve linear programming problems in two variables.

Approximate Time

30 minutes for activity
25 to 35 minutes for presentations and discussion

Classroom Organization

Small groups, followed by group presentations and whole-class discussion

Materials

Grid poster paper or transparencies (1 for each group)

Doing the Activity

This activity includes three separate problem situations, with three versions for each situation. None of the problems involve significant new concepts, so there is no need to cover them all.

We recommend that you have each group work initially on only a single version of one of the problems and have group members do individual write-ups of this problem. You may want to make these write-ups an additional part of the homework assignment, so that students can do a thorough and careful job. If groups finish early, students can begin the write-ups or look at the other versions of their problem.

You might consider alternate scenarios for student work on this activity. For instance, you can give students enough time to do all three versions of one of the problems or to do one version of each problem.

Tell students that each group will be doing a presentation on its particular problem. Point out that because each situation has several variations, groups will probably not have to present all aspects of the problem. Tell them, however, that they should all be prepared to present all aspects of the problem.

Comment: Version c of Question 2 doesn't make any sense in terms of the problem situation, and version b isn't much better. If students working on these problems point this out, have them proceed as if the problems do make sense, finding the strategy that will maximize income from sales. Assure them that this difficulty will be discussed.

Discussing and Debriefing the Activity

If the key ideas about the family of parallel lines and the role of corner points seem clear to the class, you need not go over all the problems from this activity. But if you think that these ideas are not yet clear, discuss the problems thoroughly.

Have groups make presentations to the class. Because all versions of a given problem have the same constraints and feasible region, you need not have three complete presentations on each problem. For instance, you might pick one group to give the constraints, another to present the feasible region, and a third to discuss its final solution. The first two groups can then each give their solution to their version of the problem.

Although students will need to solve pairs of linear systems to get points of intersection, you should downplay the details of this algebra in order to concentrate on two key issues:

- the "family of parallel lines" reasoning
- the fact that all the solutions are at corner points

Following are the constraints and feasible regions for each problem, with some comments on issues that might arise in discussion.

Question 1: A Nonroutine Routine

If students use x for the number of pool drop-ins and y for the number of rail slides, then the constraints are

$x + y \leq 25$

$4x + 9y \leq 180$

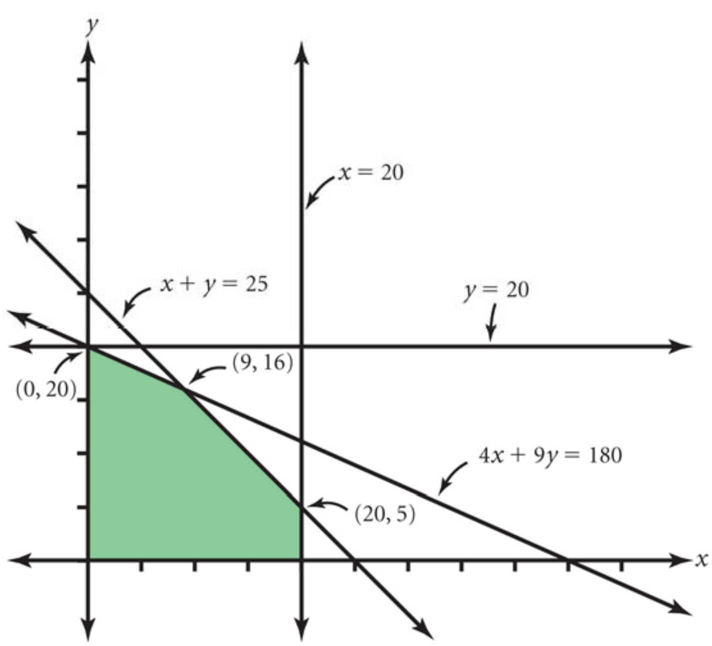
$x \leq 20$

$y \leq 20$

$x \geq 0$

$y \geq 0$

The feasible region is shaded in the diagram at right.



Each presenting group will have a different family of parallel "score lines." Here are the solutions to the different versions:

- Version a: Lance's score is maximized by doing 9 pool drop-ins and 16 rail slides, which gives him a total score of 205 points.
- Version b: Lance's score is maximized by doing 20 pool drop-ins and 5 rail slides, which gives him a total score of 165 points.
- Version c: Lance's score is maximized by doing a routine consisting exclusively of 20 rail slides, which gives him a score of 160 points.

Question 2: Planning the Prom
If students use *S* for the number of individual tickets and *C* for the number of couples tickets, then the constraints are

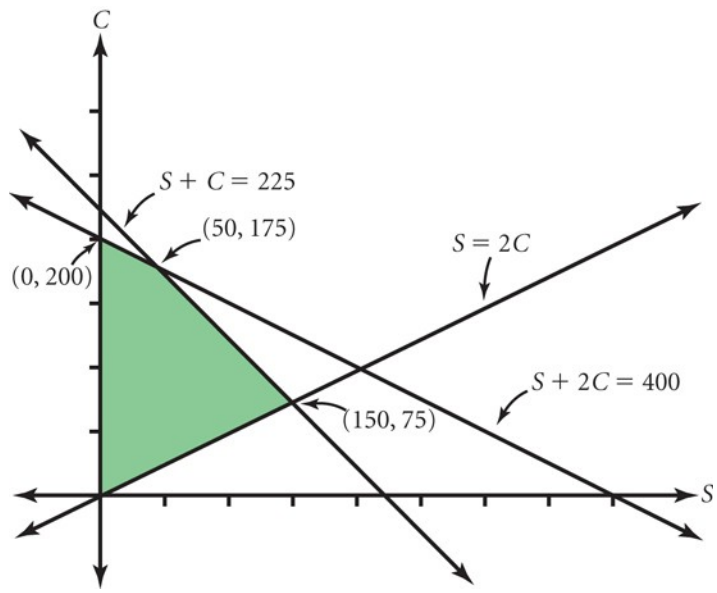
$S + 2C \leq 400$

$S \leq 2C$

$S + C \leq 225$

$S \geq 0$

$C \geq 0$



The feasible region is shaded in the diagram at right.

Here are the solutions to the different versions:

- Version a: Paige and Remy should sell 50 individual tickets and 175 couples tickets, which give a sales total of $6,350.

- Version b: Paige and Remy should sell exclusively 200 couples tickets, which give a sales total of $7,000.

- Version c: Paige and Remy should sell 150 individual tickets and 75 couples tickets, which give a sales total of $6,000.

Students will probably point out that it makes no sense to charge more for an individual ticket than for a couples ticket. They may also argue, with good reason, that it doesn't make much sense for the couples ticket to be more than twice the price of an individual ticket. If students do not raise these points, bring them up as a reminder of the importance of trying to make sense out of mathematical solutions. Bring out that under the constraints of the problem and any reasonable pricing scheme, the optimal solution would be to sell 50 individual tickets and 175 couples tickets.

Question 3: Working Two Jobs

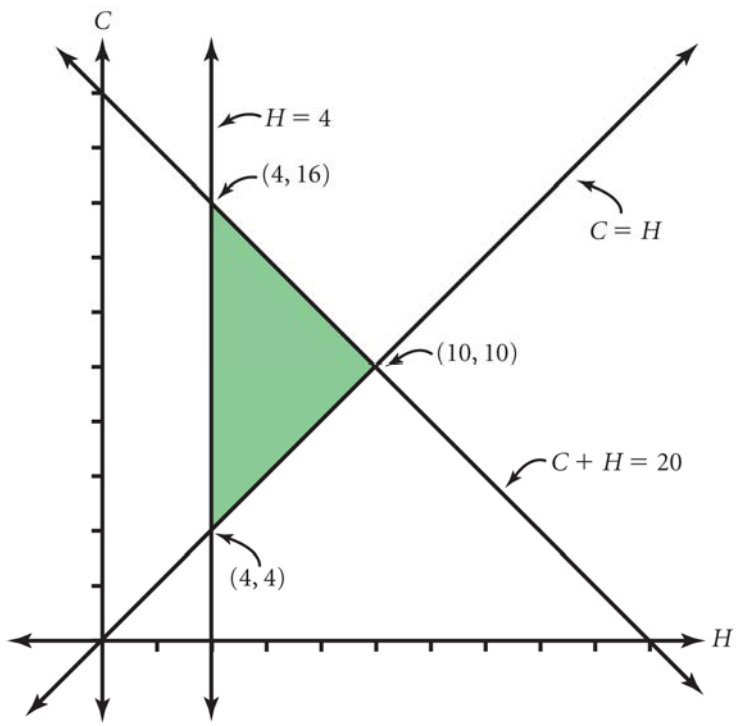
If students use H for the number of hours Raj spends at the hospital and C for the number of hours he spends at the clinic, the constraints are

$H + C \leq 20$
$H \geq 4$
$C \geq H$
$H \geq 0$
$C \geq 0$

Because of the constraint $H \geq 4$, the condition $H \geq 0$ is superfluous. In other words, any point that satisfies the condition $H \geq 4$ will automatically satisfy $H \geq 0$.

Similarly, because of the constraint $C \geq H$, the condition $C \geq 0$ is superfluous. These observations are reflected in the fact that the axes are not part of the boundary of the feasible region. Students might choose to omit these superfluous constraints.

The feasible region is shaded in the diagram at right.



As with the other problems, each group will have a different family of parallel lines. In this problem, versions a and c lead to the same solution (but different earnings). Here are the solutions to the different versions:

- Version a: Raj should work 10 hours at each job. He will earn a total of $350.
- Version b: Raj should work 4 hours at the hospital and 16 hours at the clinic. He will earn a total of $372.
- Version c: Raj should work 10 hours at each job. He will earn a total of $340.

Intuitive Explanations

Because these problems are fairly simple, students may want to explain some of them intuitively. This is fine as long as the main ideas about the linear programming method are clarified as well. For instance, a student might explain version b of Question 3 this way:

Raj gets paid more at the clinic and likes it better, so he should work there as much as possible. Therefore, he should spend the minimum time possible at the hospital—

4 hours—and spend the remaining 16 hours at the clinic.

The Importance of Corners

One of the key ideas that should come out of this discussion is the fact that the solution to a problem of this type always occurs at a corner of the feasible region.

If this hasn't yet been stated clearly in the course of the discussion, ask, **Is there any combination of pay rates that would lead Raj to a job plan different from those you got in Question 3?**

You can also ask, more specifically, **Are there any points in the middle of the feasible region that give a maximum?** Persist until students can articulate this important principle:

The profit function always achieves its maximum (or minimum) at a corner of the feasible region.

Post this principle. (We are ignoring the complication that a real-world problem may require a solution with integer coordinates. That issue is addressed in the discussion of *Finishing Off the Cookies.*)

If students raise the possibility of a "pay line" that is parallel to one of the constraint lines (for instance, in this problem, an equal pay rate for the two jobs), tell them that they will examine that issue in *Widening Woody's Woodwork*. The situation in that activity illustrates that although the maximum or minimum is always achieved at a corner, it may also be achieved along an entire edge of the feasible region (including the endpoints of that edge, which are corners).

Do not expect students to provide a proof that the maximum (or minimum) can always be achieved at a corner point. (That requires a more formal definition of "corner" and more detailed algebra than is appropriate here.) Rather, be sure they see visually that as a line "moves in a parallel way" across the plane through a polygon, it will both "enter" and "exit" the polygon at a vertex (or along an entire line, which includes two vertices).

Key Questions

Is there any combination of pay rates that would lead Raj to a job plan different from those you got in Question 3?

Are there any points in the middle of the feasible region that give a maximum?

Supplemental Activity

The Eternal Triangle (reinforcement) poses a problem similar to those in *Programming Puzzles* and can be assigned to students who seem to need more work of this type.

Donovan Meets the Beatles

Intent

This activity continues the review of two-variable linear programming.

Mathematics

This problem differs from those discussed previously in that the feasible region is unbounded and the problem asks for a minimum value of a linear expression (rather than a maximum).

Progression

This activity provides more practice with a two-variable linear programming problem.

Approximate Time

30 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity is a two-variable linear programming problem suitable for independent practice.

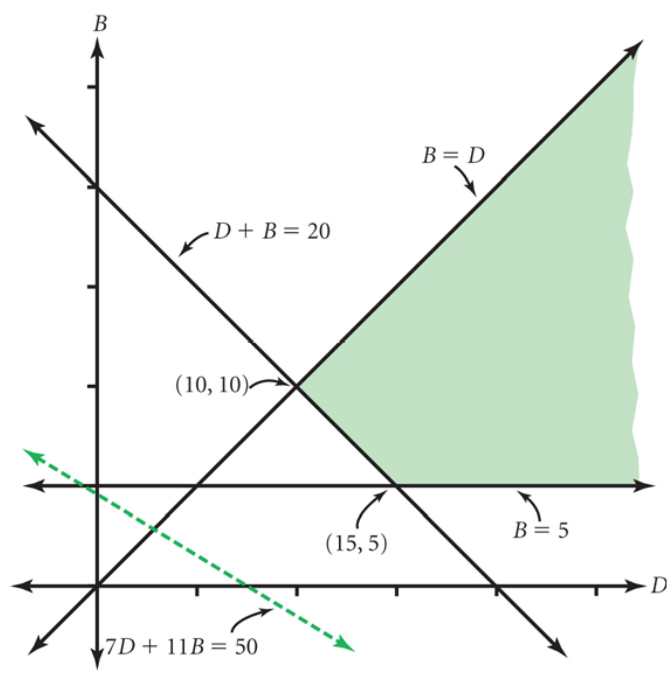
Discussing and Debriefing the Activity

You might have a volunteer present the situation from this problem. Using D for the number of Donovan songs and B for the number of Beatles songs, the constraints are

$$D \geq B$$
$$B \geq 5$$
$$B + D \geq 20$$

As with Question 3 of *Programming Puzzles*, the "nonnegativity" constraints, $D \geq 0$ and $B \geq 0$, are superfluous here because of other constraints.

The feasible region is shaded in this diagram, and the dashed line is the cost line $7D + 11B = 50$.



As the diagram suggests, the “lowest” member of the family of parallel cost lines to touch the feasible region is the cost line through (15, 5). At that point, the cost is $7 \cdot 15 + 11 \cdot 5 = \$160$. [You might have a student verify that the cost at the other corner point, (10, 10), is more than $160.]

Finding Corners Without the Graph

Intent

Students consider the task of locating corner points of a feasible region without actually drawing the region.

Mathematics

In a linear programming problem with three variables, graphing the feasible region is difficult; with more than three variables it is not possible. We know that in a two-variable problem the profit function will be maximized at a corner of the feasible region; the analogous point in an *n*-variable problem is the solution to a combination of *n* boundary equations. Without the benefit of a graphical representation, it is necessary to solve for each such "intersection," verify that the solution is feasible, and then find the corresponding profit.

This activity moves students a step closer to realizing the above strategy by having them find all of the constraint boundary intersections for a two-variable problem.

Progression

This activity is preceded by development in whole-class discussion of a strategy for solving two-variable linear programming problems without graphing. The discussion afterward brings out the need to find *all* intersections of equations from the constraints, even though some are outside the feasible region

Approximate Time

15 minutes for introduction
30 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, preceded and followed by whole-class discussion

Doing the Activity

Ask, **How might you solve a linear programming problem without drawing the feasible region?** The key dilemma is figuring out where the family of parallel lines leaves the feasible region without drawing a feasible region.

Two basic observations should come out in order for students to develop a plan for finding a maximum or minimum point without drawing a feasible region:

- The point being sought must be at a *corner point* of the feasible region.
- *Corner points* of the feasible region are places where two boundary lines meet (including the case in which one or both of the boundary lines is a coordinate axis).

Based on this discussion, have the class revise the strategy posted after *Programming and Algebra Reflections*, creating a new strategy that avoids any need for a graph of the feasible region. Students might come up with something like this (but see the warning that follows):

1. Write the constraints as a set of linear inequalities (including constraints that the variables are nonnegative).

2. Find all points where boundary lines intersect by solving all possible pairs of linear equations corresponding to constraints. (These intersection points correspond to possible corner points of the feasible region.)

3. Write down the function being maximized or minimized.

4. Evaluate the function being maximized or minimized at each of the intersection points found in step 2, and determine which of these points gives the function its maximum or minimum value in the feasible region.

This new strategy should be recorded and posted. Have students write down the strategy they create, for use in later homework activities.

Warning: The list of steps just given ignores the fact that some of the intersection points may be outside the feasible region. It is important that students recognize this difficulty on their own. If they don't see this error now, it will probably come up when they apply this strategy in *Finding Corners Without the Graph*. Therefore, we recommend that you not point out this difficulty at this time. You can bring it up later, if necessary.

As students will see in *Widening Woody's Woodwork,* even for the case in which the profit line is parallel to a boundary of the feasible region, there is still a corner point that maximizes the profit function (or minimizes the cost function). You need not bring this up now if students don't raise the issue.

Discussing and Debriefing the Activity

Begin by having a volunteer list the equations, as well as the combinations. It will help to number the constraints for reference:

I $x + 2y \leq 8$

II $2x + y \leq 13$

III $y \leq 3$

IV $x \geq 0$

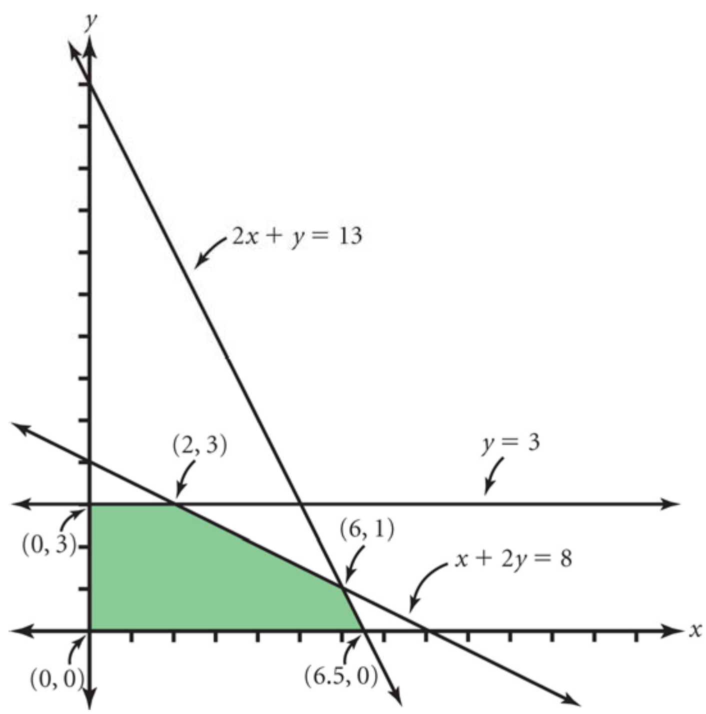
V $y \geq 0$

The presenter will probably use a list to show that there are ten ways of pairing up the corresponding equations.

Then have individual students give the points of intersection. They should get nine such points, because the equations corresponding to constraints III and V have no common solution. Using the same numbers to refer to the equations corresponding to each constraint, the points of intersection are

- Equations I and II: (6, 1)
- Equations I and III: (2, 3)
- Equations I and IV: (0, 4)
- Equations I and V: (8, 0)
- Equations II and III: (5, 3)
- Equations II and IV: (0, 13)
- Equations II and V: (6.5, 0)
- Equations III and IV: (0, 3)
- Equations III and V: no point of intersection
- Equations IV and V: (0, 0)

Finally, ask, **Are these points all vertices of the feasible region?** If necessary, remind the class that points in the feasible region (including corner points) must satisfy *all* the constraints. Have students check all nine points of intersection and identify which of them are actually in the feasible region. They should find that only (6, 1), (2, 3), (6.5, 0), (0, 3), and (0, 0) work, and they should identify which constraint or constraints the other points fail to satisfy.



Once students have completed that task, have them sketch the feasible region to confirm their work, as shown at right.

Why Find All Intersections?

Ask, **Why do you need to find all the points of intersection? Why not simply find the ones in the feasible region?**

Students should be able to explain that without the diagram, they can't tell which points are in the region and which are not. For instance, in this problem, the constraint combination I and IV gives the point (0, 4), which is outside the feasible region (because it fails to satisfy the constraint $y \leq 3$), while the combination II and V gives (6.5, 0), which is a corner point of the feasible region. Without actually graphing the individual equations, it's hard to know which pairs of equations give corner points of the feasible region.

Given this, the simplest strategy is to find *all* of the intersection points and to test to see which fit all the constraints. Students can then discard the intersection points that are outside the feasible region. (You can have students point out each of the discarded points on the diagram and use the geometry of the graph to see which constraint(s) each point fails to satisfy.)

Key Questions

How might you solve a linear programming problem without drawing the feasible region?

Are these points all vertices of the feasible region?

Why do you need to find all the points of intersection? Why not simply find the ones in the feasible region?

What Wood Would Woody Want?

Intent

Students find a point of maximum profit without drawing a feasible region.

Mathematics

This activity presents a straightforward linear programming problem in two variables for students to solve without graphing the feasible region. You may wish to have them verify the solution graphically during the follow-up discussion.

Progression

This activity applies the method discussed in *Finding Corners Without the Graph.*

Approximate Time

25 minutes for activity (at home or in class)
10 to 15 minutes for discussion (depending on whether you have students verify their solution with a graph)

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students should be able to do this activity independently with no introduction.

Discussing and Debriefing the Activity

You might begin the discussion by having several students give the constraints for the problem. It will be easier for students to be sure they have checked all intersections if they number their constraints:

I	$3C + 7T \leq 420$
II	$2C + 8T \leq 400$
III	$C \geq 0$
IV	$T \geq 0$

Have another student describe how he or she solved the problem without drawing the feasible region. The presenter will likely talk about finding the various places where the constraint lines intersect by finding the common solution to all possible pairs of equations.

If needed, elicit methods for making an organized list of pairs of equations to consider. In this case, the list might be:

- I and II
- I and III
- I and IV
- II and III
- II and IV
- III and IV

Note: Students might have omitted a combination such as I and III because they saw from a graph that the intersection is outside the feasible region. If so, remind them that they are trying to develop a method that does not depend on graphing, which means they need to list all combinations, find the points of intersection, and then check which points fit all the constraints.

In Woody's case, the combinations of equations give these points of intersection:

- I and II: $C = 56$, $T = 36$
- I and III: $C = 0$, $T = 60$
- I and IV: $C = 140$, $T = 0$
- II and III: $C = 0$, $T = 50$
- II and IV: $C = 200$, $T = 0$
- III and IV: $C = 0$, $T = 0$

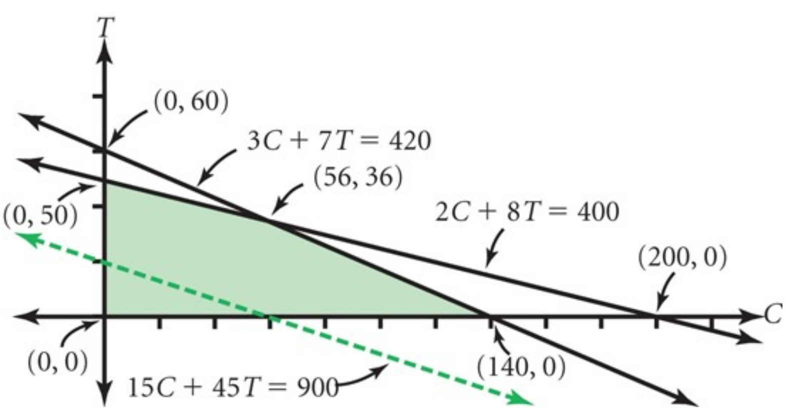
Students need to check which of these points satisfy all the inequalities. The last two inequalities ($C \geq 0$ and $T \geq 0$) are easy to check, but the first two require some arithmetic. Substitution shows that the combination $C = 0$, $T = 60$ does not fit constraint II and that the combination $C = 200$, $T = 0$ does not fit constraint I. (If students had used a graph, they would have started with the feasible region and looked only at those intersection points that were on the border of the region.)

Common sense says that the corner point $C = 0$, $T = 0$ will not maximize profit. So students need to compare profits for the three remaining possibilities. Because the profit is \$15 per chair and \$45 per table, they need to evaluate the expression $15C + 45T$ at each point, which gives these results.

C	T	$15C + 45T$
56	36	2460
140	0	2100
0	50	2250

Thus, profit is maximized by making 56 chairs and 36 tables, and the maximum profit is \$2,460.

Once students have solved the problem, you may want them to look at the graph and see what their algebra means in terms of the graph. If you have students draw individual graphs, decide in advance which variable represents the horizontal axis and which the vertical axis. The diagram here uses the horizontal axis for C and the vertical axis for T. The diagram shows all six "potential" corner points. The dashed line is the profit line $15C + 45T = 900$.



Widening Woody's Woodwork

Intent

Students solve a linear programming problem without graphing, where many points of the feasible region yield the maximum profit.

Mathematics

Students are once more presented with a two-variable linear programming problem to solve without graphing. This time they discover that the same maximum profit occurs at more than one intersection point. The discussion focuses on what happens if the profit line is parallel to an edge of the feasible region, and establishes that although the maximum profit does not occur only at a corner in this case, it can still be found by testing only the intersections.

Progression

This activity provides more practice solving a linear programming problem without graphing, but also illustrates the new situation of a profit line parallel to an edge of the feasible region.

Approximate Time

30 minutes for activity (at home or in class)
30 minutes for discussion

Classroom Organization

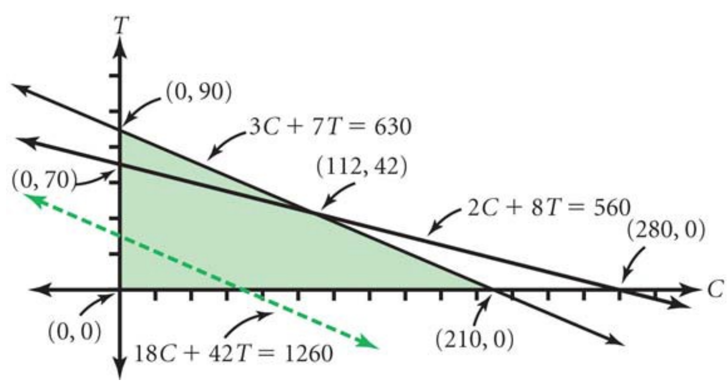
Individuals, followed by whole-class discussion

Doing the Activity

This activity is another two-variable linear programming problem. Students should be able to apply the strategy of checking the corner points even though the profit line is parallel to an edge of the feasible region.

Discussing and Debriefing the Activity

Begin by having a volunteer sketch the feasible region and give the coordinates of the corner points. Although this region will be similar to the original, the new lumber and labor-time values will lead to a new region and new corner points. This diagram shows the feasible region with one profit line.



Students should see that for the new profit expression, $18C + 42T$, the corner points (210, 0) and (112, 42) give the same profit, \$3,780, so both lie on the \$3,780 profit line. In other words, the line through these two points *is* the

$3,780 profit line. (The corner point (0, 70) gives a profit of only $2,940.)

You might ask, **What's another point along the segment connecting (210, 0) and (112, 42)?** As needed, bring out the idea that this segment is part of the constraint line $3C + 7T = 630$, so students should try to find a solution to this equation with C between 112 and 210. For instance, the point (140, 30) fits this equation, and students can verify that it also gives a profit of $3,780 (using the profit expression $18C + 42T$).

It is important to emphasize that while infinitely many points give the maximum profit in this case, there *are* corner points that maximize the profit function. That is, the strategy of checking only corner points allows one to maximize profit even in this case.

Further Refinement of the Linear Programming Strategy

Based on students' experience of using the strategy developed in the introductory discussion for *Finding Corners Without the Graph*, you may want to have them try to make their list of steps more precise. In particular, be sure that their strategy includes the step of checking whether intersection points fit all the constraints. (If students already have this detail in their strategy, then further revision may not be needed.)

This revised list replaces step 2 from the earlier list with the new steps 2a and 2b:

1. Write the constraints as a set of linear inequalities (including constraints that the variables are nonnegative).

2a. Find all points where boundary lines intersect by solving all possible pairs of linear equations corresponding to constraints.

2b. Check these intersection points against all the constraints to determine which of them actually lie in the feasible region. (Those that satisfy all the constraints are the corner points of the feasible region.)

3. Write down the function being maximized or minimized.

4. Evaluate the function being maximized or minimized at each of the intersection points found in step 2b, and determine which of these points gives the function its maximum or minimum value in the feasible region.

Be sure that the most current version of the strategy is prominently posted.

Key Question

What's another point along the segment connecting (210, 0) and (112, 42)?

Supplemental Activity

The Jewelry Business (reinforcement) also involves a profit line that is parallel to a constraint line.

More Equations

Intent

Students practice solving systems of two linear equations.

Mathematics

This activity focuses on solving pairs of linear equations, including a pair that is inconsistent and a pair that is dependent. Part II asks students to find as many definitions as they can for the word *plane*. The subsequent discussion focuses on the geometric significance of dependent and inconsistent systems of equations.

Progression

Students work on the activity individually and then discuss their results as a class.

Approximate Time

30 to 40 minutes for activity (at home or in class)
20 to 25 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity can be done with no introduction.

Discussing and Debriefing the Activity

Begin with a brief discussion of Part II and then return to Part I.

Part II

You might begin with a review of the meaning of the terms *point* and *line*. Presumably, students have an intuitive idea of what these terms mean, but you should make sure students are clear about these aspects of the terms;

- *Line* always means the same thing as *straight line*, but *straight* does not mean vertical, horizontal, or any other particular direction.

- A *line* is always "infinite in both directions."

Also review the terms *line segment* and *ray* as referring to certain portions of a line.

Then move to a geometric definition of the term *plane*, which will probably require a bit more discussion. Let students attempt to articulate a definition.

Look for intuitive phrases like "infinite flat surface." Mention that Euclid did not give rigorous definitions and that courses in formal geometry treat *point*, *line*, and *plane* as undefined terms. From the formal point of view, these terms are only given meaning by our assumptions (axioms) about how they behave.

Ask, **What are some examples of real-world objects that represent the idea of a plane? Of a line? Of a point?** Make sure students see that, for example, a sheet of paper is only a representation of a plane; it is not infinite, and it has thickness. Similarly, a stiff piece of wire or a line on a piece of paper is a representation of a line, and a dot is a representation of a point. Students should realize that such representations are important for thinking intuitively but the concept of *plane* is an abstraction.

Here are some of the other meanings that students may find when they look up *plane* in the dictionary:

- Any of several shade trees with large five-lobed leaves and flowers with globe-shaped heads
- A tool for smoothing or shaping a wood surface
- A level of existence, consciousness, or development
- An airplane

Part I

Give students a few minutes to share ideas in their groups. Then ask representatives from each of five groups to share their solutions for one of the pairs of equations in Questions 1 through 5.

From the presentations, students should see that although the pairs of equations in Questions 1 through 3 each have a unique solution, there are no solutions to the pair in Question 4 and infinitely many solutions to the pair in Question 5.

Ask, **What do you call a pair of equations such as those in Question 4?** If necessary, remind them of the term inconsistent for a system of equations that have no common solution. **Could inconsistent equations arise in a linear programming problem?**

Then ask, **What is happening geometrically in this example?** If needed, remind students that in all the questions in this activity, the graph of each equation is a straight line, and go over the fact that in Questions 1 through 3, the solution was represented by the place where two lines met. The class should be able to articulate that in Question 4, the two lines are parallel and so there is no such point.

Similarly, ask, **What do you call a pair of equations such as those in Question 5?** and review the term **dependent.** (*Note:* For two linear equations, this means that the two equations are equivalent. For larger systems, the definition is more complicated.)

Point out the geometry of a dependent system, bringing out that the equations in this example have the same graph. Therefore, any combination of values for *x* and *y* that fits one of these equations also fits the other.

Review the term **independent** to describe the pairs of equations in Questions 1, 2, and 3.

Key Questions

What are some examples of real-world objects that represent the idea of a plane? Of a line? Of a point?

What do you call a pair of equations such as those in Question 4?

Could inconsistent equations arise in a linear programming problem?

What is happening geometrically in this example?

What do you call a pair of equations such as those in Question 5?

Supplemental Activity

How Many Regions? (extension) asks students to subdivide the coordinate plane using the graphs of linear equations. They are asked to find all the possibilities for the number of regions that would be created, to give specific examples using equations for the lines, and to use inequalities to describe the regions. This problem can be used at any time in the unit.

Equations, Points, Lines, and Planes

Intent

This sequence of activities introduces graphing in the three-dimensional coordinate system and finding the equation of the line through two points.

Mathematics

The central unit problem is going to require that students understand and solve systems of equations involving many variables. In preparation for this, the activities in this cluster extend students' prior work with the graphing of linear equations into three dimensions. In order to prepare students to evaluate the solutions they will find when they solve systems of linear equations in three variables, the activities also explore how planes and lines can intersect in 3-space.

Also continuing students' work in two dimensions, these activities approach finding the equation of a line through two points in two different ways. Students do this first by using the situation from a word problem to generate a rule in the form of an equation. Later, they learn to treat the coefficients as variables, find an equation relating those two variables for the family of lines passing through each of the two points, and then solve that pair of equations.

Progression

Being Determined explores two related ideas: whether two lines uniquely determine a point and whether two points uniquely determine a line. Each of these questions launches further related exploration in other activities.

The first idea, whether two lines uniquely determine a point, is expanded into a look at the ways in which planes can intersect in 3-space in *Just the Plane Facts*. This prepares students for the later introduction of solving and interpreting systems of equations in three variables. Several activities lay the groundwork for *Just the Plane Facts*. *The Three-Variable Coordinate System* introduces the 3-D coordinate system. Students relate simple graphs to their equations in this coordinate system in *What Do They Have in Common?* and graph a more complex linear equation in the three-dimensional coordinate system in *Trying Out Triples*. *More Cookies* gives students experience in expressing constraints algebraically in three variables, and *Solving with Systems* provides practice with solving systems of linear equations in two variables in the context of word problems.

The second idea from *Being Determined*—whether two points uniquely determine a line—leads to finding the equation of the line through two given points. In *How Much After How Long?,* students generate a rule for a situation described in a word problem, then use that rule to form a linear equation that satisfies two data points. *The Points and the Equations* reverses the process, with students first finding the equation without the context of a situation, then inventing a context. *Fitting a Line* introduces a more systematic approach, one that will allow for a natural extension to the later fitting of a quadratic equation to three points in *Fitting Quadratics*.

Being Determined

How Much After How Long?

The Points and the Equations

The Three-Variable Coordinate System

What Do They Have in Common?

Trying Out Triples

More Cookies

Just the Plane Facts

Solving with Systems

Fitting a Line

Being Determined

Intent

Students relate the algebra of pairs of linear equations in two variables to the geometry of lines in the plane.

Mathematics

Being Determined requires students to examine, in two-dimensional space, whether two lines determine a unique point and whether two points determine a unique line. Students will learn the meaning of *uniquely determine*. The discussion covers the special cases of parallel lines and of identical points or lines.

Progression

It is suggested that groups be given only a few minutes to consider the questions in this activity before moving into whole-class discussion.

Approximate Time

15 to 25 minutes

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Question 1 of *Being Determined* is essentially the geometric counterpart of the discussion following *More Equations* concerning **independent, dependent**, and **inconsistent** pairs of linear equations.

You will likely find it appropriate to deal with *Being Determined* primarily as a whole-class discussion. Give groups only a few minutes to discuss each of the questions in the activity. You might bring the class together after a few minutes of work on Question 1 to discuss this question and the terminology it introduces, and then let groups return to Question 2. You may also want to emphasize again that *line* always means *straight line*.

Note: This activity specifies that l_1 and l_2 are lines *in a plane*. You will probably not need to focus now on the distinction between lines in the plane and lines in space. Issues about lines in space will be raised in connection with the discussion of *Just the Plane Facts*.

Discussing and Debriefing the Activity

Question 1

Have a student report on the group's conclusions for Question 1, and let other groups comment.

Students will probably recognize the special case of parallel lines. Ask, **What does the case of two parallel lines mean in terms of pairs of linear equations?**

They should be able to articulate that a pair of parallel lines is comparable to an inconsistent pair of linear equations.

Students may not think to deal with the case in which the two lines are identical. If they omit this, ask, **What corresponds geometrically to the case of a pair of dependent equations?** If necessary, ask explicitly what lies on both l_1 and l_2 in the case in which l_1 and l_2 are the same line. (*Note:* This is separate from the case of parallel lines. According to the definition used in many texts, a line is not considered parallel to itself.)

Students may argue that having l_1 and l_2 be the same line is not allowed, because l_1 and l_2 are different symbols or because the problem states "l_1 and l_2 are lines," which suggests that they are two different lines. Don't belabor the "rightness" or "wrongness" of this interpretation, but tell students that by convention, if they are told that *a* and *b* are two objects of some kind (two lines, two points, two numbers, and so on), they should always consider the special case in which the symbols *a* and *b* represent the same object. Tell them that if the intent was to exclude this case, the problem would have stated that l_1 and l_2 were *distinct* lines. Also, tell them that to make a general statement for Question 1 that excludes both special cases, they can say, "Two *distinct, nonparallel* lines uniquely determine a point."

Be sure students not only identify the special cases but also explain what happens in each of these cases:

- If the lines l_1 and l_2 are parallel, then there are *no points* that lie on both.
- If the lines l_1 and l_2 are identical, then there are *infinitely many points* that lie on both.

Question 2
Have a student from another group report on their conclusions for Question 2, and let other groups comment.

Because the only exception is the case in which *P* and *Q* are the same point, students might say that there are no exceptions. As with Question 1, if this special case does not come up, raise it yourself by asking, **What if *P* and *Q* are the same point?** Students should articulate that in this case, there are infinitely many lines that go through both *P* and *Q*. Ask, **How might you phrase a general statement that excludes this special case?** For example, they can say, "Two *distinct* points uniquely determine a line."

"Uniquely"—Could There Be More Than One Line or Point?
If the meanings of the phrase "one and only one" and the term "uniquely" haven't been explicitly discussed yet, take a moment to do so. For instance, for Question 2, we want students to conclude that two distinct points uniquely determine a line. Ask, **Is there a way that more than one distinct line can pass through a given pair of distinct points?** Students should see that this is impossible. Clarify that we express this impossibility by saying that a pair of distinct points *uniquely determines*

a line, focusing on the idea that *unique* means “the only one of its kind.”

Key Questions

What does the case of two parallel lines mean in terms of pairs of linear equations?

What corresponds geometrically to the case of a pair of dependent equations?

What if *P* and *Q* are the same point?

How might you phrase a general statement that excludes this special case?

Is there a way that *more than one* distinct line can pass through a given pair of distinct points?

How Much After How Long?

Intent

Students find the equation of the line that passes through two given points.

Mathematics

This activity gives students two situations involving a linear relationship between two variables and asks them to find a rule for the function in each case. They are then asked to reflect on the connection between these problems and *Being Determined* and, finally, to find the equation of a line passing through two given points without having any context.

Progression

Although most of the work in finding the equation of the line through two points is algebraic, the main goal of this activity is to bring out the connection between the algebra of linear equations and the geometry of points and lines as explored in *Being Determined*.

Approximate Time

30 minutes for activity (at home or in class)
25 to 50 minutes for discussion (depending on whether you have students create word problems of their own)

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students may find the lack of context in Question 4 difficult to deal with at this stage. They will work on more questions like this in *The Points and the Equations*.

Discussing and Debriefing the Activity

This activity is primarily intended to bring out, in broad terms, the connection between the geometry of lines and points and the algebra of linear equations.

Have students compare answers within their groups. As they do so, look for someone who was able to go from the concrete situations in Questions 1 and 2 to the abstract setting of Question 4, and assign that person's group to present Question 4. Also assign groups to present Questions 1, 2 and 3a. (Question 3b is best treated as a whole-class discussion.)

Have groups present their solutions to Questions 1 and 2 to the class. The presenters will probably begin with something like a "rate of change" analysis. For instance, they might reason in Question 1 that selling 20 additional tickets led to an additional $100 in profit, so each ticket must have cost $5. They might then figure that 100 tickets brought in $500 in sales but only $400 in profit, so the cost of putting on the show must have been $100. This likely will lead to the rule

$p = 5t - 100$. A similar approach works for Question 2.

After each presentation, be sure to ask for alternative approaches, because students may have worked through the situations in different ways.

Next ask, **Why should the situations in Questions 1 and 2 have linear rules?** Students may have various ways to articulate this, but they will probably be able to express some notion of "constant rate" to justify the linearity.

Question 3

Have a student from the chosen group present Question 3a. Students will probably not have difficulty identifying the combinations:

- 10 hours of work produced a total of $210
- 120 hours of work produced a total of $870

The most important element of Question 3 is part b, and you can simply let volunteers share ideas about this. One crucial element to elicit is that each combination can be thought of as a point in a coordinate system.

If you need to give the class a hint on Question 3b, point out that Question 2 of *Being Determined* asked whether two points determine a line. **What do points and lines have to do with Question 1?** Persist until students see that the combination of 100 tickets and $400 profit can be represented by the point (100, 400) in the coordinate plane, and the combination of 120 tickets and $500 profit by the point (120, 500). Thus, the information provided in the problem is essentially two points in the coordinate plane. To complete the connection between this activity and *Being Determined*, students should see that finding a rule for Question 1 of *How Much After How Long?* is similar to finding a line that goes through the two given points.

Without getting too technical, help students see that their conclusion in Question 2 of *Being Determined* tells them that two points in the coordinate system determine a linear rule that fits the two points.

Strictly speaking, there are many equivalent rules that describe the same line. In *Fitting a Line*, students will look at linear equations written with y expressed as a function of x, so that each (nonvertical) line has a unique equation.

You may want to bring out that knowing that a linear rule should exist is different from actually finding that rule. Assure students that they will get more experience finding such rules in *The Points and the Equations.*

The Points and the Graph

Point out that the information about tickets and profit for Question 1 has been represented in the coordinate plane by the points (100, 400) and (120, 500), and ask, **How is the rule for Question 1 related to the coordinate plane?**

Bring out that the graph of this function (with axes labeled t and p) should be the line through the two given points. Have students graph this equation either by hand or on graphing calculators (with a change of variables to *y* and *x* for the calculator) and verify that the points (100, 400) and (120, 500) lie on the graph.

You may want to go through a similar process for Question 2.

Question 4

Now move on to Question 4. Students may have some trouble with the abstraction of finding the equation of the line through two points without any context. If that is the case, you might suggest that they try to put the information from Question 4 into a more concrete setting, like those in Questions 1 and 2, or try to express the information from those questions in the abstract form of Question 4.

For instance, they might interpret the data from Question 4 as meaning that selling 2 tickets yielded a profit of \$15 and selling 7 tickets yielded a profit of \$45. Alternatively, they could restate Question 1 abstractly as, "Find the equation of a straight line that goes through the points (100, 400) and (120, 500)."

Is There a Linear Function for Any Two Points?

