Chapter 1.1

***1.** Analyze the logical forms of the following statements:

- **(a)** We'll have either a reading assignment or homework problems, but we won't have both homework problems and a test.

  Let

  R = We have a reading assignment

  H = We have homework problems

  T = We have a test

  Then (a) is

  $R \lor H \land \neg(H \land T)$

- **(b)** You won't go skiing, or you will and there won't be any snow.

  Let

  W = You will go skiing

  S = There will be snow

  Then (b) is

  $\neg W \lor (W \land \neg S)$

- **(c)** $\sqrt{7} \not\leq 2$

  Breaking down the statement into its parts, we have

  $\neg(\sqrt{7} < 2) \qquad \neg(\sqrt{7} = 2)$

  Combining both, we have

  $\neg(\sqrt{7} < 2) \land \neg(\sqrt{7} = 2) \equiv \neg[(\sqrt{7} < 2) \land (\sqrt{7} = 2)]$

2. Analyze the logical forms of the following statements:

- **(a)** Either John and Bill are both telling the truth, or neither of them is.

  Let

  J = John is telling the truth

  B = Bill is telling the truth

  Then, (a) is

  $(J \land B) \lor \neg(J \land B)$

- **(b)** I'll have either fish or chicken, but I won't have both fish and mashed potatoes.

  Let

  F = Fish

$C =$ Chicken

$M =$ Mashed potatoes

Then

$(F \vee C) \wedge \neg(F \wedge M)$

- **(c)** 3 is a common divisor of 6, 9, and 15.

  Let

  $D_1 =$ 3 is a divisor of 6

  $D_2 =$ 3 is a divisor of 9

  $D_3 =$ 3 is a divisor of 15

  Then

  $(D_1 \wedge D_2 \wedge D_3)$

3. Analyze the logical forms of the following statements:

Let

$A =$ Alice is in the room

$B =$ Bob is in the room

$C =$ Alice and Bob are in the room

- **(a)** Alice and Bob are not both in the room.

  $\neg C$

- **(b)** Alice and Bob are both not in the room.

  $\neg(A \wedge B)$

- **(c)** Either Alice or Bob is not in the room.

  $\neg A \vee \neg B$

- **(d)** Neither Alice nor Bob is in the room.

  $\neg(A \wedge B)$

4. Analyze the logical form of the following statements:

Let

$RT =$ Ralph is tall.

$ET =$ Ed is tall.

$RH =$ Ralph is handsome.

$EH =$ Ed is handsome.

- **(a)** Either both Ralph and Ed are tall, or both of them are handsome.

  $(RT \wedge ET) \vee (RH \wedge EH)$

- **(b)** Both Ralph and Ed are either tall or handsome.

  $(RT \wedge RH) \vee (ET \wedge EH)$

- **(c)** Both Ralph and Ed are neither tall nor handsome.

  $\neg(RT \wedge ET) \vee \neg(RH \wedge EH)$

- **(d)** Neither Ralph nor Ed is both tall and handsome.

  $\neg(RT \wedge RH) \vee \neg(ET \wedge EH)$

5. Which of the following expressions are well-formed formulas? - **(a)** $\neg(\neg P \vee \neg\neg R)$. (this one is) - **(b)** $\neg(P, Q, \wedge R)$. - **(c)** $P \wedge \neg P$. (this one is) - **(d)** $(P \wedge Q)(P \vee R)$.

***6.** Let P stand for the statement "I will buy the pants" and S for the statement "I will buy the shirt." What English sentences are represented by the following expressions? - **(a)** $\neg(P \wedge \neg S)$. I will not buy the pants without the shirt. - **(b)** $\neg P \wedge \neg S$. I will neither buy the pants or the shirt. - **(c)** $\neg P \vee \neg S$. I will either not buy the shirt, or not buy the pants.

7. Let S stand for the statement "Steve is happy" and G for "George is happy." What English sentences are represented by the following expressions?

- **(a)** $(S \vee G) \wedge (\neg S \vee \neg G)$. Either steve is happy or George is happy, but
- **(b)** $[S \vee (G \wedge \neg S)] \vee \neg G$. Either Steve is happy or Steve is not happy and George is happy, or George is not happy.
- **(c)** $S \vee [G \wedge (\neg S \vee \neg G)]$. Either Steve is happy or George is happy and Either Steve or George is not happy.

8. Let T stand for the statement "Taxes will go up" and D for "The deficit will go up". What english sentences are represented by the following formulas?

- **(a)** $T \vee D$

  Either the taxes will go up or the deficit will go up.

- **(b)** $\neg(T \wedge D) \wedge \neg(\neg T \wedge \neg D)$

  Both the deficit and the taxes will not go up and neither the deficit nor the taxes will go up.

- **(c)** $(T \wedge \neg D) \vee (D \wedge \neg T)$

  Either the taxes will go up and the deficit will not, or the deficit will go up and the taxes will not.

9. Identify the premises and conclusions of the following deductive arguments and analyze their logical forms. Do you think the reasoning is valid? (Although you will have only your intuition to guide you in answering this last question, in the next section we will develop some techniques for determining the validity of arguments.) - **(a)** Jane and Pete won't both win the math prize. Pete will win either the math prize or the chemistry prize. Jane will win the math prize. Therefore, Pete will win the chemistry prize. - Premises: - Jane and Pete won't both win the math prize. - Pete will win either the math prize or the chemistry prize. - Jane will win the math prize. - Conclusion -

Pete will win the chemistry prize - Validity - The reasoning is valid, because the first and second statement declare mutual exclusion for the two events.

- **(b)** The main course will be either beef or fish. The vegetable will be either peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.
  - Premises:
    * The main course will be either beef or fish.
    * The vegetable will be either peas or corn.
    * We will not have both fish as a main course and corn as a vegetable.
  - Conclusion
    * We will not have both beef as a main course and peas as a vegetable.
  - Validity
    * The reasoning is not valid, because the third statement does not exclude the events in the arguments.
- **(c)** Either John or Bill is telling the truth. Either Sam or Bill is lying. Therefore, either John is telling the truth or Sam is lying.
  - Premises:
    * Either John or Bill is telling the truth.
    * Either Sam or Bill is lying.
  - Conclusion
    * Either John is telling the truth or Sam is lying.
  - Validity
    * The reasoning is not valid, because none of the premises exclude John from lying or Sam from telling the truth (Bill could be either).
- **(d)** Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.
  - Premises:
    * Sales will go up and the boss will be happy.
    * Expenses will go up and the boss won't be happy.
  - Conclusion
    * Sales and expenses will not both go up.
  - Validity
    * The reasoning is not valid, because the premises do not exclude each other.

Chapter 1.2

*1. Make truth tables for the following formulas:

- (a) $\neg P \lor Q$.

P	Q	$\neg P$	$\neg P \lor Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

- (b) $(S \vee G) \wedge (\neg S \vee \neg G)$.

S	G	$(S \vee G) \wedge (\neg S \vee \neg G)$
F	F	F
F	T	T
T	F	T
T	T	F

2. Make truth tables for the following formulas:

- (a) $\neg[P \wedge (Q \vee \neg P)]$.

P	Q	$\neg[P \wedge (Q \vee \neg P)]$
F	F	T
F	T	T
T	F	T
T	T	F

- (b) $(P \vee Q) \wedge (\neg P \vee R)$.

P	Q	R	$(P \vee Q) \wedge (\neg P \vee R)$
F	F	F	F
F	F	T	F
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	F
T	T	T	T

3. In this exercise we will use the symbol + to mean exclusive or. In other words, $P + Q$ means "P or Q, but not both."

- (a) Make a truth table for $P + Q$.

P	Q	$P + Q$
F	F	F
F	T	T
T	F	T
T	T	F

- (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P + Q$. Justify your answer with a truth table.

P	Q	$P+Q$	$(P \vee Q) \wedge \neg(P \wedge Q)$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	F	F

4. Find a formula using only the connectives $\wedge$ and $\neg$ that is equivalent to $P \vee Q$. Justify your answer with a truth table. By DeMorgan's law: $P \vee Q \equiv \neg(\neg P \wedge \neg Q)$

P	Q	$P \vee Q$	$\neg(\neg P \wedge \neg Q)$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	T	T

***5. Some mathematicians use the symbol $\downarrow$ to mean nor. In other words, $P \downarrow Q$ means "neither P nor Q."**

- (a) Make a truth table for $P \downarrow Q$.

P	Q	$P \downarrow Q$
F	F	T
F	T	F
T	F	F
T	T	F

- (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \downarrow Q$.

  Since $P \downarrow Q$ is only true when both P and Q are false, $P \downarrow Q \equiv \neg(P \vee Q)$

- (c) Find formulas using only the connective $\downarrow$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.

P	$\neg P$	$P \downarrow P$
F	T	T
F	T	T
T	F	F
T	F	F

P	Q	$P \vee Q$	$(P \downarrow Q) \downarrow (P \downarrow Q)$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	T	T

P	Q	$P \wedge Q$	$(P \downarrow P) \downarrow (Q \downarrow Q)$
F	F	F	F
F	T	F	F
T	F	F	F
T	T	T	T

6. Some mathematicians write $P|Q$ to mean "P and Q are not both true."(This connective is called nand, and is used in the study of circuits in computer science.)

- (a) Make a truth table for $P \mid Q$.

P	Q	$P \mid Q$
F	F	T
F	T	T
T	F	T
T	T	F

- (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \mid Q$.

  $P \mid Q \equiv \neg(P \wedge Q)$

- (c) Find formulas using only the connective $\mid$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.

  $\neg P \equiv P \mid P$

  $P \vee Q \equiv (P \mid P) \mid (Q \mid Q)$

  $P \wedge Q \equiv (P \mid Q) \mid (P \mid Q)$

***7. Use truth tables to determine whether or not the arguments in exercise 7 of Section 1.1 are valid.** For (a): let

J = Jane will win the math prize P_m = Peter will win the math prize P_c = Peter will win the chemistry prize

So the premises are:

- Jane and Pete won't both win the math prize = $\neg(J \wedge P_m)$
- Pete will win either the math prize or the chemistry prize = $P_c \vee P_m$
- Jane will win the math prize = J

- Pete will win the chemistry prize = P_c

$\neg(J \wedge P_m)$

$P_c \vee P_m$

J	P_m	$\neg(J \wedge P_m)$	$P_c \vee P_m$	Conclusion: P_c
F	F	T	F	F
F	F	T	T	T
F	T	T	T	F
F	T	T	T	T
T	F	T	F	F
T	F	T	T	T
T	T	F	T	F
T	T	F	T	T

The arguments are valid, as the premises are all true on rows 2, 4 and 6, andthe last row of the truth table: P_c is true even though one of the arguments is false.

