High Dive

Circular Functions and the Physics of Falling Objects

Teacher's Guide

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High Dive Overview

Intent

In this unit, students study trigonometry in the context of a unit problem that involves a circus act. The basic problem is to determine when his fall should begin in order for the diver to land in the water.

Mathematics

Here is a summary of the main concepts and skills that students will encounter and practice in this unit:

Trigonometry

- Extending the trigonometric functions to all angles
- Reinforcing the importance of similarity in the definitions of the trigonometric functions
- Graphing the trigonometric functions and variations on those functions
- Defining the inverse trigonometric functions and principal values
- Discovering and explaining the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, and other trigonometric identities
- Defining polar coordinates and finding rectangular coordinates from polar coordinates and vice versa
- Applying the principle that the tangent to a circle is perpendicular to the radius at the point of tangency

Physics

- Developing quadratic expressions for the height of free-falling objects, based on the principle of constant acceleration
- Recognizing that a person falling from a moving object will follow a different path than someone falling from a stationary object
- Expressing velocity in terms of vertical and horizontal components
- Representing the motion of falling objects when the vertical and horizontal components of the initial velocity are both nonzero

Quadratic Equations

- Developing simple quadratic equations to describe the behavior of falling objects
- Developing the quadratic formula

- Using the quadratic formula to solve quadratic equations

- Finding a general solution for the falling time of objects with an initial vertical velocity

Complex Numbers

- Seeing the need to extend the number system to solve certain quadratic equations

- Establishing basic ideas about complex number arithmetic

- Representing complex numbers in the plane and seeing addition of complex numbers as a vector sum

Other concepts and skills are developed in connection with Problems of the Week.

Progression

The central problem of this unit concerns a circus act in which a diver is dropped from a turning Ferris wheel into a tub of water carried by a moving cart. The basic problem is to determine when his fall should begin in order for him to land in the water. Students begin by looking at the diver's height off the ground while still on the Ferris wheel, analyzing different cases in terms of the angle through which the wheel has turned, and seeing that the analysis is slightly different from one quadrant to another.

Students then develop a formula based on right-triangle trigonometry that works when the diver is in the first quadrant. They use this formula as a clue for how to extend the sine function from the familiar right-triangle context so that it is defined for all angles. The development of the general definition of the sine function involves several considerations, including the physical situation of the Ferris wheel, a graph of the diver's height while on the wheel, and a coordinate model of the wheel.

The cosine function is developed similarly and reinforces the elegance and power of definitions involving the coordinate system. In particular, students see that these definitions eliminate the need for a quadrant-by-quadrant analysis, incorporating the issue of sign quite nicely.

Students also learn about the graphs of the sine and cosine functions. In particular, they see how the graphs of the functions describing the diver's position change in response to various parameters such as the radius of the Ferris wheel and the period of its motion. As students search for angles corresponding to given values of the sine and cosine, they develop the concept of inverse trigonometric functions and their principal values.

The physics and mathematics of falling objects compose another major strand of the unit, based on the principle that falling objects have constant acceleration. Students are told to assume for simplicity that at the instant the dive begins the diver is falling as if from rest, even though this contradicts the physics of the actual situation. (The realistic solution of the circus dive, taking into account the diver's initial velocity, is found in the Year 4 unit *The Diver Returns*.) Students develop an expression for the height of an object falling from rest in terms of its time in the air, which is then used to determine the duration of the diver's fall. In the process, they review ideas about instantaneous and average speed and interpret these concepts graphically.

This work on falling objects is then combined with the analysis of the diver's position on the Ferris wheel and with information about the speed of the cart. Students synthesize these parts of the problem to develop a complex equation involving the amount of time the diver should stay on the Ferris wheel before being dropped. Then they solve this equation graphically (because it is too complex to be susceptible to algebraic manipulation) to find a solution to the unit problem, subject to the simplifying assumption described above.

Students then look at the initial velocity given to the diver through the turning of the Ferris wheel. They examine how an initial vertical component of velocity, either upward or downward, changes the diver's falling time. They see that finding the falling time requires solving a quadratic equation, which leads to an excursion into the quadratic formula.

Students also must grapple with the task of determining both the vertical and horizontal components of the diver's initial velocity, and they must determine how the horizontal component of his initial velocity affects where he lands. The issues related to finding the separate components are dealt with in a series of paired problems, with one problem in each pair involving the Ferris wheel situation and the other set in another context. Students develop a general expression, based on the physical context, for the time it takes a falling object to reach the ground, in terms of its initial height and vertical velocity.

Finally, students return to the circus act problem in its full complexity. They combine their formula for falling time with expressions for the vertical and horizontal components of the diver's velocity. This leads to a very complex expression for the diver's position when he is about to land, in terms of the time of his release from the Ferris wheel. Comparing this with the position of the moving cart leads to an equation that will solve the problem. This equation is solved graphically.

The overall organization of the unit can be summarized as follows:

Going to the Circus: Introducing the unit problem

The Height and the Sine: Analyzing the diver's height while still on the wheel and developing the sine function

Falling, Falling, Falling: Studying constant acceleration and developing a formula for the height of an object falling from rest

Moving Left and Right: Analyzing the diver's horizontal position and developing the cosine function

Finding the Release Time: Solving the unit problem, subject to the simplifying assumption that the moving diver falls as if from rest

A Trigonometric Interlude: Continuing to work with trigonometric functions, including polar coordinates and identities

A Falling Start**:** Analyzing the height and falling time for falling objects with nonzero initial vertical velocity, with a digression on the quadratic formula

Components of Velocity**:** Separating velocity into vertical and horizontal components, and analyzing these components for the diver in the unit problem

High Dive Concluded**:** Solving the unit problem, a digression about the complex plane, and compiling portfolios

High Dive and the Common Core State Standards for Mathematics

Meaningful Math—Algebra 2 is written to address the Common Core State Standards for Mathematics (CCSSM), and particularly the High School standards that Appendix A of the CCSSM recommends for inclusion in an Algebra 2 course.

Standards for Mathematical Practice

The eight Standards for Mathematical Practice are addressed exceptionally well throughout the *Meaningful Math* curriculum.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Standards for Mathematical Content

These specific content standards are addressed in the *High Dive* unit. Additional content is covered that reinforces standards from earlier grades and courses.

N-CN.1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

N-CN.3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

N-CN.4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

N-CN.6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

N-CN.7. Solve quadratic equations with real coefficients that have complex solutions.

N-CN.8. (+) Extend polynomial identities to the complex numbers.

N-CN.9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

N-VM.1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, $\boldsymbol{v}$).

N-VM.2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM.3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

N-VM.4a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

N-VM.4b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

N-VM.4c. Understand vector subtraction $\boldsymbol{v} - \boldsymbol{w}$ as $\boldsymbol{v} + (-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM.5a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(vx, vy) = (cvx, cvy)$.

N-VM.5b. Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

A-SSE.1b. Interpret complicated expressions by viewing one or more of their parts as a single entity.

F-IF.7e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-TF.2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

F-TF.7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

F-TF.8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to calculate trigonometric ratios.

Pacing Guides

50-Minute Pacing Guide (41 days)

Day	Activity	Time Estimate
1	*Going to the Circus*	0
	The Circus Act	30
	Introduce: *POW 6: The Tower of Hanoi*	15
	Homework: *The Ferris Wheel*	5
2	Discussion: *The Ferris Wheel*	15
	Discussion: *The Circus Act*	35
	Homework: *As the Ferris Wheel Turns*	0
3	Discussion: *As the Ferris Wheel Turns*	20
	The Height and the Sine	0
	At Certain Points in Time	30
	Homework: *A Clear View*	0
4	Discussion: *A Clear View*	10
	Extending the Sine	20
	Testing the Definition	20
	Homework: *Graphing the Ferris Wheel*	0
5	Discussion: *Testing the Definition*	35
	Discussion: *Graphing the Ferris Wheel*	15
	Homework: *Ferris Wheel Graph Variations*	0
6	Discussion: *Ferris Wheel Graph Variations*	15
	The "Plain" Sine Graph	35
	Homework: *Sand Castles*	0
7	Discussion: *Sand Castles*	25
	Presentations: *POW 6: The Tower of Hanoi*	20
	Introduce: *POW 7: Paving Patterns*	5
	Homework: *More Beach Adventures*	0
8	Discussion: *More Beach Adventures*	10
	Falling, Falling, Falling	0
	Distance with Changing Speed	40
	Homework: *Acceleration Variations and a Sine Summary*	0

9	Discussion: *Acceleration Variations and a Sine Summary*	10
	Free Fall	40
	Homework: *Not So Spectacular*	0
10	Discussion: *Not So Spectacular*	25
	Discussion: *Free Fall*	25
	Homework: *A Practice Jump*	0
11	Discussion: *A Practice Jump*	25
	Moving Left and Right	0
	Cart Travel Time	25
	Homework: *Where Does He Land?*	0
12	Discussion: *Where Does He Land?*	20
	First Quadrant Platform	30
	Homework: *Carts and Periodic Problems*	0
13	Discussion: *Carts and Periodic Problems*	10
	Generalizing the Platform	40
	Homework: *Planning for Formulas*	0
14	Discussion: *Planning for Formulas*	30
	Finding the Release Time	0
	Moving Cart, Turning Ferris Wheel (continue tomorrow)	20
	Homework: *Putting the Cart Before the Ferris Wheel*	0
15	Discussion: *Putting the Cart Before the Ferris Wheel*	10
	Moving Cart, Turning Ferris Wheel (continued)	40
	Homework: *What's Your Cosine?*	0
16	Discussion: *What's Your Cosine?*	20
	Discussion: *Moving Cart, Turning Ferris Wheel*	30
	Homework: *Find the Ferris Wheel*	0
17	Discussion: *Find the Ferris Wheel*	10
	A Trigonometric Conclusion	0
	Some Polar Practice	40
	A Polar Summary	0
	Homework: *Polar Coordinates on the Ferris Wheel*	0

18	Discussion: *Polar Coordinates on the Ferris Wheel*	10
	Pythagorean Trigonometry	40
	Homework: *More Pythagorean Trigonometry** (Supplemental Activity)	0
	Homework: *Coordinate Tangents*	0
19	Discussion: *Coordinate Tangents*	20
	Positions on the Ferris Wheel	30
	Homework: *More Positions on the Ferris Wheel*	0
20	Discussion: *More Positions on the Ferris Wheel*	15
	Presentations: *POW 7: Paving Patterns*	35
	Homework: *A Trigonometric Reflection*	0
21	Discussion: *A Trigonometric Reflection*	20
	Introduce: *POW 8: Which Weights Weigh What?*	10
	Homework: *Initial Motion from the Ferris Wheel*	20
22	Discussion: *Initial Motion from the Ferris Wheel*	20
	A Falling Start	0
	Look Out Below!	30
	Homework: *The Diver and the POW*	0
23	Discussion: *The Diver and the POW*	15
	Look Out Below! (continued)	35
	Homework: *Big Push*	0
24	Discussion: *Big Push*	10
	Finding with the Formula	40
	Homework: *Using Your ABC's*	0
25	Discussion: *Using Your ABC's*	20
	Imagine a Solution	30
	Homework: *Complex Numbers and Quadratic Equations*	0
26	Discussion: *Complex Numbers and Quadratic Equations*	10
	*Complex Conjugates** (Supplemental Activity)	30
	Homework: *Absolutely Complex** (Supplemental Activity)	10

27	Discussion: *Absolutely Complex** (Supplemental Activity)	10
	Complex Components	30
	Homework: *Three O'Clock Drop*	10
28	Discussion: *Three O'Clock Drop*	20
	*Vector Island** (Supplemental Activity)	25
	Homework: *Vectors and Magnitude** (Supplemental	5
29	Discussion: *Vectors and Magnitude** (Supplemental Activity)	20
	Up, Down, Splat!	25
	Homework: *Falling Time for Vertical Motion*	5
30	Discussion: *Falling Time for Vertical Motion*	15
	Components of Velocity	0
	High Noon	35
	Homework: *Leap of Faith*	0
31	Discussion: *Leap of Faith*	10
	High Noon (continued)	5
	The Ideal Skateboard	35
	Homework: *Racing the River*	0
32	Discussion: *Racing the River*	10
	The Ideal Skateboard (continued)	15
	One O'Clock Without Gravity	25
	Homework: *Swimming Pointers*	0
33	*One O'Clock Without Gravity* (continued)	10
	Discussion: *Swimming Pointers*	10
	Vector Velocities	30
	Homework: *Velocities on the Wheel*	0
34	Discussion: *Velocities on the Wheel*	20
	Release at Any Angle	30
	Homework: *A Portfolio of Formulas*	0
35	Discussion: *A Portfolio of Formulas*	10
	Moving Diver at Two O'Clock	40
	Homework: *The Danger of Simplification*	0

36	Discussion: *The Danger of Simplification*	20
	High Dive Concluded	0
	The Diver's Success	30
	Homework: *A Circus Reflection*	0
37	Discussion: *A Circus Reflection*	10
	The Diver's Success (continued)	35
	Homework: *Beginning Portfolio Selection*	5
38	Presentations: *POW 8: Which Weights Weigh What?*	35
	Homework: *"High Dive" Portfolio* (begin in class)	15
39	*The Fundamental Theorem of Algebra** (Supplemental Activity)	35
	*Trying a New Angle** (Supplemental Activity)	15
40	*Different Angles** (Supplemental Activity)	35
	Homework: *Parametric Ferris Wheel** (Supplemental Activity)	15
41	*In-Class Assessment*	50
	Homework: *Take-Home Assessment*	0

* Activities marked with an asterisk (*) are Supplemental Activities that address Common Core State Standards, and are therefore recommended for inclusion in the standard Pacing Guide.

90-minute Pacing Guide (28 days)

Day	Activity	Time Estimate
1	*Going to the Circus*	0
	The Circus Act	70
	Introduce: *POW 6: The Tower of Hanoi*	15
	Homework: *The Ferris Wheel*	5
2	Discussion: *The Ferris Wheel*	10
	As the Ferris Wheel Turns	50
	The Height and the Sine	0
	At Certain Points in Time	30
	Homework: *A Clear View*	0
3	Discussion: *A Clear View*	10
	Extending the Sine	25
	Testing the Definition	55
	Homework: *Graphing the Ferris Wheel*	0
4	Discussion: *Graphing the Ferris Wheel*	15
	Ferris Wheel Graph Variations	40
	The "Plain" Sine Graph	35
	Homework: *Sand Castles*	0
5	Discussion: *Sand Castles*	30
	More Beach Adventures	50
	Falling, Falling, Falling	0
	Homework: *Distance with Changing Speed*	10
6	Discussion: *Distance with Changing Speed*	20
	Presentations: *POW 6: The Tower of Hanoi*	25
	Introduce: *POW 7: Paving Patterns*	5
	Free Fall	40
	Homework: *Acceleration Variations and a Sine Summary*	0
7	Discussion: *Acceleration Variations and a Sine Summary*	10
	Discussion: *Free Fall*	25
	Not So Spectacular	55
	Homework: *A Practice Jump*	0

8	Discussion: *A Practice Jump*	15
	Moving Left and Right	0
	Cart Travel Time	30
	Where Does He Land?	45
	Homework: *First Quadrant Platform*	0
9	Discussion: *First Quadrant Platform*	5
	Carts and Periodic Problems	40
	Generalizing the Platform	45
	Homework: *Planning for Formulas*	0
10	Discussion: *Planning for Formulas*	40
	Finding the Release Time	0
	Putting the Cart Before the Ferris Wheel	50
	Homework: *What's Your Cosine?*	0
11	Discussion: *What's Your Cosine?*	15
	Presentations: *POW 7: Paving Patterns*	25
	Moving Cart, Turning Ferris Wheel	50
	Homework: *Find the Ferris Wheel*	0
12	Discussion: *Find the Ferris Wheel*	10
	Discussion: *Moving Cart, Turning Ferris Wheel*	30
	A Trigonometric Conclusion	0
	Some Polar Practice	50
	A Polar Summary	0
	Homework: *Polar Coordinates on the Ferris Wheel*	0
13	Discussion: *Polar Coordinates on the Ferris Wheel*	10
	Pythagorean Trigonometry	40
	Coordinate Tangents	40
	Homework: *More Pythagorean Trigonometry** (Supplemental Activity)	0
	Homework: *Positions on the Ferris Wheel*	0
14	Discussion: *Positions on the Ferris Wheel*	5
	Discussion: *Coordinate Tangents*	15
	More Positions on the Ferris Wheel	40
	A Trigonometric Reflection	30
	Homework: *Initial Motion from the Ferris Wheel*	0

15	Discussion: *A Trigonometric Reflection*	20
	Discussion: *Initial Motion from the Ferris Wheel*	20
	Introduce: *POW 8: Which Weights Weigh What?*	15
	A Falling Start	0
	Look Out Below!	30
	Homework: *The Diver and the POW*	5
16	Discussion: *The Diver and the POW*	15
	Look Out Below! (continued)	25
	Big Push	35
	Homework: *Finding with the Formula*	15
17	Discussion: *Finding with the Formula*	10
	Using Your ABC's	50
	Imagine a Solution	30
	Homework: *Complex Numbers and Quadratic Equations*	0
18	Discussion: *Complex Numbers and Quadratic Equations*	15
	Complex Components	30
	Three O'Clock Drop	45
	Homework: *Complex Conjugates** (Supplemental Activity)	0
19	Discussion: *Complex Conjugates** (Supplemental Activity)	15
	*Absolutely Complex** (Supplemental Activity)	30
	*Vector Island** (Supplemental Activity)	30
	Homework: *Scalars and Magnitude** (Supplemental Activity)	15
	Homework: *Up, Down, Splat!*	0
20	Discussion: *Up, Down, Splat!*	15
	Falling Time for Vertical Motion	35
	Components of Velocity	0
	High Noon	40
	Homework: *Leap of Faith*	0
21	Discussion: *Leap of Faith*	10
	The Ideal Skateboard	40
	Racing the River	40
	Homework: *One O'Clock Without Gravity*	0

22	Discussion: *One O'Clock Without Gravity*	10
	Swimming Pointers	25
	Vector Velocities	40
	Homework: *Velocities on the Wheel* (begin in class)	15
23	Discussion: *Velocities on the Wheel*	15
	Release at Any Angle	35
	Moving Diver at Two O'Clock	40
	Homework: *A Portfolio of Formulas*	0
24	Discussion: *A Portfolio of Formulas*	15
	The Danger of Simplification	45
	High Dive Concluded	0
	The Diver's Success	30
	Homework: *A Circus Reflection*	0
25	Discussion: *A Circus Reflection*	10
	The Diver's Success (continued)	35
	Homework: *Beginning Portfolio Selection* and *"High Dive" Portfolio* (begin in class)	35
26	*The Fundamental Theorem of Algebra** (Supplemental Activity)	30
	*Trying a New Angle** (Supplemental Activity)	30
	*Different Angles** (Supplemental Activity)	30
	Homework: *Parametric Ferris Wheel** (Supplemental Activity)	0
27	Presentations: *POW 8: Which Weights Weigh What?*	35
	In-Class Assessment	40
	Take-Home Assessment (begin in class)	15
28	Exam Discussion	45
	Unit Reflection	20

* Activities marked with an asterisk (*) are Supplemental Activities that address Common Core State Standards, and are therefore recommended for inclusion in the standard Pacing Guide.

Materials and Supplies

All Meaningful Math classrooms should have a set of standard supplies, described in the section "Materials and Supplies for the Meaningful Math Classroom" in *A Guide to Meaningful Math.*

Listed here are the supplies needed for this unit. Also available are general and activity-specific blackline masters, for transparencies or for student worksheets, in the "Blackline Masters" section.

High Dive Materials

- Items for building models of the problem, such as paper plates, pipe cleaners, and toy cars
- Coins or discs
- Poster of "As the Ferris Wheel Turns" blackline master
- (Optional) String and a small weight (such as a roll of tape)

More About Supplies

Graph paper is a standard supply for Meaningful Math classrooms. Blackline masters of
1-Centimeter Graph Paper, ¼-Inch Graph Paper, and 1-Inch Graph Paper are provided, for you to make copies and transparencies.

Assessing Progress

High Dive concludes with two formal unit assessments. In addition, there are many opportunities for more informal, ongoing assessments throughout the unit. For more information about assessment and grading, including general information about the end-of-unit assessments and how to use them, consult *A Guide to Meaningful Math*.

End-of-Unit Assessments

This unit concludes with in-class and take-home assessments. The in-class assessment is intentionally short so that time pressures will not affect student performance. Students may use graphing calculators and their notes from previous work when they take the assessments. You can download unit assessments from the *High Dive* Unit Resources.

Ongoing Assessment

One of the primary tasks of the classroom teacher is to assess student learning. Although the assigning of course grades may be part of this process, assessment more broadly includes the daily work of determining how well students understand key ideas and what level of achievement they have attained on key skills, in order to provide the best possible ongoing instructional program for them.

Students' written and oral work provides many opportunities for teachers to gather this information. We make some recommendations here of written assignments and oral presentations to monitor especially carefully that will give you insight into student progress.

- *As the Ferris Wheel Turns*
- *Testing the Definition*
- *More Beach Adventures*
- *A Practice Jump*
- *Moving Cart, Turning Ferris Wheel*
- *Big Push*
- *Complex Numbers and Quadratic Equations*
- *Three O'Clock Drop*
- *Vector Velocities*
- *The Diver's Success*

Discussion of Unit Assessments

Have students volunteer to explain their work on each of the problems. Encourage questions and alternate explanations from other students.

In-Class Assessment

In solving the equation in Question 1 by completing the square, students might first rewrite the equation as $x^2 + 4x = 3$ and then complete the square to get $(x + 2)^2 = 7$. The approximate solutions, to the nearest tenth, are 0.6 and –4.6.

For Question 2, students should sketch a parabolic graph going upward and with the two solutions from Question 1 as the x-intercepts. They might use the point (0, –3) as a guide for the sketch.

For Question 3b, students should explain that the graph can have no x-intercepts (or cannot cross or touch the x-axis), since the equation has no “ordinary number” (real number) solutions.

Take-Home Assessment

The first step in solving the problem might be to find the components of Sabrina’s initial velocity. Because the swing makes an angle of 30° with the vertical, Sabrina’s initial path makes the same angle with the horizontal. Her overall speed is 30 feet per second, so the vertical component of her velocity is $30 \cdot \sin 30°$, or exactly 15 feet per second (upward). That means her height after t seconds is $11 + 15t - 16t^2$, so for Question 1, students need to solve the equation $11 + 15t - 16t^2 = 1$. This gives a value for t of 1.4 seconds (to the nearest tenth).

For Question 2, students need to see that the horizontal component of Sabrina’s speed is $30 \cdot \cos 30°$, or approximately 26.0 feet per second. Using $t = 1.4$ means she travels about 36.4 feet in the horizontal direction. (If time and velocity are not rounded to tenths before multiplying, the value turns out to be closer to 36.1.)

For Question 3, Sabrina’s vertical velocity after t seconds is $15 - 32t$. Using the rounded-off value of $t = 1.4$ gives this component when she hits the pad as 29.8 feet per second downward (although 29.4 feet per second is more accurate, as t is actually closer to 1.388 seconds). Her horizontal velocity is assumed to remain constant at 26.0 feet per second to the right. Finding her actual velocity, then, requires the Pythagorean theorem. Using the rounded-off values produces an actual velocity of 39.5 feet per second at impact (more exact values result in 39.2 feet per second).

Supplemental Activities

The unit contains a variety of activities at the end of the student pages that you can use to supplement the regular unit material. These activities fall roughly into two categories.

Reinforcements increase students' understanding and comfort with concepts, techniques, and methods that are discussed in class and are central to the unit.

Extensions allow students to explore ideas beyond those presented in the unit, including generalizations and abstractions of ideas.

The supplemental activities are presented in the *Teacher's Guide* and the student book in the approximate sequence in which you might use them. Below are specific recommendations about how each activity might work within the unit. You may wish to use some of these activities, especially the later ones, after the unit is completed.

Mr. Ferris and His Wheel **(extension)** This activity provides interested students with an opportunity to research the general topic of Ferris wheels.

A Shifted Ferris Wheel **(extension)** This activity guides students to investigate the effect of changing the initial location of the platform from the 3 o'clock position to something else.

Prisoner Revisited **(reinforcement)** This activity can be assigned if students need further practice working with the ideas from *More Beach Adventures*.

Lightning at the Beach on Jupiter **(reinforcement)** This activity is a review of the relationship between distance, rate, and time.

The Derivative of Position **(extension)** In this activity, students use derivatives to confirm one of the formulas they developed in *Free Fall*.

A Change in Plans **(reinforcement)** This activity presents a twist on the original circus act problem, in which students don't have to take into account the angular velocity of the dive.

Polar Equations **(extension)** This activity introduces the graphing of polar equations.

Circular Sine **(extension)** This activity challenges students to transform a polar equation into a rectangular equation.

A Polar Exploration **(extension)** This activity is an open-ended exploration of the graphs of polar equations.

***A Shift in Sine* (extension)** In this activity, students examine in more depth how the graph of the cosine function can be produced by shifting the graph of the sine function.

***More Pythagorean Trigonometry* (extension)** This activity asks students to clearly explain the Pythagorean identity just developed in *Pythagorean Geometry,* and to develop similar identities for the remaining four trigonometric functions. This activity covers a Common Core State Standard.

***Complex Conjugation* (extension)** This activity introduces the concept of the conjugate of a complex number and explores how this idea can be used to divide complex numbers, as well as the role of complex conjugates in the solution of quadratic equations. This activity can be used following the discussion of *Complex Numbers and Quadratic Equations.* This activity covers a Common Core State Standard.

***Absolutely Complex* (extension)** This activity introduces the concept of the absolute value of a complex number and asks students to explore its properties. It works well used together with the preceding supplemental activity, *Complex Conjugation.* This activity covers a Common Core State Standard.

***The Polar Complex* (extension)** This activity introduces the polar form of a complex number and how it can be used. It builds on ideas in the preceding supplemental activity, *Absolutely Complex*.

***Polar Roots* (extension)** This activity extends the supplemental activity The Polar Complex to computation and verification of nth roots of complex numbers.

***Number Research* (extension)** This activity is a natural follow-up to the introduction of imaginary numbers in *Imagine a Solution* and complex numbers in *Complex Numbers and Quadratic Equations* and *Complex Components*. It can be used anytime after those activities.

***The Fundamental Theorem of Algebra* (extension)** This activity explores the Fundamental Theorem of Algebra and its application to finding imaginary roots of polynomials. This activity covers a Common Core State Standard.

***Trying a New Angle* (extension)** In this activity, students develop the radian measure. This activity covers a Common Core State Standard.

***Different Angles* (extension)** In this activity, students extend the radian measure to trigonometric functions. This activity covers a Common Core State Standard.

***A Parametric Ferris Wheel* (extension)** In this activity, students learn how to graph Ferris wheel motion using parametric functions. This activity covers a California Common Core State Standard.

***Vector Island* (extension)** This activity extends the concept of vectors introduced in *Complex Components*. Students learn about rectangular and polar forms of vectors, and addition of vectors. This activity covers a Common Core State Standard.

***Scalars and Magnitude* (extension)** This activity is a follow-up to *Vector Island.* Students learn about scalar multiplication of vectors, and magnitude. This activity covers a Common Core State Standard.

Going to the Circus

Intent

In this section, students are introduced to the central unit problem and begin to explore the relationship between the angular speed of the Ferris wheel and the height of the platform.

Mathematics

The first element of the unit problem that students will wrestle with is finding the height of the Ferris wheel's diving platform at any given time. Students begin by using right-triangle trigonometry to find the height of various clock positions on the Ferris wheel. Then the motion of the wheel is introduced, and the students use the period of revolution to find the angular velocity and thus the angular position and height of the platform at a given time.

Progression

The Circus Act introduces the central unit problem. In *The Ferris Wheel*, students find the height of the platform at given clock positions. In *As the Ferris Wheel Turns*, they work with the relationship between elapsed time and the height of the platform on the moving Ferris wheel.

The Circus Act

POW 6: The Tower of Hanoi

The Ferris Wheel

As the Ferris Wheel Turns

The Circus Act

Intent

This activity introduces the central unit problem.

Mathematics

As students explore the central unit problem, they must decide how the information they are given is relevant to the solution as they construct physical models of the situation. The activity also requires that they analyze what other information is needed that has not been provided.

Progression

The Circus Act asks students to make a model of the Ferris wheel situation that is described and to make a list of other information that will be needed in order to solve the unit problem. This is primarily an opportunity to ensure that students thoroughly understand the situation and question of the unit problem.

The discussion following the activity reveals some further basic facts about the Ferris wheel setup.

Approximate Time

30 minutes for activity

35 to 40 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Materials

Items for building models of the problem, such as paper plates, pipe cleaners, and toy cars

Transparency or poster of *The Circus Act* blackline master

Doing the Activity

You may want to have one or more students read the introduction to the unit problem out loud. Then provide materials to each group so that students can make their own physical model of the problem (as in Question 1). You might provide paper plates and pipe cleaners for the Ferris wheel itself and a toy car to represent the moving cart. Circulate among the groups to see that everyone has the right idea. (You may want to keep one or more of these models on hand for demonstration purposes during the rest of the unit.)

The cart and Ferris wheel are set up similarly to what is shown in the diagram below, with the cart passing in front of the Ferris wheel. The platform, which is not shown in this diagram, is pointed "forward" (out from the page), perpendicular to the plane in which the Ferris wheel turns. The front end of the platform is directly above the path of the cart.

Once everyone has a clear understanding of the situation, each group should compile a list of questions as indicated in Question 2.

Discussing and Debriefing the Activity

Have groups share their lists of questions from Question 2. You may want to distinguish, as described here, between questions about the Ferris wheel setup and more general questions about falling objects. As the questions are proposed, you or a student can record them on chart paper.

Here are some of the questions students might ask about the setup:

- What is the radius of the Ferris wheel?
- How high is the center of the Ferris wheel above the ground?
- How fast does the Ferris wheel turn around?
- In what direction (clockwise or counterclockwise) does the Ferris wheel turn?
- When the cart starts moving, what is its position in relation to the Ferris wheel? (That is, how far is the cart from the Ferris wheel, and in what direction?)
- How fast does the cart go?
- How high is the water level in the cart above the ground?
- Where is the diver in the Ferris wheel's cycle when the cart starts moving?

Facts About the Ferris Wheel

After students have posed their questions, present them with the following information about the circus Ferris wheel and the high-dive act. Give them all of this information even if they didn't specifically ask for it. Tell them that the description of this Ferris wheel will be used throughout the unit, though occasionally they will consider other Ferris wheels too (such as in *The Ferris Wheel*).

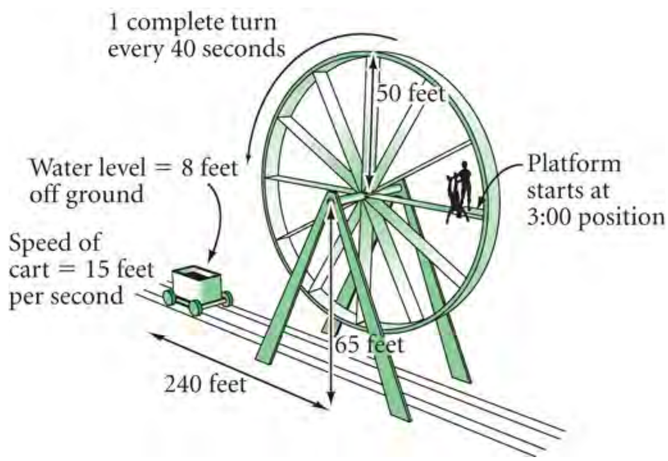
You may want to make a poster or transparency of the diagram of the Ferris wheel in *The Circus Act* blackline master) and mark in each detail as it is discussed—the diagram does not show the numerical details:

- The Ferris wheel has a radius of 50 feet.
- The center of the Ferris wheel is 65 feet off the ground.

- The Ferris wheel turns at a constant speed, making a complete turn every 40 seconds.
- The Ferris wheel turns counterclockwise.
- When the cart starts moving, its center is 240 feet to the left of the center of the base of the Ferris wheel.
- The cart travels to the right at a constant speed of 15 feet per second.
- The water level in the cart is 8 feet above the ground.
- When the cart starts moving, the diver's platform is at the 3 o'clock position in its cycle.

Students should assume that when the cart starts moving, it is immediately going at 15 feet per second.

This picture shows what the final diagram of the situation might look like:



Portions of diagram not to scale.

Post all of the Ferris wheel and cart information prominently in the classroom, together with the diagram. Some of this information will be used right away (in *As the Ferris Wheel Turns*) while other information will not be needed for a while.

Note: This is the first of many posters that you will make for this unit. Be sure to have plenty of wall space (or arrange a way to place posters in a flip-chart arrangement, which takes up less space). Also, encourage students to take notes on these facts and subsequent formulas, because they will need this information.

Students may have a variety of questions about the physics of falling objects. That is, they may wonder exactly what happens to the diver once he is released. Here are some questions that may arise:

- How fast does the diver fall?
- How long does it take the diver to reach the ground?
- Do these answers depend on the diver's weight? On his height?

Tell students that they will be learning about the mathematics and physics of falling objects later in the unit, but that they will not need to answer questions like these just yet.

For Teachers: The Diver's Initial Motion

In the early party of this unit, we will be simplifying the problem by assuming that once the diver is released, he falls straight down as if he had fallen from a motionless Ferris wheel. In *Moving Cart, Turning Ferris Wheel*, students will solve the problem based on that assumption.

Later in the unit, students will return to this situation and deal with the fact that once the diver is released, his path depends not only on where he is released but also on the direction and magnitude of the initial speed he gets from the motion of the turning Ferris wheel.

If this complication comes up in today's discussion, you should acknowledge that the diver does not fall straight down as if from rest. Tell students that they will eventually take this into account. For now, however, they will be dealing with a simplified version of the problem in which he does fall straight down as if from rest.

Supplemental Activity

Mr. Ferris and His Wheel (extension) provides interested students with an opportunity to research the general topic of Ferris wheels.

POW 6: The Tower of Hanoi

Intent

In this activity, students solve a classic mathematical puzzle involving recursion.

Mathematics

The Tower of Hanoi is a classic puzzle that challenges students to find the smallest number of moves in which 64 stacked graduated disks can be moved from one of three posts to another, one at a time, while never placing a larger disk on top of a smaller one. Students are asked for two solutions, one that is based on a recursive process and another that uses a closed formula. Students explain generalizations, which essentially requires a proof.

The discussion of the POW solution introduces recursion and inductive reasoning.

Progression

This is a difficult problem, and students should be given a week or more to work on it. As usual, several student presentations are scheduled for this POW.

Approximate Time

15 minutes for introduction

1 to 3 hours for activity (at home)

20 to 25 minutes for presentations and discussion

Classroom Organization

Individuals, followed by several student presentations and whole-class discussion

Materials

Coins or discs

Doing the Activity

Have students act out a simple version of the puzzle described in *POW 6: The Tower of Hanoi* to be sure they understand the rules. You can use the case of two discs to focus on the number of moves required. Students should see that the pile cannot be moved in fewer than three moves. You may want to let students work on the case of three discs in groups today.

Be sure students notice that the write-up categories for this POW are somewhat different from the standard categories.

Give students about a week to work on this activity. On the day before the POW is due, choose three students to make POW presentations on the following day, and give them overhead transparencies and pens to take home to use for preparing those presentations.

Discussing and Debriefing the Activity

Have the three students chosen give their presentations, and then let other students share their ideas. Students should at least find the number of moves required for specific small numbers of discs.

Finding the Minimum Number

You can use the cases involving only a few discs to bring up the issue of how to show that these results are the smallest possible number of moves. For instance, by carefully examining the case of three discs, students should note the principle that it's "wasteful" to move the same disc on two consecutive moves. This principle shows that in the first three moves, one should get the two smallest discs onto another peg and that this can't be done in fewer than three moves. You can work further with this case to see why it's impossible to move all three discs in fewer than seven moves.

Students might use the information about specific cases in various ways to develop a general formula. If no general formulas are presented, you might have students put the results from individual cases into a table like this:

Number of discs	Number of moves needed
1	1
2	3
3	7
4	15

Ask, **What patterns or rules did you find?** Two main observations are likely to come out of this table and out of students' work in generating it. If we use a_n to represent the minimum number of moves required if there are n discs, then the two key patterns can be written as:

$$a_n = 2^n - 1$$

$$a_{n+1} = 2a_n + 1$$

One proof of the first relationship involves the second relationship, together with an approach similar to mathematical induction. Use your judgment about how much of the discussion presented here will be appropriate for your students.

The Closed Form: $a_n = 2^n - 1$

Some of your students likely will see that the entries are each one less than a power of 2. In other words, the general formula for the number of moves required to move n discs is $2^n - 1$.

The Recursive Pattern: $a_{n+1} = 2a_n + 1$

Another pattern is that each successive entry is obtained by doubling the previous entry and adding 1. Ask, **Why does this pattern hold true?** Some students may have observed this pattern and seen why it works in the course of doing examples. If so, they can probably articulate fairly well why it works. If not, one way to help students get some insight into this is to have a student act out the process for, say, four discs, but to interrupt after three discs have been moved to another peg. Help them to see the process of moving four discs in three stages:

- Move three discs to the middle peg (7 moves)
- Move the biggest disc to the right peg (1 move)
- Move three discs from the middle peg to the right (7 moves)

They should notice that they cannot move the fourth disc until the other three are all on the same one of the other two pegs.

It is important to articulate that moving three discs from one peg to another is the same no matter which two pegs are involved. That's why the first and last stages in the three-stage process just described each take the same number of moves as solving the original puzzle for three discs. You simply have to be careful about which peg you move the first disc to.

To reinforce this idea, you might ask what the largest number of discs was for which anybody actually carried out the process. Suppose, for example, someone did eight discs, in 255 moves. Ask, **How can you use the answer for eight discs to help you get the answer for nine discs?** Students should see that they can move eight of the discs to the middle in 255 moves, use one move to move the largest disc to the right, and then use 255 more moves to move the pile of eight from the middle to the right. This gives a total of 255 + 1 + 255 = 511 moves.

In other words, if we use a_n to represent the number of moves required to move an n-disc pile, then we see that this relationship holds:

$$a_{n+1} = a_n + 1 + a_n$$

(Students may simplify this to $a_{n+1} = 2a_n + 1$, but you might prefer to use the unsimplified formula because that reflects the sequence in which the groups of moves are done.)

Introduce the word *recursion* to describe the process by which each term in a sequence is described in terms of a preceding term or terms. The relationship $a_{n+1} = a_n + 1 + a_n$ (or $a_{n+1} = 2a_n + 1$) is called the *recursive formula.*

Proving the Closed Formula

If you have discussed both the recursive formula and the *closed formula* $a_n = 2^n - 1$, ask, **Why does the closed formula work?** Point out that they can see that it works by example for the specific cases in the table, but that doesn't mean that it always works.

You can ask students to imagine that they know that the rule $2^n - 1$ works for a specific large value of n, such as $n = 20$. That is, they should suppose that they have verified somehow that moving 20 discs requires $2^{20} - 1$ moves. Ask, **Can you find the number of moves for 21 discs without multiplying out 2^{20}?** Tell them that they can use the recursive pattern. They should see that 21 discs will require $(2^{20} - 1) + 1 + (2^{20} - 1)$ moves.

Ask, **What do you get when you simplify $(2^{20} - 1) + 1 + (2^{20} - 1)$?** They should see that the expression simplifies to $2 \cdot 2^{20} - 1$, which is the same as $2^{21} - 1$.

Ask, **Will this work for any number of discs?** Students should see that it does, and you can ask them to try to produce a general argument that if n discs can be transferred in $2^n - 1$ moves, then $n + 1$ discs can be transferred in $2^{n+1} - 1$ moves.

As discussed earlier, the three-stage process gives the formula

$$a_{n+1} = a_n + 1 + a_n$$

so if n discs take $2^n - 1$ moves, then $n + 1$ discs take $(2^n - 1) + 1 + (2^n - 1)$ moves. But this simplifies to $2^{n+1} - 1$ moves.

In other words, this reasoning shows that if the formula $2^n - 1$ works for a particular number of discs, it also works when there is one more disc. If students get this far, tell them that this explanation is an example of a form of proof called *mathematical induction.* Tell them that this type of proof involves two elements: getting started (such as showing that the formula works in the case of only one disc) and going from one stage to the next (such as proving the recursive formula).

What About the Monks?

Oh yes, let's not forget the monks. With 64 discs, the task would take $2^{64} - 1$ moves, or approximately $1.8 \cdot 10^{19}$. A series of computations shows that $1.8 \cdot 10^{19}$ seconds is about 580 billion years. Doing only 40 discs would take a mere 35,000 years. (Siddhartha Gautama, founder of Buddhism, lived from about 563 B.C.E. to 483 B.C.E. If the monks had begun this task when he was alive and moved one disc per second, they would now have moved the thirty-seventh disc and would be rebuilding the pile of the first 36 discs on top of it.)

Key Questions

What patterns or rules did you find? Why does this pattern hold true?

How can you use the answer for eight discs to help you get the answer for nine discs?

Why does the closed formula work?

Can you find the number of moves for 21 discs without multiplying out 2^{20}?

What do you get when you simplify $(2^{20} - 1) + 1 + (2^{20} - 1)$?

Will this work for any number of discs?

The Ferris Wheel

Intent

In this activity, students recognize that trigonometry will play a role in the unit problem.

Mathematics

This activity gives students a chance to investigate a Ferris wheel's motion in a more elementary context than that of the main problem. They begin to use trigonometry to calculate the height of various points on the wheel.

Progression

Students work individually to find the height at several clock positions, then discuss their results as a class.

Approximate Time

5 minutes for introduction

25 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Optional: Transparency of *The Ferris Wheel* blackline master

Doing the Activity

Take a minute in class to go over the use of clock labels to represent Ferris wheel positions. (You may want to make a transparency of *The Ferris Wheel* blackline master.)

Discussing and Debriefing the Activity

In today's discussion, students do not need to develop any general formulas concerning the relationship between position and height. They will be working with that relationship further in *As the Ferris Wheel Turns* and several additional activities. For today, it is enough that students recognize that trigonometry will play a role. Keep in mind that from their current perspective, trigonometric functions exist only in a right-triangle context.

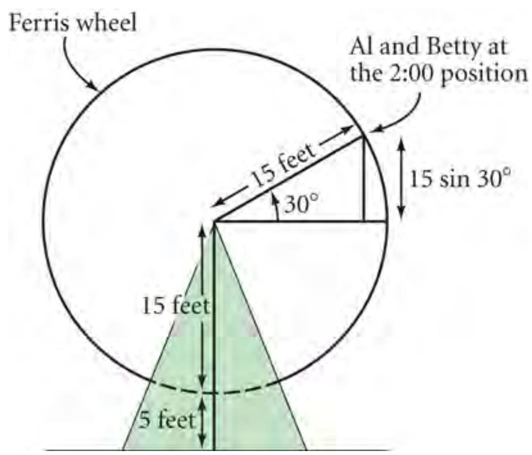
Let students discuss the problems for a few minutes in their groups, and have several groups prepare presentations for examples from Questions 2 and 3.

You can probably go over the various parts of Question 1 orally. These should be straightforward if students have read the information carefully. (They may have overlooked the fact that the low point of the Ferris wheel does not have height zero.)

Question 2

Have students do their presentations on Question 2. If they have trouble with these other positions on the Ferris wheel, you will need to review right-triangle trigonometry.

For example, for the 2 o'clock position, they should see that Al and Betty's height can be found using the triangle shown below. Thus, the height for the 2 o'clock position is given by the expression 5 + 15 + 15 sin 30°, which leads to a result of 27.5 feet.



Students sometimes mistakenly assume that the heights are equally spaced from one hour to the next. For example, because the 3 o'clock position is 20 feet high and the 12 o'clock position is 35 feet high, students may think that the 2 o'clock position is a third of the way between them, at 25 feet. If this incorrect approach is presented, acknowledge that it is a reasonable idea and then have the class discuss its merits. Be sure students see that it gives the wrong result. The different "o'clock" positions create equal central angles, but they do not have equally spaced heights. (If no one brings up this incorrect approach, you might want to present it yourself, to get students to articulate why it doesn't work.)

Question 3

If students seemed clear about the ideas involved in their discussion of Question 2, you can omit discussion of Question 3. Otherwise, use one or two more examples to clarify the process of finding the height.

As the Ferris Wheel Turns

Intent

In this activity, students start to look at the relationship between time elapsed, clock position, and height on the Ferris wheel.

Mathematics

This activity requires students to find the speed of an object moving at a constant angular speed. They also find the height, for specific times, of an object moving in a circular path.

Progression

Students work on this activity individually and then discuss their results as a class.

Approximate Time

30 minutes for activity (at home or in class)

20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

Note: The activity *At Certain Points in Time* depends on students really understanding this activity, so it's worth spending most of a class period on the discussion, even if that means that another activity is delayed.

You might begin by assigning a problem to each group to prepare for presentation. Then let different students report on their results.

Questions 1 through 3

For Question 1, students will need to use the circumference formula, $C = 2\pi r$, to see that the total distance traveled by the platform in one complete turn is 100π feet. Because the platform goes 100π feet in 40 seconds, it is going 2.5π feet per second, which is approximately 7.85 feet per second (roughly 5 miles per hour).

Post this result about the platform's speed, because it will be used later in the unit. You may want to incorporate it also into the posted diagram of the Ferris wheel:

The speed of the platform as it turns is 2.5π feet per second, which is approximately equal to 7.85 feet per second.

On Question 2, students should see that because a complete turn is 360°, the Ferris wheel must be turning $360 \div 40 = 9$ degrees each second. Review the term *angular* *speed* (introduced in the activity), which measures how fast an angle is changing. Bring out that angular speed is measured in units such as degrees per second.

On Question 3, you may want to have a student identify each angle involved. Emphasize again that angular speed does not depend on the radius of the Ferris wheel.

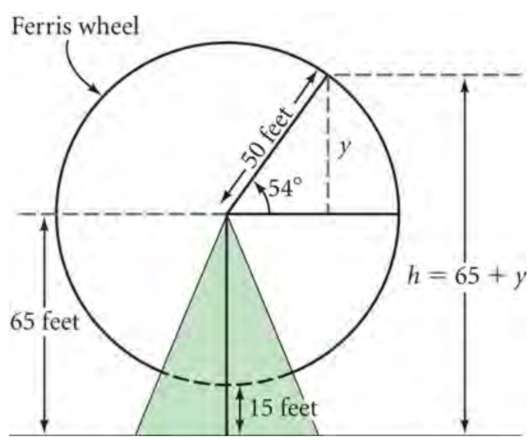
Ask students, **What word is used to describe the time interval for each complete turn of the Ferris wheel?** If necessary, remind them of the term *period.*

Question 4

Question 4 is a lead-in to *At Certain Points in Time,* so take extra care to ensure that students understand the presentations. You may want to post the results from Question 4, because students will use this information in later activities.

Questions 4a and 4b are easier because they involve angles in the first quadrant. Use the presentations of these problems to help students work out the details of getting from the time elapsed to the angle of turn, as well as the process of using trigonometry to get the platform's height from the angle of turn. As students do various examples, they should see the need to distinguish the cases, depending on the quadrant.

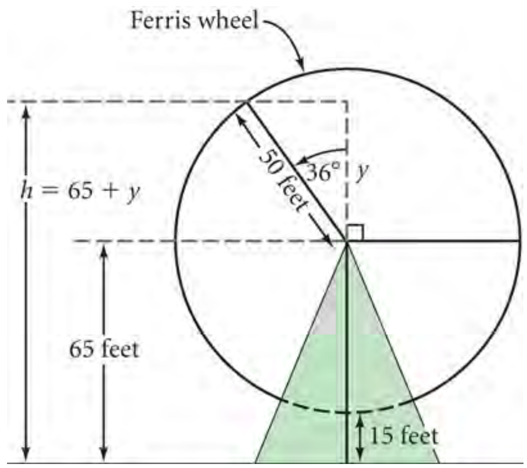
For example, for t = 6, the situation looks like the diagram shown here. The angle is $6 \cdot 9°$ because the Ferris wheel turns 9 degrees per second. Students will probably use the equation $\sin 54° = \frac{y}{50}$ to get $y = 50 \sin 54°$. Therefore, the total height off the ground is $h = 65 + 50 \sin 54°$.



Students may recognize that Question 4c is the same as Question 1b of *The Ferris Wheel* except for the conversion from time to degrees.

For Question 4d ($t = 14$), the platform will have moved through an angle of 126°. Students may recognize that this leads to the same height for the platform as for Question 4b ($t = 6$). If they do not see this, they will probably use either the supplementary angle of 54° (getting $65 + 50 \sin 54°$ for the height of the platform) or the related 36° angle (getting $65 + 50 \cos 36°$). These two approaches are illustrated in the following two diagrams. You don't need to discuss both methods, but, as always, encourage alternate approaches to problems. (*Note:* If both approaches come up, you can bring out that sin 54° = cos 36° and discuss why.)