Ask, **Does every pair of points have an associated linear function?** If a hint is needed, give students a pair of points with the same *x*-coordinate, such as (4, 9) and (4, 13)—that is, ask, **Can you get y in terms of x for the points (4, 9) and (4, 13)?**

Guide students to articulate that any two distinct points determine a line, but if that line is vertical, then it does not represent a function, and one cannot express *y* in terms of *x*. They should see, however, that there is a linear equation whose graph is that line. For instance, in the case of the points (4, 9) and (4, 13), the graph of the equation $x = 4$ is the line through the points.

Creating Word Problems for Lines

Depending on time limitations and the needs of your students, you might have them create word problems similar to those in Questions 1 and 2. If you haven't already done so, you can begin with Question 4 and have students make up a concrete setting that would require them to find the equation of the line through the two given points. (Question 2 of the next activity, *The Points and the Equations,* includes a similar task.)

Key Questions

Why should the situations in Questions 1 and 2 have linear rules?

What do points and lines have to do with Question 1?

How is the rule for Question 1 related to the coordinate plane?

Does every pair of points have an associated linear function?

Can you get *y* in terms of *x* for the points (4, 9) and (4, 13)?

The Points and the Equations

Intent

This activity continues work with finding the equation of a line through two given points.

Mathematics

The Points and the Equations reinforces both the skill of finding an equation of a line through two given points and the connection between this skill and real-world contexts

Progression

Students work on this activity on their own, reinforcing what was learned in *How Much After How Long?*

Approximate Time

20 minutes for activity (at home or in class)
5 to 10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity should be done independently, and requires little discussion.

Discussing and Debriefing the Activity

Either give students only a few minutes to share ideas or omit discussion entirely.

If you choose to discuss, have volunteers give the equations for the four lines, and let other students suggest alternatives. Keep in mind that equivalent equations are possible and that these equations need not give y in terms of x. For example, in Question 1d, students might use the equation $3x + y = -1$ as well as the equation $y = -3x - 1$.

The Three-Variable Coordinate System

Intent

This activity introduces the three-variable coordinate system.

Mathematics

Over the next several activities, students will generalize the coordinate system from two variables to three and see that the graph of a linear equation in three variables is a plane. They will then work with the geometry of planes to develop geometric and algebraic analogs of their previous work with lines and pairs of linear equations in two variables.

For this activity, the goal is to introduce the coordinate system for three variables. Introduce the orientation and representation of the three axes, ordered triples, the **coordinate planes,** and **octants.**

Progression

The Three-Variable Coordinate System is a set of reference pages for students to use in reviewing material that is presented through teacher presentation and whole-class discussion.

Approximate Time

45 minutes

Classroom Organization

Teacher presentation and whole-class discussion

Materials

- Small model of three-coordinate system, perhaps with the *x*- and *y*-axes displayed on a piece of cardboard, with a piece of spaghetti poked through a small hole at the origin to represent the *z*-axis
- *The Three-Variable Coordinate System* blackline master (1 copy per student, or a transparency, or both)
- Tape
- Three pieces of string, long enough to stretch across the classroom in all three directions

Doing the Activity

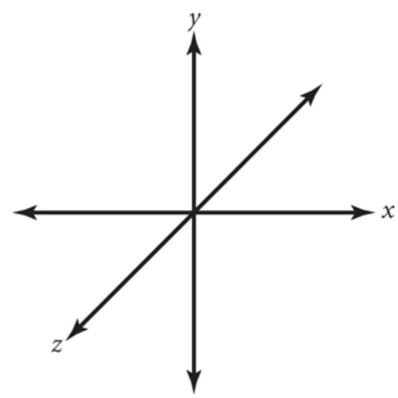
Though some students may have already been exposed to the idea of a system of three mutually perpendicular coordinate axes, you should introduce this content thoroughly, to be sure that everyone is using the same orientation and vocabulary.

Point out to students that they now have a fairly good understanding of the geometry of lines and points, and how geometry relates to the solving of pairs of linear equations in two variables. Also review the fact that solving pairs of linear equations is an important aspect of their strategy for solving two-variable linear programming problems.

Tell students that to solve linear programming problems in more than two variables, they will need to generalize the ideas from *Being Determined*, and they will start this process of generalization by using three variables rather than two.

Setting Up the Axes

Ask, **How could you represent a set of values for three variables using a coordinate system?** For instance, ask how they would represent the combination $x = 2$, $y = 1$, $z = 3$.



Most students will probably sense that they need some sort of "third axis" to go with the *x*- and *y*-axes, but it will probably not be obvious to everyone how to set up a system of three axes. A common but erroneous approach is to draw a third axis within the *xy*-coordinate system, as in this diagram, in which the *z*-axis is the bisector of the angle formed by the *x*- and *y*-axes. In this incorrect model, the diagram lies "flat" in the xy-plane.

Take time to have students illustrate why this won't work. For instance, help students to see that in this system (using the same scale on each axis), they can't find a point where $x = 2$, $y = 1$, and $z = 3$.

To help students develop a "correct" three-variable coordinate system, ask, **What important relationship do the x-axis and y-axis have in two-coordinate graphing?** Help students identify the fact that the axes are perpendicular, which means they should look for a way to set up the *z*-axis so that it is perpendicular to both the *x*-axis and the *y*-axis. It shouldn't take them long to figure out that they must leave the *xy*-plane, drawing the third axis perpendicular to this plane, through the origin.

A Physical Model

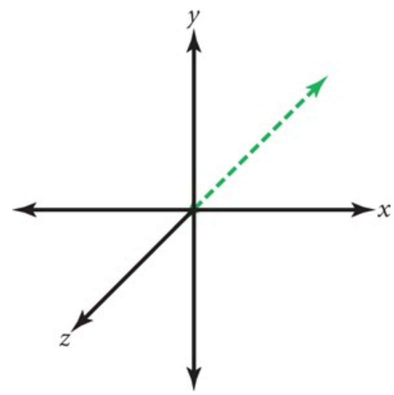
For simplicity, we suggest that you have students think of the *x*- and *y*-axes as lying within the actual plane on which they draw (for instance, the paper or the blackboard) and view the *z*-axis as coming out perpendicular to that plane. (We will use this designation of the axes; we suggest that you do the same to make the best use of this teacher guide.)

Note: There is no completely standard way of orienting and labeling the axes. For some situations in physics and mathematics, the *y*- and *z*-axes are interchanged compared with the approach we are using; this will not matter in the unit, especially given that we often won't be using the specific variables *x, y,* and *z*.

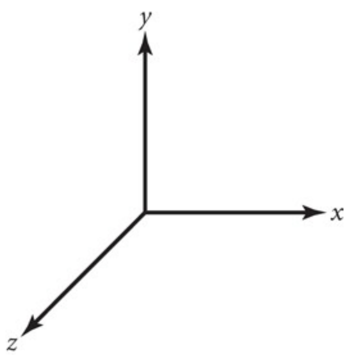
To assist students in visualizing this setup, create (or have students create) a small-scale physical model of the system of axes. For instance, you can use a piece of cardboard to represent the *xy*-plane and poke a strand of spaghetti through a small hole in the cardboard to represent the *z*-axis. Using a corner cut from a cardboard box can also work well, with the edges of the box as the three axes and the interior of the box representing the first octant.

Two-Dimensional Representation

About the best we can do in representing this system with a two-dimensional line drawing is something like the diagram at right, in which the *z*-axis is viewed as perpendicular to the paper, with the positive portion of the *z*-axis coming toward you and the negative (dashed) portion of the *z*-axis thought of as being *behind* the *xy*-plane.



An alternative representation is to draw only the positive portion of each axis, which yields this diagram:



Of course, neither diagram fully reflects the three-dimensionality of the coordinate system—it is a two-dimensional representation of three-dimensions. Visualizing this system will be difficult for many students. Therefore, it is important that you reassure them that many people have trouble visualizing things in three dimensions, even though the world we live in is three-dimensional. Much of this difficulty probably stems from the fact that we often represent things using a two-dimensional image, such as a photograph or drawing.

Representing Triples As Points

Remind students that a key purpose of developing the three-variable coordinate system is to have a way of representing combinations of values for the variables in a geometric way. Tell them that the next stage of the development is to see how to associate a triple of values (such as $x = 2$, $y = 1$, $z = 3$) with a point. Introduce the term *ordered triple* and the notation (2, 1, 3) for such a triple of values.

Coordinates in the Classroom

Students generally find it helpful to use the classroom as a model of the three-dimensional coordinate system. There are several ways to set up the classroom coordinate system (assuming that your classroom is a typical "box" shape). Here is one approach that will allow you to consider both positive and negative coordinates within the classroom.

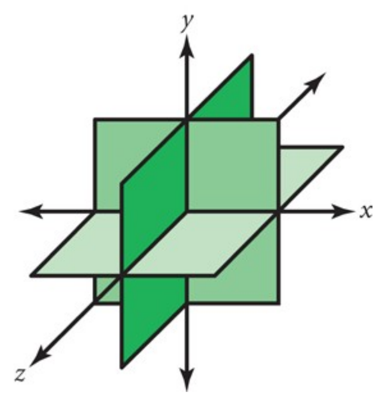
Choose a point in midair somewhere in the middle of the classroom. This point will represent the origin. Then tape three pieces of string across the room, through this point, to form straight lines in each of the three perpendicular directions (left to right, floor to ceiling, front to back). These pieces of string will represent the axes. Be sure to label the strings as the *x*-axis, *y*-axis, and *z*-axis, respectively, and to assign one end of each axis as the positive direction. We suggest that you have the *x*-axis connect the side walls, the *y*-axis connect the floor and ceiling, and the *z*-axis connect the front and rear walls. That way, the *x*-axis and *y*-axis will be parallel to their usual blackboard positions in the two-dimensional coordinate system.

Some teachers prefer to set up the axes with the origin in a corner on the floor of the classroom. In this method, all points in the classroom have positive coordinates.

Next, identify and name the three mutually perpendicular planes formed by pairs of axes, and use the term **coordinate planes** to refer to these planes. In our suggested arrangement of axes, here are the coordinate planes:

- The *xy*-plane is parallel to the front and rear walls of the classroom.
- The *xz*-plane is parallel to the floor and ceiling.
- The *yz*-plane is parallel to the side walls of the classroom.

With shading or color, you can represent the coordinate system and the coordinate planes on paper as in the next diagram, which gives more of a three-dimensional appearance than other two-dimensional representations. (A larger version of this diagram is included in *The Three-Variable Coordinate System* blackline master. You may wish to make a transparency of this, or make a copy for each student for reference.)



Once you have identified the axes ask, **What do you need besides axes to match ordered triples with points?** If necessary, remind students that they need to have *scales* for the individual axes. For your classroom model, you might want to use one foot as the unit distance on all three axes.

Based on your coordinate system setup and your designation of axes, have students locate the position of various ordered triples in the classroom.

Be sure to clarify that in each ordered triple, the first coordinate represents *x*, the second coordinate represents *y*, and the third coordinate represents *z* (regardless of how the axes are oriented).

Introduce the terms **three-dimensional coordinate system** and the shorter **3-space** to refer to the set of all points in this new system. Tell students that this set of points is the next step in the sequence *point-line-plane.*

Ask, **What do we usually call the one-dimensional coordinate system?** If necessary, remind students of the term *number line.*

Getting Comfortable with Coordinates

Once students understand the basic concept, you can play some games to help them become more comfortable using the three-dimensional coordinate system. For instance, give one student in each group an ordered triple, and have these students move so that their heads are roughly at the given positions. Be sure to include some negative coordinates. (If you have set up the model with the origin in the corner of the classroom, including negative coordinates will force students to consider portions of the coordinate system represented by points beyond the classroom itself—out in the hall, downstairs, and so on.)

You can also play games like "Guess the Object," in which a student gives the coordinates of an object in the room and the rest of the class tries to guess what the object is.

Octants

Next, remind the class that points in the two-variable coordinate system (except points on the axes) are classified into four *quadrants*. (You may want to review the signs of the coordinates in the various quadrants.)

Ask, **What corresponds to a quadrant in this three-dimension coordinate system?** Students should be able to figure out that there are eight possible sign combinations. (If necessary, have students list them all.)

Introduce the term **octant** for each of these portions of the coordinate system. Although there is no standard numbering for the complete set of octants, the set of those points whose coordinates are all positive is referred to as the **first octant.**

Students can use a copy of *The Three-Variable Coordinate System* blackline master as a reference for the ideas just discussed.

Key Questions

How could you represent a set of values for three variables using a coordinate system?

What important relationship do the *x*-axis and *y*-axis have in two-coordinate graphing?

What do you need besides axes to match ordered triples with points?

What do we usually call the one-dimensional coordinate system?

What corresponds to a quadrant in this three-dimension coordinate system?

What Do They Have in Common?

Intent

Students gain basic experience working with the three-dimensional coordinate system.

Mathematics

In this activity and the subsequent discussion, students will find algebraic descriptions of certain sets in 3-space, as well as the graphs of three-variable linear equations that have only one or two of the variables.

Progression

Students work on the activity individually and then discuss their results as a class.

Approximate Time

30 minutes for activity (at home or in class)
50 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Large towel, piece of cardboard, or flat surface to represent a plane (optional)

Transparency of *What Do They Have in Common?* blackline master

Doing the Activity

What Do They Have in Common? asks students to give the coordinates for five points in each of a number of given sets, such as the set of points in the *yz*-plane. They are then to state what characteristics the points in the set have in common.

Discussing and Debriefing the Activity

Assign one or two problems to each group to focus on and report to the class. Have individual students report for their groups, giving coordinates of five points in the set and then the common characteristic of the points. Push students to be complete with their descriptions. For instance, if a student states that all points on the *x*-axis have the *y*-coordinate equal to 0, ask what else they have in common.

Graphs of Three-Variable Linear Equations

Describing the graph of an equation in three variables is somewhat like doing the activity in reverse order. That is, in the given problems, students started with a geometric description and found a characteristic in terms of coordinates. In graphing an equation, we start with a characteristic of the coordinates (the equation) and then describe the set of points that fit the condition. We suggest that you begin with intuitive geometric descriptions before referring to the more formal concept of a graph.

Ask students, **Where do you find the points with a y-coordinate of 0?** Get specific examples, and have them locate these points in the classroom coordinate system. (This may be somewhat repetitive of their work on Question 2 of the activity.) They should see that any point in the *xz*-plane fits this condition. (According to the setup described in the discussion of *The Three-Variable Coordinate System*, this is the horizontal plane that contains both the *x*-axis and the *z*-axis. We will use that setup in the discussion here.)

Next ask, **Where do you find the points with a y-coordinate of 1?** Students should see that these points are 1 unit up from the *xz*-plane. Ask, **What sort of geometric figure does this set of points form?** Be sure they understand that these points form a plane.

Next ask, **What relationship does this plane have to the xz-plane?** As a hint, you might point out that the two planes do not intersect, and then ask what we call two *lines* (in a plane) that don't intersect. Tell students, if needed, that two *planes* that don't intersect are also called *parallel*.

Continue with similar questions using other coordinates, including negative numbers. For instance, ask the class to describe

- the set of points whose *x*-coordinate is 3
- the set of points whose *x*-coordinate is –2
- the set of points whose *z*-coordinate is 5
- the set of points whose *z*-coordinate is –6

Sets As Graphs of Equations

Take one of these examples and ask, **How can you express this geometric description using the language of graphs and equations?** For example, ask if the plane 6 units behind the *xy*-plane (the last example in the list) can be thought of as the graph of an equation. Guide them to state that "the set of points whose *z*-coordinate is –6" is simply another way of saying "the graph of the equation $z = -6$."

If necessary, do one or two more examples to bring out that the graph (in the three-dimensional coordinate system) of an equation setting one of the variables equal to a non-zero constant (such as $z = -6$ or $y = 5$) is always a plane parallel to one of the coordinate planes.

Ask the class, **What type of equation is $z = -6$ or $y = 5$?** As a hint, you can point out that the variables are not being raised to a power. Bring out that these (and the other examples in this discussion) are all linear equations, albeit simple ones.

Linear Equations with Missing Variables

Students may be confused about how the equation $z = -6$ can be a linear equation when there is no *x* or *y* in it. Whether or not they express doubts, ask, **In the *xy*-coordinate system, what did the graph of an equation without an *x* or without a *y* look like? What are some examples?** Students should see that such equations have "special" lines as their graphs, namely, lines that are parallel to one of the coordinate axes (and perpendicular to the other).

Next ask, **What's the difference between graphs of equations that include both variables and graphs that do not?** That is, how does the graph (in the *xy*-plane) of an equation like $2x + 3y = 7$ compare to the graph of an equation like $x = 5$ or $y = -2$? Students should see that all of these equations have graphs that are straight lines. But for the equation involving both variables, the graph is a line that intersects both axes, while equations like $x = 5$ or $y = -2$ have graphs that are vertical or horizontal and are parallel to one of the axes.

A Three-Variable Equation with Only One Missing Variable

Now return to the situation of three variables, but this time use an example in which exactly two of the variables are involved. For example, ask, **What do you think the graph of the three-variable equation $x + z = 2$ might look like?**

Begin by having students find ordered triples that fit this equation. You may have to point out that they need to give a *y*-coordinate for each point even though *y* is not in the equation. If needed, you can build on the prior discussion about equations like $x = 5$ or $y = -2$ graphed in the *xy*-coordinate system.

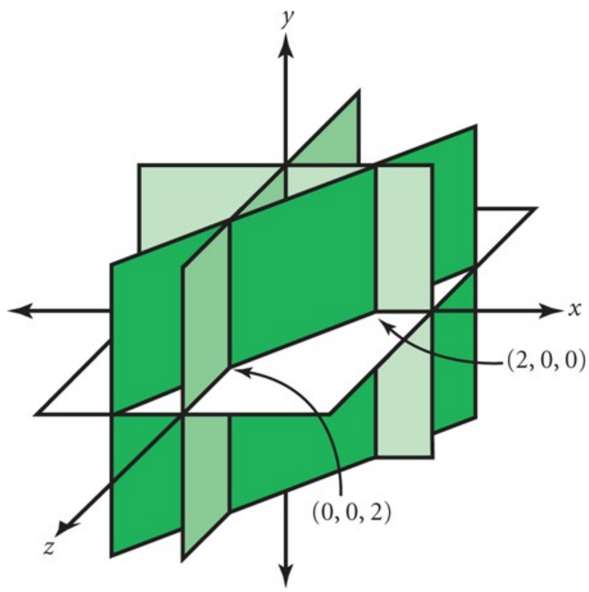
Students might give these examples somewhat at random. If the list of points seems chaotic to students, you might have different students each "act out" a point on the list, perhaps having them place their heads (approximately) at the given point in your classroom coordinate system.

Another approach is to suggest that they look first at those points that fit the equation $x + z = 2$ and whose *y*-coordinate is 0. They should see that these form a line (within the horizontal coordinate plane in our labeling system for the axes). That is, these points are like the graph of the equation $x + z = 2$ considered purely as an equation in *x* and *z* and graphed in the *xz*-plane. Be sure students see that there are points on this line with negative *x*- or *z*-coordinates, so these solutions actually form a line and not simply a line segment in the first quadrant of the *xz*-plane.

Then have them consider points on the graph whose *y*-coordinate is 1. Ask how the points of this set compare to those whose *y*-coordinate is 0. The key observation is that the points whose *y*-coordinate is 1 form a line one unit above the line formed by the points whose *y*-coordinate is 0. Bring out that, in fact, any point directly above (or below) a point on the line $x + z = 2$ in the *xz*-plane will fit the equation.

The final component of this analysis is to recognize that these lines, put together, form a vertical plane, which passes through the line in the *xz*-plane formed by the solutions to $x + z = 2$ with $y = 0$. You might use a large towel, piece of cardboard, or something similar to represent this plane in your classroom coordinate system.

Thus, the darkest plane in the diagram below is the graph of the equation $x + z = 2$. (You may wish to show a transparency of this diagram, which is provided as the blackline master *What Do They Have in Common?*)



Three-dimensional graph of the equation $x + z = 2$

Key Questions

Where do you find the points with a *y*-coordinate of 0?

Where do you find the points with a *y*-coordinate of 1?

What sort of geometric figure does this set of points form?

What relationship does this plane have to the *xz*-plane?

How can you express this geometric description using the language of graphs and equations?

What type of equation is $z = -6$ or $y = 5$?

In the *xy*-coordinate system, what did the graph of an equation without an *x* or without a *y* look like? What are some examples?

What's the difference between graphs of equations that include both variables and graphs that do not?

What do you think the graph of the equation $x + z = 2$ might look like?

How do the points whose *y*-coordinate is 1 compare to the points on the graph whose *y*-coordinate is 0?

Trying Out Triples

Intent
Students graph a linear equation involving three variables.

Mathematics
Trying Out Triples gives students a linear equation in three variables and asks them to find a dozen triples that are solutions to the equation, organize the solutions, and describe or sketch what the graph will look like. By now, students probably expect that the graph will be a plane, but this activity will solidify *why.*

In the class discussion of this activity, students construct a physical model of the graph for this equation.

Progression
Students work on this activity individually, then share their results in class discussion. Finally, they build a model of the graph of the equation.

Approximate Time
25 minutes for activity (at home or in class)
45 minutes for discussion and group activity

Classroom Organization
Individuals, followed by whole-class discussion and small-group activity

Materials
- String or yarn for each group Tape for each group
- A corner of a box (with three sides removed so the three remaining sides form a corner) (1 per group)
- Transparency of *Trying Out Triples* blackline master

Doing the Activity
Students should be able to find solutions to the equation independently.

Discussing and Debriefing the Activity
You can begin the discussion by compiling a list of points that fit the given equation, $x + y + \frac{1}{2}z = 4$. Thirty or so points will be plenty. Be sure to get some points with fractional positive coordinates. Students are especially likely to include points on the coordinate axes on this list, and they might use those points as guides in the graphing process.

Graphing the Three-Variable Linear Equation
The next phase of the discussion is to develop a physical model of the graph itself. We recommend that you do this in two ways.

- On a small scale, with each group using the corner of a cardboard box (or something similar) as a model
- On a large scale, with the whole class using the classroom as the model

Tell groups that their next task is to construct a physical model of the graph. Begin by passing out materials to groups so they can make models of the graph of $x + y + \frac{1}{2}z = 4$ (or at least the first-octant portion of the graph). You might give each group some yarn or string and the corner of a cardboard box. They should use a scale such that the box encompasses at least 8 units in the z-direction and 4 units in the x- and y-directions.

After a few minutes, ask groups to share their ideas or display their work. Based on these results, work to create a model using the classroom coordinate system. (This will work especially well if you have used the "strings as axes" method described in the discussion of *The Three-Variable Coordinate System*.)

Hints for Making the Graph

If groups are unable to make sufficient progress, you can continue as a whole class using the ideas described here. (Or you might use these suggestions as hints to individual groups.)

Ask, **How might you organize these points?** Someone may see the idea of grouping solutions with the same y-coordinate. If not, you should suggest this as a starting point. Begin by compiling all the solutions on the list whose y-coordinate is 0. These might include (4, 0, 0), (3, 0, 2), (2, 0, 4), (1, 0, 6), and (0, 0, 8). If no other solutions are given, insist on more examples, including points with noninteger (fractional) coordinates, such as (1.5, 0, 5) or (2.75, 0, 2.5).

Ask, **What does the "$y = 0$" portion of the graph look like?** Groups should plot these points on their cardboard models. You can then have individual students locate these points in the classroom coordinate system. They should see that the points form a straight line that lies in the xz-plane.

Students might also be convinced of this by looking at what happens to the equation when y is replaced with 0. They should see that they get the equation of a line, specifically, the equation $x + \frac{1}{2}z = 4$, graphed in the xz-plane.

For this and subsequent equations, you can use a piece of string to represent the particular portion of the graph. At the same time, students can use string within their cardboard models to represent this line. For the case just described (the points where $y = 0$), the string will lie in the xz-plane (which will probably be along the bottom of the box).

Next, work on the case in which $y = 1$. Again, students should start by compiling a list of points, which could include (3, 1, 0), (2, 1, 2), (1, 1, 4), and (0, 1, 6). As before, insist that they include examples with fractional coordinates, such as

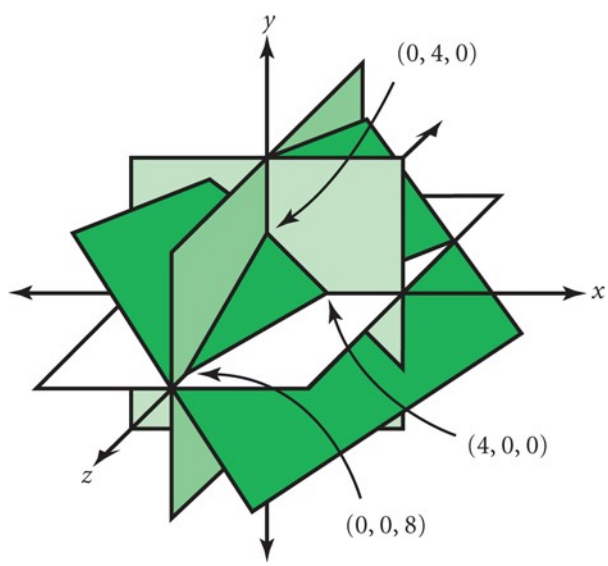
$(\frac{1}{2}, 1, 5)$.

Also as before, help students see that these points form a line by having individual students locate the points in the classroom coordinate system. They should see these points as forming a line that is one unit above the *xz*-plane. This time, the string should stretch from the point (3, 1, 0) (which is in the *xy*-plane) to the point (0, 1, 6) (which is in the *yz*-plane). You can tape the ends of the string to the appropriate points on those walls of the classroom that represent these two coordinate planes, and students can do the same with their cardboard models.

Continue with other fixed *y*-values. After doing four or five pieces of string, ask if students have a sense of what is happening. They should gradually see that these strings are combining to form a triangle in the first octant. If necessary, lay a large sheet of tagboard or some other stiff, flat material on the strings to show that the surface of the strings is flat.

Then ask, **What would happen if the strings were extended infinitely, into the other octants?** Encourage students to articulate that the triangle in the first octant is part of a plane. If necessary, remind them that the special linear equations they looked at yesterday all had graphs that were planes. Those planes did not intersect all three axes, but this one does. Bring out this detail specifically by asking, **What are the intercepts of this plane with each of the coordinate axes?** Students should be able to find each intercept directly from the equation, by setting the other variables equal to zero.

This diagram shows what the graph of the equation $x + y + \frac{1}{2}z = 4$ looks like. (An enlarged copy of this diagram is provided in *Trying Out Triples* blackline master.)



Ask, **What octants does this plane go through?** You might have students look for examples of points with various sign combinations. They can choose values for any two of the variables, but the value for the third variable will then be determined. They should see that the only sign combination that cannot be achieved is negative values for all three variables.

For another approach to graphing a plane in 3-dimensions, see the Web site "3D Model of a plane intersecting coordinate axes" at http://www2.math.uic.edu/%7Ecpmp/3d/3d.htm.

Key Questions

How might you organize these points?

What does the "$y = 0$" portion of the graph look like?

What would happen if the strings were extended infinitely, into the other octants?

What are the intercepts of this plane with each of the coordinate axes?

What octants does this plane go through?

Supplemental Activity

Special Planes (extension) gives students further experience with correlating linear equations in three variables to their graphs.

More Cookies

Intent

Students write algebraic statements to express the constraints for a situation involving three variables.

Mathematics

This activity brings back the main problem situation from the Year 1 unit *Cookies* but introduces a third variable. In this and subsequent activities, students will use this situation to test the generalizability of their strategy for solving linear programming problems, and to continue their work with three-dimensional graphing.

Progression

This activity extends a situation from the Year 1 unit *Cookies* into three variables.

Approximate Time

20 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students should be able to do this activity independently.

Discussing and Debriefing the Activity

Have individual students give the constraints that they wrote. They should come up with a list like this:

I	$1P + 0.7I + 0.9C \leq 120$
II	$0.4I \leq 32$
III	$0.15C \leq 18$
IV	$P \geq 0$
V	$I \geq 0$
VI	$C \geq 0$

Post this list of constraints, then have other students share the "cookie combinations" that they found and the profit for each combination. Make sure the class tests whether these combinations fit *all* the constraints.

Identify and post the *profit function* as well:

$$\text{profit} = 1.5P + 2I + 2.25C$$

Note: The values for the variables that maximize this profit function (while fitting the constraints) are $P = 0$, $I = 80$, and $C = 71.11$. This combination gives a profit of

$320. Someone may come up with this solution, but at this stage no one is likely to be able to prove that this solution is optimal. The issue of whether it is appropriate to consider noninteger solutions will be discussed in *Finishing Off the Cookies* (see the subsection "Whole Dozens or Not?").

Just the Plane Facts

Intent

Students examine how planes and lines intersect in 3-space.

Mathematics

Students will learn in this activity that the intersection of three planes can be a point, a line, a plane, or nothing at all, but that "usually" three planes will intersect in a single point. The term **skew lines** is introduced for lines in space that do not intersect but are not parallel.

Progression

The discussion of this activity needs to take place before assigning *The "More Cookies" Region and Strategy*.

Approximate Time

50 minutes for activity
25 minutes for presentations and discussion

Classroom Organization

Small groups, followed by presentations and whole-class discussion

Materials

Tagboard or index cards
Pipe cleaners, skewers, or pick-up sticks Scissors (1 per group)
Tape
Transparencies of *Just the Plane Facts* blackline masters

Doing the Activity

The ideas in this activity will play an important role in helping students understand what is happening geometrically when they solve systems of linear equations in three variables. Although this activity asks about several types of geometric intersections, Question 3 in particular will play a key role later in the unit.

You may want to make a variety of materials available for students' use in building models, such as tagboard or index cards for planes; pipe cleaners (called *chenille sticks* at art supply stores), skewers, or pick-up sticks for lines; and scissors and tape for putting things together.

Ask students to look over their strategy for linear programming problems (refined after *Widening Woody's Woodwork*), and ask, **How might you need to adapt your linear programming strategy to work with three-variable problems?** The goal here is primarily to motivate the upcoming discussion of the intersection of planes, and not to have students adapt their strategy fully from two dimensions to three.

Direct students specifically to the aspect of the strategy that talks about the intersection of boundary lines, and bring out that the equations corresponding to the constraints in the *More Cookies* problem are three-variable equations. You can ask what the graphs of these equations are. By now, students should recognize that each of these graphs is a plane in 3-space.

Remind students that in *Being Determined*, they examined how lines within a plane can intersect. Tell them that the next activity, *Just the Plane Facts*, will help them understand how *planes* can intersect in 3-space. This, in turn, will help them understand how to find individual "corner points" for a three-dimensional feasible region.

Tell students that after they have looked at how planes can intersect, they will return to the task of adapting the linear programming strategy to three variables. Mention that they will need to have a copy of their two-variable strategy when they work on *The "More Cookies" Region and Strategy.*

As groups near completion of the activity, they should begin preparing presentations. You may want to assign specific elements of the activity to particular groups to present. We suggest that you have two or three groups focus on Question 3, because that is the most important part of the activity for the development of the unit.

Discussing and Debriefing the Activity

Be aware that ideas from Just the Plane Facts are needed for The "More Cookies" Region and Strategy.

Presentations

Have students from different groups present each question, then let other students add ideas or show their models. Rely as much as possible on student models or drawings, but use the diagrams and blackline masters provided if needed.

Overview of the Discussion

Questions 1 and 2 should be straightforward. These questions are largely preparatory for Question 3, which is the main focus of the activity.

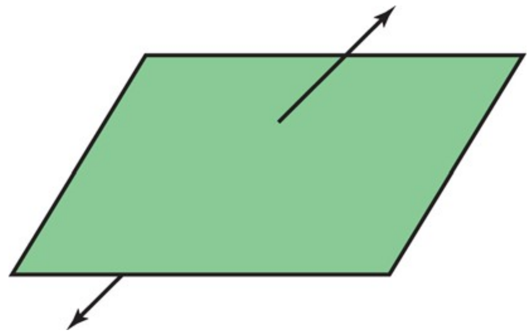
In the discussion of Question 3, students should see that the intersection of three planes can be a plane, a line, a point, or nothing. (The analysis of cases need not include the details of the presentation given here.)

In Question 4, the main ideas are the concepts of parallel and skew lines in space. In Question 5, the main idea is that four planes will generally have no points in common.

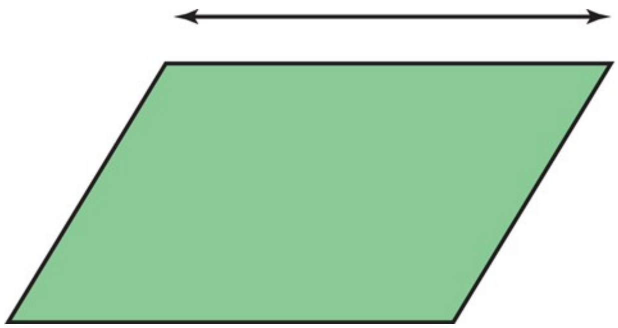
We present here some detailed discussion for each of the questions. Blackline masters for all the diagrams shown here are included in the blackline masters for *Just the Plane Facts.*

Question 1
A line and a plane considered together can:

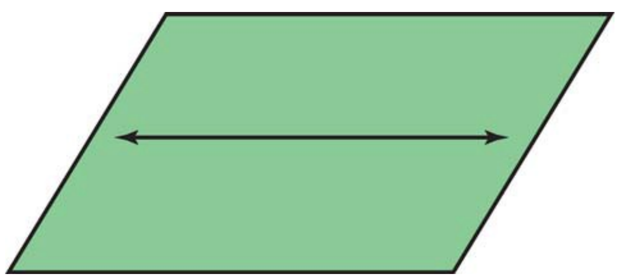
a) Intersect in a single point



b) Have no points in common (that is, the line is parallel to the plane)



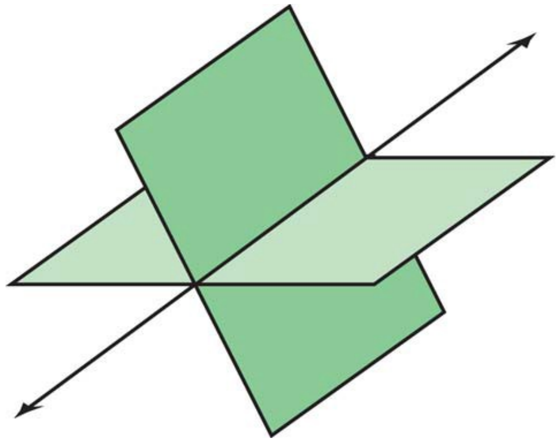
c) Intersect in an entire line (that is, the line is contained within the plane)



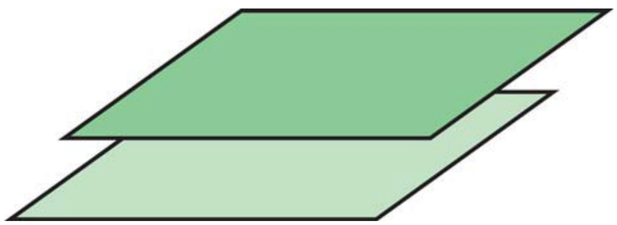
Ask students to describe Case b: **What word might we use to describe a line that doesn't intersect a plane?** Elicit that the line can be described as parallel to the plane. Point out that this is a new context for the word *parallel*. Other new contexts for parallel are discussed in this section: for two planes (Case b of Question 2), and for two lines in space (Question 4).

Question 2
Two planes considered together can:

a) Intersect in a line



b) Have no points in common (that is, the planes are parallel)

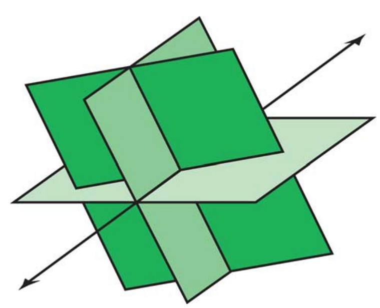


c) Intersect in an entire plane (that is, the two planes are identical)

As with Question 1, ask students how to describe Case b: **How can we describe the case in which the two planes have no points in common?** Students should describe these planes as *parallel*.

Question 3
Three planes can intersect in any of several ways. One idea that may be helpful in thinking about this is to start with two planes, consider the different cases in Question 2, and then see how the third plane might intersect the intersection from Question 2.

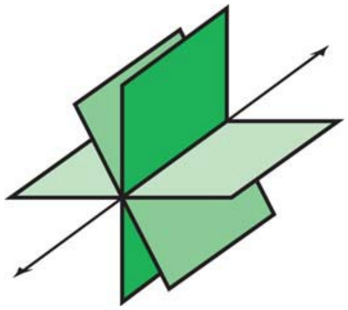
Case A: The First Two Planes Intersect in a Line
If the first two planes intersect in a line, these are the possible outcomes when a third plane is added to the system:



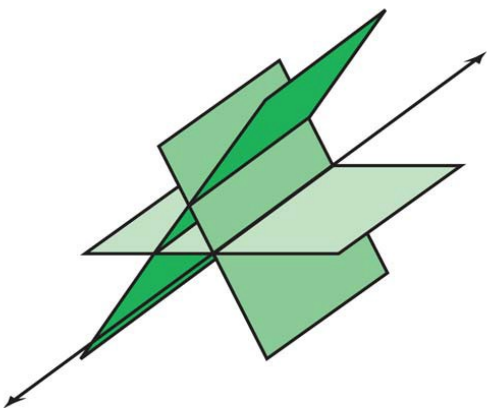
1) The three planes intersect in a point. Here, the line formed by the first two planes intersects the third plane in a point.

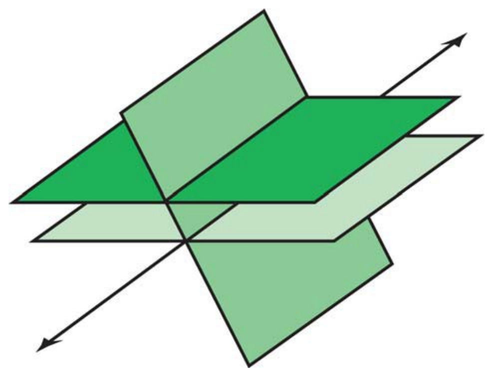
2) The three planes intersect in a line. Here, the line formed by the first two

planes is completely contained within the third plane.



3) The three planes have no points in common. This can happen in two ways. In the first diagram, the third plane is parallel to the line formed by the first two planes. In the second diagram, the third plane is parallel to one of the first two planes (including the line of intersection). The second instance is a special case of the first.

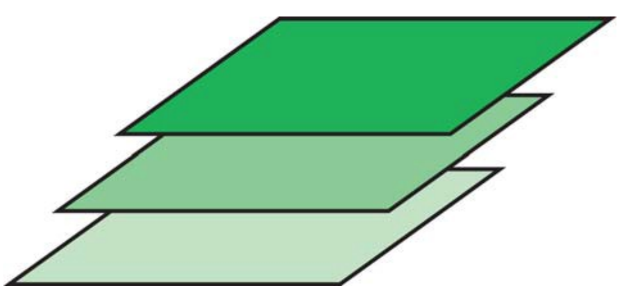




Case B: The First Two Planes Are Parallel
If the first two planes are parallel, then the three planes will certainly have no points in common. But there are still three general configurations:

- The third plane could intersect both of the first two.
- The third plane could be parallel to the first two.
- The third plane could be identical to one of the first two.

The first of these cases gives a diagram like the second diagram in Question 3, Case a, outcome 3. The last gives a diagram like Question 2 Case b. The middle case, that of three parallel planes, might look like this:



Case C: The First Two Planes Are Identical
Finally, if the first two planes are identical, then we are essentially talking about the intersection of two planes, and the resulting diagrams would be similar to those from Question 2.

Summary of Question 3
Thus, a system of three planes can have any of these intersections:

- A plane
- A line
- A point
- Nothing

What "Usually" Happens?
Ask, **What type of intersection would result if you picked the coefficients at random?** (You should acknowledge that the phrase "at random" is a bit complicated to define in this context—see the next note.)

Guide students to articulate that just as two lines in the plane will "usually" meet in a single point, so also three planes in 3-space will "usually" meet in a single point.

Note: Expressing the idea of picking three planes "at random" in terms of probability is not easy, because there is no simple way to define formally what it means to pick a plane at random. Thus, the word "usually" is used here in a colloquial or intuitive way, and not in the more mathematical sense of "with probability greater than $\frac{1}{2}$."

Question 4
The situation of two lines in space is quite different from that of two lines in a plane. (The case of two lines in a plane was discussed with *Being Determined*.) While two lines in a plane will "usually" intersect in a unique point, two lines in space will "usually" have no points in common. Students might not be aware that this does not necessarily mean that the lines are parallel.

In fact, you should use Question 4 as an opportunity to ask, **What do you think parallel means in the general case of lines in space?** If they say that *parallel* means "not intersecting," give them an example such as the edge where a side wall of the classroom meets the front wall and the edge where the back wall meets the floor. These edges do not meet (even if extended), but they are going in different directions.

With that in mind, ask again what *parallel* means. Let students know that *by definition*, two lines are considered parallel only if they are going "in the same direction." Bring out that this requires that the lines lie in a common plane and be *parallel in that plane* (in the sense of not intersecting).

Note: It might be more accurate to say that two lines are parallel if they go in the same “pair of directions.” For instance, we don’t describe the northbound lane of a highway as going “in the same direction” as the southbound lane, but the two are considered parallel. If necessary, clarify this for your students.

Introduce the term **skew lines** to refer to a pair of lines in space that do not meet but are not parallel.

Question 5
Perhaps the best way to examine the case of four planes is to start with three, consider how they might intersect, and then consider how that intersection relates to the fourth plane;

- If the first three planes have no points in common, then the four planes have no common points either.

- If the three planes intersect in a single point (the “usual” case), that point might or might not lie on the fourth plane. So the intersection of the four planes might be a single point or, as in the first case, no points.

- If the three planes intersect in a line or a plane, then students can consider how that line or plane might intersect the fourth plane.

Question 6
If students have any other combinations that they thought about, let them present their results.

Key Questions

How might you need to adapt your linear programming strategy to work with three-variable problems?

What word might we use to describe a line that doesn’t intersect a plane?

How can we describe the case in which the two planes have no points in common?

What type of intersection would result if you picked the coefficients at random?

What do you think *parallel* means in the general case of lines in space?

Supplemental Activity

Embellishing the Plane Facts (extension) asks students to provide equations of planes that intersect in various ways.

Solving With Systems

Intent

Students solve word problems using systems of linear equations.

Mathematics

This activity gives students additional experience both with writing linear systems from problem situations and with solving linear systems in two variables.

Progression

This activity reinforces students' ability to write linear equations and solve systems.

Approximate Time

25 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students should be able to do this activity independently.

Discussing and Debriefing the Activity

Let one or two students share how they created their systems for each of Questions 1 and 2 and how they solved them.