For (b); the premises are

- The main course will be either beef or fish. $= B \vee F$
- The vegetable will be either peas or corn.$= P \vee C$
- We will not have both fish as a main course and corn as a vegetable. $= \neg(F \wedge C)$

And the conclusion:

- We will not have both beef as a main course and peas as a vegetable. $= \neg(B \wedge P)$

No.	B	F	P	C	$= B \vee F$	$= P \vee C$	$= \neg(B \wedge P)$
1	F	F	F	F	F	F	T
2	F	F	F	T	F	T	T
3	F	F	T	F	F	T	T
4	F	F	T	T	F	T	T
5	F	T	F	F	T	F	T
6	F	T	F	T	T	T	T
7	F	T	T	F	T	T	T
8	F	T	T	T	T	T	T
9	T	F	F	F	T	F	T
10	T	F	F	T	T	T	T
11	T	F	T	F	T	T	F
12	T	F	T	T	T	T	F
13	T	T	F	F	T	F	T
14	T	T	F	T	T	T	T
15	T	T	T	F	T	T	F
16	T	T	T	T	T	T	F

The arguments are invalid, as demonstrated by rows 1-5, 9, 13, 15 and 16 of the truth table: P_c is true even though one of the arguments is false.

For (c); the premises are

- Either John or Bill is telling the truth. $= J \vee B$
- Either Sam or Bill is lying.$= \neg S \vee \neg B \equiv \neg(S \wedge B)$

And the conclusion:

- Either John is telling the truth or Sam is lying. $= J \vee \neg S$

J	B	S	$J \vee B$	$\neg S \vee \neg B$	Conclusion: $J \vee \neg S$
F	F	F	F	T	T
F	F	T	F	T	
F	T	F	T	T	
F	T	T	T	F	
T	F	F	T	T	
T	F	T	T	T	
T	T	F	T	T	
T	T	T	T	F	

15. How many lines will there be in the truth table for a statement containing n letters? Since there are two possible truth values for every letter, it's 2^n lines

16. Find a formula involving the connectives $\wedge$,$\vee$, and $\neg$ that has the following truth table:

P	Q	???
F	F	F
F	T	F
T	F	T
T	T	T

Chapter 1.3

***1. Analyze the logical forms of the following statements:**

- (a) 3 is a common divisor of 6, 9, and 15. (Note: You did this in exercise 2 of Section 1.1, but you should be able to give a better answer now.)

  Let $D(x)$ be "x is divisible by 3"

  then $D(6) \wedge D(9) \wedge D(15)$

- (b) x is divisible by both 2 and 3 but not 4.

Let

$D(y, z)$ be "y is divisible by z"

Then $x \in \{y | D(y, 2) \wedge D(y, 3) \wedge \neg D(y, 4)\}$

- (c) x and y are natural numbers, and exactly one of them is prime.

  Let

  $P(x)$ be "x is a prime number"

  $N(x)$ be "x is a natural number"

  "Exactly one of them is prime" can also be read as "Either x is prime, and y is not prime, or y is prime, and x is not prime"

  Then $N(x) \wedge N(y) \wedge [(P(x) \wedge \neg P(y)) \vee (\neg P(x) \wedge P(y))]$

2. Analyze the logical forms of the following statements:

- (a) x and y are men, and either x is taller than y or y is taller than x.

  Let

  $M(x)$ be "x is a man"

  $T(x, y)$ be "x is taller than y"

  Then

  $M(x) \wedge M(y) \wedge [T(x, y) \vee T(y, x)]$

- (b) Either x or y has brown eyes, and either x or y has red hair.

  Let

  $B(x)$ be "x has brown eyes"

  $R(x)$ be "x has red hair"

  Then (b) becomes

  $[(B(x) \vee B(y)) \wedge (T(x) \vee T(y))]$

- (c) Either x or y has both brown eyes and red hair.

  $[(B(x) \wedge T(x)) \vee (B(y) \wedge T(y))]$

***3. Write definitions using elementhood tests for the following sets:**

- (a) {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}.

  $y \in \big\{x \mid x \text{ is a planet of the solar system}\big\}$

- (b) {Brown, Columbia, Cornell, Dartmouth, Harvard, Princeton, University of Pennsylvania, Yale}.

  $y \in \big\{x \mid x \text{ is an Ivy League university}\big\}$

- (c) {Alabama, Alaska, Arizona, . . . , Wisconsin, Wyoming}.

  $y \in \{x \mid x \text{ is state of the United States}\}$

- (d) {Alberta, British Columbia, Manitoba, New Brunswick, Newfoundland and Labrador, Northwest Territories, Nova Scotia, Nunavut, Ontario, Prince Edward Island, Quebec, Saskatchewan, Yukon}.

  $y \in \{x \mid x \text{ is a canadian state}\}$

4. Write definitions using elementhood tests for the following sets:

- (a) {1, 4, 9, 16, 25, 36, 49, . . .}.

  $y \in \{x \in \mathbb{Z} \mid x^2\}$

- (b) {1, 2, 4, 8, 16, 32, 64, . . .}.

  $y \in \{x \in \mathbb{Z}^+ \mid 2^x\}$

- (c) {10, 11, 12, 13, 14, 15, 16, 17, 18, 19}.

  $y \in \{x \mid 9 < x < 20\}$ #### *5. Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.

- (a) $-3 \in \{x \in \mathbb{R} \mid 13 - 2x > 1\}$.

  $-3 \in \{x \in \mathbb{R} \mid x < 6\}$

  It has no free variables, as 3 is a constant; x is bound. The statement is true, as $-3 < 6$

- (b) $4 \in \{x \in \mathbb{R}^- \mid 13 - 2x > 1\}$.

  $4 \in \{x \in \mathbb{R}^- \mid x < 6\}$

  It has no free variables, x is bound. The statement is false, as $4 \notin \mathbb{R}^-$

- (c) $5 \notin \{x \in \mathbb{R} \mid 13 - 2x > c\}$.

  $5 \notin \{x \in \mathbb{R} \mid x < \frac{13-c}{2}\}$

  c is the only free variable; x is bound. The statement is true for any $c > 3$

6. Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.

- (a) $w \in \{x \in \mathbb{R} \mid 13 - 2x > c\}$

$$(w \in \mathbb{R}) \wedge (13 - 2w > c)$$

x is bound; w is free, it's true if the inequation $w < \frac{13-c}{2}$ is satisfied.

- (b)

$$4 \in \{x \in \mathbb{R} \mid 13 - 2x \in \{y \mid \text{ y is a prime number}\}\}$$

It might make this statement easier to read if we let

$$P = \{y \mid y \text{ is a prime number}\}$$

using this notation, we could rewrite the statement as

$$4 \in \{x \in \mathbb{R} \mid 13 - 2x \in P\}$$

Let $P(x)$ be "x is a prime number"

Then the statement (b) becomes

$$(4 \in \mathbb{R}) \land [P(13 - 2(4))]$$

x is bound, no free variables. The statement is true.

- (c) $4 \in \Big\{x \in \{y \mid \text{y is a prime number}\} \mid 13 - 2x > 1\Big\}$. (Using the same notation as in part (b), we could write this as $4 \in \{x \in P \mid 13 - 2x > 1\}$.)

Using the same $P(x)$ defined in part (b)

$$[P(4)] \land (13 - 2(4) > 1)$$

The statement is false, as $P(4)$ is false.

***7. What are the truth sets of the following statements? List a few elements of the truth set if you can.**

- (a) Elizabeth Taylor was once married to x.

$\{x \mid \text{Elizabeth Taylor was once married to x}\} = \{$Conrad Hilton Jr., Michael Wilding, Mike Todd, Eddie Fisher, F

- (b) x is a logical connective studied in Section 1.1.

$\{x \mid \text{x is a logical connective studied in Section 1.1}\} = \{\land, \lor, \neg\}$

- (c) x is the author of this book.

$\{x \mid \text{x is the author of this book}\} = \{\text{Daniel Velleman}\}$

8. What are the truth sets of the following statements? List a few elements of the truth set if you can.

- (a) x is a real number and $x^2 - 4x + 3 = 0$.

$x^2 - 4x + 3 = (x - 3)(x - 1)$

$\big\{x \in \mathbb{R} | x \in \{3, 1\}\big\}$

- (b) x is a real number and $x^2 - 2x + 3 = 0$.

  The truth set is ϕ (the empty set), as the solution does not involve real numbers, making the statement a contradiction.

- (c) x is a real number and $5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}$.

  $(x^2 + (5)^2 < 50) \wedge (5 \in \mathbb{R}) \wedge (x \in \mathbb{R})$

  Then the thruth set is $\{x \in \mathbb{R} \mid x < 5\}$

Chapter 1.4

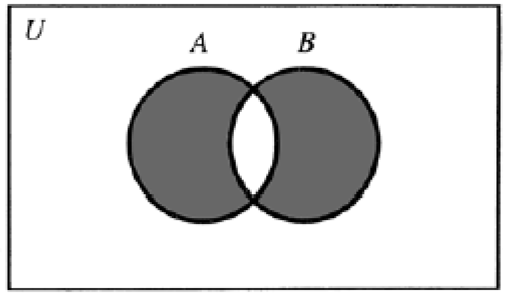
***1. Let $A = \{1, 3, 12, 35\}$, $B = \{3, 7, 12, 20\}$, and $C = \{x \mid x$ is a prime number$\}$. List the elements of the following sets. Are any of the sets below disjoint from any of the others? Are any of the sets below subsets of any others?**

- (a) $A \cap B$. $\{3, 12\}$
- (b) $(A \cup B) \setminus C$. $\{12, 20, 35\}$
- (c) $A \cup (B \setminus C)$. $\{1, 3, 12, 20, 35\}$
- Are any of the sets below disjoint from any of the others?
  - No, as no intersection $x \cap y = \phi$ for (a), (b) and (c)
- Are any of the sets below subsets of any others?
  - Yes:
    * (a) $\subseteq$ (c): $A \cap B \subseteq A \cup (B \setminus C)$
    * (b) $\subseteq$ (c): $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

2. Let $A = \{$United States, Germany, China, Australia$\}$, $B = \{$Germany, France, India, Brazil$\}$, and $C = \{x \mid x$ is a country in Europe$\}$. List the elements of the following sets. Are any of the sets below disjoint from any of the others? Are any of the sets below subsets of any others?