Alternately, students may simply follow the pattern of Questions 4a and 4b, and use the expression 65 + 50 sin 126°. They may discover that this expression yields a reasonable answer on their calculators, even though sin 126° has not yet been defined. If this comes up, remind them that so far, they only know the meaning of the trigonometric functions for acute angles. Then tell them that extending the definitions of these functions is an important element in this unit.



Here, we have $\frac{y}{50} = \cos 36°$*, so* $y = 50 \cos 36°$ *and* $h = 65 + 50 \cos 36$

For Question 4f, students will probably realize that when the Ferris wheel has turned for 49 seconds, the platform is at the same position as after 9 seconds. Use the word *periodic* to describe the motion of the platform.

Key Question

What word is used to describe the time interval for each complete turn of the Ferris wheel?

The Height and the Sine

Intent

In these activities, students explore the nature of the sine function.

Mathematics

The right-triangle definitions that were used to introduce students to the trigonometric functions are only relevant for acute angles. In this section, students extend the definition of the sine function to one that is meaningful for arbitrary angles. They graph sinusoidal functions, explore the periodic nature of the sine function, and use that periodicity to make sense of situations when their calculator returns only a principal value for the inverse sine.

Progression

Students begin by finding a general formula for the height of the central unit problem's Ferris wheel platform in the first quadrant (*At Certain Points in Time*). They obtain further familiarity with the relationship between the height of the platform and its angular position in *A Clear View*, then apply that knowledge to extend the definition of the sine function to angles beyond the first quadrant (*Extending the Sine*), including negative angles (*Testing the Definition*).

Students then explore the periodic nature of sinusoidal functions, first through the context of the Ferris wheel in *Graphing the Ferris Wheel* and *Ferris Wheel Graph Variations*, then looking at the graph of the simpler sine function in *The "Plain" Sine Graph*.

Sand Castles and *More Beach Adventures* challenge students to apply their new knowledge of sinusoidal functions to a situation that does not involve angles—the periodic rise and fall of the ocean tide. This situation becomes particularly thought provoking as students discover that even though the tides do not involve angles, they must resort to consideration of the principal angle of the inverse sine in order to make sense of the solutions to their equations.

At Certain Points in Time

A Clear View

Extending the Sine

Testing the Definition

Graphing the Ferris Wheel

Ferris Wheel Graph Variations

The "Plain" Sine Graph

Sand Castles

POW 7: Paving Patterns

More Beach Adventures

At Certain Points in Time

Intent

In this activity, students develop a formula for the platform's height when it is in the first quadrant.

Mathematics

In *As the Ferris Wheel Turns*, students found the height of the platform on the moving Ferris wheel at specific times. Now they generalize this process to find a formula for all heights in the first quadrant. At this point they are not able to extend their formula beyond the first quadrant, because they are only familiar with definitions of the trigonometric functions that are based on angles of right triangles, and thus only applicable to acute angles.

Progression

In this activity, students write a formula to generalize their findings from *As the Ferris Wheel Turns.* They'll extend their thinking beyond the first quadrant in *Extending the Sine.*

Approximate Time

25 minutes for activity

5 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

At Certain Points in Time asks students to represent the platform's height as h and to find a formula for h within the first quadrant in terms of the time in seconds (t) that the Ferris wheel has been turning. They are then asked to verify their formula using the results from Questions 4a and 4b of *As the Ferris Wheel Turns*.

Discussing and Debriefing the Activity

Let a student present and explain his or her group's formula. This should follow fairly easily from the discussion of *As the Ferris Wheel Turns*, but be sure to get a clear explanation, because this formula will play a major role in extending the sine function beyond acute angles.

Post and label the formula, which will probably look like this:

$$h = 65 + 50 \sin (9t)$$

Other quadrants will be considered in *Extending the Sine*.

A Clear View

Intent

In this activity, students continue to work with the relationship between the angular position of the platform and its height.

Mathematics

This activity uses Al and Betty's Ferris wheel and poses a question that isn't explicitly about only height or time. Students work backwards from the height of the platform to its angular position in order to determine the percentage of the time that the platform is above a fence.

Progression

Students work on this activity individually and then discuss their results as a class.

Approximate Time

30 minutes for activity (at home)

10 minutes for discussion

Classroom Organization

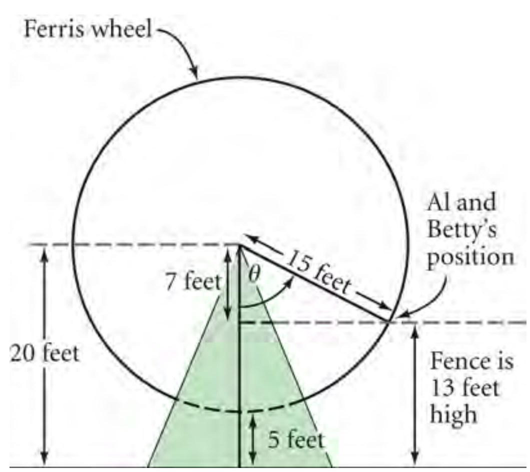
Individuals, followed by whole-class discussion

Doing the Activity

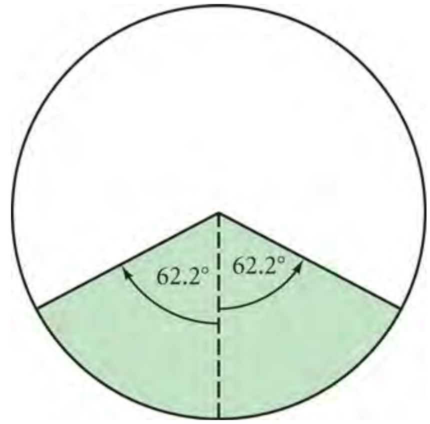
A Clear View asks students to find the percentage of the time that Al and Betty's Ferris wheel from *The Ferris Wheel* is above a 13-foot fence. They are then asked to explain how the answer would change if the period were different.

Discussing and Debriefing the Activity

Ask for a volunteer to present the problem. Be sure that the presenter includes a diagram as part of the explanation. For instance, the diagram here shows the situation when Al and Betty are exactly 13 feet off the ground (and in the fourth quadrant). At this position, they are 7 feet below the center of the Ferris wheel, so the angle θ satisfies the equation $\cos\theta = \frac{7}{15}$, which gives $\theta \approx 62.2°$. (You can use this discussion, if needed, as an opportunity to remind students of the inverse trigonometric functions and notation such as $\cos^{-1}\frac{7}{15}$.)



The presenter might use a diagram like the next one to bring out that Al and Betty are below the fence while the Ferris wheel travels through an angle of *twice* $62.2°$, using the fact that there are two places in the Ferris wheel cycle where Al and Betty are 13 feet high.



Some students may work directly with the angles. For instance, they might see that the fraction of the time during which Al and Betty are *below* the fence is equal to the ratio $\frac{2 \cdot 62.2}{360}$ or approximately 35%. Because Al and Betty are below 13 feet about 35% of the time, they can see over the fence about 65% of the time.

Some students may feel more comfortable working with the time involved. For instance, they might see that each second elapsed represents 15 degrees of turn, so 62.2 degrees represents $\frac{62.2}{15}$ seconds, or about 4.15 seconds. This means that Al and Betty are below the fence for about 8.3 seconds and above it for about 15.7 seconds. Thus, the fraction of the time that they are above the fence is about $\frac{15.7}{24}$, or approximately 65%.

You can use Question 2 to bring out that although the period of the Ferris wheel affects the *amount* of time in each cycle that Al and Betty are above the fence, it does not affect the *percentage* of time they are above the fence.

Extending the Sine

Intent

In this activity, students see how to extend the sine function to be defined for all angles.

Mathematics

Thus far, students have worked with trigonometric functions that have been defined in terms of right-triangle geometry and are thus applicable only to acute angles.

Now they will see how to redefine the sine function in a manner that will make sense for all angles.

Progression

To begin this lesson, you'll remind students that the sine function has previously been defined only for first-quadrant angles, and students reflect on how the definition of exponentiation was previously extended beyond repeated multiplication in order to make sense of negative and zero exponents. Then, using a specific Ferris wheel situation, you'll illustrate the extended definition of sine, and students verify that calculator results agree with the new definition for specific cases.

Approximate Time

20 to 25 minutes

Classroom Organization

Teacher presentation

Doing the Activity

Extending the Sine summarizes the ideas behind extending the sine function beyond the first quadrant. Lead students through the discussion described below. Then, you might have students look over the *Extending the Sine* reference pages briefly, or you might have them begin work directly on the next activity, *Testing the Definition.*

Discussing and Debriefing the Activity

Review the work in *At Certain Points in Time* developing a first-quadrant formula for the platform's height above the ground,

$$h = 65 + 50 \sin(9t)$$

Point out that it would be nice to have a formula that works no matter where the platform is, and ask, **Why can't you simply apply this formula for all values of *t*?** Bring out that the definition of the sine function is based on right triangles, so the expression $\sin(9t)$ is meaningful only if $9t$ is strictly between 0° and 90°. That is, the formula makes sense (so far) only if the platform is still in the first quadrant, which means $0 < t < 10$. (Even if students have already discovered that the formula seems to work for other angles, point out that they don't have a general definition yet for the sine function.)

Previous Experiences with Extending Functions

Tell students that their next task is to consider how to extend the sine function to be defined for all angles. Ask, **When have you extended a function or operation before?** Remind them, if necessary, that when they first learned about exponentiation, the operation was defined in terms of repeated multiplication and that the definition made sense only if the exponent was a positive integer.

Then ask, **How did you extend exponentiation in *All About Alice*?** Try to bring out these key ideas:

- The new definition extending the domain of the operation had to be consistent with the old definition.
- The situation of Alice and the cake and beverage was a useful model for thinking about exponents.
- The new definition was created so that certain patterns and algebraic rules that held true for positive integer exponents continued to hold true when the domain was extended.

Tell students that the model of the Ferris wheel for circular motion will play a role for trigonometric functions similar to that played by the Alice situation for exponentiation

A Specific Case

Tell students that a key criterion of a new, extended definition of the sine function will be whether it makes the platform height function work for all values of t.

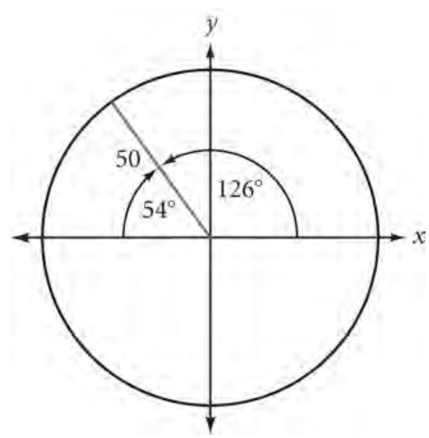
Point out that when $t = 14$, the height formula gives the expression $65 + 50 \sin (9 \cdot 14)$, which means that the angle of turn for the platform is 126°, so the platform would no longer be in the first quadrant. Ask, **What value should you assign as the definition of sin 126° so that the height formula gives the right answer for $t = 14$?** As a hint, you can ask**, What value of t gives the same height but leads to a first-quadrant angle?** Bring out that at $t = 14$, the platform has the same height as at $t = 6$, so the expression $65 + 50 \sin (9 \cdot 14)$ must evaluate to be the same as the expression $65 + 50 \sin (9 \cdot 6)$. To make this more clear, you might write this as the equation

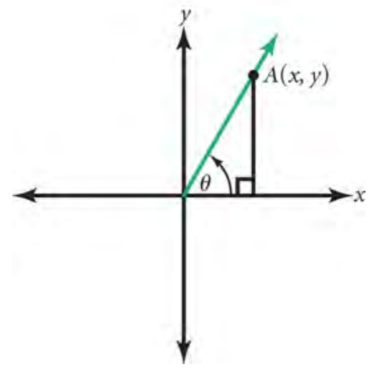
$$65 + 50 \sin 126° = 65 + 50 \sin 54°$$

Thus, for the height formula to work for $t = 14$, we must define the sine function so that sin 126° has the same value as sin 54°.

Defining the Sine Function for All Angles

Tell students that in order to create a general, extended definition of the sine function, it's helpful to replace the Ferris wheel model with the more abstract setting of the coordinate plane. You can begin with a diagram like the one shown here, suggesting that they imagine that the circle of the Ferris wheel has been placed in the coordinate plane, with its center at the origin.





Then ask the class to consider the next diagram, which corresponds to a situation in which the Ferris wheel has turned only through a first-quadrant angle. A generic point *A*, with coordinates (*x, y*), has been marked on the ray defining the angle. (The point *A* is assumed to be different from the origin.)

Ask, **How can sin θ be defined in terms of the coordinates *x* and *y*?** Students should see that in the right triangle, sin θ is equal to the ratio $\frac{y}{\sqrt{x^2+y^2}}$.

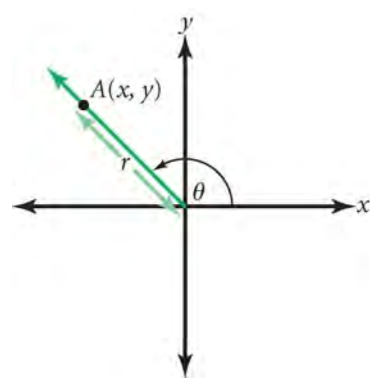
Introduce the use of *r* as a shorthand for the expression $\sqrt{x^2+y^2}$, and bring out that this corresponds to the radius of the Ferris wheel. Use the variable *r* to rewrite the expression for sin θ more simply as the ratio $\frac{y}{r}$.

Point out that the ratio $\frac{y}{r}$ makes sense for any angle. (The issue of negative angles will be discussed explicitly following *Testing the Definition*.) Tell students that this simple ratio is used for the extended definition of the sine function. Post the formal definition together with an appropriate diagram:

For any angle θ, we define sin θ by first drawing the ray that makes a counterclockwise angle θ with the positive *x*-axis and choosing a point *A* on this ray (other than the origin) with coordinates (*x, y*).

Using the shorthand $r = \sqrt{x^2+y^2}$, we then define the sine function by the equation

$$\sin\theta = \frac{y}{r}$$



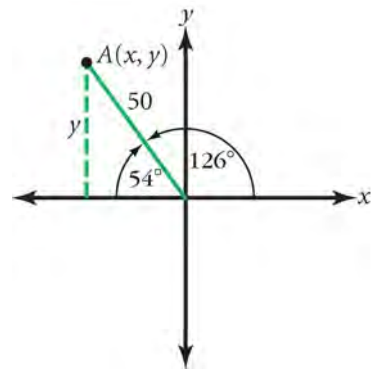
Note: The issue of why this method makes sin θ well-defined is discussed following *Testing the Definition* (see the section "Why Is the Sine Function Well Defined?").

This extension of the right-triangle sine function to the more general definition of sine for arbitrary angles is a key idea, and perhaps even the central mathematical idea of this unit, so be sure to give it appropriate emphasis.

Bring out that we have taken familiar ideas—right-triangle trigonometry and the coordinate system—and combined them in the context of a concrete situation to create a more general definition of the sine function. This new definition is consistent with the old definition of sine for acute angles. It allows us to replace a complex, quadrant-by-quadrant analysis of the platform height with a single, uniform expression.

Back to the Specific Case

Have students work in groups and ask, **How does this definition apply to the case θ= 126°?** You may need to suggest that they pick a specific value for *r* (perhaps *r* = 50, as in the Ferris wheel) and then find the corresponding value for *y,* as in this diagram.



Students should see, perhaps using the right triangle in the second quadrant, that *y* is equal to 50 sin 54°, so the ratio $\frac{y}{r}$ is equal to sin 54°. In other words, this coordinate definition will make sin 126° the same as sin 54°, as before.

Have students verify that their calculators do, in fact, give the same value for sin 126° as for sin 54°.

Key Questions

Why can't you simply apply this formula for all values of *t*? When have you extended a function or operation before?

How did you extend exponentiation in *All About Alice*?

What value should you assign as the definition of sin 126° so that the height formula gives the right answer for *t* = 14?

What value of *t* gives the same height but leads to a first-quadrant angle?

How can sin θ be defined in terms of the coordinates *x* and *y*? How does this definition apply to the case θ = 126°?

Testing the Definition

Intent

In this activity, students verify that the extended definition of the sine works in the platform height formula for all angles.

Mathematics

In *Extending the Sine*, students were presented with the development of a new definition for the sine function that makes sense for arbitrary angles rather than only for acute angles. Now they will test that definition in the formula for the height of the platform developed in *At Certain Points in Time* to see if it will successfully extend that formula to all four quadrants. The discussion will also introduce reference angles and the unit circle and will look at the case of the sine of a negative angle.

Progression

Students work on the activity in groups and then discuss their results as a class. The discussion also introduces new concepts—the unit circle, reference angles, and sine of a negative angle.

Approximate Time

20 minutes for activity

35 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Question 1 uses the special case in which the right triangle has a 45° angle. On Question 2, students will need to use right-triangle trigonometry to find the coordinates of some point A in the fourth quadrant.

You need not have all groups complete Question 2 before beginning discussion.

Discussing and Debriefing the Activity

Let students from different groups present different parts of Question 1. For Question 1a, students might pick 50 for r and see that the legs of the right triangle are both of length $25\sqrt{2}$. They might get this by using the Pythagorean theorem, by using trigonometric functions of a 45° angle, or simply by remembering the ratios for the special case of an isosceles right triangle.

Once students have the lengths of the sides, they need to see that y is negative, because point A is in the third quadrant. (*Comment:* They don't actually need to find the value of x to get the value of sin 225°, but you might have them do this anyway.)

Finally, students need to find the ratio $\frac{y}{r}$ and see that this ratio is approximately –0.707. (If they leave the ratio as $-\sqrt{2}$, that's fine, too.)

Questions 1b through 1d

When students substitute their value for sin 225° into the expression 65 + 50 sin 225°, they should get a value of approximately 29.6 feet. Their explanation that this is reasonable should certainly include the fact that the result is less than 65 feet (because the platform is below the center of the Ferris wheel) but more than 15 feet (because the lowest point in the cycle is 15 feet off the ground).

You might ask them to find the position of the platform after 5 seconds (which corresponds to an angle of 45°). Then bring out that the result there (about 100.4 feet) is about 35.4 feet above the center of the Ferris wheel, just as 29.6 feet is 35.4 feet below the center.

Be sure to have students see that their calculators give the same value for sin 225° as the value they just found by hand.

Question 2

Question 2 is similar to Question 1, except that students will likely have more difficulty finding the value of y here than in Question 1. Ask, **How can you express sin 288° in terms of the sine of a first-quadrant angle?** They should see that except for sign, sin 288° is like sin 72°, so sin 288° = –sin 72°. As with Question 1, have them see that their calculators give this result also.

The Reference Angle

Bring out that for any angle, there is a first-quadrant angle that has the same sine, except perhaps for the sign. (Be careful to distinguish between the words "sign" and "sine.")

Tell students that this first-quadrant angle is called the *reference angle.* They should see that the reference angle for 225° (from Question 1) is 45° and that the reference angle for 288° is 72°.

The General Platform Height Function

Assure students that with the coordinate definition of the sine function, their platform height function works for all values of t. Whenever the platform is above the center of the Ferris wheel, y is positive; whenever the platform is below the center of the Ferris wheel, y is negative. Thus, the platform's overall height is 65 + y, regardless of the sign of y.

By the general definition, $\sin\theta = \frac{y}{r}$, so $y = r\sin\theta$. In the case of the Ferris wheel in the unit problem, y becomes $50\sin\theta$. In other words, the height of the Ferris wheel is always $65 + 50\sin\theta$. That is the power of using the coordinate plane to understand the Ferris wheel.

Post this general result, which you might state as follows:

Suppose the Ferris wheel and platform satisfy these conditions:

- The center of the Ferris wheel is 65 feet off the ground.
- The radius of the Ferris wheel is 50 feet.
- The Ferris wheel turns counterclockwise at a constant rate with a period of 40 seconds.
- The platform is at the 3 o'clock position when $t = 0$.

Then the height of the platform off the ground after t seconds is given by the expression

$$65 + 50 \sin (9t)$$

Be sure students see how the specific details of the circus act fit into this formula. In particular, you may want to go over the fact that the coefficient 9 (in the expression $9t$) comes from dividing 360° by the period of 40 seconds. You might review the term *angular speed* by bringing out that the angular speed of the Ferris wheel is 9 degrees per second.

Negative Angles

Ask, **How might the extended definition of the sine function make sense when θ is negative?** You may want to suggest that students consider what a negative value for t would mean in the context of the Ferris wheel problem.

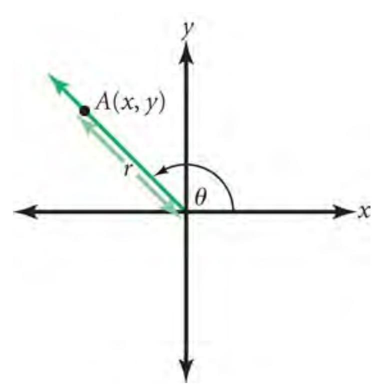
Students may be able to guess on their own, but if needed, tell them that we interpret a negative angle by going clockwise instead of counterclockwise and that a negative value of t refers to a time before the platform reaches the 3 o'clock position.

Illustrate with a specific case, such as asking students to find sin (–53°). They should see that the reference angle is 53° and that sin (–53°) = –sin 53°.

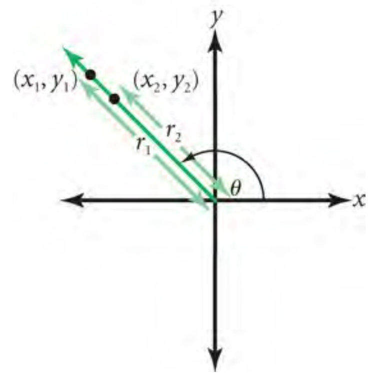
Why is the Sine Function Well Defined?

The definition of $\sin \theta$ involves the use of a point on the ray making an angle of θ measured counterclockwise from the x-axis. The question may have come up earlier as to why we can use *any* point on the ray. If it has not yet been discussed, bring it up now.

Review the fact that $\sin \theta$ is defined by the ratio $\frac{y}{r}$ for some point (other than the origin) on the ray defining θ. You might use a diagram like the one at right.



Then ask, **Why is the ratio $\frac{y}{r}$ the same for all points on the ray?** If necessary, revise the diagram to show two points on the ray, perhaps labeled (x_1, y_1) and (x_2, y_2), with r-values r_1 and r_2, like this:



If needed, restate the question in terms of the new diagram, asking, **Why is the ratio $\frac{y_1}{r_1}$ the same number as $\frac{y_2}{r_2}$?** The goal is to bring out that similarity plays a key role in this general definition, just as it did when the trigonometric functions were defined for acute angles using right triangles. You might point out that no matter what quadrant θ is in, the two ratios will have the same sign because the y-coordinates of the two points have the same sign. Students can then use similar right triangles to see that the ratios of the lengths of the sides are the same by similarity.

The Unit Circle

Tell students that because the value of the sine does not change for different values of r, they can choose whatever value is most convenient. Ask**, What value for *r* would be simplest to use?** Bring out that using 1 for r means simply that $\sin \theta = y$.

Ask, **How can you describe the set of points with *r* = 1?** Students should see that these points form a circle of radius 1, with the center at the origin. Tell them that this set of points is called the **unit circle.**

Key Questions

How can you express sin 288° in terms of the sine of a first-quadrant angle?

How might the extended definition of the sine function make sense when θ is negative?

Why is the ratio $\frac{y}{r}$ the same for all points on the ray?

Why is the ratio $\frac{y_1}{r_1}$ the same number as $\frac{y_2}{r_2}$?

What value for *r* would be simplest to use?

How can you describe the set of points with *r* = 1?

Graphing the Ferris Wheel

Intent

In this activity, students graph the Ferris wheel height function.

Mathematics

Having extended the definition of the sine beyond angles of the first quadrant, students now graph the Ferris wheel height function through two full revolutions, observing the periodicity of the graph. They then think about how changing the parameters affects the graph.

Progression

Students work on the activity individually and then discuss their results as a class.

Students will explore in more detail how changing parameters affects the graph in *Ferris Wheel Graph Variations.*

Approximate Time

30 to 40 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Transparencies of the *Graphing the Ferris Wheel* blackline masters

Doing the Activity

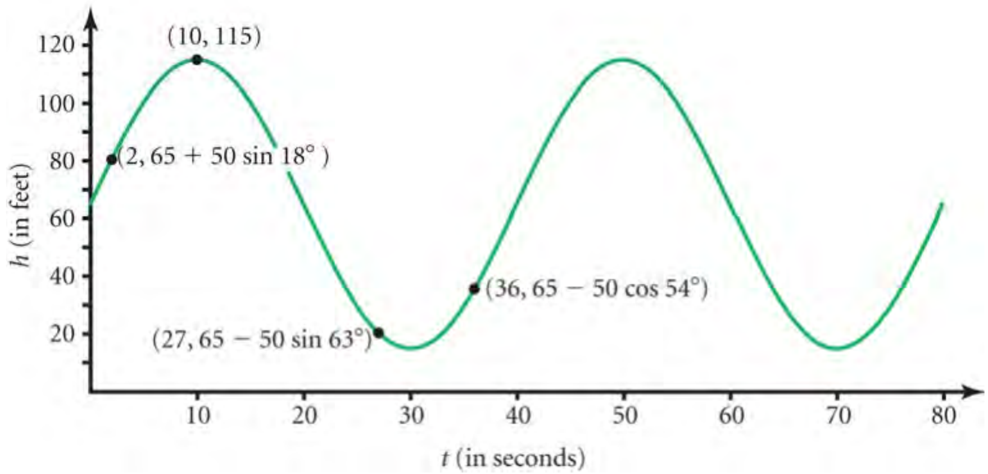
You might suggest to students that they use their data from *As the Ferris Wheel Turns* in Question 1 of this activity.

Discussing and Debriefing the Activity

Regarding the graph in Question 1, ask, **How should the axes and scales be set up?** The vertical scale should reflect the fact that the height goes from a minimum of 15 feet to a maximum of 115 feet. Students are specifically instructed to include a horizontal scale from $t = 0$ to $t = 80$.

You might display a transparency with these scales so that students can plot their points on a shared coordinate system. (A blackline master of a blank coordinate system with these scales is included in the *Graphing the Ferris Wheel* blackline masters.) You can then have students from various groups each give and explain the coordinates for one point on the graph. You can label various points with the expressions used to find the values for h.

Some students may be finding the heights in terms of right triangles, just as they did at the beginning of the unit, and perhaps even using the cosine of some angle rather than the sine. If so, you may get a diagram with points labeled as shown here. (The *Graphing the Ferris Wheel* blackline masters include a larger version of this graph without individual points labeled.)



Ask, **What single equation will describe this graph?** If needed, review the discussion from *Testing the Definition*, so students see that this is the graph of $h = 65 + 50 \sin (9t)$.

Have students graph this function on their calculators (adjusting the variables as needed for calculator entry and choosing an appropriate viewing window). They should see that the graph matches the one they created by hand.

Symmetry and Periodicity in the Graph

Ask, **Did you use any shortcuts or see any patterns?** Guide them to mention things like symmetry or periodicity here.

This is a good time to review the idea of periodicity. Ask, **What does it mean that *f* is periodic, with period 40 seconds?** Students might say something like, "The height is the same every 40 seconds." You can have a student use the graph to illustrate what this means, by showing that points whose t-coordinates differ by 40 have the same h-coordinate.

Bring out that when we say that the period for the height function is 40 seconds, this means not only that the height is the same every 40 seconds but also that there is no smaller time interval for which the height always repeats.

Question 2

The discussion of Question 2 should be limited to a qualitative description of the changes in the graphs. In the next activity, *Ferris Wheel Graph Variations*, students will look at the graphs for specific variations of each type.

In discussing Question 2a, students should be able to explain that if the radius were smaller, the new graph would not go as "high" or as "low" as the original. They might describe the new graph as "squished vertically toward the line $y = 65$." Bring out that the "midline" of the graph remains the same. That is, the graph is still as much above the line $y = 65$ as it is below this line.

Tell students that the distance from the midline to the high or low point of the graph is called the **amplitude** of the graph. In other words, the amplitude for such a Ferris wheel height graph is the same as the radius of the Ferris wheel.

On Question 2b, students should see that if the Ferris wheel turns faster, then the platform will go up and down more times during the 80-second interval shown. In other words, the height function will have a smaller period. They might describe the graph as “squished horizontally like an accordion.”

Finally, on Question 2c, students should see that the graph is simply moved down so that it is above the axis half the time and below the axis half the time. Ask, **How is the amplitude affected in Question 2c?** Students should see that the amplitude has not changed.

Key Questions

How should the axes and scales be set up?

What single equation will describe this graph?

Did you use any shortcuts or see any patterns?

What does it mean that *f* is periodic, with period 40 seconds?

How is the amplitude affected in Question 2c?

Ferris Wheel Graph Variations

Intent

In this activity, students examine how changing the specifications of the Ferris wheel affects the graph of the platform's height.

Mathematics

In *Graphing the Ferris Wheel*, students were asked to describe how various changes in the parameters of the Ferris wheel would affect the graph of the platform's height function. Now they validate their intuitive responses by creating new graphs for the height function with different specific values for the parameters. They also look at how the equation describing the function changes.

Progression

Students work on the activity individually and share their results as a class. This activity formalizes the intuitive observations that students made in *Graphing the Ferris Wheel.*

Approximate Time

30 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

The first two questions in *Ferris Wheel Graph Variations* ask students to pick a new value for a specified parameter of the Ferris wheel, draw a new graph of the height function, give an equation for the graph, and verify that the equation works for a specific value of t. The final question is similar, with the Ferris wheel in a hole such that its center is at ground level.

Discussing and Debriefing the Activity

You can have volunteers present specific graphs for each of the questions. Keep in mind that other students may have chosen different values for the radius or period, so students will probably have slightly different graphs. You can use the specific examples to review the general ideas that were included in the discussion of *Graphing the Ferris Wheel*.

Question 1

If the presenter of Question 1 does not use the term *amplitude*, ask, **How does the amplitude of this graph compare to that of the original graph?** Students should see that the amplitude of the graph for Question 1 is less than that of the "original" platform height graph (from *Graphing the Ferris Wheel*).

More generally, ask, **What does the amplitude (for a Ferris wheel height graph) depend on?** Bring out that the amplitude of the graph is equal to the radius of the Ferris wheel.

Connect the changes in the Ferris wheel and the graph to the change in the equation. For instance, if a student changed the radius to 30 feet, the equation would be $h = 65 + 30 \sin (9t)$.

Ask*,* **Has anything happened to the period?** Students should see that the period is not affected by the change in radius.

Question 2

For Question 2, ask, **Has anything happened to the amplitude?** Students should see that the amplitude is not affected by the change in period.

As with Question 1, connect the change in the Ferris wheel and the graph to the change in the equation. For instance, if a student changed the period to 20 seconds, the coefficient of t would change from 9 to 18 (found by dividing 360° by 20), and the equation would be $h = 65 + 50 \sin (18t)$.

Question 3

For Question 3, students should all have the same graph and equation. They should see that the entire graph is merely moved down so that the x-axis is its midline. The equation is simply $h = 50 \sin (9t)$.

Ask**, Has anything happened to the period or amplitude?** Students should see that they have not changed.

Key Questions

How does the amplitude of this graph compare to that of the original graph?

What does the amplitude depend on? Has anything happened to the period?

Has anything happened to the amplitude?

Has anything happened to the period or amplitude?

Supplemental Activity

A Shifted Ferris Wheel (extension) guides students to investigate the effect of changing the initial location of the platform from the 3 o'clock position to something else.

The "Plain" Sine Graph

Intent

In this activity, students graph the sine function and observe characteristics of the graph.

Mathematics

In the last two activities, students worked with graphs of "Ferris wheel situations" involving the sine function. In this activity, students will examine the graph of the sine function itself.

Progression

Students work on this activity in groups, and discuss their results as a class. The discussion following this activity also points out the graph's amplitude, period, intercepts, and maxima and minima.

Approximate Time

25 minutes for activity

10 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Materials

Transparency of *The "Plain" Sine Graph* blackline master

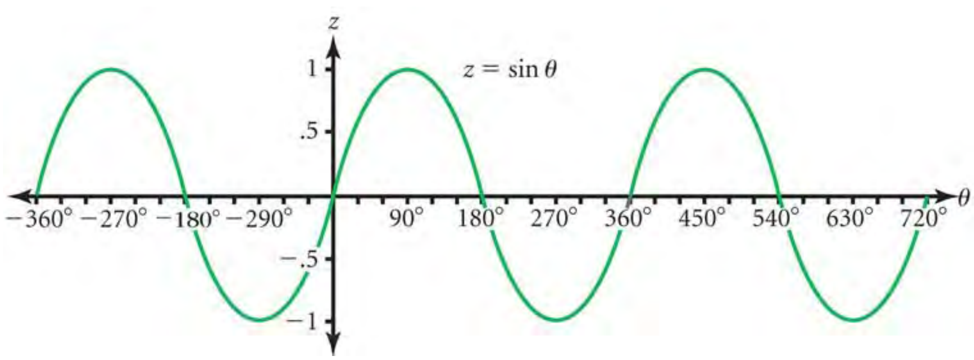
Doing the Activity

Students should be able to do this activity with no introduction.

Discussing and Debriefing the Activity

The discussion of this activity can be brief if students were comfortable answering the specific questions in their groups.

You should post a copy of the graph itself. (*The "Plain" Sine Graph* blackline master provides a graph similar to the diagram here, but with domain –180° to 540°.)



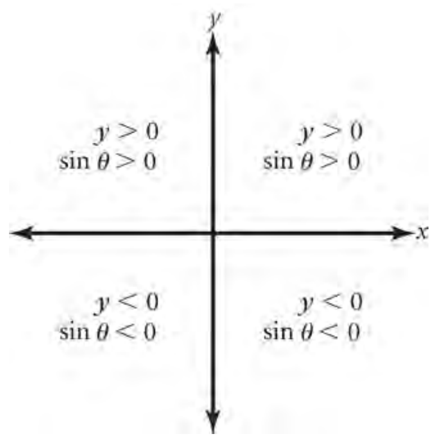
Some Further Details

Here are some specific connections and ideas to bring out, if they do not seem clear from students' group work:

- Have students explain in terms of the coordinate definition why the sine function has a maximum of 1 and a minimum of –1.
- Bring out the connection between the facts that the amplitude of the function is 1 and that the sine function has a maximum of 1 and a minimum of –1.
- Have students write an equation expressing that the period of the function is 360°. For instance, they might write $\sin(\theta + 360°) = \sin\theta$ and explain this in terms of the coordinate definition.
- Have students relate the θ-intercepts to the coordinate definition of the sine function. They should see that the sine function is zero when the "defining point" (that is, the point on the appropriate ray) has a *y*-coordinate of zero, which means that the point is on the *x*-axis.
- Bring out that the angles that are involved in the intercepts, maximum points, and minimum points are precisely the angles for which the defining point is on one of the axes, so that *y* is either zero or is equal to *r* in absolute value.

The Sign of the Sine

Ask, **For what angles is the sine function positive? When is it negative?** By looking at the sign of the *y*-coordinate, students should be able to determine that the sine function comes out positive if the angle is in the first or second quadrant and negative if the angle is in the third or fourth quadrant. (Bring out that an angle such as –53° is a fourth-quadrant angle.)



You may find it helpful to use a diagram like the one at right.

The Sine Function and the Ferris Wheel

Finally, bring out that this graph has the same basic shape as the "height functions" that students examined in *Ferris Wheel Graph Variations*. Ask, **What specifications for the Ferris wheel would give this graph?** They should see that this graph shows the height of the platform for a Ferris wheel with a radius of 1 unit and an angular speed of 1 degree per second and for which the center of the wheel is at ground level.

Key Questions

For what angles is the sine function positive? When is it negative?

What specifications for the Ferris wheel would give this graph?

Sand Castles

Intent

In this activity, students apply the extended sine function in a new problem situation.

Mathematics

This activity illustrates the use of a sinusoidal function to describe periodic motion in a context that is quite different from the Ferris wheel and that does not involve angles in any obvious way.

Progression

Sand Castles describes the tide action on a beach as fitting a given sine function and asks students to answer several questions related to the location of the waterline at various points in time. This activity is usually difficult for students and sparks substantial class discussion.

Approximate Time

40 to 50 minutes for activity (at home or in class)

25 to 30 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

You may want to give students a minute to check their work using a graphing calculator before you begin the discussion. They might use the "trace" feature or a table to check for maximum and minimum points and to confirm results for Questions 2 through 5.

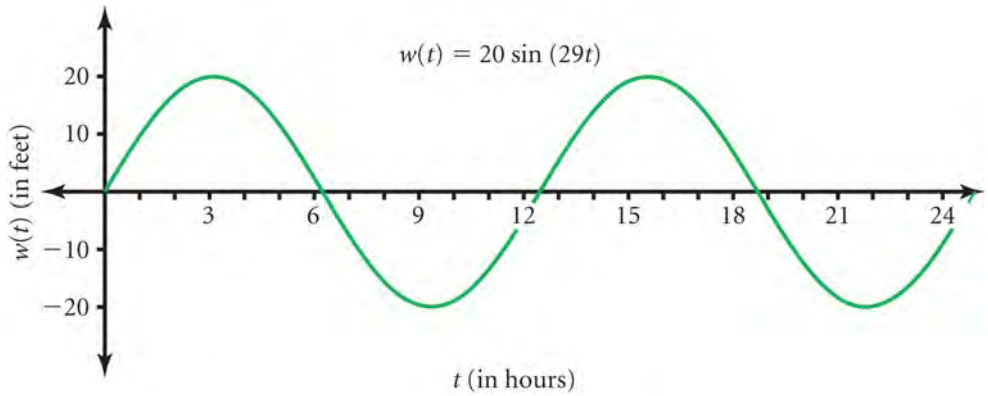
Have volunteers present their results on each question.

Question 1

In the discussion of Question 1, have the presenter explain how he or she made the graph. For instance, the student may have realized that the first maximum occurs when $29t$ is equal to 90 (which gives $t \approx 3.1$) and that the water is at its average value again when $29t$ is equal to 180 (which gives $t \approx 6.2$). Help the class to recognize the connection between the water level graph and the graph of the "plain" sine function.

Bring out that a 24-hour period will cover slightly less than two full periods for the water's motion. (You might ask at this point what the period of the function is, although that could also come out in connection with Question 3.)

The graph might look like the diagram here, although students might also label their scales to indicate where the maximum and minimum points are, rather than show whole numbers of hours. (They also might use a 24-hour interval other than from $t = 0$ to $t = 24$.)



Question 2

For Question 2, students should see that the water level will get as high as 20 feet above and as low as 20 feet below the average waterline. Have them explain how they can see this at a glance from the function, and identify the number 20 as the amplitude of the function. (*Note:* The maximum value for $w(t)$ occurs at approximately $t = 3.1$ and 15.5, and the minimum at approximately $t = 9.3$ and 21.7. In other words, high tide is at about 3:06 a.m. and 3:30 p.m., and low tide is at about 9:18 a.m. and 9:42 p.m.)

Question 3

On Question 3, have the presenter give the specific times when the waterline is at its average level and identify these times on the graph of the function. Students should see that the question refers to the duration of the "lower half" of the curve. Be sure they identify the length of this time interval (which is about 6 hours and 12 minutes) as half the period of the function.

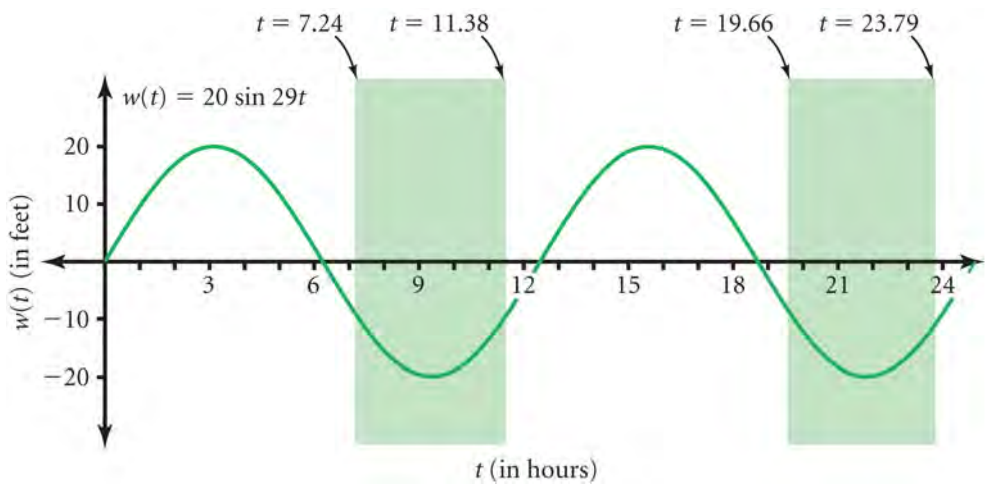
Question 4

For Question 4, the presenter will probably have looked for solutions to the equation $w(t) = -10$, which simplifies to $\sin(29t) = -0.5$. This is a good opportunity to review the idea of the inverse sine function and to discuss its limitations. (The more general inverse sine function will be discussed in *More Beach Adventures.*)

Students should see that their calculators give –30 as the value for $\sin^{-1}(-0.5)$. Discuss what this means, bringing out that the inverse sine function, as a function, can give only one of the angles whose sine is –0.5.

Ask, **How can you use the fact that sin (–30°) = –0.5?** Students might use a graph, the Ferris wheel, or a coordinate system diagram to find other angles whose sine is –0.5.

In this problem, students are looking for two times where the waterline is at –10 feet: one (as the tide goes out) on the "down side" of the graph and one (as the tide comes back in) on the subsequent "up side" of the graph, as shown in the following diagram. They should see that these two times could correspond to solutions of the equation 20 sin $(29t) = -10$ with $29t$ representing angles in the third and fourth quadrants. This leads to the conditions $29t = 210°$ and $29t = 330°$, because both sin 210° and sin 330° are equal to –0.5. (As the diagram indicates, Oceana could also use the next tide cycle, which gives the conditions $29t = 570°$ and $29t = 690°$.)



Shaded areas represent Oceana's options for the time interval in Question 4.

In other words, the water level reaches 10 feet below average about 7.24 hours after midnight (or at about 7:14 a.m.) and reaches that level again about 11.38 hours after midnight (or at about 11:23 a.m.). This gives Oceana about 4.14 hours to construct her sand castle. (The equations $29t = 570°$ and $29t = 690°$ give Oceana from about 7:39 p.m. to about 11:48 p.m., which is again a time interval of about 4.14 hours.)

Question 5

Question 5 is somewhat similar to Question 4. Again, students might use a graph or other diagram to get a sense of what's going on, perhaps seeing that Oceana needs to determine the water level one hour before low tide. They might use their knowledge about the period to see that the low tide first occurs about 9.31 hours after midnight, so Oceana should start about 8.31 hours after midnight. They then need to find 20 sin $(29 \cdot 8.31)$, which is approximately –17.5. In other words, if Oceana wants two hours in which to build her castle, the lowest she can go is 17.5 feet below the average waterline.

Sines without Angles

If it hasn't yet come up, you may want to point out that the central equation in this problem involves the sine function, but the problem as stated has nothing to do with angles. Tell students that the type of "rise and fall" motion that is involved in this problem and that is shown in the graph of the sine function occurs in many contexts that do not involve angles.

Key Question

How can you use the fact that sin (–30°) = –0.5?

POW 7: Paving Patterns

Intent

In this activity, students solve a complex problem requiring substantial exploration and write a clear justification of their solution.

Mathematics

This POW involves the Fibonacci sequence and will give students another opportunity to work with recursion.

Progression

After a brief introduction, give students about a week to work on the POW.

Approximate Time

5 minutes for introduction

1 to 3 hours for activity (at home)

25 to 35 minutes for presentations and discussion

Classroom Organization

Individuals, followed by several student presentations and whole-class discussion

Doing the Activity

POW 7: Paving Patterns asks students to find the number of ways that 1-foot by 2-foot tiles could be arranged in a 2-foot by 20-foot path, and suggests that they begin by looking at paths with shorter lengths. They are also asked to give any general formulas they found and any explanations for those formulas.

As students compile their data for paths of varying lengths, the tabulated results form a sequence similar to the Fibonacci sequence, which is most easily generalized with a recursion equation.

Allow students about a week to work on this POW. On the day before it is due, choose three students to make POW presentations on the following day, and give them overhead transparencies and pens to take home to use for preparing those presentations.

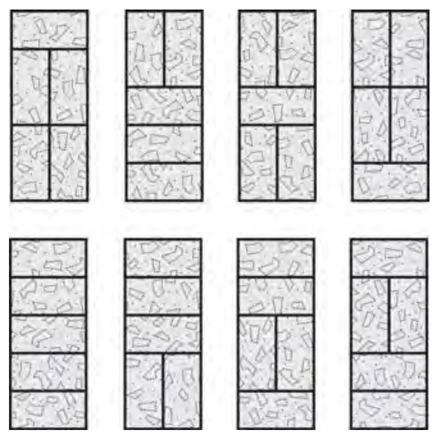
Discussing and Debriefing the Activity

Let the three students make their presentations, and have other students add their ideas.

If students make a table of results, showing the number of ways to pave an n x 2 path, it will look something like this:

Length of path	Number of paving patterns
1 foot	1
2 feet	2
3 feet	3
4 feet	5
5 feet	8
6 feet	13

For instance, here are the eight different ways in which a 5 x 2 rectangle can be paved:



There are several important aspects to this problem. The first stage is students' ability to organize their lists of paving arrangements so that they get the right data in the table. It may be productive for them to share ways in which they avoided omitting any patterns.

The second stage is recognizing the pattern in the table. The sequence 1, 2, 3, 5, 8, 13, . . . is a slight variation on the sequence known as the *Fibonacci sequence.* (The Fibonacci sequence begins with two 1's instead of one. That is, it goes 1, 1, 2, 3, 5, 8, and so on.)

Historical Note: About Leonardo Fibonacci

Leonardo Fibonacci (c. 1170–1240), also known as Leonardo of Pisa, was a major mathematician of the Middle Ages. He studied with an Arab master while his father served as consul in North Africa. In his first book, *Liber Abaci* (Book of the Abacus), published in 1202, he made the Hindu-Arabic numeral system—the base 10 place value system—generally available in Europe. Prior to that time, the system was known in Europe only to a few intellectuals who had seen translations of the writings of the ninth-century Arab mathematician and astronomer al-Khwarizmi.

Liber Abaci also contained a discussion of the number sequence that now bears Fibonacci's name, introducing the sequence in connection with this problem: A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

The Pattern in the Table

As students will probably see, each number in the right-hand column of the table is the sum of the two preceding numbers. For instance, the entry 13 is the sum of the two preceding terms, 8 and 5.

Formally, if we let a_n represent the number of ways to pave an n x 2 rectangle, then the pattern in the table can be represented by the formula

$$a_n = a_{n-1} + a_{n-2}$$

For example, the case $n = 6$ gives us the formula $a_6 = a_5 + a_4$, which is our earlier relationship 13 = 8 + 5. (Because a_n is defined only for positive values of *n,* the formula $a_n = a_{n-1} + a_{n-2}$ makes sense only if $n \geq 3$.)

Ask, **What is this type of formula called?** Students may recall this from *POW 6: The Tower of Hanoi,* but if necessary, remind them of the term *recursive formula.*

Ask*,* **What does this formula say about the paving patterns?** Bring out that it says that the number of n x 2 paths is the sum of the number of (n – 1) x 2 paths and the number of (n – 2) x 2 paths.

Students can continue the table using the recursive formula. For instance,

$$a_7 = a_6 + a_5 = 13 + 8 = 21$$

$$a_8 = a_7 + a_6 = 21 + 13 = 34$$

$$a_9 = a_8 + a_7 = 34 + 21 = 55$$

They can continue in this way to get the equation

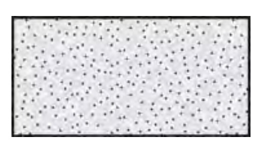
$$a_{20} = a_{19} + a_{18} = 6765 + 4181 = 10{,}946$$

Thus, there are nearly 11,000 ways for Al and Betty to pave a 20 x 2 path with their 1 x 2 paving stones.

Explaining the Recursive Formula

An important aspect of the problem is understanding, **Why does this recursive formula hold true?** Ask the class why the number of n x 2 paths should be the sum of the number of (n – 1) x 2 paths and the number of (n – 2) x 2 paths. If no one can explain this, here is a sequence of questions you can ask to lead students to understand the pattern.