Solving Systems

Solving systems of equations should be becoming more routine. For example, in Question 1, using L for the number of off-the-lip moves and C for the number of cutbacks, students will probably come up with these equations:

$$L + C = 6$$

$$6L + 8C = 40$$

If they use the "setting y's equal" method, they might solve for C to get these equations

$$C = 6 - L$$

$$C = \frac{40 - 6L}{8}$$

and set them equal, to get a single equation to solve:

$$6 - L = \frac{40 - 6L}{8}$$

(They might write the right side of this equation as $5 - 0.75L$, using decimals instead of fractions.)

Solving the equation

$$6 - L = \frac{40 - 6L}{8}$$

provides many opportunities for students to make careless mistakes. Areas that may need attention include properly using the distributive property and keeping signs correct. If your students have no trouble with these issues, then you might take this opportunity to discuss clear definition of variables and organization of work.

Fitting a Line

Intent

Students use algebra to find the equation of a line through two points. The method used here prepares them for fitting a parabola through three points later in this unit.

Mathematics

In this activity, students find a linear function that goes through two specific points by treating the coefficients as variables. The approach is developed slowly, establishing solid understanding, and laying the foundation for finding the equation of the parabola through three given points in a later activity.

Progression

In *Being Determined*, students looked at the geometry of lines through a pair of points. This activity looks at the same situation from an algebraic perspective. This activity lays a foundation for fitting a parabola to three points in *Fitting Quadratics.*

Approximate Time

5 minutes for introduction
25 to 35 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

We suggest that you take a few minutes to go over Questions 1 and 2 of this activity before students start working on their own. The key idea to bring out is that students can check whether a point is on a line simply by substituting the coordinates and seeing if they fit the equation.

Discussing and Debriefing the Activity

We expect that students will have found the relationships between a and b (in Questions 3 and 4) by examining the pairs of values that work in each case. In the discussion, you can help them begin to see a more systematic way to develop these equations, using the fact that the given points fit the equation. (See the subsection "Substituting to Get an Equation.") Students will have further opportunities to use that approach in later activities (*Fitting More Lines, Fitting Quadratics,* and *Fitting Mia's Bird Houses*).

The purpose of Question 1 is to lay a solid foundation for students. The question reviews the fundamental idea that for a point to lie on the graph of an equation means, simply, that the coordinates fit the equation.

Questions 1 and 2

Begin by having a volunteer explain why the line $y = -5x + 7$ goes through (1, 2).

The explanation can consist simply of substituting 1 for x and 2 for y and verifying that the resulting equation, $2 = -5 \cdot 1 + 7$, is a true statement.

Then ask, **What are some other linear equations that go through (1, 2)?**

Questions 3 and 4

Let other students present the values for a and b for several different equations through (1, 2). (This involves nothing more than reading off the coefficients from the equation.)

Then ask, **Did you find a relationship between the two coefficients that always held true?** Because Question 3 uses a point with $x = 1$, this leads to a fairly simple relationship between a and b that most students will see, namely, $a + b = 2$. (If students did not see this relationship, have them put the combinations for a and b into a table, as suggested in the activity.)

After students finish discussing Question 3, have them reverse the process, finding one or two new number pairs for a and b that fit the equation $a + b = 2$ and verifying that these lead to linear functions of y in terms of x that go through the point (1, 2). For instance, $a = 3$, $b = -1$ fits $a + b = 2$. Students should verify that the line $y = 3x - 1$ goes through (1, 2).

One purpose of doing both Question 3 and the "reverse" is to bring out that the condition $a + b = 2$ holds true *if and only if* the line $y = ax + b$ goes through (1, 2). You may want to bring this idea out explicitly.

The work with Question 4 should be similar. In this case, students will probably see the relationship as either $a = b + 6$ or $b = a - 6$.

Question 5

Have a volunteer present Question 5.

You may want to ask why, for a linear function to go through both points, the coefficients a and b need to fit the conditions from both Question 3c and Question 4b: **Why do a and b need to fit these two conditions?** Thus, to find the correct linear function, students should have solved the system

$$a + b = 2$$

$$a = b + 6$$

or an equivalent one. (If students seem to have had a hard time getting an appropriate system of equations, give them a few minutes to solve the system the class developed in its discussion of Questions 3 and 4.)

Be sure students verify that the solution to the system ($a = 4$, $b = -2$) leads to a linear equation, $y = 4x - 2$, that does indeed go through both (1, 2) and (−1, −6). Because only one line passes through these two points, this should be the only linear function that works (though the same equation could be written in different forms). Connect this with the fact that the equations $a + b = 2$ and $a = b + 6$ have only one common solution, so $a = 4$, $b = -2$ is the only choice for the coefficients in the equation $y = ax + b$.

Substituting to Get an Equation

Tell students that there is a systematic way to get the equations asked for in Questions 3c and 4b.

Ask, **What does it mean that the equation y = –5x + 7 goes through (1, 2)?** They will probably respond with a statement like, "If you substitute $x = 1$ and $y = 2$, you get a true statement."

Then ask, **What does it mean that the equation y = ax + b goes through (1, 2)?** Help them to see that it should mean essentially the same thing, and have them substitute $x = 1$ and $y = 2$ in the equation $y = ax + b$.

This gives a relationship between a and b, namely, the equation $2 = a \cdot 1 + b$. Ask students where they have seen that equation in connection with this activity, prompting them to remember that this is the equation they got for Question 3c (or at least is equivalent to that equation). In other words, they could have gotten the equation for Question 3c simply by substituting, rather than by looking for a relationship among the various combinations for a and b that worked.

Have students verify that the same approach works for Question 4b. That is, if they substitute $x = -1$ and $y = -6$ in the equation $y = ax + b$, they get the equation $-6 = -a + b$, which is equivalent to their answer for Question 4b.

You might point out that this approach *proves* that their equations for Questions 3c and 4b are correct.

Key Questions

What are some other linear equations that go through (1, 2)?

Did you find a relationship between the two coefficients that always held true?

Why do *a* and *b* need to fit these two conditions?

What does it mean that the equation $y = -5x + 7$ goes through (1, 2)?

What does it mean that the equation $y = ax + b$ goes through (1, 2)?

Cookies, Cookies, Cookies

Intent

In this sequence of activities, students solve a three-variable linear programming problem.

Mathematics

Students apply what they've learned about planes, linear equations in three variables, and the general strategy for solving linear programming problems to solve the *More Cookies* problem introduced in the previous section.

Progression

The POW presents a number game called SubDivvy. Students are asked to investigate whether there are strategies for the game that can guarantee that a particular player will win.

In *More Cookies*, students wrote algebraic statements for the constraints in a three-variable linear programming problem. Now, in *The "More Cookies" Region and Strategy*, they will adapt their strategy for solving linear programming problems to problems having three variables. They will recognize that the feasible region is a polyhedron and, in *Finishing Off the Cookies*, will interpret the solutions of systems of three equations as vertices of the polyhedron or as other intersections of planes representing solutions that are not feasible.

POW 2: SubDivvy

The "More Cookies" Region and Strategy

Finishing Off the Cookies

POW 2: SubDivvy

Intent

Students explore what constitutes a winning strategy in a number game and justify their strategies.

Mathematics

POW 2: SubDivvy involves a number game called SubDivvy and the analysis of the strategy for the game. It has no direct mathematical connection to the unit. An important element of the POW write-up, as always, is the justification that students provide for their solutions.

Progression

Students will work on this POW for about a week with a partner, and then several individual students will make presentations.

Approximate Time

10 minutes for introduction
1 to 3 hours for activity (at home)
15 to 20 minutes for presentations and discussion

Classroom Organization

Pairs, followed by several presentations and whole-class discussion

Doing the Activity

Introduce this game by having two students or two sides of the class play several games to be sure that everyone understands the rules. Also make sure everyone understands what is meant by "a winning strategy." Students need to know that a strategy is a *complete plan* that spells out exactly what a player should do in each situation. A winning strategy is one that guarantees that the player will win no matter what her or his opponent does.

Have students work on this POW with a partner. Either assign partners or let students choose on their own. You can allow each pair to turn in a joint report (although each student may want a copy to include in his or her portfolio).

Allow students at least a week to work on *POW 2: SubDivvy.* On the day before the POW is due, choose three students to make presentations on the following day, and give them overhead transparencies and pens to take home to use for preparing the presentations.

Discussing and Debriefing the Activity

Have the three students make presentations, and then ask other students to contribute things they discovered. Try to help students formulate good arguments and/or proofs for the statements they make. If no one comes up with a complete solution to the winning strategy question, leave it as an open problem.

Winning Strategies

If the starting number is even, then there is a winning strategy for the first player. The first player can always win by subtracting 1 to leave the second player with an odd number. Because odd numbers only have odd divisors, the second player must subtract an odd number, which gives the first player an even number again. If the first player again subtracts 1, the second player must subtract an odd number, giving the first player an even number. Play continues this way until the second player is forced to give the first player the number 2. The first player then wins by subtracting 1.

Note: At each turn, the first player can choose to subtract any odd divisor, not simply 1. However, the only odd number that is guaranteed to be a divisor is 1.

If the starting number is odd, the there is a winning strategy for the *second* player. As previously, the first player must subtract an odd number, leaving the second player with an even number. Then the second player can simply follow the strategy for the first player just outlined.

For a starting number n, the longest game would have $n - 1$ moves (by subtracting 1 at each move). For a starting number of 2^n, the shortest game would have n moves, with each player subtracting half of the given number each time. There is no general formula for the number of turns for the shortest game.

Students might think that the way to end the game in the fewest number of moves is for each player to subtract the biggest possible divisor. But this isn't always the case. You may want to ask students to look for an exception.

Supplemental Activity

The Shortest Game (extension) challenges students to continue exploring the question of what the shortest path will be in SubDivvy.

The *More Cookies* Region and Strategy

Intent

Students adapt the strategy for solving linear programming problems without graphing to problems with three variables.

Mathematics

Students identify the feasible region for the *More Cookies* problem, and adapt their linear programming strategy to three variables. Students will see that the feasible region of a three-variable linear programming problem is a polyhedron, and they'll refine a non-graphical method for solving linear programming problems.

Progression

In this activity, students describe and sketch the feasible region for the set of three-variable constraints from the linear programming problem in *More Cookies*.

Discussion following the activity will cover:

- Articulating that the feasible region is a polyhedron
- Reviewing the terms *face, edge,* and *vertex*
- Developing a general sketch of the feasible region
- Adapting the general strategy to a three-variable problem, focusing on
  - the reason the profit is maximized at a corner point
  - the principle that corner points are found as the intersections of three boundary planes

Approximate Time

30 minutes for activity (at home or in class)
30 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

An assortment of polyhedron blocks (optional)
Transparencies of The "More Cookies" Region and Strategy blackline masters

Doing the Activity

If assigning this activity as homework, tell students that they should take home a copy of the general linear programming strategy developed so far, so they can adapt it as they work on this activity.

Discussing and Debriefing the Activity

Ask for volunteers to share ideas about what the feasible region looks like.

By now, students should recognize that the graph of a linear equation in three variables is a plane. They should be able to see, by analogy to the two-variable case from *Cookies*, that the graph of a linear *inequality* in three variables is the set of all points on some side of the plane. (Deciding which side isn't necessarily easy; don't get sidetracked by this.) Thus, they should recognize that the feasible region for the *More Cookies* problem is the region bounded by the six planes that are the graphs of the equations corresponding to the constraints.

Ask if students can be any more specific about the nature of this region. If a hint is needed, ask, **What sort of figure is bounded by flat surfaces?** As a further hint, go back to two dimensions, and ask what sort of figure is bounded by line segments.

Then go from the answer to that hint—polygons—to the three-dimensional case. Someone should be able to say something like "a solid figure with flat sides." If necessary, remind students that such a figure is called a *polyhedron.* Review that each surface of such a figure is called a *face,* that the lines where faces meet are called *edges,* and that the points where edges meet are called *vertices.*

It may be useful to have some odd-shaped polyhedra, such as assorted irregular blocks, to illustrate the concept of a polyhedron.

To get at a more specific analysis of the *More Cookies* feasible region, it's simplest to begin with constraints IV, V, and VI (using the numbering of constraints given in the discussion of *More Cookies*):

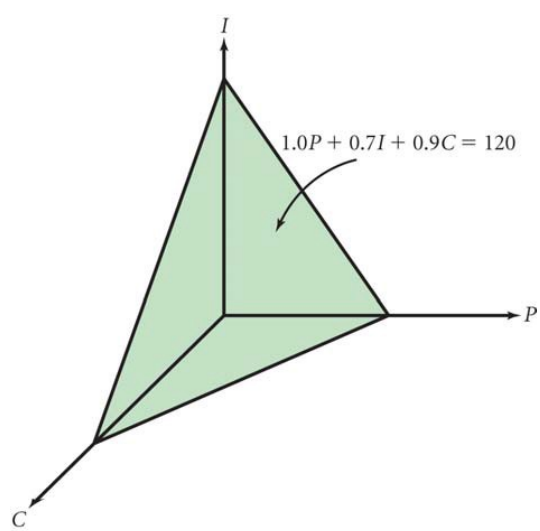
I	$1P + 0.7I + 0.9C \leq 120$
II	$0.4I \leq 32$
III	$0.15C \leq 18$
IV	$P \geq 0$
V	$I \geq 0$
VI	$C \geq 0$

Students should see that the planes corresponding to constraints IV, V, and VI are the coordinate planes, so that these three constraints define the "all-positive" first octant.

Introduce the term *constraint plane* for the graph of an equation corresponding to one of the constraints of a three-variable linear programming problem.

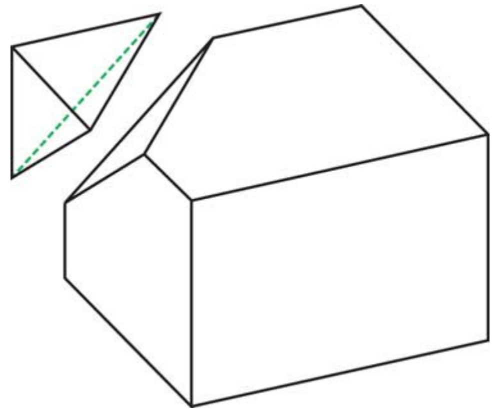
Ask students, **What would the region look like if only constraints I, IV, V, and VI were considered?** That is, they should consider the first-octant portion of the inequality $1P + 0.7I + 0.9C \leq 120$.

Help them to see that this region looks like the diagram at right.

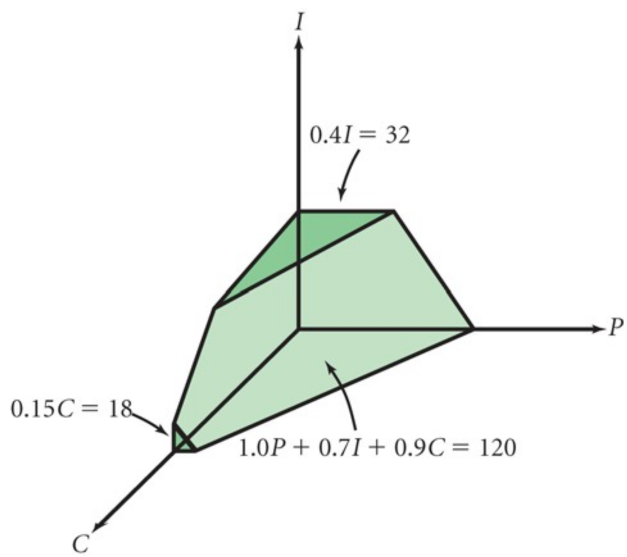


Note: A diagram showing all the constraint planes appears later in this section. All three diagrams shown for this discussion are included in *The More Cookies Region and Strategy* blackline masters.

Students should see that this region is similar to the shape one would get by cutting off the corner of a box. This diagram shows such a corner, although for the sake of perspective, a different corner is illustrated.



Ask, **What does the complete feasible region look like?** If any students made a three-dimensional model of the region or developed a good two-dimensional drawing of it, have them display their work. If not, you can show them the next diagram, which shows the first diagram cut by the planes $0.4I = 32$ and $0.15C = 18$, corresponding to constraints II and III. (These two planes are parallel, respectively, to the *PC*-plane and the *PI*-plane.) Emphasize that the feasible region is the solid bounded by the constraint planes, and not simply the surface of that solid.



Note: Two other diagrams, giving the points of intersections of various constraint planes, are included in the activity notes for *Finishing Off the Cookies*.

Reviewing the General Strategy

Before discussing Question 2, you may want to review the strategy components developed for two-variable problems. Here are the steps as given following *Widening Woody's Woodwork*:

1. Write the constraints as a set of linear inequalities (including constraints that the variables are nonnegative).

2a. Find all points where boundary lines intersect by solving all possible pairs of linear equations corresponding to constraints.

2b. Check these intersection points against all the constraints to determine which of them actually lie in the feasible region. (Those that satisfy all the constraints are the corner points of the feasible region.)

3. Write down the function being maximized or minimized.

4. Evaluate the function being maximized or minimized at each of the intersection points found in step 2b, and determine which of these points gives the function its maximum or minimum value in the feasible region.

The only part of this strategy that needs adaptation is Step 2a.

Question 2: The Three-Variable Strategy for Finding Corner Points

Begin the discussion of Question 2 by having volunteers share ideas on how to adapt Step 2a of the strategy. As a hint, you can remind the class that the corner of a polygon is the intersection of two lines, then ask, **How do you find the corner of a polyhedron?**

Students will probably recognize that each corner point is the intersection of some of the faces of the polyhedron. They should also see that each of these faces is a portion of the plane corresponding to a constraint.

Help them to build on their work in *Just the Plane Facts* to see that they will probably use three constraints at a time to get each corner point, because they found that "usually" three planes determine a point. Thus, for a three-variable linear programming problem, they should replace Step 2a with something like this;

2a. Find all points where boundary planes intersect by solving all possible triples of linear equations corresponding to constraints.

Why Corner Points?

Before leaving the issue of finding corner points, you may want to ask students **Why are corner points important?** If necessary, have them review the two-variable situation. Then have them adapt that reasoning to the three-variable situation. You want them to come up with the idea of a *parallel family of profit planes,* analogous to the parallel family of profit lines of two-variable problems.

If necessary, ask what the set of points that give a particular profit (in the *More Cookies* problem) would look like. As needed, help students to see that setting the profit expression $1.5P + 2I + 2.25C$ equal to a given value gives a graph that is a plane.

Students can visualize one of these profit planes moving through space, forming a parallel family, as the profit increases. They should see that their goal is to find the place where this family of planes "leaves the feasible region." You want students to visualize, or at least to accept as reasonable by analogy, that this will take place at a corner point of the feasible region.

One useful method for picturing this family of profit planes is to imagine the feasible region as a polyhedral wooden block, partly submerged in a container of water. The plane of the water represents the profit plane. We can gradually add water to the container, holding the block fixed, so that the profit plane moves through the feasible region.

The last part of the block to go under will be a vertex, or a whole edge, or a whole face, but in any case, at least one vertex will be "the last to go." You may want to actually demonstrate this with an irregular polyhedral block.

You may wish to point out that the profit function must be linear for this method to work.

Key Questions

What sort of figure is bounded by flat surfaces?

What would the region look like if only constraints I, IV, V, and VI were considered?

What does the complete feasible region look like?

How do you find the corner of a polyhedron?

Why are corner points important?

Finishing Off the Cookies

Intent

Students complete the solution to a three-variable linear programming problem.

Mathematics

As students implement their strategy for solving a linear programming problem in three variables without graphing, they will solve a number of systems of three equations. The discussion will focus on:

- Sharing ideas for systematic ways to compile a list of systems of equations that must be solved
- Recognizing that not every combination of three constraints leads to a corner point
- Recognizing that students will eventually need to solve more complex linear systems than those in the *More Cookies* problem
- Beginning to think about generalizing beyond three variables
- Discussing issues related to noninteger maximum points

Progression

Students work in small groups to form a plan for dividing up work on the solution to *More Cookies.* They then work individually on their assignment. Finally, they work further in their groups to finalize the solution. This activity completes the solution to *More Cookies.*

Approximate Time

15 to 20 minutes for introduction and for groups to form plans
30 minutes for activity (at home or in class)
45 to 50 minutes for presentations and discussion

Classroom Organization

Small groups to formulate plans
Individuals to complete assignments from group
Individuals or groups to make presentations
Whole-class discussion

Materials

Transparencies of the *Finishing Off the Cookies* blackline masters

Doing the Activity

As the text of the activity states, groups will need to divide the work for this activity between the group members. This activity works well as a homework assignment, but if you are using it as homework you will need to allow some time for the groups to devise their plans and assign tasks to individuals. Inform them that groups will be given some time in the next class period to combine their work and that one or more groups will make presentations. Then let groups begin work on the activity. As they do so, circulate around the room to see if they seem to know how to proceed.

Presumably, their general plan will be to examine the linear equations corresponding to the constraints in sets of three in order to find possible corner points of the feasible region, check which of these corner points actually fit all the constraints, and then find the one that maximizes the profit function $1.5P + 2I + 2.25C$. There are 20 sets of three constraints to look at.

This may seem like a huge amount of work, but each system is fairly easy to solve because the equations are so simple. (Only the equation corresponding to constraint I actually uses more than one variable.) Furthermore, students will not actually have to check all 20 sets of three if they realize that certain combinations of equations can't have a common solution. For example, the equations corresponding to constraints II and V are inconsistent, so any combination including these two will have no common solution. The same applies to combinations involving constraints III and VI. The issue of inconsistent systems will be discussed after students have completed their work.

If groups are having trouble getting started, you can bring the class together to discuss the general principles further and perhaps to talk about how to get an organized list of triples of equations that includes all possible combinations.

Discussing and Debriefing the Activity

Begin by talking about the general ideas of the strategy, focusing on the step of compiling a list of combinations of equations that must be checked. Discuss how students can tell if their list of combinations is complete. As noted above, students may eliminate some combinations from consideration immediately. (For instance, if a combination involves the equations corresponding to constraints II and V, then it gives conflicting values for the variable I.)

Once the class has agreed on a set of systems, you can assign a group to each system and have them prepare presentations.

Once again, here are the constraints:

I	$1P + 0.7I + 0.9C \leq 120$
II	$0.4I \leq 32$
III	$0.15C \leq 18$
IV	$P \geq 0$
V	$I \geq 0$
VI	$C \geq 0$

Presentations

When presenters are ready, let them begin. One issue that should come up in the discussion is that not every set of three planes leads to a corner point of the feasible region.

Bring out the two reasons a particular set of three equations might not lead to a corner point:

- The three equations might not have a unique solution. This is the algebraic analog of the geometric fact that three planes don't always intersect in a unique point. Three linear equations in three variables might have no solution or infinitely many solutions, just as is the case for two linear equations in two variables.
- A point in which three given planes meet might actually be outside the feasible region. Use this fact to remind students that they always need to check potential corner points with *all* of the constraints.

Students may not know what to do about the first of these issues. Guide them to realize that they are looking for the corner points and that every corner point *is* the intersection of three constraint planes. Tell them that they can ignore sets of planes that do not give a unique solution.

Note: Systems of linear equations in three variables are addressed further in this discussion after students complete the *More Cookies* problem.

Details of the Solution

As noted previously, certain combinations of equations corresponding to constraints are inconsistent. Eliminating cases involving either both II and V or both III and VI leaves 12 cases. The table below gives these combinations and the common solution in each case (to the nearest hundredth in the case of noninteger solutions).

Combination	*P*	*I*	*C*
I, II, III	−44	80	120
I, II, IV	0	80	71.11
I, II, VI	64	80	0
I, III, IV	0	17.14	120
I, III, V	12	0	120
I, IV, V	0	0	133.33
I, IV, VI	0	171.43	0
I, V, VI	120	0	0
II, III, IV	0	80	120
II, IV, VI	0	80	0
III, IV, V	0	0	120
IV, V, VI	0	0	0

Once students find the common solution for each of these systems, they need to check which of these solution points are actually in the feasible region—that is, which of the solutions actually satisfy all of the original inequalities.

The first combination in the table is easily ruled out because of the negative value for *P*. Three other combinations can also be ruled out for violating other inequalities:

- The combination I, IV, V violates constraint III.

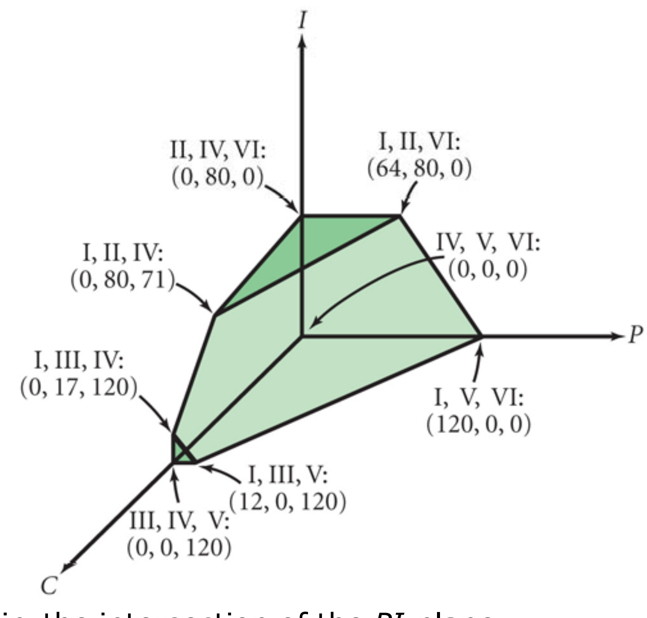
- The combination I, IV, VI violates constraint II.
- The combination II, III, IV violates constraint I.

The remaining candidates must then be checked, using the profit function $1.5P + 2I + 2.25C$, to see which gives the greatest profit. This table shows these eight combinations, their solutions, and the associated profits:

Combination	*P*	*I*	*C*	Profit
I, II, IV	0	80	71.11	320
I, II, VI	64	80	0	256
I, III, IV	0	17.14	120	304.29
I, III, V	12	0	120	288
I, V, VI	120	0	0	180
II, IV, VI	0	80	0	160
III, IV, V	0	0	120	270
IV, V, VI	0	0	0	0

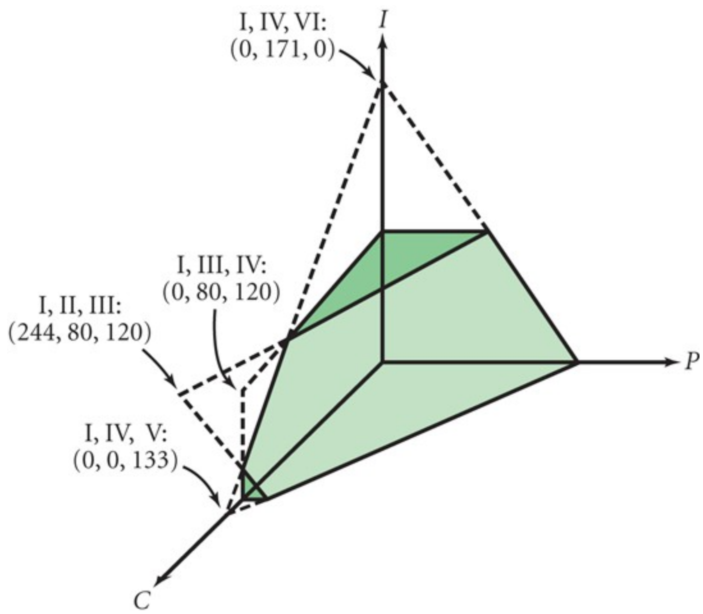
Thus, the maximum profit comes from the common solution to the equations corresponding to constraints I, II, and IV, where $P = 0$, $I = 80$, and $C = 71.11$. So the Woos' maximum possible profit is $320. (See the section "Whole Dozens or Not?" for discussion of what to do about the fact that *C* is not a whole number.)

This diagram shows the eight corner points of the feasible region, listed in the table above. (A copy of this diagram and the next are included in the *Finish Off the Cookies* blackline master.)



You might have students identify the various edges of the feasible region as the intersection of two constraint planes. For instance, the edge connecting the points (0, 80, 0) and (64, 80, 0) is the intersection of the planes from constraints II and VI. Students could deduce this from the fact that these two points have those constraints in common, but they could also see geometrically that this edge lies in the intersection of the *PI*-plane (constraint VI) and the horizontal plane $0.4I = 32$ (constraint II).

The next diagram shows the four points of intersection that were not corner points because they failed to satisfy one of the constraints. In this diagram, certain edges have been extended beyond the feasible region to show the points of intersection.



Systems of Three Linear Equations in Three Variables
As students saw in their work on *Just the Plane Facts,* a set of three planes can intersect in various ways. They have now seen some of the algebraic analog of those ideas, and this is a good time to discuss the algebraic concepts, first for three variables and then more generally.

Ask students to summarize what they saw in the *More Cookies* problem, asking, **What can happen when you try to solve a set of three linear equations in three variables?** Presumably, they will point out that "usually" such a system has a unique solution. Help them connect this with the "usual" case of how three planes intersect.

If students don't mention it in connection with the *More Cookies* problem, ask, **What do you call a system of equations with no common solution?** Then review the term *inconsistent* for a system of equations with no solution. Again, relate this to the geometry. For instance, help them to see that the combination corresponding to constraints I, II, and V had no common solution because the constraint planes for constraints II and V were parallel.

Generalizing to More Than Three Variables
Remind students that the unit problem, *Meadows or Malls?*, involves six variables, not three. You can suggest that they think about how they would have to further adapt and generalize their strategy to accommodate more variables.

Ask, **Can you create a graph for a six-variable problem?** No one has found a convenient way to visualize the graph of a linear equation in more than three variables, so students will not have a four-dimensional version of the activity *Just the Plane Facts*. Beginning with *Easy Does It!*, they will be investigating the algebra of linear equations further, and they will come back to the issue of solving linear programming problems in more than three variables later in the unit.

Whole Dozens or Not?
There is legitimate reason to question whether the solution to the *More Cookies* problem (*P* = 0, *I* = 80, *C* = 71.11) is acceptable, because it represents fractional dozens of cookies. (The exact value for *C* is $71\frac{1}{9}$, which represents 71 dozen plus $1\frac{1}{3}$ cookies, so this doesn't even represent whole cookies, let alone whole dozens.)

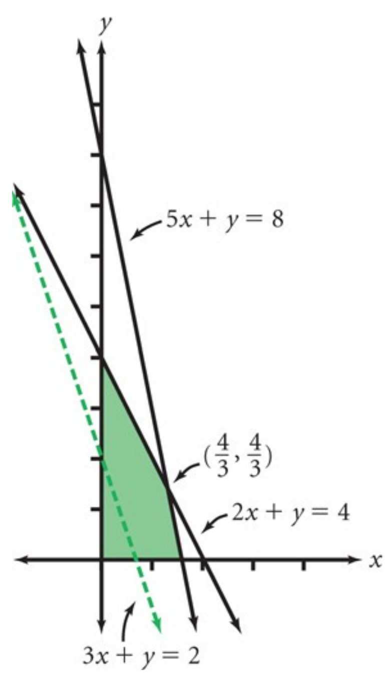
If the problem is interpreted to allow these fractions, then there is no difficulty. (One such interpretation would be that the constraints on ingredients are *per day* and the Woos are in constant production. So, in the long run, these amounts can turn out to be whole dozens or whole cookies.)

On the other hand, if the problem is interpreted to require whole-number values for *P*, *I*, and *C*, then the situation becomes quite complicated. The geometric argument no longer works, because the corner points are not necessarily *lattice points.*

In the *More Cookies* problem, the optimal solution occurs at the point (0, 80, 71.11). We can reasonably expect that if we consider only lattice points in the feasible region, the best choice would be to reduce *C* from 71.11 to 71 and use the point (0, 80, 71). Among all lattice points in the feasible region, this is the one closest to (0, 80, 71.11).

As the example below shows, however, there is no guarantee that the lattice point closest to the corner that maximizes profit is actually the lattice point that maximizes profit.

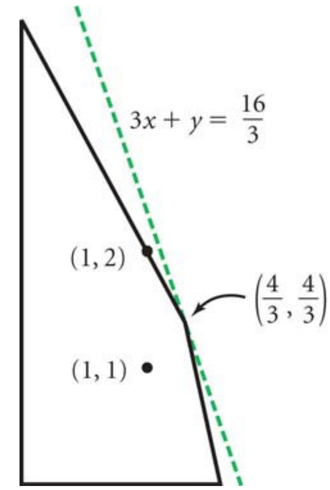
A Strange Example in Two Dimensions
As an illustration of the preceding discussion, here is an example (in two dimensions) in which the lattice point that maximizes profit is *not* the lattice point in the feasible region closest to the corner point that maximizes profit. For simplicity, the example is formulated in terms of the mathematics, rather than in a real-world context:



Consider the feasible region defined by the inequalities $5x + y \le 8$, $2x + y \le 4$, $x \ge 0$, and $y \ge 0$, and consider the "profit expression" $3x + y$. (In other words, the goal is to maximize this expression within the region.) This diagram shows the feasible region and the "profit line" $3x + y = 2$. One can verify that the expression $3x + y$ is maximized in the feasible region at the point $\left(\frac{4}{3}, \frac{4}{3}\right)$.

The lattice point within the feasible region that is closest to $\left(\frac{4}{3}, \frac{4}{3}\right)$ is the point (1, 1), and it would be reasonable to expect that this would be the lattice point with the greatest "profit." But the expression $3x + y$ actually has a greater value at (1, 2) than it has at (1, 1). Because (1, 2) is in the feasible region (actually, on the boundary), (1, 1) is *not* the best lattice point to choose.

In the next diagram, which shows a close-up of the feasible region, one can see that although the point (1, 2) is farther than (1, 1) from the "optimal point" $\left(\frac{4}{3}, \frac{4}{3}\right)$, the point (1, 2) is closer than (1, 1) to the "profit line" $3x + y = \frac{16}{3}$, which goes through that optimal point.



Unfortunately, there is no simple algorithm for finding the best lattice point when the best corner point is not itself a lattice point, and mathematicians recognize that finding the optimal point is a complex and challenging problem.

Key Questions

What can happen when you try to solve a set of three linear equations in three variables?

What do you call a system of equations with no common solution?

Can you create a graph for a six-variable problem?

Equations, Equations, Equations

Intent

In this sequence of activities, students learn to solve systems of linear equations in three variables.

Mathematics

The activities now extend students' use of the elimination method to solving three-variable systems of equations. As students increase their facility with this method, they are also moving toward the realization that solving still larger systems of equations by this method requires a lot of work, motivating the eventual introduction of matrices. Recall, the central unit problem has six variables!

Progression

The elimination method is initially introduced using two-variable systems of linear equations in *Easy Does It!* Then *Get Rid of Those Variables!* helps students see how to apply the method to a greater range of equations by forming equivalent equations, and they discover in *Eliminating More Variables* that they can add equations in the same way that they have been subtracting them.

Elimination in Three Variables expands the elimination method to systems of equations in three variables.

Easy Does It!

Get Rid of Those Variables!

Eliminating More Variables

Gardener's Dilemma

Elimination in Three Variables

More Equation Elimination

Easy Does It!

Intent

This activity is a lead-in to the elimination method for solving systems of linear equations.

Mathematics

As students continue to work with linear programming problems, particularly with those having more than two variables, they will be constantly encountering the need to solve systems of linear equations. This activity introduces the elimination method for solving those systems. This method will often be easier to use than the substitution method, which students are already familiar with. The activity begins with intuitive ideas about comparing equations, then students make the transition to a more formal algebraic representation of those ideas using the elimination method.

Progression

Easy Does It! presents several word problems that lead students toward solving systems of linear equations by subtracting one equation from another.

Approximate Time

15 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

Begin by asking for volunteers to explain Question 1. Presumably, students will say something like, "Tania bought one additional package of batteries and paid an additional $3, so the package of batteries costs $3."

The purpose of this simple problem is for students to explain their reasoning in terms of equations, motivating the technique of adding or subtracting one equation from another. Get students to give a pair of equations that describe the problem, such as

$$4b + c = 20$$

$$5b + c = 23$$

where b is the cost of a package of batteries and c is the cost of a package of CDs.

If necessary, use the word *difference* in a dual sense as a hint as to how to manipulate the equations. You can ask, **What is the difference between what**

Will bought and what Tania bought? If needed, suggest that the word *difference* has a specific mathematical meaning related to subtraction. Students should see that they can subtract the first equation from the second to get $b = 3$. Ask them to relate this subtraction process to the informal reasoning used previously.

You can also point out that it's easier to handle the mechanics if the problems are set up in a more natural way for subtraction, with the "bigger" values on top and with some indication of subtraction.

For instance, the problem could be set up like this:

$$\begin{array}{r} 5b + c = 23 \\ -(4b + c = 20) \\ b \qquad = 3 \end{array}$$

Be sure to get a conceptual justification for this subtraction process. That is, students should see that they are subtracting *equal amounts*—namely, $4b + c$ and 20—from the two sides of the first equality. **Why can you subtract $4b + c$ from one side and 20 from the other side?** If necessary, remind them that they know they could subtract a fixed number, such as 13, or a particular variable, such as c, from both sides. Similarly, they could subtract 8 from one side and 5 + 3 from the other side. Here, they know that $4b + c$ and 20 represent the same amount, so they can legitimately subtract $4b + c$ from one side and 20 from the other.

Discuss the mechanics of "subtracting a quantity." That is, when students subtract $4b + c$ from the left side of the first equation, they are subtracting both $4b$ and c. (This may require some review of ideas related to the subtraction of a sum.)

Questions 2 and 3

Question 2 is only slightly more complicated. As in the first example, students can probably give an intuitive explanation. Tanisha purchased three more pencils and spent 75¢ more, so each pencil must cost 25¢.

Question 3 is more complicated because there is no matching purchase. Students need to think up *new* purchases that will help them find the price of an apple or a pear. They can probably give an intuitive explanation, such as saying that Ursula bought almost exactly twice what Sanji did but added an additional apple. Because 10 pears and 6 apples would cost $11.60, the extra apple must cost 60¢.

As in Questions 1 and 2, have students articulate this reasoning algebraically. They can begin by expressing the two given conditions in terms of variables, which will probably give the equations

$$5p + 3a = 5.80$$

$$10p + 7a = 12.20$$

where a is the cost of an apple and p is the cost of a pear.

Have them express the "double what Sanji bought" situation in terms of these variables and justify this new equation, $10p + 6a = 11.60$, as the result of multiplying both sides of the first equation by 2.

Emphasize the role of the distributive property in this process. That is, be sure students see that they must multiply the entire expression $5p + 3a$ by 2.

Students can then justify the comparison between Ursula's purchase and twice Sanji's purchase in terms of subtraction of equations.

Before going on to the next activity, tell students that this method of solving systems of linear equations is called the **elimination method** (or *Gaussian elimination*). As with the substitution method, the goal is to create a new equation that has fewer variables than the initial system.

Key Questions

What is the difference between what Will bought and what Tania bought?

Why can you subtract $4b + t$ from one side and 20 from the other side?

Get Rid of Those Variables!

Intent

This activity lays the foundation for using equivalent equations in order to solve a system of linear equations by the elimination method.

Mathematics

The elimination method for solving systems of equations requires that the coefficients for one of the variables in both equations be equal (or opposite). *Get Rid of Those Variables!* helps students to see that they can easily create this situation by forming new equations that are equivalent to those given.

Progression

Students first develop a family of equations from two initial linear equations. They then observe the pairs of equations that they have created and find a pair that makes it particularly easy to see the value of one of the variables.

The purpose of *Get Rid of Those Variables!* is for students to explore the generalizability of the elimination method. Students will be coming back to linear programming problems once they develop a few more skills.

Approximate Time

20 to 25 minutes for activity
10 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Students work on this activity in groups, then share their results in class discussion.

Students may need help getting started with Question 2. Guide them to see that they can start with any set of values for their variables, make up any linear expression in their variables, and then simply compute the constant term by plugging in the values chosen for the variables.

Discussing and Debriefing the Activity

Ask students, **How does your work on Get Rid of Those Variables! relate to your discussion of the elimination method?** They should see that using the elimination method involves generating new equations in a similar manner.

Have volunteers share some of the equations they created and explain how they got them. Basically, the explanations should suggest some combination of multiplying the initial equations by constants and adding or subtracting the results.

Then ask, **What is the relationship between the solution to the original pair**

of equations and the new pairs of equations you are generating? It is essential that students understand that the solution to the original equations must also satisfy all of these new equations (because of the way these equations are being created) even though they don't yet know what that original solution is.

Students should see that it is reasonably easy to generate these new equations. Their goal in creating these new equations is to make them come out with some matching coefficients. That way, it will be easy to find the (unique) common solution to the new equations. It isn't necessarily important that students find the fastest or easiest way to eliminate a variable.

Key Questions

How does your work on *Get Rid of Those Variables!* relate to your discussion of the elimination method?

What is the relationship between the solution to the original pair of equations and the new pairs of equations you are generating?

Eliminating More Variables

Intent

Students continue to work with the elimination method for solving systems of linear equations, now adding equations as well as subtracting them.

Mathematics

This activity is mostly a straightforward exercise in the elimination method for solving systems of two linear equations in two variables. However, the examples students have seen so far had only positive coefficients, so the technique involved the *difference* between equations. Questions 4 and 5 are intended to suggest the idea of *adding* two equations. (Students may find other ways to do these problems, and the technique of adding equations need not be discussed in advance of the activity.)

Progression

After students have completed the activity independently, discuss it thoroughly and provide additional time for practice if needed. The next activity, *Gardener's Dilemma,* continues this work and also has students set up linear equations for a situation involving three variables.

Approximate Time

30 minutes for activity (at home or in class)
20 minutes for discussion
20 to 30 minutes for additional practice with the elimination method

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Eliminating More Variables presents students with six pairs of equations to be solved using the elimination method.

Discussing and Debriefing the Activity

You might give groups a few minutes to compare ideas on the activity, then assign specific problems to groups to present.

Maintain a relaxed pace in the discussion, allowing lots of opportunity for questions and presentations of alternative methods for each problem.

Questions 1 through 3

Here are the solutions to these problems:

- Question 1: $a = 1$, $b = 2$
- Question 2: $r = 3$, $s = 2$
- Question 3: $w = -19$, $z = 16$

Questions 1 through 3 of the activity are very similar to the problems students did previously in *Get Rid of the Variables!* Use the discussion of these problems to help students articulate how they can find new, equivalent equations that will allow them to match up coefficients.

For instance, on Question 1, the presenter will probably start by doing one of two things:

- Multiplying the first equation by 2, giving $4a + 6b = 16$, which has the same a-coefficient as the second equation
- Multiplying the first equation by 3, giving $6a + 9b = 24$, which has the same b-coefficient as the second equation

Whichever the presenter does, ask, **Why did you decide to multiply that equation by that factor?** The presenter should be able to articulate the idea of getting the coefficients to match so that subtraction eliminates a variable.