- (a) $A \cup B$. {United States, Germany, China, Australia, France, India, Brazil}
- (b) $(A \cap B) \setminus C$. {Germany}
- (c) $(B \cap C) \setminus A$. {France}
- Are any of the sets below disjoint from any of the others?
  - Yes:
    * (b) $\subseteq$ (c): $\Big[(A \cap B) \setminus C\Big] \cap \Big[(B \cap C) \setminus A\Big] = \phi$
- Are any of the sets below subsets of any others?
  - Yes, both (b) and (c) are subsets of (a)

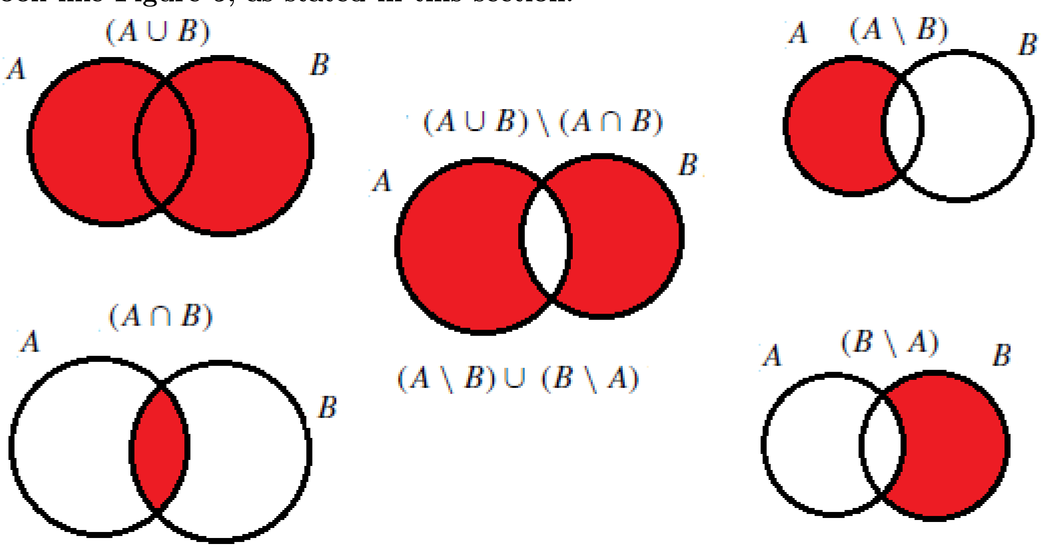
3. Verify that the Venn diagrams for $(A \cup B) \setminus (A \cap B)$ and $(A \setminus B) \cup (B \setminus A)$ both



$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Figure 5

look like Figure 5, as stated in this section.



***4. Use Venn diagrams to verify the following identities:**

- (a) $A \setminus (A \cap B) = A \setminus B$.
- (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

5. Verify the identities in exercise 4 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.

$$A \setminus (A \cap B) = A \setminus B$$

$\Big(x \in A\Big) \wedge \neg\Big(x \in A \wedge x \in B\Big)$

$\Big(x \in A\Big) \wedge \Big(x \notin A \vee x \notin B\Big)$ (DeMorgan's law)

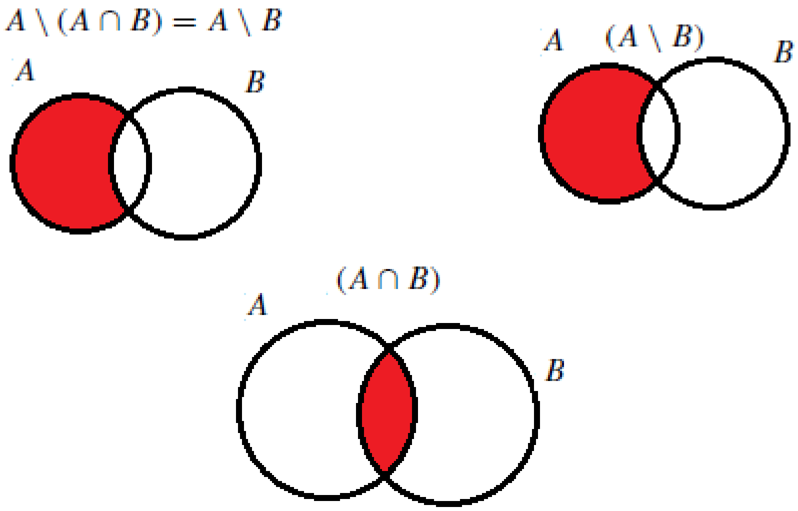


Figure 1: Exercise 1.4-4a

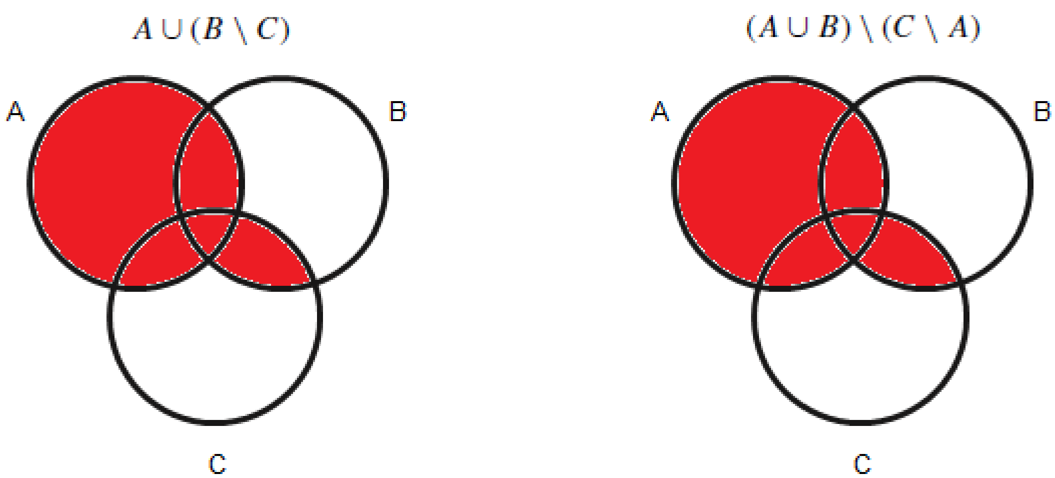


Figure 2: Exercise 1.4-4b

$\Big(x \in A \wedge x \notin A\Big) \vee \Big(x \in A \wedge x \notin B\Big)$ (Distributive property)

$\Big(x \in A \setminus A\Big) \vee \Big(x \in A \setminus B\Big)$ (Definition of $\setminus$)

$A \setminus A = \phi$, therefore

$\Big(x \in \phi\Big) \vee \Big(x \in A \setminus B\Big)$

$x \in \phi \cup \Big(A \setminus B\Big)$ (Definition of $\cup$)

Since $\phi \cup S \equiv S$ for any set S

$\phi \cup \Big(A \setminus B\Big) \equiv A \setminus B$

$x \in A \setminus B$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$x \in A \cup (B \cap C)$ is equivalent to

$x \in A \vee (x \in B \wedge x \in C)$ (Definitions of $\cup$, $\cap$)

$\Big(x \in A \vee x \in B\Big) \wedge \Big(x \in A \vee x \in C\Big)$ (Distributive law)

$\Big(x \in A \cup B\Big) \wedge \Big(x \in A \cup C\Big)$ (Definition of $\cup$)

$x \in \Big(A \cup B\Big) \cap \Big(A \cup C\Big)$ (Definition of $\cap$)

6. Use Venn diagrams to verify the following identities:

- (a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
- (b) $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$.

7. Verify the identities in exercise 6 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$$

$x \in (A \cup B) \setminus C$

$\Big(x \in A \vee x \in B\Big) \wedge \Big(x \notin C\Big)$ (Definitions of $\cup$, $\setminus$)

$\Big(x \in A \wedge x \notin C\Big) \vee \Big(x \in B \wedge x \notin C\Big)$ (Distributive law)

$\Big(x \in A \setminus C\Big) \vee \Big(x \in B \setminus C\Big)$ (Definition of $\setminus$)

$x \in \Big(A \setminus C\Big) \cup \Big(B \setminus C\Big)$ (Definition of $\cup$)

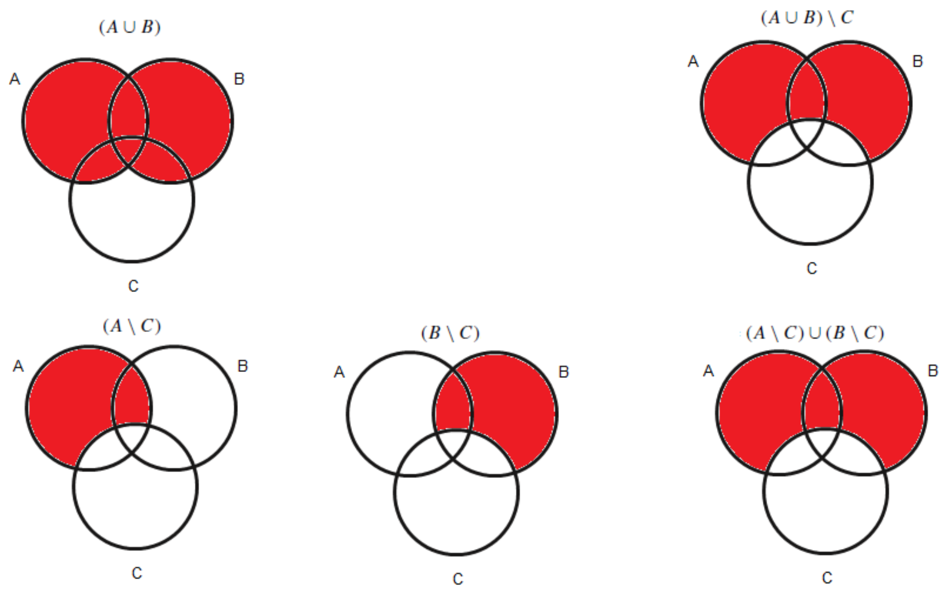


Figure 3: Exercise 1.4-6a

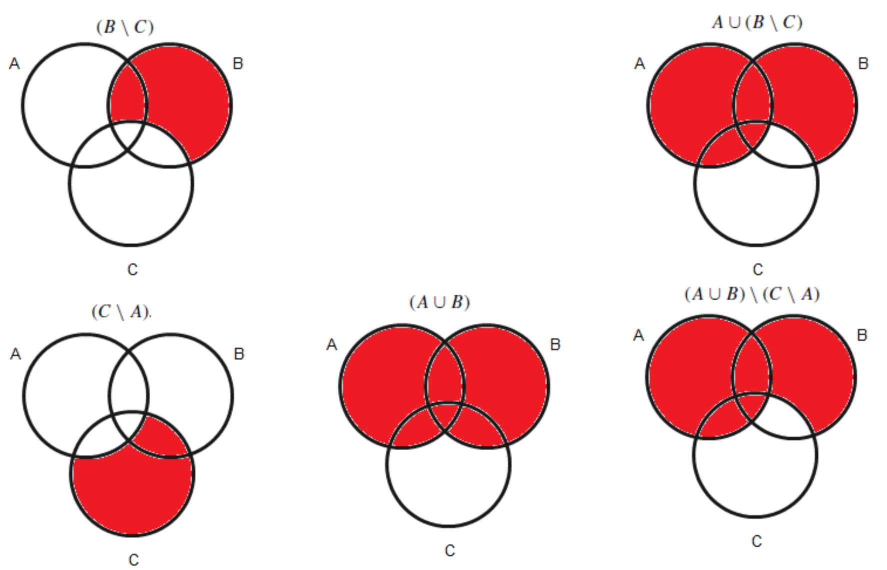


Figure 4: Exercise 1.4-6b

$$A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$$

$x \in A \cup (B \setminus C)$

$x \in A \vee (x \in B \wedge x \notin C)$ (Definition of $\setminus$)