First, ask*,* **What can the 'beginning' of the paving pattern look like?**, bringing out that there are two options. One option is to place a single stone sideways, like this:



The other option is to place two stones "vertically" adjacent to each other, as shown at right.

Ask, **In each case, how many more feet of path are needed to build a path that is *n* feet long?** In the first case, the initial part of the path is only 1 foot long, so an additional $n - 1$ feet are needed. In the second case, the initial part of the path is 2 feet long, so only an additional $n - 2$ feet are needed.

Ask, **How many ways are there to complete each possible 'beginning'?** Help students to see that the number of $n \times 2$ paths that begin with a horizontal paving stone is a_{n-1}, because the remainder of such a path is simply an $(n - 1) \times 2$ path. Similarly, the number of $n \times 2$ paths that begin with a pair of vertical paving stones at the top is a_{n-2}, because the remainder of the path is simply an $(n - 2) \times 2$ path.

For example, of the eight patterns shown earlier for a 5 x 2 path, there are five with a horizontal paving stone at the top, and three with a pair of vertical paving stones at the top.

Some students may develop the recursive formula from this general analysis, rather than from the numerical data. That's fine; there is no right or wrong order in which to think about this.

The Closed Form

There is a closed-form expression for the number of $n \times 2$ paths, but finding this expression requires knowing (or guessing) that it should be the sum of two exponential expressions. The number of $n \times 2$ paths is given by the expression

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Amazingly, this gives a positive integer for every positive integer value of n. To get the nth term of the Fibonacci sequence, simply replace $n + 1$ with n.

Key Questions

What is this type of formula called?

What does this formula say about the paving patterns?

Why does this recursive formula hold true?

What can the 'beginning' of the paving pattern look like?

In each case, how many more feet of path are needed to build a path that is *n* feet long?

How many ways are there to complete each possible 'beginning'?

More Beach Adventures

Intent

In this activity, students continue to work with the new extended definition of the sine function.

Mathematics

As students manipulate the equation 20 sin $(29t) = -14$, they will find that $29t = \sin^{-1}\left(\frac{-14}{20}\right)$, but solving that for t yields a solution that is not within a meaningful range relative to the problem situation. This introduces the concept of the *principal value* of the inverse sine function. Students will use their knowledge of the periodic nature of the sine function to find relevant values for t using the principal value for the inverse sine.

Progression

The first problem of this activity uses the situation from *Sand Castles*. While many students will have settled for approximate graphical solutions to that activity, they should now be encouraged to find more exact solutions to the question in the current activity. Question 2 gives the students further practice with using the principal value for the inverse sine and the periodicity of the sine function.

Approximate Time

30 to 35 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

You may want to briefly review the use of the inverse sine function for this activity.

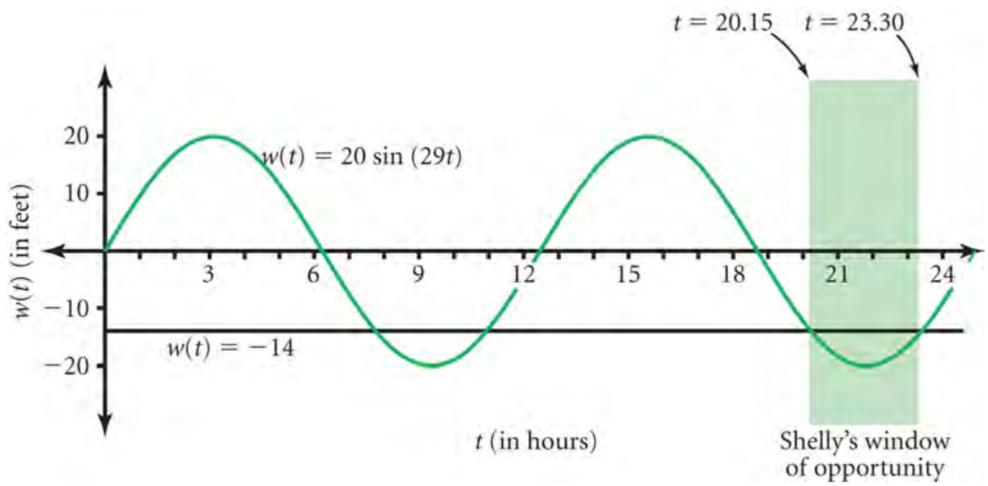
Discussing and Debriefing the Activity

Question 1 is somewhat similar to Question 4 of *Sand Castles.* You may want to give students a few minutes to compare ideas on Question 1, and then have a student explain his or her solution.

The problem involves solving the equation $w(t) = -14$, which means finding values of t that fit the condition 20 sin $(29t) = -14$. But students need to consider the periodicity of the function in order to find the two solutions for t that correspond to the evening hours. For instance, if they consider “evening” to mean a time between 6 p.m. and midnight, then they need t to be a value between 18 and 24.

By using the inverse sine function, students will come up with the "basic" solution to the equation. The equation $29t = \sin^{-1}\left(\frac{-14}{20}\right)$ yields $t \approx -1.53$, which corresponds to about an hour and a half before midnight on the previous day. There are several ways that students might find a solution for the evening of the day in question.

One approach is to use the condition $29t \approx -44.4$ to get other values for the expression $29t$. Using the graph of the function $w(t)$ (shown here) may help in understanding what's happening.



Given that the basic solution is $29t \approx -44.4$, students can use the graph to see that Oceana and her friend can pass by where the rocks jut into the water between $29t = 540° + 44.4°$ and $29t = 720° - 44.4°$. This gives the time interval from $t \approx 20.15$ and $t = 23.30$, which means from about 8:09 p.m. to about 11:18 p.m.

Question 2

Questions 2a through 2d involve a more abstract version of the issues in Question 1, although these questions are simpler because they involve merely $\sin\theta$ instead of $\sin(29t)$. You might have several volunteers give an answer for each of the questions. Some students may point out that each question has not just three but four angles that work. If so, you could ask whether every given value of the sine will produce four angles on this interval, and have students justify their responses using their "plain" sine graph.

Principal Value of the Inverse Sine Function

For Question 2d, many students will have used the "$\sin^{-1}$" key on their calculators. Point out that the calculator has to somehow choose one of many answers as its output for $\sin^{-1}(-0.71)$, and so it is important to standardize the process for doing this.

You can let students explore for a few minutes to see how the calculator's "$\sin^{-1}$" key works. They should discover these facts:

- If x is between 0 and 1, $\sin^{-1} x$ is between 0° and 90°.
- If x is between 0 and –1, $\sin^{-1} x$ is between 0° and –90°.

Of course, if the calculator is given a number with absolute value greater than 1, it gives an error message when you use $\sin^{-1}$; see the next subsection, "The Domain of Inverse Sine."

Note: With most scientific calculators, you enter the number (–0.71, for instance) and then push the "$\sin^{-1}$" key. With most graphing calculators, you do the reverse.

Tell students that although each of the answers to Question 2d is an inverse sine for –0.71, the value that the calculator gives (approximately –45.2°) is called the principal value for the inverse sine of –0.71. We write this using notation such as $\sin^{-1}(-0.71) \approx -45.2°$. Bring out that the general definition of the inverse sine involves a convention for choosing these principal values.

Note: Some textbooks introduce the notation "Sin^{-1}" (with a capital "S") to represent the principal value, and use "$\sin^{-1}$" to be "multivalued." This is not standard mathematical notation, however.

You may want to point out the analogy between this distinction and a similar distinction with square roots. For example, the number 9 has two square roots, 3 and –3, but the symbol $\sqrt{9}$ represents only the number 3, which is sometimes called the *principal square root* of 9.

The Domain of Inverse Sine

Ask, **What's $\sin^{-1}(2)$?** Have students find it on their calculators. They should get some kind of error message.

Ask students to explain this. As a hint, ask them to express the problem in terms of the sine function (rather than in terms of inverse sine). They should be able to describe this as looking for a solution to the equation $\sin x = 2$. You may want to let them discuss this equation in their groups until someone can explain why this problem has no solution.

Use this example to review terminology by asking, **What is the domain of the inverse sine function? What is its range?** Bring out that the domain is the interval from –1 to 1, because the equation $\sin x = c$ has a solution only if $-1 \leq c \leq 1$ or, in other words, because the range of the sine function is the interval from –1 to 1.

The range of $\sin^{-1}$ is the set of angles from –90° to 90°, inclusive, although this is the result of the conventions for principal values.

Key Questions

What's $\sin^{-1}(2)$?

What is the domain of the inverse sine function? What is its range?

Supplemental Activity

Prisoner Revisited (reinforcement) is similar to this activity and can be assigned if students need further practice working with the ideas from this activity.

Falling, Falling, Falling

Intent

In this section, students consider the falling motion of the diver in the central unit problem.

Mathematics

Analysis of the falling motion of the diver is complicated by the fact that he accelerates as he falls. The activities in this section use an area model for distance to help students develop general formulas for freely falling bodies. But the analysis is further complicated by the fact that in order to apply the formula for falling time to the unit problem, it must be expressed in terms of the diver's height when he begins his fall, which depends in turn on the time he spends on the Ferris wheel. So students must merge their formula for the time it takes a body to fall with the one they developed earlier for the height of the platform.

Progression

In *Distance with Changing Speed* and *Acceleration Variations and a Sine Summary,* students learn that they can find the average speed for an interval of constant acceleration by averaging the speeds at the beginning and ending of the interval.

They apply this principle to the constant acceleration of falling bodies in *Free Fall*, developing formulas for the height of a falling body at any point in time and for the time it will take a body to fall. In *A Practice Jump*, students express this latter formula in terms of the time the diver spends on the moving Ferris wheel platform.

Distance with Changing Speed

Acceleration Variations and a Sine Summary

Free Fall

Not So Spectacular

A Practice Jump

Distance with Changing Speed

Intent

In this activity, students develop a method for finding the total distance traveled in situations involving constant acceleration.

Mathematics

In the introduction to this activity, students see that they can express total distance traveled in terms of area under the graph of the speed function. This leads to a discovery that under constant acceleration the average speed for a time interval is the average of the initial and final speeds for that interval.

Progression

Introduce this activity with a brief discussion about using an area model to represent distance in terms of speed. Students apply this concept as they work on the activity individually, then share their results in class discussion.

Approximate Time

5 to 10 minutes for introduction

20 to 25 minutes for activity (at home or in class)

10 to 20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Transparency of *Distance with Changing Speed* blackline master

Doing the Activity

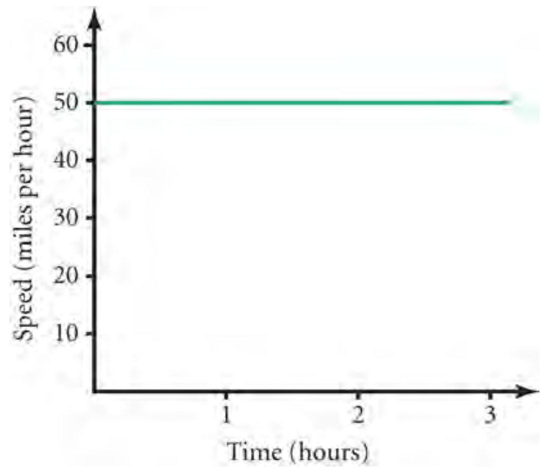
Remind students that one type of motion they will be considering will be that of the falling diver, whose speed changes as he falls. Tell them that because the speed is changing, the relationship among the variables of distance, speed, and time is more complex than if the speed were constant. Inform them that in order to understand this complex situation, they will develop a simple model, using a graph, for representing the distance a moving object (or person) travels in terms of its speed.

Begin by posing this straightforward question:

Suppose a person drives for 3 hours at a constant speed of 50 miles per hour. How far does the person go?

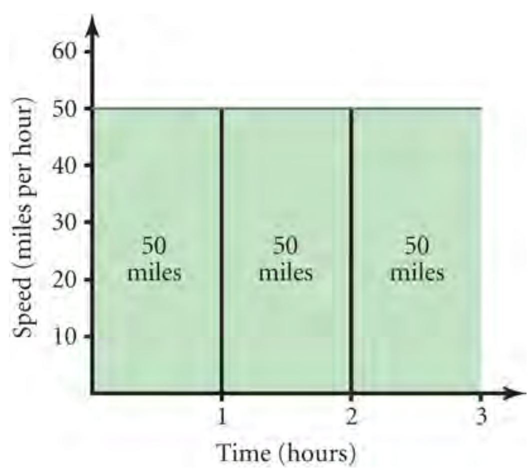
All students need to do here is multiply the speed (50 miles per hour) by the time (3 hours) to get the distance (150 miles).

Have students make a graph showing speed as a function of time for this situation. Their graphs should look like this:



Then ask, **How can you represent the distance geometrically?** As a hint, remind students that for constant speed, distance is simply the product of the speed and the time. As a further hint, remind them that the area of a rectangle is often a good model for multiplication.

These hints should lead the class to form a rectangle as in the next diagram and to see that the area of this rectangle gives the numerical value of the distance traveled. You may want to suggest that students subdivide the rectangle, as shown here, to indicate the distance covered each hour.



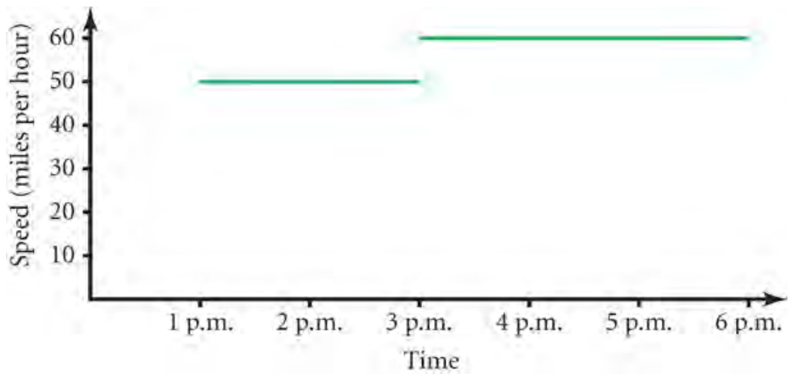
Tell the class that this activity continues the use of this area model for finding total distance.

Comment: It may seem strange to use an area model to represent a linear measurement, as the discussion here does. But because *distance* is the product of *rate* and *time*, it is appropriate to use a two-dimensional model for distance. (You may want to acknowledge this anomaly to students.)

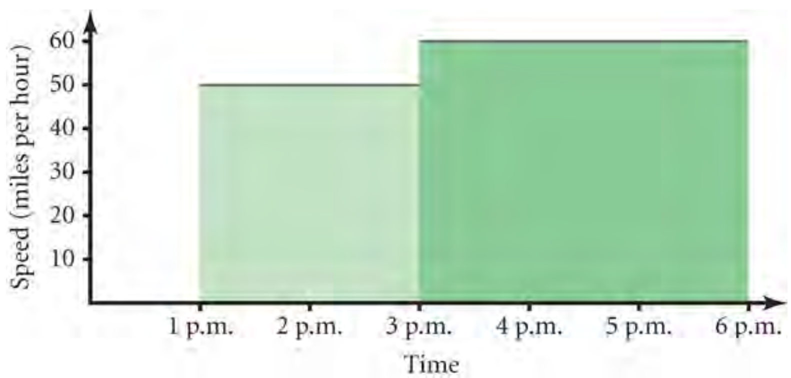
With the preceding introduction to the idea of an area model for finding distance, have groups begin work on the activity.

Discussing and Debriefing the Activity

Question 1 extends the use of the area model beyond the case of constant speed. For Question 1a, students should get a diagram something like this, showing speed as a function of time:

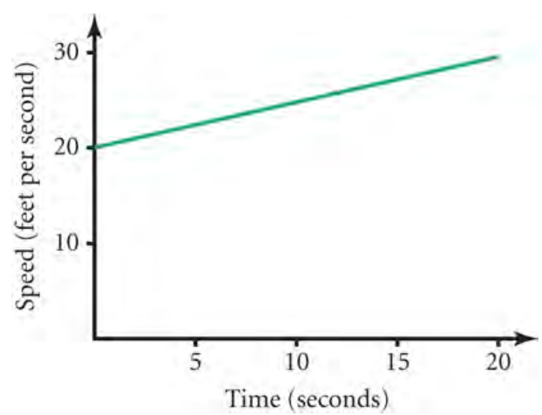


For Question 1b, students might draw in the rectangles, as shown in the next diagram, to illustrate that the distance traveled from 1 p.m. to 3 p.m. is the area of the first rectangle and the distance traveled from 3 p.m. to 6 p.m. is the area of the second rectangle. Thus, the total area under the graph is equal to the total distance traveled.



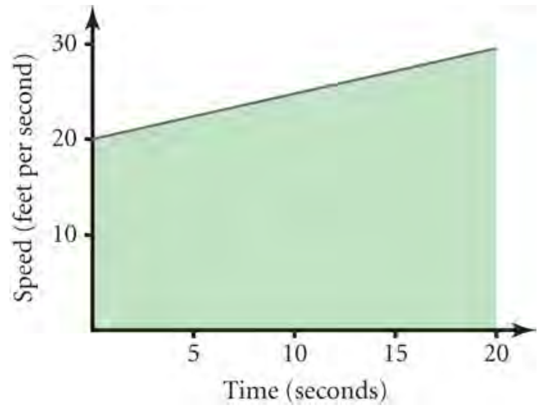
Question 2

Question 2 applies the area model to a situation in which speed is changing at a constant rate. For Question 2a, students should get a graph like the one at right. (You may want to use a transparency of this graph, provided in the *Distance with Changing Speed* blackline master, to aid in the discussion.)

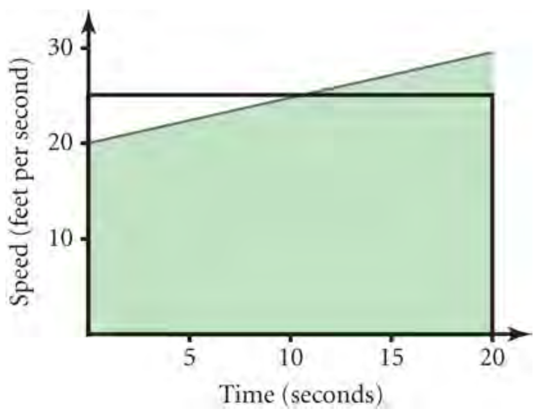


For Question 2b, students will likely take a purely intuitive approach, saying that because the speed increases at a constant rate from 20 feet per second to 30 feet per second, the average speed is simply 25 feet per second.

The main goal of this activity is to confirm this intuitive approach by using the area model. Based on the earlier examples, students should accept that the total distance traveled ought to be equal to the area under the graph, as shown at right.



Ask, **How do you find the area of this shaded figure?** Students may recall that the figure is called a trapezoid, and they might even recall the formula for the area of a trapezoid. But don't get sidetracked by area formulas here. One intuitive approach is to ask, **What rectangle would have the same area, using the same base?** Students should see that the rectangle shown in this diagram has the same area as the trapezoid:



Students should also be able to see that the height of this rectangle is 25—which is the average of the heights of the two ends of the trapezoid—and that the area of the rectangle, and hence of the trapezoid, is 500. You may want to have them express the area in the form $\left(\frac{20+30}{2}\right)\cdot 20$ and review that this is an illustration of the general formula for the area of a trapezoid.

Ask, **What does the area mean in terms of the runner?** Bring out that it means that the runner goes a total of 500 feet.

Finally, verify that this result is consistent with the intuitive answer to Question 2b. That is, if the runner averaged 25 feet per second, as in Question 2b, and went at this speed for 20 seconds, he would travel 500 feet, as found in Question 2c using area.

Ask, **What do we call the rate at which speed is changing?** (If no one comes up with the term, you might ask what we call it when a car speeds up.) If no one knows, introduce the term *acceleration.* Use the phrase *constant acceleration* to describe the situation in Question 2, in which the speed is changing at a constant rate.

Averaging the Endpoints

Bring out that the trapezoid approach will work for any situation of constant acceleration and that it gives a simple way to find the total distance traveled, even though the speed is not constant.

Ask students to summarize the principle for finding average speed in situations of constant acceleration. They should be able to articulate something like this, which you should post:

If an object is traveling with constant acceleration, then its average speed over any time interval is the average of its beginning speed and its final speed during that time interval.

We will refer to this principle as the "averaging the endpoints" method for finding average speed. Emphasize to students that it applies only to situations with constant acceleration.

You might also ask how this principle can be used to find the total distance traveled. Students should see that, as always, they can multiply the average speed by the length of the time interval to get the total distance.

Key Questions

How can you represent the distance geometrically?

How do you find the area of this shaded figure?

What rectangle would have the same area, using the same base?

What does the area mean in terms of the runner?

What do we call the rate at which speed is changing?

Supplemental Activity

Lightning at the Beach on Jupiter (reinforcement) is a review of the relationship between distance, rate, and time.

Acceleration Variations and a Sine Summary

Intent

In this activity, students continue to work with the area model for distance, and they also summarize ideas about extending the sine function.

Mathematics

The first part of this activity looks at how the principle of averaging the beginning and ending speeds to find the average speed of a time interval is affected if we abandon the requirement that the acceleration be constant. The second part asks students to summarize their work in extending the sine function and to connect this new definition with the unit problem. The main purpose of Part II is to get them thinking about the issues as preparation for a class discussion.

Progression

Students work on the activity individually and share their results in class discussion.

Approximate Time

30 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

You may want to give some students transparencies to prepare graphs for the discussion of Part I of this activity.

Important: If the class has not completed the discussion of *Distance with Changing Speed,* you should postpone Part I of this activity.

Discussing and Debriefing the Activity

Part I: Acceleration Variations

Let a different volunteer share an example for each of the variations called for in Part I of the activity. You might give presenters transparencies, and have them duplicate the axes and graph shown in the activity and then sketch their graphs along with it. Like the graph in the activity, the graphs students create should show the speed going from 20 feet per second to 30 feet per second. Have each presenter explain how he or she knows that the graph fits the given condition.

More about Acceleration

Point out that the units for acceleration are somewhat complicated. Speed itself is measured here in feet per second, and acceleration measures how speed changes over time. In the example given in the problem, the runner's speed increases by 10 feet per second over a 20-second interval, so it increases by 0.5 feet per second for each second that elapses. Tell students that we express this by saying that the acceleration is 0.5 feet per second per second.

Emphasize the distinction between *speed,* which in this case tells the rate at which the runner's *position* is changing, and *acceleration,* which tells how fast the runner's *speed* is changing. Here, the *speed* is not constant (the runner's *position* is not changing at a constant rate), but the *acceleration* is constant (the runner's *speed* is changing at a constant rate).

Part II: A Sine Summary

Let students share their ideas with the class about the extension of the sine function and how it fits with the problem. Students should feel fairly comfortable at this stage of the unit with the idea that the expression $65 + 50 \sin (9t)$ gives the height of the platform after t seconds (starting from the 3 o'clock position).

Free Fall

Intent

In this activity, students examine the behavior of falling objects.

Mathematics

In the discussion that introduces this activity, students recognize that falling objects accelerate, and they learn that their acceleration is constant. They then use the "averaging the endpoints" method to find formulas for the height of an object falling from rest in terms of time, and for the time it takes for an object to fall h feet.

Progression

This activity is introduced with a brief discussion of falling objects. Students then work in groups, and discuss their results as a class.

Approximate Time

10 minutes for introduction

30 minutes for activity

25 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Falling Objects

Quickly review the principle of "averaging the endpoints." Then ask, **What does 'averaging the endpoints' have to do with the main unit problem?** If students don't see a connection, ask, **What happens to an object as it falls freely?** Let students share their own ideas and experiences about what happens as objects fall.

If no one has a convincing argument that falling objects gain speed, then you may want to bring up a situation like this for them to consider:

Which would hurt more, a fall from your roof or a fall from your bed?

(Although this example relates more directly to the force of impact than to speed, most students will attribute the added force of falling from a roof to moving faster at impact.)

Tell students that although our experience shows us that objects go faster and faster as they fall, physicists actually know precisely how falling objects behave. Specifically, from experimental data and from theoretical considerations, they know that falling objects have *constant acceleration.* You may want to post this statement, perhaps adjacent to the description of the "averaging the endpoints" method:

Falling objects have constant acceleration (under ideal circumstances). That is, the speed of a falling object changes at a constant rate.

Discuss that the phrase "under ideal circumstances" means that there is no wind, air resistance, or other complicating factor to interfere with the object's fall. That is, the principle describes the behavior of *free-falling* objects. (This assumption is mentioned in the activity.) You might also discuss the fact that this assumption is reasonable for some types of objects (such as rocks) and not for others (such as feathers).

Free Fall

Have students read the introduction to the activity *Free Fall* and go over the details in the section "Starting from Rest." Then have them work on the questions. If necessary, suggest that for Question 1, they use the "averaging the endpoints" method.

Note: Question 5 explicitly states that the diver falls "from rest." If the issue of the effect of the Ferris wheel's motion has come up before (see the subsection "For Teachers: The Diver's Initial Motion" in the discussion notes for *The Circus Act*), you may want to remind students that they are assuming for now, in their work on the central unit problem, that the diver falls from rest. See the subsection "But There's More to the Problem!" in the discussion notes for *Moving Cart, Turning Ferris Wheel*.

Discussing and Debriefing the Activity

Let one or two students present their analysis for Question 1. Question 1a should be straightforward. That is, if students understood the introduction to the activity, they should see that the speed at $t = 5$ is simply $5 \cdot 32 = 160$ ft/s.

To find out how far an object falls in 5 seconds, students should reason that the object's average speed for that interval is equal to the average of its instantaneous speeds at the endpoints of the interval. These endpoints are $t = 0$ and $t = 5$. The information in the activity tells them that the instantaneous speed at $t = 0$ is 0 ft/s and the instantaneous speed at $t = 5$ is 160 ft/s.

The average of 0 and 160 is 80, so the object has an average speed over the 5-second interval of 80 ft/s. Therefore, the object falls $5 \cdot 80 = 400$ feet during this interval.

Question 2

The key element in this activity is for students to generalize the reasoning from Question 1 to develop the general formula asked for in Question 2.

The approach we expect is for students to see that the instantaneous speed at the end of t seconds is $32t$ and the instantaneous speed at the start is 0. Thus, the average speed over the first t seconds is $16t$ ft/s. Students can then multiply this average speed by the length of the time interval, which is t seconds, to get a total distance traveled of $16t^2$ feet.

Post this conclusion, because it will play a critical role throughout the rest of the unit:

> **If an object falls freely from rest, it will fall $16t^2$ feet in its first t seconds.**

Note: Physicists often use the term "displacement" and the letter s for the distance an object has traveled. Some students may have seen the formula $s = 16t^2$ in a physics class to describe the displacement of an object falling from rest.

Questions 3 and 4

For Question 3, students simply need to subtract $16t^2$ from the initial height h to get the expression $h - 16t^2$ for the object's height after t seconds. You can add this additional conclusion to the statement just posted:

> **If the object's initial height is h feet, then its height after t seconds is $h - 16t^2$ feet.**

Question 4 may seem straightforward once the formula $h - 16t^2$ has been found, but be sure to go over the transition carefully. For some students, it may be a substantial step from the idea of "reaching the ground" to the step of setting $h - 16t^2$ equal to 0. Help them to understand that answering Question 4 is equivalent to solving the equation $h - 16t^2 = 0$ for t in terms of h. Have a volunteer show the details of solving the equation to get $t = \sqrt{\frac{h}{16}}$ seconds.

Once students have developed this additional generalization, you should post it along with the previous formula. This new statement might say something like this:

> **If an object falls freely from rest, it will take $\sqrt{\frac{h}{16}}$ for the object to fall h feet.**

Question 5

Question 5 provides an important variation, in which students need to see that the diver is actually falling 82 feet. They might set this up with the equation $90 - 16t^2 = 8$, or they might simply set $16t^2$ equal to 82. In either case, they should get the expression $\sqrt{\frac{82}{16}}$, which means that it takes approximately 2.26 seconds for the diver to fall to the water level.)

The Number 32 Is an Approximation

At some point, bring out that the number 32, which appears in *Free Fall*, is a numerical approximation based on experiments. (The activity does say "approximately" in giving the value of 32 feet per second for each second of the object's fall, but students may overlook this.)

Also point out that this number is specific to the use of feet as the unit of length. For example, if we instead measure length in meters, then we use approximately 9.8 instead of 32 (because 32 feet is about 9.8 meters).

Key Questions

What does ‘averaging the endpoints’ have to do with the main unit problem?

What happens to an object as it falls freely?

Supplemental Activity

In *The Derivative of Position* (extension), students use derivatives to confirm one of the formulas they developed in this activity.

Not So Spectacular

Intent

In this activity, students again work with the fact that there are many angles with the same sine.

Mathematics

In this activity, students develop a general expression for the times at which the diver will be at a given height.

Progression

This activity resembles *A Clear View,* but deals with the periodicity of the Ferris wheel's motion. This activity also puts some of the ideas from *More Beach Adventures* into the context of the Ferris wheel.

Approximate Time

30 minutes for activity (at home or in class)

25 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

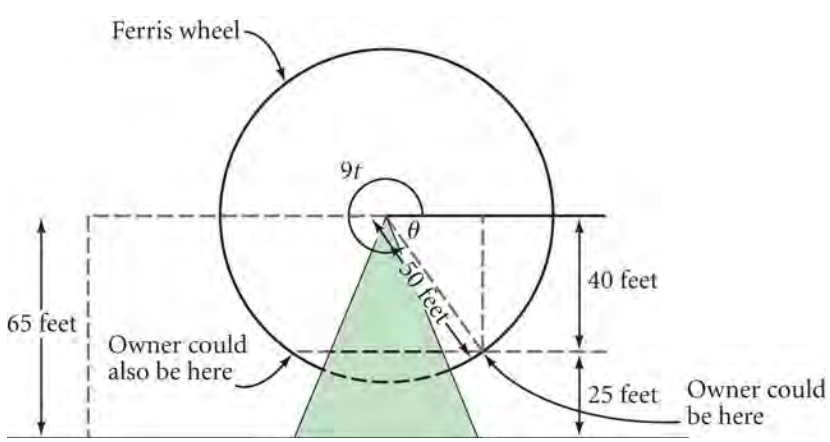
Discussing and Debriefing the Activity

Give students a few minutes to compare ideas, and perhaps select a student at random to present a solution for the activity. Up to a point, this problem is essentially the same as *A Clear View*. In both problems, students must find the points in the cycle of the Ferris wheel where a person is a given height off the ground. The difference in this problem is the focus on obtaining the general solution. Use this discussion to review the ideas about the inverse sine function and principal values discussed with *More Beach Adventures.*

Initial Solutions

The diagram here shows the two positions at which the owner would be 25 feet off the ground and gives details about one of these locations. This shows that the angle θ must satisfy the condition

$\sin\theta = \frac{40}{50}$ so $\theta = \sin^{-1} 0.8$,



which is approximately 53.1°. This means that the large angle shown as $9t$ is approximately equal to $360° – 53.1° = 306.9°$. In other words, the Ferris wheel could have been turning for about $306.9 \div 9 = 34.1$ seconds. Another approach is to find $\sin^{-1}(–0.8) = –53.1°$, then add 360°, leading to the same result.

For the other time at a height of 25 feet, we have

$$9t \approx 180° + 53.1° = 233.1°,$$

which means $t \approx 233.1 \div 9$ or 25.9 seconds.

To this point, the problem is basically a repetition of the approach in *A Clear View*.

Generalizing the Solutions

The new aspect of this problem is that each of these two locations actually leads to an infinite list of possible answers to the question of how long the Ferris wheel has been turning, because the Ferris wheel may have gone around several times before reaching the given point. (If students did not realize that the activity called for this additional stage, let them work on it in groups, trying to find the general solutions.)

Two main approaches are likely to come about. Here are descriptions of these two approaches for the position at the lower right of the diagram:

- Find different possibilities for the total angle $9t$ by adding multiples of 360° to the basic angle of 306.9°, and then divide each total angle by 9 (degrees per second).
- Find the time of 34.1 seconds for the basic angle of 306.9° and then add multiples of 40 seconds to that time.

In either case, one gets 34.1 seconds, 74.1 seconds, 114.1 seconds, and so on, as the sequence of possible times. Students should be able, perhaps with some prodding, to express this general solution as something like $34.1 + 40n$, where n can be any positive integer.

For the position at the lower left of the diagram, the same steps apply, but the general solution is $25.9 + 40n$ seconds.

Bring out that all of these values satisfy the equation $\sin(9t) = –0.8$, but none of them correspond directly to the principal value of the inverse sine function, which gives a negative value for t. Ask whether the two expressions together provide *all* possible solutions to $\sin(9t) = –0.8$. Help students see that this is true if n is allowed to be any *integer*, but that the solutions generated by negative values of n (e.g., –5.9 seconds, –45.9 seconds, or –14.1 seconds, –54.1 seconds, and so on) do not fit the physical realities of this problem.

A Practice Jump

Intent

In this activity, students combine principles about falling objects with the general sine function in connection with the unit problem.

Mathematics

This activity asks students to combine their new formula for falling time with their earlier work on the height of the platform at a given position, and to generalize this in terms of the amount of time the Ferris wheel has been turning.

Progression

Students combine the principles from several previous activities as they work on this activity. They work individually, and then share their results in class discussion.

Approximate Time

25 to 30 minutes for activity (at home or in class)

15 to 25 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students find the time it will take the platform to reach the 11 o'clock position, then find the diver's height at that position, and finally find the time it will take him to fall from that height. Students then generalize this to a formula for the diver's falling time for a release time of *W* seconds after the Ferris wheel starts turning.

Discussing and Debriefing the Activity

Presentations of this activity will be a good indicator of how well students understood their work in previous activities. Let different volunteers present different parts of the problem. As usual, ask for alternate approaches after the presentations.

Questions 1 and 2

One way to find the "let go" time (Question 1) is for students to see that the 11 o'clock position is one-third $\left(\frac{4}{12}\right)$ of the way around, so it should take one-third of the time of a full rotation to get there. Another approach is to find the angle of rotation from the 3 o'clock position to the 11 o'clock position (120°) and divide this by the angular speed of 9 degrees per second.

In either case, students should see that the assistant should let go after about 13.33 seconds (or that value plus any multiple of 40 seconds).

To solve Question 2, students can substitute the time just found, about 13.33 seconds, into the general height formula, $h = 65 + 50 \sin (9t)$. This will give them the height of 108.3 feet. They can also use the angle directly, getting the height from the expression $65 + 50 \sin 120°$.

Question 3

For Question 3, students need to subtract 8 feet from the height of 108.3 feet (found in Question 2) to get the distance the diver will fall before he hits the water in the cart. That is, the diver falls 100.3 feet.

To complete Question 3, students can substitute the value $h = 100.3$ into the expression $\sqrt{\frac{h}{16}}$. They should find that the diver's falling time is about 2.50 seconds.

Question 4

For Question 4, students need to combine the general formula for the diver's height after t seconds on the Ferris wheel with the formula from *Free Fall* for the time required to fall h feet. Essentially, this is merely substitution, but it is difficult for some students to make this transition. Go slowly, and be sure that presenters explain themselves clearly.

You may want to refer repeatedly to the specifics of Question 3 as this generalization is developed. Here are the key steps of the generalization:

- After W seconds, the diver is at a height of $65 + 50 \sin (9W)$ feet.
- The diver needs to fall 8 feet less than this, for a total falling distance of $57 + 50 \sin (9W)$ feet.
- The time it takes an object at rest to fall $57 + 50 \sin (9W)$ feet is $\sqrt{\frac{57 + 50\sin(9W)}{16}}$ seconds.

Falling Time is a Function of Turning Time

Bring out that students are expressing the time the diver spends in the air in terms of the amount of time the Ferris wheel turns before the assistant lets go. You might do this through a series of questions like these:

- First ask, **What does the diver's falling time depend on?** Students will probably say that it depends on his height when the assistant lets go.
- Then ask, **What does this height depend on?** Students will probably say it depends on what the diver's position is when the assistant lets go.
- Finally, ask, **What does this position depend on?** They should see that it depends on how long the Ferris wheel is turning before the assistant lets go.

Students may shortcut one or more of these steps. For example, they may say that the falling time depends on where the diver is, or even that it depends on how long the Ferris wheel is turning. The important thing is that they trace the falling time back to the amount of time the Ferris wheel turns before the assistant lets go.

Then ask, **What are you trying to find in the main unit problem?** They should see that they are being asked to find out how long the assistant should let the Ferris wheel turn before letting go.

In other words, what they are looking for is what was called *W* in Question 4 of the activity and in the formula they found for that question. That is, they have just found a formula for the diver's falling time in terms of the variable that they are trying to find in the unit problem.

This can be summarized like this:

> **If the Ferris wheel passes the 3 o'clock position at $t = 0$ and turns for W seconds more before the assistant lets go, then the diver will fall for $\sqrt{\frac{57+50\sin(9W)}{16}}$ seconds before reaching the level of the water in the cart.**

Post this conclusion prominently. It synthesizes two major ideas—the time required for a free-falling object to fall a given distance and the diver's height at the time of release—into a single important formula. You may want to add it on to the poster for the height of the platform after *W* seconds (see the discussion notes for *Testing the Definition*).

Because this expression for falling time is so complex, you may find it helpful to use a single letter to represent it. We will use *F* ("falling time"), so *F* is given by the formula

$$F = \sqrt{\frac{57+50\sin(9W)}{16}}$$

If you use this abbreviation, be sure students keep in mind that *F* is a function of *W*. You may want to have several students express in their own words what *F* represents.

Key Questions

What does the diver's falling time depend on?

What does this height depend on?

What does this position depend on?

What are you trying to find in the main unit problem?

Moving Left and Right

Intent

In this section, students focus on the horizontal dimension of the central unit problem.

Mathematics

In the previous sections of this unit, students explored the vertical dimension of the unit problem. They have looked at the vertical position of the platform as the Ferris wheel turns and they have examined the diver's falling motion after he leaves the platform. Now they will focus on horizontal motion.

As the Ferris wheel rotates, the horizontal position of the platform changes, too. Determining the x-coordinate of the platform will involve the cosine function, so, as was done previously with the sine function, the definition of the cosine function will need to be expanded to allow for angles beyond the first quadrant.

While the falling motion of the diver has only a vertical component in this simplified version of the unit problem, the length of time that the cart moves horizontally is in part determined by the time it takes the diver to fall. So an analysis of the cart's motion will include elements of both the horizontal and vertical movement of the platform, as well as of the diver's fall.

Progression

The cart begins to move at the same instant that the Ferris wheel begins to turn, so its motion spans both the time that the diver is on the moving platform and the time that the diver is falling. Students fit these facts together into an expression for the cart's travel time in *Cart Travel Time*. They then return to consideration of the horizontal position of the diver on the platform, first looking at specific times in *Where Does He Land?*, then developing a general formula in *First Quadrant Platform*, and finally using that formula as a basis for extending the cosine function to arbitrary angles in *Generalizing the Platform*. Students return to building a formula for the cart's position in *Carts and Periodic Problems*. In *Planning for Formulas* they consolidate all of the formulas they have developed in preparation for solving a simplified version of the unit problem (in which the initial motion of the Ferris wheel is not yet considered).

Cart Travel Time

Where Does He Land?

First Quadrant Platform

Carts and Periodic Problems

Generalizing the Platform

Planning for Formulas

Cart Travel Time

Intent

In this activity, students begin to explore the horizontal dimension of the unit problem.

Mathematics

The solution to the unit problem requires that the cart and the diver end up in the same place at the same time. So far, students have considered the movement of the diver. Now they will turn their attention to the movement of the cart. In this activity, they develop an expression for the time that the cart will travel, including both the time that the diver is on the platform and the time that the diver is falling from the platform to the water level.

Progression

Students work in groups to write an expression in terms of *W* for the time that the cart will travel, from the moment it starts until the moment the diver reaches the level of the water. In the subsequent discussion, you'll introduce the horizontal axis of the Ferris wheel coordinate system.

Approximate Time

15 to 20 minutes for activity

10 to 15 minutes for discussion and introduction of horizontal axis

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

In this activity, students use their conclusion about how long the diver is falling to reach a conclusion about the amount of time the *cart* is traveling. The key is to look at the cart's travel time in two parts:

- The time between when the platform passes the 3 o'clock position and when the diver is released from the platform
- The time while the diver is falling from the platform

In a later activity, *Carts and Periodic Problems,* students will find the cart's *position* at the moment when the diver reaches the water level.

Discussing and Debriefing the Activity

Only a brief presentation should be needed here, because students already have expressions for each of the two parts of the cart's travel time.

- The cart travels for *W* seconds from when the platform passes the 3 o'clock position until the diver is released.

- The cart travels for F seconds while the diver is falling (where F is given in terms of W by the expression $F = \sqrt{\frac{57+50\sin(9W)}{16}}$).

Post a summary of the conclusion from this activity:

If the cart begins moving when the Ferris wheel passes the 3 o'clock position, and the diver is released *W* seconds later, then the cart will travel for *W* + *F* seconds before the diver reaches the level of the water in the cart, where *F* is given by the formula

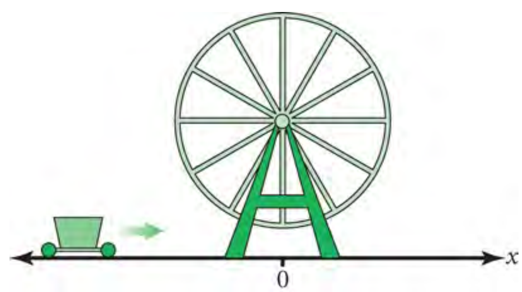
$$F = \sqrt{\frac{57+50\sin(9W)}{16}}$$

The Horizontal Dimension

Bring out that most of students' work on the unit problem has focused on vertical motion—the height of the platform as it turns and the falling motion of the diver.

Ask, **What will determine the success or failure of the circus act?** The cart needs to be in the right place along its horizontal path when the diver reaches the water level. So point out to students that they must explore the horizontal dimension as well.

Introduce the horizontal axis as shown in the diagram here. The center of the base of the Ferris wheel represents zero, positions to the right of the Ferris wheel are considered to have positive x-coordinates, and distances are measured in feet. Tell students that we will identify an object's horizontal position within this system using its x-coordinate.



To clarify this system, ask, **What is the cart's *x*-coordinate at the start of the circus act?** They should see that the cart's initial x-coordinate is –240, because the cart is 240 feet to the left of the center of the base of the Ferris wheel.

Similarly, ask, **What about the platform?** Students should see that the platform's initial x-coordinate is 50, because the platform starts at the 3 o'clock position and the radius of the Ferris wheel is 50 feet.

Key Questions

What will determine the success or failure of the circus act?

What is the cart's *x*-coordinate at the start of the circus act?

What about the platform?

Where Does He Land?

Intent

In this activity, students find the platform's x-coordinate for specific cases.

Mathematics

This activity is like a combination of *As the Ferris Wheel Turns* and *Graphing the Ferris Wheel,* except that it deals with horizontal instead of vertical position.

Students use the cosine function in different quadrants to find the diver's x-coordinate when he falls and to graph the x-coordinate as a function of time.

Progression

Students work individually as they explore the diver's horizontal position. Students will extend this activity to develop a general formula in *First Quadrant Platform*.

Approximate Time

30 minutes for activity (at home or in class)

15 to 20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Transparency of the *Where Does He Land?* blackline master

Doing the Activity

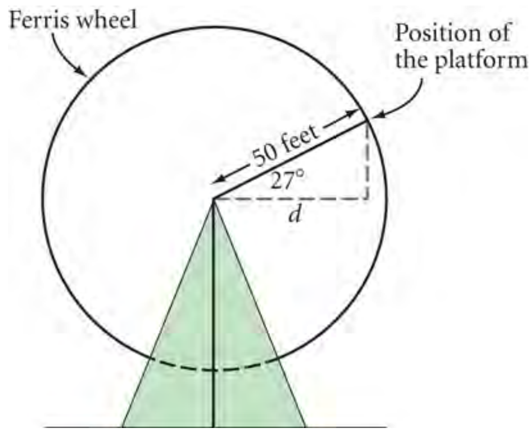
Question 1 asks students to find the diver's x-coordinate at each of five given release times. Question 2 asks them to graph the x-coordinate as a function of t.

Discussing and Debriefing the Activity

This unit has adopted the informal convention of using t as a general time variable and W as the value of t when the diver is released. But both t and W are often used to represent the same quantity: the number of seconds the Ferris wheel turns before release. Clarify this if students express confusion over the use of these two variables.

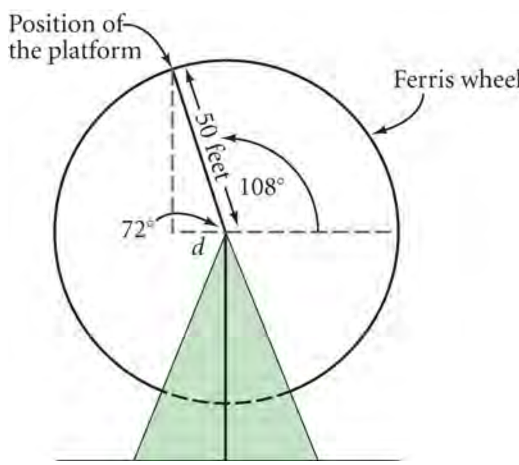
For Question 1, let several students present their results for specific values of t, using diagrams to explain their answers. As these presentations are discussed, bring out that the x-coordinate of the diver's landing position is the same as the x-coordinate of the platform at the moment the diver is released.

For instance, for $t = 3$ (Question 1a), the Ferris wheel has turned 27°, and the presenter might give a diagram like this:



This gives $\cos 27° = \frac{d}{50}$, so the platform's x-coordinate is $50 \cos 27°$, or approximately 44.55. This means that while the diver is falling, his x-coordinate is also approximately 44.55. (Students may simply say that the diver lands about 44.55 feet to the right of the center of the Ferris wheel. Help them make the transition from this description to the use of the coordinate terminology.)

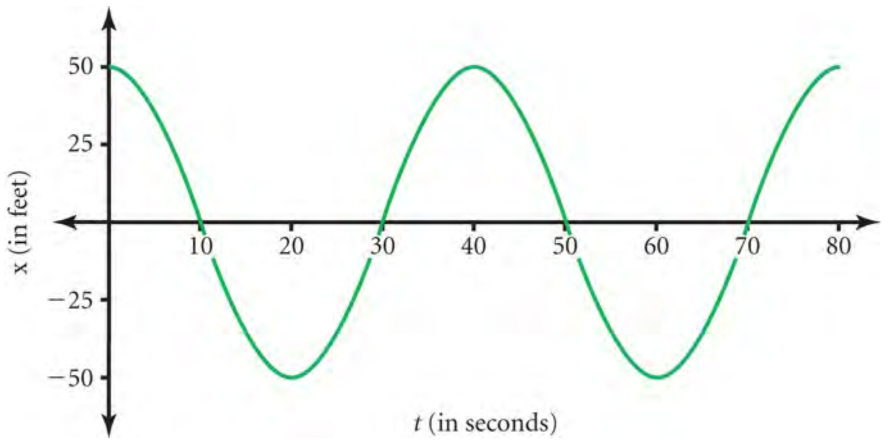
For the case $t = 12$ (Question 1c), the platform is in the second quadrant when the diver is released, and students are likely to express their common x-coordinate as $-50 \cos 72°$, as illustrated in the next diagram. It is important to bring out that although the segment labeled d has length $50 \cos 72°$ (because $\cos 72° = \frac{d}{50}$), the x-coordinate must be negative. That is, $x = -50 \cos 72°$.



Similarly, for $t = 26$, students are likely to get $x = -50 \cos 54°$, while for $t = 37$, they will probably get $x = 50 \cos 27°$. (Students might express these answers using the sine function instead, but ultimately, the goal is to get a general formula in terms of cosine.)

Question 2

You can use a transparency of the blank coordinate system for this activity, provided in the *Where Does He Land?* blackline master, or you can have students develop the scales for the axes themselves. In either case, you or the students can plot individual points as they are suggested. Be sure to get a variety of points from $t = 0$ through $t = 80$. The graph should look roughly like this:



There are two main observations to make about this graph:

- This graph makes sense in terms of the Ferris wheel problem. For example, it is periodic with period 40, and it shows the platform being farthest to the right at $t = 0$, 40, and 80, farthest to the left at $t = 20$ and 60, and having a zero x-coordinate at $t = 10$, 30, 50, and 70.
- This graph is similar to the graph of vertical position that students made in *Graphing the Ferris Wheel*, with two key differences:
  a) This graph has its maximum at $t = 0$ seconds, while the graph of the platform's height had its maximum at $t = 10$ seconds.
  b) This graph is "balanced" around the horizontal axis, while the graph of the platform's height was "balanced" around the height of 65 feet.