On Question 3, the presenter needs to multiply both equations by some factor (or else use fractions) to get a pair of matching coefficients. Students are also more likely to have had trouble with the arithmetic of this problem, because it involves both larger numbers and negative numbers.

Help students try to maintain some connection between the intuitive ideas used in *Easy Does It!* and the formal mechanics needed to solve these equations.

For instance, on Question 2, if the presenter multiplied the second equation by 4 to get $4r + 8s = 28$, bring out that this equation has "an extra $5s$" on the left and "an extra 10" on the right (compared to the first equation, $4r + 3s = 18$), so the expression $5s$ must be equal to 10, giving $s = 2$.

Question 4

The presenter on Question 4 might not use the "obvious" approach of simply adding the two equations together. An alternative approach is to multiply the second equation by -1 to get $-5p + 4q = -6$, so that the q-coefficients match. One could then say that the first equation ($3p + 4q = 10$) has $8p$ more on the left than this new equation, and 16 more on the right, so $8p = 16$. This method is perfectly correct, even if it may seem rather complicated.

If no student suggests adding the two equations as they are presented in the problem, you can simply illustrate the idea directly. That is, show students how to add $5p - 4q$ to the left of the first equation while adding 6 to the right, and illustrate visually how to set this up so it looks like simply "adding the two equations."

You may want to refer back to the mystery bags model of balancing equations from the Year 1 unit *Overland Trail*. Point out that although you are not adding "the same thing" to both sides of the first equation, you are adding "equal things," which is just as good.

The solution to Question 4 is $p = 2$, $q = 1$.

Questions 5 and 6
Questions 5 and 6 do not involve any new concepts, but Question 6 does involve fractional answers.

The solution to Question 5 is $c = 3$, $d = 3$; the solution to Question 6 is $f = 2\frac{1}{2}$, $g = 3\frac{1}{2}$.

Additional Practice with Elimination
In whatever time remains, let students continue working on using elimination for two-variable linear systems. As they become more comfortable with the concepts and techniques, they should be able to handle more cumbersome examples, working with larger numbers and "uglier" fractional answers.

Key Question

Why did you decide to multiply that equation by that factor?

Supplemental Activity

A Linear Medley (reinforcement) describes several situations involving either one or two variables in which students must answer questions using systems of linear equations.

Gardener's Dilemma

Intent

This activity provides students with reinforcement of the elimination method for solving systems of linear equations.

Mathematics

Students will develop (but not solve) a system of three linear equations in three variables and also solve some fairly difficult two-variable systems.

Progression

Part I of *Gardener's Dilemma* asks students to write a system of three linear equations to describe a real-world situation involving three variables. Part II provides four pairs of equations in two variables to be solved. Students will solve this system they write in Part I using matrix inverses and graphing calculators in *Fitting Mia's Bird Houses*.

Approximate Time

30 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

Begin the discussion with Part II, so that you can use Part I as a lead-in to the activity *Elimination in Three Variables*.

Part II

There are no new ideas in these examples, so the amount of time you spend on Part II will depend on how comfortable students are with the mechanics and concepts of the elimination method.

On Question 3, some students may notice that the system of equations can be simplified by dividing both sides of the second equation by 2. If they don't see this, point out that this shortcut can be helpful whenever the coefficients and the constant term have a numerical factor in common.

Here are the solutions to the problems:

- Question 1: $c = -3$, $d = 4$
- Question 2: $u = 8$, $v = -3$
- Question 3: $x = 10$, $y = 15$

- Question 4: $r = \frac{11}{7}$, $s = \frac{3}{7}$

At this point, students should be fairly comfortable with the mechanics of solving two-variable linear systems by substitution.

Part I

Let a volunteer present the set of three equations from the data. Insist that all variables be defined clearly. We will use L for the amount of water needed per square foot of lawn, F for the amount of water needed per square foot of flowers, and S for the amount of water needed per square foot of shrubs. This leads to these equations:

$$\begin{aligned} 900L + 120F + 40S &= 1865 \\ 160F + 800S &= 180 \\ 120L + 80F + 240S &= 310 \end{aligned}$$

Developing this system will probably be fairly straightforward, but provide an opportunity for questions anyway.

Supplemental Activity

In *The General Two-Variable System* (extension), students develop a formula for the solution to a system of two linear equations.

Elimination in Three Variables

Intent

Students solve several systems of linear equations in three variables.

Mathematics

In *Elimination in Three Variables,* students extend the elimination method to solve three systems of linear equations in three variables.

Progression

The central unit problem involves solving many systems of linear equations in six variables. In this activity, students extend the elimination method to systems of equations in three variables, and they move toward the realization that they'll need to find a more efficient method before they can tackle the unit problem.

Approximate Time

10 minutes for introduction
20 to 25 minutes for group work
10 minutes for discussion

Classroom Organization

Whole-class discussion, followed by small groups, then presentations

Doing the Activity

The three equations from Part I of *Gardener's Dilemma* were a rather intimidating linear system (even if the equations can be simplified by dividing through by a constant).

Elimination in Three Variables presents a more gradual introduction to using the elimination method for three-variable systems. We suggest that you begin by having the whole class work together on Question 1.

Ask, **How might you solve this system of linear equations?**:

$$4x + 2y + z = 9$$

$$2x - y - z = 2$$

$$x + 2y + 3z = 9$$

If a hint is needed, ask, **How can you combine two of these equations to eliminate one of the variables?**

There are two "obvious" options to begin with, based on cases in which corresponding coefficients have the same absolute value:

- Adding the first two equations

- Subtracting the third equation from the first

Suppose, for instance, that students use the first of these methods, getting the equation $6x + y = 11$. Ask, **How can you create a second equation that has only *x* and *y*?** That is, ask if there is another way to eliminate z.

Let students work on this in their groups until someone comes up with an answer, and have that group share its result. For instance, students might triple the second equation and add the result to the third equation to get $7x - y = 15$.

Ask, **What might you do next?** Without getting into the numerical details, guide them to articulate that they now have a system of two linear equations in two variables, x and y, and so they should be able to solve these two equations to get x and y.

With that preparation (or a similar sequence of steps), let students work in their groups to finish solving the system. If groups are stuck after finding x and y, you can suggest that they think about how to use the fact that they know what x and y are, perhaps reminding them of the simple systems they solved for the *More Cookies* problem.

You may want to bring the class together to share results for Question 1 or let students continue within their groups on the other two problems.

Discussing and Debriefing the Activity

Have students from different groups make presentations. If time allows, you might have more than one presentation for each problem, because different groups may have used different combinations of equations. If necessary, this discussion can be combined later with the discussion of *More Equation Elimination*.

Here are the solutions to the systems:

- Question 1: $x = 2$, $y = -1$, and $z = 3$
- Question 2: $u = 3$, $v = 1$, and $w = -1$
- Question 3: $r = -2$, $s = 3$, $t = 1$

Key Questions

How might you solve this system of linear equations?

How can you combine two of these equations to eliminate one of the variables?

How can you create a second equation that has only *x* and *y*?

What might you do next?

More Equation Elimination

Intent
Students practice solving systems of linear equations in three variables.

Mathematics
This activity continues students' work with the elimination method for three-variable systems. Question 4 involves a dependent system; discussion of this question helps students connect the algebra of a dependent system with the geometry of planes in 3-space. The discussion also explores when to use substitution rather than elimination.

Progression
This activity culminates student work on solving systems of equations in three-variables.

Approximate Time
30 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization
Individuals, followed by whole-class discussion

Materials
Transparencies of *Just the Plane Facts* blackline masters

Doing the Activity
Students do this activity independently.

Discussing and Debriefing the Activity
If students seemed comfortable in their work on *Elimination in Three Variables*, you might skip discussion of Questions 1 and 2. (If you did not have time to discuss that activity previously, you can begin with discussion of that activity, and then proceed to this one.)

On the other hand, if you think your students would benefit from seeing more examples in which elimination produces a unique solution, then have students present Questions 1 and 2 carefully.

Here are the solutions to the first two problems:

- Question 1: $a = 3$, $b = -2$, $c = -1$
- Question 2: $u = 5$, $v = 0$, $w = -4$

Question 3
You might make a point of having a presentation of Question 3 because, like the

equations in the central unit problem, the coefficients in these equations are all 0's and 1's.

The solution is $x = 2$, $y = -3$, and $z = 4$.

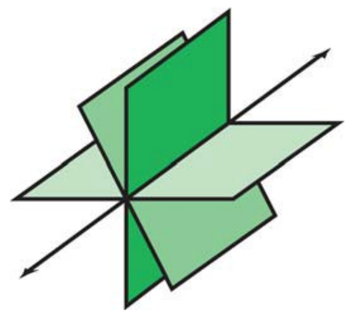
Question 4

Question 4 involves a dependent system, because the third equation is the sum of the first two. If students did not realize this, have a volunteer describe what happened when he or she tried to solve the system. Most likely, the student kept getting a dependent pair of two-variable equations.

For example, the student may have added the first and third equations to get $5d + 3f = 13$ and also added twice the first equation to the second equation to get the same result.

You might tell students to think back to *Just the Plane Facts*, and ask, **What might be happening here geometrically?**

You might also show them transparencies (the *Just the Plane Facts* blackline masters) of the various ways that three planes can intersect, as discussed for Question 3 of *Just the Plane Facts*. Ask, **Which diagram illustrates how the graphs in this problem intersect?** Because no two of the graphs are dependent, there are no two parallel planes. Therefore, the situation is roughly like this diagram:



Remind students that a system like that in Question 4 is called *dependent*. Point out that, in fact, there are infinitely many solutions to this system.

Substitution or Elimination

After discussion of the activity, ask, **Can the substitution method be used for three-variable systems?** You might focus students on any of the three questions in *Elimination in Three Variables*, because the equations in those examples have several coefficients of either 1 or −1. For instance, in Question 1 of *Elimination in Three Variables*, one could use the first equation to write $z = 9 - 4x - 2y$.

Emphasize that students need to substitute this expression for z in *both* of the other equations, giving a new system of two equations in two variables.

Point out to students that they can "mix and match" approaches. For instance, they can start with substitution as just described and then solve the resulting two-variable system using elimination.

Comparing Approaches

You may want to lead a brief discussion comparing different methods, talking generally about solving linear systems and the role of linear systems in analyzing problem situations. **What do you think about different methods of solving**

linear systems? Here are some ideas that might come out:

- Simplifying equations, such as by dividing both sides by a constant, can sometimes make life much easier.
- Sometimes it's easier to work with fractions than with decimals. (This probably isn't true for all students, so tell them to use what's easiest for them. But point out that decimal values from a calculator may yield approximate rather than exact solutions.)
- It's a good practice to check the answers on problems like these that involve a lot of computation.
- Checking that an answer fits the equations is not a check that the answer fits the original problem. The equations may not perfectly represent the problem.
- Solving linear systems can be a lot of work, especially if there are more than two variables. Tell students that later in the unit, they will see a way to get the graphing calculator to do the "dirty work" for them.

Key Questions

What might be happening here geometrically?

Which diagram illustrates how the graphs in this problem intersect?

Can the substitution method be used for three-variable systems?

What do you think about different methods of solving linear systems?

Equations and More Variables in Linear Programming

Intent

In this sequence of activities, students expand their strategy for solving linear programming problems to encompass problems in *n* variables. As they realize that more effort is required to solve linear programming problems with increasing numbers of variables, they learn that another tool is needed to help them reduce the workload. Thus the stage is set for the following activity cluster, *Saved by the Matrices!*

Mathematics

The activities in this unit are focused around two linear programming problems, each more involved than those encountered so far. These problems require that students modify their strategy for solving linear programming problems in order to accommodate three and then four variables (and to include equations as well as inequalities for the constraints). Students then have enough experience to define a strategy for linear programming problems in *n* variables.

Students address related issues as they work with these two problems. They explore how to create systematic lists of combinations of elements, which they will need to identify all the systems of equations necessary for locating all possible corners of the feasible region. They learn how these lists can be pared down by restricting them to combinations that include any constraint equations, as well as how to solve systems of four equations by elimination.

Progression

Ming's New Maneuver is a linear programming problem in three variables. Through this problem, students see the usefulness of reducing the number of systems of equations to be solved by only considering those combinations that include all constraint *equations* (as opposed to inequalities). *Constraints Without a Context* gives students more practice with that concept. *Let Me Count the Ways* helps students see how to create systematic lists of combinations of equations, and *Three Variables, Continued* requires them to think again about how the solutions of systems of three linear equations relate to the geometry of intersecting planes, as in the *More Cookies* problem.

Eastside Westside Story is a linear programming problem in four variables. *Grind It Out* prepares them for this by giving them some experience in solving systems of four linear equations.

Other activities give students more practice with the previously learned skills of fitting a line through two points (*Fitting More Lines*) and using systems of equations to solve word problems (*Ages, Coins, and Fund-Raising*).

Ming's New Maneuver

Let Me Count the Ways

Three Variables, Continued

Grind It Out

POW 3: Crack the Code

Constraints Without a Context

Eastside Westside Story

Fitting More Lines

Ages, Coins, and Fund-Raising

Ming's New Maneuver

Intent

Students practice applying their method for solving linear programming problems without graphing, and discover that a constraint that is an equation simplifies the solution process.

Mathematics

In the central unit problem, some of the constraints are actually equations, and not inequalities. *Ming's New Maneuver* has one such constraint. The main purpose of this activity is for students to see that they can use the presence of an equation among the constraints to reduce the number of combinations they need to check.

Progression

Ming's New Maneuver presents a linear programming problem in three variables with six constraints, one of which is an equation. We suggest that the time allotted for students to work on this problem be split into two parts, with a preliminary discussion in-between regarding how to use the constraint that is an equation to reduce the number of combinations that need to be checked. Having students complete *Let Me Count the Ways* beforehand will facilitate that discussion.

Approximate Time

30 to 35 minutes for initial student work
10 minutes for preliminary discussion
30 minutes for remaining student work
5 to 10 minutes for final discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

We will use this list of constraints in our discussion of the activity:

I $L + C + T = 20$
II $L \geq 3$
III $C + 2T \leq 24$
IV $L \geq 0$
V $C \geq 0$
VI $T \geq 0$

(The choice of variables (L, C, and T) is spelled out in the assignment.)

Although there are 20 ways to choose three constraints from among constraints I through VI, the insistence on including constraint I reduces the number of combinations students need to examine to 10.

Also, because constraint II requires that $L \geq 3$, there is no need for constraint IV. Thus, each set of three constraint equations that students need to check will consist of equation I and two equations from those corresponding to constraints II, III, V, and VI. This further reduces the number of combinations they need to check down to only 6.

Students may or may not see on their own how to reduce the number of combinations. Initially, we urge you to let students explore and muddle along. You will probably find it useful to let students work for about half an hour, have a class discussion concerning the combinations of equations that need to be checked (see "Preliminary Discussion," below), and then allow students to work for another half hour before the final discussion. It will be beneficial if students have completed and discussed *Let Me Count the Ways* before the preliminary discussion takes place.

Discussing and Debriefing the Activity

Preliminary Discussion

Begin the discussion by getting a list of combinations of equations that students need to check to find corner points of the feasible region.

Here are the constraints again:

III $L + C + T = 20$
IV $L \geq 3$
III $C + 2T \leq 24$
IV $L \geq 0$
VII $C \geq 0$
VIII $T \geq 0$

Students, on their own, may have developed the idea that the *equation* constraint (constraint I in the list) should be part of every combination. If so, be sure they explain the reason for this.

If this idea has not been developed yet, make it the focus of the discussion now. You might begin by reminding students that whenever they solve some combination of equations, they need to check that the solution fits the other constraints. For instance, the solution to the combination of constraints I, II, and III is $L = 3$, $C = 10$, $T = 7$. Because these numbers are all nonnegative, they fit the remaining constraints IV, V, and VI, and so (3, 10, 7) is a corner point.

Ask, **Does this point fit the equations corresponding to these constraints? Is that okay?** Be sure students understand that this point does not fit the *equations* corresponding to constraints IV, V, and VI, and that this is fine, but it does fit the constraints themselves.

Constraint I is itself an equation, however, and not an inequality. Because corner points must satisfy all constraints, they must satisfy this equation. That is, if the solution for some combination of constraints fails to satisfy the equation

$L + C + T = 20$, then that solution will not fit the conditions of the problem. So there is no point in looking at combinations of constraints that do not include this equation. (You may find it helpful to consider a two-dimensional analogy in discussing this issue.)

As noted previously, including constraint I in all combinations reduces the number of systems to check from 20 to 10.

In terms of the geometric analysis, constraint I states that the feasible region is not actually a polyhedron, but instead is a region within the particular plane defined by the equation $L + C + T = 20$. The vertices of this region will be places where this plane intersects two other planes defined by the constraints.

Ask students, **How does compiling this list relate to *Let Me Count the Ways*?** Their list of 10 combinations is like the list of 10 committees that could be formed in Question 2b. Both lists start with six items (constraints or students) and have one item that must be included, and in both cases, they want combinations that have three items altogether (including the required one).

Optional: Further Reducing the List of Combinations

Also noted previously, any point that fits constraint II will automatically fit constraint IV. Because one must check constraint II anyway, there is no point in working with constraint IV at all. Thus, students may see that they only need to check these six combinations:

- (I, II, III)
- (I, II, V)
- (I, II, VI)
- (I, III, V)
- (I, III, VI)
- (I, V, VI)

Note: If students do not see this further reduction, there's no harm in that. It simply means that they will have to work with ten combinations instead of six.

Finishing the Problem

Once the list has been shortened as discussed here, have students work in their groups to complete the problem.

Final Discussion

Once several groups are done, you can finish the discussion, having one or two students present their conclusions.

The accompanying table gives the values for L, C, and T that result from considering each of the six combinations in the reduced list. Two of the combinations violate other constraints, as indicated. For the rest of the cases, the final column of the table gives Ming's points, computed using the expression $L + 4C + 5T$.

Combination	*L*	*C*	*T*	Points
I, II, III	3	10	7	78
I, II, V	3	0	17	violates III
I, II, VI	3	17	0	71
I, III, V	8	0	12	68
I, III, VI	−4	24	0	violates II
I, V, VI	20	0	0	20

Thus, the first combination is the best, and the highest score Ming can get is 78 points.

Key Questions

Does this point fit the *equations* corresponding to these constraints? Is that okay?

How does compiling this list relate to *Let Me Count the Ways*?

Let Me Count the Ways

Intent

Students find methods for creating systematic lists to count combinations.

Mathematics

As students begin to work with linear programming problems having larger numbers of constraints and variables, they will need to be able to carefully create lists of all possible combinations of constraints, taken n at a time (where n is the number of variables).

Progression

Let Me Count the Ways asks students to create lists of possible combinations in several different contexts. It will be useful to discuss this activity prior to the preliminary discussion of *Ming's New Maneuver*, which takes place after the initial student work on that activity.

Approximate Time

25 minutes for activity (at home or in class)
5 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students will work on this activity independently, and then share their ideas in class discussion.

Discussing and Debriefing the Activity

Let one or two volunteers share their methods for each of the problems in the activity. Be sure the presenters explain how they were sure they hadn't left out any possibilities. Try to elicit at least a couple of different methods for creating systematic lists.

Supplemental Activity

Playing Forever (reinforcement) gives students further practice with this topic.

Three Variables, Continued

Intent

Students examine the relationship between intersections of planes and solutions to systems of three-variable linear equations.

Mathematics

Three Variables, Continued gives students several systems of linear equations in three variables. Students are asked to state what type of intersection the graph of each system has and to find any unique solutions.

Progression

This activity continues the work on three-variable systems, with a particular focus on connecting the algebra to the geometry of *Just the Plane Facts*.

Approximate Time

30 minutes for activity (at home or in class)
20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

3-D graphing software, such as *Grapher*, which is installed in the Applications: Utilities folder of many Apple computers (optional)

Doing the Activity

Students continue their work solving three-variable systems of equation, but now also first identifying the type of intersection—no points, one, or infinite.

Discussing and Debriefing the Activity

You may want to have volunteers make presentations on each of these problems. Each presenter should not only state what type of intersection the three graphs have but also explain how he or she reached that conclusion. You might also have presenters describe what the intersections are for a given *pair* of planes from the problem.

3-D graphing software can provide a vivid visual confirmation of students' findings.

Question 1

Students should see that this system is inconsistent, and so the graphs have no points in common. They might realize this by noticing that the left side of the third equation is the sum of the left sides of the first two equations, but that observation isn't essential. Geometrically, the line formed by the intersection of the first two planes is parallel to the third plane, as in Diagram 3 for case a in *Just the Plane Facts*.

Question 2

Students should see that this system has infinitely many solutions. (If they try to work with this system using elimination, they will probably end up with the equation $0 = 0$.)

As with Question 1, students might realize what's going on with these planes from simply examining the left sides. Here, the three equations are equivalent, and so all three equations have the same plane for their graph.

Question 3

This is, in a sense, the simplest example to work with, because it has a unique solution: $u = 1$, $v = 2$, $w = 3$. In other words, the graphs intersect in a single point.

Question 4

The simplicity of the first equation in this system may be difficult for students to work with. If students need a hint, ask, **What does the first equation tell you about *x*? How could you use that information?**

Guide them to realize that they can substitute 3 for x in the third equation. This gives $18 + 6y + 2z = 4$, which simplifies to $6y + 2z = -14$. Students can then see that this is equivalent to the second equation.

Students should conclude that the system is dependent and must have infinitely many solutions.

Key Questions

What does the first equation tell you about *x*? How could you use that information?

Grind It Out

Intent
Students solve a system of linear equations in four variables. In the introductory discussion, students also complete the development of a general strategy for solving linear programming problems with several variables.

Mathematics
Students have seen that in order to solve linear programming problems with two variables they must look at solutions of pairs of equations, and that to solve linear programming problems with three variables they must examine solutions of triples of equations. Led by the discussion that introduces this activity, students will now generalize their strategy to solving linear programming problems in n variables by considering the solutions to systems of n equations.

One goal of this activity is to motivate students to want to use graphing calculators and matrices to solve systems of equations. Another is to get students to think about why a system of n equations in n variables "usually" has a unique solution.

Progression
Students will generalize the strategy for solving systems of linear equations to encompass problems having n variables, including the principle that all constraint equations (as opposed to inequalities) should be included in every system being solved. They'll then solve a system of equations in four variables.

Approximate Time
10 minutes for introduction
25 to 35 minutes for activity (at home or in class)
10 to 15 minutes for discussion

Classroom Organization
Individuals, preceded and followed by whole-class discussion

Doing the Activity

The New General Strategy
Tell students that the idea of including all constraint *equations* in every combination they check is an important element to add to their strategy for solving linear programming problems.

Remind them again, however, that the unit problem has six variables. When they last discussed the general strategy (in the discussion following *The "More Cookies" Region and Strategy*), they talked about how to find corners for a three-variable problem. Now ask, **How can you generalize the process of finding corners from two or three variables to *n* variables?**

They should build on the cases of two and three variables by seeing that for two variables, they solved *pairs* of linear equations, and for three variables, they solved *triples* of linear equations. They will probably generalize to the fact that to get "corner points" for *n* variables, they need to solve systems of *n* linear equations.

Help them to see, in terms of the algebra of solving systems of linear equations, that a system of *n* equations in *n* variables will usually have a unique solution. The repeated use of either the substitution method or the elimination method should help make this clear.

Students' new strategy should also incorporate the observation (discussed in *Finishing Off the Cookies*) that if a system does not have a unique solution, it can be ignored.

You may want to generate several iterations of the strategy as students refine and amend one another's ideas. Use the problems that students have already worked as a "testing ground."

A Strategy for n Variables
The strategy should evolve into something like this:

1. List the constraints, both equations and inequalities (including constraints that make each variable nonnegative).

2. Make a list of all combinations of constraints, *n* at a time, that include every constraint equation (where *n* is the number of variables).

3. For every such combination, look for a common solution to the corresponding system of linear equations. If a system has a unique solution, that solution becomes a "potential corner point". (If a system does not have a unique solution, ignore that combination of constraints.)

4. Test all the "potential corner points" to see if they fit *all* the constraints. Those that do become the "actual corner points".

5. Evaluate the profit or cost function for every "actual corner point".

6. Identify the corner point that maximizes or minimizes the given function.

Post this latest version of the general strategy. (You may want to have each class compare the strategy it develops with those developed by other classes so they can see different ways to express these ideas.)

Be sure students are convinced that their strategy works for two-variable and three-variable problems, and that they can explain it graphically. You may want to assure them that they aren't expected to be able to visualize graphs in four or more dimensions, but they should have some confidence that the same general process will work.

You should also point out that students have not *proved* that this method works in general. Without proof, it’s hard to be sure about anything geometric involving more than three dimensions, but they should agree that it seems reasonable. You can tell them that a mathematical proof is possible.

Identifying the Trouble Spots

Ask, **What do you think will be the hardest parts of carrying out this plan for the central unit problem?** They will probably point to step 3—solving *n* linear equations in *n* unknowns—as the most difficult component. Tell them that they are going to learn an easy way to do this, in which graphing calculators will do most of the work.

They may also think that keeping track of all the different combinations of equations and inequalities might present a problem. You can remind them that they can rule out many of the cases through common sense or close observation. They shouldn’t hesitate to do this, and they’ll see examples of this soon.

Discussing and Debriefing the Activity

Let several students present their work for the system in Question 1. Probably, they used a variety of sequences of equations. The solution to the system is $x = 2$, $y = 3$, $z = -1$, $w = -4$.

One of the purposes of this problem is to impress on students that this can be unpleasant work, so that they are motivated to learn how to use matrices and graphing calculators. You should assure them that in the real world, people don’t often have to do this sort of thing by hand.

Then let students share their ideas on Question 2. You can allow this discussion to proceed as long as it seems fruitful, but there is no particular explanation that you should expect.

One possible explanation involving elimination is:

You can combine one of the equations with each of the other three, eliminating the same variable in each case. (This is especially easy to do using the fourth equation in this problem.) This will give three equations in the remaining three variables, and this new system will “usually” have a unique solution.

Key Questions

How can you generalize the process of finding corners from two or three variables to *n* variables?

What do you think will be the hardest parts of carrying out this plan for the central unit problem?

POW 3: Crack the Code

Intent

Students solve a complex logic problem and explain their reasoning in a POW write-up.

Mathematics

This POW has no direct connection to the mathematics of this unit, but provides students with practice in careful reasoning, applying problem-solving strategies, and communicating in a thorough and convincing manner. *Crack the Code* asks students to decipher the codes in a number of problems in which letters have been substituted for the digits in arithmetic problems.

Progression

As you assign this POW, you will need to sure that students understand how this type of problem works.

Give students about a week to work on the POW at home, and then have several students present the solutions.

Approximate Time

15 minutes for introduction
1 to 3 hours for activity (at home)
20 minutes for presentations

Classroom Organization

Individuals, followed by several student presentations

Doing the Activity

Let students read the POW through the statement of the rules, and then have them propose possible solutions to this problem:

$$\begin{array}{r} AB \\ AB \\ \underline{+\ AB} \\ CD \end{array}$$

They should see that there is more than one solution. (For instance, this letter problem could represent the equation 18 + 18 + 18 = 54 or the equation 21 + 21 + 21 = 63.) However, they should also see that not every replacement of the letters by numbers will work. For example, A cannot be more than 3, because that would lead to a three-digit sum.

They also need to be careful not to use the same digit for two different letters. For example, the equation 13 + 13 + 13 = 39 will not work because that would set both B and C equal to 3.

Proving That You Have All the Solutions

Emphasize to students that they must find *every* solution to the problems in the POW and must *prove* that they have all the solutions. To get them started on this type of reasoning, you might have them find the complete set of solutions for the preceding problem and give a clear explanation of how they know that they have them all. (There are ten solutions.)

On the day before the POW is due, select students to make presentations on the following day. Because Question 4 is particularly difficult, you may want to ask for a volunteer to make a presentation on that problem. You can then choose three other students to make presentations, with each student focusing on one of Questions 1 through 3. Give the presenters overhead transparencies and pens to take home to use for preparing presentations.

Discussing and Debriefing the Activity

Let the selected students make their presentations. You should focus the discussion on clarity of reasoning, because the numerical solutions themselves are not that interesting. In particular, students should be proving that they have found all the solutions.

Also, have students describe how they actually found their solutions. In some cases, students' descriptions of how they came up with the solutions will essentially be a proof that they have found all the solutions.

Questions 1 and 2 are mainly intended to get students started and to ensure that they understand the rules. These problems both have unique solutions.

Question 3 requires that A = 1, B = 8, and D = 0, but it leaves several possibilities for C and E. Be sure that students cover all the cases and explain why no others exist.

Question 4 has a unique solution, but it requires some work and careful organization. The fact that different letters have to represent different numbers is important in this problem.

Supplemental Activity

Cracking Another Code (reinforcement) gives students another code problem of this type, but this time with a multiplication problem involved.

Constraints Without a Context

Intent
Students solidify their understanding of the concepts and mechanics of finding corner points for a three-variable feasible region.

Mathematics
Constraints Without a Context presents a set of eight constraints in three variables. Students list the systems of equations that would need to be solved in order to find all the corner points for the feasible region, and then explain how they would use their list to do so.

Progression
Students work on this activity individually, then share their results in class discussion. The discussion reiterates these points:

- Corner points must fit any constraints that are equations
- Some systems are "obviously" inconsistent
- Potential corner points should be checked in all the constraints

Approximate Time
25 minutes for activity (at home or in class)
10 to 15 minutes for discussion

Classroom Organization
Individuals, followed by whole-class discussion

Doing the Activity
Students work on this activity independently.

Discussing and Debriefing the Activity
Here is the list of constraints:

a. $x + 2y + z \le 20$
b. $3x + 2z \le 12$
c. $y + z = 8$
d. $x \le 10$
e. $y \le 7$
f. $x \ge 0$
g. $y \ge 0$
h. $z \ge 0$

Let volunteers offer combinations of equations that they would check to find the corner points of the region. All systems should include constraint (c), because this is an equation. Therefore, as a first step, students should consider all combinations involving constraint (c) and the equations corresponding to any two of the other constraints. (This gives 21 possibilities.)

Be alert for systems that are "obviously" inconsistent, such as (c), (d), and (f). If someone suggests including this as a system that might yield a corner point and no one objects, ask students to solve it. They should see that the equations $x = 10$ and $x = 0$ are inconsistent. The combination (c), (e), and (h) is a bit more subtle, because no two of the corresponding equations are inconsistent, but students should see that the system as a whole is inconsistent.

If students eliminate cases like (c), (e), and (h), they might end up with this list of 17 systems:

- (a), (b), and (c)
- (a), (c), and (d)
- (a), (c), and (e)
- (a), (c), and (f)
- (a), (c), and (g)
- (a), (c), and (h)
- (b), (c), and (d)
- (b), (c), and (e)
- (b), (c), and (f)
- (b), (c), and (g)
- (b), (c), and (h)
- (c), (d), and (e)
- (c), (d), and (g)
- (c), (d), and (h)
- (c), (e), and (f)
- (c), (f), and (g)
- (c), (f), and (h)

We recommend that you *not* take the time to solve each system, but emphasize (in the discussion of Question 2) that any solution to one of these systems should be checked to see if it fits all the constraints. If you think you need to illustrate this principle with an example, the system (c), (f), and (g) is a good one to use, because it is easy to solve but its solution, (0, 0, 8), violates constraint (b).

Eastside Westside Story

Intent

Students solve a linear programming problem involving four variables.

Mathematics

It requires considerable time to find the solution to this problem, but it is class time well spent. As students repeatedly work through the mechanics of creating a list of systems of equations to be solved, solving each system of four equations, and then checking each solution against the constraints and against the profit function, they will thoroughly internalize this procedure. Afterward, as they consider the amount of effort that was required, they will realize that they do not want to do this for the central unit problem, with six variables. This realization will motivate the study of matrices.

Progression

The discussion following the activity includes consideration of the issue of fractional solutions.

Approximate Time

60 minutes for activity
15 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

This problem involves checking a lot of cases. Initially, students write the constraints and the cost expression (Questions 1 and 2) and decide what combinations of equations they need to check (Question 3). The remaining work, finding solutions to combinations of equations (Question 4) and finding the feasible solution with minimal cost, may need to be split between two days due to the length of this assignment.

If students are having difficulty, you might hold a whole-class discussion about the constraints and the process of listing all the cases so that no combinations are omitted. We will number the constraints like this:

I	$N_e + R_e = 300$
II	$N_w + R_w = 250$
III	$N_e + N_w \leq 350$
IV	$R_e + R_w \leq 225$
V	$N_w \geq N_e$
VI	$N_e \geq 0$
VII	$R_e \geq 0$
VIII	$N_w \geq 0$

IX $R_w \geq 0$

Students should know from the general strategy that they need to examine the corresponding linear equations in sets of four. But based on their experience with *Ming's New Maneuver*, they should also realize that because constraints I and II are equations, each set of four linear equations to be examined must include those two equations, as well as the equations corresponding to two of the remaining seven inequalities.

This gives a total of 21 systems to consider, each consisting of four linear equations. Have students compile a list of all 21 cases. They may be able to rule out some of these combinations without actually solving the systems. (For example, the equations corresponding to constraints VI and VII cannot both hold true, because that would contradict constraint I.)

Discussing and Debriefing the Activity

As stated previously, there are potentially 21 sets of four equations that must be checked—namely, all combinations of the first two constraints with any two of the remaining seven. If they haven't already done so, have the class create this list. As noted above, some cases can easily be ruled out as impossible, and students may choose to do this.

With whatever cases remain, ask different groups for the solution to each system of four linear equations.

Also, ask for the linear function that expresses the cost to the city for busing ($1.2N_e + 2R_e + 3N_w + 1.5R_w$) and compute the cost in each case.

For your convenience, the table that follows lists all 21 combinations of constraints. For each combination, the table either gives the solution to the corresponding system of linear equations or states that the system is inconsistent (that is, that there is no solution).

For those systems that have a solution, the last column of the table either states that the solution violates one or more of the remaining constraints or gives the busing cost for the solution.

Combination	N_e	R_e	N_w	R_w	Constraint(s) violated or cost
I, II, III, IV	inconsistent				
I, II, III, V	175	125	175	75	$1,097.50
I, II, III, VI	0	300	350	−100	violates IX
I, II, III, VII	300	0	50	200	violates V
I, II, III, VIII	350	−50	0	250	violates V, VII
I, II, III, IX	100	200	250	0	$1,270
I, II, IV, V	162.5	137.5	162.5	87.5	$1,088.75

I, II, IV, VI	0	300	325	−75	violates IX
I, II, IV, VII	300	0	25	225	violates V
I, II, IV, VIII	325	−25	0	250	violates V, VII
I, II, IV, IX	75	225	250	0	$1,290
I, II, V, VI	0	300	0	250	violates IV
I, II, V, VII	300	0	300	−50	violates III, IX
I, II, V, VIII	0	300	0	250	violates IV
I, II, V, IX	250	50	250	0	violates III
I, II, VI, VII	inconsistent				
I, II, VI, VIII	0	300	0	250	violates IV
I, II, VI, IX	0	300	250	0	violates IV
I, II, VII, VIII	300	0	0	250	violates IV, V
I, II, VII, IX	300	0	250	0	violates III, V
I, II, VIII, IX	inconsistent				

The table shows there are only four corner points for the feasible region for this problem. The apparent best solution to the problem is $N_e = 162.5$, $R_e = 137.5$, $N_w = 162.5$, and $R_w = 87.5$, which gives a cost of $1,088.75. (*Reminder:* The goal is to *minimize* the cost.)

Of course, you can't send one-half of a student to one school and the other half to another, so this isn't really the answer. Clearly, rounding off the optimal solution is still better than any of the other corner points, but it is not at all obvious how best to round off that solution.

Because of constraints I and II, if N_e goes up, then R_e goes down, and vice versa, and similarly for N_w and R_w. So there are four cases to consider. The values $N_e = 162$, $R_e = 138$, $N_w = 162$, and $R_w = 88$ violate constraint IV; and the values $N_e = 163$, $R_e = 137$, $N_w = 162$, and $R_w = 88$ violate constraint V. So the two cases that fit all the constraints are

$N_e = 162$, $R_e = 138$, $N_w = 163$, $R_w = 87$

and

$N_e = 163$, $R_e = 137$, $N_w = 163$, $R_w = 87$

The first of these gives a cost of $1,089.90 and the second gives a cost of $1,089.10 so the latter is the least expensive arrangement.

You should point out that the difference in cost between the two cases that fit the constraints is fairly trivial. You may also want to review the ideas discussed following *Finishing Off the Cookies* (in the section "Whole Dozens or Not?"). The essence of that discussion is that the general theory may break down when we are restricted to whole numbers. But here, the apparently best whole-number solution costs $1,089.10 and the best of all solutions (if we consider all other lattice points) costs $1,088.75, so the whole-number solution we seem to have found would at worst cost an extra 35¢.

In real life, the cost of searching for a possible improvement would be far greater than the savings if an improvement were found.

Supplemental Activities

Producing Programming Problems—More Variables (extension) asks students to invent and solve a linear programming problem in four or more variables.

Fitting More Lines

Intent

Students find a linear function that goes through two specific points by treating the coefficients as variables.

Mathematics

This activity continues the ideas of *Fitting a Line*, but with points for which finding a relationship between a and b is more difficult. Although some students may recall the "substitution" approach that was part of the discussion of *Fitting a Line*, you can expect that most will look for a pattern among specific examples again. The difficulty of finding the pattern should further motivate students to use the more systematic method.

Progression

This activity continues the ideas of *Fitting a Line*. Students will apply these techniques further in *Fitting Quadratics* and *Fitting Mia's Bird Houses*.

Approximate Time

30 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students should work on this activity independently.

Discussing and Debriefing the Activity

Let volunteers share their work on Questions 1 and 2, describing how they found the relationship. Unless they used the substitution approach, they are likely to express b in terms of a and get this pair of equations:

$$b = 4 - 3a$$

$$b = 1 - 5a$$

(If students used the substitution approach, they will probably express the equations in a different form, though the equations should be equivalent to those above.)

Question 3

In Question 3, the main idea to bring out is that solving the pair of equations from Questions 1 and 2 leads to a function that goes through both (3, 4) and (5, 1). In this case, the desired equation is $y = -\frac{3}{2}x + \frac{17}{2}$ (or $y = -1.5x + 8.5$). Be sure to have students verify that both (3, 4) and (5, 1) fit the equation.

Also emphasize that $a = -\frac{3}{2}$, $b = \frac{17}{2}$ is the *only* pair of numbers that fits the equations from Questions 1 and 2 (that is, the equations $b = 4 - 3a$ and $b = 1 - 5a$ or their equivalents). Connect the uniqueness of this solution to the fact that there is a unique line through the points (3, 4) and (5, 1).

Question 4
Question 4 involves the identical process used in Questions 1 through 3. Use your judgment about how much discussion is needed.

The equations for a and b should be equivalent to the system $-1 = 4a + b$ and $7 = -2a + b$. The solution is $a = -\frac{4}{3}$, $b = \frac{13}{3}$, so the desired equation is

$y = -\frac{4}{3}x + \frac{13}{3}$.

Using Substitution Again
If no one used the substitution approach on this assignment, ask, **Is there a simpler way to get the equations for *a* and *b*?**

As needed, review the substitution approach. Bring out, for instance, that having $y = ax + b$ go through (3, 4) (in Question 1) means that substituting 3 for x and 4 for y must give a true statement. Help students see that it's easy to simply write down the result of this substitution (that is, $4 = a \cdot 3 + b$) and that this equation is equivalent to the relationship they found in Question 1a.

You might give students a new pair of points on which to try out this method. Or you can simply let them verify that this works on the points in Question 4.

The Purpose of This Approach
The approach of treating coefficients as variables may seem somewhat artificial to students. To bring out its purpose, ask, **Why might you do equation fitting this way?** Remind them of examples in which they did something like this (or ask them for examples).

Students have seen many situations in which they knew that a linear function was appropriate. The principal concept is that in solving real-world problems, we often have a general idea of the *type* of equation that should be used to fit the data.

Tell students that coefficients such as a and b are often called *parameters,* which represent the specific details within a certain "family" of functions. In this case, the family is the set of linear functions, and the parameters a and b tell us which line we are considering. The key idea is to treat these parameters as the unknowns and to use the known data—here, the two given points—to create equations involving the parameters.

Key Questions

Is there a simple way to get the equations for *a* and *b*?

Why might you do equation fitting this way?

Supplemental Activity

Fitting a Plane (extension) extends the idea of this activity to three variables and equations of planes. You might assign this activity now, or wait until students have completed *Fitting Quadratics.*

Ages, Coins, and Fund-Raising

Intent
Students solve word problems by setting up and solving systems of linear equations.

Mathematics
Students gain more experience both in setting up equations to describe problems and in solving systems of linear equations.

Progression
Students work on this activity individually, then share their results as a class if needed.

Approximate Time
25 minutes for activity (at home or in class)
5 minutes for discussion

Classroom Organization
Individuals, followed by whole-class discussion (if needed)

Doing the Activity
Students should work on this activity independently.

Discussing and Debriefing the Activity
Use your judgment about how much time to spend discussing this activity. No new ideas are introduced.

Here are the answers to the problems:

- Paige is 6, Quiana is 12, and Ryan is 15.
- Uncle Ralph has 5 quarters, 4 dimes, and 9 nickels.
- They sold 20 pies, did 50 car washes, and sold 400 raffle tickets.

Supplemental Activity
Your Own Three-Variable Problem (extension) asks students to make up a problem that can be solved using three variables and three linear equations.

Saved by the Matrices!

Intent

In this section, students learn to solve systems of linear equations using matrices.

Mathematics

Solving a system of six linear equations by substitution or elimination is quite daunting, and doing so repeatedly to solve the central unit problem would be particularly challenging. Having wrestled with this task in four variables in *Eastside Westside Story*, students are now thoroughly motivated to look at another method. Fortunately, matrices—combined with the power of the graphing calculator—offer the much-needed shortcut.

Students learn to use matrices to represent information, develop the definitions of matrix addition and multiplication, and examine whether matrix operations have properties such as associativity and commutativity. They learn that a system of linear equations can be represented with a matrix equation, and they recognize the role of identity and inverse elements in solving certain types of matrix equations. Students then learn to use a graphing calculator to perform operations with matrices, including using matrix inversion to solve systems of linear equations.

This section also continues the theme of curve fitting, as students use the method introduced in *Fitting a Line* and *Fitting More Lines* to fit a quadratic equation to a set of points, in *Fitting Quadratics* and *Fitting Mia's Bird Houses*.

Progression

The activities initially define matrices and establish much of the related vocabulary in *Matrix Basics*, and then develop matrix addition in *Inventing an Algebra* and matrix multiplication in *Flying Matrices*, *Matrices in the Oven*, *Fresh Ingredients*, and *Make It Simple*. *Calculators to the Rescue* introduces students to performing matrix multiplication on the calculator, and *Things We Take for Granted* establishes that matrix multiplication differs from numerical multiplication in that it is noncommutative.