$\Big(x \in A \vee x \in B\Big) \wedge \Big(x \in A \vee x \notin C\Big)$ (Distributive law)

$\Big(x \in A \vee x \in B\Big) \wedge \neg\Big(x \in C \wedge x \notin A\Big)$ (Reverse the second statement with a DeMorgan's equivalent)

$\Big(x \in A \cup B\Big) \wedge \neg\Big(x \in C \setminus A\Big)$ (Definitions of $\cup$, $\setminus$)

$\Big(x \in A \cup B\Big) \wedge \Big(x \notin C \setminus A\Big)$ (Apply the $\neg$)

$x \in \Big(A \cup B\Big) \setminus \Big(C \setminus A\Big)$ (Definition of $\setminus$)

***8. For each of the following sets, write out (using logical symbols) what it means for an object x to be an element of the set. Then determine which of these sets must be equal to each other by determining which statements are equivalent.**

- (a) $(A \setminus B) \setminus C. = \Big(x \in A \wedge x \notin B\Big) \wedge x \notin C$
- (b) $A \setminus (B \setminus C). = x \in A \wedge \Big(x \in B \wedge x \notin C\Big)$
- (c) $(A \setminus B) \cup (A \cap C). = \Big(x \in A \wedge x \notin B\Big) \vee \Big(x \in A \wedge x \in B\Big)$
- (d) $(A \setminus B) \cap (A \setminus C). = \Big(x \in A \wedge x \notin B\Big) \wedge \Big(x \in A \wedge x \notin C\Big)$
- (e) $A \setminus (B \cup C) = x \in A \wedge \Big(x \in B \vee x \in C).$

9. It was shown in this section that for any sets A and B, $(A \cup B) \setminus B \subseteq A$. Give an example of two sets A and B for which $(A \cup B) \setminus B \neq A$. Our goal is to satisy the expression $(A \cup B) \setminus B \neq A$

In order for this to happen, $A \cap B \neq \phi$, the examples below illustrate this

1. Example with $A \cap B = \phi$

   Let $A = \{1, 2, 3, 4\} \quad B = \{5, 6, 7, 8\}$, then

   $(A \cup B) \setminus B = \{1, 2, 3, 4\} \cup \{5, 6, 7, 8\} \setminus \{5, 6, 7, 8\}$

   $= \{1, 2, 3, 4, 5, 6, 7, 8\} \setminus \{5, 6, 7, 8\} = \{1, 2, 3, 4\} = A$

2. Example with $A \cap B \neq \phi$

   Let $A = \{1, 2, 3, 4\} \quad B = \{2, 4, 6, 8\}$, then

   $(A \cup B) \setminus B = \{1, 2, 3, 4\} \cup \{2, 4, 6, 8\} \setminus \{2, 4, 6, 8\}$

$= \{1,2,3,4,6,8\} \setminus \{2,4,6,8\} = \{1,3\} \neq A$

***10. It is claimed in this section that you cannot make a Venn diagram for four sets using overlapping circles.**

- (a) What's wrong with the following diagram? (Hint: Where's the set $(A \cap D) \setminus (B \cup C)$?)

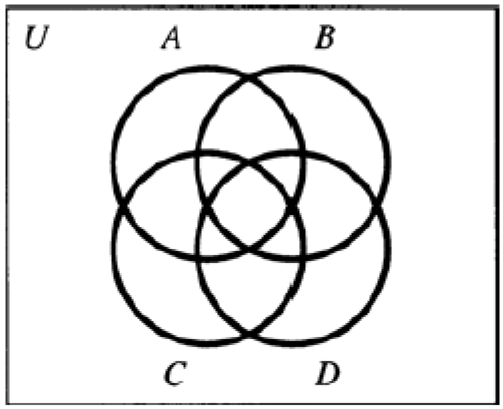


Figure 5: Exercise 9 figure

The attempt below shows the set $(A \cap D) \setminus (B \cup C)$ cannot be displayed using circles.

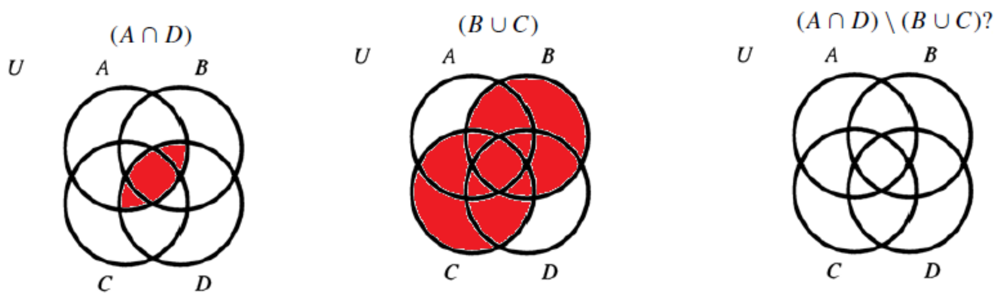


Figure 6: Section 1.4: Exercise 9-a

- (b) Can you make a Venn diagram for four sets using shapes other than circles?

Yes, below is a succesful attempt of depicting $(A \cap D) \setminus (B \cup C)$ using triangles.

11.

- (a) MakeVenn diagrams for the sets $(A \cup B) \setminus C$ and $A \cup (B \setminus C)$. What can you conclude about whether one of these sets is necessarily a subset of the other?

$(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

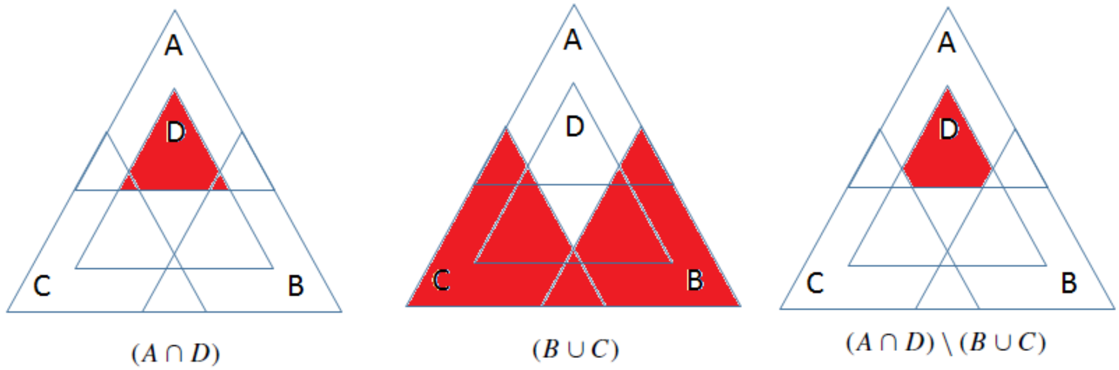


Figure 7: Section 1.4: Exercise 9-b

- (b) Give an example of sets A, B, and C for which $(A \cup B) \setminus C \neq A \cup (B \setminus C)$.

  If there is an element x such that $x \in A$, $x \in B$ and $x \in C$, the equation $(A \cup B) \setminus C \neq A \cup (B \setminus C$ will be satisfied.

  In an elementwise notation, this is expressed as:

  $(A \cup B) \setminus C \neq A \cup (B \setminus C$ for $y \in \{x | x \in A \cap B \cap C\}$

  Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{3, 10\}$, then

  $(A \cup B) \setminus C \neq A \cup (B \setminus C)$ becomes

  $(\{1, 2, 3\} \cup \{3, 4, 5\}) \setminus \{3, 10\} \neq \{1, 2, 3\} \cup (\{3, 4, 5\} \setminus \{3, 10\})$

  $\{1, 2, 3, 4, 5\} \setminus \{3, 10\} \neq \{1, 2, 3\} \cup \{4, 5\}$

  $\{1, 2, 4, 5\} \neq \{1, 2, 3, 4, 5\}$

***12. Use Venn diagrams to show that the associative law holds for symmetric difference; that is, for any sets A, B, and C, $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.** Recalling the symmetric difference equation:

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

Chapter 1.5

***1. Analyze the logical forms of the following statements:**

- (a) If this gas either has an unpleasant smell or is not explosive, then it isn't hydrogen.

  $(S \wedge \neg E) \longrightarrow \neg H$ where S stands for "has an unpleasant smell", E stands for "it's explosive" and H stands for "it's hydrogen"