First Quadrant Platform

Intent

In this activity, students generalize the platform's *x*-coordinate for the first quadrant, in preparation for extending the cosine function definition to one that is meaningful for arbitrary angles.

Mathematics

In *Where Does He Land?* students calculated the platform's *x*-coordinate at specific points in time. Now they generalize this to develop a formula for the platform's *x*- coordinate in the first quadrant. They are not yet able to extend this to the other quadrants because the definition of the cosine that they are working with is based upon right-triangle geometry and therefore is only meaningful for acute angles.

Students will develop a more general definition of the cosine function for arbitrary angles in *Generalizing the Platform*.

Progression

First Quadrant Platform asks students to generalize their work from *Where Does He Land?* to find a formula for the position of the platform in terms of *t* in the first quadrant. This activity is very similar to *At Certain Points in Time* and should require little discussion.

Approximate Time

15 to 25 minutes for activity (at home or in class)

5 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

In this activity, students develop a general formula for the platform's *x*-coordinate for the first quadrant (as they did for the platform's height in *At Certain Points in Time*). They will then use that formula to motivate the general definition of the cosine function. No introduction is needed, and probably only a brief discussion will be required.

Discussing and Debriefing the Activity

Let a student present his or her result. The class should be able to see that the platform's *x*-coordinate is given by the equation $x = 50 \cos (9t)$.

Carts and Periodic Problems

Intent

In this activity, students find a formula for the cart's position and consider examples of periodic behavior.

Mathematics

Students combine their earlier formula for the cart's travel time with the newly introduced horizontal coordinate system to find a formula for the position of the cart. They also think further about situations that involve periodic motion.

Progression

Students work on the activity individually or in groups, then discuss their results as a class.

Approximate Time

25 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

In *Cart Travel Time*, students found an expression for the cart travel time in terms of W, the time that the Ferris wheel has been turning. Now they combine that with the information about the speed and starting coordinate of the cart to come up with a formula for the cart's x-coordinate when the diver reaches water level.

Part II asks students to describe, state the period, and sketch graphs of several situations that they believe are periodic.

Discussing and Debriefing the Activity

Part I: Where's the Cart?

Have a volunteer present Part I. In *Cart Travel Time*, students saw that from $t = 0$ until the diver reaches the water level, the cart travels for $W + F$ seconds (where F represents the diver's falling time and is given by the expression $\sqrt{\frac{57+50\sin(9W)}{16}}$).

Students need to combine that information with the facts about the cart's speed and initial position to find that the cart's x-coordinate when the diver reaches the water level is $-240 + 15(W + F)$.

Post this conclusion:

> **Suppose the diver is dropped after W seconds on the Ferris wheel (starting from the 3 o'clock position). When the diver reaches the water level, the cart's x-coordinate is**
>
> $$-240 + 15(W + F) \text{ where } F = \sqrt{\frac{57 + 50\sin(9W)}{16}}.$$

Part II: Periodic Problems

You can give overheads to a couple of groups and have them choose one or two of their most interesting examples to share with the class. They should present the situation, give the period, and show a sketch of the graph.

Here are some of the many ideas they might mention:

- Phases of the moon
- Menstrual cycles
- The movement of the hands of a clock
- The height of the sun in the sky

Generalizing the Platform

Intent

In this activity, students extend the cosine function to be defined for all angles.

Mathematics

Similar to what students saw in *Extending the Sine*, they are here asked to redefine the cosine in a way that makes sense for arbitrary angles.

The subsequent discussion emphasizes that there is only one way to extend the cosine function that will allow the first-quadrant formula for the platform's x-coordinate to work in all quadrants. After the cosine function has been formally defined, students graph the function. The discussion connects the cosine function to the Ferris wheel problem, bringing out that students now have general formulas for the diver's vertical and horizontal positions as functions of time.

Progression

Students work in groups to define the cosine function beyond the first quadrant. In the discussion, they graph this function, and connect it to the Ferris wheel problem.

Approximate Time

20 to 30 minutes for activity

15 to 25 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Tell students that this activity is basically a cosine version of earlier work with the sine function. To extending the cosine function beyond the right-triangle definition, students seek to define the cosine function for non-acute angles so that the equation $x = 50 \cos (9t)$ will give the platform's horizontal position for all values of t.

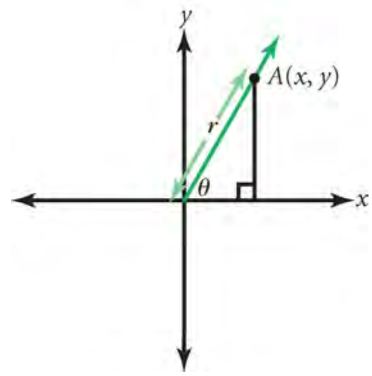
Discussing and Debriefing the Activity

Have a student present Question 1. Discussion of this problem should give you a good sense of how well students understand the process they used for generalizing the sine function (in *Extending the Sine* and *Testing the Definition*). They should see that the platform's x-coordinate is $-50 \cos 72°$ for $t = 12$. For the equation $x = 50 \cos (9t)$ to give this value when $t = 12$, they need to define $\cos 108°$ to be equal to $-\cos 72°$.

Use your judgment about whether you need to discuss Question 2 as well.

Question 3

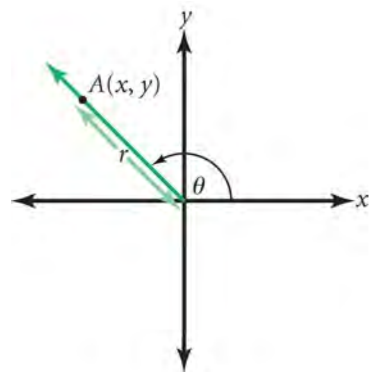
Let another student present Question 3, and provide additional hints and review as needed. The goal is to have students see that in a diagram such as the one shown here, with θ a first-quadrant angle, cos θ is equal to $\frac{x}{r}$, and that using this ratio as the general definition for arbitrary angles is consistent with their work in Questions 1 and 2. (If needed, you can go through details such as those used in *Extending the Sine* for the sine function.)



Defining the Cosine Function

When students seem clear about the process, you can simply tell them that this is the basis for the formal definition of the cosine function for arbitrary angles. You should post this together with an appropriate diagram (such as the one that follows):

> **For any angle θ, we define cos θ by first drawing the ray that makes a counterclockwise angle θ with the positive x-axis and choosing a point A on this ray (other than the origin) with coordinates (x, y).**
>
> **Using the shorthand $r = \sqrt{x^2 + y^2}$, we then define the cosine function by the equation**
>
> $$\cos \theta = \frac{x}{r}$$



As with the discussion of the sine function, emphasize how well this works, particularly in terms of sign: the extended cosine function is positive when x is positive and negative when x is negative. These two cases correspond exactly to the sign of the platform's x-coordinate, which is positive to the right of center and negative to the left of center.

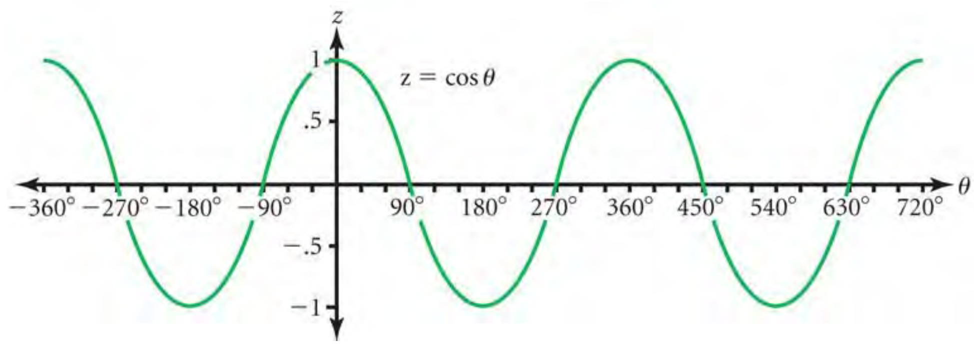
The Cosine Function Is Well Defined

Briefly point out that the ratio $\frac{x}{r}$ does not depend on the specific point chosen on the defining ray. You can discuss the fact that this can be proved using similarity, just as was done for the sine function.

Cosine Graphs

Have students graph the function defined by the expression 50 cos (9t) on their calculators, adjusting the viewing window to include all values from t = 0 to t = 80. Have them compare the result to their graphs from Question 2 of *Where Does He Land?* They should see that the graphs are the same.

Also have students graph the "plain" cosine function on their calculators, including negative values for the angle, and compare it with the graph of the "plain" sine function. They should see that the two graphs are identical in shape, but one is “shifted” from the other. You can post and label a graph like the one shown here near the graph of the sine function for future reference.



Cosine and the Ferris Wheel
Ask students, **What does this general definition of the cosine function mean in terms of the x-coordinate of the platform as the Ferris wheel turns?** They should be able to articulate a conclusion like this, which you should post:

> **If a Ferris wheel of radius 50 feet makes a complete turn every 40 seconds, starting from the 3 o’clock position, then the *x*-coordinate of the platform, after *t* seconds, is given by the function**
>
> $$x = 50 \cos (9t)$$

The Diver’s Position When He Reaches the Water Level
Ask how this conclusion fits into the solution of the unit problem. Bring out that (at least for the current simplified version of the problem), the diver’s *x*-coordinate as he falls is the same as the *x*-coordinate of the platform when the diver is dropped. In other words:

If the diver is dropped after the Ferris wheel has been turning for W seconds, starting from the 3 o’clock position, then his x-coordinate as he falls is given by the function

$$x = 50 \cos (9W)$$

Key Question

What does this general definition of the cosine function mean in terms of the *x*-coordinate of the platform as the Ferris wheel turns?

Planning for Formulas

Intent

In this activity, students sum up and explain their work thus far in the unit.

Mathematics

In preparation for fitting all of the pieces together to solve the unit problem in *Moving Cart, Turning Ferris Wheel*, in this activity students review each of the formulas that they have developed in the unit. They explain how the facts of the unit problem are reflected in each formula and what each piece of the formula represents.

Progression

Students work on this activity individually. The discussion reviews each of the formulas in depth, but not how they fit together.

Approximate Time

30 to 40 minutes for activity (at home or in class)

30 to 40 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students work on this activity independently.

Discussing and Debriefing the Activity

Have several students present elements of each formula. For each formula, you might have one student give the formula itself and have others explain how the facts about the cart and Ferris wheel, the definitions of the sine and cosine functions, and the principles about falling objects fit in. This discussion involves a thorough review of the unit so far, and so it is likely to take a significant amount of time.

Finding the Release Time

Intent

In this section, students solve the central unit problem.

Mathematics

Students now put together the various pieces of the solution that they have accumulated throughout the unit to write an equation whose solution will answer the unit problem. Since the equation cannot be solved through algebraic manipulation, they must also consider what other avenues they have for approximating a solution. Each of the several primary strategies for solving the equation highlights the power of the graphing calculator.

Progression

Students solve the simplified unit problem in *Moving Cart, Turning Ferris Wheel*. *Putting the Cart Before the Ferris Wheel* helps them to reason through the pieces that were necessary to complete the former activity as they solve a variation on the unit problem. They establish firmly their understanding that the diver and the cart must be in the same place at the same time.

The remaining activities in this section complete the development of the cosine function. Students graph the function and examine what happens in various quadrants in *What's Your Cosine?* They look again at an application of the cosine function in *Find the Ferris Wheel*, as they examine the connection between the parameters of the Ferris wheel and the coefficients in the formula for the x-coordinate of the platform.

Moving Cart, Turning Ferris Wheel

Putting the Cart Before the Ferris Wheel

What's Your Cosine?

Find the Ferris Wheel

Moving Cart, Turning Ferris Wheel

Intent

In this activity, students solve the central unit problem.

Mathematics

In *Carts and Periodic Problems*, students developed an expression for the position of the cart when the diver reaches water level in terms of *W*, the time at which the diver is dropped. In *Generalizing the Platform*, students arrived at a definition of the cosine function that would allow them to apply their formula for the *x*-coordinate of the platform (and therefore of the diver) to all four quadrants. The key to creating an equation for the unit problem is recognizing that the diver and the cart must both be in the same place when the diver reaches water level.

Setting the expressions from these two activities equal to each other creates an equation that will solve the simplified unit problem, but that equation can't be solved by algebraic manipulation. It can be solved by an organized guess-and-check approach or, more elegantly, by using the graphing calculator to approximate the intersection of the functions from the two earlier activities.

Progression

Students work in groups to solve the simplified unit problem. They will need plenty of time both for formulating the equation, and for thinking through how to solve it. Class discussion will be particularly important for sharing ideas about how to solve the equation, and looking at the issue of just how accurate the answer needs to be.

Approximate Time

50 to 60 minutes for activity

30 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

No introduction is needed for this activity.

This is the culmination of a long effort to solve the simplified unit problem. It is important that you allow students sufficient time to work through this problem. Although this activity is "nothing more than" a combination of previous ideas, the process of bringing these varied ideas together may be difficult for many students.

Although some groups may need to start by guessing specific numbers, you should help them move toward the formulation of a specific equation that they can use to find *W*.

Hints as Students Work

As students work on this, you can remind them to focus on the idea of having the cart and the diver be in the same place when the diver reaches the water level. You can also suggest that they look around the room (or in their notes, or at their work on *Planning for Formulas*) for summaries of key elements of the analysis.

There are several forms for an equation that states that the cart is in the right position when the diver reaches the water level. This is one possibility:

$$-240 + 15\left(W + \sqrt{\frac{57 + \sin(9W)}{16}}\right) = 50 \cos(9W)$$

Once students get this or an equivalent equation, they will need some time to come to the realization that algebraic manipulation is not going to work. The discussion below suggests several approaches for getting a numerical estimate of the solution.

Discussing and Debriefing the Activity

Let several students present their analyses of the problem. Give further hints as needed to get an equation that expresses the fact that the cart is in the right place. Again, here is one possibility:

$$-240 + 15\left(W + \sqrt{\frac{57 + \sin(9W)}{16}}\right) = 50 \cos(9W)$$

Make sure students can articulate that the left side of the equation gives the x-coordinate of the cart at the time the diver is at water level, and the right side of the equation gives the x-coordinate of the diver at that same time. In other words, the equation is saying that the diver is landing in the water.

Students may come up with variations on this equation. Let students present any alternate approaches on the overhead, and ask them to articulate their meaning.

Help students appreciate that this analysis is "quadrant-free." That is, it works no matter where the platform is when the diver is released. You might remind them that in the initial activities of the unit (such as *As the Ferris Wheel Turns*), they did not have the general sine and cosine functions, and their analysis was slightly different for each quadrant.

Solving the Equation

Elicit presentations by groups that have solved the equation by different methods, including guess-and-check and graphing. Here are three ways students might solve the complicated equation just developed:

- Guess-and-check: That is, pick a value for W, evaluate both sides, and then repeatedly adjust W to bring the two sides of the equation closer together.
- Graphing: For instance, graph the two functions defined by the expressions on the two sides of the equation and then look at where the graphs meet. This will require adjusting the window settings in order to locate the point of intersection.

- Using a “solve” feature on a calculator.

This may be a good time to discuss how to approach entry of complicated equations into the calculator (perhaps by breaking them into smaller pieces) and how to use the “solve” feature on the calculator.

It turns out that the assistant should release the diver about 12.28 seconds after the cart starts moving.

Ta-Da!

Give a cheer! The problem is solved! If students haven’t already done so, you will probably want to have them substitute $W = 12.28$ into both sides of the equation to confirm that this value really is correct.

Also, be sure to answer the question posed in *Moving Cart, Turning Ferris Wheel* of where the diver is on the Ferris wheel when the assistant releases him. After 12.28 seconds, the Ferris wheel will have turned $9 \cdot 12.28 \approx 110.5°$, which will place the platform between the 11 o’clock and 12 o’clock positions.

The diver’s height off the ground when released is given by the expression $65 + 50 \sin(9W)$, which gives a value of about 112 feet. The diver’s x-coordinate is given by the expression $50 \cos(9W)$, which comes out to about –17.5, which means he is about 17.5 feet to the left of center.

Students will probably want to work out some more of the stages in the process for $W = 12.28$. For example, the diver must fall about $112 - 8 = 104$ feet, which will take about $\sqrt{\frac{104}{16}} \approx 2.55$ seconds. (That is, $F = 2.55$.) Thus, the cart must travel a total of about $12.28 + 2.55 = 14.83$ seconds.

How Much Accuracy Is Needed?

Because the value 12.28 is an approximate solution to a practical problem, you should raise the question, **How accurate does the assistant need to be?** For instance, if he drops the diver after 12.3 seconds, what will happen?

Students can explore this simply by substituting other values for W into the two sides of the equation to see how much effect changes in W have on the x-coordinates of the cart and the diver. For instance, if $W = 12.3$ seconds, then at the moment the diver reaches the water level, the cart’s x-coordinate is approximately –17.3 and the diver’s x-coordinate is approximately –17.7. As long as the tub is of a reasonable size, these few inches should not matter. On the other hand, for $W = 12.4$, the cart’s x-coordinate is approximately –15.9 and the diver’s x-coordinate is approximately –18.4, so this represents a difference of several feet, which might be problematic.

But There’s More to the Problem!

As noted earlier, the problem as solved just now involves a simplification—assuming that the diver falls as if from rest. The more complex version of the problem will be pursued in the remainder of the unit.

If students are interested, you might encourage them to speculate on the more complex version of the problem now.

Key Question

How accurate does the assistant need to be?

Putting the Cart Before the Ferris Wheel

Intent

In this activity, students look at a question that is similar to the central unit problem but much simpler.

Mathematics

Working on this simpler problem may help some students with *Moving Cart, Turning Ferris Wheel.* It will particularly assist them in realizing that the key to the unit problem is to ensure that the cart and the diver are in the same place when the diver reaches water level.

Progression

Students work on the activity individually and then discuss their results as a class.

Approximate Time

25 to 35 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Putting the Cart Before the Ferris Wheel asks where the cart would need to start in order to catch the diver if he is released 25 seconds after the Ferris wheel began turning.

Discussing and Debriefing the Activity

Let volunteers explain each of Questions 1 and 2. Question 1 should be a review of familiar ideas. The diver's x-coordinate as he falls (and when he lands) is $50 \cos (9 \cdot 25)$, or approximately –35.4.

Question 2

Students need to put several ideas together to answer Question 2. First, they need to find out how long the diver is in the air. If he is released after 25 seconds on the Ferris wheel, his height off the ground (in feet) will be $65 + 50 \sin (9 \cdot 25)$, or approximately 29.6 feet, and so he will fall 21.6 feet to reach the water level. This means that his falling time (in seconds) will be $\sqrt{\frac{21.6}{16}}$, or approximately 1.16 seconds.

Once they have found this falling time, students can think about the cart's movement. The cart will travel a total of approximately 26.16 seconds (the initial 25 seconds plus 1.16 seconds for the diver's falling time). At 15 feet per second, this means the cart will travel approximately 392.4 feet. Therefore, to be in the right place when the diver reaches the water level, the cart must start 392.4 feet to the left of –35.4, which means the cart must start 427.8 feet to the left of center.

What's Your Cosine?

Intent

In this activity, students graph the cosine function and consider its periodicity.

Mathematics

This activity gives students experience with the new extended definition of the cosine function. The questions reinforce the vocabulary associated with periodic functions, make students think about the sign of the cosine in each quadrant, and give them practice working with the symmetry of the graph.

Progression

Students work on the activity individually and then discuss their results as a class.

Approximate Time

30 minutes for activity (at home or in class)

15 to 20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Question 1 asks students to draw a graph of the cosine function and to name its amplitude, period, intercepts, and angles for which it has maxima and minima. Question 2 has students find multiple angles that have the same or opposite cosine as a given angle.

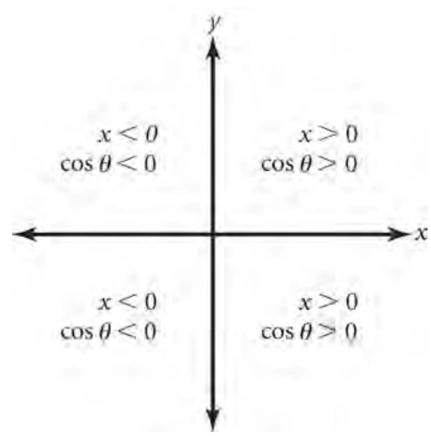
The subsequent discussion includes consideration of the sign of the cosine function in each quadrant.

Discussing and Debriefing the Activity

Let volunteers each answer a component of the activity. The discussion of Question 1 can be similar to that for *The "Plain" Sine Graph.*

Question 2

By this time students should be able to argue with confidence why each part of Question 2 has four possible answers. Use the discussion of Question 2 to get students to generalize about the sign of the cosine function. By looking at the sign of the *x*-coordinate, they should be able to determine that the cosine function is positive if the angle is in the first or fourth quadrant, and negative if the angle is in the second or third quadrant. You may find it helpful to use a diagram like the one at right.



Find the Ferris Wheel

Intent

This activity offers an opportunity to confirm that students understand the connections between the parameters of the Ferris wheel problem and the coefficients in the formula for the x-coordinate of a rider on the Ferris wheel.

Mathematics

In this activity, students examine the connection between the function describing a rider's horizontal position and the parameters of the Ferris wheel. They also look at the effect of changes in the function on its graph.

Progression

Students work on the activity individually and share their results in class discussion.

Approximate Time

15 to 25 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Question 1 presents two variations of the equation for the x-coordinate of the platform on the Ferris wheel and asks students to explain what values for the physical parameters of the Ferris wheel are represented by each equation. Question 2 asks students to write a similar expression for a Ferris wheel with a smaller radius but greater angular speed than that in Question 1a, and to explain how the graphs of the two expressions would differ.

Discussing and Debriefing the Activity

There may be some confusion over period versus angular speed. The latter is the number that appears in the equation. For example, in Question 1a, the coefficient 10 in the expression $25 \cos (10t)$ means that the Ferris wheel turns 10 degrees per second. To get the period, students need to divide 360° by this coefficient (or do some equivalent arithmetic process).

Question 2

Students will probably all have different answers for Question 2, and you can use two or three of these answers to illustrate the options. Be sure to get at least a verbal description of how the graph for such a function would differ from that in Question 1a.

After discussion of Question 2, use your judgment about whether to take the time to have students actually graph examples involving the variations described. If students seem able to articulate that a smaller radius will lead to a smaller amplitude of the graph and that a faster angular speed will lead to a more "scrunched up" graph, then they probably have a sufficient understanding of the ideas.

A Trigonometric Interlude

Intent

In this section, students study additional trigonometric topics.

Mathematics

While students have solved the central unit problem at this point, the Ferris wheel situation is a useful context for introducing several more concepts in trigonometry. Describing the diver's position on the Ferris wheel platform in terms of the angle of rotation lends itself nicely to the introduction of polar coordinates, and the analogy is equally helpful in developing several trigonometric identities that can be visualized using the symmetry of the Ferris wheel. This section also extends the definition of the tangent function to all angles, as was already done for the sine and cosine functions.

Progression

Polar coordinates are introduced in *Some Polar Practice* and *A Polar Summary*, and students gain further insights into this topic through their work in *Polar Coordinates on the Ferris Wheel*.

Students develop an important Pythagorean identity in *Pythagorean Trigonometry*. In *Positions on the Ferris Wheel*, they discover that the sines of supplementary angles are equal, and in *More Positions on the Ferris Wheel* students consider the sine and cosine of a negative angle.

Coordinate Tangents extends the tangent function to arbitrary angles. Students summarize what they have learned in this unit so far in *A Trigonometric Reflection*.

Some Polar Practice

Reference: A Polar Summary

Polar Coordinates on the Ferris Wheel

Pythagorean Trigonometry

Coordinate Tangents

Positions on the Ferris Wheel

More Positions on the Ferris Wheel

A Trigonometric Reflection

POW 8: Which Weights Weigh What?

Some Polar Practice

Intent

This activity is an introduction to polar coordinates.

Mathematics

The Ferris wheel situation lends itself to consideration using polar coordinates. In this activity, students are introduced to polar coordinates, and they find rectangular coordinates from polar coordinates and vice versa. This will be connected to the Ferris wheel in *Polar Coordinates on the Ferris Wheel*.

Progression

Begin this activity by introducing polar coordinates, bringing out that a point has many representations in polar coordinates. Students will then work on the activity in groups. The subsequent discussion develops general equations for expressing rectangular coordinates in terms of polar coordinates, and includes an introduction to the use of negative angles, angles greater than 360°, and negative values for r in polar coordinates.

Approximate Time

10 to 15 minutes for introduction

20 to 30 minutes for activity

10 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Polar Coordinates

Ask, **How have you been describing the platform's** (or diver's) **position on the Ferris wheel?** Emphasize that students have described this position in terms of the platform's height off the ground and its horizontal position relative to the center of the Ferris wheel.

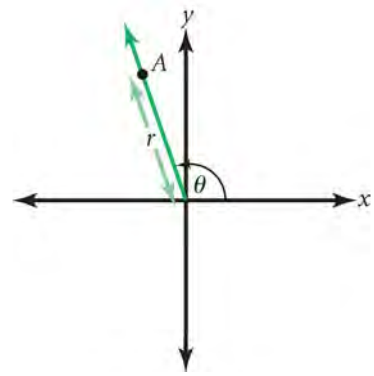
Ask, **Where is the origin of the coordinate system?** Help students articulate that they have been working with an *xy*-coordinate system whose origin is at ground level, directly below the center of the Ferris wheel.

Then ask, **What information have you been using to get these coordinates?** Bring out that both coordinates are expressed in terms of the radius of the Ferris wheel and the angle through which the platform has turned. Tell them that because the turning occurs at the center of the Ferris wheel, rather than at ground level, it makes more sense to treat the center of the Ferris wheel itself as the origin.

Next, inform students that in the coordinate plane, the measurements corresponding to the radius of the Ferris wheel and the angle of turn are called the

polar coordinates of a point and are usually represented by the variables r and θ. (We suggest that you wait until after *Some Polar Practice* before introducing the fact that points in the plane have many polar representations. See the subsection "Multiple Answers" in the discussion below.)

Illustrate the idea of polar coordinates with a diagram as shown here, clarifying the role of each variable:



- The polar coordinate r represents the distance from point A to the origin.
- The polar coordinate θ represents the counterclockwise angle made between the positive direction of the x-axis and the ray from the origin through point A.

Tell students that in giving the polar coordinates of a point, we give the r value first, by convention. Illustrate this with an example. For instance, ask, **What are the polar coordinates for a rider on the 50-foot Ferris wheel at the 11 o'clock position?** Students should see that this position would be represented as (50, 120°).

Point out that with two systems of coordinates under discussion, it's important to be clear which system is being used. Review the term *rectangular coordinates* as another name for the system of x- and y-coordinates.

Some Polar Practice

With the preceding introduction of the basic definitions, have groups work on *Some Polar Practice.*

Discussing and Debriefing the Activity

Let several students present individual problems. Have the class discuss whether the answers seem reasonable. For instance, on Question 1a, students should see that both the x-coordinate and the y-coordinate should be positive (because the point is in the first quadrant) and that the x-coordinate should be larger than the y-coordinate (because the angle is less than 45°). On Question 1b, they should see that the x-coordinate is negative and the y-coordinate is positive because the point is in the second quadrant.

To the nearest hundredth, the answers are

- For Question 1a: (1.73, 1)
- For Question 1b: (–3.83, 3.21)

The General Equations

Ask, **In general, how can you find the rectangular coordinates of a point from its polar coordinates?**

If students need a hint, you might suggest that they look at the general definitions of the sine and cosine functions. Students should see that they can simply multiply each defining equation by r. That is, $\cos \theta = \frac{x}{r}$ (by definition), so $x = r \cos \theta$. Similarly, $y = r \sin \theta$. Point out that these relationships work in all quadrants, and post these equations.

Post this result:

> **If a point has polar coordinates r and θ, then its rectangular coordinates x and y can be found by the equations**
>
> $$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

Question 2

For Question 2, students are likely to give only the "obvious" answers for the two examples, which are approximately (8.25, 14°) for Question 2a and (9.85, 294°) for Question 2b. (If they give other answers, that will lead smoothly into the next subsection.)

Multiple Answers

Tell students that by convention, we allow *any* angle, not merely those from 0° to 360°, to be considered as the angle in polar coordinates. Bring out that an angle of more than 360° simply represents more than one complete turn around the origin. Also clarify that negative angles are measured clockwise (from the positive direction of the x-axis).

Have students provide at least a couple of alternate solutions for each of Questions 2a and 2b, one with an angle of more than 360° and one with a negative angle. For instance, the answer to Question 2a could be (8.25, 374°) or (8.25, –346°), and the answer to Question 2b could be (9.85, –66°) or (9.85, 654°). Tell students that because a given point has more than one representation, we sometimes speak of "a polar representation" of a point rather than "the polar coordinates" of the point.

You can use the Ferris wheel model to illustrate this idea of multiple representations, pointing out that a rider will pass the same place many times, which gives many different polar coordinate representations of that position. Students will probably see that these representations all have the same r-coordinate. But they should realize that there are infinitely many values for θ that correspond to a given point.

Tell students that we also allow negative values for r, by going in the opposite direction from the ray defined by θ. Either give an example or elicit one from the class. For instance, help students see that the point in Question 2a could also be represented in polar coordinates by the pair (–8.25, 194°).

Without getting into details, we suggest that you point out that because each point has many polar coordinate representations, there are no simple formulas for getting a polar representation of a point from its rectangular coordinates.

Direct students to the *A Polar Summary* reference pages for a summary of the topics of this discussion.

Key Questions

How have you been describing the platform's position on the Ferris wheel?

Where is the origin of the coordinate system?

What information have you been using to get these coordinates?

What are the polar coordinates for a rider on the 50-foot Ferris wheel at the 11 o'clock position?

What's another name for the system of *x*- and *y*-coordinates?

In general, how can you find the rectangular coordinates of a point from its polar coordinates?

Supplemental Activities

Polar Equations (*extension*) introduces the graphing of polar equations.

Circular Sine (*extension*) challenges students to transform a polar equation into a rectangular equation.

A Polar Exploration (*extension*) is an open-ended exploration of the graphs of polar equations.

Reference: A Polar Summary

Intent

A Polar Summary is reference material that introduces polar coordinates.

Mathematics

These reference pages summarize the representation of points in polar coordinates and the conversion of rectangular coordinates to polar coordinates. It includes discussion of polar coordinates with angles greater than 360°, negative angles, and negative values for *r*.

Progression

The material in this activity is simply a summary of the discussion notes from *Some Polar Practice*. Direct students to this reference material after the discussion following that activity.

Approximate Time

0 to 10 minutes for discussion

Classroom Organization

For reference and/or discussion only

Doing the Activity

After the class has completed and discussed *Some Polar Practice*, refer students to the summary of polar coordinates in *A Polar Summary.*

Discussing and Debriefing the Activity

You may want to review these ideas briefly again, referring to the text of *A Polar Summary*.

Polar Coordinates on the Ferris Wheel

Intent

In this activity, students look at the relationship between polar coordinates and the Ferris wheel problem.

Mathematics

This activity gives students further experience working with polar and rectangular coordinates while enabling them to tie their new knowledge of the polar coordinate system to the familiar context of the Ferris wheel. This context helps to reinforce the concept that every point in the coordinate plane has many ways to be represented in polar coordinates.

Progression

Students work on the activity individually and then share their results as a class.

Approximate Time

20 to 30 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Polar Coordinates on the Ferris Wheel asks the student to find both polar and rectangular coordinates for a rider on a Ferris wheel at a given time, to find a different time at which the rider would be in the same position and the associated polar coordinates, and then to find a general expression for each set of coordinates at time t.

Discussing and Debriefing the Activity

This discussion will show you how well students are making the connection between polar coordinates and the Ferris wheel. For Question 1, they should see that 3 seconds represents a turn of 54° (based on the period of 20 seconds), so the "obvious" polar coordinates are (40, 54°). To get the rectangular coordinates, students can think in terms of either "coordinate formulas" or "Ferris wheel formulas" to see that the rectangular coordinates can be expressed as (40 cos 54°, 40 sin 54°), which is approximately (23.5, 32.4).

Question 2a illustrates the periodicity of the motion, and students might give values such as 23 seconds, 43 seconds, and so on. For Question 2b, they might use angles of 414°, 774°, and so on. For Question 3, the only "work" needed to get the polar coordinates is the computation that each second represents 18° of turn, so the rider's polar coordinates after t seconds are $(40, 18t°)$. The rectangular coordinates are simply a variation on the formulas students found for the rider's position in the main unit problem.

Pythagorean Trigonometry

Intent

In this activity, students develop a Pythagorean identity.

Mathematics

The Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ is the single most important relationship among the trigonometric functions. This activity will help students discover this relationship.

Progression

Students work in groups to discover the Pythagorean identity. The subsequent discussion introduces the term *identity* to describe a general relationship such as the equation just developed, and reviews the identity $\sin\theta = \cos(90° - \theta)$.

Approximate Time

25 minutes for activity

15 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

This activity asks students to combine the coordinates of a point on the unit circle (expressed in terms of *x, y,* and trigonometric functions) and the equation of the unit circle to write an equation relating $\sin\theta$ and $\cos\theta$. Students should develop the identity $\sin^2\theta + \cos^2\theta = 1$. Students then choose an angle in each quadrant to verify that the relationship holds true.

You may want to review the idea that the definitions of sine and cosine are independent of the choice of point along the appropriate ray, and that we sometimes use a point on the unit circle for the definitions.

Discussing and Debriefing the Activity

Let different students present results for each question. For Question 1, presenters will probably substitute 1 for r in $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, and come up with $y = \sin\theta$ and $x = \cos\theta$. For Question 2, they should get $x^2 + y^2 = 1$. On Question 3, if students replace x and y as intended, they will get the equation $(\cos\theta)^2 + (\sin\theta)^2 = 1$.

A Notation Convention

Tell students that by convention, the square of $\cos\theta$ is written as $\cos^2\theta$ and the square of $\sin\theta$ is written as $\sin^2\theta$ (and similarly for the other trigonometric

functions). Thus, we usually write the equation from Question 3 as
$\cos^2 \theta + \sin^2 \theta = 1$.

You may want to check occasionally to verify that students understand that $\cos^2 \theta$ really means $(\cos \theta)^2$, and so on. In particular, if students try to use the "$\cos^2 \theta$" notation on their calculators, they will discover that the calculator will not accept it. For instance, if they want to find $\cos^2 20°$, they will need to enter the expression $(\cos 20)^2$.

Beyond the Unit Circle

Point out that Questions 1 through 3 are set up using points on the unit circle, and ask, **Does the relationship cos2 θ + sin2 θ = 1 require you to use points on the unit circle?** Students might simply point out that the equation doesn't involve *r*, so once the equation has been established (using the point for which $r = 1$), the value of *r* no longer matters.

Question 4

Use your judgment about whether to take class time to go over the verification of the equation for specific values.

The Pythagorean Identity

Tell students that the relationship

$$\cos^2 \theta + \sin^2 \theta = 1$$

is known as a *Pythagorean identity.* (There are other Pythagorean identities.)

Explain that this equation is called an **identity** because it is true for all values of the variable, that is, for all angles θ. The "Pythagorean" part of the name comes from its connection with the Pythagorean theorem.

Post this relationship with its name and a statement that this equation is true for all values of θ.

A Familiar Trigonometric Identity

Students have previously seen the trigonometric identity $\sin \theta = \cos (90° - \theta)$ and may have reviewed it earlier in this unit. But in previous discussions of this relationship, they knew only the right-triangle definitions of the trigonometric functions. In the next part of the discussion, students are reminded of this equation as another illustration of an identity, and find that it is true for all angles. .

Ask, **What other trigonometric identity have you seen involving the sine and the cosine?** As a hint, tell students to think of an identity that expresses the sine of an angle in terms of the cosine of a related angle. As a further hint, draw a right triangle and ask how the sine of one base angle might be expressed as the cosine of another angle, and what the relationship is between the two angles. Use the fact that the angles are complementary to review the equation
$\sin \theta = \cos (90° - \theta)$.

Then ask, **Does this relationship hold true for all angles?** You may simply have students verify it for specific values of θ in different quadrants.

One possible proof of this relationship uses a quadrant-by-quadrant analysis. A more intuitive approach uses the idea that the graph of the function $y = \cos \theta$ can be obtained by reflecting the graph of the function $y = \sin \theta$ about the line $\theta = 45°$.

Key Questions

Does the relationship $\cos^2 \theta + \sin^2 \theta = 1$ require you to use points on the unit circle?

What other trigonometric identity have you seen involving the sine and the cosine?

Does this relationship hold true for all angles?

Supplemental Activity

A Shift in Sine (*extension*) has students examine in more depth how the graph of the cosine function can be produced by shifting the graph of the sine function.

More Pythagorean Trigonometry (*extension*) asks students to clearly explain the Pythagorean identity just developed and to develop similar identities for the remaining four trigonometric functions.

Coordinate Tangents

Intent

This activity continues the process of extending the trigonometric functions beyond their right-triangle definitions.

Mathematics

In this activity, students extend the definition of the tangent function to all angles and construct its graph.

Progression

Students work on the activity individually. The discussion highlights the difference between the period of the tangent function and that of the sine and cosine functions, and emphasizes that the tangent is not defined for all values.

Approximate Time

25 to 40 minutes for activity (at home or in class)

15 to 20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

In the first question of *Coordinate Tangents*, students see that the right-triangle-based definition of the tangent as $\frac{y}{x}$ works for angles in all four quadrants.

Question 2 leads to the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Students test this identity in the third question, and then sketch a graph of $z = \tan\theta$.

Discussing and Debriefing the Activity

Let a volunteer present Question 1, but be sure to determine how well students were able to handle this question. Some may have considered it easy after the work with sine and cosine, but others may have had trouble. Use a first-quadrant example to illustrate that the ratio $\frac{y}{x}$ comes from the right-triangle definition and that it makes sense to define $\tan\theta$ in general using this ratio.

Ask, **Are there any difficulties that could arise from using this ratio to define the tangent function?** If a further hint is needed, ask what happens if $x = 0$. Bring out that the ratio is undefined in that case, and tell students that the tangent function is thus undefined for certain angles.

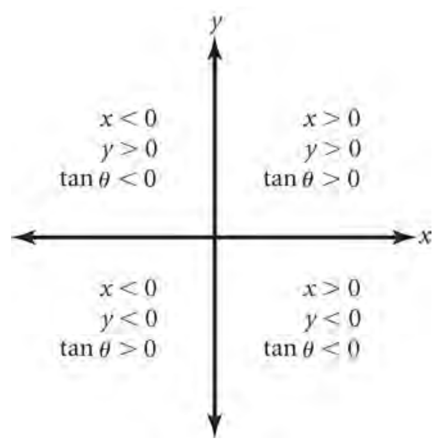
Have the class determine for which angles the tangent function is undefined. They should see that it is undefined for 90° and 270° in the "first cycle." They might see, more generally, that it is undefined for any odd multiple of 90°. Bring out that for right triangles, the ratio $\frac{\textit{opposite}}{\textit{adjacent}}$ gets larger and larger as the base angle gets closer to 90°, so it makes sense that there would be a problem at 90°.

On Question 2, the goal is to develop the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Students might get this by writing x and y in terms of r and θ, so that the ratio $\frac{y}{x}$ becomes $\frac{r\sin\theta}{r\cos\theta}$ which simplifies to $\frac{\sin\theta}{\cos\theta}$.

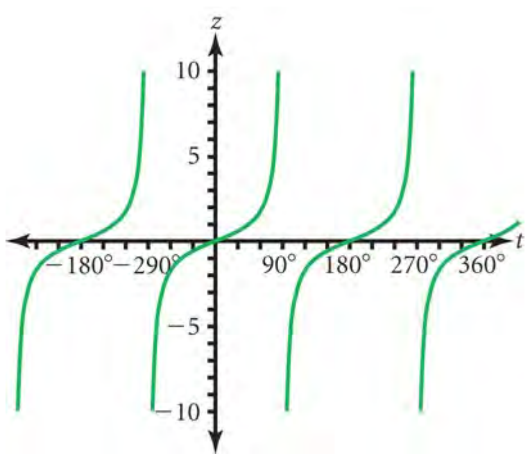
The examples in Question 3 should be fairly direct applications of the definition. If some students were unable to develop the general definition for the tangent function, you may want to have the class work on Question 3 now. The intent is for students to use right-triangle diagrams and reference angles to find the rectangular coordinates of appropriate points, and to apply the general definition. You can have them verify that their calculators give the same answers.

Question 4

On Question 4, the main goal is to have students see the general pattern of the graph. Take this opportunity to review that the tangent function is undefined for certain angles. Also focus on the sign of the tangent function in the different quadrants. You might make a diagram like the one at right to go with similar diagrams for sine and cosine.



The graph itself should look something like this:



Ask, **What is the period of the tangent function?** Students may assume that the period is 360°, as with sine and cosine. Bring out that the tangent function actually has a period of only 180° because of the way the signs work out.

Key Questions

Are there any difficulties that could arise from using this ratio to define the tangent function?

What is the period of the tangent function?

Positions on the Ferris Wheel

Intent

In this activity, students develop another trigonometric identity.

Mathematics

Students use the Ferris wheel analogy again in this activity to develop the identity $\sin\theta = \sin(180° - \theta)$.

Progression

Students work on the activity individually or in groups, then share their results as a class.

Approximate Time

20 to 25 minutes for activity (at home or in class)

5 minutes for discussion

Classroom Organization

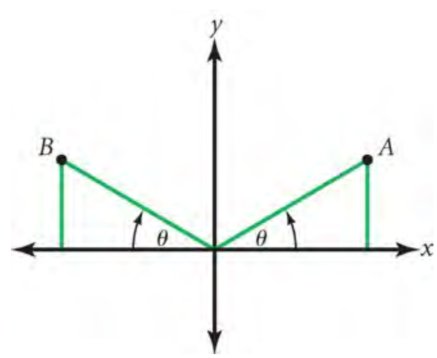
Small groups or individuals, followed by whole-class discussion

Doing the Activity

You may want to go over the "reading" portion of this as a whole class, and then have students begin work on the specific questions.

The activity first asks students to express the angle of rotation for a Ferris wheel position in the second quadrant in terms of the angle of rotation θ for the position in the first quadrant that is at the same height. Students then use the fact that both positions are at the same height to develop the identity $\sin\theta = \sin(180° - \theta)$.

Discussing and Debriefing the Activity



For Question 1, students need to see that the angle for point *B* is $180° - \theta$. Be sure to get an explanation for this conclusion. For instance, students might see that the angle between the ray through *B* and the negative end of the *x*-axis must be equal to θ, as shown here. (They might explain this using the two right triangles in the diagram.)

For Question 2, students must set the expressions $50 \sin\theta$ and $50 \sin(180° - \theta)$ equal to each other. Presumably, they will then divide by 50 to get the identity

$$\mathbf{\sin\theta = \sin(180° - \theta)}$$

You may want to post this result for reference in later units.

More Positions on the Ferris Wheel

Intent

This activity continues the work of *Positions on the Ferris Wheel* in developing trigonometric identities.

Mathematics

In this activity, students develop the identities $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Progression

Students work on the activity individually and then share their results as a class.

Approximate Time

25 to 30 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Part I of this activity has students explain why the Ferris wheel positions for angles of rotation of θ and $-\theta$ have the same x-coordinate, first for a specific example and then for the general case. They use this information, in turn, to explain why $\cos(-\theta) = \cos\theta$.

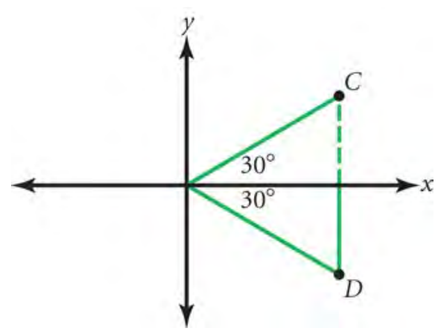
In Part II, students are given the identity $\sin(-\theta) = -\sin\theta$. They are asked to check the identity using specific values, and then to explain it using both the Ferris wheel situation and a more general coordinate system diagram.

Discussing and Debriefing the Activity

You can have different volunteers present their ideas on each of the questions in the activity.

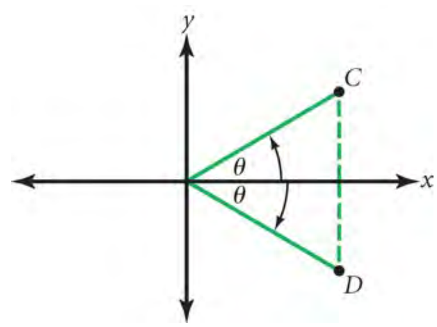
Part I: Clockwise and Counterclockwise

Part I is similar to the previous activity, *Positions on the Ferris Wheel*. For Question 1a, students might use a diagram like the one at right, with C and D representing the positions of the two riders. They can show that the two right triangles are congruent (or use a more intuitive argument) to explain why C and D have the same x-coordinate.



To answer Question 1b, they should reason that the formula gives 50 cos 30° and 50 cos (–30°) as the two x-coordinates. Because these x-coordinates are equal, cos 30° and cos (–30°) must be equal.

The diagram at right simply generalizes the previous one.



The explanation for Question 2a is essentially the same as that for Question 1a. For Question 2b, students will probably have little difficulty moving from their work in deriving the equation cos (–30°) = cos 30° to the more general identity

$\cos(-\theta) = \cos\theta$

Ask, **How is this equation connected to the graph of the cosine function?** Help students see that the equation says that the graph of the function $y = \cos\theta$ is symmetric about the y-axis.

Part II: From Identity to the Ferris Wheel

For Part II, begin by having students illustrate their work from Question 3, explaining the identity $\sin(-\theta) = -\sin\theta$. Be sure to include an example involving a negative angle, because the issue of signs is important here. Specifically, they should see that if θ itself is negative, then $-\theta$ is positive.

The main task of Part II is finding a Ferris wheel situation that expresses the given identity. Let volunteers offer ideas. For example, a student might use the diagram from Part I and point out that a rider going counterclockwise is just as high above the center of the Ferris wheel as a rider going clockwise is below the center.

Post all of the identities discussed in this activity.

Key Question

How is this equation connected to the graph of the cosine function?

A Trigonometric Reflection

Intent

In this activity, students summarize ideas about trigonometry.

Mathematics

A Trigonometric Reflection asks students to compile a summary, complete with diagrams and explanations, for the trigonometric concepts from this unit. Those ideas include extension of the trigonometric functions beyond first-quadrant angles, graphs and periodicity of the functions, identities, and polar coordinates.

Progression

This activity will be included in the students' portfolios for this unit.

Approximate Time

30 minutes for activity (at home or in class)

20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Tell students that this summary will be included in their portfolios for this unit.

Discussing and Debriefing the Activity

You may want to focus on the process students used in extending the trigonometric functions from the right-triangle definitions to the complete functions defined in this unit.

POW 8: Which Weights Weigh What?

Intent

Students solve a difficult logic problem and communicate their reasoning in writing.

Mathematics

This POW involves looking for a pattern in a problem situation. Students are asked to find a procedure for choosing weights to be used on a balance scale that will allow the user to verify the largest number of sequential integer weights, beginning with 1. They are to do this both for a method in which the weights must all go on the opposite side of the scale from the item to be weighed and for a method in which some weights may be placed on the same side of the scale as the item. The problem can be extended by having students develop a proof for the sum of powers of 2 or by relating the first method to base 2 numeration.

Progression

Give students about a week to work on this POW. Part II of *The Diver and the POW* will help them get started. Presentations will follow.

Approximate Time

10 to 15 minutes for introduction

3 to 4 hours for activity (at home)

30 to 35 minutes for presentations and discussion

Classroom Organization

Individuals, followed by whole-class presentations and discussion

Doing the Activity

You might have volunteers read the POW aloud. You may want to clarify what the task is, emphasizing that students are to consider each method separately.

Part II of *The Diver and the POW* will help students get started on the problem for a review of how to write up their results.

On the day before the POW is due, select several students to make presentations of their solutions the following day.

Discussing and Debriefing the Activity

Discuss each method separately, beginning with presentations on the first (and simpler) method, letting other students contribute, and then moving to the second method.

Method 1: Combining Weights on One Side

If the presentations on this method do not examine specific examples in detail, discuss at least a couple of cases, such as two weights and then three weights.

For all cases, the first weight must be 1 ounce, because otherwise it's impossible to verify the weight of a 1-ounce package. For two weights, the second weight must be 2 ounces (the only other option is another 1-ounce weight, which is less useful); the king can then verify the weights of packages up to 3 ounces.

Similarly, for three weights, the first two weights must be 1 and 2 ounces, so the third should be 4 ounces (the least amount that can't be verified with the first two). The king can then verify the weights of packages up to 7 ounces.