With *Back and Forth* and *Matrices and Linear Systems*, students begin to represent a system of linear equations with a single matrix equation. Understanding how to solve these equations requires development of the concepts of the multiplicative identity matrix and the inverse matrix. *Solving the Simplest* introduces the multiplicative identity, and *Finding an Inverse* introduces the inverse.

Of course, students quickly discover that finding the inverse of the coefficient matrix for a system of linear equations is not much quicker than solving the system without matrices. Once more, calculators come to the rescue as students discover in *Calculators Again* the simplicity of using the calculator to find the inverse matrix and then solve the matrix equation.

Matrix Basics

Inventing an Algebra

Fitting Quadratics

Flying Matrices

Matrices in the Oven

Fresh Ingredients

Calculators to the Rescue

Make It Simple

Back and Forth

Matrices and Linear Systems

Solving the Simplest

Things We Take for Granted

Finding an Inverse

Inverses and Equations

Calculators Again

Fitting Mia's Bird Houses

Matrix Basics

Intent

This activity introduces matrices.

Mathematics

Motivated by their experiences with the time-consuming process of solving linear programming problems in three and four variables, students are now introduced to matrices. Matrix algebra will prove to be a tremendous time-saving tool for solving large systems of linear equations. In this activity, students are presented with the basic principles of expressing equations as matrices, and related vocabulary.

Progression

Matrix Basics provides a page of reference material to be introduced by the teacher. In the discussion, students see that solving systems of linear equations involves working with the coefficients and constants, so it is more efficient to express those equations in a form that displays only those elements. They then learn how to write matrices and use related vocabulary.

Approximate Time

5 minutes for discussion

Classroom Organization

Teacher presentation

Doing the Activity

Students have seen the importance of being able to solve systems of linear equations and should recognize that this gets progressively harder as more variables are involved. They have also seen that some problems require solving many systems of equations. *Grind It Out* and the numerous combinations to work through on *Eastside Westside Story* should convince them of the value of finding an easier way to solve such systems.

Ask the class for suggestions about how to solve this system of equations:

$$x + y + z = 6$$

$$x + 2y - z = 1$$

$$2x + 3y + z = 10$$

After students have suggested a few steps to take in this process, ask, **What are you primarily working with as you solve the system?** (That is, are they doing anything to the variables x, y, and z?) As needed, help them to see that they are working only with the coefficients. (Students do not need to finish solving this system—simply have them go far enough to see that they are working with the coefficients, not the variables.)

Inform the class that there is a way to represent the coefficients of a system, using an array:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}$$

You can also represent the constant terms (from the right side of each equation) in their own array, like this:

$$\begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix}$$

Then, rather than work with the entire equations, they would simply do the same arithmetic on the numbers in these arrays.

Inform the class that rectangular arrays like these are called **matrices** (the singular is **matrix;** point out that this is similar to *vertex* and *vertices*) and that systems of linear equations can be written using matrices and solved using *matrix arithmetic.*

Tell them that they will now start learning matrix arithmetic, which is usually called *matrix algebra,* and that this will turn out to be a much more efficient way to solve systems of linear equations.

Discussing and Debriefing the Activity

Matrix Basics introduces some further terminology for matrices. You should discuss these terms, especially the way of denoting the dimensions of a matrix.

The activity and associated discussion introduce these vocabulary and concepts:

- The matrix as a rectangular array
- Rows and columns
- Matrix dimensions
- Square matrices
- Row and column vectors
- Matrix entries
- The matrix of coefficients and the matrix of constant terms
- Matrix algebra

Key Question

What are you primarily working with as you solve the system?

Inventing an Algebra

Intent

Students develop the definition of matrix addition.

Mathematics

The contexts of recording student grades and cookies in the Woos' bakery (from *More Cookies*) lend themselves to a natural development of matrix addition.

Progression

This activity uses problem contexts to lead students to a definition of matrix addition. The discussion helps students realize that representing data with a matrix often involves arbitrary decisions about row and column assignment, and that only matrices of the same dimensions can be added.

Approximate Time

25 minutes for activity
10 to 15 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

In this activity, students will look at some familiar situations in which matrices can be useful. Based on these examples, they will develop general rules for working with matrices.

Discussing and Debriefing the Activity

Have students from various groups give their results for different problems.

During this discussion, review the terms *row* and *column* and the terminology and notation for the dimensions of a matrix (for instance, "3 x 4," read as "3-by-4"). You might suggest the idea of an architectural column, which is vertical, as a way of remembering the difference between rows and columns.

Question 1

On Question 1, ask, **Why does Ana get the first row? Is there another way to set up this matrix? Why might you choose to set up the data one way or the other?** For instance, the data could appear like this, with the rows and columns reversed:

	Ana	Ben	Cass	Devon
Homework	18	35	46	60
Reports	54	23	15	60
POWs	30	52	60	60

Help students realize that the setup of the matrix is somewhat arbitrary. You can ask why one might choose to set up the data one way or another. For instance, a teacher might decide based on how some computer software works or on how a grade book is formatted. Other factors might include the number of people in the class, the number of grading categories, and even the size and shape of the paper on which the data is to be recorded.

Point out that it would be confusing to switch systems midway through a marking period, so once the decision is made, one should be consistent. Therefore, the matrices in Questions 1a and 1b should have the same shape and organizing principle as the matrix for the original data.

Be sure that students actually write out the formal matrix sum equation:

$$\begin{bmatrix} 18 & 54 & 30 \\ 35 & 23 & 52 \\ 46 & 15 & 60 \\ 60 & 60 & 60 \end{bmatrix} + \begin{bmatrix} 10 & 60 & 0 \\ 52 & 35 & 58 \\ 42 & 20 & 48 \\ 60 & 60 & 60 \end{bmatrix} = \begin{bmatrix} 28 & 114 & 30 \\ 87 & 58 & 110 \\ 88 & 35 & 108 \\ 120 & 120 & 120 \end{bmatrix}$$

Ask, **How could the sum be found directly in terms of the matrices?** (that is, without reference to the original tables of data). Students should see that the arithmetic done with the individual entries reflects the arithmetic done with each student's scores, and that each number in the final matrix—the sum—has a meaning.

For example, ask, **What does the number 108 in the sum represent?** Students should note that this is the total of Cass's points for POWs in the first and second marking periods combined. Without any explicit labels, there is no way to know what the numbers mean, so a matrix out of context can be useless.

Question 2

On Question 2, you might again ask, **Could you have the rows represent 'day of the week' and the columns represent 'kind of cookie'?** Students should see that it was an arbitrary choice, but they should also recognize the importance of keeping the same method throughout the problem. That way, the matrices "match up," and the entries can be added as in Question 1. (It also makes sense to keep the days of the week in their natural order, although the order of the types of cookies is arbitrary.)

Question 3

There may be some disagreement on Question 3. A student might claim, for example, that we can think of both matrices in Question 3b as 3 x 4, with a missing row or column of zeroes, or that the row and column vectors in Question 3d can be added because they have the same number of entries.

A good counter-argument is that the definitions are ultimately a matter of convention and usefulness, and it would be ambiguous or confusing to add matrices of different shapes. For example, in Question 3b, we couldn't be sure where the missing zeroes belong, and in Question 3d, we wouldn't know what the shape of the sum should be.

Question 4

Question 4 summarizes the ideas that likely arose in discussing Question 3. Take this opportunity to get students to articulate the principle:

Two matrices can be added if and only if they have the same dimensions.

Question 5

Students may have already articulated the idea of "adding corresponding entries" through their discussion of Questions 1 through 3, but it's worth having them summarize it again here.

Subtracting Matrices

Ask, **When can you subtract matrices? How do you do it?** Students will probably say that the conclusions they have reached about adding matrices of the same dimensions apply equally well to subtraction. Tell them that this is indeed the case.

Key Questions

Why does Ana get the first row? Is there another way to set up this matrix?

Why might you choose to set up the data one way or the other?

How could the sum be found directly in terms of the matrices?

What does the number 108 in the sum represent?

Could you have the rows represent 'day of the week' and the columns represent 'kind of cookie'?

When can you subtract matrices? How do you do it?

Fitting Quadratics

Intent

In this activity, students find the quadratic function that defines the parabola that will pass through specific points.

Mathematics

This activity continues the theme of *Fitting a Line* and *Fitting More Lines*. Students now use the strategy of treating the coefficients as variables to fit a quadratic function to a set of given points.

Progression

After a brief introduction, students work on this activity individually, then share their results in class discussion.

Approximate Time

5 minutes for introduction
25 to 30 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Have students look at Question 1 of this activity, and examine how this activity is similar to *Fitting a Line* and *Fitting More Lines*. You may want to review the substitution method described in the discussions of those activities.

Discussing and Debriefing the Activity

The mechanics of these questions are quite similar to students' previous work with fitting lines.

Have volunteers present the various parts of Questions 1 through 3. They should come up with this linear system (or equivalent equations):

$$a + b + c = 6$$

$$9a + 3b + c = 8$$

$$25a + 5b + c = 2$$

If many students had trouble finding the three conditions, you may want to give groups some additional time to work on finding the solution to the system. (The solution is $a = -1$, $b = 5$, and $c = 2$.)

Once students find the values for a, b, and c and combine them to create the quadratic function $y = -x^2 + 5x + 2$, have them verify that it actually fits the three given points. Discuss the fact that the system of linear equations has a unique solution (Question 4b). This illustrates the principle that "usually" there is a unique quadratic function that goes through a given set of three points.

You might have students think about what geometric relationship among the three points wouldn't form a quadratic function. (If two of the points have the same *x*- coordinate but different *y*-coordinates, then there is no quadratic function that passes through all three points—in fact, by the definition of *function,* there is no function of any kind that passes through all of them.)

If the three points have distinct *x*-coordinates, then the linear system for the coefficients *a*, *b*, and *c* will have a unique solution. If the points have distinct *x*-coordinates and are also collinear, then that unique solution will have $a = 0$, which means that the resulting function is linear rather than quadratic. Therefore, there is no quadratic function passing through the three points. If the points have distinct *x*-coordinates and are not collinear, then there is a unique quadratic function that passes through them.

Supplemental Activities

Fitting a Plane (extension) has students use a similar technique to find the equation of a plane defined by three points.

Flying Matrices

Intent

Students begin to develop the definition of multiplication of matrices.

Mathematics

The definition of matrix addition is fairly straightforward, while matrix multiplication is more complicated and somewhat harder to motivate. The activity *Flying Matrices* motivates the definition of matrix multiplication by having students examine the arithmetic involved through the context of a familiar problem.

Progression

Through this activity and *Matrices in the Oven,* students will establish the formal definition of matrix multiplication.

Approximate Time

25 to 30 minutes for activity
15 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

No introductory discussion is needed for this activity. Students will likely come up with nonstandard ways to set up these problems. That's okay— they'll learn the standard procedure in the discussion of the next activity, *Matrices in the Oven*. They should see that the standard procedure gives the same results as their alternate methods.

Bring the class together for discussion when all groups have at least finished Questions 1 through 4, and some are finished with the entire activity.

Discussing and Debriefing the Activity

For each of Questions 1 through 3 of *Flying Matrices,* have two or three students report on how they set up the matrices. As you get various answers, record them on overhead transparencies or chart paper so you can refer to them later.

Be prepared for a wide variety of methods for setting up matrices, perhaps including methods that are internally inconsistent. Although this variety may create some confusion, it will help students see that some steps they take in the setup process are arbitrary, while others need to be done a certain way. As students gain experience, they will see what makes sense.

Here is one *standard* way to set up Questions 1 through 3. (Labels have been added for clarity.)

For Question 1:

$$\begin{array}{c} \\ \text{feed} \\ \text{calcs} \end{array}\begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 40 & 2 \\ 50 & 3 \end{bmatrix} \end{array}$$

For Question 2:

$$\begin{array}{c} \\ \text{Mon} \end{array}\begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 500 & 200 \end{bmatrix} \end{array}$$

For Question 3:

$$\begin{array}{c} \\ \text{Mon} \end{array}\begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 30{,}000 & 1600 \end{bmatrix} \end{array}$$

But students certainly could have arranged the numbers differently. For example, Question 1 could be set up this way.

$$\begin{array}{c} \\ \text{wt} \\ \text{vol} \end{array}\begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 40 & 50 \\ 2 & 3 \end{bmatrix} \end{array}$$

Also, at this point, the only reason the *feed* column (or row) comes before the *calculator* column (or row) is that the information was presented in that order in the problem. As long as the four numbers are included in an organized fashion, the answer should be considered correct at this point. Similarly, the answers to Questions 2 and 3 could be presented as column vectors instead of row vectors.

Presumably, students will all come up with the same numerical totals for Question 3, even if they display the information differently.

Question 4

The explanations in Question 4 are a crucial part of this activity, so have several students describe what they did.

No matter how students displayed the information from Questions 1 through 3, they should have multiplied the appropriate entries of the matrix for Question 2 by the appropriate entries of the matrix for Question 1 and added the products. For example, they should see that the entry "30,000" in the matrix for Question 3 comes from the computation

$$(500 \cdot 40) + (200 \cdot 50)$$

Before proceeding further with the analysis of this computation, identify the matrix answer to Question 2 as the special type of matrix called a *vector*. **What do you call a matrix with either only one row or only one column?** This is either a row vector or a column vector, depending on how each student expressed the answer. We expressed this answer as the **row vector** $\begin{bmatrix} 500 & 200 \end{bmatrix}$.

Similarly, the two entries comprising the weight information also form either a row or column vector. According to the form we used for Question 2, this information gives the **column vector** $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$.

Next ask students, **What was done with the two vectors $\begin{bmatrix} 500 & 200 \end{bmatrix}$ and $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$ to produce the entry '30,000' in Question 3?** Help students to articulate an answer something like this:

You multiply the first entry of one by the first entry of the other, then multiply the second entry of one by the second entry of the other, and then add the two products.

Now move on to Questions 5 and 6. Again, there are likely to be a variety of answers. But you may want to suggest that the answer to Question 5 should be similar to that for Question 2. For example, if a student gave the answer to Question 2 as a 1 x 2 row vector, then the answer to Question 5 should contain two rows, one for each day, with the first row like the answer from Question 2. The answers might look like this:

For Question 5:

$$\begin{array}{c} \\ \text{Mon} \\ \text{Tues} \end{array} \begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 500 & 200 \\ 400 & 300 \end{bmatrix} \end{array}$$

For Question 6:

$$\begin{array}{c} \\ \text{Mon} \\ \text{Tues} \end{array} \begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 30{,}000 & 1600 \\ 31{,}000 & 1700 \end{bmatrix} \end{array}$$

Again, the key element of the discussion is explaining how the matrix answer to Question 6 comes from the matrix answers for Questions 1 and 5. For example, the entry "1700" represents the volume transported on Tuesday and comes from the computation

$$(400 \cdot 2) + (300 \cdot 3)$$

Using our notation, this can be seen as the product of the *Tuesday row vector* $\begin{bmatrix} 400 & 300 \end{bmatrix}$ (from the matrix of Question 5) and the *volume column vector* $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (from the matrix of Question 1).

Be sure students know what each of the numbers in the answer to Question 6 represents. For example, they should be able to express that the entry "31,000" represents the weight of Linda Sue's load on Tuesday.

Tell students at this point that the matrix that results from combining the answers for Questions 1 and 5 in this way is called the *product* of the matrices. Let them know that they will be seeing a standard way to set up matrices so that this multiplication can be done routinely on calculators or computers.

Key Questions

What do you call a matrix with either only one row or only one column?

What was done with the two vectors $\begin{bmatrix} 500 & 200 \end{bmatrix}$ **and** $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$ **to produce the entry '30,000' in Question 3?**

Matrices in the Oven

Intent

Students continue to develop a definition of matrix multiplication.

Mathematics

Matrices in the Oven returns to the Woos' bakery (from *More Cookies*) to help students develop the logic behind matrix multiplication. The discussion leads to the realization that while there are many ways to define matrix multiplication that might make sense, a standard convention is needed in order for us to be able to understand one another's work.

Progression

This activity is similar to *Flying Matrices*. The discussion following the activity formally defines matrix multiplication. Students should record their results from this activity for use in *Fresh Ingredients*.

Approximate Time

30 minutes for activity (at home or in class)
40 to 50 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students do this activity independently, with no introduction. The discussion will establish the procedure for multiplying matrices

Discussing and Debriefing the Activity

Give groups some time to discuss the activity, and then have students present each problem. Again, save the answers for use later.

Here is a possible set of answers for Questions 1 through 3:

For Question 1:

$$\begin{array}{r c} & \begin{array}{ccc} \text{plain} & \text{iced} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{array}{r} \text{dough} \\ \text{icing} \\ \text{choc chip} \end{array} & \begin{bmatrix} 1 & 0.7 & 0.9 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.15 \end{bmatrix} \end{array}$$

For Question 2:

$$\begin{array}{r l} & \begin{array}{ccc} \text{plain} & \text{iced} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{array}{r}\text{Wed}\\ \text{Thurs}\end{array} & \begin{bmatrix} 30 & 45 & 30 \\ 28 & 32 & 25 \end{bmatrix} \end{array}$$

For Question 3:

$$\begin{array}{r l} & \begin{array}{ccc} \text{dough} & \text{icing} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{array}{r}\text{Wed}\\ \text{Thurs}\end{array} & \begin{bmatrix} 88.5 & 18 & 4.5 \\ 72.9 & 12.8 & 3.75 \end{bmatrix} \end{array}$$

Again, emphasize that the information in Questions 1 and 2 could be set up differently but that the computations that create the entries for the Question 3 matrix are the same no matter how the matrices are set up. For example, the entry "88.5" in the Question 3 matrix represents the amount of cookie dough used on Wednesday and comes from the computation $(30 \cdot 1) + (45 \cdot 0.7) + (30 \cdot 0.9)$.

Using the set-up given above for the matrices in Questions 1 and 2, students should see that they have taken the *Wednesday row* $\begin{bmatrix} 30 & 45 & 30 \end{bmatrix}$ from the matrix for Question 2 and multiplied its entries by the corresponding entries of the *cookie dough row* $\begin{bmatrix} 1 & 0.7 & 0.9 \end{bmatrix}$ from the matrix for Question 1. This computation is the same no matter how the matrices are set up. All that changes is the arrangement of the data in the matrix.

Be sure that students can identify what the numbers in their matrices represent, especially for Question 3.

Coordinating the Matrix Labels

Students should see that in both *Flying Matrices* and *Matrices in the Oven*, they found each entry of the final matrix by multiplying the entries of one row or column by the corresponding entries of another row or column and then adding the products. In other words, they multiplied two vectors.

At this point, tell students that mathematicians have agreed on a system for setting up matrices so that they can take unlabeled matrices and know how to combine them to get the product. The only question is whether to combine a row with a row, a row with a column, or a column with a column. Students should see that depending on how they set up their matrices, different combinations are appropriate.

To give some sense to this system, ask, **What categories of labels were used in the activity?** Help students to see that the labels are of three types:

- Type of cookie
- Type of ingredient

- Day of the week

Point out that for each matrix, they must decide on a category of label to assign to either row or column.

You might comment that it would be nice if we could assign either *row* or *column* to each of these throughout the problem. Point out that this is impossible, because different combinations occur at different stages of the problem.

For example, the category *type of cookie* occurs in the answers to both Question 1 and Question 2. Therefore, if this were set up as a row category in Questions 1 and 2, then *day of the week* and *type of ingredient* would have to be column categories. But these two occur together in Question 3, so we have an impasse.

Ask, **What relationship is there among the categories when two matrices are multiplied?** Guide students to articulate that whenever two matrices are multiplied, they have a *common category*. In *Matrices in the Oven,* the common category in Questions 1 and 2 is *type of cookie*. Point out that this category "drops out" in the final result.

Optional Approach

There is an informal way to see that it makes sense for *type of cookie* to be a row category in one of these two problems and a column category in the other. The reasoning is this:

Question 1 asks for *amount of a type of ingredient per type of cookie*. In a sense, the "units" for this matrix are like $\frac{\text{ingredients}}{\text{cookie type}}$, with *cookie type* in the denominator. But Question 2 asks for *amount of a type of cookie per day of the week*, so the units are $\frac{\text{cookie type}}{\text{day of the week}}$, with *cookie type* in the numerator.

Because *cookie type* is a numerator in one case and a denominator in the other, it makes some sense that *cookie type* should be a row category in one case and a column category in the other. When the matrices are multiplied, the "units" are also multiplied, with *cookie type* canceling out, resulting in $\frac{\text{ingredients}}{\text{day of the week}}$, which gives the units for Question 3. Thus, the computation can be represented in terms of units as

$$\frac{\text{ingredients}}{\text{cookie type}} \cdot \frac{\text{cookie type}}{\text{day of the week}} = \frac{\text{ingredients}}{\text{day of the week}}$$

Though this is not a formal mathematical explanation, it does have a mathematical basis, and it may be helpful in making sense of the formal definition of matrix multiplication.

Ask students what the categories were in *Flying Matrices*. They might refer to them as *day of the week*, *type of load*, and *type of plane limitation* (that is, weight or volume). They should also see that the common category was *type of load*. Therefore, again, the final answer simply involves the other two categories, namely, amount of weight or volume carried on each day.

The Definition of Matrix Multiplication

Tell students that for the sake of standardization, it is useful to have a rule for multiplying matrices so they can work with the numbers without thinking about the labels. The trick is to set up the matrices properly so that the arithmetic does what they want.

Give students this rule for the actual multiplication:

To get each individual entry of the product of two matrices, multiply a row in the first matrix by a column in the second matrix.

Point out that as always when multiplying vectors, certain dimensions must match up. Here, in order for this computation to make sense, the *length* of a row in the first matrix must be the same as the *height* of a column in the second matrix.

Illustrate how this works in context using the *Matrices in the Oven* problem. You can begin by going back to an individual entry of the product in which students found the *cookie dough on Wednesday* entry using this computation:

$$(30 \cdot 1) + (45 \cdot 0.7) + (30 \cdot 0.9)$$

For this computation to match the definition, we need to make the information "30, 45, 30" (the amount of each cookie type for Wednesday) into a row of the first matrix, and the information "1, 0.7, 0.9" (the amount of cookie dough needed for each cookie type) into a column of the second matrix.

Ask students how to set up the matrices to fit this requirement. After a few minutes' work, they should see that the matrices can be set up like this:

$$\begin{array}{r} \\ \text{Wed} \\ \text{Thurs} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{plain} & \text{iced} & \text{choc chip} \end{array} \\ \begin{bmatrix} 30 & 45 & 30 \\ 28 & 32 & 25 \end{bmatrix} \end{array} \times \begin{array}{r} \\ \text{plain} \\ \text{iced} \\ \text{choc chip} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{dough} & \text{icing} & \text{choc chip} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0.7 & 0.4 & 0 \\ 0.9 & 0 & 0.15 \end{bmatrix} \end{array}$$

It turns out that the labels for the common category, *type of cookie*, go across in the first matrix and down in the second. Each row of the first matrix has three entries, because there are three types of cookies. For the same reason, each column of the second matrix has three entries.

The product of the two matrices has the row labels of the first matrix as its row labels, and the column labels of the second matrix as its column labels:

$$\begin{array}{r|ccc|} & \text{dough} & \text{icing} & \text{choc chip} \\ \hline \text{Wed} & 88.5 & 18 & 4.5 \\ \text{Thurs} & 72.9 & 12.8 & 3.75 \\ \hline \end{array}$$

With these matrices to refer to, illustrate again how, when we find the product matrix, we take a row from the first matrix, multiply its entries by the corresponding entries from a column in the second matrix, and add the products.

Point out that all students are doing is finding what was previously called the product of two vectors.

In this case, they are multiplying a row vector from the first matrix by a column vector from the second matrix. Tell them that this is referred to as *multiplying the row times the column.*

They should see that the position of the row and column being used tells us where to put the result in the final matrix. Thus, if we multiply the second row of the first matrix by the third column of the second matrix, the result goes in the second row, third column of the resulting matrix.

In our example, this means that when we multiply the *Thursday row* $\begin{bmatrix} 28 & 32 & 25 \end{bmatrix}$ by the *chocolate chip column* $\begin{bmatrix} 0 \\ 0 \\ 0.15 \end{bmatrix}$, we compute

$$(28 \cdot 0) + (32 \cdot 0) + (25 \cdot 0.15) = 3.75$$

This becomes the *Thursday, chocolate chip* entry of the result, which goes in the second row of the third column of the product matrix.

Ask, **What does the number 3.75 represent?** Students should see that it tells us that the Woos used 3.75 pounds of chocolate chips on Thursday.

Turning the Computation Around

Note that the original matrices could have been written with the rows and columns interchanged, resulting in this computation:

$$\begin{array}{r|ccc|} & \text{plain} & \text{iced} & \text{choc chip} \\ \hline \text{dough} & 1 & 0.7 & 0.9 \\ \text{icing} & 0 & 0.4 & 0 \\ \text{choc chip} & 0 & 0 & 0.15 \\ \hline \end{array} \times \begin{array}{r|cc|} & \text{Wed} & \text{Thurs} \\ \hline \text{plain} & 30 & 28 \\ \text{iced} & 45 & 32 \\ \text{choc chip} & 30 & 25 \\ \hline \end{array} = \begin{array}{r|cc|} & \text{Wed} & \text{Thurs} \\ \hline \text{dough} & 88.5 & 72.9 \\ \text{icing} & 18 & 12.8 \\ \text{choc chip} & 4.5 & 3.75 \\ \hline \end{array}$$

There is no theoretical reason to prefer one of these representations to the other.

Practice As Time Allows
If you have time available, give students some matrices to multiply. (Be sure to give them matrices whose dimensions match appropriately for multiplication.)

Key Questions

What categories of labels were used in the activity?

What relationship is there among the categories when two matrices are multiplied?

What does the number 3.75 represent?

Fresh Ingredients

Intent

Students use matrix multiplication to express answers to real-world problems.

Mathematics

This activity provides practice with setting up matrices for meaningful multiplication, and with the mechanics of matrix multiplication.

Progression

Fresh Ingredients returns to the Woos' bakery from *More Cookies*. This time, however, the questions require that the student determine how to set up the matrices so that multiplying them will yield the information that is requested. Students should save the results from the activity for purposes of comparison when they do the next activity, *Calculators to the Rescue*.

Approximate Time

25 minutes for activity (at home or in class)
25 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students do this activity independently.

Discussing and Debriefing the Activity

Let groups discuss how best to set the matrices up, and what the results are. Then have different students present the matrix of costs and the matrix of overall results at the different markets.

As given in the activity, the *baking-plan* matrix is

$$\begin{array}{r c} & \begin{array}{ccc} \text{plain} & \text{iced} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{array}{r} \text{Wed} \\ \text{Thurs} \end{array} & \begin{bmatrix} 30 & 45 & 30 \\ 28 & 32 & 25 \end{bmatrix} \end{array}$$

and the *ingredient* matrix is

$$\begin{array}{r c} & \begin{array}{ccc} \text{dough} & \text{icing} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{array}{r} \text{plain} \\ \text{iced} \\ \text{choc chip} \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0.7 & 0.4 & 0 \\ 0.9 & 0 & 0.15 \end{bmatrix} \end{array}$$

Because the ingredients are used as column labels in the ingredient matrix, they need to be used as row labels for the *cost* matrix (Question 1). The *market names* are used as labels for the columns of the cost matrix. Therefore, the cost matrix is

$$\begin{array}{r c} & \begin{array}{cc} \text{Frmr Mkt} & \text{Dntn Groc} \end{array} \\ \begin{array}{r} \text{dough} \\ \text{icing} \\ \text{choc chip} \end{array} & \begin{bmatrix} 30 & 29 \\ 20 & 28 \\ 32 & 22 \end{bmatrix} \end{array}$$

There are various ways to proceed to get the information the Woos need. One way is to build on students' work in *Matrices in the Oven*, in which they found this *amounts-needed-per-day* matrix:

$$\begin{array}{r c} & \begin{array}{ccc} \text{dough} & \text{icing} & \text{choc chip} \end{array} \\ \begin{array}{r} \text{Wed} \\ \text{Thurs} \end{array} & \begin{bmatrix} 88.5 & 18 & 4.5 \\ 72.9 & 12.8 & 3.75 \end{bmatrix} \end{array}$$

Students can then multiply this result by the cost matrix. In other words, they should find this product:

$$\begin{array}{r c} & \begin{array}{ccc} \text{dough} & \text{icing} & \text{choc chip} \end{array} \\ \begin{array}{r} \text{Wed} \\ \text{Thurs} \end{array} & \begin{bmatrix} 88.5 & 18 & 4.5 \\ 72.9 & 12.8 & 3.75 \end{bmatrix} \end{array} \times \begin{array}{r c} & \begin{array}{cc} \text{Frmr Mkt} & \text{Dntn Groc} \end{array} \\ \begin{array}{r} \text{dough} \\ \text{icing} \\ \text{choc chip} \end{array} & \begin{bmatrix} 30 & 29 \\ 20 & 28 \\ 32 & 22 \end{bmatrix} \end{array}$$

to get this result:

$$\begin{array}{r c} & \begin{array}{cc} \text{Frmr Mkt} & \text{Dntn Groc} \end{array} \\ \begin{array}{r} \text{Wed} \\ \text{Thurs} \end{array} & \begin{bmatrix} 3159 & 3170 \\ 2563 & 2555 \end{bmatrix} \end{array}$$

For example, the first entry in the result means that the total of all ingredients needed for Wednesday would cost \$31.59 at the Farmer's Market. (*Note:* Total costs have been rounded to the nearest penny.)

Alternate Approach

Another way to do the computation would be to first find the cost per dozen of each type of cookie at each market, from the matrix product

$$\begin{array}{r|ccc|}
 & \text{dough} & \text{icing} & \text{choc chip} \\
\text{plain} & 1 & 0 & 0 \\
\text{iced} & 0.7 & 0.4 & 0 \\
\text{choc chip} & 0.9 & 0 & 0.15
\end{array}
\times
\begin{array}{r|cc|}
 & \text{Frmr Mkt} & \text{Dntn Groc} \\
\text{dough} & 30 & 29 \\
\text{icing} & 20 & 28 \\
\text{choc chip} & 32 & 22
\end{array}$$

This gives the *cookie-cost-per-market* matrix

$$\begin{array}{r|cc|}
 & \text{Frmr Mkt} & \text{Dntn Groc} \\
\text{plain} & 30 & 29 \\
\text{iced} & 29 & 31.5 \\
\text{choc chip} & 31.8 & 29.4
\end{array}$$

Entries are in cents and represent costs per dozen. (*Note:* The values are not rounded off because another computation remains to be done.) For example, the entry "30" means that the ingredients for a dozen plain cookies cost 30¢ at the Farmer's Market. Using this method, this matrix could then be multiplied by the baking-plan matrix to get the total costs for each day at each market:

$$\begin{array}{r|ccc|}
 & \text{plain} & \text{iced} & \text{choc chip} \\
\text{Wed} & 30 & 45 & 30 \\
\text{Thurs} & 28 & 32 & 25
\end{array}
\times
\begin{array}{r|cc|}
 & \text{Frmr Mkt} & \text{Dntn Groc} \\
\text{plain} & 30 & 29 \\
\text{iced} & 29 & 31.5 \\
\text{choc chip} & 31.8 & 29.4
\end{array}$$

This gives the same product as before (now rounded again to the nearest penny):

$$\begin{array}{r|cc|}
 & \text{Frmr Mkt} & \text{Dntn Groc} \\
\text{Wed} & 3159 & 3170 \\
\text{Thurs} & 2563 & 2555
\end{array}$$

For instance, the Woos would spend $31.59 if they shopped at the Farmer's Market on Wednesday.

Summary

Whichever way one does the computation, it looks as if the Woos should shop at the Farmer's Market on Wednesday and at the Downtown Grocery on Thursday. But, of course, the difference in cost is minimal. If they shopped at the more expensive market each day, they would spend only 19¢ more.

An Algebraic Representation

When we represent matrices by variables, we will use brackets around the letter, as in the notation [A]. There are various ways to notate matrices, but this notation is common, and matches how most calculators display matrices.

As with variables representing numbers, multiplication can be indicated by juxtaposing the matrices, as in the notation [A][B] (omitting the multiplication sign itself). You can also omit the multiplication sign when matrices are given numerically, because the brackets serve to separate one matrix from another. For instance,

$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

represents the product of the two matrices.

If we label the baking-plan matrix as [A], the ingredient matrix as [B], and the cost matrix as [C], then with the first method discussed previously, we compute the product [A][B] and then multiply this by [C]. With the second method, we compute [B][C] and then multiply this by [A].

In other words, one method computes ([A][B])[C], and the other computes [A]([B][C]). Students will learn in the discussion of *Things We Take for Granted* that multiplication of matrices is associative. In the context of *Fresh Ingredients*, the associativity of multiplication simply means that the two methods just discussed give the same result. Because of the associativity, we can unambiguously write the product as [A][B][C].

Supplemental Activity

Surfer's Shirts (reinforcement) provides two other situations in which students have to set up matrices and matrix expressions that will provide certain information.

Calculators to the Rescue

Intent

Students use a graphing calculator to perform matrix computations.

Mathematics

Working with matrices by hand can be quite involved. In this activity, students are introduced to the use of the graphing calculator for matrix arithmetic.

Progression

After a brief presentation by the teacher of the mechanics of entering matrices into the calculator and performing matrix arithmetic, students use the calculator to check their solutions to *Fresh Ingredients*.

Approximate Time

10 minutes for introduction
15 to 20 minutes for activity

Classroom Organization

Individuals, preceded by teacher presentation

Materials

Calculator Note "Entering Matrices and Doing Matrix Arithmetic" (1 copy per student, optional)

Doing the Activity

Tell students that they can use graphing calculators to do matrix arithmetic quickly and easily. Show the mechanics of entering a matrix into the calculator and multiplying two matrices. The Calculator Note "Entering Matrices and Doing Matrix Arithmetic" will help; you might use this for your own review, or provide it to students for reference. You may also wish to use an overhead calculator to assist with this presentation.

We recommend that you begin by helping students enter the baking-plan matrix (see Question 1). Next, let them try to enter the ingredient matrix on their own (Question 2). Then bring the class back together and show how to multiply the two together.

If students question whether the time to enter matrices into their calculators is worth it, remind them that the unit problem contains six variables.

Now you can let students work in their groups on the rest of *Calculators to the Rescue.*

As stated at the end of the discussion of *Fresh Ingredients*, the answer is the product [A][B][C], which can be calculated in that form, or as either ([A][B])[C] or

[A]([B][C]).

While groups work, you can provide assistance as needed with either graphing calculator mechanics or ways to make the matrices represent the problem.

Discussing and Debriefing the Activity

No formal discussion of the activity is necessary, other than perhaps to confirm that the graphing calculator computations give the same result as the pencil-and-paper method.

Make It Simple

Intent

Students reflect on the mechanics of matrix multiplication.

Mathematics

This activity keeps students thinking about the definition of matrix multiplication, even as they learn to do the actual computations on a graphing calculator.

Progression

In *Make It Simple,* students write clear instructions explaining how to multiply two 3 x 3 matrices, test those instructions on someone, and then revise the instructions as necessary.

Approximate Time

25 minutes for activity (at home or in class)
5 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity can be done independently.

Discussing and Debriefing the Activity

Have one or two volunteers share their instructions and experiences.

Back and Forth

Intent
Students see that systems of linear equations are equivalent to a certain type of matrix equation.

Mathematics
Back and Forth introduces the representation of linear systems using matrices. Students write an equivalent matrix equation given a system of linear equations, and vice versa.

Progression
In the introductory discussion, students are reminded that solving the central unit problem will entail solving many systems of linear equations in six variables. The discussion then introduces how to write a matrix equation that is equivalent to a system of linear equations.

Approximate Time
10 minutes for introduction
35 minutes for activity
10 minutes for discussion

Classroom Organization
Small groups, preceded and followed by whole-class discussion

Doing the Activity

Matrices, Linear Equations, and the Unit Problem
Ask, **What do you think this work with matrices has to do with the central unit problem, *Meadows or Malls*?**

Students may recall that they first encountered matrices as a shorthand for representing work with systems of linear equations. (If they don't remember, remind them of this.)

Tell them that they are about to see in more detail how matrices—and specifically, matrix multiplication—can be used to express systems of linear equations. Within a few days, they will see how they can use matrices to solve complicated systems of linear equations, with the help of graphing calculators.

Writing Linear Equations with Matrices
Begin making the connection between matrix multiplication and linear equations by starting with a single linear equation, such as

$$5x + 3y + 7z = 10$$

Ask students, **What's the connection between the expression on the left and**

matrix multiplication? The connection that should be made is that the expression $5x + 3y + 7z$ is a sum of products, just like the kind of computation we use to obtain the entries for the product of two matrices. If students don't see the connection, ask, **Suppose I give you three numbers, such as 2, 4, and 9, for *x*, *y*, and *z*. What would you do to find the value of $5x + 3y + 7z$?**

Get students to describe the arithmetic they would do. They may say, "Simply substitute," but insist on more detail, such as, "Multiply 5 times 2, 3 times 4, and 7 times 9, and add the results." You can then ask, **What arithmetic would you do to get this answer?** They should be able to connect this to the process for multiplying vectors.

Once students make this connection, ask **How can $5x + 3y + 7z$ be written as a product of vectors?** They should see that it is the product of the vectors $\begin{bmatrix} 5 & 3 & 7 \end{bmatrix}$ and $\begin{bmatrix} x & y & z \end{bmatrix}$.

One important technical step is to ask, **Which matrix should be on the left and which should be on the right?** As needed, review the procedure for multiplication of matrices—multiply a row of the first matrix by a column of the second. Thus, the expression is the product of these vectors:

$$\begin{bmatrix} 5 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Finally, ask students to restate the original equation using this matrix product:

$$\begin{bmatrix} 5 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 10$$

Note: Technically, we should write this as

$$\begin{bmatrix} 5 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}$$

using the 1 x 1 matrix [10] instead of the number 10, because the terms on the left are matrices, but we often blur the distinction. Students may write the right side as simply 10, and that's fine for now.

With this introduction, let students work on the activity *Back and Forth*. Point out that all they have to do in this activity is express the systems of linear equations as matrix equations, and vice versa.

If students seem stuck on Question 1, ask, **How can you state the original equation using matrices?** Then ask, **If you set up each equation using matrices, what do the matrix equations have in common?** This may help them see that the coefficients in the different linear equations should make up rows of a single matrix. Thus, the system is equivalent to the matrix equation

$$\begin{bmatrix} 5 & 3 & 7 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 4 \end{bmatrix}$$

As an alternative hint for Question 1, have students move on to Question 2 and begin by multiplying the two matrices on the left. They should see that the product

$$\begin{bmatrix} 3 & -2 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

is equal to

$$\begin{bmatrix} 3u - 2v \\ u - 6v \end{bmatrix}$$

Thus, for the left side to be equal to the column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, they must have $3u - 2v = 2$ and $u - 6v = 1$. These are the two desired equations. In other words, the matrix equation

$$\begin{bmatrix} 3 & -2 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is equivalent to the pair of linear equations

$$3u - 2v = 2$$

$$u - 6v = 1$$

With the help of this example connecting matrix equations and systems of linear equations, they should be able to tackle Question 1.

Discussing and Debriefing the Activity

Have one or two students present their ideas on Question 1. As needed, use the hints in the introductory discussion to help clarify the relationship between matrix equations and systems of linear equations.

Next, have a student present results on Question 2.

You don't necessarily have to go over each of the problems from this activity.

However, you should probably discuss Questions 4 and 5 so that students see how to deal with missing terms and how this relates to zeroes in the matrix.

The main point of this activity is for students to recognize that a system of linear equations can be expressed by a matrix equation and that a certain type of matrix equation can be solved by a system of linear equations.

Point out that these matrix equations can all be written in a general form as

[A][X] = [B]

where [A] and [B] are two specific matrices and [X] is a matrix of variables. In Questions 2 through 5, [A] is 2 x 2, [B] is 2 x 1, and [X] is 2 x 1; in Questions 6 and 7, [A] is 3 x 3, [B] is 3 x 1, and [X] is 3 x 1.

In general, a system of *n* linear equations in *n* variables is equivalent to a matrix equation of the form

[A][X] = [B]

where [A] is a given *n* x *n* square matrix (the **coefficient matrix**), [B] is a given *n* x 1 column matrix (the **constant term matrix**), and [X] is the *n* x 1 column matrix whose entries are the individual variables in the system of linear equations. Review that [A] needs to be a square matrix because we want the number of equations to be equal to the number of variables. This may require some discussion about intersections of planes and related ideas.

Key Questions

What do you think this work with matrices has to do with the central unit problem, *Meadows or Malls?*

What's the connection between the expression on the left and matrix multiplication?

Suppose I give you three numbers, such as 2, 4, and 9, for *x*, *y*, and *z*. What would you do to find the value of 5*x* + 3*y* + 7*z*?

What arithmetic would you do to get this answer?

How can 5*x* + 3*y* + 7*z* be written as a product of vectors?

Which matrix should be on the left and which should be on the right?

How can you state the original equation using matrices?

If you set up each equation using matrices, what do the matrix equations have in common?

Matrices and Linear Systems

Intent

This is reference material about using matrices to represent systems of linear equations.

Mathematics

Matrices and Linear Systems summarizes the basic ideas about writing a system of linear equations using matrix multiplication.

Progression

Although *Matrices and Linear Systems* is largely a summary of the information presented in the introductory discussion to *Back and Forth*, it is recommended that students not be directed toward this summary until after they have completed that activity.

Approximate Time

10 minutes for student review (at home)

Classroom Organization

Individuals

Doing the Activity

Assign students to read this summary as part of their homework activity.

Discussing and Debriefing the Activity

No discussion of this reference material is necessary.

Solving the Simplest

Intent

This activity uses the connection between matrix equations and systems of linear equations to focus attention on the special character of the identity matrix.

Mathematics

This activity defines the identity element and applies that concept to the operation of matrix multiplication.

Progression

Solving the Simplest helps students realize that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ leads to the simplest possible system of two linear equations. The whole-class discussion generalizes this to systems of three linear equations.

Approximate Time

25 minutes for activity (at home or in class)
35 to 40 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students do this activity independently.

Discussing and Debriefing the Activity

You don't need to go over Questions 1a through 1c unless students are having difficulty understanding the relationship between a matrix equation and the corresponding system of linear equations.

Instead, go straight to Questions 2 and 3, and ask for a volunteer to share ideas on which matrix equation was easiest to solve. If students don't share an opinion, you might ask them to go over what was involved in solving the matrix equations in Questions 1a and 1c. That is, the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \end{bmatrix}$$

led to the system

$$w + 2z = 6$$

$$3w + 4z = 16$$

while the matrix equation

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

led to the system

$$2u = 8$$

$$3v = 15$$

Students should agree that the second system is easier to solve. (The solution to the first is $w = 4$, $z = 1$.)