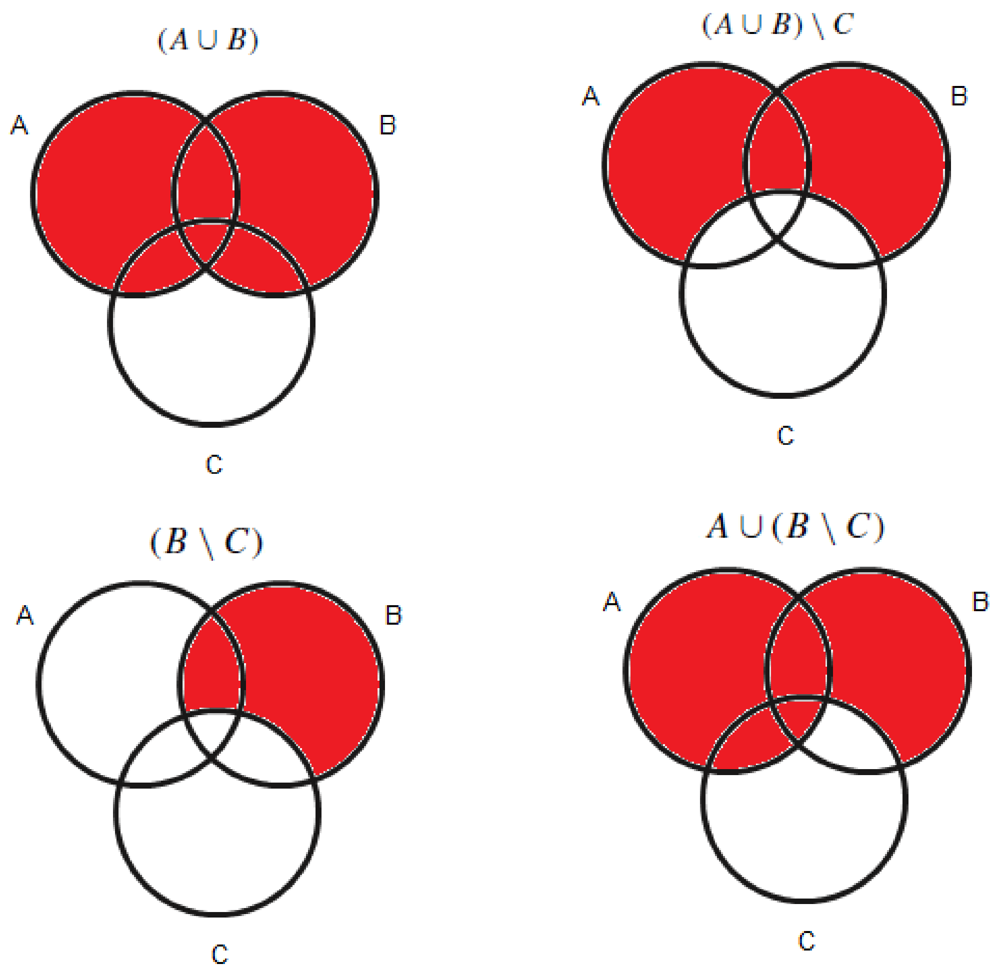


Figure 8: Section 1.4: Exercise 11

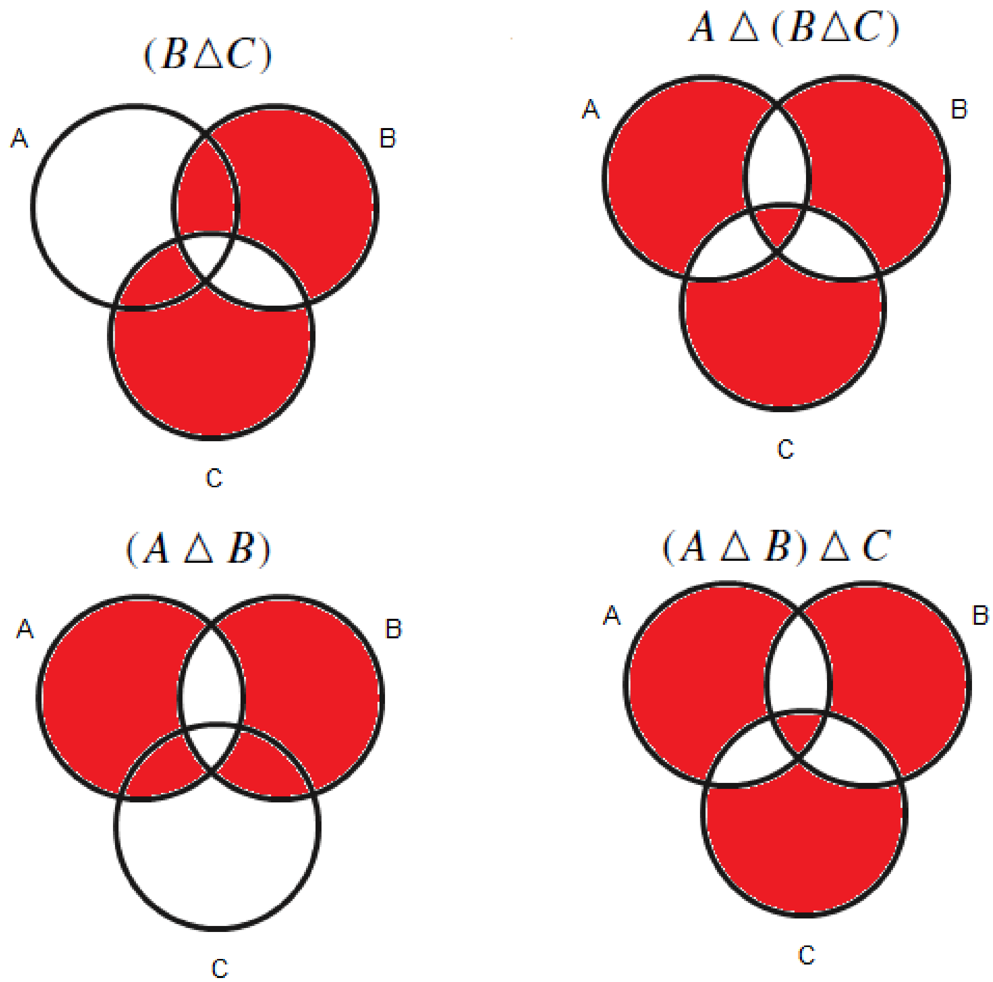


Figure 9: Section 1.4: Exercise 12

- (b) Having both a fever and a headache is a sufficient condition for George to go to the doctor.

  $(F \wedge H) \longrightarrow D$

- (c) Both having a fever and having a headache are sufficient conditions for George to go to the doctor.

  $(F \longrightarrow D) \vee (H \longrightarrow D)$

- (d) If $x \neq 2$, then a necessary condition for x to be prime is that x be odd.

  $(x \neq 2) \longrightarrow P(x) \longrightarrow O(x)$

2. Analyze the logical forms of the following statements:

- (a) Mary will sell her house only if she can get a good price and find a nice apartment.

  $(G \wedge N) \longrightarrow S$

- (b) Having both a good credit history and an adequate down payment is a necessary condition for getting a mortgage.

  $(C \wedge D) \longrightarrow M$

- (c) John will kill himself, unless someone stops him. (Hint: First try to rephrase this using the words if and then instead of unless.)

  Let

  S be "someone stopped John"

  K be "John will kill himself"

  $S \longrightarrow \neg K = K \longrightarrow \neg S$

- (d) If x is divisible by either 4 or 6, then it isn't prime.

  $D \longrightarrow \neg P$

***4. Use truth tables to determine whether or not the following arguments are valid:**

- (a) Either sales or expenses will go up. If sales go up, then the boss will be happy. If expenses go up, then the boss will be unhappy. Therefore, sales and expenses will not both go up.

  Let S be "sales will go up", E be "expenses will go up", H be "the boss will be happy.

  $S \vee E$

  $S \longrightarrow H$

  $E \longrightarrow \neg H$

  $-----$

  $\therefore \neg S \wedge \neg E$

No.	S	E	H	$S \vee E$	$S \longrightarrow H$	$E \longrightarrow \neg H$	$\neg(S \wedge E)$
1	F	F	F	F	T	T	T
2	F	F	T	F	T	T	T
3	F	T	F	T	T	T	T
4	T	T	T	T	T	F	F
5	T	F	F	T	F	T	T
6	T	F	T	T	T	T	T
7	T	T	F	T	F	T	F
8	T	T	T	T	T	F	F

- (b) If the tax rate and the unemployment rate both go up, then there will be a recession. If the GNP goes up, then there will not be a recession. The GNP and taxes are both going up. Therefore, the unemployment rate is not going up.

Let T be "the tax rate goes up", U be "the unemployment rate goes up", R be "there will be a recession", G be "GNP goes up"

$(T \wedge U) \longrightarrow R$

$G \longrightarrow \neg R$

$G \wedge T$

$- - - - - - - -$

$\therefore \neg U$

No.	T	U	G	R	$(T \wedge U) \longrightarrow R$	$G \longrightarrow \neg R$	$G \wedge T$	$\neg U$
1	F	F	F	F	T	T	F	T
2	F	F	F	T	T	T	F	T
3	F	F	T	F	T	T	F	T
4	F	F	T	T	T	F	F	T
5	F	T	F	F	T	T	F	F
6	F	T	F	T	T	T	F	F
7	F	T	T	F	T	T	F	F
8	F	T	T	T	T	F	F	F
9	T	F	F	F	T	T	F	T
10	T	F	F	T	T	T	F	T
11	T	F	T	F	T	T	T	T
12	T	F	T	T	T	F	T	T
13	T	T	F	F	F	T	F	F
14	T	T	F	T	T	T	F	F
15	T	T	T	F	F	T	T	F
16	T	T	T	T	T	F	T	F

- (c) Thewarning light will come on if and only if the pressure is too high and the relief valve is clogged. The relief valve is not clogged. Therefore, the warning light will come on if and only if the pressure is too high.

5.

- (a) Show that $P \iff Q$ is equivalent to $(P \land Q) \lor (\neg P \land \neg Q)$.

  $P \iff Q$

  $(P \implies Q) \land (Q \implies P)$ (Definition of $\iff$)

  $(\neg P \lor Q) \land (\neg P \implies \neg Q)$ (Definition of $\implies$, contrapositive law)

  $(\neg P \lor Q) \land \neg(\neg P \land \neg(\neg Q))$

  $(\neg P \lor Q) \land (P \lor \neg Q)$

- (b) Show that $(P \implies Q) \lor (P \implies R)$ is equivalent to $P \implies (Q \lor R)$.

Chapter 2.1

***1. Analyze the logical forms of the following statements.**

- (a) Anyone who has forgiven at least one person is a saint.

  $\exists x$(if x has forgiven a person, then x is a saint)

  Let $S(x)$ be "x is a saint", $F(y, x)$ be "y has been forgiven by x", then

  $\exists x(\exists y F(y, x) \longrightarrow S(x))$

- (b) Nobody in the calculus class is smarter than everybody in the discrete math class.

  "Nobody" can be rewritten as "not somebody".

  Let $C(x)$ be "x is in the calculus class", $D(x)$ be "x is in the discrete math class", and $S(x, y)$ be "x is smarter than y"

  $\neg\exists x$(x is in the calculus class and is smarter than everyone in the discrete math class)

  $\neg\exists x\Big[C(x) \land \text{if y is in the discrete math class and x is smarter than y}\Big]$

  $\neg\exists x\Bigg[C(x) \land \forall y\Big(D(y) \longrightarrow S(x, y)\Big)\Bigg]$

- (c) Everyone likes Mary, except Mary herself.

  Let $L(x)$ be "x likes Mary", and $M(x)$ be "x is Mary"

  Rephrasing it for simplicity as "If you are not Mary, you like Mary", we have

  $\forall x\Big(\neg M(x) \longrightarrow L(x)\Big)$

  Which, by contrapositive could be written as "If you do not like Mary, you are Mary"

  $\forall x\Big(\neg L(x) \longrightarrow M(x)\Big)$

- (d) Jane saw a police officer, and Roger saw one too.