Help students see that they can proceed systematically, at each stage adding a new weight that is 1 ounce more than the sum of the previous weights. At each stage, no matter what weights the king has, the largest weight he can verify is the sum of his weights.

Students will probably realize that the desired weights are all powers of 2 (starting with 2^0, which equals 1) and that with these n weights, the king can verify the weights of packages up to $2^n - 1$ ounces.

Optional: Powers of 2 and the Base 2 Numeration System

This problem offers the opportunity for some interesting digressions. Here are two ideas you might pursue:

- Have students prove that the sum of the first n powers of 2 (starting with 2^0) is $2^n - 1$.
- Use the idea of sums of powers of 2 to introduce the base 2 numeration system. The general principle for Method 1 is essentially the same as the idea that every positive integer can be expressed using 0s and 1s in this numeration system.

Method 2: Using Weights on Both Sides

Method 2 is more complex, as it essentially involves both sums and differences.

For the case of two weights, students should see that if the king's weights are 1 ounce and 2 ounces, he can verify packages of 1, 2, or 3 ounces, but if his weights are 1 and 3, he can verify packages of 1, 2, 3, or 4 ounces.

Some students may think that using weights of 2 and 3 ounces might be even better, because it is still possible to verify the weight of a 1-ounce package and this combination allows the king to verify the weight of a 5-ounce package. But this pair doesn't allow for weighing a 4-ounce package. You might ask students to try to prove in detail that no pair of weights is better than 1 and 3.

In general, with n weights, the optimal selection is to have weights that are the first n powers of 3, starting from 3^0. For instance, if the king is selecting four weights, he should choose 3^0, 3^1, 3^2, and 3^3 ounces.

Whatever the weights are, the largest weight the king can verify is the sum of his weights. If the weights are 1, 3, 9, . . . , 3^{n-1}, the sum turns out to be $\frac{3^n - 1}{2}$. For example, when n is 4, the weights 1, 3, 9, and 27 can be used to verify the weight of any package from 1 through $\frac{3^4 - 1}{2}$; that is, from 1 through 40 ounces.

A Falling Start

Intent

In these activities, students examine the effects of an initial vertical speed on a falling body.

Mathematics

Students now begin to look at the effect of the turning of the Ferris wheel on the diver's fall. They examine the simplest cases, those in which the initial motion is purely vertical. Along the way, they consider the use of the quadratic formula for solving the quadratic equations that arise in situations involving falling objects. This leads to a short digression concerning imaginary and complex numbers.

Progression

In *The Diver and the POW,* students consider, in a general way, the complexity of the combined effects of the vertical and horizontal motion imparted to the diver by the Ferris wheel.

A more exact analysis of situations involving only initial vertical motion begins with *Look Out Below!* and *Big Push,* where the initial motion is downward. The quadratic expressions that emerge motivate the introduction of the quadratic formula in *Finding with the Formula*. Students look at some of the difficulties that arise in applying the formula in *Using Your ABCs.* One of those difficulties—that of a negative radicand—leads to the introduction of imaginary and then complex numbers in *Imagine a Solution, Complex Numbers and Quadratic Equations,* and *Complex Components,* which also introduces vectors.

Three O'Clock Drop and *Up, Down, Splat!* turn the focus back to falling bodies having initial vertical motion, this time when that initial motion is upward.

Finally, *Falling Time for Vertical Motion* asks students to generalize their work so far by using the quadratic formula to develop an expression for the falling time of a freely falling body with initial vertical motion.

Initial Motion from the Ferris Wheel

Look Out Below!

The Diver and the POW

Big Push

Finding with the Formula

Using Your ABCs

Imagine a Solution

Complex Numbers and Quadratic Equations

Complex Components

Three O'Clock Drop

Initial Motion from the Ferris Wheel

Intent

Students summarize their progress on the unit problem, reflecting upon the formulas they have developed, and begin to consider the effects of the Ferris wheel's motion on the diver.

Mathematics

Students compile the formulas they have used so far and speculate on how they will need to adjust them for the more complex version of the problem. They then look at a gravity-free analogy to the diver's release to help them understand how the circular motion of the Ferris wheel platform affects the diver's motion.

Progression

An optional demonstration will help students get a feel for how the Ferris wheel's motion affects the path of the diver. Students work on the two parts of the activity individually and share findings in a class discussion.

Approximate Time

10 minutes for introduction

30 minutes for activity (at home or in class)

20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Transparency of the *Initial Motion from the Ferris Wheel* blackline master

Optional: String and a small weight (such as a roll of tape) for demonstrating a release from circular motion

Doing the Activity

You may want to introduce the activity with this optional demonstration. For safety reasons, it might be best to conduct it outdoors. Test this demonstration before doing it for the class.

Tie an object, such as a roll of masking tape or a chalkboard eraser, to the end of a piece of string about 4 feet long. The object should be heavy enough to create minimal air resistance, but soft enough to prevent serious injury. Spin around, swinging the object around yourself in a circle at a more or less constant height. Then let go of the string and watch the object fly.

Help students notice two things about the object's path of flight:

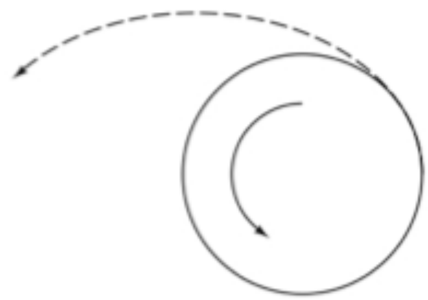
- The object continues in a straight line (except that it sinks to the ground). It

does not continue in its original circular path, although people's intuition often suggests that it should.
- That straight line is tangent to the circle. For example, if you are turning counterclockwise and facing north when you let go, the object will fly west.

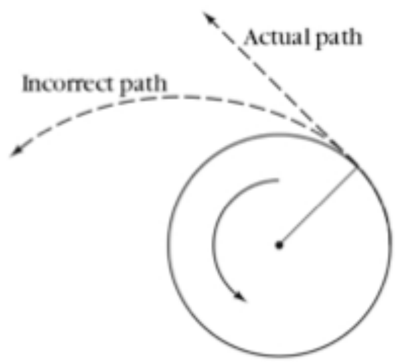
If you allow students to experiment, keep in mind that they may not have good control over timing the release of the string.

Discussing and Debriefing the Activity

Let students share ideas about Question 1. Many may think that the skateboarder will continue in a circular, or at least curving, path, as shown here.



In fact, a person traveling in a circular path and then released would travel in a direction tangent to the circular path, as shown at right. (You may want to project this diagram.)



Help students understand that the tangent to a circle is perpendicular to the circle's radius at the point of tangency. This fact will be useful in determining certain angles involved in the diver's initial motion as he leaves the Ferris wheel. Post the principle for students' reference:

> **In the absence of external forces such as gravity, an object released from a circular path will travel in a straight line. That straight-line path forms a right angle with the circle's radius to the point of release.**

In Part I of *The Diver and the POW,* students will need to consider the implications of the principle just stated. Because of the role of gravity, the Ferris wheel situation is more complex, and in fact, the diver does actually move along a curved path (unless he is initially traveling either straight up or straight down).

Ask, **How fast would an object be going when first released?** Students may have various ideas on this, and you may have to simply tell them this principle:

> **When an object is released from a circular path, it will have an initial speed equal to the speed at which it was already traveling.**

Key Questions

How might these formulas need to change for the more complex version of the problem?

How fast would an object be going when first released?

Look Out Below!

Intent

Students examine a situation involving an object falling with initial downward speed.

Mathematics

In *Free Fall,* students developed formulas to assist in the analysis of the motion of an object falling freely from rest. The diver in the unit problem, however, has an initial speed imparted by the turning Ferris wheel. Now students consider how to modify one of their equations to allow for an initial downward speed.

In this activity, they develop an expression for the height of a pillow falling downward as a function of time using the “averaging the endpoints” method of finding average speed. Solving for the time the pillow takes to reach the ground leads to a quadratic equation, which is used to review the method of completing the square.

Progression

The class reviews the concept of constant acceleration and the “averaging the endpoints” method before students work in groups on the activity. The follow-up discussion includes a review of the “completing the square” technique and a lead-in to a more substantial discussion of quadratic equations in conjunction with *Finding with the Formula*.

Approximate Time

55 to 65 minutes

Classroom Organization

Small groups, preceded and followed by whole-class discussion

Doing the Activity

Before groups begin, review the idea of constant acceleration and the method of “averaging the endpoints” by revisiting Question 1 of Free Fall. For instance, ask, **If an object falls freely from rest, what is its instantaneous speed after 5 seconds? What is its average speed for those 5 seconds? How far does it fall in that time?**

Review the principle that this object’s average speed is the average of its initial speed (0, because the object is falling from rest) and its final speed ($32 \cdot 5$ feet per second). Emphasize that this principle holds because acceleration is constant. Once students have the average speed, they can get the distance traveled by multiplying average speed by time spent traveling.

Point out that although this new activity involves another falling object, the rate of acceleration here is different, because air resistance has a significant effect on the pillow's motion. (In fact, as noted in the activity, air resistance will cause the pillow to accelerate at a nonconstant rate.)

You may want to work through Question 1 as a class to be sure the situation is clear.

If students need assistance with Question 4, have them suppose that the fall took t seconds. They should then find expressions for the pillow's instantaneous speed t seconds after Maxine sees it, for its average speed for the first t seconds, and for the distance it travels in those t seconds. This should lead to a quadratic expression for the distance the pillow falls in t seconds. Students can use a guess-and-check or graphing approach to solve the related equation.

Discussing and Debriefing the Activity

For Question 1, students should see that after 1 second, the pillow must be traveling 50 ft/s and, after 2 seconds, 70 ft/s.

For Question 2, the intent is for students to use the "averaging the endpoints" method. Thus, they should average the beginning speed (30 ft/s) with the instantaneous speed after 2 seconds (70 ft/s) to get an average speed of 50 ft/s for the interval from 0 to 2 seconds.

For Question 3, get an explicit statement of how to obtain the distance from this average speed, as this is a key step in the generalization required for Question 4. Students need to see that they must multiply the average speed (50 ft/s) by the time interval (2 seconds) to find that the pillow falls 100 feet during the 2 seconds.

For Question 4, using t for the time the fall took, students will likely begin the explanation by noting that the instantaneous speed after t seconds is $30 + 20t$ ft/s, so the average speed for the first t seconds is $\frac{30+(30+20t)}{2}$ ft/s. You might ask a volunteer to simplify this expression and then have the class verify that the simplified expression, $30 + 10t$, gives the value 50 when $t = 2$, consistent with the answer to Question 2.

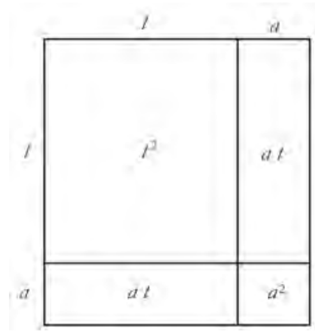
The next step is to multiply this average speed by the length of the time interval, which gives the expression $t(30 + 10t)$, or $30t + 10t^2$, as the distance traveled in t seconds. Again, you might have students check that for $t = 2$, this matches their earlier result. They will probably then simply set this expression equal to 200 feet (the distance the pillow needs to fall) to get the equation $30t + 10t^2 = 200$.

Once this equation is found, ask, **What type of equation is this? What is its standard form?** Have the class identify this as a quadratic equation and put it into standard quadratic form, perhaps simplifying it to

$$t^2 + 3t - 20 = 0$$

Solving by Completing the Square

Although students may have solved the equation using a guess-and-check approach (getting 3.2 seconds, to the nearest tenth), have them look at the solution algebraically as well.



As needed, review the technique of completing the square. Students might begin by writing the equation as $t^2 + 3t = 20$ and then look for a constant term to add to both sides so that the left side becomes a perfect square. You might review the use of a diagram like the one at right for this process.

Because we want a square, the two sides are the same. The two rectangles with area *at* combine to give 3*t,* so $a = 1.5$, and the small section has an area of 2.25. Adding 2.25 to both sides gives $(t + 1.5)^2 = 22.25$, so $t + 1.5 = \pm\sqrt{22.25}$, or approximately ±4.7. Since we are looking for the number of seconds, only a positive solution to this equation is needed. Thus, $t \approx 3.2$.

Question 5

Students should see that the pillow's height *t* seconds after Maxine sees it is given by the equation

$$h = 200 - 30t - 10t^2$$

Post this for comparison with similar equations in the discussion following *Big Push.*

One goal of Question 5 is for students to see the role played by the initial height, the initial speed, and the gravitational acceleration in determining the height of a falling object. To bring this out, ask, **Where do the coefficients in the expression 200 – 30t – 10t2 come from?** Students should be able to identify 200 as the pillow's initial height and 30 as its initial speed.

It may be more difficult for students to state where the coefficient 10 comes from. Have them retrace their steps to see that it is half of 20, which is the rate of acceleration. (In the discussion of *Using Your ABCs,* students will develop a general equation for a falling object's height after *t* seconds.)

As noted previously, the pillow will meet substantial air resistance. This resistance increases as the pillow goes faster, so the rate of acceleration will not be constant. You may want to point out that the mathematical model described in the problem is actually not a good one for this situation.

Another Approach

Another method for solving this problem involves visualizing the pillow as having been dropped from rest from some higher altitude. One can determine, based on the acceleration rate, that the pillow has been falling for 1.5 seconds when Maxine sees it. During that time, it averages 15 feet per second (halfway between 0 and 30), so it has already traveled 22.5 feet. Thus, its initial height was 222.5 feet (adding in the 200 feet it drops after Maxine sees it). So *t* seconds after Maxine sees the pillow, it has been traveling $t + 1.5$ seconds, falling from rest from a height of 222.5 feet. The total fall takes $\sqrt{\frac{222.5}{10}}$ seconds (replacing 16 in the usual expression by 10), and the part of the fall after Maxine sees the pillow takes 1.5 seconds less.

Key Questions

If an object falls freely from rest, what is its instantaneous speed after 5 seconds? What is its average speed for those 5 seconds? How far does it fall in that time?

What type of equation is this? What is its standard form?

Where do the coefficients in the expression $200 - 30t - 10t^2$ come from?

The Diver and the POW

Intent

Students consider the general effects on the diver of being released from the Ferris wheel at several clock positions.

Mathematics

In Part I, students apply ideas about release from circular motion to the Ferris wheel situation, from an intuitive perspective. In Part II, they consider a problem similar to the situation in the POW to help them develop ideas for the POW.

Progression

Students work on the activity individually and share ideas in a class discussion.

Approximate Time

25 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

Part I

For Question 1, bring out that if the diver is moving partly upward at the moment of release, his falling time will be increased by the effect of the motion of the Ferris wheel's motion. If he is moving partly downward when released, his falling time will be decreased by the effect of the Ferris wheel's motion. Students might answer the question using a description like this or, more specifically, in terms of Ferris wheel positions. For instance, they might say that the diver's falling time is increased if he is on the right side of the wheel (moving upward from the 6 o'clock to the 12 o'clock position, counterclockwise).

Similarly, for Question 2, students should see that if the diver is moving to the left (right) when he is released, he will land to the left (right) of his release position.

There is no purely intuitive way to answer Question 3. With the diver being released between the 12 o'clock and 11 o'clock positions, here are the issues:

- The wheel's motion will cause the diver to reach the water sooner. This means the cart will not have traveled as far when the diver reaches the water. This effect, taken by itself, suggests that the assistant should hold onto the diver longer to allow the cart more time to travel.
- The wheel's motion will cause the diver to be farther to the left when he

reaches the water, which means the cart doesn't have to go as far. This effect, taken by itself, suggests that the assistant should hold onto the diver for less time.

You may want to let students discuss this in their groups for a few minutes and then share their ideas. There is no simple way to see how to balance these two effects. Tell students they will have to work out the numerical details of the situation to answer the question. (Students will actually answer this question in *The Danger of Simplification.*)

Part II

You can begin by having a student answer the specific questions in the problem. Students should see that to get 10 ounces, they can begin by filling an 8-ounce cup and pouring from it to fill the 6-ounce cup. This leaves 2 ounces in the 8-ounce cup. They can use these 2 ounces and add a full 8-ounce measure to get 10 ounces. More generally, they should see also that variations on this procedure allow them to get any multiple of 2 ounces. On the other hand, they should see that they cannot measure amounts that are an odd number of ounces.

Ask for volunteers to discuss the similarities and differences between this problem and the situation in the POW. They should see, for example, that just as filling the 6-ounce cup from the 8-ounce cup leaves 2 ounces in the 8-ounce cup, so also putting an 8-ounce weight on one side of the scale and a 6-ounce weight on the other will allow the king to verify the weight of a 2-ounce object.

On the other hand, they should see that in the POW, if the king has just two weights, of 6 ounces and 8 ounces, then he cannot confirm the weight of anything above 14 ounces.

Big Push

Intent

Students calculate how the diver's falling time would be affected if he were released from the Ferris wheel with a purely downward initial speed.

Mathematics

This problem is basically a Ferris wheel version of *Look Out Below!* Students calculate the falling time for the diver to reach the water level from the 9 o'clock position of both a moving and a stationary Ferris wheel. They compare the answers to see how far the diver would miss by if he neglected to take into account the initial speed imparted by the wheel's motion. Finding the falling time involves solving a quadratic equation.

Progression

Students work on the activity individually or in groups. The follow-up discussion briefly reviews the technique of solving an equation by completing the square.

Approximate Time

25 to 30 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

The analysis is essentially the same as that for *Look Out Below!,* except for these details:

- The initial height is 65 feet instead of 200 feet.
- The diver needs to reach a height of only 8 feet rather than reach the ground.
- The initial speed is 7.85 ft/s (2.5π ft/s) instead of 30 ft/s.
- The acceleration is 32 ft/s per second instead of 20 ft/s.

For Question 1, students should find that the height after t seconds is given by the equation

$$h = 65 - 7.85t - 16t^2$$

Save this equation for comparison with the equations from *Look Out Below!* and from Question 2c of *Using Your ABCs* in the discussion that introduces *Three O'Clock Drop.*

For Question 2, students need to take into account that the water level is 8 feet off the ground. They should come up with an equation equivalent to

$65 - 7.85t - 16t^2 = 8$ and find (probably by guess-and-check) that it takes approximately 1.66 seconds for the diver to reach the water.

In Question 3, students should see that if the diver is dropped from the 9 o'clock position of a stationary wheel, it will take $\sqrt{\frac{57}{16}} \approx 1.89$ seconds for him to reach the water. Therefore, the initial motion from the turning wheel shortens the time by about 0.23 second.

Question 4 asks the question, **How far would the cart move in this amount of time?** At 15 ft/s, it would travel about $0.23 \cdot 15 \approx 3.5$ feet (or 3.4 feet if students do not round at intermediate steps), so failure to take the diver's initial speed into account might cost him his life!

Key Question

How far would the cart move in this amount of time?

Finding with the Formula

Intent

This activity introduces the quadratic formula.

Mathematics

As students have seen in several recent activities, the height function for a falling object will always involve a quadratic equation. The class reviewed the technique of solving these equations by completing the square with the equation from *Look Out Below!* The introductory discussion of this new activity develops the quadratic formula using the technique of completing the square. The activity provides two practice problems to be solved with the quadratic formula, one of which has a negative value for the coefficient b and one of which is not presented in standard form.

Progression

The teacher introduces this activity with a quick look at how quadratic equations are related to the unit problem and then assists the class in developing the quadratic formula by solving the general quadratic equation. Students then work on the activity individually.

Approximate Time

15 to 20 minutes for introduction

15 to 20 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, preceded and followed by whole-class discussion

Doing the Activity

Introduce the activity by pointing out that *Look Out Below!* and *Big Push* each involved finding the solutions to a quadratic equation. Get students to state those two equations, and display them for today's discussion. It isn't essential that the equations be in standard form. For example, they might look like this:

$$10t^2 + 30t = 200$$

$$65 - 7.85t - 16t^2 = 8$$

Ask, **Why do problems of this type lead to quadratic equations?** Students will probably recognize that average speed is a linear function of t, so multiplying the average speed by the time interval will give a quadratic expression.

Bring out that the coefficient 7.85 comes from a particular release position and that, in general, this coefficient will depend on when the diver is released. That means that to create a general equation for the falling time, students will have to solve an equation like the one from *Big Push,* but with an expression involving W in place of the specific value 7.85. Tell them that their next task will be to find a way to write the solution to any quadratic equation as an expression in terms of its coefficients.

Sketching a Solution by Completing the Square

Before students undertake that formidable task, remind them that they previously solved the equation $200 - 30t - 10t^2 = 0$ by simplifying and completing the square. Although the numbers in the equation from *Big Push* are messier than those from *Look Out Below!,* it's worthwhile to ask students to outline how they would solve $65 - 7.85t - 16t^2 = 8$ by that method.

For instance, they might begin by rewriting the equation as $16t^2 + 7.85t = 57$. Without having them do the arithmetic, have the class put into words what the remaining steps might be. You might have them redo the solution to $200 - 30t - 10t^2 = 0$ as a model.

Have students make the steps fairly explicit and write them down. They might come up with an outline like this:

1. Divide both sides by 16.
2. Take half the coefficient of t and think of that as the constant part of the squared term.
3. Add the square of that "half-coefficient" to both sides.
4. Write the left side as a perfect square.
5. Take the square root of both sides.
6. Subtract to get the value of t.

Try to get students to agree that this process would work even though the computations might be pretty messy.

The General Quadratic Equation

Ask, **What might the general quadratic equation look like?** If needed, review that the standard form is $ax^2 + bx + c = 0$.

Then ask students to apply their outline to this equation, doing the work in terms of the coefficients. (This may actually be easier than working with numbers.) Although you may need to help with the details, urge students to rely as much as possible on their outline. They might get this sequence of steps. (The issue of the two square roots is avoided in this sequence, but is addressed next.)

Subtract c from both sides to prepare for using the outline.	$ax^2 + bx = -c$
Step 1: Divide both sides by a.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
Step 2: Identify the "half-coefficient" as $\frac{b}{2a}$.	

Step 3: Add the square of the half-coefficient to both sides.	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$
Step 4: Write the left side as a perfect square.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$
Step 5: Take the square root of both sides.	$x + \frac{b}{2a} = \sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2}$
Step 6: Subtract to get the value of *x.*	$x = \sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}$

The last step shows an unsimplified version of the quadratic formula, which is fine for now. (Be sure it's clear that the subtraction of $\frac{b}{2a}$ is outside the square–root sign.) The important thing is for students to see that there is a general process—in fact, a formula—that they can use to solve *any* quadratic equation (at least any quadratic equation with real roots). They will look at equations with complex roots more closely in *Imagine a Solution* and *Complex Numbers and Quadratic Equations.*

What About Two Roots?

Raise the issue that the equations students have been examining have generally had two solutions, even if only one made sense in the context, and ask where the other solution is. If this issue hasn't been raised yet, you might use a graph to illustrate why a quadratic function generally has two roots.

If needed, return to a numerical example to bring out that Step 5 is a bit subtler than just described. Help students to see that, in fact, there are two possibilities for the value of $x + \frac{b}{2a}$:

$$\sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2} \quad \text{or} \quad -\sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2}$$

You might suggest that students can write the pair of solutions with a single equation:

$$x = \pm\sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}$$

Applying the Formula

Have students apply the formula to the equation from *Look Out Below!* That is, have them identify *a*, *b*, and *c* for the equation $200 - 30t - 10t^2 = 0$ and verify that the formula gives the same answers they found earlier.

Students might rewrite the equation as $10t^2 + 30t - 200 = 0$, but they can also use it as is, with $a = -10$, $b = -30$, and $c = 200$. Either approach gives $\frac{b}{2a} = 1.5$ [so $\left(\frac{b}{2a}\right)^2 = 2.25$] and $-\frac{c}{a} = 20$, and leads to $x = \pm\sqrt{22.25} - 1.5$. The positive square root equals 3.2, which answers the question from *Look Out Below!*

The Standard Quadratic Formula

Lead the class through the algebra of transforming their work into the standard version of the quadratic formula, or simply give them the standard version and tell them that it is equivalent to what they found. In either case, post the official result:

$$\textbf{If } \boldsymbol{ax^2 + bx + c = 0}\textbf{, and } \boldsymbol{a \neq 0}\textbf{, then } x = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{b}^2 - 4\boldsymbol{ac}}}{2\boldsymbol{a}}.$$

Tell the class that this equation, which gives the solution in terms of the coefficients, is called the **quadratic formula**.

You might point out that in this version of the formula, the various fractions have been combined into a single “master fraction” using a common denominator that is outside the square-root sign. You might have students verify this version of the formula using the equation from *Look Out Below!*

After this introduction, students can work on the activity. Caution them to be careful about parentheses when entering complicated expressions into their calculators.

Discussing and Debriefing the Activity

For Question 1, students should come up with $x = -4$ and $x = 7$ as solutions. Go over the details carefully, clarifying issues about signs and how to deal with the $\pm$ symbol. For instance, bring out that the coefficient b is -3, so the term $-b$ in the numerator is “positive three.” (Students often think of $-b$ as “negative b” rather than as “the opposite of b” and may reason that if b is already negative, there is no need to change the sign.)

The expression inside the square-root sign comes out to 121, so students can eliminate the square root by recognizing that $\sqrt{121} = 11$. Help them see that they can rewrite the expression $\frac{3 \pm 11}{2}$ as two separate expressions, $\frac{3+11}{2}$ and $\frac{3-11}{2}$, and simplify to get 7 and -4. Have them verify that both values actually fit the equation. You might want to point out that this equation can be solved by factoring as well.

Then have students graph the function given by the equation $y = x^2 - 3x - 28$ and interpret the two solutions in terms of the graph. They should see that the graph crosses the x-axis at two places, $x = -4$ and $x = 7$, which are the points on the graph where $y = 0$.

For Question 2, students will need to put the equation in standard form. They should then be able to see that $a = 3$, $b = 7$, and $c = -5$. According to the quadratic formula, there are two solutions, given by the expression $\frac{-7 \pm \sqrt{109}}{6}$. This problem involves a value of *a* different from 1 and a different sign for *b,* so review the details of the solution.

Students may have some difficulty with the computation needed to check these exact answers. As a second check, they could use numerical approximations and see that these values are very nearly correct. This is another opportunity to reinforce the distinction between exact and approximate solutions.

Key Questions

Why do problems of this type lead to quadratic equations?

What might the general quadratic equation look like?

Where's the other solution?

Using Your ABCs

Intent

Students use the quadratic formula to solve problems.

Mathematics

Applying the quadratic formula to equations that arise from the context of a situation, students consider the significance of having two solutions. They are also introduced to the issue of the nonexistence of real solutions.

Progression

Students work individually to solve equations using the quadratic formula and then to write and solve quadratic equations for problem situations. The follow-up discussion focuses on the mechanics of applying the quadratic formula and examines the significance of an equation having no solution or two solutions.

Approximate Time

30 minutes for activity (at home or in class)

20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

You can facilitate a smooth transition from this activity into the work with complex numbers (beginning with *Imagine a Solution*) by discussing Question 2 before Question 1.

Question 2

For the various parts of Question 2, students may make different choices about what their variables represent, which may lead to different equations. Have presenters show how to use the quadratic formula in each case to get exact solutions. If there are common errors, take time to discuss them. Be sure to talk about what the solution to each equation means in terms of the problem.

For part a, using x to represent the length of the shorter side leads to the equation $x(x + 5) = 126$. This is equivalent to $x^2 + 5x - 126 = 0$, which has $x = 9$ and $x = -14$ as solutions. If $x = 9$, the longer side of the rectangle is 14, and the rectangle is 9 feet by 14 feet.

Students should recognize that $x = -14$ does not make literal sense as a side length. They may also notice that this solution has the same absolute value as the long side and interpret this solution as representing a rectangle with sides whose "lengths" are –14 feet and –9 feet. Such a rectangle could be interpreted as lying in the third quadrant of a coordinate system, in which both x and y are negative.

If students set up their equations using the variable to represent the longer side of the rectangle, the equation and its solutions will be different, but the rectangle's dimensions will be the same.

For part b, if x represents the longer leg, the equation is $x^2 + (x - 6)^2 = 13^2$. In standard form, this is $2x^2 - 12x - 133 = 0$, and students can use the quadratic formula to get the two solutions $\frac{12 \pm \sqrt{1208}}{4}$, which yields approximately $x = -5.7$ and $x = 11.7$. Only one solution makes literal sense, and the triangle has legs of length 11.7 and 5.7. As with part a, students might give an interpretation to the negative solution and might set up the equation differently.

For part c, students should get the equation simply by setting the expression $90 + 50t - 16t^2$ equal to 120. In standard form, the equation is $16t^2 - 50t + 30 = 0$, which has two solutions, $\frac{50 \pm \sqrt{580}}{32}$. This yields values of approximately $t = 0.81$ and $t = 2.32$.

Before discussing whether both solutions make sense, ask, **What does the equation $h = 90 + 50t - 16t^2$ mean in terms of the situation?** Students will probably be able to interpret this as meaning that the object started with a height of 90 feet and an upward speed of 50 feet per second.

With this in mind, ask, **Which of the solutions to the equation make sense in the problem?** One solution ($t = 0.81$) indicates the time when the object reaches 120 feet on the way up. The other solution ($t = 2.32$) indicates the time when the object reaches 120 feet on the way down.

Interpreting the Negative Solution to "Look Out Below!"

After students have interpreted the two solutions to Question 2c in terms of the object going up and then down, you may want to return to *Look Out Below!*, in which the equation had two solutions, $t = 3.2$ and $t = -6.2$. If they did not come up with an interpretation then for the negative solution, you might ask whether it makes more sense now.

They might imagine that someone threw the pillow upward with just the right force so that it would pass Maxine on the way down at a speed of 30 feet per second. The solution $t = -6.2$ means the pillow would have to have been thrown upward 6.2 seconds before Maxine saw it. (As noted earlier, the issue of air resistance makes the mathematical model for this problem a poor one. That issue may also make this interpretation seem somewhat unrealistic.)

The General Height Formula

Ask students to develop a general formula for the height of a freely falling or rising object in terms of its initial height and velocity. They should come up with a principle like this one, which generalizes the similar statement from the discussion following *Free Fall*:

> **If a freely falling or rising object (straight down or up) has an initial height of h feet and an initial velocity of v feet per second (where a positive value for v means upward motion), its height after t seconds is $h + vt - 16t^2$ feet.**

Post this new principle for reference.

The Case v = 0

Ask students to compare the statement just developed with the principle posted for *Free Fall* giving the expression $h - 16t^2$ for the height of an object falling *from rest* from a height of h feet. They should see that the "from rest" formula, for the case $v = 0$, is a special case of the general formula. Remind them, if needed, that the initial velocity of the object in *Free Fall* was zero.

Question 1

Part a has integer solutions ($x = -3$ and $x = -4$), and students should be able to describe how they used the quadratic formula to get them.

Part b and c are a bit more difficult, because the exact solutions, given below, require square roots. You may need to caution students to pay particular attention to signs.

- Question 1b: $\frac{3+\sqrt{41}}{2}$ and $\frac{3-\sqrt{41}}{2}$, or approximately 4.7 and –1.7
- Question 1c: $\frac{-5+\sqrt{33}}{4}$ and $\frac{-5-\sqrt{33}}{4}$, or approximately 0.2 and –2.7

Part d presents a new challenge, because the equation has no real roots. Students should have found that the quadratic formula yields the expression $\frac{3 \pm \sqrt{-23}}{4}$.

Ask, **What does this expression mean?** Be sure students articulate that there is no number (that they know of) whose square is –23 and that this means the equation has no solutions (within the number system as they probably know it so far).

Also ask, **What does this lack of a solution mean in terms of the graph of $y = 2x^2 - 3x + 4$?** Students should be able to articulate that the graph has no x-intercepts and give a rough sketch of what such a graph might look like. (It's sufficient if they merely sketch a parabola opening upward that is entirely above the x-axis.) Students might want to verify, using their calculators, that the graph of $y = 2x^2 - 3x + 4$ has no x-intercepts.

Key Questions

What does the equation $h = 90 + 50t - 16t^2$ mean in terms of the situation?

Which of the solutions to the equation make sense in the problem?

What does this lack of a solution mean in terms of the graph of $y = 2x^2 - 3x + 4$?

Imagine a Solution

Intent

This activity introduces imaginary numbers.

Mathematics

Students saw in Question 1d of *Using Your ABCs* that some quadratic equations have no real solutions, as evidenced by a negative radicand when they apply the quadratic formula. This new activity introduces i as the square root of -1.

Progression

Students work on the activity individually or in groups and then discuss their discoveries.

Approximate Time

30 minutes

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

Tell students that in the next two activities, they will learn about two new kinds of numbers: **imaginary numbers** and **complex numbers**. These new numbers will enable them to solve equations that don't seem to have any solutions.

You might introduce the topic by asking for the simplest quadratic function students can create that has no x–intercepts. They may generate such functions as $f(x) = x^2 + k$ with $k > 0$, which lead to seemingly impossible equations like $x^2 + k = 0$. To resolve the dilemma, students could then read the explanation in the student book.

As students work through the questions, you might clarify that an expression like $2i$ is shorthand for $2 \cdot i$. In this respect, i is treated as a variable rather than as a number.

You might also give an example of multiplication with imaginary numbers, such as rewriting $2i \cdot 3i$ as $2 \cdot 3 \cdot i \cdot i$ and then simplifying, first as $6i^2$ and finally as -6.

Discussing and Debriefing the Activity

For Question 1, ask students how they could multiply to check their solutions. You might note that each problem has two answers and that neither is the "positive" square root.

For Question 2c, you may need to reassure students that they can rewrite a negative radicand like $\sqrt{-5}$ as the product $\sqrt{5} \cdot \sqrt{-1}$.

For Question 3, students should see that the powers of i form a repeating pattern with a cycle of length 4. For part c, they might articulate the process of dividing the exponent by 4 and finding the remainder.

Complex Numbers and Quadratic Equations

Intent

Students' work with imaginary numbers is extended to include the full system of complex numbers.

Mathematics

Students see that a combination of a real and an imaginary number is needed to express the solutions to some quadratic equations, and that these complex numbers are typically expressed in the form $a + bi$.

Progression

Students work on the activity individually and share discoveries in a class discussion.

Approximate Time

25 minutes for activity (at home or in class)

10 to 15 minutes for discussion

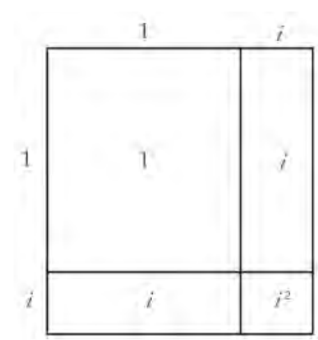
Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Tell students they are about to encounter a system of numbers called the **complex numbers,** a system that includes both the imaginary numbers and the real numbers (the numbers represented on the number line). You may want to have students read the explanation in the student book and ask questions for clarification before they tackle the activity.

Discussing and Debriefing the Activity



Question 1 involves substitution, the distributive property, and the definition of *i.* Students might use an area diagram similar to those they have seen for squaring binomials to find that $(1 + i)^2$ is equal to $1 + 2i + (-1)$, or simply $2i$.

This gives $(1 + i)^2 - 2(1 + i) + 2 = 2i - (2 + 2i) + 2 = 0$.

In Question 2, substituting into the quadratic formula (with $a = 1$, $b = -2$, and $c = 2$) yields the expression $\frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2}$, which simplifies to $\frac{2 \pm \sqrt{-4}}{2}$. Students should see that $\sqrt{-4}$ is equal to $2i$ and then simplify the fraction to $1 \pm i$.

For Question 3, they need only substitute $1 - i$ for x.

For Question 4, they will have to use radical multiplication to separate $\sqrt{-12}$ into $\sqrt{4} \cdot \sqrt{3} \cdot \sqrt{-1}$.

Supplemental Activities

Complex Conjugation (extension) introduces the complex conjugate, how it is used in the division of complex numbers, and how it arises in solving quadratic equations.

Absolutely Complex (extension) explores the absolute value of a complex number.

Complex Components

Intent

Students explore the representation and addition of complex numbers on the complex plane using vectors.

Mathematics

By examining the representation of two complex numbers and their sum as vectors in the complex plane, students discover that vectors can be added graphically by moving one vector so that it begins where the other ends. Students will notice that the real and imaginary parts of the two numbers being added are treated separately, much like the vertical and horizontal components of velocity will be treated in the unit problem. (The polar form of complex numbers is introduced and explored in the supplemental activities *The Polar Complex* and *Polar Roots.*)

Progression

Students work on the activity individually and share their findings with the class.

Approximate Time

30 minutes

Classroom Organization

Individuals, followed by whole-class discussion

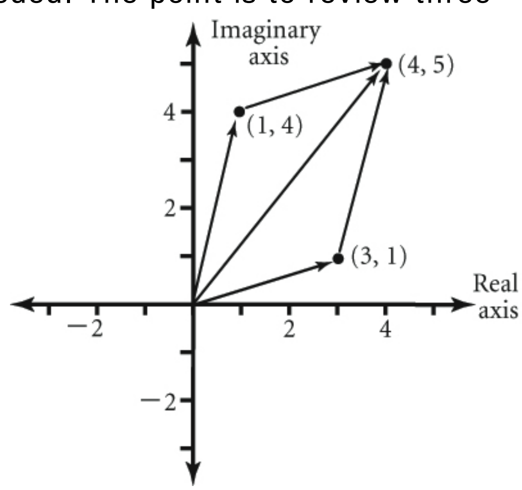
Doing the Activity

Introduce the activity by explaining that the class will now start considering the fact that the diver is already moving at the moment he is released.

Discussing and Debriefing the Activity

Questions 1 and 2 should be straightforward.

Have two or three students present their answers to Question 3, and ask others to contribute any other ways they might have proceeded. The point is to review three distinct ways of constructing the vector representing the sum $4 + 5i$:



- Have the vector for $1 + 4i$ begin where the vector for $3 + i$ ends, moving the second vector so that its length and direction do not change.
- Follow the same procedure by starting with the vector for $1 + 4i$.
- Notice that the four points form a parallelogram and that the “sum vector” is a particular diagonal of the parallelogram.

Also point out that diagonal vectors can be added in the same way as complex numbers, that is, by adding their horizontal (real) components and their vertical (imaginary) components separately.

Follow up this discussion by asking, **What are the key characteristics of a vector?** The goal is to develop an intuitive understanding that in a two-dimensional geometric context, a vector

- is a line segment
- has a fixed length
- has a fixed orientation (that is, a fixed angle with regard to some reference direction)
- has a fixed direction (that is, it begins at one end and ends at the other)
- can be moved to any position in the plane as long as its length, orientation, and direction remain the same

Conclude by telling students that later in the unit, they will use vectors to represent velocities for the diver on the Ferris wheel.

Key Question

What are the key characteristics of a vector?

Supplemental Activities

The Polar Complex (extension) introduces the polar form of a complex number.

Polar Roots (extension) has students use the polar form to find and test roots of complex numbers.

Number Research (extension) asks students to research the names given to different sets of numbers and to make a poster illustrating their findings.

Vector Island (extension) extends the concept of vectors. Students learn about rectangular and polar forms of vectors, and addition of vectors. This activity covers a Common Core State Standard.

Scalars and Magnitude (extension) is a follow-up to *Vector Island.* Students learn about scalar multiplication of vectors, and magnitude. This activity covers a Common Core State Standard.

Three O'Clock Drop

Intent

After the interlude with complex numbers, students now return to the main topic and examine a situation in which a falling diver has initial upward motion.

Mathematics

The opening discussion introduces the distinction between velocity and speed and establishes the convention that vertically upward motion will be described as having a positive velocity. Students then use this convention in evaluating a Ferris wheel situation in which the diver has initial upward motion.

Progression

The teacher introduces this activity by discussing the fact that velocity involves both a speed and a direction and the convention that positive velocity in the vertical direction means upward motion.

Approximate Time

10 minutes for introduction

25 minutes for activity (at home or in class)

10 to 20 minutes for discussion

Classroom Organization

Individuals, preceded and followed by whole-class discussion

Doing the Activity

Have the class compare the height equations from the last several problems:

- $h = 200 - 30t - 10t^2$ for *Look Out Below!*
- $h = 65 - 7.85t - 16t^2$ for *Big Push*
- $h = 90 + 50t - 16t^2$ for Question 2c of *Using Your ABCs*

Bring out that in Question 2c of *Using Your ABCs,* the coefficient of t is positive, while in the other problems, it is negative. Ask, **What does the sign of the coefficient of t say about the situation?** Students will probably recognize that in Question 2c, the object was going up to begin with, while in the other problems, the object was going down initially.

In the context of vertical motion, it usually makes sense to think of "up" as the positive direction. Tell students that they should use this convention for the remainder of the unit.

With this convention in mind, return to the three problems and ask, **Why does the coefficient of the linear term in the height function have the sign it does in each case?** Help students reach these conclusions:

- In *Look Out Below!,* the pillow had a velocity of –30 feet per second when Maxine first saw it.
- In *Big Push,* the diver had a velocity of –7.85 feet per second when he was first released from the Ferris wheel.
- In Question 2c of *Using Your ABCs,* the object had an initial velocity of 50 feet per second.

Velocity Has Sign

Tell students that **velocity** is the term used in physics to indicate the combination of speed and direction. Thus, the velocity of an object tells both the speed at which the object is traveling and the direction of the motion. Velocity may be positive or negative, but speed is never negative—in fact, it's the absolute value of velocity.

Acceleration and the Sign of Velocity

There is potential for confusion about the idea of constant acceleration in a context in which the sign of the velocity is changing. We have established the convention that positive velocity represents upward motion. While an object is moving upward, the effect of gravity is to slow it down, decreasing its speed. On the other hand, when an object is moving downward, the effect of gravity is to increase its speed.

Discussing and Debriefing the Activity

For Question 1, students will probably develop an equation for the time the diver is falling from a formula for an object's height in terms of time. They might simply write down this equation, based on earlier examples, without going through the analysis of finding the average velocity.

The diver's height t seconds after release is given by the expression $65 + 7.85t - 16t^2$. As a follow-up to the discussion of the sign convention for velocity, ask why this expression is identical to that for *Big Push* except for the sign of the coefficient of t. Students should see that the diver is being released from the Ferris wheel at the same height and with the same speed in both cases. The distinction is that here his initial motion is upward, while in *Big Push* it was downward.

To answer Question 1, students need to determine when the diver is 8 feet off the ground, so they need to solve the equation $65 + 7.85t - 16t^2 = 8$. The solutions to the equation are $t \approx 2.15$ and $t \approx -1.66$. Encourage students to use the quadratic formula to get the exact values in terms of square roots. (In *Falling Time for Vertical Motion,* they will need to use the formula to get a general expression for the falling time of a free-falling object.)

You might discuss once again that the equation only imperfectly represents the problem, because only one of its solutions makes sense in this context.

For Question 2, students might simply substitute 57 for h in the expression $\sqrt{\frac{h}{16}}$ (using the expression developed in Question 4 of *Free Fall*). Or they might point out that the time is the same as that for Question 3 of *Big Push,* namely, $t \approx 1.89$. Students can also find this value by solving the equation obtained by eliminating the term $7.85t$ from the equation used in Question 1; that is, by solving $65 - 16t^2 = 8$.

Next, ask, **How much of a difference does the diver's upward motion from the movement of the Ferris wheel make in his falling time?** Students can calculate that it takes approximately an additional 0.26 second. You might compare this to the decrease in falling time resulting from the effect of the Ferris wheel's motion in *Big Push*, which was approximately 0.23 second. Although the amounts are roughly the same, they are not identical.

Key Questions

What does the sign of the coefficient of t say about the situation?

Why does the coefficient of the linear term in the height function have the sign it does?

How much of a difference does the diver's upward motion from the movement of the Ferris wheel make in his falling time?

Up, Down, Splat!

Intent

Students continue to work with a falling body having an initial upward motion.

Mathematics

This activity puts initial upward motion in a different context. As in *Three O'Clock Drop*, students need to recognize that they can find the time until the object hits the ground by setting the height variable to zero and solving the resulting quadratic equation. In this activity, they are also asked the object's speed at the moment of impact, which can be found by using the initial velocity, falling time, and acceleration.

Progression

Students work on the activity individually or in groups and share findings in a class discussion.

Approximate Time

15 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

Question 1 involves essentially the same process as used in the discussion of *Three O' Clock Drop*, but with different numbers. The height is given by the expression $25 + 35t - 16t^2$, and the solutions to the equation $25 + 35t - 16t^2 = 0$ are $t \approx 2.75$ and $t \approx -0.57$. Students presumably will recognize that the solution $t \approx 2.75$ is the one they want.

You might take this opportunity to make sure students are connecting their algebra to the situation by asking, **What does the value 2.75 represent?** They should see that it gives the number of seconds from when Melissa throws the egg container upward until it hits the ground.

Question 2 brings up an issue that has not yet been discussed in this sequence of problems: speed at impact. The simplest approach is to use the fact that velocity is decreasing by 32 feet per second for each second, so velocity after t seconds is given by the expression $35 - 32t$. Students can then substitute $t \approx 2.75$ into this expression to get a value of about -53 feet per second for the velocity when the egg container hits the ground. Be sure to discuss the significance of the negative sign.

Key Question

What does the value 2.75 represent?

Falling Time for Vertical Motion

Intent

Students use the quadratic formula to develop a general expression for the falling time of an object in terms of its initial velocity upward or downward and its initial height.

Mathematics

In the discussion of *Using Your ABCs*, students determined that the height of a freely falling object after t seconds is defined by the expression $h + vt - 16t^2$. Now they will use that information to derive an expression for the falling time in terms of h and v. That is, they will use the quadratic formula to solve the equation $h + vt - 16t^2 = 0$ for t.

Progression

Students work on the activity individually or in groups. The follow-up discussion includes an exploration of whether the expression that students develop is consistent with that developed earlier for a body falling from rest. This exploration provides a nice review of simplifying radical expressions.

Approximate Time

5 minutes for introduction

20 to 25 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

Little or no introduction is needed for this activity.

Discussing and Debriefing the Activity

For Questions 1 and 2, students should rewrite the equation, likely as $16t^2 - vt - h = 0$, and then apply the quadratic formula, to get

$$t = \frac{v \pm \sqrt{(-v)^2 - 4 \cdot 16 \cdot (-h)}}{32}$$

which simplifies to

$$t = \frac{v \pm \sqrt{v^2 + 64h}}{32}$$

The task in Question 3 is to decide which sign to use. Students may tap their experience with previous problems, or they may be able to analyze the situation. For example, because h is assumed to be positive, the expression inside the square-root sign is greater than v^2, so using the negative sign would result in a negative value for t.

Post this solution, with a description of what it represents:

> **If an object is put into the air with an initial vertical velocity of v feet per second and falls a net distance of h feet, the time for this fall is given by the expression**
>
> $$\frac{v + \sqrt{v^2 + 64h}}{32}$$

Explain that *net distance* is the distance from the object's initial position to the ground. You might bring out that this expression is smaller if v is negative than if v is positive with the same absolute value. That is, the object will reach the ground faster if it is initially moving downward.

What if v = 0?

This complex expression may intimidate students. One way to make it less mysterious, as well as to review previous ideas, is to ask, **What does this expression say about the case $v = 0$?** That is, does this expression agree with what students learned earlier about objects falling from rest?

Let students investigate this question for a few minutes. Their first step should be to replace v with 0, which simplifies the expression considerably. But they may not immediately connect the resulting expression, $\frac{\sqrt{64h}}{32}$, with their earlier result, $\sqrt{\frac{h}{16}}$.

Have groups examine whether these two expressions are equivalent. If they get stuck, you might ask whether they can write $\frac{\sqrt{64h}}{32}$ as simply a radical or whether they can write $\sqrt{\frac{h}{16}}$ with a radical in the numerator only. In other words, **Are the expressions $\frac{\sqrt{64h}}{32}$ and $\sqrt{\frac{h}{16}}$ equivalent?** Another option is to square and then simplify both expressions and see that the results are the same.

Key Questions

What does this expression say about the case $v = 0$?

Are the expressions $\frac{\sqrt{64h}}{32}$ and $\sqrt{\frac{h}{16}}$ equivalent?

Components of Velocity

Intent

In these activities, students learn to work with falling motion that is not purely vertical.

Mathematics

Up to this point, students have considered falling motion in which the initial velocity was nonexistent, vertically upward, or vertically downward. Unless the diver is released at the 3 o'clock or 9 o'clock positions, however, he will have an initial velocity that is not purely vertical.

In these activities, students will learn that the diver's motion can be split into two components, vertical and horizontal velocity. These two components can be calculated separately: the diver falls as if he were not moving horizontally, and moves horizontally as if he were not falling. The two components are tied together only in that the horizontal motion is limited by the falling time.