Ask, **What is it about the coefficient matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ that leads to such an easy system to solve?** Students should see that the presence of zeroes is the key.

Moving on to Question 3, students should realize that the systems absolutely easiest to solve come from the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (If someone argues for $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ instead, you can point out that it makes sense to have the first equation involve the first variable and the second equation involve the second variable.)

Push this idea one step further by asking, **What would be the coefficient matrix for the easiest-to-solve three-variable linear system?** If necessary, let groups work on this for a few minutes. They should see that the answer is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ordinary Equation Solving

Tell students that there is a way to solve a matrix equation of the form [A][X] = [B] that is actually quite similar to solving an ordinary "number equation" of the form $ax = b$. Give students an example of a simple number equation such as $5x = 27$, and ask **How would you solve the numerical equation $5x = 27$?**

You are likely to get a response like "divide both sides by 5." If so, give students a pair of matrices and ask how they would divide one by the other. After they ponder this, tell them that at this point, they don't have anything like division for matrices, so this won't help them find an analogous method. The only matrix operations currently available are addition and multiplication.

As needed, ask, **How can you solve $5x = 27$ using multiplication?** The goal here is to elicit a response like "multiply both sides by $\frac{1}{5}$." Students should see that this transforms the equation into the equivalent equation $x = \frac{1}{5} \cdot 27$.

The point is that multiplying both sides by $\frac{1}{5}$ solves the equation, because it leads to an equation of the form "x = some number."

There is one more step in the analogy, concerning the "disappearance" of the coefficient 1. You can introduce this idea by asking students what would happen if they multiplied both sides of $5x = 27$ by 2. They should say that this gives $10x = 54$. Get them to explain the coefficient 10 as $2 \cdot 5$.

We want them to see that, by analogy, they should have gotten $1 \cdot x$ on the left side when they multiplied both sides by $\frac{1}{5}$. Ask, **What happened to the coefficient of *x* when you multiplied 5x = 27 by $\frac{1}{5}$?** The purpose of this brief excursion is to bring out two key ideas:

- The expression $1 \cdot x$ is the same as x, so once you get an equation into the form "$1 \cdot x$ = something," you have really already solved the equation.
- You can transform an equation of the form $ax = b$ into an equivalent equation of the form "$1 \cdot x$ = something" by multiplying both sides of $ax = b$ by the reciprocal of the coefficient a. (You might note that this method fails if $a = 0$.)

Back to Matrices
Now ask if there is a matrix analogy to the way the number 1 works in numerical equations. That is, **Is there a matrix that you could use for [A] so that the equation [A][X] = [B] is 'already solved'?**

If students need help, ask them to think about the activity they just completed. As a hint, you might point out that they have just seen that $1 \cdot x = b$ is the simplest example of a number equation of the type $ax = b$, and now they want an analogous matrix equation of the form [A][X] = [B].

These should be sufficient clues to the fact that we are looking for matrices like $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Ask, **Why does the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ lead to such an easy system to solve?**

Try to get someone to articulate that it has the property $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$[X] = [X] for any column vector of length 2.

Ask, **What if [X] were a matrix of a different size?** Students should see that $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$[X] doesn't make any sense unless [X] has exactly two rows.

Then ask, **What happens if you multiply $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ by a matrix with two rows?** For instance, have them find the product

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}5 & -4\\3 & 7\end{bmatrix}$$

They should see that the result is $\begin{bmatrix}5 & -4\\3 & 7\end{bmatrix}$. The same sort of thing happens with a product with more than two columns, such as $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2 & -5 & 1\\3 & -7 & 4\end{bmatrix}$.

Introduce the term *identity element for multiplication* to describe the special role that the number 1 and the matrix $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ play in multiplication.

Ask, **What does it mean for something to be an identity element for multiplication?** Students should be able to explain that when any particular number (or matrix) is multiplied by 1 (or by $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$), the result is that same particular number (or matrix).

More specifically, we might say that 1 is the identity element for multiplication of *numbers* and that $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is the identity element for multiplication of *two-row matrices*.

Ask students, **What would be the identity element for multiplication of 'three-row matrices'?** It is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

Ask, **Do you know of any other identity elements?** If necessary, suggest that they consider operations other than multiplication. More specifically, you can ask them to think about ordinary addition of numbers. They should recognize that the number 0 is the identity element for addition of numbers. They should also be able to explain that this means that any number plus 0 is that number.

Next ask, **What would be the identity element for addition of 2 x 2 matrices?** If necessary, ask students for an example of how this would work. They should be able to come up with a description like, "If you add it to another 2 x 2 matrix, you get that other matrix."

If needed, let them work on this in groups until someone comes up with the answer $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. They should then be able to find the identity element for addition of 3 x 3 matrices, 4 x 4 matrices, and so on. In fact, they should see that there are identity elements for addition for matrices of any shape, not just square matrices.

Key Questions

What is it about the coefficient matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ that leads to such an easy system to solve?

What would be the coefficient matrix for the easiest-to-solve three-variable linear system?

How would you solve the numerical equation 5*x* = 27?

How can you solve 5*x* = 27 using multiplication?

What happened to the coefficient of *x* when you multiplied 5*x* = 27 by $\frac{1}{5}$?

Is there a matrix that you could use for [A] so that the equation [A][X] = [B] is 'already solved'?

Why does the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ lead to such an easy system to solve?

What if [X] were a matrix of a different size?

What happens if you multiply $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by a matrix with two rows?

What does it mean for something to be an identity element for multiplication?

What would be the identity element for multiplication of 'three-row matrices'?

Do you know of any other identity elements?

What would be the identity element for addition of 2 x 2 matrices?

Things We Take for Granted

Intent

This activity asks students to consider some of the possible general algebraic properties of matrix operations.

Mathematics

Students recognize the noncommutativity and the associativity of matrix multiplication, through searching for counterexamples.

Progression

Students work on this activity individually, then share their results in class discussion.

Approximate Time

25 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students do this activity independently.

Discussing and Debriefing the Activity

Question 1

Students should have no trouble coming up with counterexamples to the commutative property for multiplication of 2 x 2 matrices. That is, they should find two specific 2 x 2 matrices [A] and [B] such that the products [A][B] and [B][A] are different. After looking at one or two such examples, tell students that we summarize this by saying that multiplication of matrices is *noncommutative*.

While on the subject of noncommutativity, remind students about the special matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ discussed yesterday, which was described as the *identity element for multiplication* of two-row matrices. Specifically, they saw that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$[Y] = [Y] for any two-row matrix [Y].

Ask, **If you multiply a two-row matrix [Y] on the right by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, will you get the matrix [Y]?**. Guide them to articulate that [Y]$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ doesn't make sense unless

[Y] has two columns, but if it does, then we get [Y] $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = [Y].

Tell students that by definition, an identity element must "work" on both sides. So they have just seen that we can truly describe $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as the multiplicative identity for 2 x 2 matrices.

You can illustrate the importance of checking "both directions" using the example of ordinary subtraction (of numbers). We *do not* consider 0 to be an identity for subtraction, even though $x - 0 = x$ for all x, because it is not true that $0 - x = x$.

Matrices Other Than 2 x 2

Although students were asked to consider only 2 x 2 matrices in this activity, you might ask, **Why might matrix multiplication be noncommutative for matrices of other sizes?** Emphasize that it's possible for a matrix product [A][B] to be defined but for the reverse product [B][A] to be undefined (because the dimensions don't fit).

For example, the product $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ makes sense, but the product $\begin{bmatrix} 2 \\ 5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ does not.

Question 2

Students should not have found any counterexamples in Question 2, because multiplication of matrices is **associative.** It may not be worth making a big deal out of this, because associativity is something that most students take for granted.

The fact that no one found a counterexample does not prove that matrix multiplication is associative. Tell students, however, that multiplication of matrices *is* associative (as long as the matrices "fit"), and that this can be proved.

Key Questions

If you multiply a two-row matrix [Y] on the right by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, will you get the matrix [Y]?

Why might matrix multiplication be noncommutative for matrices of other sizes?

Supplemental Activity

An Associative Proof (extension) asks students to prove that multiplication is associative for 2 x 2 matrices. The problem also asks about commutativity and associativity of matrix addition.

Finding an Inverse

Intent

This activity introduces matrix inverses.

Mathematics

Through the analogy of solving the equation $5x = 27$ by multiplying both sides of the equation by $\frac{1}{5}$, students come to understand the principle of using a matrix inverse to solve a matrix equation. They find that the solution to [A][X] = [B] will be $[A]^{-1}[B]$, where $[A]^{-1}$ represents the inverse of [A].

The activity guides students through the process of finding the multiplicative inverse of a 2 x 2 matrix, and the subsequent discussion introduces the notation $[A]^{-1}$ for the inverse of matrix [A].

Progression

After introductory discussion, students find the inverse of a 2 x 2 matrix, and learn the notation for inverse matrices. This activity concludes the development of using the inverse to solve a linear system.

Approximate Time

10 minutes for introduction
15 to 20 minutes for activity (at home or in class)
15 to 25 minutes for discussion

Classroom Organization

Small groups or individuals, preceded and followed by whole-class discussion

Doing the Activity

***Solving* [A][X] = [B]**

Review the idea that we are trying to solve a matrix equation, [A][X] = [B]. This should be done similarly to the way we solve an "ordinary" equation, $ax = b$. Remind the class that with an equation like $5x = 27$, we simply multiply both sides by $\frac{1}{5}$, which gives $1 \cdot x = \frac{1}{5} \cdot 2$. Because 1 is the identity for multiplication of numbers, this tells us x = some number.

Give students a specific matrix equation, such as

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and ask what the analogous process would be. **What if you applied this process**

to a matrix equation? What would that look like?

The goal here is for students to come up with the idea of multiplying both sides by a matrix [C] that will "get rid of" the matrix [A]. As a hint, you can ask, **Why did we use $\frac{1}{5}$ in the number problem?** They should respond that it "cancels out" the 5.

More specifically, try to get them to see that $\frac{1}{5}$ worked because $\frac{1}{5} \cdot 5 = 1$, which is the identity element for multiplication. Thus, multiplying by $\frac{1}{5}$ gives $1 \cdot x$, or simply x.

The key idea to elicit from this discussion is that we want to find a matrix [C] such that

$$[C]\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we find this matrix [C], then we can multiply both sides of our specific matrix equation by it, which gives

$$[C]\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} = [C]\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Because of the special choice of [C], the left side is simply $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix}$, and by the special property of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, this is simply $\begin{bmatrix} w \\ z \end{bmatrix}$. In other words, if $[C]\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\begin{bmatrix} w \\ z \end{bmatrix} = [C]\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. That is, once we have [C], we have only to find the product $[C]\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and we get w and z. (You may want to point out that this last step is similar to actually multiplying 27 by $\frac{1}{5}$ in the number equation.)

Note: The associativity of matrix multiplication does play a role in the reasoning here, but you need not bring this out unless students ask about it.

Ask, **What should the dimensions of [C] be?** They should reason that because the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2×2 matrix, [C] must be a 2 x 2 matrix as well. Thus, you can identify the final stage of the process of solving [A][X] = [B] as "trying to find [C]." Before getting into that, however, you should introduce some terminology.

Multiplicative Inverses
Tell students that when two numbers multiply together to give 1 (which is the multiplicative identity for numbers), each is called the **multiplicative inverse** of the other. For example, $\frac{1}{5}$ is the multiplicative inverse of 5. (Students may be familiar with the term *reciprocal* for this concept in the context of fractions.)

Tell them that similarly, the matrix [C] that they are looking for is called the multiplicative inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, because the product of [C] and $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is the multiplicative identity for 2 x 2 matrices.

Note: Formally, the statement that matrix [U] is the inverse of matrix [V] reflects the fact that the matrices are both square and that the products [U][V] and [V][U] are *both* the multiplicative identity matrix of the appropriate size.

The activity explains the meaning of multiplicative inverse but omits these details. One can prove, however, that if [U] and [V] are square matrices and [U][V] is the multiplicative identity matrix, then [V][U] must also be the multiplicative identity matrix. Thus, the statement in the activity is actually true, though incomplete.

Finding an Inverse
Have students begin the activity *Finding an Inverse*. No further introduction is needed, but you may want to use ideas from the discussion as hints for groups that get stuck.

Discussing and Debriefing the Activity

By following the directions in the activity, students should see that the problem involves this system of four linear equations:

$$r + 3s = 1$$

$$2r + 5s = 0$$

$$t + 3u = 0$$

$$2t + 5u = 1$$

They should also see that these four equations can be viewed as two systems of two equations each:

$$r + 3s = 1$$

$$2r + 5s = 0$$

and

$$t + 3u = 0$$

$2t + 5u = 1$

(If students needed help getting this far, have them again work in their groups to solve these two systems of linear equations.)

Students should be able to solve these two systems and find that $r = -5$, $s = 2$, $t = 3$, and $u = -1$. In other words, they should conclude that $[C] = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. Identify this matrix as the *multiplicative inverse* of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

Note: Students may notice that except for a couple of signs, this is simply a rearrangement of the entries of the original matrix, $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. This is no coincidence; rather, it happens for all matrices whose determinant is 1 or −1. We do not define determinants or explain this phenomenon in the unit, but students can explore it on their own if interested in the Supplementary Problem *Determining the Determinant*.

After students have found [C], they should check that it "works." That is, they should compute the product $[C]\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and verify that it really is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Notation for Matrix Inverse

Introduce the notation $[A]^{-1}$ for the multiplicative inverse of a matrix [A]. Ask, **Why do you think we use the notation [A]−1 for the multiplicative inverse of a matrix [A]?** Students might remember that 5^{-1} is defined as the fraction $\frac{1}{5}$. You can tell students that based on that definition, it is standard practice to use the notation of an exponent of −1 to represent multiplicative inverses.

Using the $\boxed{x^{-1}}$ key (which appears on most scientific and graphing calculators) in connection with a number will give the reciprocal of that number. For example, a key sequence like $\boxed{5}$ $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$ will usually give the result 0.2 (which is equal to $\frac{1}{5}$). Students will see in *Calculators Again* that using the $\boxed{x^{-1}}$ key with a matrix gives the inverse of the matrix (if it exists).

Warning: A Different Use of the Exponent –1

In the context of looking at this notation, you might point out that the same notation is used for inverse trig functions, and it *does not* mean multiplicative inverse in that context. That is, $\sin^{-1} x$ is not the same thing as $\frac{1}{\sin x}$.

More Examples of Inverses

Remind students of the nonmatrix identities mentioned yesterday, and discuss their matrix analogs. For instance, −5 is the *additive inverse* of 5 because when it is added to 5, the result is 0, which is the additive identity for numbers. Likewise, $\begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix}$ is the additive inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, because their sum is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is the additive identity for 2 x 2 matrices.

Using the Inverse

Now that students have found the multiplicative inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ (usually just called its "inverse"), have groups use this inverse to solve the original equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

They should recognize that they want to multiply both sides of this equation by the multiplicative inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ —that is, by $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

After a few minutes of group work on this, have a volunteer explain the process. Multiplying both sides of the equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

by $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ gives the equation

$$\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

But $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ is the multiplicative inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, so this simplifies to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or simply

$$\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(because $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the multiplicative identity).

Multiplying out the right side gives

$$\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$$

In other words, $w = -8$ and $z = 5$.

Putting It All Together

Connect this result to the question of linear systems by asking, **What pair of linear equations have we solved?** They should be able to return to the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and express it as the pair of equations

$$w + 2z = 2$$

$$3w + 5z = 1$$

They should then verify that $w = -8$, $z = 5$ is the solution to this system.

It may help to review the whole process symbolically. Students should see that in general, they can take a system of linear equations and express it as a matrix equation

$$[A][X] = [B]$$

They can then multiply both sides of this equation by the multiplicative inverse of the matrix [A], that is, by $[A]^{-1}$, to get

$$[A]^{-1}[A][X] = [A]^{-1}[B]$$

But $[A]^{-1}[A] = [I]$ (the symbol for the identity matrix), and $[I][X] = [X]$, so the equation simplifies to

$$[X] = [A]^{-1}[B]$$

Thus, $[A]^{-1}[B]$ is the solution to the matrix equation. In other words, the entries of the column vector $[A]^{-1}[B]$ are the values of the variables that fit the system of linear equations.

To summarize:

> **To solve a system of linear equations, find the inverse of the coefficient matrix [A] and multiply it by the constant term column vector [B]. The product $[A]^{-1}[B]$ shows the solutions to the system.**

This process is one of the reasons that matrix inverses are so useful.

Finding inverse matrices by hand can be cumbersome, but students will soon learn how to use graphing calculators to simplify the work.

Which Side to Multiply On?

Ask, **Why is [A]–1 on the left side of [B]? Could it be on the right?** Remind students as needed of the noncommutative nature of matrix multiplication. Students may also point out that if $[A]^{-1}$ were on the right side of [B], then in cases like those discussed here, the product couldn't be computed.

Key Questions

What if you applied this process to a matrix equation? What would that look like?

Why did we use $\frac{1}{5}$ in the number problem?

What should the dimensions of [C] be?

Why do you think we use the notation $[A]^{-1}$ for the multiplicative inverse of a matrix [A]?

What pair of linear equations have we solved?

Why is $[A]^{-1}$ on the left side of [B]? Could it be on the right?

Inverses and Equations

Intent

Students recognize the connection between a system of linear equations having a unique solution and the matrix of coefficients having a multiplicative inverse.

Mathematics

In this activity, students learn that the coefficient matrices associated with dependent or inconsistent systems of linear equations will be noninvertible.

Progression

Students solve four simple pairs of linear equations and explore why there is not a unique solution in some cases. The subsequent discussion introduces the terms *invertible* and *noninvertible,* and establishes that the coefficient matrix for an unsolvable linear system (either dependent or inconsistent) is noninvertible, and that this occurs in 2 x 2 matrices when one row is a multiple of another.

Approximate Time

30 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students do this activity independently.

Discussing and Debriefing the Activity

Ask students to compare their results in groups. They may need reassurance from one another about the impossibility of solving certain problems.

Have several students report on their results for the different systems of equations. They should have discovered that Questions 1a and 1b have unique solutions, but that Questions 1c and 1d do not. Question 1c has infinitely many solutions, and students should describe this system as *dependent;* Question 1d has no solutions, and students should describe this system as *inconsistent*. You might also review that a pair of linear equations in two variables with a unique solution is called *independent*.

The solution to Question 1a is $x = 0$, $y = 2$; the solution to Question 1b is $x = -3$, $y = 2.5$.

Have other students explain their work on Question 2. For Question 2a, they may have written the problem as

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and then set this up as a system of equations (as in the activity *Back and Forth*), like this:

$r + 3s = 1$

$2r + 4s = 0$

$t + 3u = 0$

$2t + 4u = 1$

The solution to this system is $r = -2$, $s = 1$, $t = 1.5$, and $u = -0.5$. In other words, the multiplicative inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$.

For Question 2b, the linear system is

$r + 2s = 1$

$2r + 4s = 0$

$t + 2u = 0$

$2t + 4u = 1$

which has no solution. The first pair of equations is inconsistent, and so is the second pair. Be sure students understand that this means that the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no multiplicative inverse.

Tell students that a matrix that has an inverse is called *invertible,* and a matrix that does not have an inverse is called *noninvertible.*

Question 3: Connecting Invertibility of Matrices to Solvability of Systems

Ask for volunteers to discuss Question 3. The main goal is for students to see that the two systems whose coefficient matrix was the invertible matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ had unique solutions, while the two systems whose coefficient matrix was the noninvertible matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ were either dependent or inconsistent.

Question 4: Which 2 x 2 Matrices Are Invertible?

Have volunteers share ideas about which 2 x 2 matrices have inverses. As needed, remind students that finding the inverse of a matrix involves solving a system of equations. Ask, **Which coefficient matrices give unique solutions and which do not?**

Students should note that 2 x 2 matrices in which one row is a multiple of another give either dependent or inconsistent systems of equations and also do not have inverses. If students don't articulate this, that's okay; this understanding is not needed elsewhere in the unit.

Key Question

Which coefficient matrices give unique solutions and which do not?

Supplemental Activities

When Can You Find an Inverse? (extension) extends the investigation of when matrix inverses exist to 3 x 3 matrices.

The General Two-Variable System (extension) involves solving a two-variable linear system in terms of coefficients using the elimination method. If students complete that problem, they will see that whether the system has a unique solution depends on whether a certain expression involving the coefficients is nonzero—this expression is called the *determinant* of the coefficient matrix.

Determining the Determinant (extension) explores the concept of a determinant.

Calculators Again

Intent

Students use the graphing calculator to solve systems of linear equations.

Mathematics

Calculators Again introduces the use of the calculator to find the inverse of a matrix. Students then apply the use of the inverse to solve matrix equations, tapping the capabilities of their graphing calculators.

Progression

The teacher introduces this activity by demonstrating how to get the inverse of a matrix using a graphing calculator. The activity then gives students four systems of linear equations to solve, in up to six variables. The subsequent discussion reinforces that if a coefficient matrix has no inverse, then the linear system does not have a unique solution, and in solving linear programming problems, one can ignore systems that do not have a unique solution.

Approximate Time

10 minutes for introduction
20 to 25 minutes for activity
15 to 20 minutes for discussion

Classroom Organization

Small groups, preceded by teacher presentation and followed by whole-class discussion

Materials

Calculator Note "Finding the Inverse of a Matrix" (1 copy per student, optional)

Doing the Activity

Demonstrate to students how to use their graphing calculators to find the inverse of a matrix. You might illustrate this first using a matrix for which students already know the inverse, such as $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ from *Inverses and Equations*. The Calculator Note "Finding the Inverse of a Matrix" will help; you can use this for your own review, or provide it to students for reference.

Next use an example that has no inverse, such as $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, so students see what the calculator does in response.

Finally, have them do an example of a 3 x 3 matrix. Ask them how they can verify that the answer that the calculator gives is actually the inverse. They should see that they need to multiply the supposed inverse by the original matrix and check that they get the 3 x 3 matrix that is the identity for multiplication, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Discussing and Debriefing the Activity

Have students from different groups report on their results. Although Question 1 may not seem worth using graphing calculators to solve, Question 3 should definitely make students appreciate them.

For Question 2, students will have found that the coefficient matrix has no inverse (unless they entered a coefficient incorrectly). Ask, **What should you do if the coefficient matrix has no inverse?** Emphasize that this means that the system is either inconsistent or dependent (although students haven't proved this).

In terms of solving linear programming problems, the issue is straightforward. As noted in the discussion of *Finishing Off the Cookies*, when students are looking for "corner points," they can ignore linear systems that do not have a unique solution:

- If the system is inconsistent, then there is no solution, so there can't be a corner point that fits the system.
- If the system is dependent, then the solution set is an "edge," or a "face," or whatever the six-dimensional equivalents are. Geometrically, a "corner point" is a place where "the right number" of planes intersect. (We will use the term "corner point" for arbitrary dimensions.)

The solution to Question 4 is $a = 0$, $b = 2$, $c = 4$, $d = 6$, $e = 8$, and $f = 10$.

A Calculator Warning

At least one model of graphing calculator gives $2 \cdot 10^{-12}$ as the value of a for the system in Question 4, although the actual value for a is 0. You may want to mention to students that they should be wary of round-off problems whenever they use calculators and should always verify solutions. Some common sense may help as well.

Revisiting "Gardener's Dilemma"

In *Gardener's Dilemma,* students developed, but did not solve, a system of linear equations for determining the amount of water needed for each of three categories of plants. Here are those equations:

$$900L + 120F + 40S = 1865$$

$$160F + 800S = 180$$

$$120L + 80F + 240S = 310$$

To give students another example of the use of matrix inverses to solve a system of linear equations, you can have them solve this system on their graphing calculators.

The solution is $L = 2$, $F = 0.5$, and $S = 0.125$.

Key Question

What should you do if the coefficient matrix has no inverse?

Fitting Mia's Bird Houses

Intent

This activity gives students more experience with using coefficients in polynomial functions as the variables for curve fitting.

Mathematics

Fitting Mia's Bird Houses asks students to solve a problem like that discussed in *Fitting Quadratics*. The last part of this problem touches on one of the great stumbling blocks in curve fitting: It is necessary not only to find a function that appears to fit the data well, but also to have some reason to suspect that the particular family of functions (that is, a function of that order) is appropriate for the situation from which the data were extracted.

Progression

The activity gives students data for a situation and asks them to find a quadratic function that fits the data. The subsequent discussion leads to the realization that the quadratic function does not make sense in the context of the given situation.

Approximate Time

30 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students do this activity independently.

Discussing and Debriefing the Activity

Questions 1 and 2 should be fairly straightforward, and Question 3 is an application of the ideas from *Fitting Quadratics*.

You can have students practice the use of matrix inverses for solving linear systems by having them enter the appropriate matrices and solve Question 3 on graphing calculators.

The quadratic function whose graph goes through the points (1, 2), (3, 6), and (5, 9) is $y = -\frac{1}{8}x^2 + 2\frac{1}{2}x - \frac{3}{8}$.

Question 4: Does the Function Make Sense?

Perhaps the most interesting part of this activity is Question 4. Be sure students see that the function they found does not make much sense, because y has become negative by the time x reaches 20. Emphasize that there is no particular reason to expect the problem situation to fit a quadratic function. The method used in *Fitting a*

Line, Fitting More Lines, and *Fitting Quadratics* is appropriate only if there is good reason to think that the situation calls for a particular type of function.

Solving Meadows or Malls?

Intent

In this sequence of activities, students solve the unit problem.

Mathematics

Students should now be ready to put together all of the bits and pieces that they have learned during this unit to solve the rather formidable unit problem—a linear programming problem in six variables with a dozen constraints. As students work on that task, they also summarize what they have learned for their portfolios.

Progression

Getting Ready for *Meadows or Malls?*

Meadows or Malls? Revisited

Beginning Portfolios—Part I

Beginning Portfolios—Part II

Meadows or Malls? Portfolio

Getting Ready for *Meadows or Malls?*

Intent

This activity prepares students for the final stage of solving the central unit problem.

Mathematics

This activity focuses on simplifying work on a complex problem. Looking at the constraints for the central unit problem, students develop a reduced list of systems whose solutions need to be checked.

Progression

Students revisit their constraints for the *Meadows or Malls?* unit problem, and begin to solve this linear programming problem.

Approximate Time

30 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

For the activity discussion and for students' work on the upcoming activity *"Meadows or Malls?" Revisited,* you should have the constraints and the most recent version of the general strategy posted in the classroom. Here are the constraints again:

I	$G_R + G_D = 300$
II	$A_R + A_D = 100$
III	$M_R + M_D = 150$
IV	$G_D + A_D + M_D \geq 300$
V	$A_R + M_R \leq 200$
VI	$A_R + G_D = 100$
VII	$G_R \geq 0$
VIII	$A_R \geq 0$
XI	$M_R \geq 0$
X	$G_D \geq 0$
XI	$A_D \geq 0$
XII	$M_D \geq 0$

Begin by soliciting a list of combinations that would need to be checked if no special insights were used—that is, the list of all combinations of constraints that consist of the four equations and two of the inequalities. This gives 28 combinations.

Ask, **How could you reduce the number of combinations that need to be checked?** Have volunteers share ideas. If necessary, ask, **Are there any variables that can't be zero?**

If you want to give students a further hint, point out that $G_D \leq 100$ (by constraint VI) and ask, **Does constraint I and the fact that GD ≤ 100 tell you anything about GR?** This should help them see that $G_R \geq 200$, so $G_R \neq 0$, so they can ignore the equation corresponding to constraint VII in looking for corner points. The fact that this reduces the number of six-variable systems they need to solve should motivate them to look for other facts.

Here are some conclusions that students might reach, with possible explanations:

- G_R cannot be zero.

Explanation (as just discussed): Constraint VI says $A_R + G_D = 100$, so G_D is at most 100. On the other hand, constraint I says $G_R + G_D \geq 300$, so G_R must be at least 200.

- G_D cannot be zero.

Explanation: Constraint II says $A_R + A_D = 100$ and constraint III says

$M_R + M_D = 150$. Together, these facts guarantee that $A_D + M_D$ is at most 250. But constraint IV says that $G_D + A_D + M_D$ is at least 300, so G_D must be at least 50.

- A_D cannot be zero.

We've just seen that $G_D \geq 50$, but constraint VI says $A_R + G_D = 100$, so A_R must be at most 50. But constraint II says $A_R + A_D = 100$, so A_D must be at least 50.

Because the variables G_R, G_D, and A_D cannot be zero, students can ignore the equations corresponding to constraints VII, X, and XI in their search for corner points. This cuts the number of cases that must be checked down to only ten. That is, students need to check the following ten combinations. (For simplicity, the four *required* constraints are listed first here.)

- I, II, III, VI, IV, V
- I, II, III, VI, IV, VIII
- I, II, III, VI, IV, IX
- I, II, III, VI, IV, XII
- I, II, III, VI, V, VIII
- I, II, III, VI, V, IX
- I, II, III, VI, V, XII

- I, II, III, VI, VIII, IX
- I, II, III, VI, VIII, XII
- I, II, III, VI, IX, XII

Key Questions

How could you reduce the number of combinations that need to be checked?

Are there any variables that can't be zero?

Does constraint I and the fact that $G_D \leq 100$ tell you anything about G_R?

Meadows or Malls? Revisited

Intent

Students solve the central unit problem.

Mathematics

Students are now ready to completely solve the unit problem—a linear programming problem in six variables with 12 constraints.

Progression

Students work in groups to solve the unit problem and to prepare a group report for the city manager. You may want to have them prepare this report individually, summarizing the work they completed as a group. Several group presentations cap off the activity.

Approximate Time

5 minutes for introduction
65 to 70 minutes for activity
40 minutes for presentations

Classroom Organization

Small groups, followed by several group presentations

Doing the Activity

Students now solve the unit problem.

Level of Detail for Reports

Before students begin work, discuss what they should include in their reports. Specifically, go over how much explanation, and at what level, they need to include. You might frame this discussion as a clarification of what they can assume the city manager knows. For instance, clarify whether reports should include any or all of these items:

- A statement of the group's overall strategy
- An explanation of why the solution must be at a corner point
- A discussion of why finding the corner points involves solving systems of six equations
- An explanation of how to use matrix inverses to solve systems of linear equations
- A discussion of the meaning of matrix multiplication

You may want to establish your own criteria on these issues, or you may prefer to let the class resolve them according to what it considers to be reasonable.

Hints on the Activity

If students get stuck, you can review the general strategy they developed earlier in the unit. As they begin to find corner points of their feasible region, you can suggest that they try to use matrices as a shortcut for finding the cost to the city in each case.

For example, if they represent a system using the matrix equation [A][X] = [B] (where [A] is the coefficient matrix and [B] is the constant term matrix), they can get the solution as the matrix product $[A]^{-1}[B]$. They can then get the cost for this allocation as $[C][A]^{-1}[B]$, where [C] is the row vector of cost function coefficients. They can do these matrix multiplications on the graphing calculator.

(You may need to remind students that the order of these matrices is important. If they write $[A]^{-1}[C][B]$ or some other sequence, they will probably get either an error message or the wrong answer.)

Give overhead transparencies and pens to one or two groups that are the first to finish so that they can prepare presentations.

Discussing and Debriefing the Activity

If you have not already selected presenters, you may want to pick a group at random for the first presentation and then let other groups add any further insights or present alternative methods.

As discussed in the notes for *Getting Ready for Meadows or Malls?*, students need only check the systems of equations corresponding to the following combinations of constraints in their search for corner points. The constraints themselves are listed in *Getting Ready for Meadows or Malls?*

- I, II, III, VI, IV, V
- I, II, III, VI, IV, VIII
- I, II, III, VI, IV, IX
- I, II, III, VI, IV, XII
- I, II, III, VI, V, VIII
- I, II, III, VI, V, IX
- I, II, III, VI, V, XII
- I, II, III, VI, VIII, IX
- I, II, III, VI, VIII, XII
- I, II, III, VI, IX, XII

Much of the work can be done on the graphing calculator, as students have seen. For example, to check the first combination, they would enter, as matrix [A], the matrix of coefficients for the equations that go with these constraints.

Ordering the variables as in constraints VII through XII and ordering the equations as shown in the first list of combinations (I, II, III, VI, IV, V) gives this coefficient matrix:

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and this matrix of constant terms:

$$[B] = \begin{bmatrix} 300 \\ 100 \\ 150 \\ 100 \\ 300 \\ 200 \end{bmatrix}$$

To solve this system, simply have the graphing calculator compute $[A]^{-1}[B]$:

$$[A]^{-1}[B] = \begin{bmatrix} 50 \\ -150 \\ 350 \\ 250 \\ 250 \\ -200 \end{bmatrix}$$

Because this product has negative values for two of the variables, it violates constraints VIII and XII, and so this potential corner point is outside the feasible region.

To look at the other cases, students need only change the last row or two of the matrices [A] and [B]. They also need to check whether their solutions satisfy the constraints not used in the given combination. In four cases (including the one just used to illustrate the process), at least one of the variables comes out negative, and in two more cases, constraint IV is violated. So these six cases give points outside the feasible region.

In one of the remaining four cases, the matrix [A] has no inverse, so students can ignore this system. (The equations corresponding to constraints IX and XII—$M_R = 0$

and $M_D = 0$—are inconsistent with constraint III, which says $M_R + M_D = 150$. In other words, this system has no solution.)

This table gives the results in each case:

Combination	G_R	A_R	M_R	G_D	A_D	M_D	Cost
I, II, III, VI, IV, V	50	−150	350	250	250	−200	violates VIII and XII
I, II, III, VI, IV, VIII	200	0	50	100	100	100	365,000
I, II, III, VI, IV, IX	225	25	0	75	75	150	353,750
I, II, III, VI, IV, XII	150	−50	150	150	150	0	violates VIII
I, II, III, VI, V, VIII	200	0	200	100	100	−50	violates XII
I, II, III, VI, V, IX	400	200	0	−100	−100	150	violates X and XI
I, II, III, VI, V, XII	250	50	150	50	50	0	violates IV
I, II, III, VI, VIII, IX	200	0	0	100	100	150	410,000
I, II, III, VI, VIII, XII	200	0	150	100	100	0	violates IV
I, II, III, VI, IX, XII	[A] has no inverse						

This leaves three cases that satisfy all the constraints. Thus, the feasible region actually has only three corner points. The last column in the table above gives the cost to the city for these cases, based on the cost expression

$$50G_R + 200A_R + 100M_R + 500G_D + 2000A_D + 1000M_D$$

Thus, the least expensive solution for the city is given by this allocation of land:

$G_R = 225$

$A_R = 25$

$M_R = 0$

$G_D = 75$

$A_D = 75$

$M_D = 150$

This allocation costs the city $353,750.

Note for Teachers: The Simplex Method

Although the *Meadows or Malls?* problem is somewhat cumbersome, the ideas are mostly the same as those involved in other problems, such as *More Cookies*. You might think that with a computer, you could easily solve any linear programming problem using this method. In fact, however, real-world linear programming problems often involve dozens, or even hundreds, of variables and constraints. The number of possible combinations of constraints quickly becomes astronomical, so that even the fastest computer cannot find and test them all.

In the immediate post–World War II era, a mathematician named George Dantzig (1914–2005), who later became a professor of operations research and computer science at Stanford University, developed an algorithm called the *simplex method* for efficiently moving through the feasible region in search of the optimal corner point.

The simplex method has since been made more efficient, but it remains the basis for practical solution of highly complex linear programming problems.

Beginning Portfolios—Part I

Intent

This activity is one of two parts of students' initial work in preparing their portfolios.

Mathematics

This activity focuses on the connection between possible intersections of planes in 3-space and possible results from solving a system of three linear equations in three variables.

Progression

Beginning Portfolios–Part I asks students to summarize how three ideas are related: graphing linear equations in three variables, solving systems of linear equations in three variables, and finding intersections of planes in 3-space. This activity will be included in the unit portfolio.

Approximate Time

20 to 25 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

Ask for volunteers to read their summaries. Other students can add their ideas to those presented. Remind students that this activity will be part of their portfolios. They should add to it as necessary to make it complete and useful.

Beginning Portfolios—Part II

Intent

This activity will constitute the second part of students' portfolios.

Mathematics

Beginning Portfolios—Part II asks students to summarize what they learned about matrices and about solving equations with matrices.

Progression

As with *Beginning Portfolios—Part I*, you can follow this up by letting volunteers share their ideas.

Approximate Time

20 to 25 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

You can have a whole-class discussion with students offering their ideas. Students should revise this activity as necessary for inclusion in the portfolio.

Meadows or Malls? Portfolio

Intent

In this activity, students complete their unit portfolios.

Mathematics

Students will have done part of the selection process in *Beginning Portfolios—Part I* and *Beginning Portfolios—Part II*, so their main task in this activity is to write their cover letters.

Progression

As with previous unit portfolios, students write a cover letter in which they describe the main mathematical ideas of the unit and how they were developed, select and include key papers from the unit, and reflect on personal growth during the unit.

Approximate Time

0 to 5 minutes for introduction
30 to 40 minutes for activity (at home or in class)

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

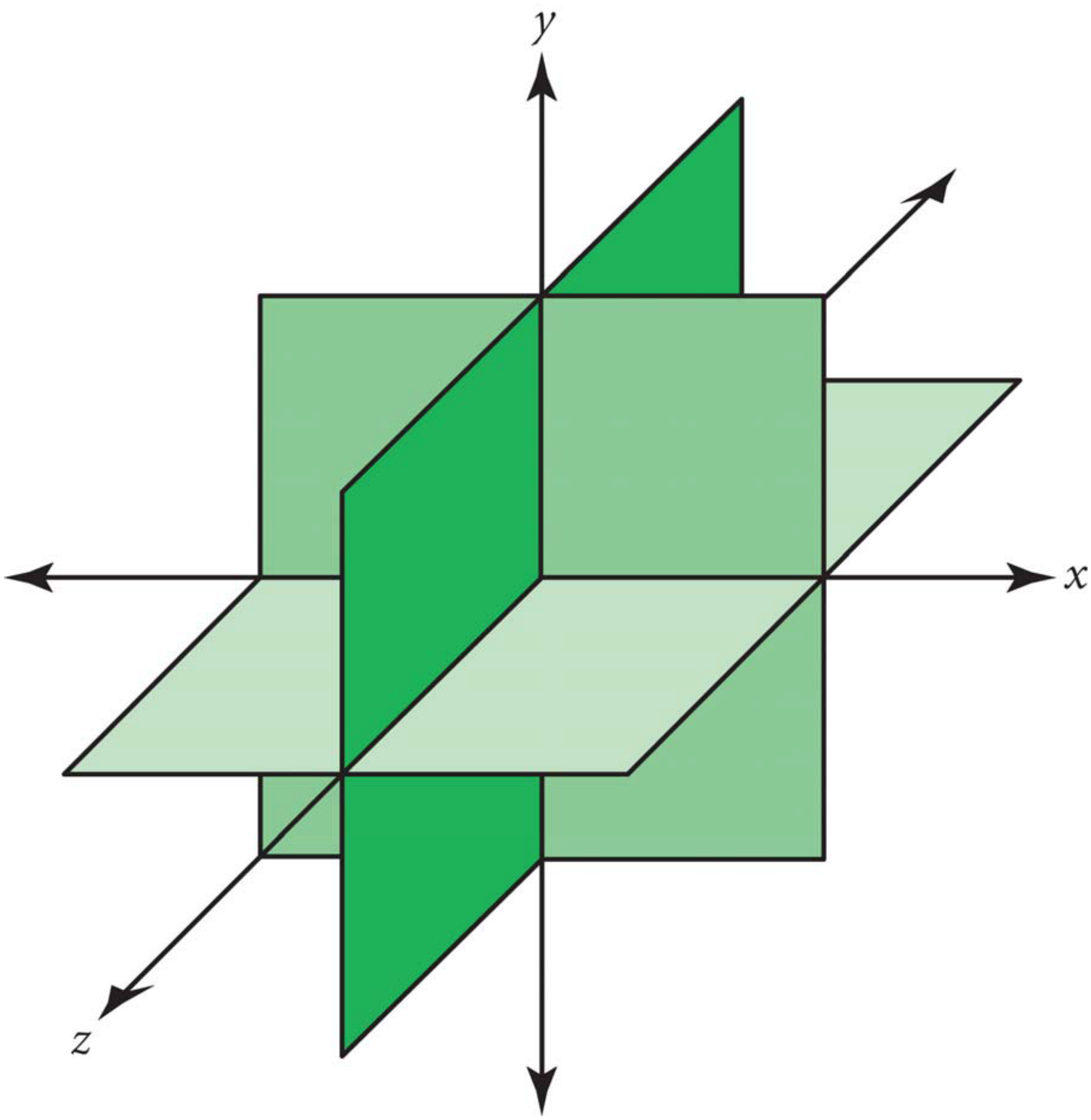
Students work independently writing their cover letter and selecting portfolio activities that reflect the mathematical ideas learned in the unit.