  Could be rewritten as a relationship clause, as $\exists y\Big[P(y) \wedge S(y,x)\Big]$ where $P(y)$ stands for “y is a police officer” and $S(y,x)$ stands for “y has been seen by x”

  $\exists y\Big[P(y) \wedge S(y, Jane)\Big] \wedge \exists x\Big[P(x) \wedge S(x, Roger)\Big]$

- (e) Jane saw a police officer, and Roger saw him too.

  Could be rewritten as "There was a police officer that was seen by Jane and by Roger

  $\exists x\Big[P(x) \wedge S(x, Jane) \wedge S(x, Roger)\Big]$

2. Analyze the logical forms of the following statements.

- (a) Anyone who has bought a Rolls Royce with cash must have a rich uncle.

  Anyone means everyone, in the sentence above.

  “If x has bought a Rolls Royce with cash, then x has a rich uncle”

  Let $B(x)$ stand for “x has bought a Rolls Royce with cash, $R(x)$ b.”x has a rich uncle", then

  $\forall x\Big[B(x) \longrightarrow R(x)\Big]$

- (b) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.

  Anyone means “somebody” in the sentence above, rewriting it we have:

  If somebody in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.

  Let

Statement	Be
$D(x)$	x is in the dorm
$M(x)$	x has the measles
$Q(x)$	x will be quarantined
$F(x,y)$	x is a friend of y

  then

  $\exists x\Bigg[D(x) \wedge M(x)\Bigg] \longrightarrow \forall y\Bigg[D(z) \wedge F(y,z) \longrightarrow Q(y)\Bigg]$

- (c) If nobody failed the test, then everybody who got an A will tutor someone who got a D.

  $\neg\exists x\Big[F(x)\Big] \longrightarrow \forall x \forall y\Bigg[\Big(A(x) \wedge D(y)\Big) \longrightarrow T(x,y)\Bigg]$

where

Statement	Be
$F(x)$	x failed the test
$A(x)$	x got an A
$D(x)$	x got a D
$T(x,y)$	x will tutor y

- (d) If anyone can do it, Jones can.

  Let $C(x)$ be "x can do it", then

  $\forall x \exists y \Big[C(x) \longrightarrow (y = Jones) \land C(y)\Big]$

- (e) If Jones can do it, anyone can.

  $\forall x \exists y \Big[(y = Jones) \land C(y) \longrightarrow C(x)\Big]$

3. Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?

- (a) Every number that is larger than x is larger than y.

  $\forall z \in \mathbb{R}(z > x \longrightarrow z > y)$, z is bound, as x and y have to be provided (they are free).

- (b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution $iff\ a \geq -2$.

  $\forall a \Big[S(ax^2 + 4x - 2 = 0) \iff a \geq -2\Big]$, where $S(z)$ states that "z has at least one solution". The free variable is x

  $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 + 8a}}{2a}$

  Pluging in to stay in $\mathbb{R}$ we need the term $16 + 8a$ to always be greater or equal than 0, which means $16 + 8a \geq 0$, then $a \geq -2$

- (c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.

  "If y is a solution to the inequality $x^3 - 3x < 3$, then y is smaller than 10"

  $\forall y \Big[S(y, x^3 - 3x < 3) \longrightarrow x < 10\Big]$, where $S(x, y)$ stands for "x is a solution to the y"

- (d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is between -10 and 10.

  $\exists w \left[\exists x \big(x^2 + 5x = w\big) \land \exists y \big(4 - y^2 = w\big)\right] \longrightarrow -10 \leq w \leq 10$

***4. Translate the following statements into idiomatic English.**

- (a) $\exists x \left[\Big(H(x) \land \neg \exists y M(x,y) \Big) \longrightarrow U(x) \right]$, where $H(x)$ means "x is a man", $M(x,y)$ means "x is married to y", and $U(x)$ means "x is unhappy."

  Any man who is not married is unhappy.

- (b) $\exists z \left(P(z,x) \land S(z,y) \land W(y) \right)$, where $P(z,x)$ means "z is a parent of x", $S(z,y)$ means "z and y are siblings", and $W(y)$ means "y is a woman".

  Someone is a parent and has a sister

5. Translate the following statements into idiomatic mathematical English.

- (a) $\forall x \left[\Big(P(x) \land \neg(x=2) \Big) \longrightarrow O(x) \right]$, where $P(x)$ means "x is a prime number" and $O(x)$ means "x is odd."

  For all x, if x is a prime number and is not equal to two, then prime is an odd number.

- (b) $\exists x \left[P(x) \land \forall y \Big(P(y) \longrightarrow y \leq x \Big) \right]$, where $P(x)$ means "x is a perfect number".

  If there is an x such that x is a perfect number, and for all y that is also a perfect number, then y is less than or equal to x.

6. Are these statements true or false? The universe of discourse is the set of all people, and P(x, y) means "x is a parent of y."

- (a) $\exists x \forall y P(x,y)$.

  There is some person x for which the statement $\forall y P(x,y)$ is true, that is to say, that for every person y, y would be a child of x. This statement is $false$.

- (b) $\forall x \exists y P(x,y)$.

  For all people, there are some other people who are their children. True.

- (c) $\neg \exists x \exists y P(x,y)$.

  There is not somebody for which some other people are their child. False

- (d) $\exists x \neg \exists y P(x,y)$.

  There is some person for which there is not someone who is their child. True.

- (e) $\exists x \exists y \neg P(x,y)$.

  There is some person x for which there is some other person y,whom x is not their parent. True.

***7. Are these statements true or false? The universe of discourse is $\mathbb{N}$.**

- (a) $\forall x \exists y(2x - y = 0)$.
  - For all natural numbers, there is a number y for which the equation $2x - y = 0$ is true.
- (b) $\exists y \forall x(2x - y = 0)$.
  - There is some number $y \in \mathbb{N}$ for which every natural number x satisfies the equation $2x - y = 0$. This is false, as for any $y \in \mathbb{N}$, there is only one $x \in \mathbb{N}$ which satisfies the equation.
- (c) $\forall x \exists y(x - 2y = 0)$.
  - For all $x \in \mathbb{N}$ there exists at least one number $y \in \mathbb{N}$ that satisfies the equation $x - 2y = 0$, this is false, as for $x < 2$, the equation cannot be satisfied.
- (d) $\forall x\Big(x < 10 \longrightarrow \forall y(y < x \longrightarrow y < 9)\Big)$.
  - For all $x \in \mathbb{N}$, if $x < 10$, then for all $y \in \mathbb{N}$, if $y < x$ then $y < 9$. True.
- (e) $\exists y \exists z(y + z = 100)$.
  - There is at least one $y \in \mathbb{N}$ and one $z \in \mathbb{N}$ that satisfy the equation $y + z = 100$. True.
- (f) $\forall x \exists y\Big(y > x \land \exists z(y + z = 100)\Big)$.
  - This is false, as for any $x > 100$, there is no z that can satisfy the equation $y + z = 100$.

Chapter 2.2

***1. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)**

- (a) Everyone who is majoring in math has a friend who needs help with his homework.

  Let $M(x)$ be "x is majoring in math", $H(x)$ be "x needs help with his homework", and $F(x, y)$ be "x is a friend of y".

  Then we have the statement in (a) expressed as

$$\forall x\Big(M(x) \longrightarrow \exists y\big(F(x, y) \land H(y)\big)\Big)$$

Negating this statement:

$$\neg\forall x\Big(M(x) \longrightarrow \exists y\big(F(x, y) \land H(y)\big)\Big)$$

$$\exists x \neg\Big(\neg M(x) \lor \exists y\big(F(x, y) \land H(y)\big)\Big)$$

$$\exists x\Big(M(x) \wedge \neg\exists y\big(F(x,y) \wedge H(y)\big)\Big)$$

$$\exists x\Big(M(x) \wedge \forall y\neg\big(F(x,y) \wedge H(y)\big)\Big)$$

$$\exists x\Big(M(x) \wedge \forall y\big(\neg F(x,y) \vee \neg H(y)\big)\Big)$$

There is someone majoring in math, and everyone who is not their friend, does not need help with their homework.

- (b) Everyone has a roommate who dislikes everyone.

  Let $R(x,y)$ be "x is a roommate of y", $L(x,y)$ be "x likes y"

  $\forall x\exists y(R(x,y) \longrightarrow \forall z\neg L(y,z))$

  Then

  $\neg\Big[\Big(\forall x\exists y(R(x,y) \longrightarrow \forall z\neg L(y,z)\Big)\Big]$

  $\exists x\neg\exists y(R(x,y) \longrightarrow \forall z\neg L(y,z))$

  $\exists x\forall y\neg(\neg R(x,y) \vee \forall z\neg L(y,z))$

  $\exists x\forall y(R(x,y) \wedge \neg\forall z\neg L(y,z))$

  $\exists x\forall y(R(x,y) \wedge \exists z L(y,z))$

- (c) $A \cup B \subseteq C \setminus D$

  $\forall x\Big(x \in (A \cup B) \longrightarrow x \in (C \setminus D)\Big)$

  $\forall x\Big(\big(x \in A \vee x \in B\big) \longrightarrow \big(x \in C \wedge x \notin D\big)\Big)$

  $\forall x\Big(\neg\big(x \in A \vee x \in B\big) \vee \big(x \in C \wedge x \notin D\big)\Big)$

  Negating:

  $\neg\forall x\Big(\neg\big(x \in A \vee x \in B\big) \vee \big(x \in C \wedge x \notin D\big)\Big)$

  $\exists x\neg\Big(\neg\big(x \in A \vee x \in B\big) \vee \big(x \in C \wedge x \notin D\big)\Big)$

  $\exists x\Big(\big(x \in A \vee x \in B\big) \wedge \neg\big(x \in C \wedge x \notin D\big)\Big)$

  $\exists x\Big(\big(x \in A \vee x \in B\big) \wedge \big(x \in D \vee x \notin C\big)\Big)$

  $\exists x\Big(\neg\big(x \notin A \wedge x \notin B\big) \wedge \big(x \in D \vee x \notin C\big)\Big)$

$\exists x\Big(x \notin A \cap B \longrightarrow x \in D \setminus C\Big)$

- (d) $\exists x \forall y[y > x \longrightarrow \exists z(z^2 + 5z = y)]$.

  $= \neg\exists x \forall y[y > x \longrightarrow \exists z(z^2 + 5z = y)]$

  $\forall x \neg\forall y[y > x \longrightarrow \exists z(z^2 + 5z = y)]$

  $\forall x \exists y \neg[y < x \vee \exists z(z^2 + 5z = y)]$

  $\forall x \exists y[y > x \wedge \neg\exists z(z^2 + 5z = y)]$

  $\forall x \exists y[y > x \wedge \forall z \neg(z^2 + 5z = y)]$

  $\forall x \exists y[y > x \wedge \forall z(z^2 + 5z \neq y)]$

2. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)

- (a) There is someone in the freshman class who doesn't have a roommate.