Progression

The analysis begins with a pair of activities in which the initial motion is purely horizontal, one involving the Ferris wheel (*High Noon*) and one set in a different context (*Leap of Faith*). Then students look at another pair of activities (*The Ideal Skateboard* and *Racing the River*) where the initial velocity is neither purely vertical nor purely horizontal, but in which they do not have to deal with the complication posed by acceleration due to gravity as they determine and work with the velocity's horizontal and vertical components. In *One O'Clock Without Gravity,* they apply this idea to the Ferris wheel.

Swimming Pointers and *Vector Velocities* extend students' brief introduction to vectors (in *Complex Components*) to vertical and horizontal components of velocity. Students combine the two components to find the resultant and also resolve a given vector into components.

Students return to the Ferris wheel with a group of activities (*One O'Clock Without Gravity, Velocities on the Wheel, Release at Any Angle,* and *Moving Diver at Two O'Clock*) that get them ready to complete the unit problem. In *A Portfolio of Formulas*, students compile all the formulas they've learned that may be helpful in solving the unit problem.

Finally, in *The Danger of Simplification*, students explore whether it was really necessary to consider the effect of the diver's initial motion.

High Noon

Leap of Faith

The Ideal Skateboard

Racing the River

One O'Clock Without Gravity

Swimming Pointers

Vector Velocities

Velocities on the Wheel

Release at Any Angle

A Portfolio of Formulas

Moving Diver at Two O'Clock

The Danger of Simplification

High Noon

Intent

Students begin to consider the motion of falling objects that have a horizontal component to the initial velocity.

Mathematics

It often surprises students to realize that horizontal motion does not affect the length of time it takes an object to fall. This realization can then lead to confusion over the fact that the horizontal motion, on the other hand, is limited by the falling time. The horizontal and vertical components of falling motion can be treated completely separately, but the vertical motion must be analyzed first to determine the length of time the object will be moving horizontally.

Progression

The teacher introduces this activity with a brief discussion of what needs to come next in solving the main unit problem. Once students realize that the Ferris wheel will almost certainly impart an initial horizontal motion to the diver, discussion continues with an observation that horizontal motion and gravitational fall work independently of each other. Students then work through the activity individually or in groups, and the class reviews the findings.

Approximate Time

35 to 40 minutes

Classroom Organization

Individuals or small groups, preceded and followed by whole-class discussion

Doing the Activity

Now that the class has dealt with the falling motion of a body having vertical initial motion, ask, **What do you think comes next in the solution of the unit problem?** Point out that in the 3 o'clock and 9 o'clock positions, the diver's initial motion either shortens or lengthens the time it takes him to fall to the water level (as compared to the time it would take if he fell from rest).

Tell students to suppose now that the diver is released at some position other than 3 o'clock or 9 o'clock. Ask, **What other effect might the motion of the Ferris wheel have on the diver, besides shortening or lengthening the time of his fall?** Students should recognize that he would also be moving to one side or the other and that this would cause him to land in a different place than if he were dropped from rest. Tell students that the next aspect of the unit is to learn how to combine this sideways motion with vertical motion.

Horizontal Motion and Gravitational Fall Are Independent
Before dealing with these other cases of the unit problem, students need to learn one more principle about the physics of motion.

Explain that the key idea for combining horizontal and vertical motion is to treat them completely separately. Acknowledge that this principle may not fit students' intuition, but explain that it can be verified experimentally. You might summarize and post the principle as follows:

If an object is moving sideways as it falls, then

- **the rate at which its height changes is the same as if it were falling straight down**
- **the rate at which it moves sideways is the same as if there were no gravity**

Bring out that the object's overall speed is a blend of the vertical and horizontal speeds and is greater than either of these separate speeds. (The term used in physics for this combined motion is *resultant*.)

Discussing and Debriefing the Activity

If the class has already discussed *Leap of Faith,* you may wish to give groups more time to work on this activity. This will give you an opportunity to determine whether the presentation of this activity will be fruitful for the whole class or will simply be a repetition of the ideas in the earlier discussion.

At the 12 o'clock position, the diver is 107 feet above the water level, so his falling time (down to the water level) is $\sqrt{\frac{107}{16}} \approx 2.59$ seconds. His horizontal speed at the moment of release is 7.85 ft/s, so he will travel $7.85 \cdot 2.59 \approx 20.3$ feet to the left as he falls.

Key Questions

What do you think comes next in the solution of the unit problem?

What other effect might the motion of the Ferris wheel have on the diver besides shortening or lengthening the time of his fall?

Leap of Faith

Intent

Students continue to analyze falling motion for a body having an initial horizontal velocity.

Mathematics

This activity applies the ideas from *High Noon* in a context other than the Ferris wheel.

Progression

Students work on the activity individually. The follow-up discussion emphasizes that an object's falling time can be found independently of its horizontal motion.

Approximate Time

25 to 30 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

The key to this activity is to recognize, based upon the ideas in the discussion introducing *High Noon*, that the falling time in this situation is the same as that for an object falling 30 feet from rest: $\sqrt{\frac{30}{16}}$ seconds, or about 1.37 seconds.

By applying that previous discussion's second principle, students will see that the jumper will cover the same horizontal distance as if he or she had continued moving horizontally at a constant rate. The task is to find that rate. Because the jumper wants to travel 15 feet in 1.37 seconds, the appropriate rate is $\frac{15}{1.37}$ ft/s, or about 11 ft/s.

Question 2 addresses an interesting side issue. If the net is 10 feet across, the jumper can travel anywhere from 10 feet to 20 feet and still land in the net (although it's probably best to land somewhere near the middle). Therefore, the jumper's initial speed can be as little as $\frac{10}{1.37}$ ft/s (about 7.3 ft/s) and as much as $\frac{20}{1.37}$ ft/s (about 14.6 ft/s).

The Ideal Skateboard

Intent

Students analyze velocity in terms of its vertical and horizontal components in a context that does not involve the complication of gravity.

Mathematics

Until this point, students have worked with initial velocities that are either purely horizontal or purely vertical. In *High Noon*, the combination of the horizontal initial velocity and the vertical pull of gravity resulted in motion that was both horizontal and vertical. Students are now reassured that they can always split motion into perpendicular components for separate treatment, enabling them to work with situations in which the initial motion is neither purely horizontal nor purely vertical.

Progression

The teacher introduces this activity with a discussion of the principle that velocities can always be broken into perpendicular components that can be analyzed separately. Students then work through the activity individually or in groups, and the class reviews the findings.

Approximate Time

40 to 50 minutes

Classroom Organization

Individuals or small groups, preceded and followed by whole-class discussion

Doing the Activity

Ask students to list the specific cases of the diver's release that they have looked at.

- The 9 o'clock position, in *Big Push*
- The 3 o'clock position, in *Three O'Clock Drop*
- The 12 o'clock position, in *High Noon*

Ask, **What is special about these cases?** Someone will probably recognize that these are all situations in which the diver's initial velocity is either purely vertical or purely horizontal.

Remind students that in *High Noon*, although the diver's initial velocity was horizontal, his overall movement was both vertical and horizontal. **How did you work with this combination of horizontal motion and the downward force of gravity?** Students should be able to articulate that they treated the two parts of the diver's motion separately, using these two principles:

- The diver moves sideways as if there were no gravity.
- The diver moves down as if there were no sideways motion.

Review that according to both theoretical and experimental physics, we can always treat these two parts of motion as if they were completely separate. Introduce the phrases *vertical component of velocity* and *horizontal component of velocity* for these two parts of an object's motion.

Go over how this idea works in the context of *High Noon.* Ask, **What was the vertical component of the diver's velocity 1 second after he was released? What was the horizontal component?** Students should see that the vertical component would be the same as if he had fallen from rest, –32 ft/s. (This velocity is negative because he is moving downward.) The horizontal component doesn't change during the diver's fall, so it is the same as when he started, –7.85 ft/s. (This velocity is negative because he is moving to the left.)

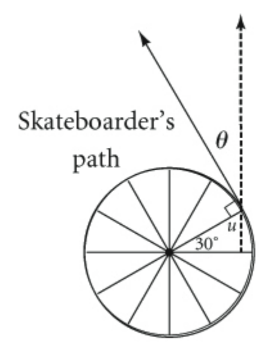
Make sure students realize that for the vertical component, they need to be thinking about instantaneous velocity, because this component is changing at each second. You might point out that this change in the vertical component explains why the diver's path gets steeper as he falls.

For Question 2, you might remind students that the skateboarder's path is tangent to the circle, meaning it is perpendicular to the circle's "2 o'clock" radius.

Discussing and Debriefing the Activity

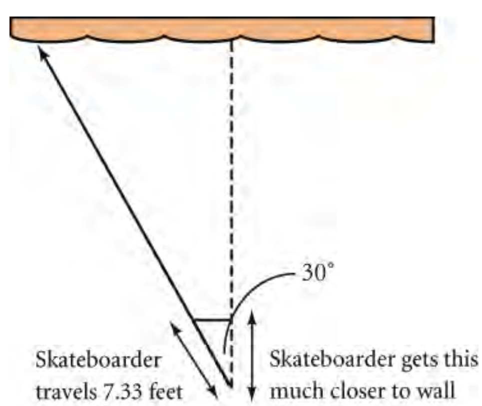
Question 1 is like a similar question for the Ferris wheel. The skateboarder's speed is approximately 7.33 ft/s.

Question 2 contains an important element: the determination of the angle. Students may find it helpful to use a diagram of an overhead view of the situation, like the one at right.



One way to find angle θ is to first find angle u and then solve the equation $u + 90° + \theta = 180°$ (which uses the fact that the tangent is perpendicular to the radius). Once students see that θ is 30°, mention that it's no coincidence that θ is equal to the angle made between the radius to the 3 o'clock position and the radius to the point of release.

Students might then use a diagram like the next one to see that the skateboarder moves 7.33 cos 30° feet, or approximately 6.35 feet, closer to the wall each second. Identify this as the "toward-the-wall" component of the skateboarder's velocity.



Students will probably answer Question 4 by dividing the distance (30 feet) by the toward-the-wall velocity component (6.35 ft/s). This gives a time of approximately 4.7 seconds for the skateboarder to reach the wall.

Question 5 is intended as a confirmation of the component concept. Students should see that the total distance *d* that the skateboarder travels fits the equation $\cos 30° = \frac{30}{d}$, so $d = \frac{30}{\cos 30°} \approx 34.6$ feet. Dividing this result by 7.33 ft/s again gives about 4.7 seconds.

Whichever way students find this time, point out that cos 30° is part of the computation. The computation for Question 4 is $\frac{30}{7.33 \cos 30°}$, while the computation for Question 5 is $\frac{\left(\frac{30}{\cos 30°}\right)}{7.33}$.

Now have students find the "parallel-to-the-wall" component of the skateboarder's velocity. They will probably see that in magnitude, this component is $7.33 \sin \theta$ ft/s. You might use the small triangle in the diagram above to point out that because the two components are perpendicular, they and the overall speed satisfy the Pythagorean theorem. That is, the values satisfy the equation $(7.33 \sin \theta)^2 + (7.33 \cos \theta)^2 = 7.33^2$.

Key Questions

What is special about these cases (the 3, 9, and 12 o'clock drops)?

How did you work with this combination of horizontal motion and the downward force of gravity?

What was the vertical component of the High Noon diver's velocity 1 second after he was released? What was the horizontal component?

Racing the River

Intent

Students express velocity in terms of horizontal and vertical components.

Mathematics

This activity, which is quite similar to *The Ideal Skateboard,* poses a situation in which two swimmers are racing across a river, one swimming straight across and the other at an angle to the shoreline. Students need to recognize that the speed of the swimmer who is crossing at an angle can be broken into two components: one perpendicular to the shoreline and one parallel to the shoreline. The activity presents the same situation with two different angles so that students can observe that each of the two perpendicular components of the speed is affected by the angle.

Progression

Students work on the activity individually or in groups, followed by a class discussion.

Approximate Time

30 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Materials

Transparency of *Racing the River* blackline master

Doing the Activity

You may want to clarify that students are to assume there is no river current in this situation.

Discussing and Debriefing the Activity

The key idea here is recognizing that a River High swimmer's speed has a "toward-shore" component that depends on the angle at which the swimmer is moving.

Questions 1 and 2

These questions should be fairly straightforward. Students can find the distance for the River High swimmer using either the Pythagorean theorem or trigonometry. This distance (approximately 283 meters) can then be divided by the swimmer's speed (1.5 m/s) to get the time it will take the swimmer to finish the race (approximately 189 seconds). For Question 2, students will probably divide the distance by the time to get the rate (200 m $\div$ 189 s $\approx$ 1.06 m/s).

Once the rate for the New High swimmer has been determined, get at the idea of the River High swimmer's toward-shore component with a pair of questions like these: **How far from the starting shore is the New High swimmer after 50 seconds? How far from the starting shore is the River High swimmer after 50 seconds?**

The first question can be answered by multiplying the New High swimmer's rate (1.06 m/s) by 50 seconds. Help students see that the second question must have the same answer as the first. Use that observation to establish that the River High swimmer is swimming toward the shore at a speed of 1.06 m/s, even though he is actually swimming at 1.5 m/s. Identify this value, 1.06 m/s, as the "toward-shore" component of the River High swimmer's speed.

Question 3

Question 3 is essentially a repeat of Questions 1 and 2, with a different angle. The goal is to elicit the idea that for a given speed, the toward-shore component of the River High swimmer's rate depends on the angle at which the swimmer is moving. You can illustrate this idea using a transparency of the diagram by placing a ruler along the starting shore and moving it gradually toward the finishing shore.

The activity *Swimming Pointers* will revisit this question, using vector diagrams to represent the toward-shore and parallel-to-shore components of the swimmer's velocity.

Key Questions

How far from the starting shore is the New High swimmer after 50 seconds?

How far from the starting shore is the River High swimmer after 50 seconds?

One O'Clock Without Gravity

Intent

Students find the vertical and horizontal components of velocity at a particular point on the Ferris wheel.

Mathematics

The Ideal Skateboard and *Racing the River* introduced students to the concept of breaking a speed into perpendicular components. Now they apply this to a simple case of the Ferris wheel situation, under the simplifying assumption that there is no gravity. The activity asks what the vertical and horizontal components of the diver's initial speed would be if he were released from the wheel at the 1 o'clock position, which requires students to determine the angle of the diver's initial motion at that clock position.

Progression

The activity is introduced with a brief discussion of how the angle of release might affect the diver's motion. Students then work on the activity individually or in groups. The follow-up discussion brings out that the angle between the diver's path and the vertical direction is equal to the angle through which the Ferris wheel turns before the diver is released.

Approximate Time

15 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

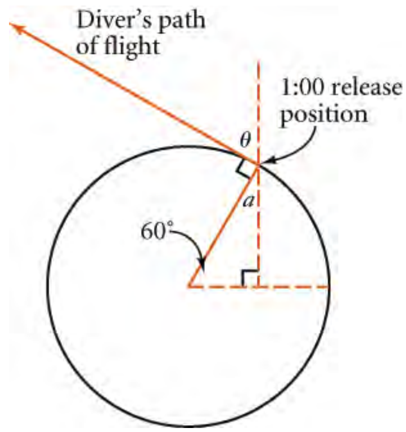
Doing the Activity

In Question 2 of *The Ideal Skateboard*, students found the angle (labeled θ) between the skateboarder's path and a perpendicular to the wall. Ask, **How does an angle like θ figure into the Ferris wheel problem?** Bring out that the diver's release point determines the angle of his initial motion, so it affects the way his motion breaks up into vertical and horizontal components. (You might raise the idea that his initial speed is the same no matter when he is released. Similarly, students could find the skateboarder's speed without first finding θ.)

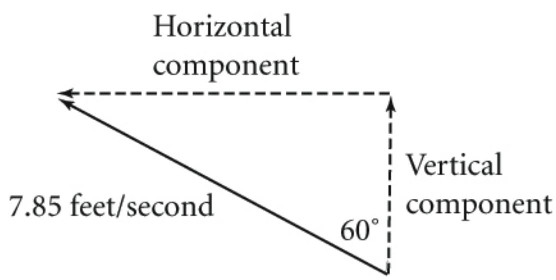
With this introduction, have students work on the activity.

Discussing and Debriefing the Activity

Have volunteers share their ideas. The key to the problem is recognizing that the angle between the diver's path and the vertical direction (labeled θ in the diagram) is the same as the angle through which the diver has turned, which is the angle between the horizontal radius and the radius to his release point. (That angle of turn is 60°, because the diver is released at the 1 o'clock position.) The reasoning is the same as in *The Ideal Skateboard*, and you might again note that the angle between the path and the vertical is the same as the angle between the radius and the horizontal.



Once students find angle θ, the work is similar to what they did in *The Ideal Skateboard*. They might use a diagram like the next one to find the components.



Students should see that the vertical component has a magnitude of 7.85 cos 60°, so the diver is rising at about 3.9 ft/s. The horizontal component has a magnitude of 7.85 sin 60°, so the diver is moving to the left at about 6.8 ft/s.

Ask, **How do we show that horizontal travel is to the left or right?** Students should recognize that this horizontal motion is to the left, so the horizontal component of velocity is –6.8 ft/s.

You may want to save the values for these components of velocity for comparison with the results from Question 1 of Velocities on the Wheel.

Key Questions

How does an angle like θ figure into the Ferris wheel problem?

How do we show that horizontal travel is to the left or right?

Swimming Pointers

Intent

Students express a velocity vector as a vector sum of its vertical and horizontal components.

Mathematics

This activity recasts Question 3 from *Racing the River* in the context of vectors, which students first encountered in *Complex Components*. Students use vector diagrams and terminology together with trigonometry to resolve a vector into its vertical and horizontal components and then show how these combine to reconstruct the original vector.

Progression

Students work on the activity individually or in groups and share findings in a class discussion.

Approximate Time

15 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

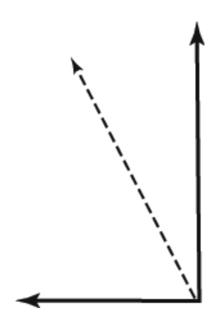
Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

Discussion of Questions 1 and 2 can be fairly brief. In Question 1a, the toward-shore component can be found by multiplying the overall speed by cos θ, where θ is the angle between the actual path and the direct route. That is, for each 1.5 meters the swimmer moves at an angle θ from the direct route, he or she gets closer to shore by 1.5 cos θ meters. You might have students consider the special cases $\theta = 0°$ and $\theta = 90°$ to confirm this result.

The main point of Question 3 is the vector diagram. Use this opportunity to review the various ways of constructing a diagram to show a vector sum. Students will probably construct a right triangle showing the tail of one component affixed to the head of the other. Ask, **Does it matter which vector you use first in the diagram for adding two vectors?** Students should be able to create two congruent triangles, thus suggesting that vector addition is commutative. Also bring out that the vector representing the sum of two other vectors can be seen as the diagonal of a rectangle, as in this diagram.



This is a more specific case of the sum of any two vectors (not necessarily perpendicular) being the diagonal of a parallelogram, which students found in *Complex Components*. If the vertical and horizontal velocity components are rounded to the nearest hundredth (1.30 and 0.75 m/s, respectively), the Pythagorean theorem gives 1.50 m/s for the resultant. The next activity, *Vector Velocities*, will generalize these results for any vector pointing northeast.

Key Question

Does it matter which vector you use first in the diagram for adding two vectors?

Vector Velocities

Intent

Students generalize their work in decomposing vectors and then develop the reverse process in the first-quadrant case.

Mathematics

Students develop a general procedure for finding the vertical and horizontal components of any vector featuring an acute angle in standard position. After considering a specific case based on a Ferris wheel, they generalize the reverse process of finding the magnitude and direction of a northeast-pointing vector given its horizontal and vertical components.

Progression

Students work on the activity in groups and share discoveries in a class discussion. In the next activity, *Velocities on the Wheel*, they explore vector decomposition for angles in any quadrant in the context of the Ferris wheel unit problem.

Approximate Time

30 to 40 minutes

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

This activity should need no introduction. Question 1b may stump some students who don't think to factor out v or don't recall the Pythagorean identity from trigonometry.

Students have solved problems like Questions 2 and 3 using triangles but not with vectors. Suggesting that they draw accurate diagrams should help those having trouble.

Question 2c is the most difficult, requiring first some angle matching and then some sort of proportion to convert degrees into hours and minutes.

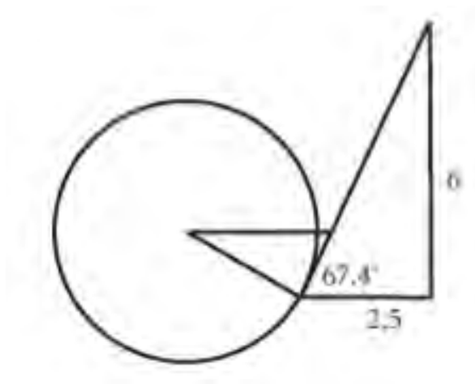
Discussing and Debriefing the Activity

Have a student present Question 1, including a diagram. For part a, you might ask, **Do we know whether the same formulas will work for angles in other quadrants?** (No, we don't yet know; this is one focus of the next activity.) For part b, ask, **For what angles θ is $\sin^2 \theta + \cos^2 \theta = 1$ a true statement?** Be sure students recall that the identity works for all angles θ.

Question 2 involves some new ideas. In part a, the Pythagorean theorem yields exactly 6.5 feet per second. For part b, students will probably evaluate $\cos^{-1}\left(\frac{2.5}{6.5}\right)$

to get 67.4°, though some may use the inverse sine to find the complementary angle, or even the inverse tangent. Encourage students to give a clear description of the direction, not just to name an angle.

Part c involves reasoning about angles similar to that in *The Ideal Skateboard*. A diagram, such as the one at right, is essential.



The rider’s path is tangent to the Ferris wheel and thus perpendicular to the radius at the point of tangency. Drawing a horizontal radius past the circle to the rider’s path creates a triangle similar to the larger triangle, since the path acts as a transversal cutting two parallel (horizontal) lines. Students should explain how this determines the central angle as 22.6°. They might use a variety of proportions to show that this angle corresponds to about 45 clock minutes, so the clock position is 3:45. A totally different approach might involve imagining that you’re riding on the Ferris wheel looking straight ahead (in the direction of motion) and thinking how far you’ve turned (from 3 o’clock) to get to this position.

Question 3 gives students an opportunity to retrace their specific solutions to Questions 2a and 2b and so to describe a general procedure. You might point out that the process of finding the angle becomes more complicated outside the first quadrant (we will not pursue that idea further in this unit).

Key Question

For what angles θ is $\sin^2 \theta + \cos^2 \theta = 1$ a true statement?

Velocities on the Wheel

Intent

Students develop formulas for the horizontal and vertical components of the diver's initial velocity upon release from the Ferris wheel.

Mathematics

This activity requires students to find the relationship between the time that the Ferris wheel has been moving and the horizontal and vertical components of the diver's initial velocity.

Progression

Students work on the activity individually. The follow-up discussion will produce the final key formulas for the central unit problem. Students will confirm in *Release at Any Angle* that these formulas work in all four quadrants.

Approximate Time

20 to 30 minutes for activity (at home or in class)

15 to 20 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Discussing and Debriefing the Activity

Except for the choice of release position, Question 1 is identical to Questions 1 and 2 of *One O'Clock Without Gravity.* Students should show that the platform has turned through a 72° angle. Thus, the vertical component of velocity is 7.85 cos 72°, so the diver is initially moving upward at approximately 2.43 ft/s. The horizontal component of velocity is –7.85 sin 72°, so the diver is initially moving to the left at approximately 7.47 ft/s.

By comparison, in the discussion of *One O'Clock Without Gravity*, students found the vertical component to be 3.9 ft/s and the horizontal component to be –6.8 ft/s. Ask, **Do the differences between this activity's results and the results in *One O'Clock Without Gravity* make sense?** Students should see that the larger angle in *Velocities on the Wheel* means the diver's initial velocity is "more horizontal" and recognize that this is consistent with the results. Sketching vector diagrams may help illustrate the effect.

For Question 2, students should be able to generalize their work from Question 1 and from *One O'Clock Without Gravity* to get the following expressions for the case in which the diver is released within the first quadrant ($0 < W < 10$):

- Vertical component of velocity: 7.85 cos (9W)
- Horizontal component of velocity: –7.85 sin (9W)

The expression $9W$ represents the angle of rotation after traveling for W seconds, as used earlier in the unit.

As with the specific examples, be sure students recognize the significance of the minus sign for the horizontal component. They should see that for angles in the first quadrant, the diver's horizontal movement will be to the left, so this component of velocity needs to be negative.

For Question 3a, students should see (from a diagram or similar analysis) that the vertical component of the diver's velocity is positive if the diver is released in either the first or fourth quadrant, which means W is either less than 10 or more than 30. Similarly, the vertical component of the diver's velocity is negative if W is between 10 and 30, and zero for both $W = 10$ and $W = 30$.

For Question 3b, students should see that the horizontal component of the diver's velocity is positive if W is greater than 20, negative if W is less than 20, and zero at 0, 20, and 40.

You may want to post these conclusions, as they will be referred to in the next activity, *Release at Any Angle*.

Key Question

Do the differences between this activity's results and the results in *One O'Clock Without Gravity* make sense?

Release at Any Angle

Intent

Students find general expressions for the vertical and horizontal components of the diver's velocity in the main unit problem.

Mathematics

In *Velocities on the Wheel,* students developed expressions for the vertical and horizontal components of the diver's initial velocity as he is released from a first-quadrant position on the Ferris wheel. Now they test whether those expressions work in all four quadrants.

Progression

Students work on the activity in groups. In the follow-up discussion, the class confirms that the expressions for the vertical and horizontal components of the diver's initial velocity work in all four quadrants.

Approximate Time

30 to 35 minutes

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

In Question 2 of *Velocities on the Wheel,* students developed expressions for the components of the diver's initial velocity. Thus far, they have considered only the case in which the diver is released within the first quadrant—that is, for $0 < W < 10$. Tell them that they will now examine whether these expressions work for all points of release.

Let students work on the activity in groups. The goal of Questions 1b and 1c is to verify that a graph based on the expression is consistent with an intuitive, qualitative analysis of the situation.

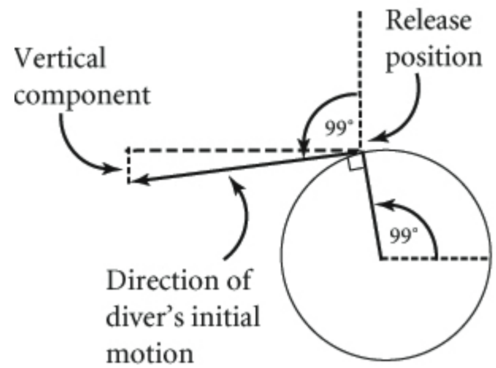
Even if groups have not finished the activity, allow a few minutes at the end of class to discuss at least through Question 1c and to assure the class that the expressions do, in fact, work for all values of W.

Discussing and Debriefing the Activity

Have a student sketch a graph based on the proposed equation for the vertical component of velocity (Question 1a). As a class, verify (see Question 1b) that the graph is consistent with the sign analysis from *Velocities on the Wheel.*

For Question 1c, have another student describe, based on the situation, how the vertical component of velocity changes in size as W increases from 10 to 20. The presenter should note that at $W = 10$, the diver's initial motion is horizontal, so the vertical component is zero, and that as W increases, his motion becomes more directly downward so that the size (absolute value) of the vertical component increases (becoming “more negative”). Again, confirm as a class that this is consistent with the graph.

If time allows and you feel students need the discussion, continue with Question 1d and then Question 2. The key in Question 1d is to confirm that the vertical component of velocity does, in fact, increase in absolute value as *W* increases from 11 to 12.



For instance, for *W* = 11, the angle of turn is 99° (which is 9*W*), and students might use a diagram like the one here. If they view this diagram sideways so that the upward line is like the positive direction of the *x*-axis, then the vertical component of the diver's initial motion is like the *x*-coordinate of a point on the ray showing that motion. Students have already seen, in discussing the Ferris wheel problem, that they can find a point's *x*-coordinate by multiplying its *r*-value by cos θ. Here, the overall speed, 7.85 ft/s, is like that *r*-value, and the vertical component of the diver's initial motion is 7.85 cos 99°. Bring out that this is negative but not very large (in absolute value), just as cos 99° is negative but not very large.

Before concluding, assure students that the formulas do work for all angles, and post the formulas along with the others around the room. To have more precise values available for the final solution of the unit problem, use the exact value of 2.5π ft/s (rather than 7.85 ft/s) as the platform's speed as it turns around on the Ferris wheel.

If the diver is released *W* seconds after passing the 3 o'clock position, his initial velocity will have these components:

- **Vertical component of velocity: 2.5π cos (9*W*)**
- **Horizontal component of velocity: –2.5π sin (9*W*)**

A Portfolio of Formulas

Intent

Students summarize the formulas related to the unit problem.

Mathematics

All formulas related to the unit problem are compiled.

Progression

Students work individually to compile their lists of formulas. In the subsequent discussion, they share formulas but do not talk about how to use them to solve the unit problem. *A Portfolio of Formulas* will be part of the unit portfolio.

Approximate Time

30 minutes for activity (at home or in class)

10 to 15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

If assigning this activity as homework, you may want to give students some time to take notes on what's posted around the classroom.oik.

Point out to students that this activity will be included in their portfolios for this unit.

Discussing and Debriefing the Activity

Here is the list of the key formulas for the simplified problem. (Your class may have a somewhat different list.)

- Diver's height at time of release: $65 + 50 \sin(9W)$
- Diver's horizontal coordinate at time of release: $50 \cos(9W)$
- Distance the diver falls: $57 + 50 \sin(9W)$
- Time the diver falls: $\sqrt{\dfrac{57+50 \sin(9W)}{16}}$
- Time the cart travels until the diver reaches the water level:
  $W + \sqrt{\dfrac{57+50 \sin(9W)}{16}}$
- Cart's horizontal coordinate when diver reaches the water level:
  $-240 + 15\left(W + \sqrt{\dfrac{57+50 \sin(9W)}{16}}\right)$

The first three formulas can be used without change, because they relate only to what happens before the diver is released.

The formula for the time for the diver's fall will change in order to take into account the diver's initial vertical velocity. Students might recognize that they need to combine two new ideas to get the new formula:

- The vertical component of the diver's velocity is $2.5\pi \cos(9W)$, as found in *Release at Any Angle*.
- If the vertical component of the diver's velocity is v, then his falling time is $\frac{v + \sqrt{v^2 + 64h}}{32}$, as found in the discussion of *Falling Time for Vertical Motion.* A better description for this quantity might be "duration of the dive after release," since the diver may actually be rising for the first portion of the dive. Likewise, the expression $57 + 50 \sin(9W)$ is the vertical distance from the diver's point of release to the water level, but he may actually travel a greater total vertical distance during his dive.

The change in the time of the diver's fall will also affect the last two formulas in the list, requiring that the expression $\sqrt{\frac{57 + 50 \sin(9W)}{16}}$ be replaced by $\frac{v + \sqrt{v^2 + 64h}}{32}$, with v replaced by $2.5\pi \cos(9W)$. (Students might identify the quadratic formula as a "new formula" they need, because they used this to develop the falling-time formula.)

Finally, students will need to take into account the horizontal component of the diver's initial velocity. In *Release at Any Angle*, they found that this horizontal component is $-2.5\pi \sin(9W)$.

This horizontal velocity in turn affects the diver's horizontal coordinate when he reaches the water level. In the simplified version of the problem, this was the same as his horizontal coordinate at the time of release. Now, the diver's horizontal coordinate when he reaches the water level depends on his coordinate at the time of release, the horizontal component of his velocity upon release, and the duration of his fall. Students might combine these elements into a single formula or simply give a verbal description of how to combine them.

Moving Diver at Two O'Clock

Intent

Students put together much of what they have learned to analyze a specific situation related to the unit problem.

Mathematics

Students do a complete analysis for a Ferris wheel situation involving both vertical and horizontal initial velocity, finding the diver's falling time and both the diver's and the cart's horizontal positions when the diver reaches the water level.

Progression

Students work in groups to apply the formulas just developed to another Ferris wheel situation. The subsequent discussion provides an opportunity to review the general principles of the unit.

Approximate Time

40 minutes

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Assure students that they can use all the formulas and principles that have been posted around the classroom, as students are sometimes tempted to develop these formulas again.

You will likely want fairly detailed presentations on Questions 1 and 2. You may want to give groups materials to use for preparing these presentations.

Discussing and Debriefing the Activity

For Question 1, students will probably go through steps like these:

- The diver's height when he is released is $65 + 50 \sin 30° = 90$ feet.
- The vertical component of his initial velocity is $2.5\pi \cdot \cos 30° \approx 6.80$ ft/s.
- The diver's height t seconds after release is given by the expression $90 + 6.80t - 16t^2$.
- The answer to Question 1 is the positive solution to the equation $90 + 6.80t - 16t^2 = 8$. Students will probably put this equation in standard form as $16t^2 - 6.80t - 82 = 0$ and solve it using the quadratic formula. The solution is given by the expression

  $$\frac{6.80 + \sqrt{6.80^2 + 64 \cdot 82}}{32}$$

  which gives $t \approx 2.49$ seconds.

Each part of this analysis represents the application of a formula that has been developed at some point in the unit. You may want to identify the relevant posted formula or result as each part is discussed.

For Question 2, the task is a bit simpler, because the falling time (more accurately, the length of time after release) has already been found.

- The diver's *x*-coordinate when he is released is 50 cos 30° ≈ 43.3 feet (to the right of center).
- The horizontal component of his initial velocity is –2.5π · sin 30° ≈ –3.93 ft/s, which means he is moving to the left.
- The diver takes 2.49 seconds to fall to the water level.
- The diver moves about 2.49 · (–3.93) ≈ –9.8 feet while falling, so his position when he reaches the water level is about 43.3 – 9.8 = 33.5, or about 33.5 feet to the right of center.

The analysis for Question 3 will probably go something like this:

- The diver is on the Ferris wheel for 30 ÷ 9 ≈ 3.33 seconds before being released.
- The diver takes 2.49 seconds to fall to the water level.
- Altogether, the cart is moving for 3.33 + 2.49 = 5.82 seconds.
- The cart starts 240 feet to the left of center, so its initial position is –240.
- The cart travels 15 ft/s, so it travels a total of about 5.82 · 15 = 87.3 feet to the right.
- The cart's position when the diver reaches the water level is –240 + 87.3 = –152.7, so the cart is 152.7 feet to the left of center.

Have students summarize what would happen if this scenario actually took place. The diver would land about 33.5 feet to the right of center, while the cart would be 152.7 feet to the left of center, so the cart would miss the diver by 186.2 feet.

The Danger of Simplification

Intent

Students examine the effect of the diver's initial velocity using the value of *W* that worked for the simplified unit problem.

Mathematics

Now students consider what will happen to the diver if they use the result from *Moving Cart, Turning Ferris Wheel* but take into account the diver's initial velocity due to the motion of the Ferris wheel.

Progression

Students work on the activity individually or in groups. The subsequent discussion emphasizes that the diver's initial velocity has a significant effect on the result.

Approximate Time

25 minutes for activity (at home or in class)

20 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

Students can begin on the activity with no introduction. Note that although they may have found a more precise answer, such as 12.28 seconds, for *Moving Cart, Turning Ferris Wheel,* the difference between that value and the value of 12.3 seconds used here does not have a significant impact on the outcome. The "performance error" that emerges in the latter analysis is virtually all due to the diver's initial velocity.

Discussing and Debriefing the Activity

The main task in this activity is finding the answer to Question 1. The analysis is given in some detail here for your reference. Students will need to generalize this process to solve the unit problem, however, so they need to develop the ideas themselves.

Question 1

To answer Question 1, students first need to find the diver's height upon release and the vertical component of his initial velocity. (We also present here his horizontal position upon release and the horizontal component of his initial velocity. Students do not need these values until Question 2, but are likely to find them as part of this preliminary stage of the analysis.)

Although students have formulas for these components, this may be the first time they have used them for a non-first-quadrant case, so you may want to discuss the details. Students should reach these conclusions for $W = 12.3$:

- The vertical velocity component is given by the expression $2.5\pi \cos(9 \cdot 12.3)$, which yields a value of approximately –2.78 ft/s. This is consistent with the fact that in the second quadrant, the diver is moving downward.
- The horizontal velocity component is given by the expression $-2.5\pi \sin(9 \cdot 12.3)$, which yields a value of approximately –7.35 ft/s. This is consistent with the fact that in the second quadrant, the diver is moving to the left.

You may want to bring out that the horizontal component is larger in magnitude than the vertical component, which reflects the fact that for $W = 12.3$, the diver is on the way from the 12 o'clock to the 11 o'clock position, so his movement is primarily horizontal.

Students also need to determine the diver's position at the moment of release.

- His height above the center of the Ferris wheel is given by the expression $50 \sin(9 \cdot 12.3)$, which yields a value of approximately 46.8 feet. This means he is $46.8 + 57 = 103.8$ feet above the water level.
- His x-coordinate (relative to the center of the wheel) is given by the expression $50 \cos(9 \cdot 12.3)$, which yields a value of approximately –17.7. This means he is about 17.7 feet to the left of center.

Next, students need to use these results to find the time for the diver to fall to the water level. They might explicitly write down the expression for his height t seconds after release, $103.8 - 2.78t - 16t^2$, and use the quadratic formula to solve the equation obtained by setting this expression equal to 0. Or they might apply the equation for falling time found in *Falling Time for Vertical Motion:*

$$t = \frac{v + \sqrt{v^2 + 64h}}{32}$$

where v is the vertical component of velocity (–2.78 ft/s, in this case) and h is the distance the diver falls (103.8 feet).

In either case, students should reach this conclusion for Question 1: If the diver is released after 12.3 seconds, he will reach the water level in approximately 2.46 seconds.

Now might be a good time to introduce the variable F to distinguish the falling time from W, the length of time on the Ferris wheel, giving $W = 12.3$ seconds and $F = 2.46$ seconds.

Questions 2 and 3

Once the falling time is determined, the remaining analysis is fairly straightforward. After release, the diver moves to the left at 7.35 ft/s for 2.46 seconds, starting at

$x = –17.7$, so his x-coordinate when he reaches the water level can be found from the expression $–17.7 + 2.46 \cdot (–7.35)$. This comes out to approximately $–35.8$. That is, the diver reaches the water level approximately 35.8 feet to the left of the Ferris wheel's center.

For Question 3, the cart moves to the right at 15 ft/s both while the diver is on the Ferris wheel (12.3 seconds) and while he is falling (2.46 seconds). The cart starts at $x = –240$, so its x-coordinate at the time the diver reaches the water level is found from the expression $–240 + (12.3 + 2.46) \cdot 15$. This comes out to approximately $–18.6$, so at the moment the diver reaches the water level, the cart is approximately 18.6 feet to the left of the Ferris wheel's center.

Question 4

You may want to play up the drama of the conclusion from Question 4. Students should have discovered that the diver misses the cart by approximately 17.2 feet! This demonstrates the significance of the work students have been doing in considering the diver's initial velocity. (Of course, the dimensions of the tub are never specified. Perhaps the tub is wide enough to deal with this level of miscalculation.)

Looking Back

You may want to compare these results with those from *Moving Cart, Turning Ferris Wheel.* In that simplified version of the problem, the diver's x-coordinate throughout his fall was $–17.5$, and his falling time was about 2.55 seconds.

The vertical component of the diver's initial velocity shortened his falling time slightly (from 2.55 seconds to 2.46 seconds), which means the cart is not as far along the track. (It's at $–18.6$ instead of $–17.5$.) But the horizontal component of the diver's initial velocity had a substantial effect. He ends up at $x = –35.8$, instead of at $x = –17.5$, more than 18 feet farther to the left. Ouch!

High Dive Concluded

Intent

In these activities, students solve the unit problem.

Mathematics

These activities relate to solving the unit problem and wrapping up the unit.

Progression

Students solve the unit problem in the culminating activity, *The Diver's Success*. The unit concludes with reflections and unit portfolios.

The Diver's Success

A Circus Reflection

Beginning Portfolio Selection

High Dive Portfolio

The Diver’s Success

Intent

Students solve the unit problem.

Mathematics

Students now have knowledge of all the concepts necessary to solve the unit problem. Their first task will be to synthesize what they have learned in order to formulate an equation that represents the problem situation. The more formidable task may be to enter this complex equation into a calculator and solve it—probably graphically, as the equation defies solution by way of algebraic manipulation.

Progression

Students will work in groups on this activity. They will likely need some assistance, particularly in remembering the mechanics of how to split a large function into several manageable pieces in the calculator.

Approximate Time

65 minutes

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

Have students work in groups, but ask that they complete individual reports. You may want to offer assistance as groups proceed, especially about how to define some functions in terms of others to avoid excessively complex expressions on the calculator.

Discussing and Debriefing the Activity

You may wish to have different groups present various parts of the solution to maintain interest in the process. You might have a group that did not finish the problem present first, and then have other groups follow up to describe how they continued from that point.

If students did not get the correct answer, you might let them rework their reports for homework. Students may want to use their reports in studying for the assessment or preparing their portfolios. The report should be included in everyone’s portfolio.

An Outline of the Solution

The following possible approach to solving the problem is a synthesis of ideas developed over the course of the unit.

As indicated in the activity, $t = 0$ represents the time when the cart begins moving and $t = W$ represents the time at which the diver is released. These variables represent other components of the solution:

- *h:* the diver's height above the water level at the time of release
$$h = 57 + 50 \sin (9W)$$
- *c:* the diver's *x*-coordinate at the time of release
$$c = 50 \cos (9W)$$
- v_y: the vertical component of the diver's velocity when he is released
$$v_y = 2.5\pi \cos (9W)$$
- v_x: the horizontal component of the diver's velocity when he is released
$$v_x = -2.5\pi \sin (9W)$$
- *F:* the duration of the diver's fall from the time of release until he reaches the water level
$$F = \frac{v_y + \sqrt{{v_y}^2 + 64h}}{32}$$

  This expression comes from using the expression $h + v_y t - 16t^2$ for the diver's height above the water level t seconds after he is released and solving the equation $h + v_y t - 16t^2 = 0$. In standard form, this equation is $16t^2 - v_y t - h = 0$, so for the purposes of the quadratic formula, $a = 16$, $b = -v_y$, and $c = -h$. In applying the quadratic formula in this context, we want the "+" portion of the ± sign.

Based on these variables, students can find expressions for other aspects of the problem:

- $W + F$: the total time the cart is moving
- $-240 + 15(W + F)$: the *x*-coordinate of the cart at the time the diver reaches the water level
- $c + v_x F$: the *x*-coordinate of the diver when he reaches the water level

The task is to find the value of W that puts the cart in the right place at the right time. Based on the two expressions just given for the positions of the cart and the diver at the time the diver reaches the water level, we need to find the value of W that solves the equation

$$-240 + 15(W + F) = c + v_x F$$

It turns out that the desired value is $W \approx 11.45$ seconds. If the diver is released 11.45 seconds after the cart starts, he lands in the water about 30.54 feet to the left of the Ferris wheel's base (or 30.53 feet if you use the unrounded value of W).

You may want to ask students to trace this value of W through the problem.

For your convenience, here are the values of the different pieces of the puzzle. These results were found using a more precise value for W of 11.449 seconds. Final values were rounded to the nearest hundredth, but values for expressions such as

$W + F$, $c + V_X F$, and $-240 + 15(W + F)$ were initially found using pre-round-off values for the variables in those expressions.

- $h = 105.71$ (The diver is 105.71 feet above the water level when released.)
- $c = -11.28$ (The diver is 11.28 feet to the left of center when released.)
- $v_y = -1.77$ (The diver has an initial vertical component of velocity of 1.77 ft/sec downward.)
- $v_x = -7.65$ (The diver has an initial horizontal component of velocity of 7.65 ft/sec to the left.)
- $F = 2.52$ (The diver is in the air for 2.52 seconds.)
- $W + F = 13.96$ (The cart travels for 13.96 seconds.)
- $c + v_x F = -30.54$ (The diver is 30.54 feet to the left of center when he reaches the water level.)
- $-240 + 15(W + F) = -30.54$ (The cart is 30.54 feet to the left of center when the diver reaches the water level.)

You may want to have students compare this result with the analyses they did in *Moving Cart, Turning Ferris Wheel* and *The Danger of Simplification.*

Finally, you might want to point out that we have ignored a whole separate problem, which is more a question of physiology than mathematics or physics: **Could the diver survive the dive into the water?**

Key Question

Could the diver survive the dive into the water?

A Circus Reflection

Intent

Students consider situations in which oversimplification might lead to serious consequences.

Mathematics

Simplifying assumptions are often valuable and necessary tools for modeling a real-life situation mathematically. However, as students just saw in *The Danger of Simplification*, an oversimplification can lead to a solution that is not realistic or useful.

Progression

Students answer the questions on their own and share ideas in a class discussion.

Approximate Time

15 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity requires no introduction.

Discussing and Debriefing the Activity

Have students share some of the scenarios they imagined in Question 2.

Beginning Portfolio Selection

Intent

Students begin to select material for the unit portfolio.

Mathematics

The focus of the activity is on the connections among quadratic equations, complex numbers, and vectors.

Progression

Students select activities that helped them see connections among major topics in the unit and then share their ideas in class.

Approximate Time

30 minutes for activity (at home or in class)

Classroom Organization

Individuals

Doing the Activity

This activity requires no introduction.

Discussing and Debriefing the Activity

You might have students discuss their selections and reflections in small groups. You may also want to have a couple of volunteers share which activities they chose and the connections they made.

High Dive Portfolio

Intent

Students reflect upon the key concepts of the unit as they compile their unit portfolios and write their cover letters.

Mathematics

As students pull together all the information they have studied in the unit, they consider how it all fits together to develop the unit's main ideas and to solve the unit problem.

Progression

Students complete their unit portfolios, selecting key activities and writing their cover letters.

Approximate Time

5 minutes for introduction

30 to 40 minutes for activity (at home or in class)

Classroom Organization

Individuals

Doing the Activity

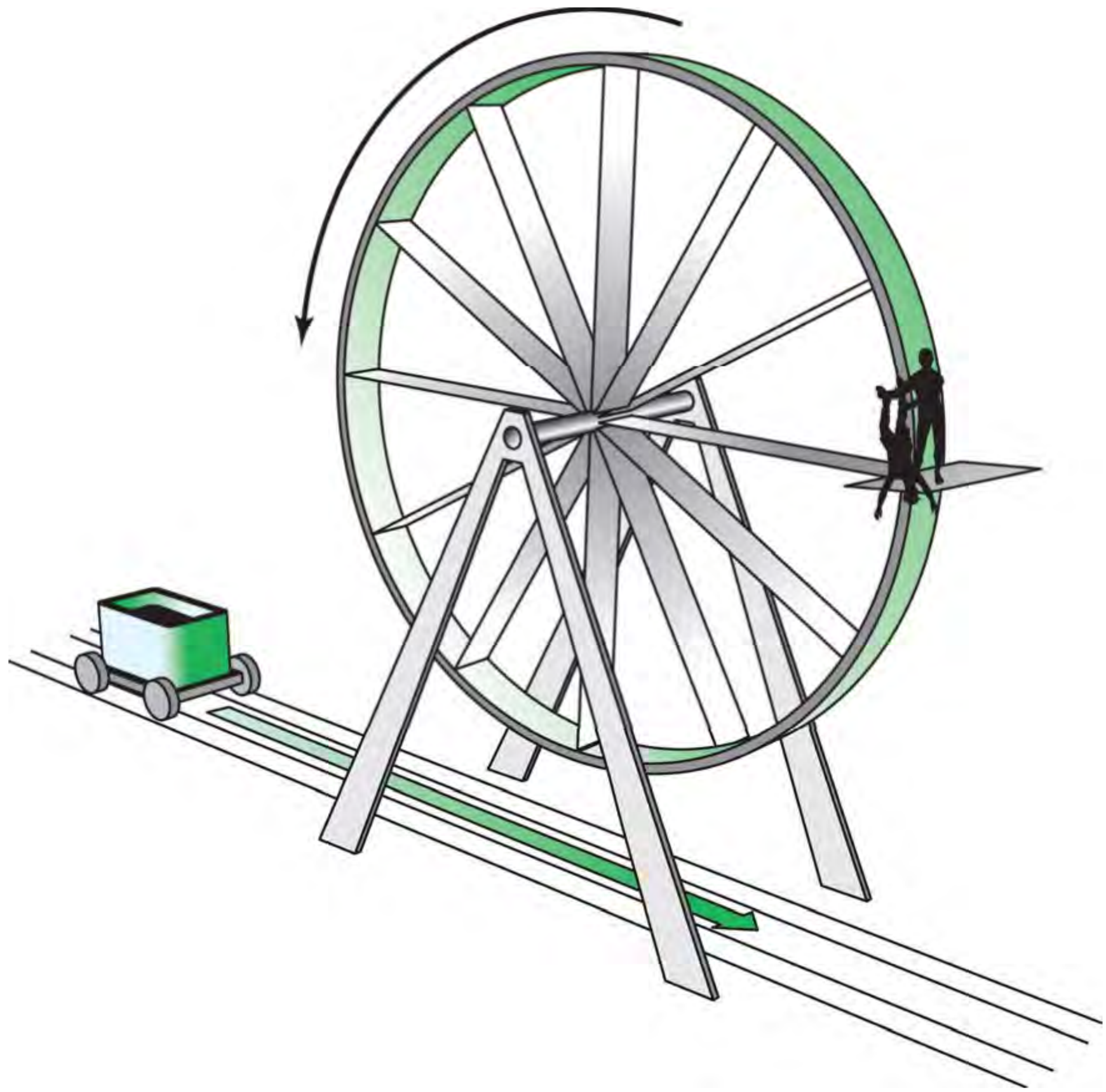
Have students read the instructions in the student book carefully.