Discussing and Debriefing the Activity

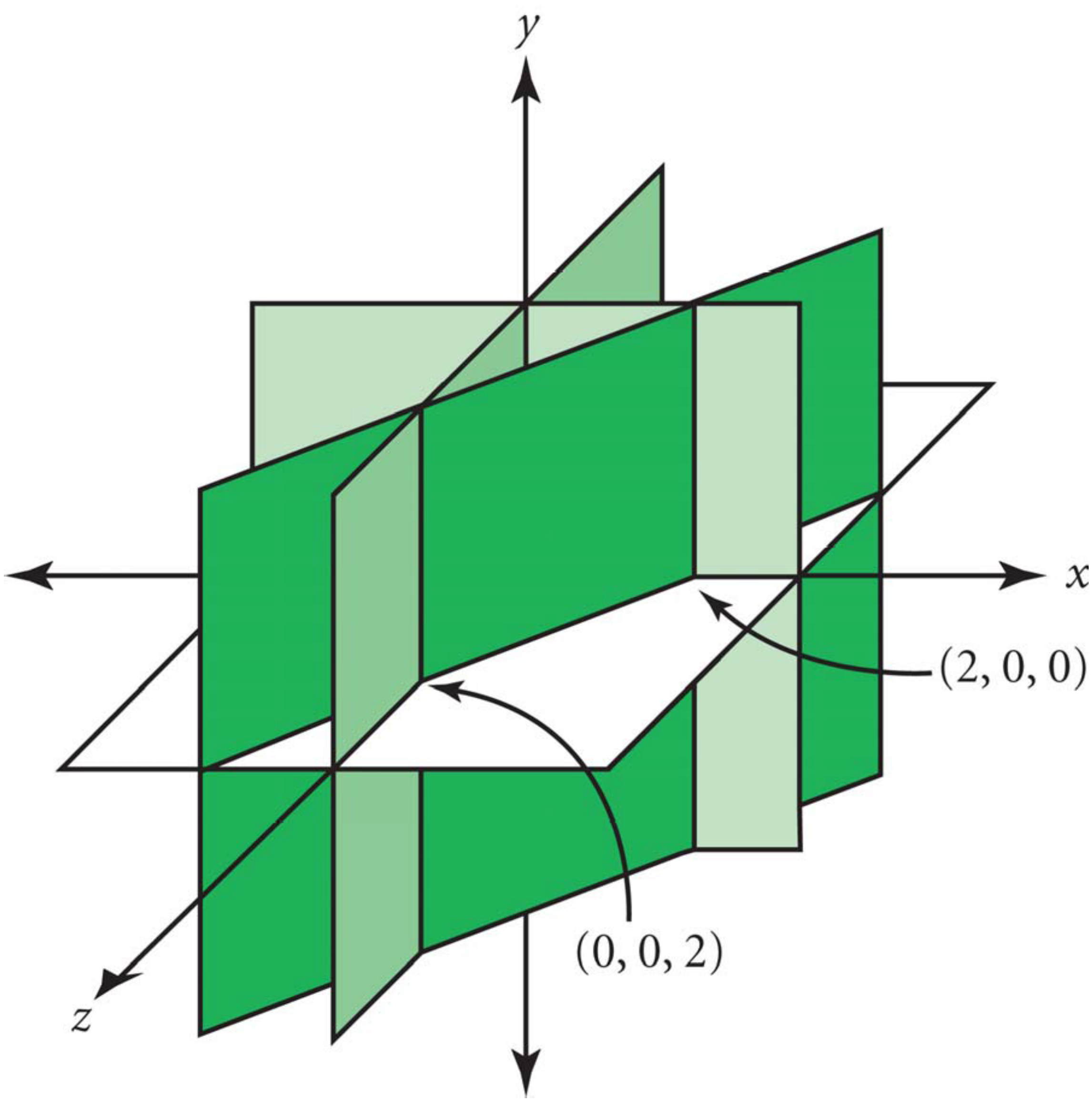
Have volunteers share their portfolio cover letters as a way to start a discussion to summarize the unit.

Blackline Masters

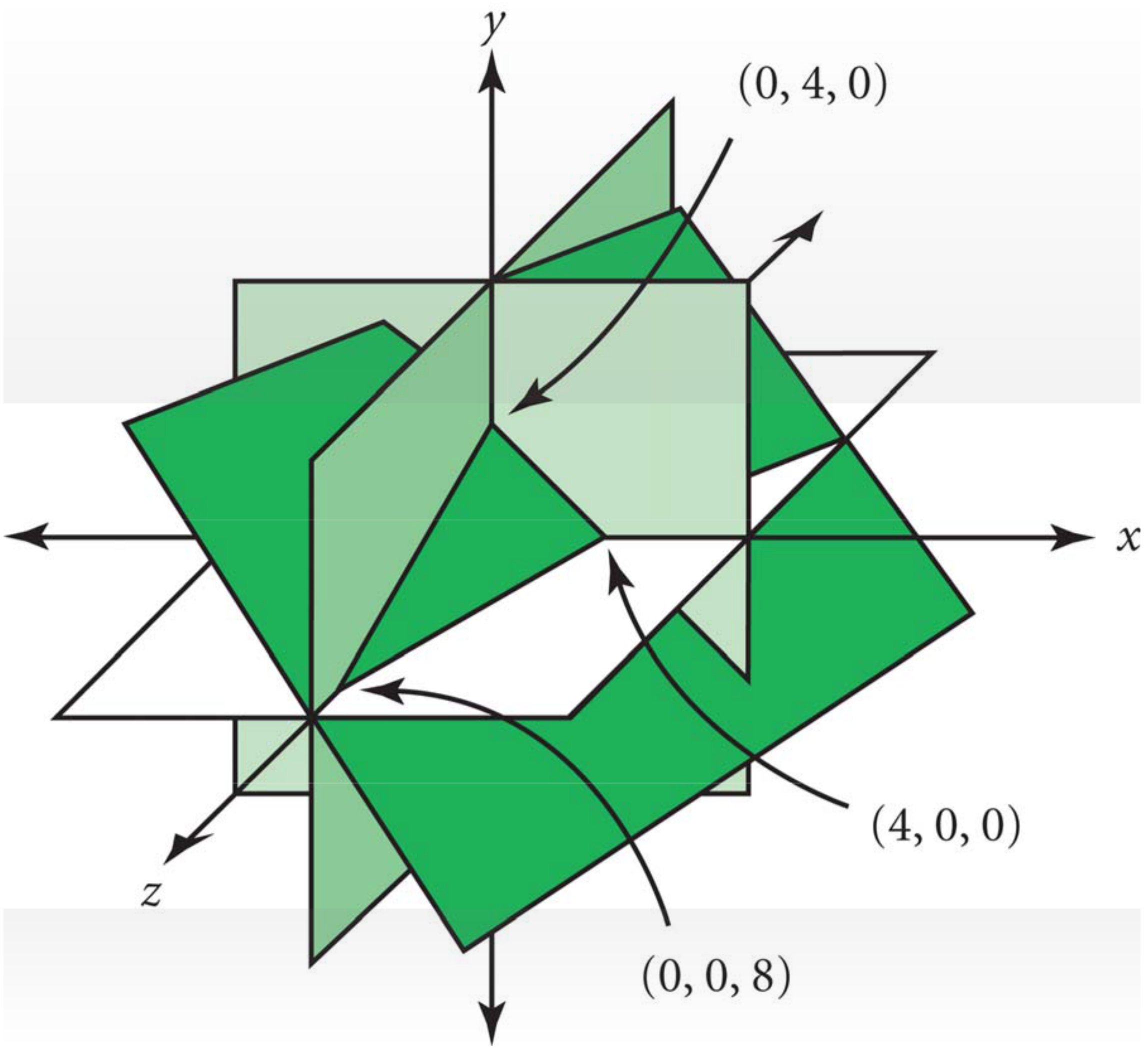
The Three-Variable Coordinate System



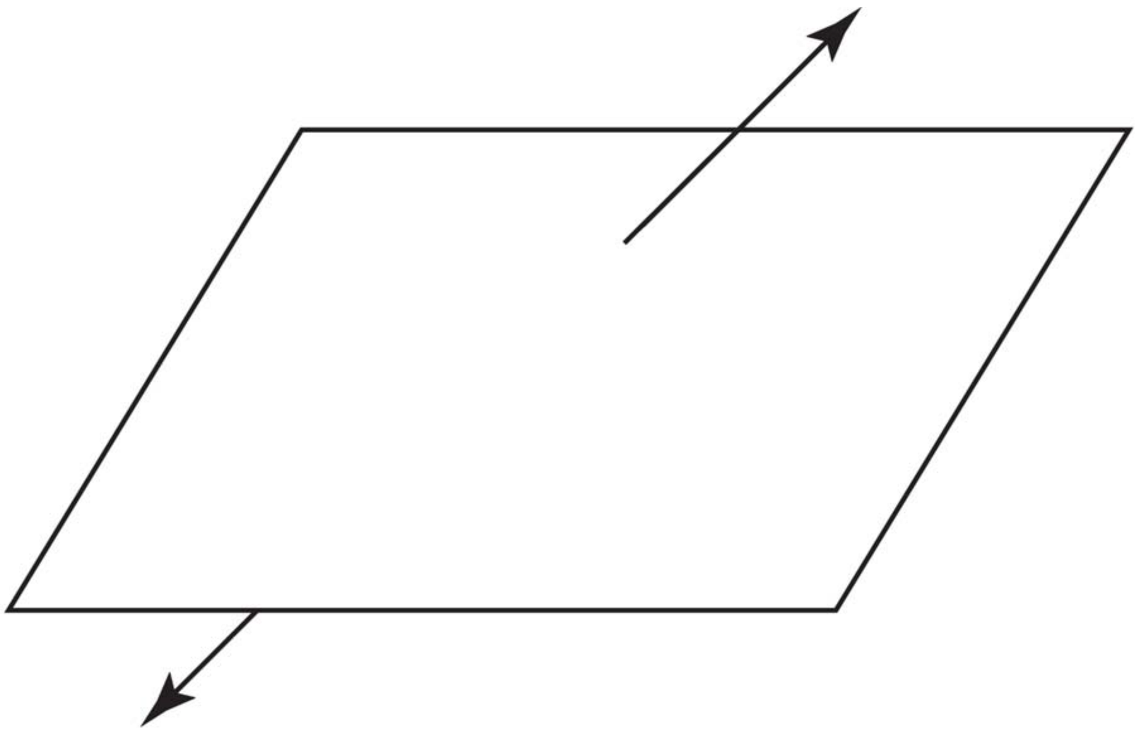
What Do They Have in Common?

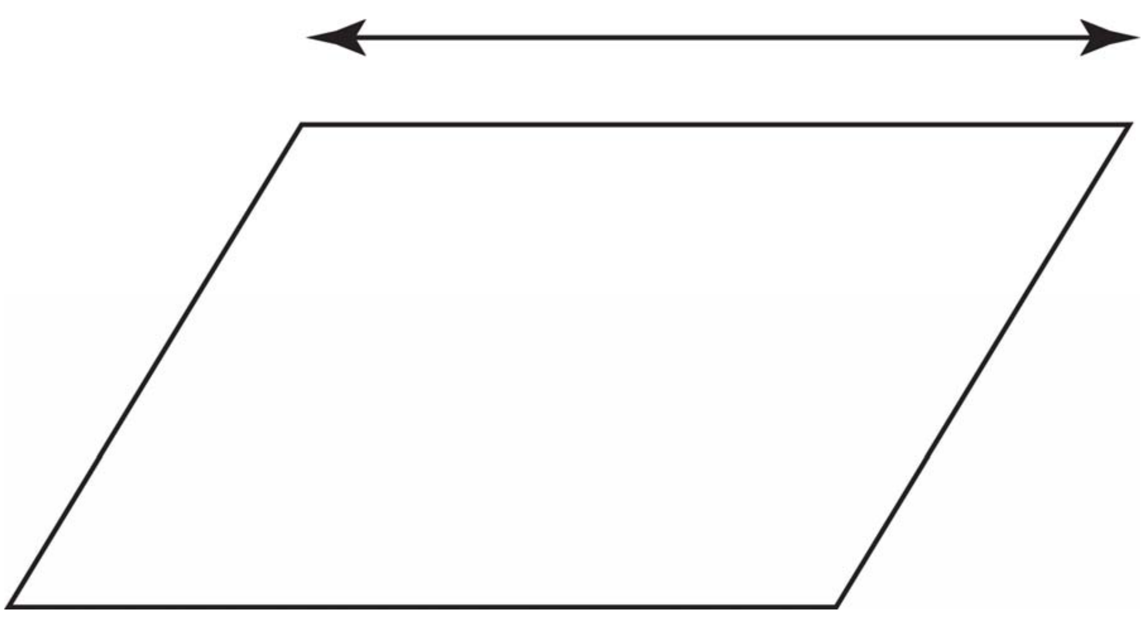


Trying Out Triples



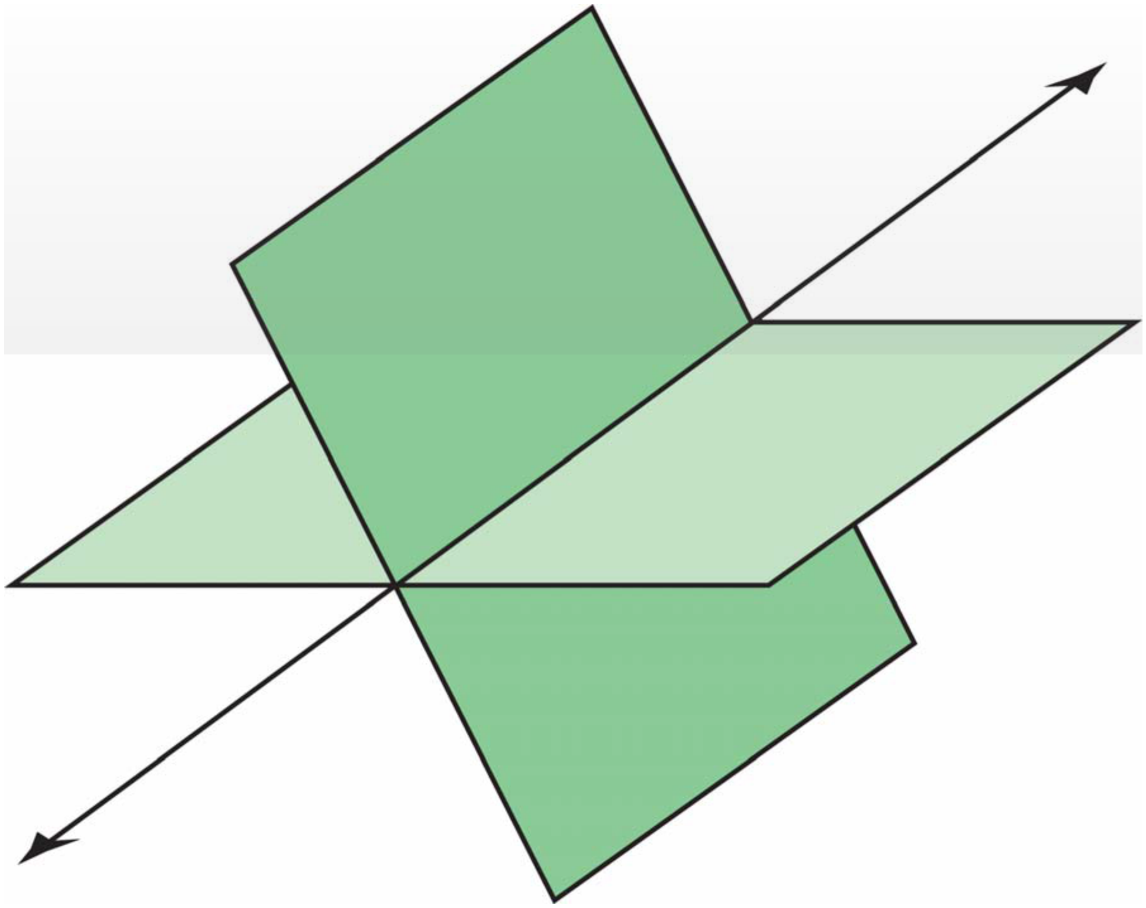
Just the Plane Facts



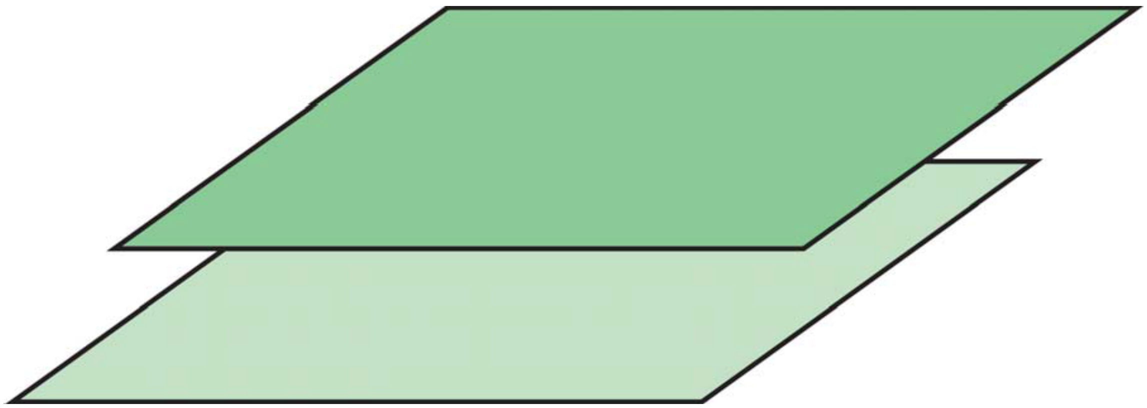


Just the Plane Facts (continued)

Just the Plane Facts (continued)

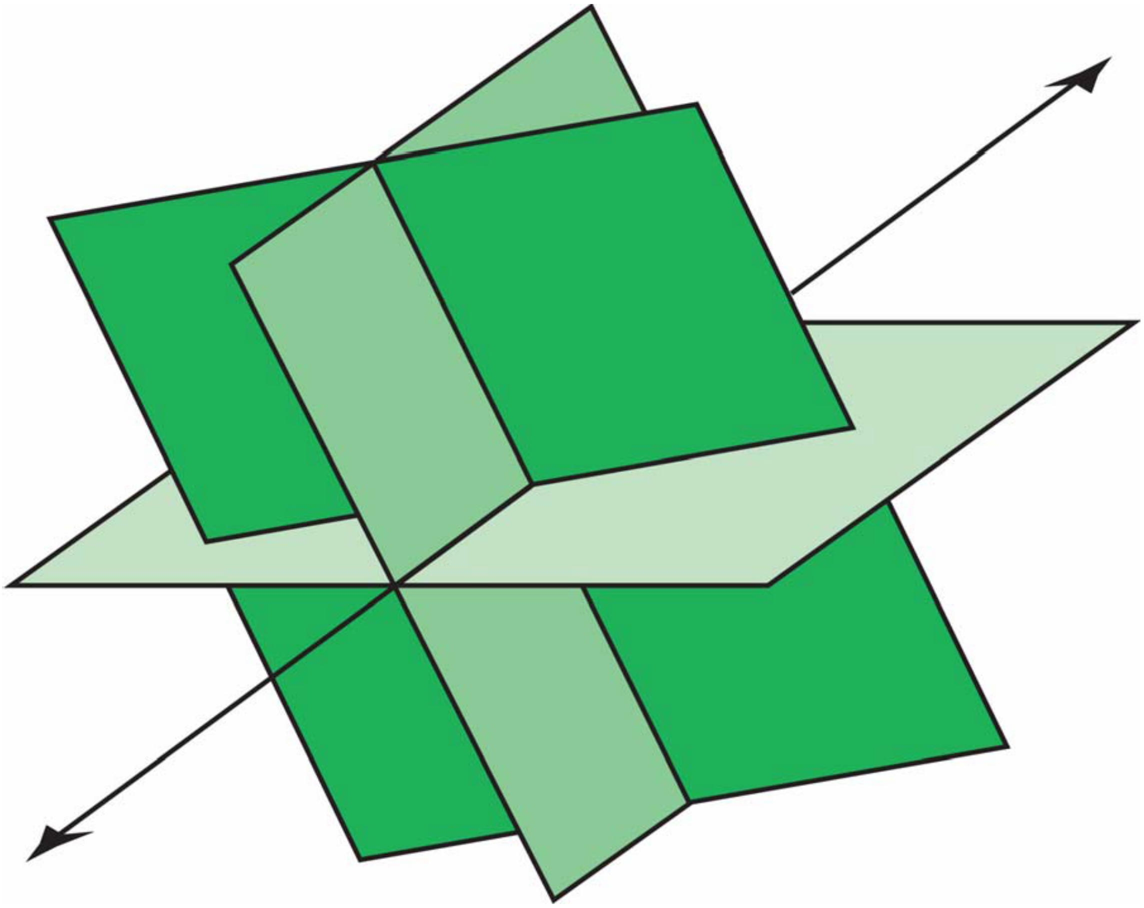


Just the Plane Facts (continued)

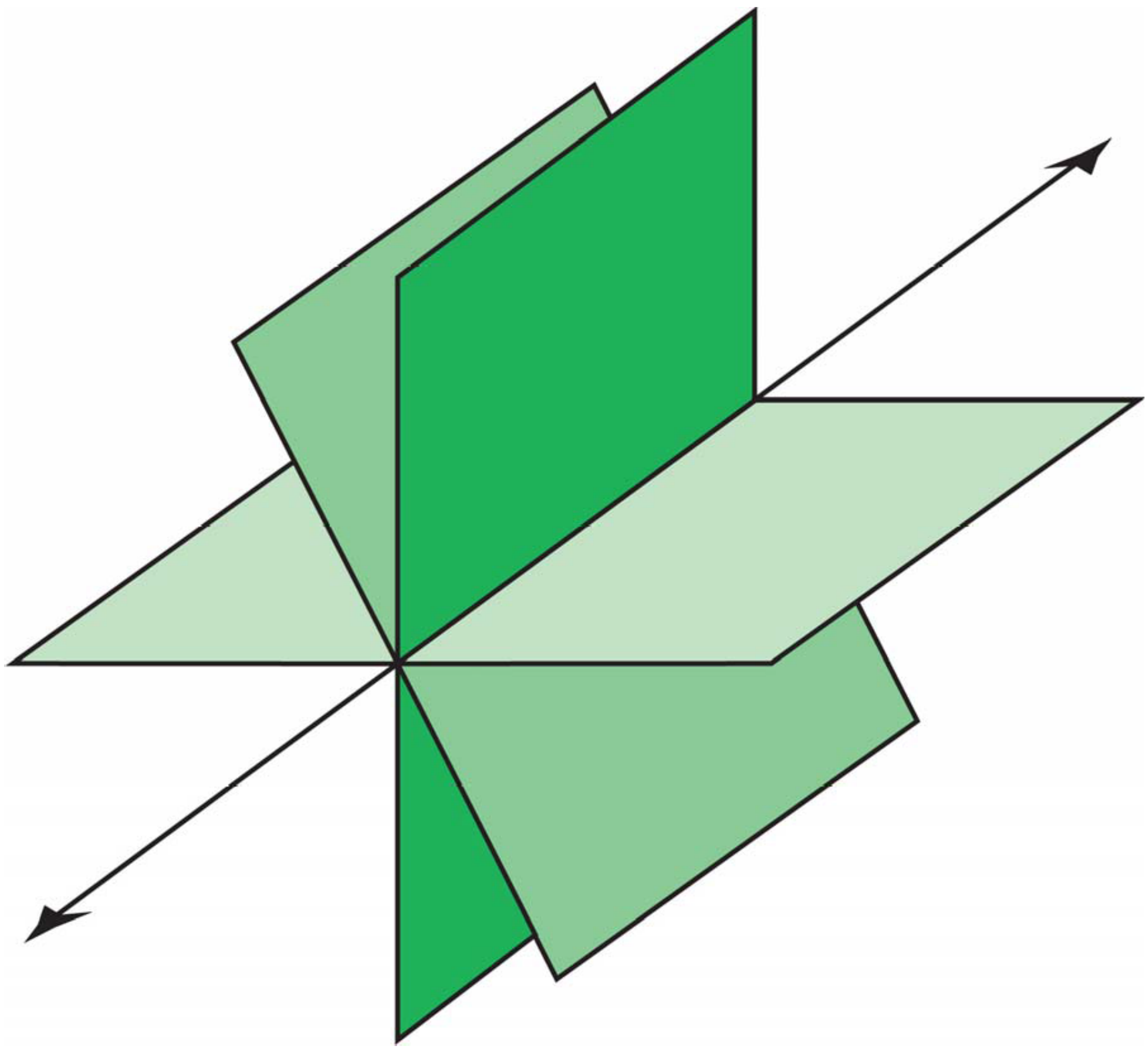


Just the Plane Facts (continued)

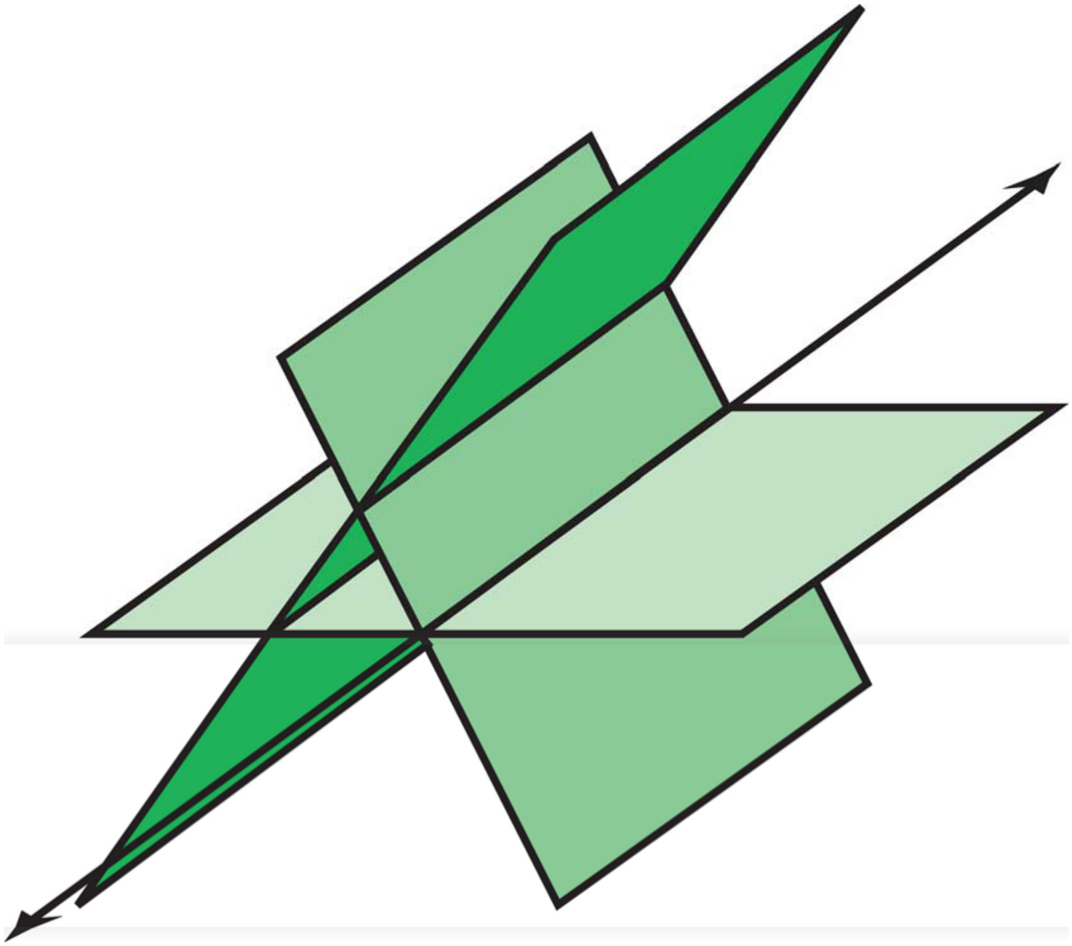
Just the Plane Facts (continued)



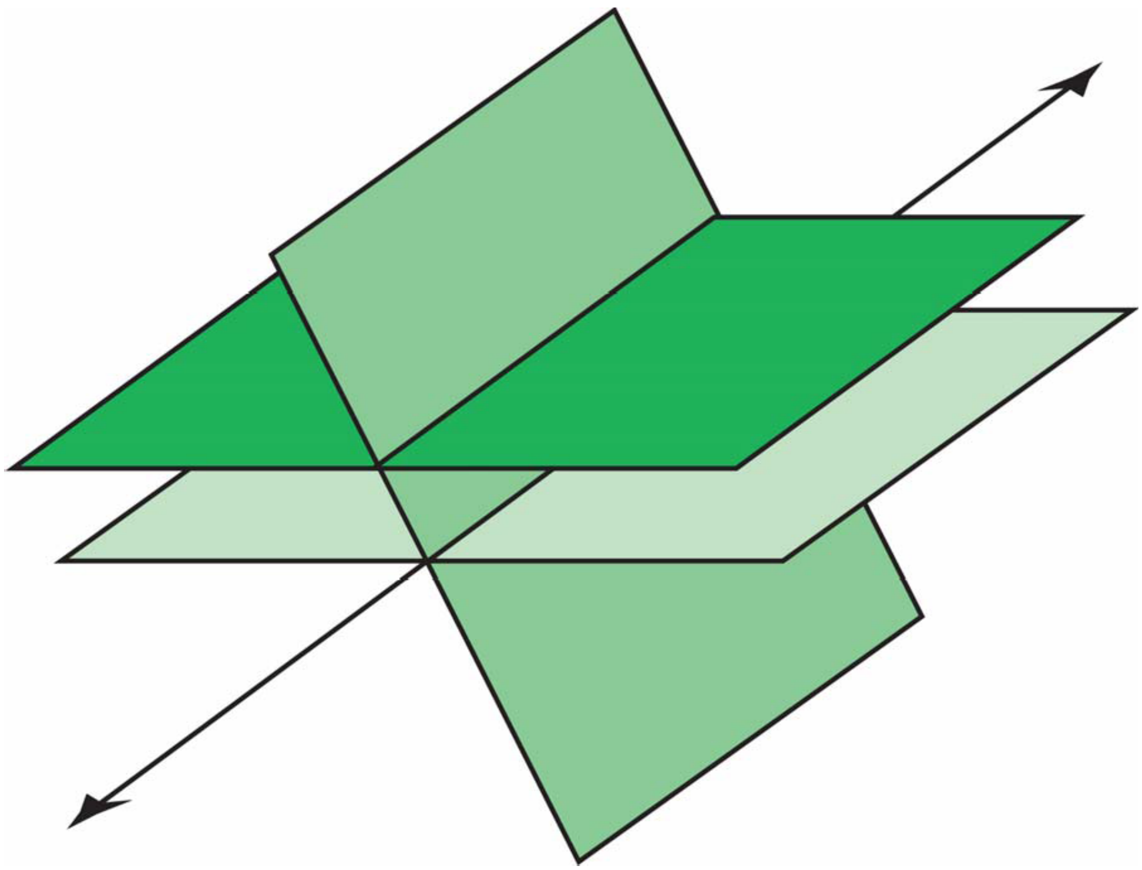
Just the Plane Facts (continued)



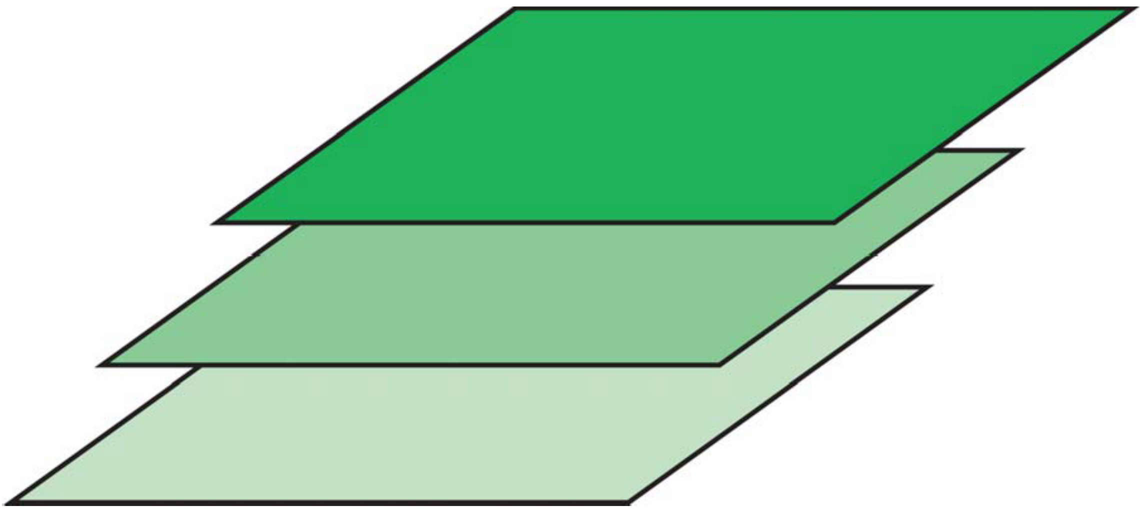
Just the Plane Facts (continued)



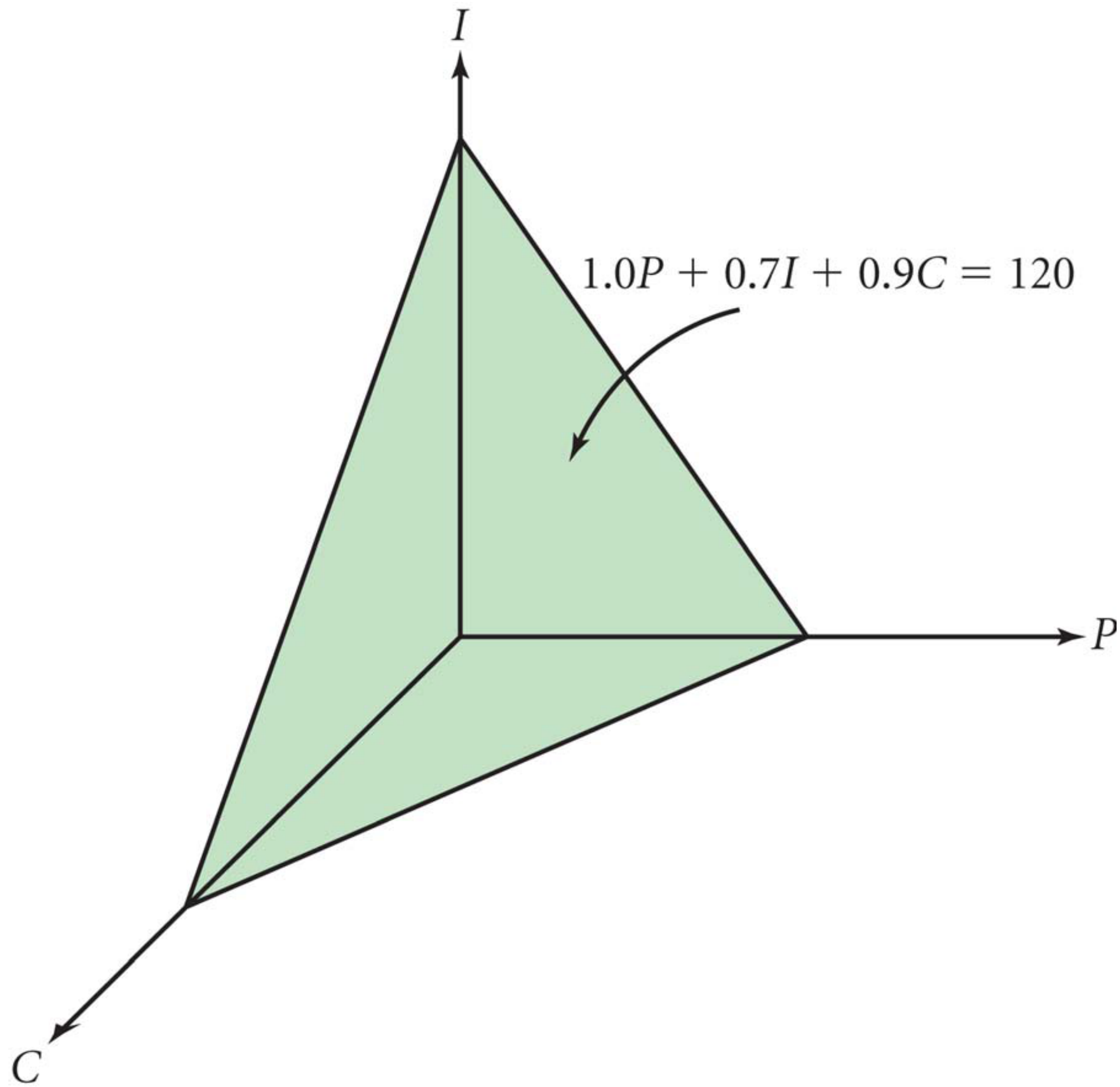
Just the Plane Facts (continued)

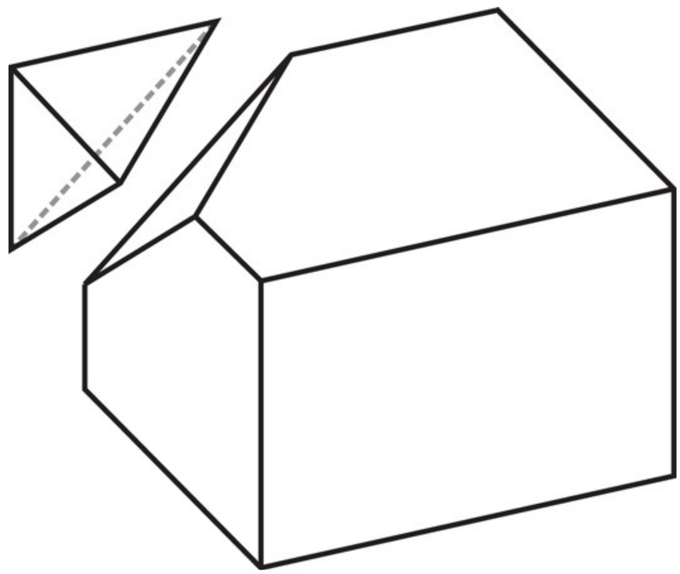


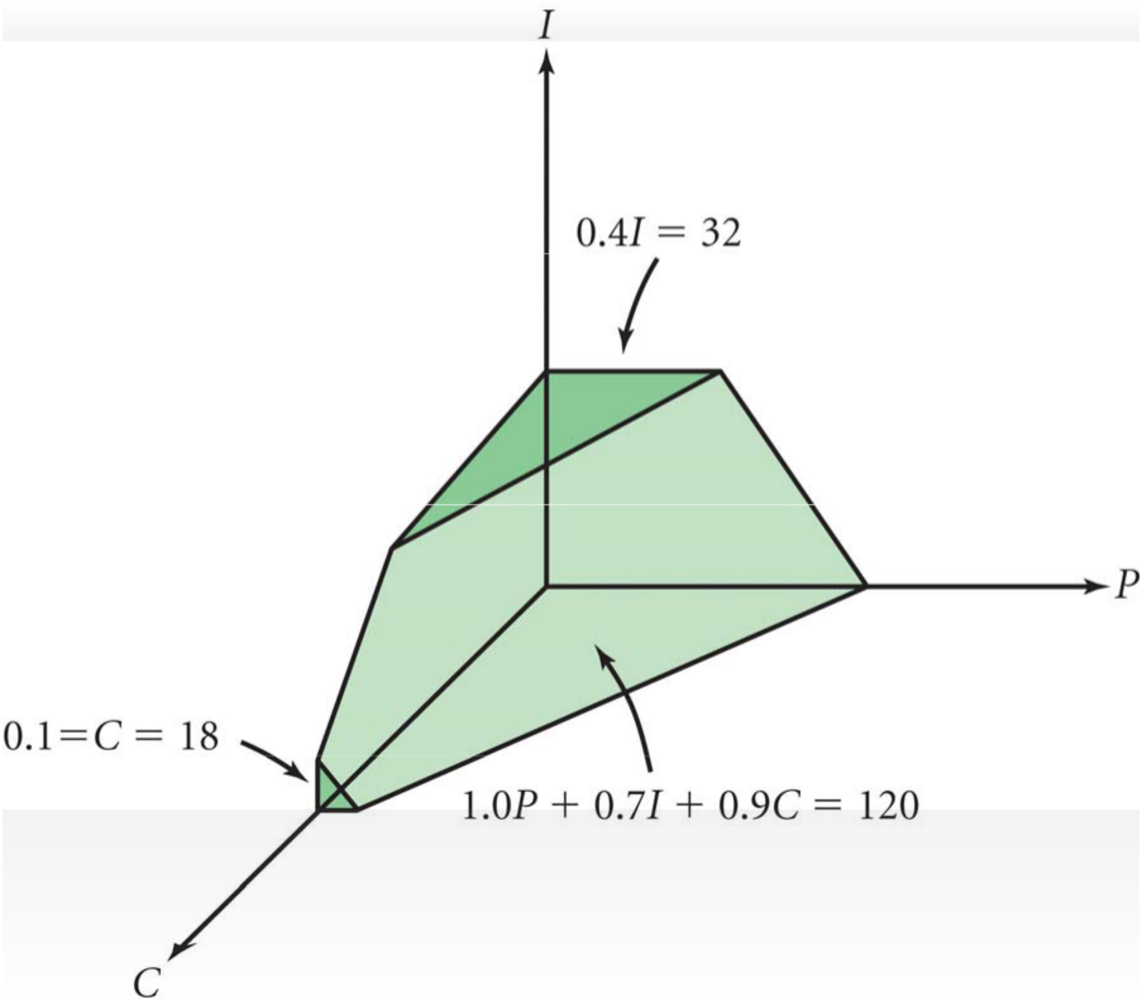
Just the Plane Facts (continued)