  Let $F(x)$ be "x is in the freshman class", $R(x, y)$ be "x and y are roommates"

  Then we have $\exists x\Big(F(x) \wedge \forall y \neg R(x, y)\Big)$

  $\neg\exists x\Big(F(x) \wedge \forall y \neg R(x, y)\Big)$

  $\forall x\Big(\neg F(x) \vee \exists y R(x, y)\Big)$

  $\forall x\Big(F(x) \implies \exists y R(x, y)\Big)$

  For everyone who is in the freshman class, there is at least one person who has a roommate.

- (b) Everyone likes someone, but no one likes everyone.

  Let $L(x, y)$ be "x likes y", then

  $\forall x \exists y\Big(L(x, y)\Big) \wedge \neg\exists x \forall y\Big(L(x, y)\Big)$

  $\neg\Bigg[\forall x \exists y\Big(L(x, y)\Big) \wedge \neg\exists x \forall y\Big(L(x, y)\Big)\Bigg]$

  $\neg\forall x \exists y\Big(L(x, y)\Big) \vee \neg\neg\exists x \forall y\Big(L(x, y)\Big)$

  $\exists x \neg\exists y\Big(L(x, y)\Big) \vee \exists x \forall y\Big(L(x, y)\Big)$

  $\exists x \exists y \neg L(x, y) \vee \exists x \forall y L(x, y)$

  Either there is at least one person x, who dislikes a different person y, or that person x likes everyone.

- (c) $\forall a \in A \exists b \in B(a \in C \iff b \in C)$.

  $$\forall a\left(a \in A \implies \exists b\Big(b \in B \implies (a \in C \iff b \in C)\Big)\right)$$

  $$\forall a\left(a \in A \implies \exists b\Big(b \in B \implies (a \in C \implies b \in C) \wedge (B \in C \implies a \in C)\Big)\right)$$

  $$\forall a\left(a \notin A \vee \exists b\Big(b \notin B \vee (a \notin C \vee b \in C) \wedge (b \notin C \vee a \in C)\Big)\right)$$

  Then

  $$\neg\forall a\left(a \notin A \vee \exists b\Big(b \notin B \vee ((a \notin C \vee b \in C) \wedge (b \notin C \vee a \in C)\Big)\right)$$

  $$\exists a\neg\left(a \notin A \vee \exists b\Big(b \notin B \vee ((a \notin C \vee b \in C) \wedge (b \notin C \vee a \in C))\Big)\right) \text{ (quantifier negation)}$$

  $$\exists a\left(a \in A \wedge \neg\exists b\Big(b \notin B \vee ((a \notin C \vee b \in C) \wedge (b \notin C \vee a \in C))\Big)\right) \text{ (DeMorgan's law)}$$

  $$\exists a\left(a \in A \wedge \forall b\neg\Big(b \notin B \vee ((a \notin C \vee b \in C) \wedge (b \notin C \vee a \in C))\Big)\right) \text{ (quantifier negation)}$$

  $$\exists a\left(a \in A \wedge \forall b\Big(b \in B \wedge ((a \in C \wedge b \notin C) \vee (b \in C \wedge a \notin C))\Big)\right) \text{ (DeMorgan's law)}^3$$

- (d) $\forall y > 0 \exists x(ax^2 + bx + c = y)$.

  $\forall y(y > 0 \implies \exists x(ax^2 + bx + c = y))$ (expand)

  Then

  $\neg\forall y(y > 0 \implies \exists x(ax^2 + bx + c = y))$

  $\exists y\neg(\neg(y > 0) \vee \neg\exists x(ax^2 + bx + c = y))$

  $\exists y\neg(y < 0 \vee \forall x\neg(ax^2 + bx + c = y))$

  $\exists y\neg(y < 0 \vee \forall x(ax^2 + bx + c \neq y))$

  $\exists y(\neg(y < 0) \wedge \neg\forall x(ax^2 + bx + c \neq y))$

  $\exists y(y > 0 \wedge \exists x\neg(ax^2 + bx + c \neq y))$

  $\exists y(y > 0 \wedge \exists x(ax^2 + bx + c = y))$

3. Are these statements true or false? The universe of discourse is $\mathbb{N}$.

- (a) $\exists x(x < 7 \implies \exists a \exists b \exists c(a^2 + b^2 + c^2 = x))$.

  $x \in \{-\infty, \ldots, 0, 1, 2, 3, 4, 5, 6\}$, since our statement does not state that a, b or c have to be different, with $x = 6$ and $a = b = 1, c = 2$, the equation $a^2 + b^2 + c^2 = x$ is true $(1 + 1 + 4 = 6)$.

- (b) $\exists! x((x - 4)^2 = 9)$.
  - True, since 9 is a perfect square, there is only one value in $\mathbb{N}$ for which $(x - 4)^2 = 9$, which would be $x = 7$
- (c) $\exists! x((x - 4)^2 = 25)$.
  - True, for the same reason as (b): only one value for $x \in \mathbb{N}$ satisfies $(x - 4)^2 = 25$
- (d) $\exists x \exists y((x - 4)^2 = 25 \wedge (y - 5)^2 = 2)$.
  - False, as there is no $y \in \mathbb{N}$ that satisfies $(y - 5)^2 = 2$

***4. Show that the second quantifier negation law, which says that $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$, can be derived from the first, which says that $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$. (Hint: Use the double negation law.)** $\neg\exists x P(x) \equiv \forall x \neg P(x)$

$\neg\Big(\neg\exists x P(x)\Big) \equiv \neg\Big(\forall x \neg P(x)\Big)$

$\neg\neg\exists x P(x) \equiv \neg\forall x \neg P(x)$

$\neg\neg\exists x P(x) \equiv \exists x \neg\neg P(x)$

$\exists x P(x) \equiv \exists x \neg\neg P(x)$

$\exists x P(x) \equiv \exists x P(x)$

***6. Show that the existential quantifier distributes over disjunction. In other words, show that $\exists x\Big(P(x) \vee Q(x)\Big)$ is equivalent to $\exists x P(x) \vee \exists x Q(x)$. (Hint: Use the fact, discussed in this section, that the universal quantifier distributes over conjunction.)** Show that

$\exists x\Big(P(x) \vee Q(x)\Big) \equiv \exists x P(x) \vee \exists x Q(x)$

Negating both sides of the equivalence

$\neg\exists x\Big(P(x) \vee Q(x)\Big) \equiv \neg\Big(\exists x P(x) \vee \exists x Q(x)\Big)$

$\forall x \neg\Big(P(x) \vee Q(x)\Big) \equiv \neg\Big(\exists x P(x) \vee \exists x Q(x)\Big)$

$\forall x\Big(\neg P(x) \wedge \neg Q(x)\Big) \equiv \neg\Big(\exists x P(x) \vee \exists x Q(x)\Big)$

$\forall x \neg P(x) \wedge \forall x \neg Q(x) \equiv \neg\Big(\exists x P(x) \vee \exists x Q(x)\Big)$

Using the quantifier negation laws once more

$\neg\exists x P(x) \wedge \neg\exists x Q(x) \equiv \neg\Big(\exists x P(x) \vee \exists x Q(x)\Big)$

Through DeMorgan's law

$\neg\Big(\exists x P(x) \vee \exists x Q(x)\Big) \equiv \neg\Big(\exists x P(x) \vee \exists x Q(x)\Big)$

$\neg\neg\Big(\exists x P(x) \vee \exists x Q(x)\Big) \equiv \neg\neg\Big(\exists x P(x) \vee \exists x Q(x)\Big)$

$\exists x P(x) \vee \exists x Q(x) \equiv \exists x P(x) \vee \exists x Q(x)$

7. Show that $\exists x\Big(P(x) \implies Q(x)\Big)$ **is equivalent to** $\forall x P(x) \implies \exists x Q(x)$**.** $\exists x\Big(P(x) \implies Q(x)\Big) \equiv \forall x P(x) \implies \exists x Q(x)$

$\exists x\Big(\neg P(x) \vee Q(x)\Big) \equiv \forall x P(x) \implies \exists x Q(x)$

Substituting $\neg P(x) \vee Q(x)$ by its DeMorgan equivalent $\neg(P(x) \wedge \neg Q(x))$

$\exists x \neg\Big(P(x) \wedge \neg Q(x)\Big) \equiv \forall x P(x) \implies \exists x Q(x)$

$\neg\forall x\Big(P(x) \wedge \neg Q(x)\Big) \equiv \forall x P(x) \implies \exists x Q(x)$

Since the universal quantifier distributes over conjunction

$\neg\Big(\forall x P(x) \wedge \forall x \neg Q(x)\Big) \equiv \forall x P(x) \implies \exists x Q(x)$

$\neg\Big(\forall x P(x) \wedge \neg\exists x Q(x)\Big) \equiv \forall x P(x) \implies \exists x Q(x)$

DeMorgan's law:

$\neg\forall x P(x) \vee \neg\neg\exists x Q(x) \equiv \forall x P(x) \implies \exists x Q(x)$

$\neg\forall x P(x) \vee \exists x Q(x) \equiv \forall x P(x) \implies \exists x Q(x)$

Conditional law ($P \implies Q \equiv \neg P \vee Q$)

$\forall x P(x) \implies \exists x Q(x) \equiv \forall x P(x) \implies \exists x Q(x)$

***8. Show that** $(\forall x \in A P(x)) \wedge (\forall x \in B P(x))$ **is equivalent to** $\forall x \in (A \cup B) P(x)$**. (Hint: Start by writing out the meanings of the bounded quantifiers in terms of unbounded quantifiers.)** $(\forall x \in A P(x)) \wedge (\forall x \in B P(x)) \equiv \forall x \in (A \cup B) P(x)$.