Discussing and Debriefing the Activity

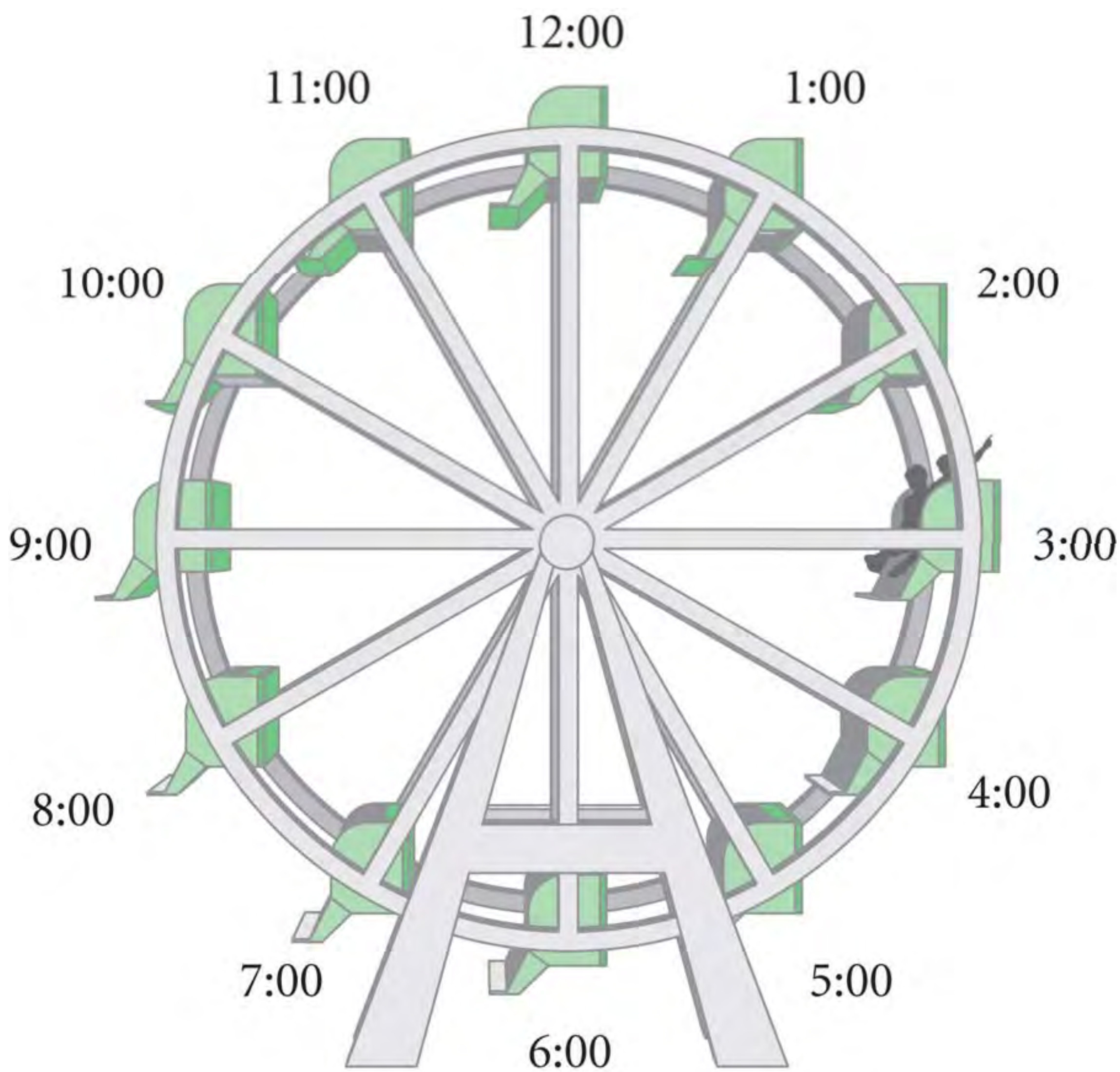
You may want to have a few volunteers read their cover letters to start a discussion summarizing the key ideas of the unit.

Blackline Masters

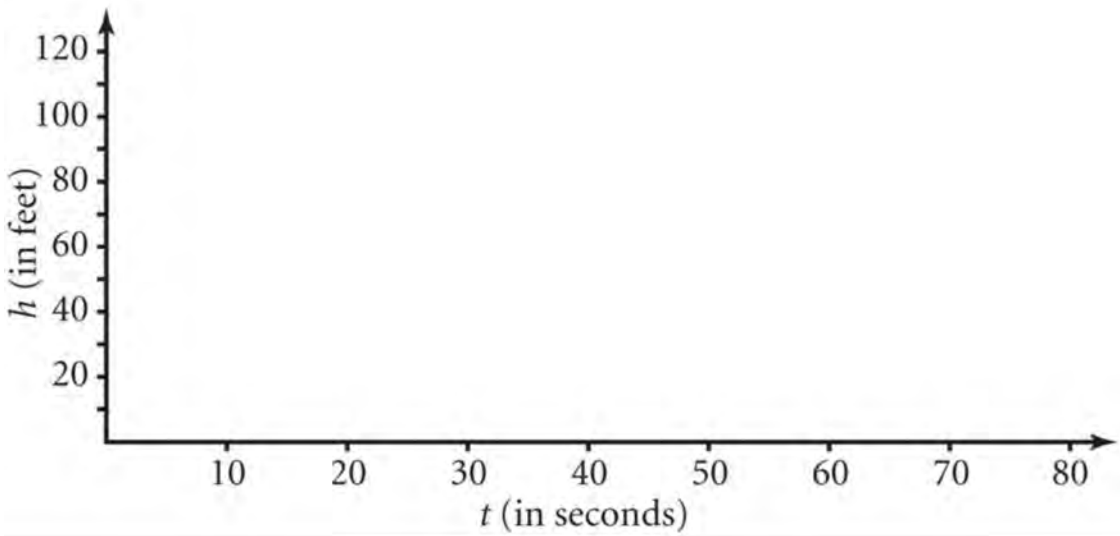
The Circus Act

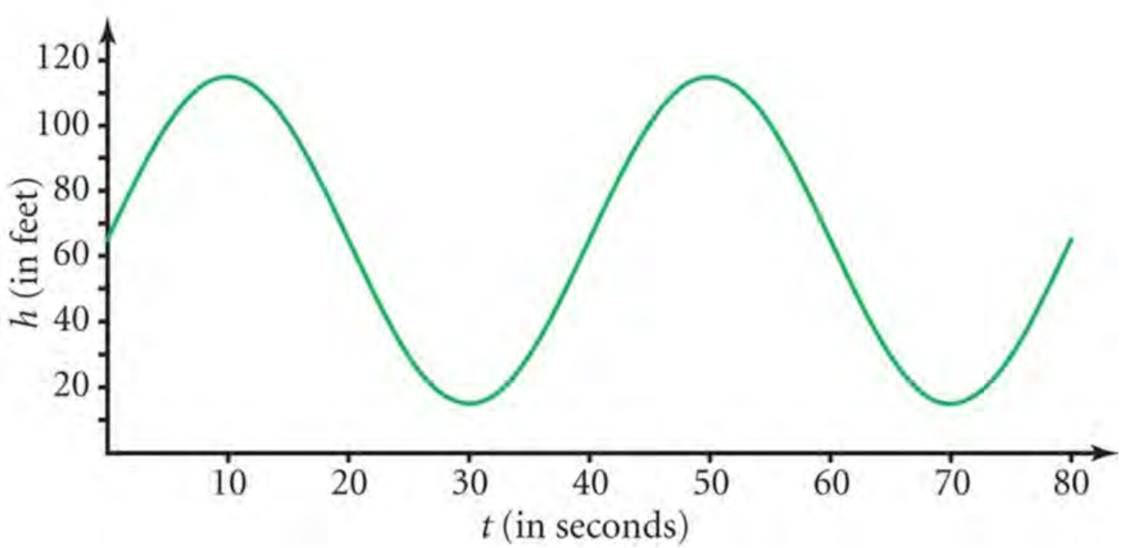


The Ferris Wheel

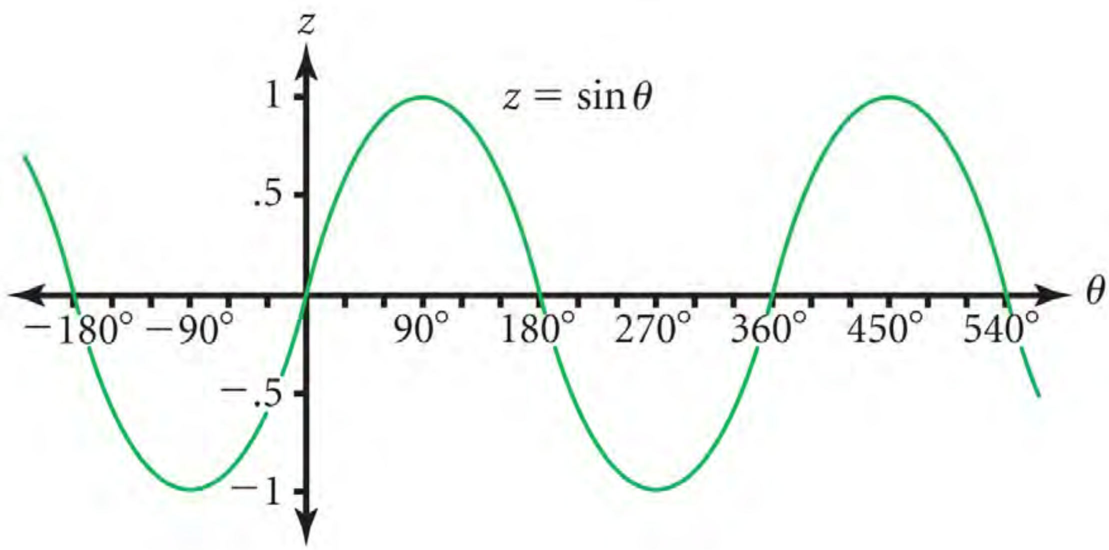


Graphing the Ferris Wheel

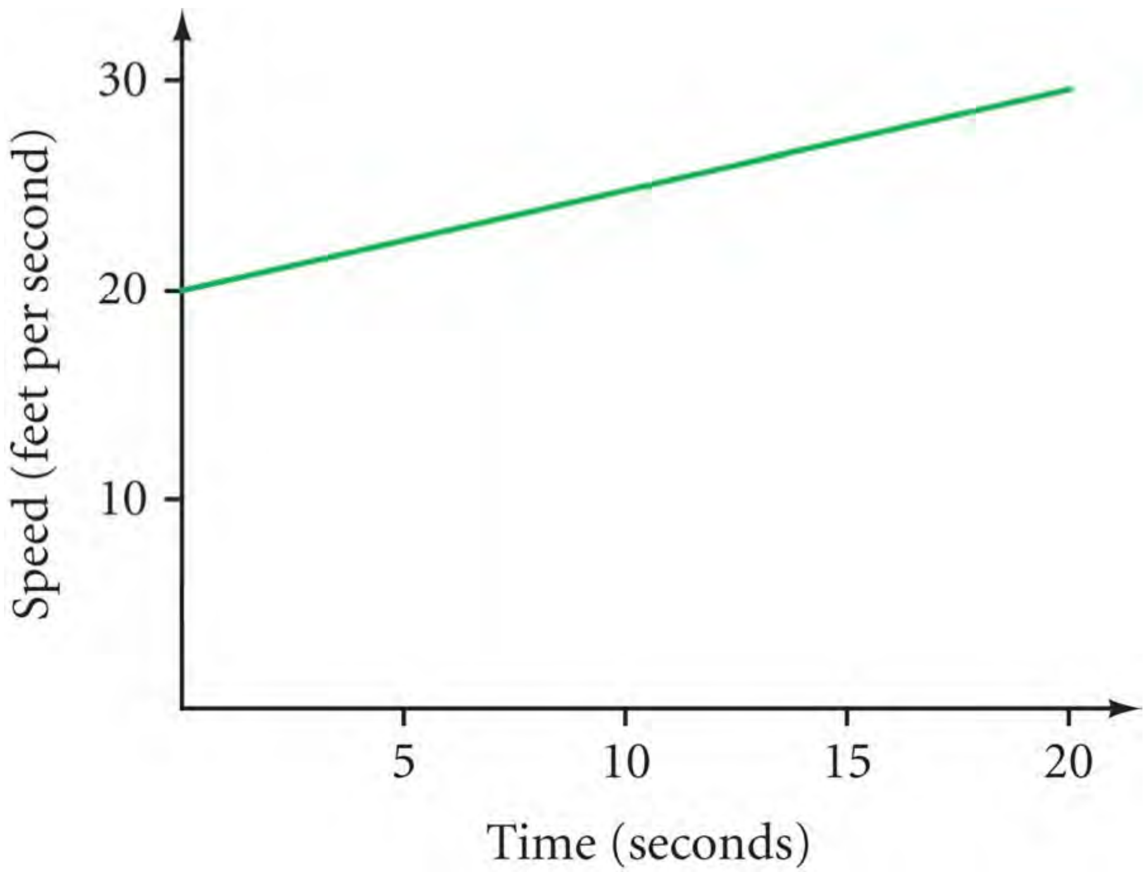




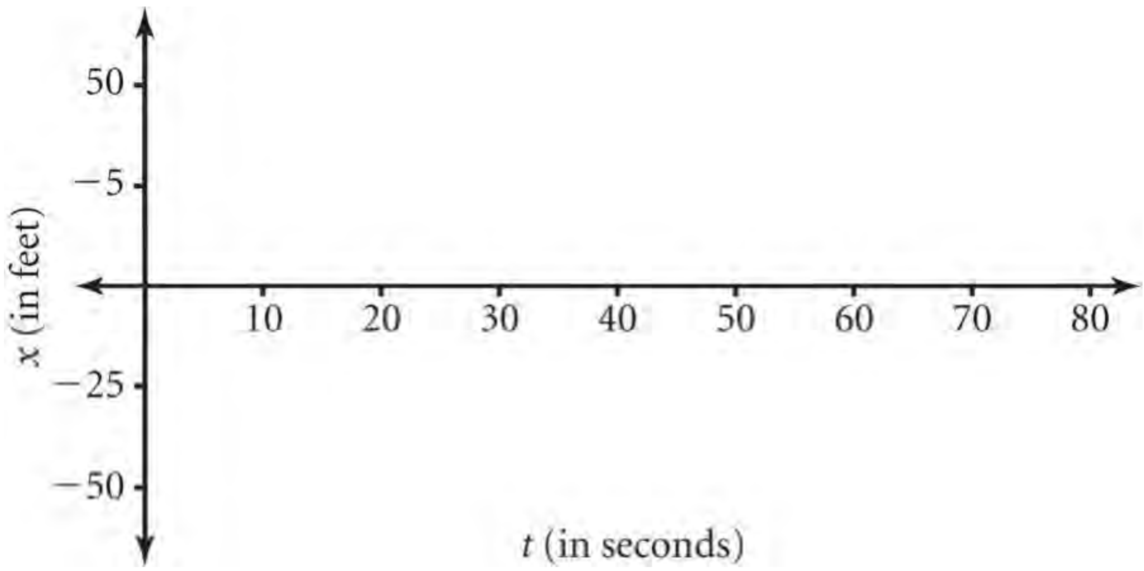
The "Plain" Sine Graph



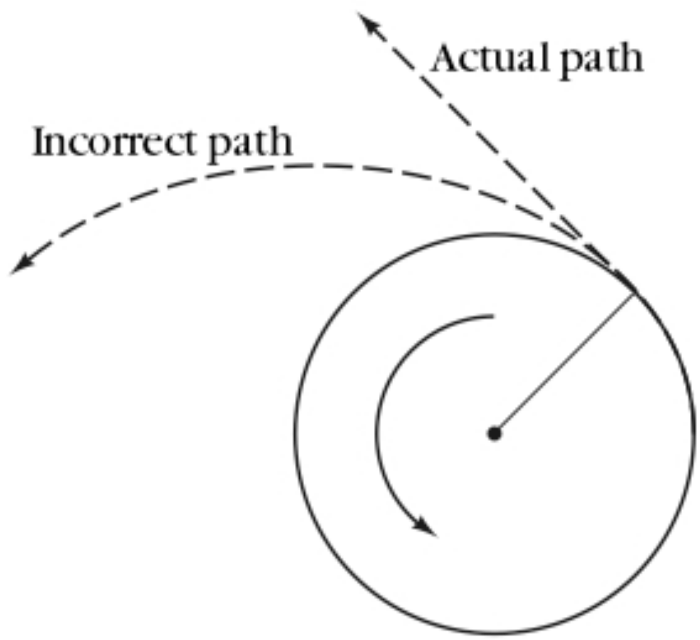
Distance with Changing Speed



Where Does He Land?



Initial Motion from the Ferris Wheel



Racing the River

¼-Inch Graph Paper

1-Centimeter Graph Paper

1-Inch Graph Paper

Assessments

In-Class Assessment

1. Solve the quadratic equation $x^2 + 4x - 3 = 0$ in each of these two ways.

   a. By completing the square

   b. By using the quadratic formula

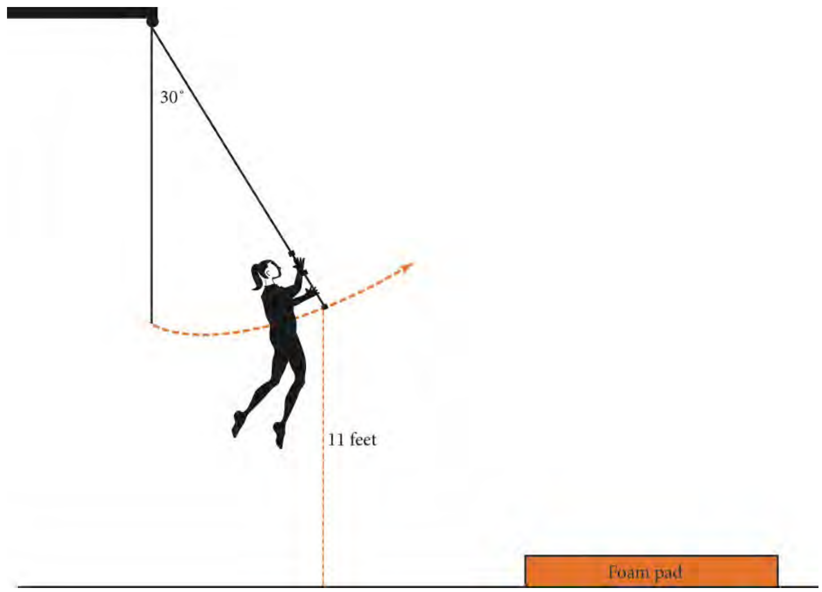
   Show your work in each case. Give your answers in two ways: (1) in exact form using square roots and (2) to the nearest tenth.

2. Sketch a graph of the function $y = x^2 + 4x - 3$, using information from Question 1. Explain how you used that information.

3. a. Show by substitution that both $4 + 3i$ and $4 - 3i$ are solutions to the equation $x^2 - 8x + 25 = 0$.

   b. Explain what this tells you about the graph of the function $y = x^2 - 8x + 25$.

Take-Home Assessment



Sabrina does a swing act in the circus. At the end of her act, she gets the swing going really fast. As she is on the upward part of her swing forward, she lets go and flies up and forward from the swing.

At the moment she lets go, she is moving at 30 feet per second, she is 11 feet off the ground, and the swing is at an angle of 30° from vertical. She will land on a foam pad that is 1 foot thick.

Assume that Sabrina's path after she lets go is based only on her initial velocity and the usual gravitational acceleration rate, with no air resistance or other complications.

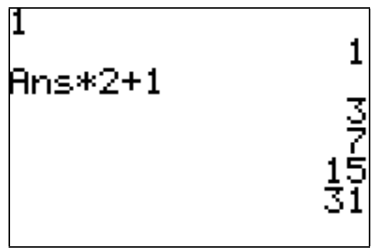
1. From the moment Sabrina lets go, how long will it take until she hits the foam pad?
2. How far forward will Sabrina fly from the point where she leaves the swing until she hits the pad?
3. How fast will she be going when she hits the pad?

Give your answers to the nearest tenth of a second, the nearest tenth of a foot per second, or the nearest tenth of a foot, as appropriate.

High Dive Calculator Guide for the TI-83/84 Family of Calculators

High Dive makes use of what students already know about the graphing calculator and introduces several new techniques as well. The unit problem itself leads to an equation that is far too complex to solve by algebraic manipulations and that instead is solved graphically with the calculator. Even a graphical solution without the calculator would be awkward. Students should gain a tremendous sense of satisfaction at having solved an impossible-looking equation, as well as an appreciation for the power of the graphing calculator.

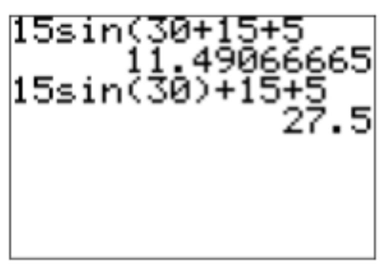
Throughout the unit, students can use the calculator to check graphically what they discover with algebra and trigonometry. The second week provides an appropriate context in which to reveal the CALC menu full of helpful features. As students begin, near the end of the unit, to pull together the many elements of the unit problem, they will learn how to break lengthy functions into smaller pieces for entry into the calculator. After the unit problem has been solved, a parametric demonstration on the calculator can provide both a vivid illustration of their solution and an introduction to parametric equations.



POW 6: The Tower of Hanoi: The recursive pattern for *POW 6: The Tower of Hanoi*, $a^{n+1} = 2a^n + 1$, can easily be illustrated on the calculator. Enter the number of moves needed for a single disc by pressing $\boxed{1}$ $\boxed{\text{ENTER}}$. Find the number of discs for the second move by pressing $\boxed{\text{x}}$ $\boxed{2}$ $\boxed{+}$ $\boxed{1}$ $\boxed{\text{ENTER}}$. Press the $\boxed{\text{ENTER}}$ key again to calculate the number of moves for a tower of three discs. Each time the $\boxed{\text{ENTER}}$ key is pressed, the total for the next larger tower will be displayed.

The Ferris Wheel: When reviewing *The Ferris Wheel,* you might remind students to select $\boxed{\text{MODE}}$ and to check that their calculators are in degree mode. The calculator defaults to radian mode whenever the calculator is reset. This can happen when the batteries are low or are changed. Students should develop the habit of checking the mode if they get results that seem odd, or whenever they use trigonometric functions.

Because students will be using trigonometric functions for many of the activities in this unit, encourage them to explore how the functions work on their personal calculators. For example, on the graphing calculator, you find sin 80° by pressing $\boxed{\text{SIN}}$ $\boxed{8}$ $\boxed{0}$ $\boxed{\text{ENTER}}$, in that order. On most scientific calculators, the order is reversed; that is, the angle must be entered before the sine function is selected.

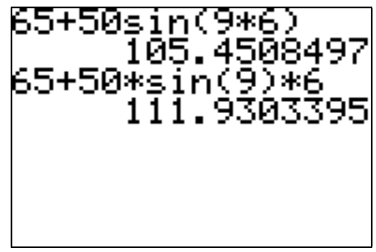


You might also remind students to be careful with placement of parentheses. As shown here, the graphing calculator automatically opens a set of parentheses when a trigonometric function is selected. Unless those parentheses are closed, the calculator assumes closure at the end of the line. Compare the results to the two expressions entered here.

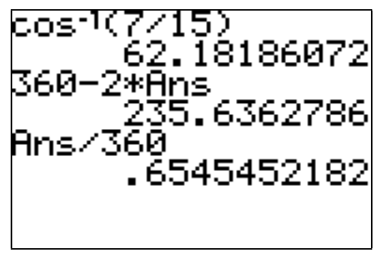
As the Ferris Wheel Turns: Although an approximation of 3.14 for π may be close enough for this activity, remind students that π is an irrational number and can be obtained more accurately by using $\boxed{\text{2ND}}$ $\boxed{\pi}$ on the calculator. The activity *Falling Bridges,* in the unit *Do Bees Build It Best?,* illustrated that small round-off errors can produce dramatic differences in the final result under some circumstances. Encourage students to develop calculator habits that minimize those differences.

In both *As the Ferris Wheel Turns* and the subsequent activity, *At Certain Points in Time,* placement of parentheses may again become an issue.

Students need to remember to close parentheses that the calculator opens automatically. Thus, in Question 4b of *As the Ferris Wheel Turns,* the parentheses cannot be omitted from **65+50sin(9*6)**. Compare the results of the calculations shown in this display.



A Clear View: Discussion of *A Clear View* offers an excellent opportunity to remind students of the ease with which previous answers can be recalled for use in the calculator, rather than working with rounded approximations. It will make only a slight difference in this activity, but these habits will be important in other instances, particularly for students pursuing careers in science or engineering.



Graphing the Ferris Wheel: After bringing out that individual points for Question 1 of *Graphing the Ferris Wheel* can best be found with the equation $h = 65 + 50 \sin 9t$, ask students how they went about constructing their graphs. Most probably created an In-Out table in preparation for graphing. Ask how they might have used their calculators to create this table more quickly.

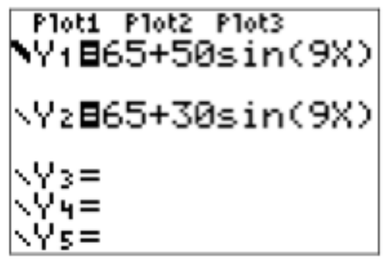
Instructions for using the TABLE feature is provided in the Calculator Note "Creating Tables on the Calculator." In Step 4, students should discover that it is necessary to set **TblStart** to zero. A value of 5 for **ΔTbl** will yield a sufficient number of values at convenient intervals for graphing by hand.

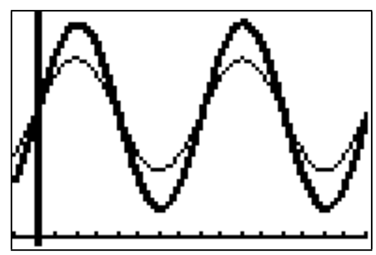
After viewing students' graphs for Question 1, have them graph this function on their calculators. (But we recommend that calculator graphs for the situations in Question 2 not be generated until the discussion of *Ferris Wheel Graph Variations*.) Students should see that the graph matches the one they created by hand. The procedure for graphing this on the calculator should be familiar to most students, but instructions follow for those who need review or may not have experience with the graphing calculator. See the Calculator Note "Graphing Functions."

The set of values **Xmin=–5, Xmax=80, Ymin=–5,** and **Ymax=120** will yield a nice window for display of this graph for Question 1, but resist the urge to simply give these values to students. They need the experience that comes from thinking about what the two variables represent and deciding on an appropriate range of values. If necessary, remind them that they made this same decision when they decided how to scale the graphs they drew by hand.

While discussing the periodicity of the graph in Question 1, you might have a student use the graph to illustrate what a period of 40 seconds means. This can be seen approximately on the calculator simply by pressing TRACE and using the left and right arrows to move the cursor along the graph. You may not be able to find *x*-values that are exactly 40 units apart in this way, due to the size of the increment assigned to each pixel on the screen, but a good approximation should suffice. Students can also use the number keys to enter the exact *x*-coordinate in which they are interested, after pressing TRACE. The Calculator Note “Tracing Tips” presents instructions for obtaining values for the window variables that yield a friendlier window for tracing.

Ferris Wheel Graph Variations: The calculator provides an ideal way to compare the graphs of the functions students develop in *Ferris Wheel Graph Variations* with the graph of the original function. Have students check their work on this activity by graphing each of the functions on the calculator, as described in the Calculator Note “Graphing Functions.” Tell them to enter the original function from *Graphing the Ferris Wheel* as **Y1** in the Y= screen. They should then enter the equation as modified as **Y2**.





If you like, you can select a heavier line for the graph of the original function to more easily distinguish the two graphs. Do this while still at the Y= screen by using the left arrow to move the cursor to the “\” symbol, which is to the left of **Y1**. Press ENTER to cycle through the various line styles. Pressing ENTER only once will select the heavier “****” symbol.

You may also turn off an individual function without erasing it by moving the cursor to the equal sign and pressing ENTER to remove the highlighting. The deselected function will still be evaluated, but it will not be graphed.

The “Plain” Sine Graph: When working on this activity, students will find it helpful to use the TABLE feature of their calculators to obtain the data set to graph. See the Calculator Note “Creating Tables on the Calculator.”

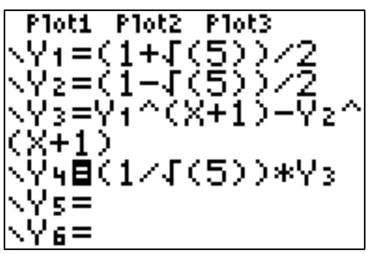
Sand Castles: Graphing the function from this activity will be useful for the discussion. Begin by having students graph $w(t) = 20 \sin(29t)$ on their calculators, with a viewing window using scales similar to those they chose when graphing the function by hand. See the Calculator Note “Graphing Functions.” Point out again that the TABLE feature of the calculator could have been useful for generating the data set for their graphs.

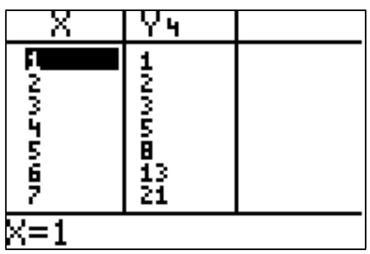
For Question 2, after (and only after) students are able to explain how they can see from the function that the minimum and maximum values must be –20 and 20, let them verify that with the graphing calculator CALC features. Note that we have avoided helping students discover these features until now, because we did not want to short-circuit the learning of the mathematics behind them. But now, an awareness of these calculator tools will be useful as students move beyond high school. See the Calculator Note “Solving ‘Sand Castles’ with the CALC Menu.”

Portions of Questions 3 and 4 are also covered in the Calculator Note “Solving ‘Sand Castles’ with the CALC Menu.” Once again, use this material only after a thorough discussion focusing on a solution based on the properties of the sine function.

Question 5 requires students to find the coordinates of the point 1 unit to the left of the minimum vertex of the curve. Trying to obtain sufficient accuracy while tracing the graph of the function can be frustrating. The instructions in the Calculator Note “Tracing Tips” are worth discussing even if you elected not to introduce the CALC features. They provide several tips for tracing to an exact point with minimal fuss.

POW 7: Paving Patterns: You can use the calculator to demonstrate that the closed form of the equation from *POW 7: Paving Patterns* really does, amazingly, yield the correct sequence. The function has been entered here using the techniques described in the Calculator Note “Graphing a Complicated Equation.” See also the Calculator Note “Creating Tables on the Calculator” for instructions on using the calculator’s TABLE feature.



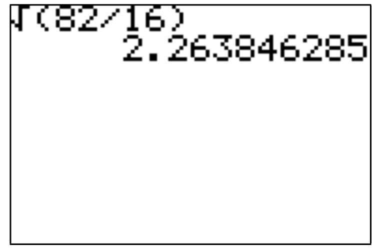


More Beach Adventures: Though the CALC features are useful tools, do not let them become a substitute for working with the powerful mathematics in this activity. The goal here is for students to become thoroughly familiar with the properties of the sine function. Learning to use the CALC features must be secondary. Require students to show the mathematics that supports their solution, even if they check it with the help of the CALC features.

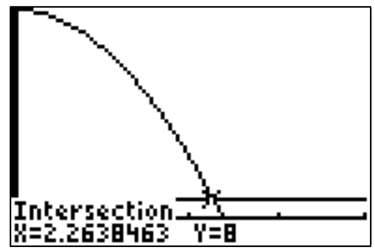
The inverse trigonometric functions will be used frequently throughout the remainder of this unit. Remember that $\sin^{-1} 0.4$, for example, is entered as $\boxed{\text{2ND}}$ [SIN^{-1}] $\boxed{.}$ $\boxed{4}$ $\boxed{\text{ENTER}}$.

The *Teacher’s Guide* suggests that near the end of the discussion of *More Beach Adventures,* you have students see what happens when they ask the calculator to find $\sin^{-1}(2)$. The result is shown at right. What an excellent opportunity to reinforce the vocabulary of *domain* and *range*! Selecting **Quit** will return you to a new line on the home screen, ready for a new operation. Selecting **Goto** will allow you to edit the command that caused the error to occur.

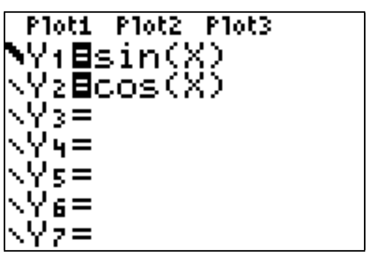
Free Fall: Question 5 of this activity requires students to find an approximate value for $\sqrt{\frac{82}{16}}$. The calculator will automatically open a set of parentheses when the square-root function is selected. If the parentheses are not closed, the calculator will assume closure is at the end of the line.

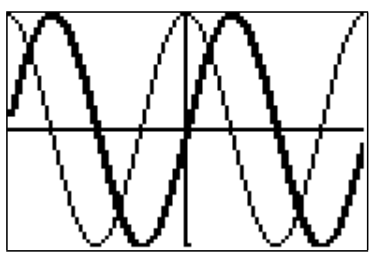


If some students find the answer to this question by using the equation $90 - 16t^2 = 8$, you might ask how they could solve this graphically. If **Y1=90–16X^2** represents the height, they can trace this graph to find the value of **X** that yields a height of 8, or they can find the intersection between this function and **Y2=8**. (See *Question 4: Finding Intersections* in the Calculator Note “Solving “Sand Castles” with the CALC Menu.”)



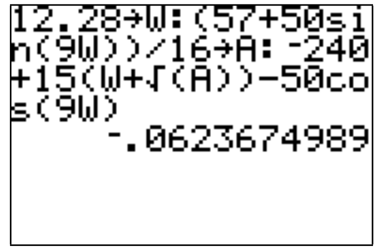
Generalizing the Platform: The *Teacher’s Guide* suggests that you have students graph the cosine function on their calculators and compare it with the posted graph of the sine function (see the Calculator Note “Defining the Cosine Function”). The similarity of the two curves can be demonstrated vividly by having students graph both functions at once on the calculator, entering one as **Y1** and the other as **Y2**. You might also move the cursor to the left of **Y1** in the Y= screen and press ENTER to change the “\” symbol to the thicker “****” symbol. This will cause one function to be drawn with a thicker line, making it easier to distinguish between the two overlapping curves.





Moving Cart, Turning Ferris Wheel: The solution for this activity will involve some complex calculator work, whether it is solved by guess-and-check, graphically, or with the calculator’s SOLVE feature. One of the biggest stumbling blocks will be the correct nesting of parentheses.

Students using guess-and-check will have a difficult time due to the need to enter this complicated equation repeatedly. The Calculator Note “Guess-and-Check Without the Pain” offers some suggestions that involve using multiple commands separated by colons (to avoid the confusion of nested parentheses) and 2ND [ENTRY] (to avoid having to reenter the complex commands for each guess).



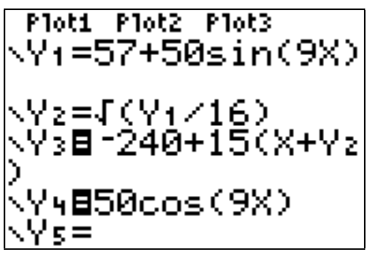
The screen demonstrates the results of a guess of $W = 12.28$ for the equation

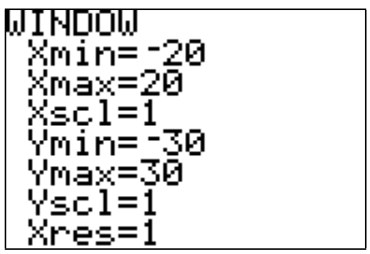
$$-240 + 15\left(W + \sqrt{\frac{57 + 50\sin 9W}{16}}\right) - 50\cos 9W = 0.$$

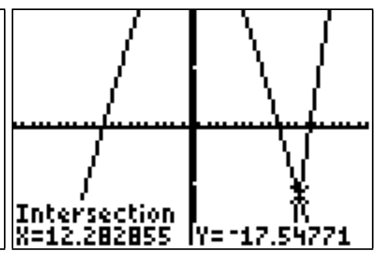
Notice that the approximate guess does not yield a result of exactly zero, but is essentially correct.

Students using a graphical approach will be aided by the techniques described in the Calculator Note "Graphing a Complicated Function," which explains how to break a large function into smaller pieces to reduce confusion. The screens shown here illustrate one way to solve the equation

$$-240 + 15\left(W + \sqrt{\frac{57 + 50\sin 9W}{16}}\right) - 50\cos 9W = 0$$ by looking for the intersection of the graphs of the left and right halves of the equation, represented by **Y3** and **Y4**. Only these two functions are activated for graphing. Once the two functions are graphed, the intersection can be found by tracing or by using the **intersect** feature on the 2ND [CALC] menu.

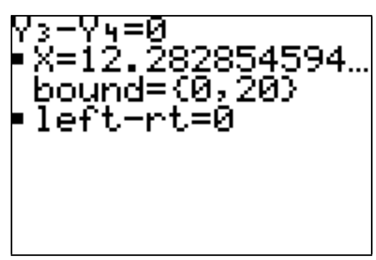






Interpreting the results may be trickier than is initially apparent. Once students have found the coordinates of the intersection, ask them what these two variables represent. Bring out that **X** represents the diver's time on the wheel, in seconds, and is not his *x*-coordinate, and that **Y** represents the *x*-coordinate of the diver when he is released from the Ferris wheel. There will likely be some initial confusion.

This activity is likely to require more than a single class period, so you may need to consider how to avoid having the equations from one class revealed to (or destroyed by) students from another class. One solution is to have students save their functions as a graph database, as described in the Calculator Note "Storing and Recalling a Graph Database." After saving the functions as a graph database, have students clear their functions from the Y= screen.



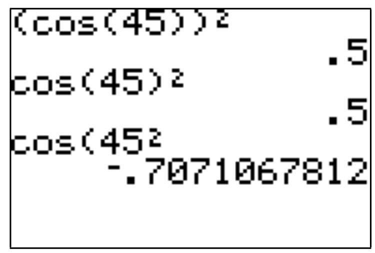
The equation can also be solved using the SOLVE feature. Because the hardest part of this approach is entering the complicated equation, we suggest that you have students do that first, using the instructions from the Calculator Note "Graphing a Complicated Function." Then proceed to the instructions in the Calculator Note "Using the Equation Solver."

You might discuss with other teachers at your school whether the SOLVE feature is something you want to introduce at all. There is no significant danger in familiarizing students with this calculator feature, as long as you require them to show sufficient work on activities. With that requirement, the worst that can happen is that they have another way to check their answers.

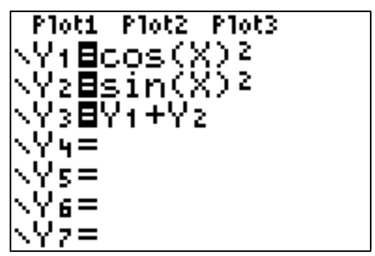
Find the Ferris Wheel: Graphing the two equations from Question 2b on the calculator at the same time is an easy way to verify predictions of the difference. You might use differing line thicknesses, as described earlier, to differentiate between the two curves.

***Some Polar Practice**:* As polar coordinates are introduced, some students are bound to find the ▶**Polar** command (under the **CPX** menu on the MATH key). However, this calculator feature will not do their rectangular-to-polar coordinate conversion for them; it works only when converting complex numbers to polar form. The calculator does have a coordinate conversion feature, which students are less likely to locate, on the **ANGLE** menu. The Calculator Note "Converting Between Coordinate Systems" describes this feature; but, carefully consider the level of understanding your students are demonstrating before introducing this shortcut.

Pythagorean Trigonometry: As the screen here demonstrates, students need to be careful with parentheses when verifying the Pythagorean identity for Question 4. Students should develop the habit of using parentheses to make the desired order of operations clear, as shown in the first line. Note that the calculator will not accept **cos²45**.



The TABLE and GRAPH features of the calculator provide a convenient way of vividly illustrating the Pythagorean identity. Begin by entering the function $\cos^2 x + \sin^2 x$ at the Y= screen in three pieces, as shown here. (**Y1** and **Y2** are entered by selecting them from the **Function** menu, found by pressing VARS and then using the right arrow to highlight **Y-VARS**.)



To use a table to verify the identity, set up the table by pressing 2ND [TBLSET] and then setting the variables to these values:

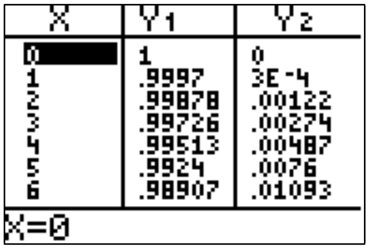
TblStart=0

ΔTbl=1

Indpnt: Auto

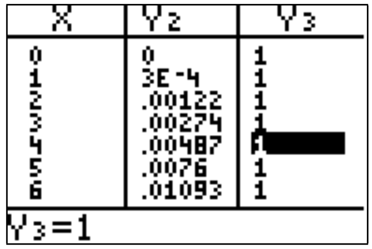
Depend: Auto

Press 2ND [TABLE] to view the table. The angles will be visible in the first (**X**) column and the values for $\cos^2 x$ and $\sin^2 x$ will be visible in the second (**Y1**) and third (**Y2**) columns, as in the first screen. Then use the right arrow to move the cursor to the fourth (**Y3**) column, which is to the right of the columns that are initially visible. This will produce the second screen below. It is quite dramatic to see that $\cos^2 x$ and $\sin^2 x$, with their lengthy decimal values, always add to exactly 1.



X	Y1	Y2
0	1	0
1	.9997	3E-4
2	.99878	.00122
3	.99726	.00274
4	.99513	.00487
5	.9924	.0076
6	.98907	.01093

X=0



X	Y2	Y3
0	0	1
1	3E-4	1
2	.00122	1
3	.00274	1
4	.00487	1
5	.0076	1
6	.01093	1

Y3=1

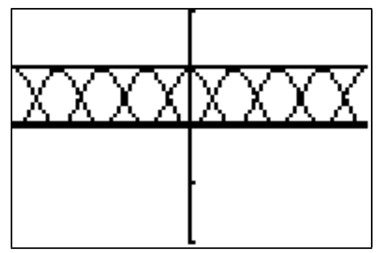
To use a graph to further verify the identity, press WINDOW and set the variables to these values:

Xmin=-360

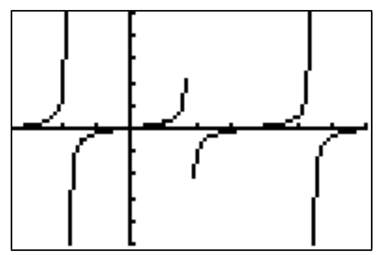
Xmax=360

Ymin=–2

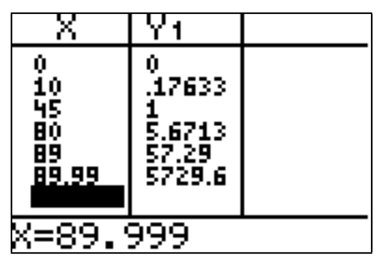
Ymax=2



Press GRAPH to observe the graphs of the three functions being drawn, in the sequence in which they were entered at the Y= screen. Note that the sum of the two sinusoidal curves is always 1. Ask students to explain why both curves are drawn entirely above the x-axis, when they have learned previously that the sine and cosine curves are both centered about the x-axis.



Coordinate Tangents: If students use the calculator to view the graph of the tangent function from Question 4 of, they are likely to be misled. The calculator may show some, but not necessarily all, of the asymptotes as solid lines. At the requested scale, it also truncates the tangent curve as it gets very close to the asymptote.



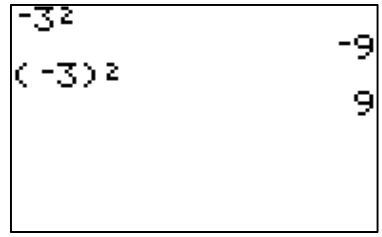
X	Y1
0	0
10	.17633
45	1
80	5.6713
89	57.29
89.99	5729.6

X=89.999

In discussing what happens to the tangent function as it approaches an asymptote, you can use the calculator table after entering the function into the Y= screen. Press 2ND [TBLSET] and set **Indpnt** to **Ask** and **Depend** to **Auto**, using the arrow and ENTER keys. (In this mode, the other settings will not matter.) Press 2ND [TABLE] and enter the angles at which you wish to evaluate the function.

Have students enter **tan 90** into their calculators and see what happens. They will get a "domain error" warning. Reinforce the terminology of *domain* and *range* by asking what the calculator is telling them in this error message.

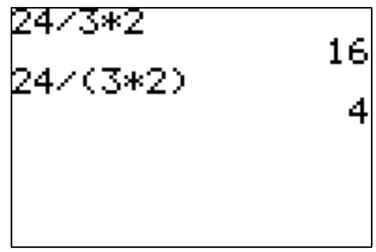
Finding with the Formula: As students begin work on evaluating the quadratic formula, they will need to be careful in three areas relative to their use of the calculator.



First, if $b = -3$, for example, they need to realize that b^2 is not equal to -3^2. As shown here, the calculator correctly performs exponentiation before negation and interprets -3^2 as $-(3^2)$. Encourage students to perform substitution by replacing the variable with an empty set of parentheses and then "dropping" the value of the variable into the parentheses.

If students choose to enter large sections of the formula at a time, they need to be very careful with parentheses. When evaluating $\sqrt{b^2 - 4ac}$, users must close the parentheses that the calculator opens.

When dividing by $2a$, students need to place parentheses around the denominator of the formula. For example, if the formula has been simplified to $\frac{24}{3 \cdot 2}$, a brief review of the order of operations will confirm that entering **24/3*2** will not give the correct result.



If students enter the entire formula at once, they also need to use parentheses around the entire numerator. Even then, it is easy to forget to close both sets of parentheses—those around the numerator and those around the radicand—before moving on to the denominator.

If you have time, let students talk you through the process of generating a simple calculator program for evaluating the quadratic formula. Then have them enter either the programs they have generated or the program given in the Calculator Note "Programming the Quadratic Formula." Basic programming instructions are also provided in the Calculator Note "Entering a Program into the Calculator," and can be used before students enter either their own program or the one provided.

At first, it might not seem like a good idea for students to have a program for evaluating the quadratic formula at this point. However, as long as you continue to require them to show their work for each problem, there are actually several benefits from doing so. First, it is a wonderful demonstration of the power of the quadratic formula compared to factoring or completing the square, one that lends itself to programming and works with numbers that may not be "friendly." Second, the simplicity of the program, combined with the powerful result, often ignites students' interest in programming. It is eye opening to discover that one can write and enter a program that will evaluate the formula in a fraction of a second, in about the same amount of time it takes to evaluate the formula once by hand. Finally, students who have the program available to check their work actually tend to get more practice with the formula while doing their assignments, not less. As long as they are still required to show their work, the program merely gives them a way to check their answers. When their answers are wrong, the program makes them aware that there is an error in their work, and they usually spend more time locating the error, instead of simply quitting.

The general outline of the program should look something like this:

- Enter the coefficients *a, b,* and *c,* and assign them to variables.
- Evaluate the radical expression.
- Evaluate the entire formula, using a + sign in front of the radical expression.
- Evaluate the entire formula again, using a – sign in front of the radical expression.
- Display the answers.

The program presented in the Calculator Note will give an error message and stop if it encounters a negative value for the radicand. After students learn more about complex numbers later in this unit, you may want to suggest that they modify the program to handle that situation more gracefully. In the meantime, it is better to leave the program simple, impressing students with the relative ease of adapting the quadratic formula to a calculator program.