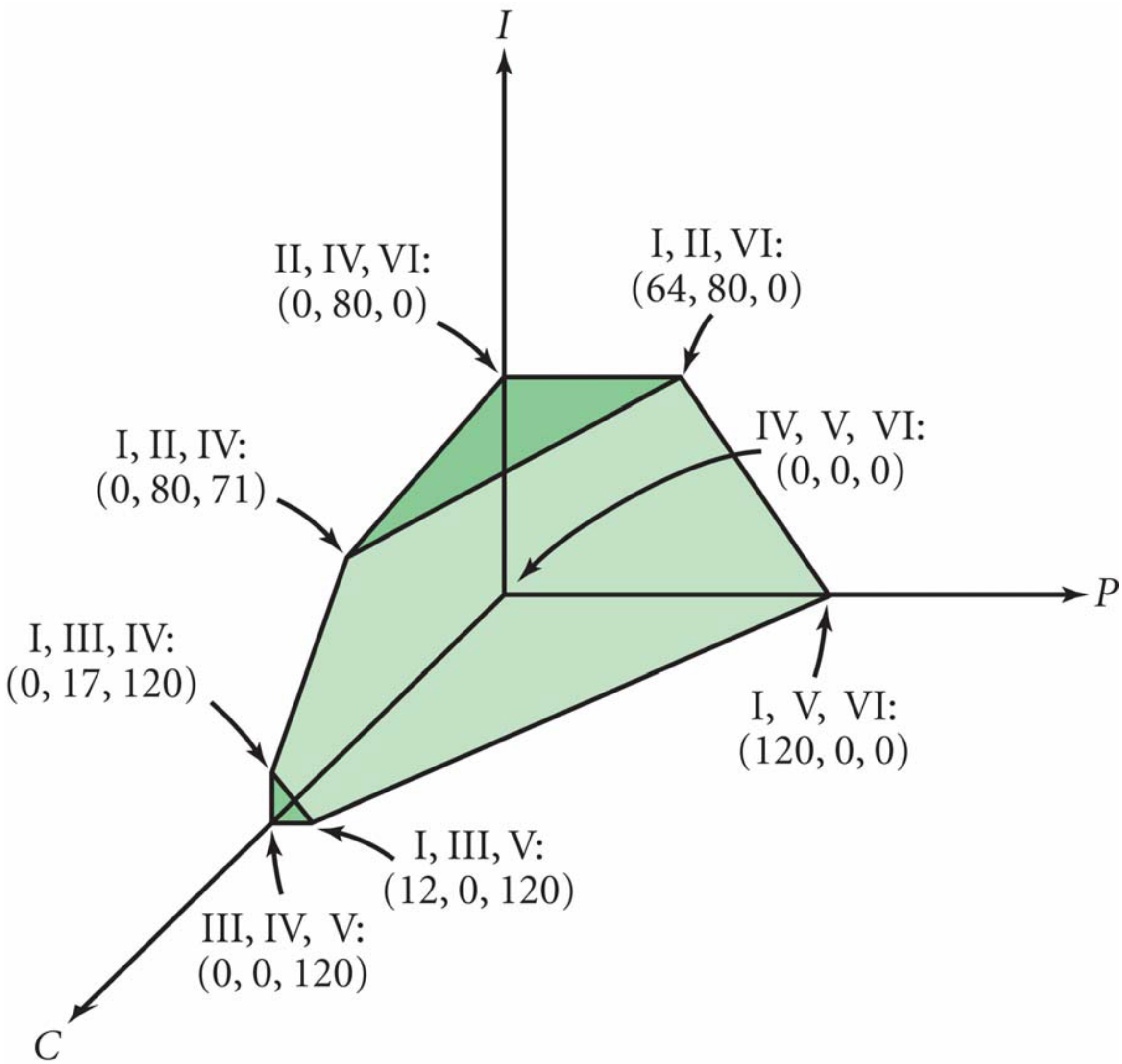
The More Cookies Region and Strategy



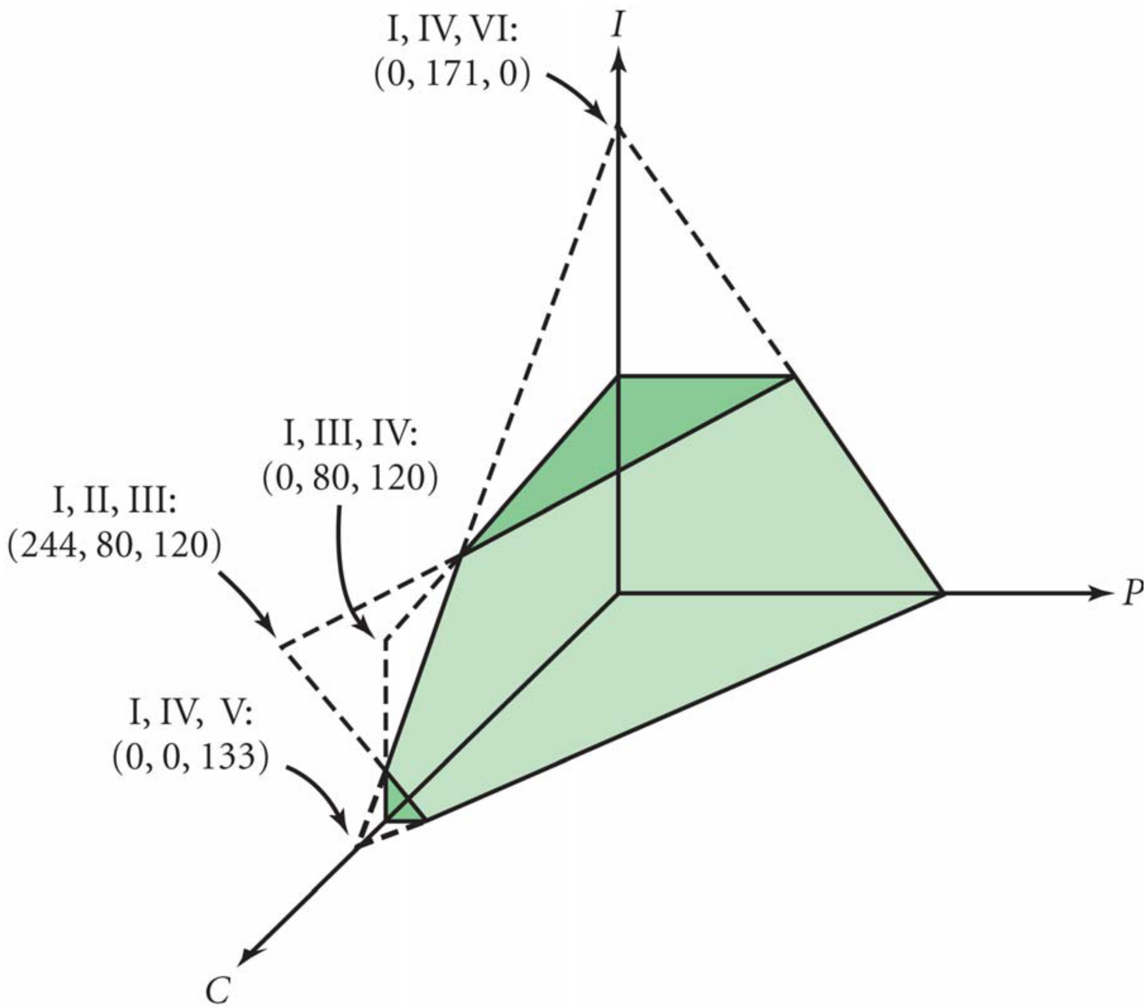



I
0.4I = 32
P
0.1=C = 18
1.0P + 0.7I + 0.9C = 120
C

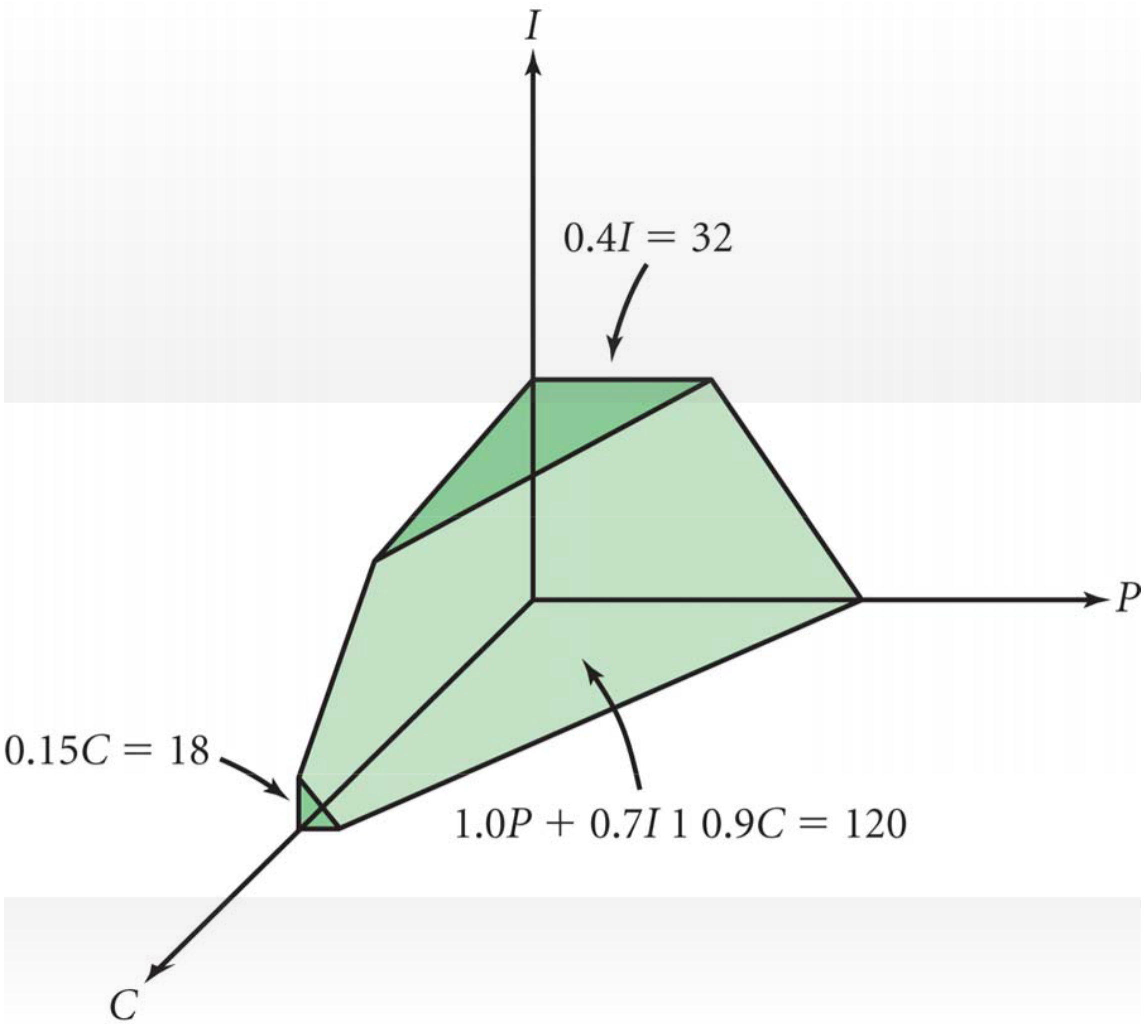
Finishing Off the Cookies



Finishing Off the Cookies (continued)



Finishing Off the Cookies (continued)



¼-Inch Graph Paper

1-Centimeter Graph Paper

1-Inch Graph Paper

Meadows or Malls? In-Class Assessment

You have worked with several methods for solving systems of linear equations, including

- graphing
- substitution
- elimination
- matrix algebra

Solve these systems of equations using a different method for each system. Explain why you are choosing the particular method in each case.

1. $2f + 3g = 40$
   $4f - 5g = -30$

2. $5w - 4z = 12$
   $3w = 6z - 18$

3. $7a + 4b = 17$
   $3a - 2b = 11$

4. $3x - y + z + 2w = 10$
   $2x + 2y - z - w = -4$
   $x + 3y - 3z + 4w = 10$
   $-x + 3y + 2z - 3w = -18$

Meadows or Malls? Take-Home Assessment

1. Write a matrix equation that represents this system of equations.

   $13a + 12b - 15c = 20$

   $24a - 3b + 36c = 71$

   $2a + 4b + 7c = 36$

   (You do not need to solve the matrix equation.)

2. Aki is making vases and cups out of clay. Each vase requires 0.5 pounds of clay and 0.1 pounds of glaze. Each cup requires 0.3 pounds of clay and 0.07 pounds of glaze. Aki plans to make 40 vases and 20 cups this week and 30 vases and 45 cups next week.

   Show how to use matrices to figure out how much clay and how much glaze Aki will need each week. You need to set up the appropriate matrices and explain how to combine them to get the answer.

3. Marilyn recently scored 27 points in a basketball tournament. Her points came from a combination of free throws (worth one point each), two-point field goals, and three-point field goals.

   Marilyn made a total of 13 shots (including free throws). She made half as many free throws as two-point field goals.

   Set up and solve a system of linear equations to determine how many of each type of shot she made.

Meadows or Malls? Calculator Guide for the TI-83/84 Family of Calculators

Meadows or Malls? is a true reflection of how the nature of important mathematics is affected by today's powerful technology. This unit presents a complex linear programming problem involving six variables and six equations. The graphing calculator's ability to manipulate matrices is essential—through the use of the calculator, solving systems of equations becomes a much simpler task. It is possible to use paper and pencil, but the calculator allows students to concentrate on more sophisticated questions rather than on repetitive mechanics.

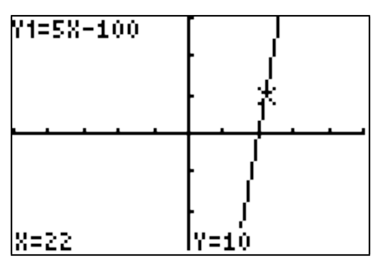
Once students learn how to enter matrices into the calculator, they can do matrix operations, such as addition, multiplication, and inversion, on the calculator. The matrix inversion feature of the graphing calculator will help students solve the unit problem.

Like *Orchard Hideout*, much of this unit further develops coordinate geometry's connection to algebraic ideas, focusing especially on three- dimensional geometry. Few calculator skills are needed during these stages. However, students may recall that the calculator has graphing capabilities that can be used to solve two-variable systems of equations.

Heavy Flying: Several students likely will recall working with the calculator to graph the linear equations defined by the constraints of *Heavy Flying*. If not, there is no particular need to push students to remember the techniques. You may choose to provide students with the Calculator Note "Solving Systems by Graphing" following the discussion of *Ideas for Solving Systems*. You may also want to review the Calculator Note "Function Graphing" from the *Calculator Basics* in the Year 3 general resources.

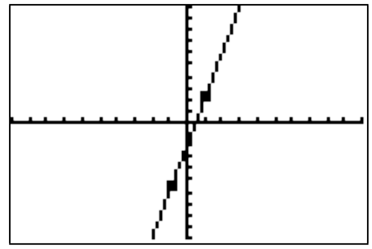
Ideas for Solving Systems: Students may want to use a graphing calculator to help them solve two-variable linear programming problems. The Calculator Note "Solving Systems by Graphing" summarizes the graphing calculator techniques for solving these types of problems. You may wish to use this Calculator Note in conjunction with the *Ideas for Solving Systems* reference pages in the student text. Note that the examples in the Calculator Note use the equations from *Heavy Flying*.

How Much After How Long?: While discussing this activity, use the graphing and tracing features of the calculator to explore the relationships between the situations and the ordered pairs of the associated functions. Refer to the Calculator Note "Function Graphing" from the *Calculator Basics* in the Year 3 general resources to review these features. Be sure to replace the independent variable with **X** and the dependent variable with **Y** on the calculator. After setting up the graph, press $\boxed{\text{ZOOM}}$ and select **8:ZInteger**— doing so will allow you to trace only integer values for **X**,

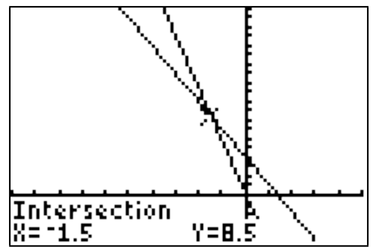


which may be helpful in this activity. This screen shows an ordered pair found while tracing **Y1=5X–100**.

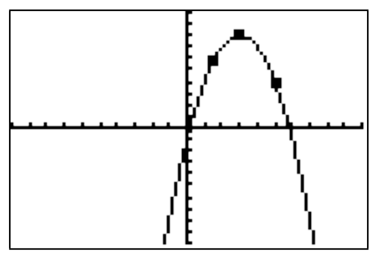
Fitting a Line: During the discussion of this activity, you may find that some students have used the regression capabilities of the graphing calculator, although that is not the expectation here. Even if students come up with such a calculator technique for solving the problem, they should still be expected to understand the non-calculator technique being developed in the activity.



Fitting More Lines: Again, some students may use the calculator to help solve problems like those in this activity. In addition to using the regression method mentioned in *Fitting a Line*, students might graph the equations from Questions 1a and 2a to determine the point of intersection. Then, they would use these coordinates for *a* and *b* when completing Question 3a. (The same method could be used for Question 4 as well.) The emphasis in this activity (and similar activities) is on the idea that solving the pair of equations from Questions 1 and 2 leads to a linear function that passes through both (3, 4) and (5, 1).



Fitting Quadratics: Once again, some students may solve this activity using the quadratic regression capabilities of the graphing calculator. Do not bring this approach to their attention. However, if any student points it out, give them a pat on the back for having connected ideas from statistics and algebra within the mathematics of this problem.



Calculators to the Rescue: The Calculator Note "Entering Matrices and Doing Matrix Arithmetic" describes the mechanics of working with matrices on the calculator. It includes instructions on how to enter a matrix into the calculator and do simple matrix arithmetic. These instructions will assist students with this activity.

Students will work with matrices throughout the remainder of the unit. You might encourage them to save this Calculator Note for later reference.

Finding an Inverse: The Calculator Note "Solving Linear Systems Using Matrices" summarizes the entire process for translating a system of linear equations into a matrix equation and solving the matrix equation using a matrix inverse, which you'll discuss after this activity. The Calculator Note uses the system from *Gardener's Dilemma*. Because it may be more valuable for students to summarize their steps by writing these instructions themselves, you might hold off on making this Calculator Note available to students.

Refer to the Calculator Note "Matrices: Shortcuts and Tips" for more suggestions about working with matrices on the graphing calculator.

As you wrap up the discussion and introduce the notation $[A]^{-1}$, some curious students are likely to investigate the calculator command $\mathbf{[A]^{-1}}$. They will discover that it does work. Although the *Teacher's Guide* suggests delaying this discussion until the following day, that may prove difficult if students exhibit a strong interest in this feature. Clearly, this is a powerful capability of the graphing calculator.

Inverses and Equations: For this activity, we recommend that you have students try to compute the inverses by hand. It will contribute to their understanding of how Question 2 relates to Question 1. (It is also the case that every student may not have access at home to graphing calculators.)

In the discussion of this activity, demonstrate how to determine the inverse of a matrix using a graph calculator. The Calculator Note "Finding the Inverse of a Matrix" may be helpful for students to use during your demonstration or as a reference while they work.

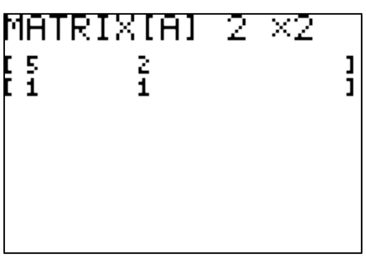
Calculators Again: These comments and screen shots are a reference for you in your class discussion of this activity.

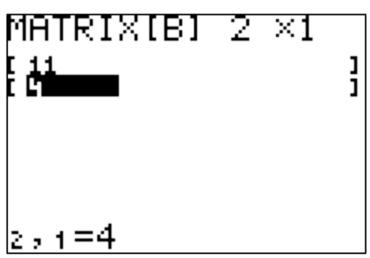
Each system of linear equations can be set up in the equivalent matrix equation [A][X] = [B]. Manipulating and then solving $[X] = [A]^{-1}[B]$ will calculate the values of the variables in the original system of linear equations. Use the calculator to evaluate $[A]^{-1}[B]$.

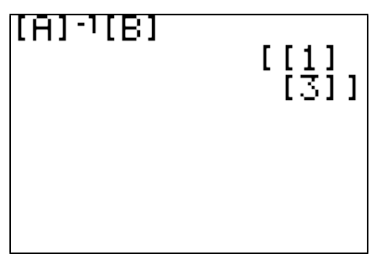
For the sake of convenience, each example here will store the coefficient matrix in [A] and the constant term matrix in [B].

First, set up Question 1 as the equivalent matrix equation $\begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} d \\ e \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$.

Enter $\begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$ as **[A]** and $\begin{bmatrix} 11 \\ 4 \end{bmatrix}$ as **[B]**. Evaluating $\mathbf{[A]^{-1}[B]}$ will solve for $\begin{bmatrix} d \\ e \end{bmatrix}$. As seen here, $d = 1$ and $e = 3$.



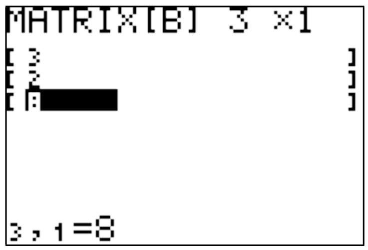


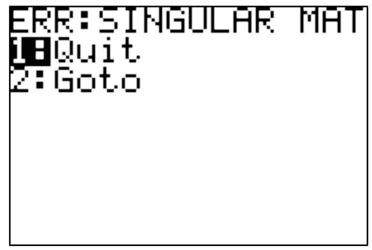


For Question 2, solve the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \\ 4 & -1 & 7 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}$$

The first two screens below show the process of entering matrices [A] and [B]. The third screen shows the result of attempting to evaluate $[A]^{-1}[B]$. The calculator gives this error message because [A] has no inverse.

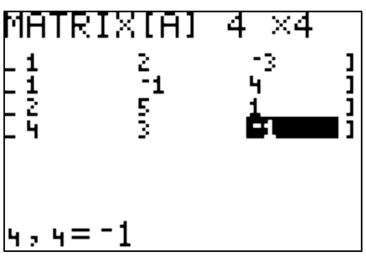


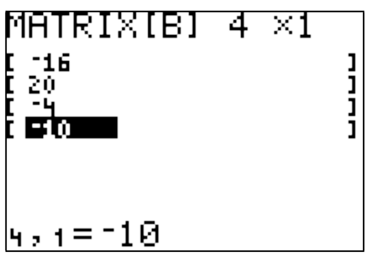


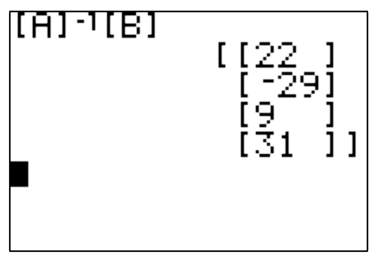
For Question 3, solve the matrix

$$\begin{bmatrix} 4 & 1 & 2 & -3 \\ -3 & 1 & -1 & 4 \\ -1 & 2 & 5 & 1 \\ 5 & 4 & 3 & -1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16 \\ 20 \\ -4 \\ -10 \end{bmatrix}$$

The first two screens below show the process of entering matrices [A] and [B]. The third screen shows the solution.



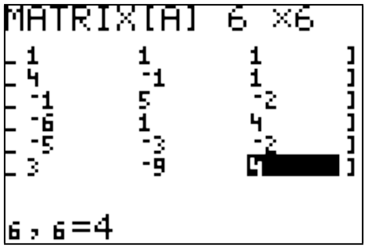


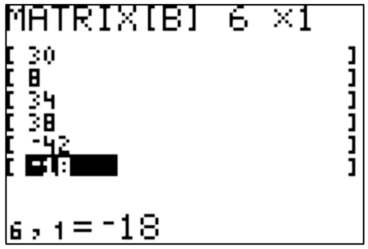


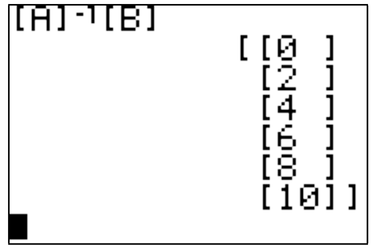
For Question 4, solve the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & -6 & 4 & -1 & 1 \\ 5 & 4 & 3 & -1 & 5 & -2 \\ 2 & -3 & 8 & -6 & 1 & 4 \\ 6 & 2 & 7 & -5 & -3 & -2 \\ -5 & 8 & -5 & 3 & -9 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 30 \\ 8 \\ 34 \\ 38 \\ -42 \\ -18 \end{bmatrix}$$

Again, the first two screens show the process of entering the matrices, and the third screen shows the solution.







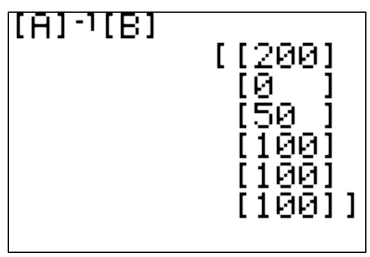
Students may encounter some difficulties while working on this activity. They must recognize that when variables have no coefficient written, it is equivalent to a coefficient of 1, so a **1** should be placed in the matrix. Remind students also that if a variable is omitted in a particular equation, they will need to enter a 0 in the matrix. Furthermore, when subtraction is used in one of the linear equations instead of addition, they must translate that to be equivalent to adding the opposite. Thus, the opposite of the coefficient must appear in the coefficient matrix. Finally, after inverting a matrix, a calculator may display some very small number, such as $2 \cdot 10^{-12}$. If this occurs, it is not due to student error. Rather than tell students that the actual value is 0, use it as an opportunity to ask if a number of this magnitude makes sense given the original problem.

"Meadows or Malls?" Revisited: Students must rely on the ability of their graphing calculators to invert matrices to solve the unit problem. Thus, you can expect to do some "calculator debugging" in the next couple of days.

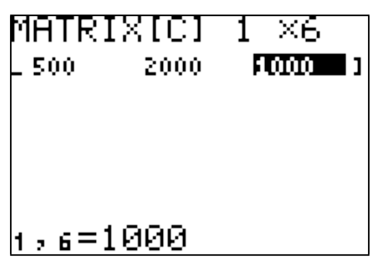
As mentioned previously, expect the calculator to occasionally report very small numbers, such as $2 \cdot 10^{-12}$, when the actual value is 0. Also, because the constraints have coefficients of 0 and 1 only, students must carefully organize their data entry into the calculator's matrix editor. The example below shows the matrix equation to solve for the corner point of constraints

I, II, III, VI, IV, and VIII (as numbered in the *Teacher's Guide*). The solution as found by the calculator is shown in the accompanying screen.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_R \\ A_R \\ M_R \\ G_D \\ A_D \\ M_D \end{bmatrix} = \begin{bmatrix} 300 \\ 100 \\ 150 \\ 100 \\ 300 \\ 200 \end{bmatrix}$$

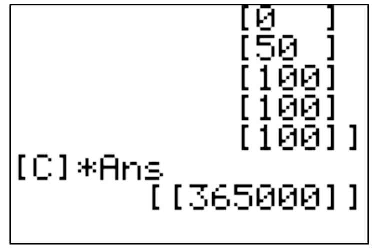


If the cost function coefficients are entered as a 1 x 6 row vector into **[C]**, the evaluation of **[C][A]$^{-1}$[B]** will figure the cost for this solution. First, enter the coefficients into **[C]**. Press MATRX, highlight the **EDIT** menu, and select **3:[C]**. Enter the coefficients (in the proper order) for the cost expression



$50G_R + 200A_R + 100M_R + 500G_D + 2000A_D + 1000M_D$.

Return to the home screen by pressing 2ND [QUIT]. Then, enter **[C][A]$^{-1}$[B]**. Or, if the last answer is the result of finding **[A]$^{-1}$[B],** simply enter **[C]*Ans**. Note that you must follow this order (**[C]*Ans**) because matrix multiplication is not commutative.



The screen shows that this allocation has a cost to the city of $365,000.

"Meadows or Malls?" Portfolios: In reflecting on the activities that helped them develop their understanding of matrices, students may want to include in their portfolios printouts of calculator screens or an entire matrix.

Use TI Connect™ (available free at education.ti.com) and a connecting cable to connect the calculator to the computer and print. If students are aware they have this option, they may also want to incorporate calculator screen captures or other data from the calculator into their write-up of the unit problem.

In-Class Assessment: Students should have access to graphing calculators during in-class assessments. Despite the possibility that using graphing calculators for the *In-Class Assessment* may result in students discovering how other students have set up one of the matrices to solve one of the systems, this does not warrant clearing all work from calculators between classes. However, if you do wish to remove all previous student work, press 2ND [MEM] 5 1 2. (*Warning:* This will erase *all* stored memory.)

Supplemental Problems

Fitting a Plane: This activity becomes much less cumbersome once students have the ability to solve systems of linear equations using matrices and their graphing calculators. Students who continue to pursue the work from *Fitting Quadratics* usually end up with a very profound appreciation for the power of mathematics.

You may also challenge students to continue exploring the pattern from *Fitting Quadratics* by selecting four points and fitting a cubic equation. Again, the process is significantly less cumbersome with the power of matrices at their disposal.

Surfer's Shirts: Students set up matrices and evaluate matrix expressions in this activity. This can also serve as reinforcement or practice in entering matrices and performing addition of matrices on the calculator.

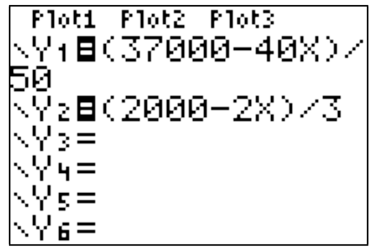
When Can You Find an Inverse?: Students continue practicing finding inverses on the calculator while investigating this activity.

Determining the Determinant: If students pursue this extension activity, you may suggest that they consult their calculator guidebook to learn how to have the calculator compute the determinant for a given matrix.

Calculator Notes for the TI-83/84 Family of Calculators

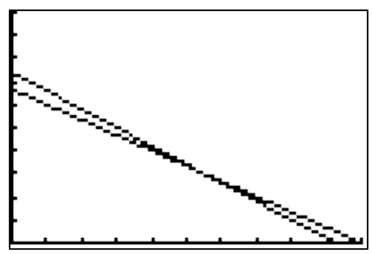
Solving Systems by Graphing

You can use a graphing calculator to estimate the coordinates of the point at which the graphs of two equations intersect. This is the same as finding the common solution to a pair of equations.



If you used other variables when writing your equations, you will need to replace them with **X** and **Y** and solve for **Y** in order to enter the equations into the Y= screen of the calculator. For example, in working with the situation for *Heavy Flying*, you might use **X** to represent the number of containers of chicken feed and **Y** to represent the number of boxes of calculators. If so, the equations are entered as shown here.

You will also need to set the calculator's viewing screen appropriately using WINDOW, considering the values of the variables that are meaningful in the situation.

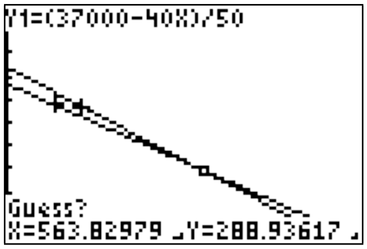


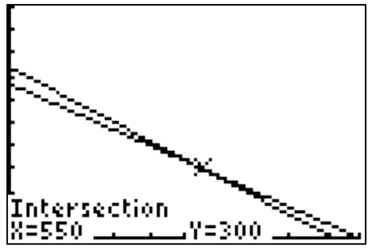
Graph the equations you have entered by pressing GRAPH. Using a combination of TRACE and ZOOM, you can determine with reasonable accuracy the *x*- and *y*-coordinates at which these equations intersect.

A convenient tracing feature is the **ZoomInteger** command, found in the ZOOM menu. After selecting this command, you'll be returned to the graph screen. Press ENTER. Now your calculator will trace only for integer values of **X**. You can also use the **ZoomDecimal** command, which enables tracing by tenths. Using these tracing features may make it simpler to determine a point of intersection.

However, a more accurate method of finding the coordinates of the point of intersection is to calculate the point of intersection. To do this, press 2ND [CALC], highlight **5:intersect**, and press ENTER.

You are returned to the graph, and you must answer each prompt that appears. You first need to identify the equations whose intersection you will find. The equation for the first equation you graphed will appear at the top of the screen, and you select it by pressing ENTER. The other equation will then appear, and you again press ENTER. (If there are more than two graphs on the screen, you can move among them using the up- and down-arrow keys.)





To reply to the **Guess?** prompt, simply place the cursor near the point of intersection. (The left- and right-arrow keys will move it along the graph of the equation that is currently displayed.) Then press $\boxed{\text{ENTER}}$ one more time to have the calculator display the point of intersection.

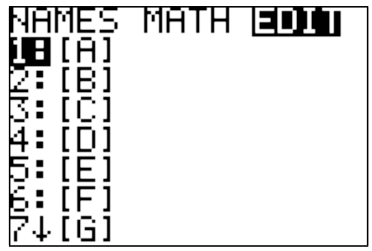
Entering Matrices and Doing Matrix Arithmetic

Using your graphing calculator, you can enter matrices and perform many operations with matrices. You display or edit a matrix in the matrix editor. You can then use many of the calculator's mathematics functions with matrices, as long as the dimensions of the matrix are appropriate.

Your calculator has ten matrix variables, **[A]** through **[J]**, and each can have up to 99 rows and columns, depending on whether enough memory is available.

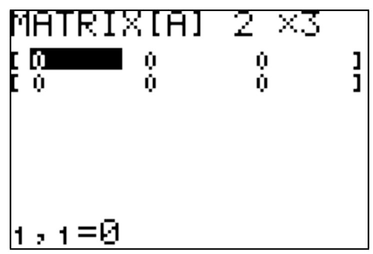
These instructions will guide you through *Calculators to the Rescue*, which uses the matrices written in *Fresh Ingredients*.

To enter a matrix into your calculator, go to the calculator's matrix editor. To do this, press MATRX and highlight **EDIT**. Press ENTER if you would like to store the matrix in **[A]**.

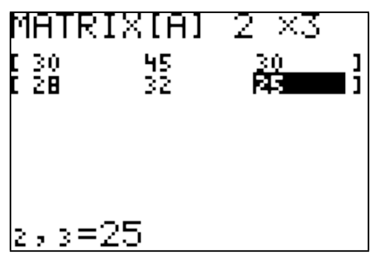


Before filling in your matrix entries, you must define the matrix dimensions. The Woos' baking-plan matrix in Question 1 of *Calculators to the Rescue* is a 2 x 3 matrix, so press 2 ENTER 3 ENTER. You can edit any mistakes by using the arrow keys to highlight what you want to change and typing over it.

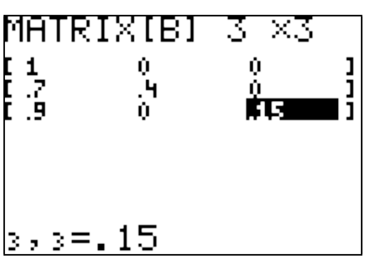
Unless used previously, all entries in the matrix will appear as **0**. As you move through the matrix, notice that the calculator identifies the row and column at the bottom of the screen.

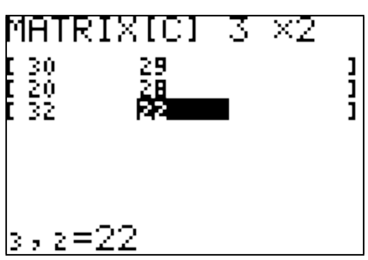


Enter the Woos' baking-plan matrix by keying in each entry and then pressing ENTER. You will move to the right through the first row, then continue on to the second row, and so on.

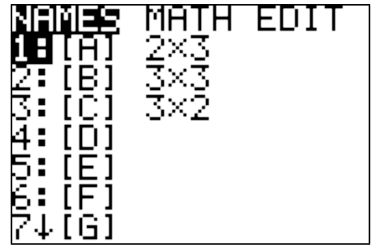


Press MATRX and highlight **EDIT** to enter the next matrix. Follow the same steps to enter the Woos' ingredient matrix and cost matrix. Be sure to enter the ingredient matrix as matrix **[B]** and the cost matrix as matrix **[C]**.



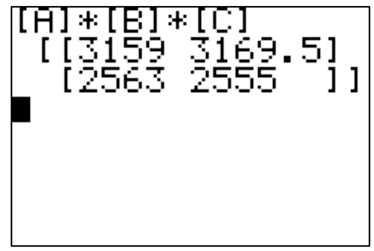


You can instruct the calculator to do arithmetic with matrices as you would give any other arithmetic command. Go to the home screen by typing 2ND [QUIT]. Enter the matrix variables by pressing MATRX and selecting from the **NAMES** menu. Press ENTER after highlighting the matrix name you wish to place in the home screen.



(*Note:* The brackets found as the secondary function above the x and – keys *cannot* be used in conjunction with letters to enter matrix names. The calculator will report a syntax error if a matrix variable name is entered in this manner.)

To calculate the total cost matrix for the Woos, you must calculate **[A]*****[B]*****[C]**. (The calculator will also multiply correctly if enter **[A][B][C]**.) This screen shows the final step of *Calculators to the Rescue*.

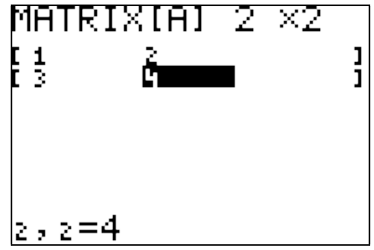


Investigate how the calculator responds when you try to multiply matrices in which the column and row dimensions do not match.

Finding the Inverse of a Matrix

You have seen that you can determine the inverse of a matrix by solving a system of equations. You can also use your graphing calculator to find the inverse of a matrix. To do so, follow these steps.

1. Enter the matrix you wish to invert into one of the calculator's matrix variables. This example shows matrix variable **[A]** defined as $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

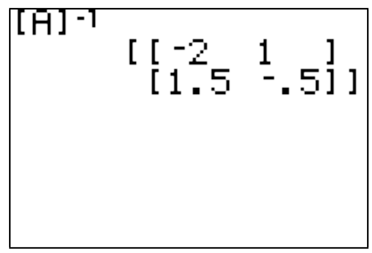


2. Press $\boxed{\text{2ND}}$ [QUIT] to return to the home screen.

3. Press $\boxed{\text{MATRX}}$. The matrix **[A]**, which you wish to use, will already be highlighted, so just press $\boxed{\text{ENTER}}$.

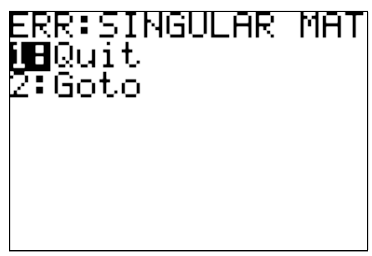
4. To enter the $^{-1}$, press $\boxed{x^{-1}}$.

5. Complete the command by pressing $\boxed{\text{ENTER}}$.

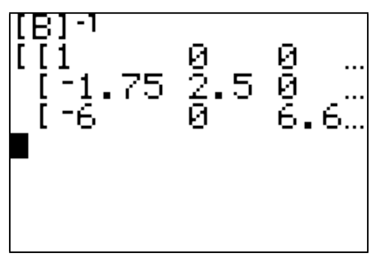


Notice how the results are displayed like a 2 x 2 matrix on the calculator's screen. A **[** indicates the beginning of each row, and a **]** indicates the end. An additional set of brackets indicates the beginning and end of the matrix.

Attempt to ask the calculator for the inverse of a noninvertible matrix, such as $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Note the error message, and then select **Goto** to observe what the calculator identifies as the cause.



Next, invert a 3 x 3 matrix. The screen here shows the inverse of the matrix **[B]** defined as $\begin{bmatrix} 1 & 0 & 0 \\ 0.7 & 0.4 & 0 \\ 0.9 & 0 & 0.15 \end{bmatrix}$.



The ellipses to the right indicate additional information. Use the arrow keys to scroll through the matrix.

Solving Linear Systems Using Matrices

These instructions will guide you through the steps needed to solve a system of linear equations using the calculator's matrix capabilities.

Recall that a system of linear equations can be written as a matrix equation in the form [A][X] = [B], where [A] is the *coefficient matrix*, [X] is a column matrix made up of the variables in the system, and [B] is a column matrix called the *constant term matrix.* Solving for [X] will solve for the variables of the system of linear equations.

This matrix equation can be solved by multiplying both sides by the inverse of matrix [A]. The resulting equation, $[X] = [A]^{-1}[B]$, states that the variables that make up column matrix [X] are equal to the entries of the column matrix that is a result of the multiplication $[A]^{-1}[B]$.

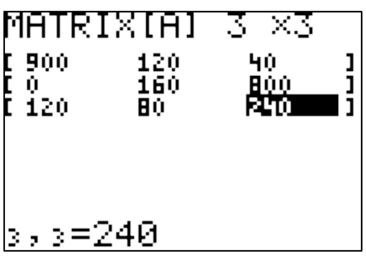
To solve the system of linear equations, you can use your calculator to find $[A]^{-1}[B]$. In this example, we'll use these equations from *Gardener's Dilemma*.

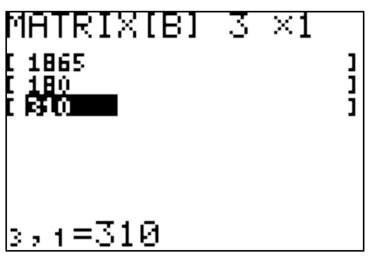
$900L + 120F + 40S = 1865$
$160F + 800S = 180$
$120L + 80F + 240S = 310$

The matrix equation to be solved is:

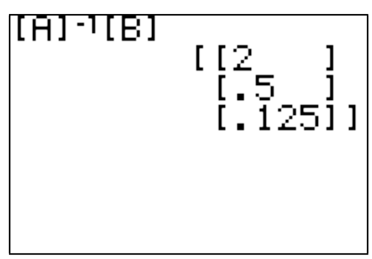
$$\begin{bmatrix} 900 & 120 & 40 \\ 0 & 160 & 800 \\ 120 & 80 & 240 \end{bmatrix} \begin{bmatrix} L \\ F \\ S \end{bmatrix} = \begin{bmatrix} 1865 \\ 180 \\ 310 \end{bmatrix}$$

Begin by entering the coefficient matrix and column matrix into two separate matrix variables. In this example, we'll use matrix variables **[A]** and **[B]**. Be sure to place a 0 in the second row of [A] because L is not part of the second equation.





Press 2ND [QUIT] to return to the home screen. Press MATRX (**1:[A]** will be highlighted already) and press ENTER. Press x^{-1}, highlight **2:[B]**, and press ENTER. Your home screen should look like the first line of the screen shown below. Now, simply press ENTER, and the calculator will display the result of multiplying the inverse



of the coefficient matrix and the constant term matrix. Each entry of the column matrix displayed is equal to the corresponding variable of the original system of equations.

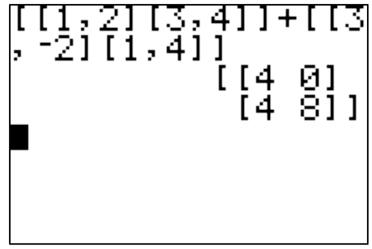
This shows that $\begin{bmatrix} L \\ F \\ S \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \\ 0.125 \end{bmatrix}$.

Therefore, the solution to the system of equations is $L = 2$, $F = 0.5$, and $S = 0.125$.

Matrices: Shortcuts and Tips

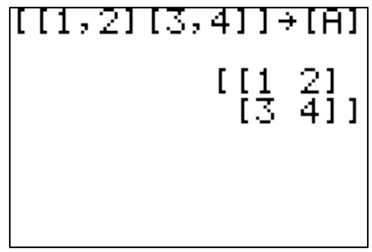
You can enter, edit, and store matrices in the matrix editor. You can also do these tasks in the home screen.

For example, if you wish to add $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} + \begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}$ directly on the home screen, enter as shown here.

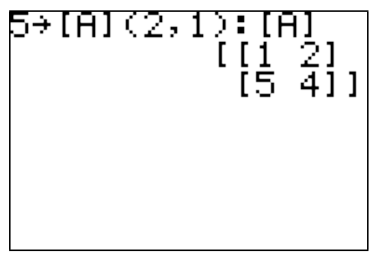


For brackets, press 2ND [[ ] and 2ND [ ]]. Notice that the first **[** indicates the start of a matrix and the second **[** indicates the start of a row. Each row must end with a **]**, and each matrix must end with a **]** as well. Separate row entries with a comma (,). *Reminder:* You cannot use these brackets to enter the name of a matrix variable, such as **[A]**.

The ability to enter matrices in the home screen can be helpful when storing a matrix to a matrix variable. Enter the matrix, being careful to place the brackets in the correct places. Press STO→ and then select the matrix variable from the MATRX **NAMES** menu.



If you wish to change an entry in a matrix that has already been entered, enter the new entry (**5** in the example at right), press STO→ MATRX, highlight the matrix name (**[A]** in our example), and press ENTER. This returns you to the home screen. Now, designate the row and column affected by the change (**2, 1** in our example) by pressing (2 , 1) ALPHA [:]. Press MATRX (**1:[A]** is already highlighted) and press ENTER ENTER to display matrix **[A]**.



Sometimes a matrix is too large to be seen on only one screen. Use the calculator's arrow keys to scroll through the matrix.

If the matrix rolls off the screen because of repeating decimals, you can always ask the calculator to round to one decimal place (or any other number of places). Press MODE and then highlight **1** instead of **Float**.