$(\forall x(x \in A \implies P(x))) \wedge (\forall x(x \in B \implies P(x))) \equiv \forall x \in (A \cup B) P(x)$. (expand)

$\forall x(x \notin A \vee P(x)) \wedge \forall x(x \notin B \vee P(x)) \equiv \forall x \in (A \cup B) P(x)$. (conditional law)

$\forall x\Big((x \notin A \vee P(x)) \wedge (x \notin B \vee P(x))\Big) \equiv \forall x \in (A \cup B) P(x)$. (distribute over conjunction)

$\forall x\Big(\Big(x \notin A \wedge x \notin B\Big) \vee P(x)\Big) \equiv \forall x \in (A \cup B) P(x)$. (distribute law)

$\forall x\Big(\neg\Big(x \in A \vee x \in B\Big) \vee P(x)\Big) \equiv \forall x \in (A \cup B) P(x)$. (DeMorgan's law)

$\forall x\Big(\neg\Big(x \in (A \cup B)\Big) \vee P(x)\Big) \equiv \forall x \in (A \cup B)P(x)$. (Definition of $\cup$)

$\forall x\Big(x \in (A \cup B) \implies P(x)\Big) \equiv \forall x \in (A \cup B)P(x)$. (Definition of $\implies$)

$\forall x \in (A \cup B)P(x) \equiv \forall x \in (A \cup B)P(x)$. (abbreviate)

9. Is $\forall x(P(x) \vee Q(x))$ equivalent to $\forall xP(x) \vee \forall xQ(x)$? Explain. (Hint: Try assigning meanings to $P(x)$ and $Q(x)$.) Let $P(x)$ be "x flag has the color red", and $Q(x)$ be "x flag has the color blue", then our statements become

(i) $\forall x(P(x) \vee Q(x))$ = all flags have either the color red or the color blue

(ii) $\forall xP(x) \vee \forall xQ(x)$ = Either all flags have the color red, or all flags have the color blue

The two statements suggest very different things: (i) says that any flag can have any one of the colors red or blue, or both, so long as they have one, while (ii) is an absolute statement about all the flags: they all have red, or they all have blue.

10. (a) Show that $\exists x \in AP(x) \vee \exists x \in BP(x)$ is equivalent to $\exists x \in (A \cup B)P(x)$.

$\exists x \in AP(x) \vee \exists x \in BP(x) \equiv \exists x \in (A \cup B)P(x)$.

$\exists x\big(x \in A \implies P(x)\big) \vee \exists x\big(x \in B \implies P(x)\big) \equiv \exists x \in (A \cup B)P(x)$ (expand)

$\exists x\neg\big(x \in A \wedge \neg P(x)\big) \vee \exists x\neg\big(x \in B \wedge \neg P(x)\big) \equiv \exists x \in (A \cup B)P(x)$ (conditional law)

Since the existential quantifier distributes over disjunction (see exercise 6)

$\exists x\Big[\neg\big(x \in A \wedge \neg P(x)\big) \vee \neg\big(x \in B \wedge \neg P(x)\big)\Big] \equiv \exists x \in (A \cup B)P(x)$ ($\exists$ disjunction distribution)

$\exists x\neg\Big[\big(x \in A \wedge \neg P(x)\big) \wedge \big(x \in B \wedge \neg P(x)\big)\Big] \equiv \exists x \in (A \cup B)P(x)$ (DeMorgan's law)

$\exists x\neg\Big[\big(x \notin A \vee P(x)\big) \wedge \big(x \notin B \vee P(x)\big)\Big] \equiv \exists x \in (A \cup B)P(x)$ (DeMorgan's law)

$\exists x\neg\Big[P(x) \vee \big(x \notin A \wedge x \notin B\big)\Big] \equiv \exists x \in (A \cup B)P(x)$ (Distribuitve property)

$\exists x\neg\Big[P(x) \vee \neg(x \in A \vee x \in B)\Big] \equiv \exists x \in (A \cup B)P(x)$ (DeMorgan's law)

$\exists x\neg\Big[P(x) \vee \neg(x \in A \cup B)\Big] \equiv \exists x \in (A \cup B)P(x)$ (union definition)

$\exists x\Big[\neg P(x) \wedge x \in A \cup B\Big] \equiv \exists x \in (A \cup B)P(x)$ (DeMorgan's)

Change $\neg P(x) \wedge x \in A \cup B$ for its Demorgan's equivalent

$\exists x\Big[P(x) \vee x \notin A \cup B\Big] \equiv \exists x \in (A \cup B)P(x)$ (DeMorgan's)

$\exists x\Big[x \notin A \cup B \vee P(x)\Big] \equiv \exists x \in (A \cup B)P(x)$ (reordering)

$\exists x\Big[x \in A \cup B \implies P(x)\Big] \equiv \exists x \in (A \cup B)P(x)$ (conditional law)

$\exists x \in (A \cup B)P(x) \equiv \exists x \in (A \cup B)P(x)$ (abbreviate)

(b) Is $\exists x \in AP(x) \land \exists x \in BP(x)$ equivalent to $\exists x \in (A \cap B)P(x)$?

No, because the two terms in $\exists x \in AP(x) \land \exists x \in BP(x)$ could refer to different entities. It suffices to change the bound variable in one of the terms to see it:

$\exists x \in AP(x) \land \exists y \in BP(y)$

***11 Show that the statements $A \subseteq B$ and $A \setminus B = \varnothing$ are equivalent by writing each in logical symbols then showing that the resulting formulas are equivalent.** $A \subseteq B \equiv A \setminus B = \varnothing$

$\forall x(x \in A \longrightarrow x \in B) \qquad \equiv \qquad \forall x(x \in A \land x \notin B) = \varnothing$

$\forall x(x \notin A \lor x \in B) \qquad \equiv \qquad \forall x(x \in A \land x \notin B) = \varnothing$

Changing the term $x \notin A \lor x \in B$ for its DeMorgan equivalent, we get

$\forall x(x \in A \land x \notin B) \qquad \equiv \qquad \forall x(x \in A \land x \notin B) = \varnothing$

12. Show that the statements $C \subseteq A \cup B$ and $C \setminus A \subseteq B$ are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent.

Chapter 2.3

***1. Analyze the logical forms of the following statements. You may use the symbols $\in$, $\notin$, $=$, $\neq$, $\land$, $\lor$, $\implies$, $\iff$, $\forall$ and $\exists$ in your answers, but not $\subseteq$, $\nsubseteq$, $\mathscr{P}$, $\cap$, $\cup$, $\setminus$, $\{$, $\}$, or $\neg$. (Thus, you must write out the definitions of some set theory notation, and you must use equivalences to get rid of any occurrences of $\neg$.)**

- (a) $\mathcal{F} \subseteq \mathscr{P}(A)$.

  By the definition of the subset, every element of $\mathcal{F}$ is also an element of $\mathscr{P}(A)$, then

  $\forall x(x \in \mathcal{F} \implies x \in \mathscr{P}(A))$

  Rewriting $x \in \mathscr{P}(A)$ as $x \subseteq A$

  $\forall x(x \in \mathcal{F} \implies x \subseteq A)$

  $\forall x(x \in \mathcal{F} \implies \forall y(y \in x \implies y \in A))$

- (b) $A \subseteq \{2n+1 \mid n \in \mathbb{N}\}$.

  $\forall x(x \in A \implies x \in \{2n+1 \mid n \in N\})$

  $\forall x(x \in A \implies \exists n \in \mathbb{N}(x = 2n+1))$

- (c) $\{n^2+n+1 \mid n \in \mathbb{N}\} \subseteq \{2n+1 \mid n \in \mathbb{N}\}$.

  $\forall n\Big(\{n^2+n+1 \mid n \in \mathbb{N}\} \implies \{2n+1 \mid n \in \mathbb{N}\}\Big)$

  $\forall n\Big(\exists m \in \mathbb{N}(\{n^2+n+1 \mid n \in \mathbb{N}\} \implies \{2n+1 \mid n \in \mathbb{N}\}\Big)$

$\exists n \in \mathbb{N} \exists m \in \mathbb{N}(n^2 + n + 1 = 2m + 1)$

- (d) $\mathscr{P}(\cup_{i \in I} A_i) \nsubseteq \cup_{i \in I} \mathscr{P}(A_i)$.

  Solution: $\exists x(\forall y(y \in x \implies \exists i \in I(y \in A_i)) \land \forall i \in I \exists y(y \in x \land y \notin A_i))$

  Quoting example 2.3.6: to say that one set is not a subset of another means that there is something that is an element of the first but not the second.

  $$\exists x\Bigg(x \in \mathscr{P}(\cup_{i \in I} A_i) \land x \notin \cup_{i \in I} \mathscr{P}(A_i)\Bigg)$$

  $$\exists x\Bigg(\forall y\Big(y \in x \implies y \in \cup_{i \in I} A_i\Big) \land \neg\Big(x \in \cup_{i \in I} \mathscr{P}(A_i)\Big)\Bigg)$$

  $$\exists x\Bigg(\forall y\Big(y \in x \implies \exists i \in I(y \in A_i)\Big) \land \neg\Big(\exists i \in I(x \subseteq A_i)\Big)\Bigg)$$

  $$\exists x\Bigg(\forall y\Big(y \in x \implies \exists i \in I(y \in A_i)\Big) \land \forall i \in I \neg \forall y(y \in x \implies y \in A_i)\Bigg)$$

  $$\exists x\Bigg(\forall y\Big(y \in x \implies \exists i \in I(y \in A_i)\Big) \land \forall i \in I \exists y \neg(y \notin x \lor y \in A_i)\Bigg)$$

  $$\exists x\Bigg(\forall y\Big(y \in x \implies \exists i \in I(y \in A_i)\Big) \land \forall i \in I \exists y(y \in x \land y \notin A_i)\Bigg)$$

2. Analyze the logical forms of the following statements. You may use the symbols $\in$, $\notin$, $=$, $\neq$, $\land$, $\lor$, $\implies$, $\iff$, $\forall$ and $\exists$ in your answers, but not $\subseteq$, $\nsubseteq$, $\mathscr{P}$, $\cap$, $\cup$, $\setminus$, $\{$, $\}$, or $\neg$. (Thus, you must write out the definitions of some set theory notation, and you must use equivalences to get rid of any occurrences of $\neg$.)

- (a) $x \in \cup\mathcal{F} \setminus \cup\mathcal{G}$.

  $x \in \cup\mathcal{F} \land \neg(x \in \cup\mathcal{G})$ (definition of $\setminus$)

  $\exists A(A \in \mathcal{F} \land x \in A) \land \neg(\exists A(A \in \mathcal{G} \land x \in A))$ (definition of $\cup\mathcal{F}$)

  $\exists A(A \in \mathcal{F} \land x \in A) \land \forall A \neg(A \in \mathcal{G} \land x \in A)$ (quantifier negation)

  $\exists A(A \in \mathcal{F} \land x \in A) \land \forall A(A \notin \mathcal{G} \lor x \notin A)$ (DeMorgan's law)

- (b) $\{x \in B \mid x \notin C\} \in \mathscr{P}(A)$.

  $\exists x(x \in B \land x \notin C) \subseteq A$

  $\forall y\Big(y \in \exists x(x \in B \land x \notin C) \implies y \in A\Big)$

- (c) $x \in \cap_{i \in I}(A_i \cup B_i)$.

- (d) $x \in (\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$.