Release at Any Angle: The TABLE feature of the calculator will again be useful for obtaining the data items to be graphed in *Release at Any Angle*. If your students need review of this feature, see the Calculator Note “Creating Tables on the Calculator.”

The Diver’s Success: Function-graphing techniques will be useful in this activity. If students need review, see the Calculator Note “Graphing a Complicated Function”

The screens shown here illustrate one possible arrangement of the equations to solve the unit problem. The functions are arranged in the same order in which they are presented in the Discussion section of *The Diver’s Success* in the *Teacher’s Guide*. Using the notation from that discussion, we have these functions:

$\mathrm{Y}_1 = h$

$Y_2 = c$

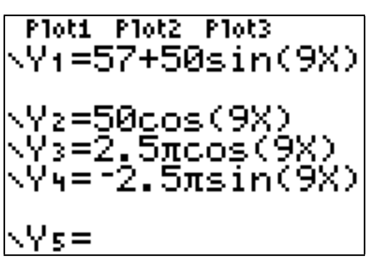
$Y_3 = vy$

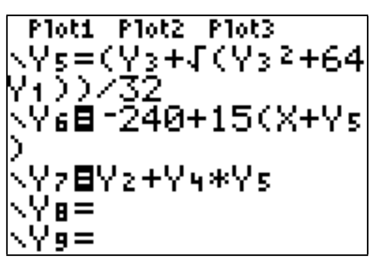
$\mathrm{Y}_4 = \mathrm{vx}$

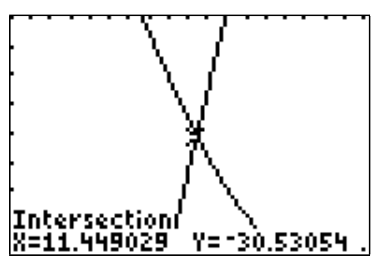
$Y_5 = F$

$Y_6 = -240 + 15(W + F)$

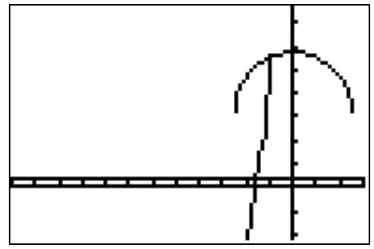
$Y_7 = c + v_xF$







As your class summarizes the unit, you may wish to demonstrate the situation using the following calculator program, which uses parametric equations to show in “real time” what happens with various release times.



There’s no need for students to enter this program or even to fully understand how the program works. You might briefly introduce the concept of parametric equations and let students see the equations. Basically, the position of a moving object is broken down into two functions, the *x*-coordinate as a function of time and the *y*-coordinate as a function of time.

In the program, Function 1 represents the platform’s position, Function 2 represents the cart’s position, Function 3 represents the diver’s position as if falling at that moment, and Function 4 represents the diver’s position as either on the platform or falling, depending on the release time.

To start, press PRGM, use the right arrow to select **NEW**, press ENTER to select **Create New**, and enter the name of your program. We suggest **HIGHDIVE** as a name. If students need help with the basics of program entry, have them review the instructions in the Calculator Note “Entering a Program into the Calculator.”

Press ENTER after your program name and enter these commands. Of course, you can always simply download the program onto your computer and then onto your calculator, or transfer between calculators by linking.

Instruction Explanation

:Degree Sets the calculator to degree mode. Press MODE, highlight **Degree**, and press ENTER. This inserts the **Degree** command into your program. Press ENTER again to advance to the next program line.

:Param Sets the calculator to parametric mode. Select **Par** from the MODE screen as described previously.

:Simul Tells the calculator to graph all functions simultaneously, rather than sequentially. Select **Simul** from the MODE screen.

:Input "DROP TIME?",D Prompts the user to enter the time at which the diver is dropped (in seconds) and stores this drop time as variable *D*. To find **Input**, press PRGM and use the right arrow to move to the **I/O** menu. Press ENTER to select **1:Input**. The space between **DROP** and **TIME** is an ALPHA character located above the 0 key. If you use [A-LOCK], press ALPHA to shift back out of the ALPHA mode in order to enter the comma near the end of the line.

:Input "SPEED .1 TO .5?",Tstep Prompts the user to select the speed at which the graph will be drawn, and assigns this value to the variable **Tstep**. (This will be the time increment at which the functions are evaluated and graphed.) To enter **Tstep**, press VARS. From the menu, press

$\boxed{\text{ENTER}}$ to select **1:Window**. Use the right arrow to highlight **T/θ** and then press 3 to select **3:Tstep**.

:0→Tmin Sets the lower time limit for which the graph will be drawn. The → represents the $\boxed{\text{STO>}}$ key. To find **Tmin**, press $\boxed{\text{VARS}}$ $\boxed{\text{ENTER}}$ $\boxed{>}$, similar to **Tstep**.

:20→Tmax Sets the upper time limit for which the graph will be drawn. To find **Tmax,** start by pressing the $\boxed{\text{VARS}}$ key, as described previously.

:-240→Xmin This group of commands set the window range and axes scale for the graph. Be sure to use the $\boxed{\text{(-)}}$ key for the negative sign. **Xmin** and the related variables that follow are found under the **Window** menu after pressing $\boxed{\text{VARS}}$.

:60→Xmax

:20→Xscl

:-10→Ymin

:120→Ymax

:20→Yscl

:FnOff Turns all functions off to keep them from graphing prematurely when the **ZSquare** command is selected in the next line. To find **FnOff**, press $\boxed{\text{VARS}}$, use the right arrow to select **Y-VARS**, and then press $\boxed{4}$ followed by $\boxed{2}$.

:ZSquare Adjusts the window range to ensure that the circle of the Ferris wheel will not be distorted. Press $\boxed{\text{ZOOM}}$ $\boxed{5}$.

:"50cos(9T)"→X_{1T} Specifies the *x*-coordinate of the platform. Notice that the left side of the command must be enclosed in quotation marks. Use $\boxed{\text{ALPHA}}$ [T] to enter the **T**. To enter X_{1T}, press $\boxed{\text{VARS}}$ and then use the right arrow to select **Y-VARS**. Then select **2:Parametric** and select the variable from the resulting menu.

:"65+50sin(9T)"→Y_{1T} Specifies the *y*-coordinate of the platform.

:"-240+15T"→X_{2T} Specifies the *x*-coordinate of the cart.

:"8"→Y_{2T} Specifies the *y*-coordinate of the cart.

:"50cos(9D)-(2.5πsin(9D))*(T-D)"→X_{3T} Specifies the *x*-coordinate of the diver as if falling.

:"65+50sin(9D)+(2.5πcos(9D))*(T-D)-16(T-D)2"→Y_{3T} Specifies the *y*-coordinate of the diver as if falling. Use the $\boxed{x^2}$ key for the exponent.

:"X_{1T}***(T<D)+**X_{3T}***(T≥D)"→**X_{4T} Specifies the *x*-coordinate of the diver. The two tests (in parentheses) return a value of 1 if true and 0 if false. Thus, if $T < D$, X_{4T} is set equal to X_{1T}. If not, it is set equal to X_{3T}. Do not omit the multiplication symbols in this line. Implied multiplication (such as X_{1T}**(T<D)**) will not work in this case. To find the inequality symbols, press $\boxed{\text{2ND}}$ [TEST] and then select the appropriate symbol.

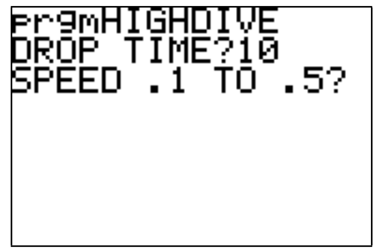
`:"Y1T*(T<D)+Y3T*(T≥D)"→Y4T` Specifies the *y*-coordinate of the diver.

:FnOff 3 Turns off graphical display of Function 3, so that Function 4 controls the display of the diver's position.

:DispGraph Displays the graph. Press PRGM, use the right arrow to display the **I/O** menu, and then select **4:DispGraph**.

Press 2ND [QUIT] to exit the editing mode. To run your program, press PRGM and select your program under the **EXEC** menu by highlighting the program name and pressing ENTER. When the program name appears on the home screen, press ENTER again.

When prompted, enter the desired time at which the diver is to be released (in seconds). The next prompt asks you to enter a value between 0.1 and 0.5, which will control the speed at which the graph is drawn. Larger numbers will yield faster animation, but smaller numbers will make it easier to see if the diver and the cart arrive at the same moment.



After running the program, press Y= to display the parametric equations and lead a brief discussion of how parametric equations work. When you are finished, you may want to press MODE and return the calculator to the function and sequential modes.

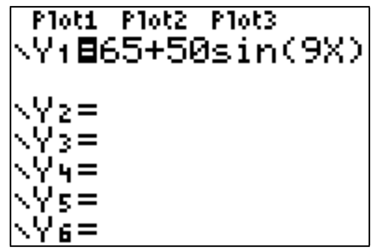
Supplemental Activities:
A Polar Exploration: This activity asks students to investigate graphs of polar equations. This exploration can be accomplished very nicely on the graphing calculator. The instructions in the Calculator Note "Graphing Polar Equations" will help students get started.

High Dive Calculator Notes for the TI-83/84 Family of Calculators

Creating Tables on the Calculator

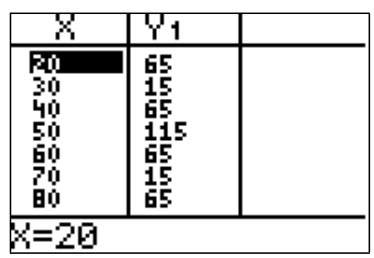
Throughout this unit, you will find it helpful to create In-Out tables as preparation for constructing graphs by hand. The calculator's TABLE feature can make this process much easier.

These instructions use as an example Question 1 of *Graphing the Ferris Wheel*. In that question, you graph the first 80 seconds of a Ferris wheel platform's movement, which is defined by the equation $h = 65 + 50 \sin 9t$.

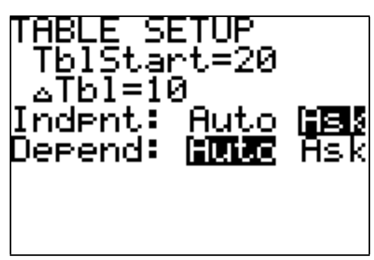


1. Press Y= and enter the equation to be graphed. The calculator uses the variables **X** and **Y** in the Y= screen, so you must substitute these variables for *t* and *h*.
2. Because you are using a trigonometric function, press MODE and select **Degree**.
3. Press 2ND [TBLSET] to access the table setup screen.

4. Enter the first input value for your table at **TblStart**. Enter a value for **ΔTbl** (read "delta table"), which will be the size of the steps between entries in the input column of your table. For example, if you select **TblStart=20** and **ΔTbl=10**, the entries on the left side of your table will be 20, 30, 40, and so on. Select values for these variables that will yield a sufficient number of values for a smooth graph over the range specified in the activity. (The values 20 and 10 for **TblStart** and **ΔTbl** are not necessarily the best choice.)
5. Use the arrow and ENTER keys to highlight **Auto** for both **Indpnt** (the independent variable) and **Depend** (the dependent variable).

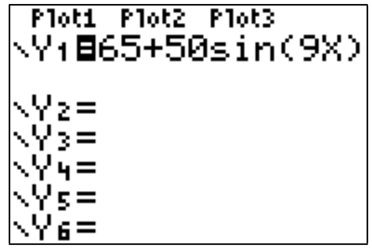


6. Press 2ND [TABLE] to view your table. Use the up and down arrows to scroll through the table to values that lie above or below the values visible on the screen. (The cursor must be in the **X** column to scroll upward.)
7. If you want to view only a few individually selected values, return to 2ND [TBLSET] and set **Indpnt** to **Ask**. Now press 2ND [TABLE] once more. Enter a value for **X** and press ENTER, and the corresponding value for $\mathbf{Y_1}$ will appear. The values for **TblStart** and **ΔTbl** will have no effect in this mode.



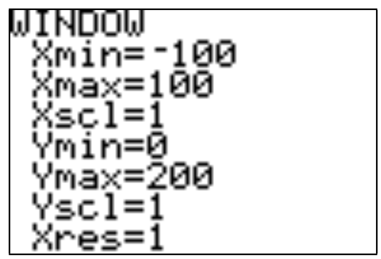
Graphing Functions

The following instructions describe how to graph functions, using as an example the function developed for Question 1 of *Graphing the Ferris Wheel*. For that question, you graph the height of a particular Ferris wheel platform for the first 80 seconds of movement. That graph is defined by the equation
$h = 65 + 50 \sin 9t$.



1. Press Y= and enter the equation to be graphed. Remember that the calculator uses the variables **X** and **Y** in the Y= screen, so you must first substitute these variables for *t* and *h*.
2. Because you are using a trigonometric function, press MODE and be sure to select **Degree**.
3. Press WINDOW to adjust the viewing window.

    a. Set **Xmin** and **Xmax** to the smallest and largest values of **X** in which you are interested. (These are defined by the question.) Choosing **Xmin** to be negative and **Xmax** to be positive will allow you to view the *x*-axis with your graph. Set **Ymin** and **Ymax** to include the smallest and largest values that you expect to see for **Y1**. This will require a little more thought. Don't be afraid to guess at the values; you can always make adjustments after viewing your graph. The values shown at right are not your best choice.



    b. **Xscl** and **Yscl** determine the intervals at which the calculator places the tick marks along the axes. Because no numerical scale will appear beside these marks, it usually makes little difference what values you enter here.

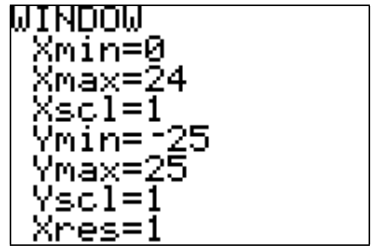
    c. The setting, **Xres**, should be set to **1** to obtain maximum accuracy in your graph. Higher values, up to **8**, allow the graph to be drawn more quickly, but less accurately. (For example, a value of **5** for **Xres** would cause the calculator to evaluate and draw the function only at every fifth pixel.)

4. Press GRAPH to view your graph. If you find stray dots or lines on the screen or encounter a **DIM MISMATCH** error, you probably need to turn off STAT PLOT. This is most easily done by pressing 2ND [STAT PLOT] and selecting **4:PlotsOff**. Press ENTER when **PlotsOff** appears on your home screen.

Tracing Tips

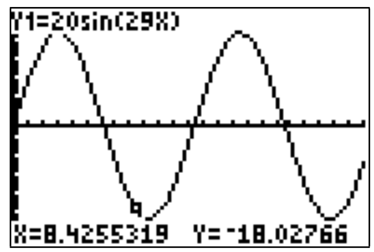
Tracing a curve on a calculator graph to obtain coordinates with the desired degree of accuracy can be challenging. These instructions offer several suggestions, using Question 5 from *Sand Castles* as an example. In that question, you need to find the coordinates of the point that is on the function **Y1=20sin(29X)** and is one unit to the left of a point where **Y1** has a minimum.

1. Graph the function, with the window settings shown here.

   In Question 2, you found that a minimum occurs at $29t = 270$, because this makes sin $(29t)$ equal to -1. Solving that equation reveals that t is approximately 9.31 at the minimum. What you need to find, then, is where $t = 8.31$.

2. Press TRACE. The cursor will appear on the graph of your function, and coordinates for the cursor's location will appear at the bottom of the screen. (If the coordinates do not appear, press 2ND [FORMAT], use the down arrow to move the cursor to **CoordOn**, press ENTER to select it, and then press TRACE to return to your graph.)

3. Using the left or right arrow key, try to find **x=8.31**. You will find that you can get close but cannot obtain exactly 8.31. Due to the size of the pixels, the values entered for the window range caused each pixel to represent a step of about 0.255. This is an awkward increment with which to work. You may have solved this difficulty in the past by repeatedly selecting **ZoomIn**.

4. However, there is a much easier way. Press the TRACE key, then press 8 . 3 1 ENTER. The cursor will move to that exact value, and the coordinates will be displayed.

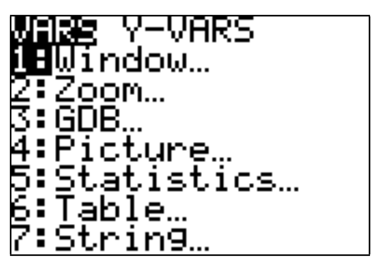
5. Alternately, here is an approach that will give you a friendlier window within which to trace.

   a. Press 2ND [QUIT] to return to the home screen.

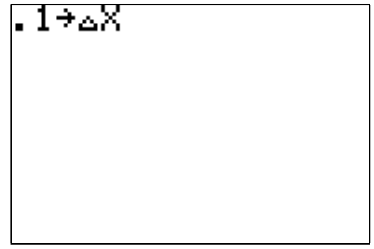
   b. Press . 1 STO> but do not press ENTER yet.

   c. Press VARS to bring up the **VARS** menu. Press ENTER to select **1:Window**.

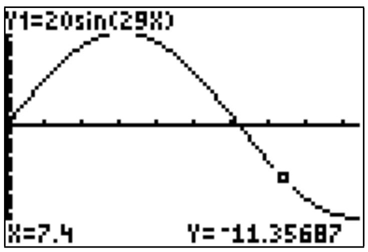
   d. Select **8:ΔX** by pressing the 8 key or by scrolling down and highlighting it and then pressing ENTER.

e. Your home screen will now look like the one shown here. Press $\boxed{\text{ENTER}}$. You have just told the calculator to change the **Xmax** window setting such that each pixel represents an increment of exactly 0.1.



f. Press $\boxed{\text{TRACE}}$ and use the left or right arrow key to locate **X=8.3**. Experiment with tracing to 8.31 instead of 8.3. You could do this by setting **Xmin** to 0.01 before changing **ΔX** to 0.1. You could also do it by changing **ΔX** to 0.01, but this will enlarge the graph so much that you will have to set **Xmin** much closer to 8.3 to be able to view the desired point.



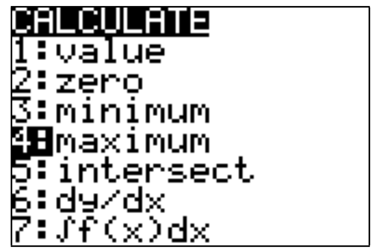
Solving "Sand Castles" with the CALC Menu

Sand Castles provides an excellent opportunity to learn to use some powerful calculator features to which you have not previously been exposed. Work through the suggestions below.

Question 1: Graphing the Function

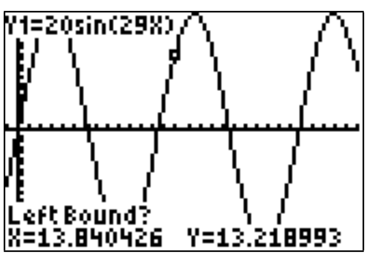
Graph the function $w(t) = 20 \sin(29t)$ on your calculator. Set up the viewing window to match the one you used in drawing the graph by hand. Verify that the calculator graph looks like the one you drew by hand.

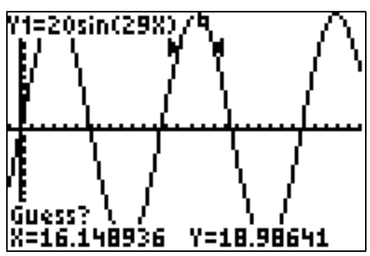
Question 2: Finding the Maximum and Minimum



To find the maximum value of the function, press 2ND [CALC]. Select **maximum** by pressing 4 or by highlighting it with the down arrow key and then pressing ENTER.

The calculator will prompt you to select the left boundary. Use the left or right arrow to move the cursor to any point on the curve to the left of one of the peaks, and press ENTER. The calculator will then prompt you to select a right boundary. Move the cursor to the right of the same peak and press ENTER. Finally, the calculator will ask you to make a guess. Move the cursor near the peak of the curve and press ENTER once more. In a moment, the calculator will display the coordinates of the maximum that you selected. Verify that this matches what you obtained from the equation.



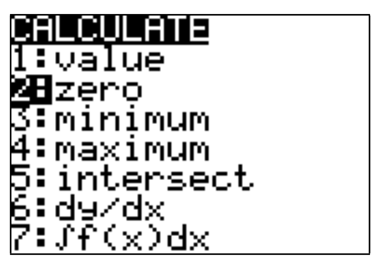


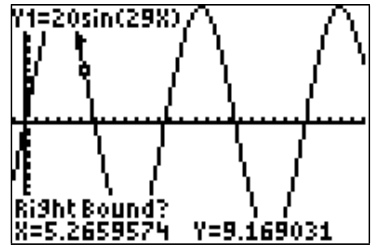
Use 2ND [CALC] to find the minimum in a similar manner.

Question 3: Finding the Zeros

To answer Question 3, you need to find the coordinates of two points where the function has a value of zero. These points need to be immediately to the left and right of a portion of the curve where the y-value is negative, so that the interval between them will represent the time in which the tide is below the average water line. Once again, the calculator provides a shortcut.

Go to the **CALCULATE** menu by pressing 2ND [CALC] and select **2:zero**.

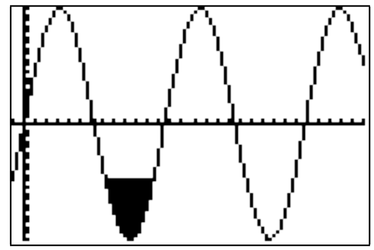




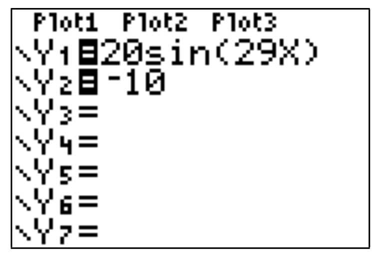
As before, you will be asked to select a left and a right boundary and to make a guess. Find the first zero by selecting boundaries on either side of the point where the curve first drops below the *x*-axis. Guess a value that is near the point of intersection with the axis.

Find the second zero in a similar manner. Verify that these times agree with your calculations or estimations from Question 3.

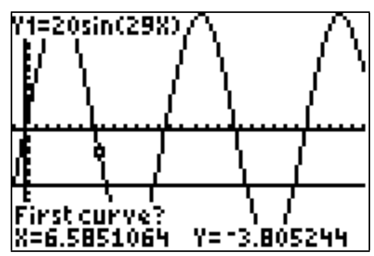
Question 4: Finding Intersections



Question 4 requires that you find the time interval during which the function yields values less than –10. This will be the width of the shaded area in the illustration at right.



Return to the Y= screen and enter a second equation, **Y2=-10**. Press GRAPH, and both functions will be drawn on the same screen. The points you need to find are the two intersections of the curve and the line, which correspond to the left and right edges of the shaded area in the illustration above.



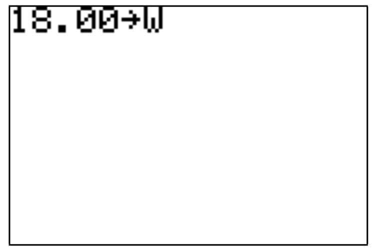
Press 2ND [CALC] and select **5:intersect**. The calculator will return to the graphing screen and will prompt you to select **First curve**. Your cursor will be located on the curve for **Y1**. Press ENTER. You will then be prompted to select **Second curve**. Your cursor will be located on the line for **Y2**. Press ENTER. Finally, you will be asked to make a guess. Place the cursor near one of the desired intersections and press ENTER again. The coordinates of the intersection will be displayed. Repeat this process to find the second intersection.

Guess-and-Check Without the Pain

In trying to solve *Moving Cart, Turning Ferris Wheel,* you have developed a complicated equation that defies algebraic solution. One way to solve it is through guess-and-check, but even that may be quite difficult. Placing parentheses within parentheses on the calculator quickly gets confusing, and having to reenter the entire equation for each new guess is tedious. The tips here will help you deal with these two difficulties.

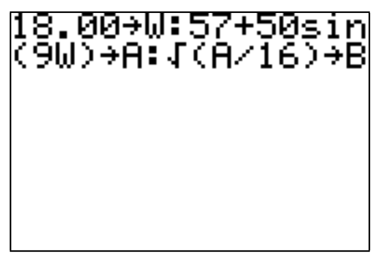
Your equation probably includes the expression $\sqrt{\frac{57+50\sin 9W}{16}}$ or something similar. These instructions use this expression as an example. You will need to expand what is explained here to include your entire equation.

1. Enter an initial guess for the value of your variable into your calculator's home screen. Use as many digits as needed to give the level of accuracy you want for your answer. Do not press ENTER. Instead, after your guess, press STO> ALPHA [W]. This stores your guess as the variable *W*. (You can use any variable to store information, but be aware that graphing a function will cause a value stored as *X* to change, so you should not use *X* here.)



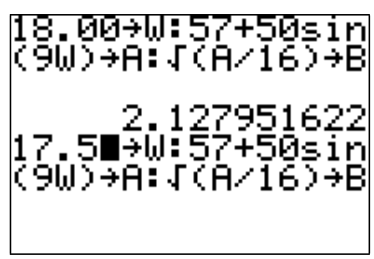
2. Separate the command you entered in step 1 from the next one with a colon by pressing ALPHA [:].

3. Enter a portion of your function that does not require complicated parentheses and store it as variable *A*. Follow that with a colon. Continue to enter your function a piece at a time, without pressing the ENTER key, using variables to replace expressions as needed for simplicity. An example for the expression $\sqrt{\frac{57+50\sin 9W}{16}}$ is shown here (with *B* representing the full expression); your function will be more complex. You may find it helpful to put your equation into a form in which all expressions containing the variable are on one side of the equation. In that way, you will only have to do a single calculation to check your guess.



4. Press ENTER and check the result against the other side of your equation.

5. To adjust your guess, you do not need to reenter the entire function. Simply press 2ND [ENTRY] to recall the function to the screen, and then use the arrow keys to move to your guess. Edit the guess and press ENTER. Repeat as necessary.



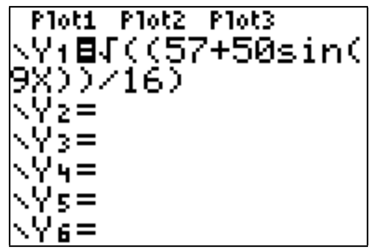
Graphing a Complicated Function

The hardest part about graphing a complicated function is entering it into your calculator without order-of-operation mistakes. If you try to enter the function you developed for *Moving Cart, Turning Ferris Wheel,* you may find it difficult to determine where the many sets of parentheses that are needed should be opened and where they should be closed. These instructions use the expression

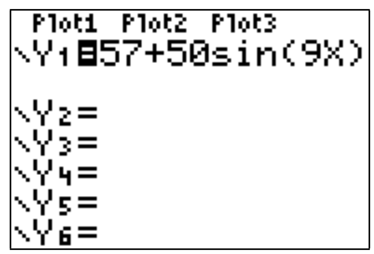
$$\sqrt{\frac{57+50\sin 9W}{16}}$$

as an example of how to do this without confusion. You will need to expand what is explained here to include your entire equation.

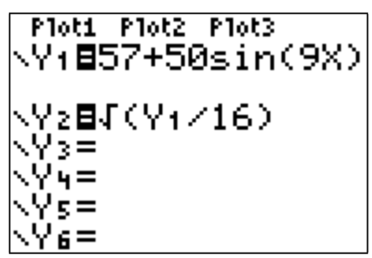
We could enter the expression as shown here, but notice how confusing the nested parentheses become. And this is only a part of the function you need to graph for *Moving Cart, Turning Ferris Wheel*. Fortunately, the calculator provides an easier way.



1. Press [Y=]. Choose a part of your function that can be entered without using more than one set of parentheses. If your function already contains parentheses, the expression inside of those parentheses is a good place to start. Enter this expression as **Y_1**. In the example, we have started with the numerator of the expression within the radical symbol. Remember that the calculator requires that you substitute **X** for your variable.



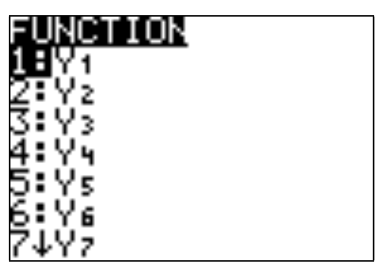
2. Enter another piece of your function as **Y_2**. This can either be a completely separate piece or involve doing something more to the expression you entered as **Y_1**. In our example, we can now simplify the original function by replacing **57+50sin(9X)** with **Y_1**. The complicated expression



$$\sqrt{\frac{57+50\sin 9W}{16}}$$

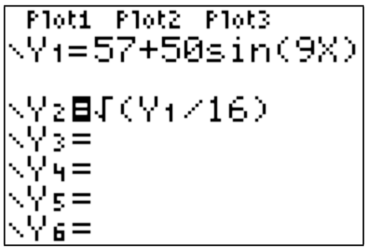
which was shown earlier as **√((57+50sin(9X))/16)**, can now be entered more simply as **√(Y_1/16)**.

To insert one function within the definition of another function, press [VARS] and use the right arrow to select the **Y-VARS** menu. From the **Y-VARS** menu, select **1:Function** by pressing the [ENTER] key. Finally, select **Y_1**, or whichever other variable you need.



For a more complicated function, you may have to split it into three or four pieces, using $\mathbf{Y_3}$ and $\mathbf{Y_4}$.

3. As the functions are defined now, pressing the $\boxed{\text{GRAPH}}$ key would cause a curve to be drawn for each of the two functions we have used. But $\mathbf{Y_1}$ is merely a part of our original function; it is not something we want to graph. To tell the calculator not to graph $\mathbf{Y_1}$, use the arrow keys to move the cursor onto the equal sign in front of $\mathbf{Y_1}$. Press $\boxed{\text{ENTER}}$ to remove the highlighting from that equal sign, making this function inactive with respect to being displayed on the graph. Repeat this for any other functions that you do not wish to see on your graph. Only those functions that have highlighted equal signs will be graphed.

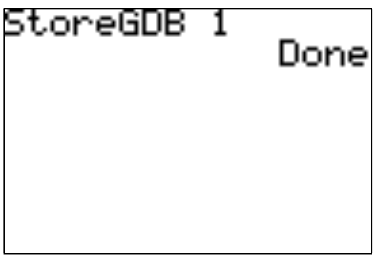


Storing and Recalling a Graph Database

These instructions describe how to store and recall a graph database. This feature can help you to avoid the need to repeatedly enter complicated functions into your calculator by storing them for future recall. A single command will allow you to store or recall this information:

- All functions in the Y= screen and the display status of each
- The graphing mode
- All viewing window variables
- Format settings
- The line style for each Y= function

To store a graph database (GDB) press 2ND [DRAW], use the right arrow to display the **STO** menu, and press 3 to select **StoreGDB.** The **StoreGDB** command will be copied to the home screen. Press any number key to select one of ten variables to which this GDB can be stored, and then press ENTER.



Use a similar procedure to recall the stored GDB, but select **RecallGDB** from the 2ND [DRAW] **STO** menu. Again, enter the variable number from which you wish to recall the GDB and press ENTER.

Caution: Recalling the GDB replaces all Y= functions. Any functions that are entered at the Y= screen will be erased when the GDB is recalled.

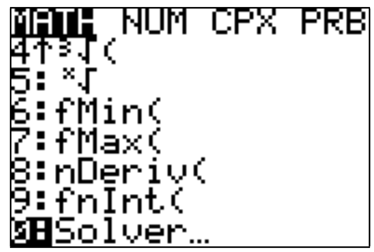
Using the Equation Solver

To use EQUATION SOLVER, your equation needs to be in a form with one side of the equation equal to zero. If your equation is complicated, as in *Moving Cart, Turning Ferris Wheel,* you will want to use the Y= screen first to break it up into smaller pieces. This process is described in the Calculator Note “Graphing a Complicated Function.”

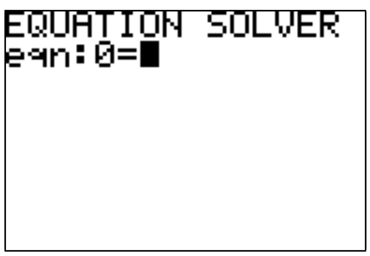
For an example, we will assume that you have broken your equation up and entered it into the Y= screen so that one side of your equation is represented by the variable **Y3** and the other side by **Y4**. So, your equation could now be expressed as **Y3=Y4**.

1. Manipulate your equation to create an equivalent equation in which one side of the equation is equal to zero. The equation **Y3=Y4** would become **Y3–Y4=0**. You will enter this equation into the calculator in a moment.

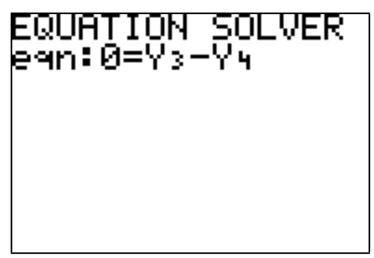
2. If you need to, press 2ND [QUIT] to return to the home screen. Press MATH. Use the down arrow to scroll all the way down to **0:Solver** and press ENTER.



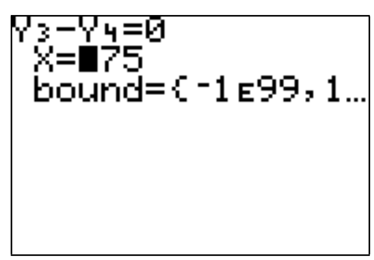
3. You will enter your equation on the **EQUATION SOLVER** editing screen, which appears as shown here. If your screen does not say **EQUATION SOLVER** at the top, press the up arrow to get to this screen.



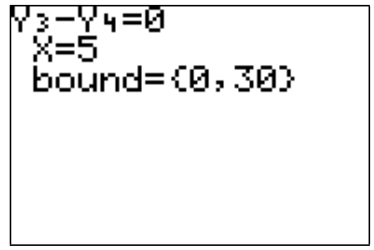
4. Enter your equation after the **0=** prompt. To enter **Y3**, press VARS, use the right arrow to select the **Y-VARS** menu, select **1:Function** by pressing the ENTER key, and select **Y3**. Do the same to select **Y4**.



5. Press ENTER to obtain a screen similar to that shown here. The displayed value for the variable **X** will be whatever value was last stored under that variable, and it therefore has no significance.



6. Change the displayed value of your variable to a reasonable guess of its actual value (the time at which the diver should be released from the Ferris wheel). Then use the down arrow to move to the boundaries that are displayed. The … on the right side means there are more digits beyond the right edge of the screen, which can be viewed using the right arrow. These are the lower and upper boundaries, between which you want the calculator to find a value for your variable that makes your equation true. By default, the calculator chooses the smallest and largest numbers with which it can work as boundaries. Change these numbers to realistic lower and upper boundaries for your variable. For example, we know that *X* cannot be smaller than zero, because we are not interested in negative time. The two boundaries must be separated by a comma and enclosed in braces. Don't confuse braces ({,}) with brackets ([,]). Braces are found above the parentheses keys—press $\boxed{\text{2ND}}$ [{] and $\boxed{\text{2ND}}$ [}].



7. Use the up arrow to move your cursor back to the guess you made for your variable. Press $\boxed{\text{ALPHA}}$ [SOLVE] (above the $\boxed{\text{ENTER}}$ key); in a moment, the calculator will display the solution to **$Y_3 - Y_4 = 0$**. The value displayed after **left–rt=** is the difference between the two sides of your equation, using the solution's value for the variable. If there is no round-off error, this should be zero.

Converting Between Coordinate Systems

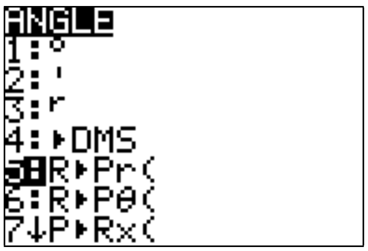
The graphing calculator can perform conversions between rectangular and polar coordinates.

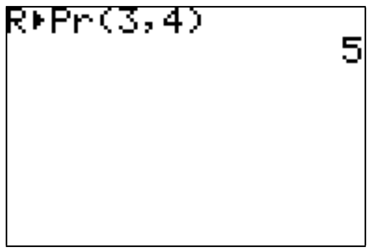
Converting from Rectangular to Polar Coordinates

1. Press MODE and make sure your calculator is set in the mode in which you wish to have angle measurements reported. This will usually be the **Degree** mode.

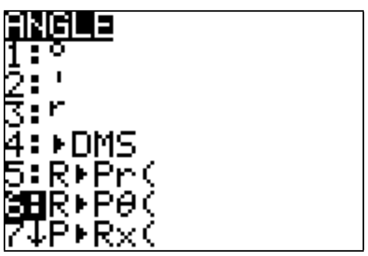
2. Press 2ND [ANGLE] (above the APPS key) and then press 5 to select **R▶Pr(**. The command will be copied to the home screen.

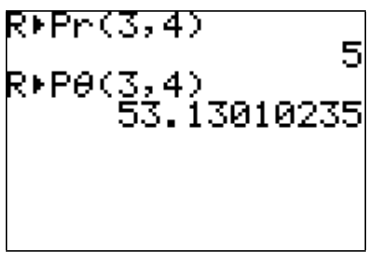
   Enter the rectangular coordinates, separated by a comma. Press ENTER. The calculator will display the *r*-value of the polar coordinates.





3. Press 2ND [ANGLE] again and then press 6 to select **R▶Pθ(**. The command will be copied to the home screen. Enter the rectangular coordinates again and press ENTER. The calculator will display the θ-value of the polar coordinates. This screen shows a conversion of the rectangular coordinates (3, 4) to the polar coordinates (5, 53.13).

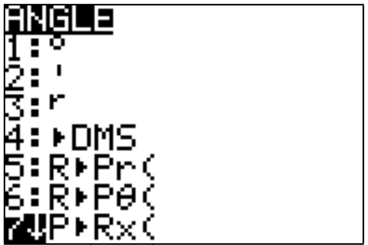


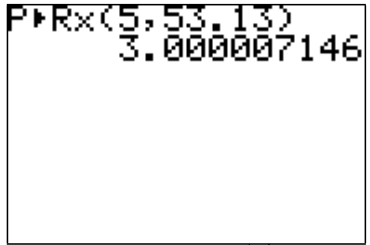


Converting from Polar to Rectangular Coordinates

1. Press MODE and make sure your calculator is set in the mode in which you have the angle measurement. This will usually be the **Degree** mode.

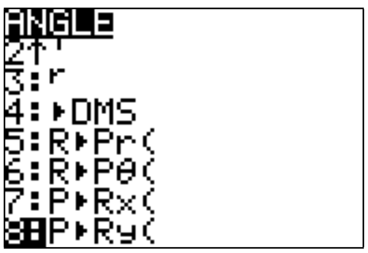
2. Press 2ND [ANGLE] (above the APPS key) and then press 7 to select **P▶Rx(**. The command will be copied to the home screen. Enter the polar coordinates, separated by a comma, with the r-value before the θ-value. Press ENTER. The calculator will display the x-value of the rectangular coordinates.

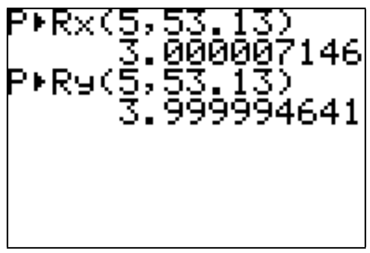




3. Press 2ND [ANGLE] again and then press 8 to select **P▶Ry(**. (You can view this command by using the down arrow to scroll downward beyond the last menu option visible on the screen.) The command will be copied to the home screen.

   Enter the polar coordinates again and press ENTER. The calculator will display the *y*-value of the rectangular coordinates. This screen shows a conversion from polar coordinates of (5, 53.13) to rectangular coordinates of approximately (3, 4).

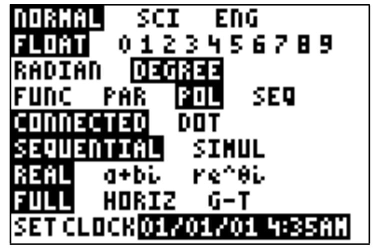




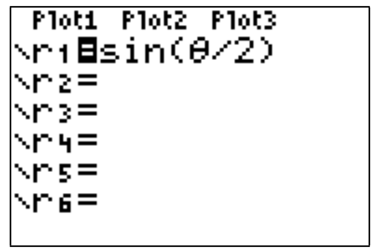
Graphing Polar Equations

These instructions explain how to graph polar equations on the graphing calculator.

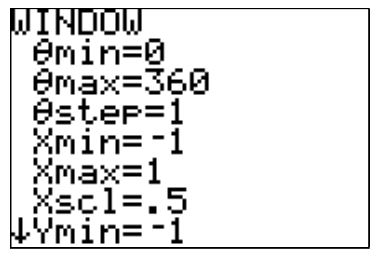
1. Press MODE, use the arrow keys to highlight **Pol** and press ENTER. This shifts the calculator into polar mode. While you are at the MODE screen, be sure that **Degree** is highlighted.



2. Press Y= and enter your equation. The equation must first be in a format with *r* alone on one side. Use the X,T,θ,n key to enter θ.

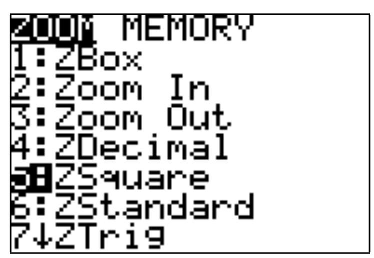


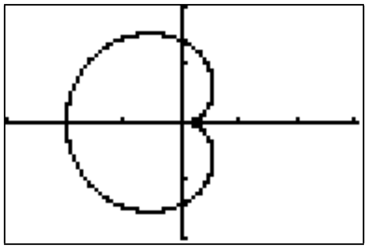
3. Press WINDOW and enter appropriate values for the various variables. In most cases, you will want to graph from ***θ*min=0** to ***θ*max=360**. The variable ***θ*step** controls how finely the data items for the graph are calculated. Setting ***θ*step=1** will cause the calculator to determine the coordinates at every integer number of degrees. A larger number will draw more quickly but will also yield a coarser approximation of the graph.



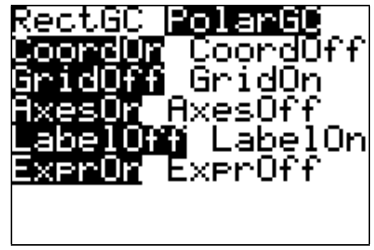
   Notice that even though this will be a polar graph, the window is still dimensioned in terms of *x* and *y*. While it may be difficult to predict what your graph will look like, it is usually fairly simple to at least estimate the greatest possible value of *r* by using what you know about trigonometric functions. For example, in the equation shown in the preceding illustration, we know that the sine always ranges between –1 and 1. A safe window to use would extend one unit in each direction, so **Xmin** and **Ymin** are set to –1 and **Xmax** and **Ymax** are set to 1. **Xscl** and **Yscl** simply control the distance between tick marks on the axes.

4. Because we are using a coordinate system based on the circle, the graph will be distorted if the window is not proportioned correctly. That is, circles will look more like ovals. The easiest way to avoid this is to use **Zoom Square** after setting up the window variables. This feature will enlarge your window in one direction to make each step in the horizontal direction equal in size to a step in the vertical direction. Press ZOOM and then 5 to select **5:ZSquare**. The graph will be drawn.





5. If you wish to trace the function using polar coordinates, you must select polar grid coordinates in the window format menu. To do this, press $\boxed{\text{2ND}}$ [FORMAT], highlight **PolarGC**, and press $\boxed{\text{ENTER}}$. Press $\boxed{\text{TRACE}}$ and use the left and right arrow keys to trace the function.



Entering a Program into the Calculator

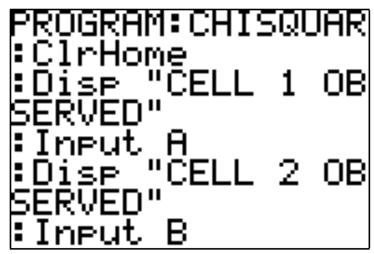
Starting the Program

If you press the PRGM key, you will see three menu headings across the top of the screen: **EXEC**, from which to execute (run) a program; **EDIT**, from which to edit (change) an existing program; and **NEW**, from which to begin entering a new program.

To start a new program, press PRGM, use the right arrow to highlight the **NEW** menu heading, and press ENTER to select **Create New**. Notice that the cursor is now a flashing **A**. This means that the calculator is locked in ALPHA mode. Enter a program name of eight characters or less and press ENTER.

Entering a Program

Every program consists of a series of commands that the calculator executes in order. The calculator marks the beginning of each command with a colon, which the calculator inserts when you press ENTER at the end of the previous line. You can also enter several commands on a single line by separating them with colons.

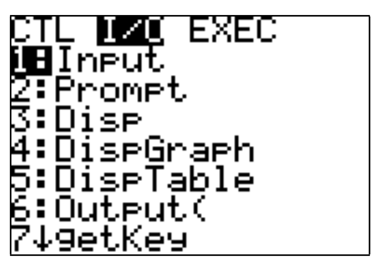


You enter most commands into your program by selecting them from menus that are displayed when you press various keys. You cannot enter these commands by spelling them out in ALPHA mode.

Occasionally, you will accidentally select the wrong menu when looking for a program command to insert into your program. When that happens, press CLEAR to return to your program editing. (If you press 2ND [QUIT], the calculator will exit the program editing mode altogether. You will then have to once again select your partially completed program for editing, as described at the end of these instructions.)

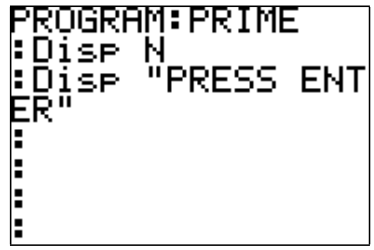
Using Input/Output Commands

From within a new program, press the PRGM key and use the right arrow to highlight the **I/O** menu heading. (This will only work if your cursor is within a program. The PRGM key causes a different menu to be displayed when you are editing a program than when you are at the home screen.)



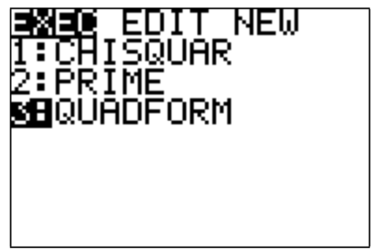
The **Input** command causes the program to display a question mark and then to pause until the user enters a value. **Input** is followed on the same line by the name of the variable to which the value is to be saved, as shown here.

The **Disp** (display) command is followed by the number, variable, expression, or string of text that is to be displayed. Text to be displayed must be enclosed in quotation marks.



Running a Program

When you finish entering a program, exit the program editing mode by pressing $\boxed{\text{2ND}}$ [QUIT]. Select a program to run by pressing $\boxed{\text{PRGM}}$, highlighting the name of the program in the **EXEC** menu, and then pressing $\boxed{\text{ENTER}}$. The program name will appear on the home screen. Press $\boxed{\text{ENTER}}$ again to run the program. If you need to interrupt your program while it is running, press $\boxed{\text{2ND}}$ [QUIT].



Editing a Program

Don't be surprised if your program does not operate as intended the first time. It is usually necessary to return to the program editing mode to revise a new program. These glitches in the program are know as "bugs," and the process of correcting them as "debugging."

To edit your program, press $\boxed{\text{PRGM}}$, highlight the **EDIT** menu heading, highlight the name of your program, and press $\boxed{\text{ENTER}}$.

Programming the Quadratic Formula

These instructions explain how to write a program to evaluate the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Begin by giving your program a name. Press PRGM, use the right arrow to highlight **NEW**, and press ENTER. Enter a name of no more than eight letters and press ENTER again. See the Calculator Note “Entering a Program into the Calculator” if you need more detail.

Enter the program as described here.

Instruction	Explanation
Input "A?",A	Causes the program to display the prompt **A?** and then to store the value the user enters as variable *A*. Press PRGM, use the right arrow to highlight the **I/O** menu heading, and press ENTER to select **1:Input**. Enter **"A?",A** (using the ALPHA key for all but the comma) and then press ENTER.
:Input "B?",B	Enter as described previously.
:Input "C?",C	
:√(B2-4AC)→D	Evaluates the portion of the formula that involves the square root and assigns this value to the variable *D*. (The value within parentheses is known as the discriminant.) Use the x^2 key to get the exponent 2 and the STO> key to get the →.
:Disp (-B+D)/(2A)	Displays the answer that involves adding the value of the radical expression. Find **Disp** as you did **Input**, under the **I/O** menu after pressing PRGM. Be sure to use the negative key, (-), for the negative symbol in front of *B*.
:Disp (-B-D)/(2A)	Displays the answer that involves subtracting the value of the radical expression. Enter as described previously.

Press 2ND [QUIT] to exit the editing mode. To run your program, position your cursor on an empty line on the home screen, press PRGM, highlight your program on the **EXEC** menu, and press ENTER. When your program name appears on the home screen, press ENTER again. Enter the coefficients when prompted to do so, pressing ENTER after each one.